VARIOUS NUMERICAL INVARIANTS FOR ISOLATED SINGULARITIES

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1. Introduction. In the theory of isolated singularities, one always wants to find invariants associated to the isolated singularities. Hopefully with enough invariants found, one can distinguish between isolated singularities. However, not many invariants are known.

Definition 0. Let V be a Stein analytic space with x as its only singular point. Let $\pi : M \to V$ be a resolution of the singularity of V. We shall denote dim $H^i(M, \mathcal{O})$, $1 \leq i \leq n - 1$ by $h^{(i)}$, and dim $H^q(M, \Omega^p)$ for $1 \leq p \leq n$, $1 \leq q \leq n$ by $h^{p,q}(M)$.

So far as the classification problem is concerned, $h^{(n-1)}$ is one of the most important invariants. In this paper, we shall introduce a bunch of invariants (cf. Definition 2.6, Definition 4.1, and Definition 5.1) which are naturally attached to isolated singularities. These invariants are used to characterize the different notions of sheaves of germs of holomorphic differential forms on analytic spaces. Various formulae which relate all these invariants are proved. We also show how to calculate these invariants explicitly.

Our paper is organized as follows. In section 2 we discuss the relationship between three different kinds of sheaves of germs of holomorphic forms introduced by Grauert-Grothendieck, Noether and Ferrari respectively, and the dualizing sheaf on a complex analytic space which admits only isolated singularities. We remark that the torsion sheaf of the sheaf of germs of holomorphic p-forms in the sense of Grauert-Grothendieck was studied by Brieskorn [9], Greuel [48], Kantor [19], Suzuki [38], Vetter [39] and the author [43].

In section 3 we relate the invariant $g^{(n-1)}$ of a hypersurface isolated singularity with analytic invariants and topological invariants of any resolution of the singularity. The Noether's formula for the rank two bundle

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Ω^1 on the strongly pseudo-convex 2-dimensional manifold is proved. We give an example to show how to use this formula.

In section 4 we prove that for the 2-dimensional strongly pseudo-convex manifold M, $\dim H^1(M, \Omega^1) \geq b_2$ where b_2 is the second betti number of M. The equality holds for rational singularities (cf. [51] and [41]) and simple elliptic singularities (Theorem 4.6 below). Theorem 3.2 actually gives the general formula for $\dim H^1(M, \Omega^1)$ in case the singularity is hyper-surface. We study the invariants $h^{(n-1)}$ and $s^{(n-1)}$ (cf. Definition 4.1) for negative line bundles over compact complex manifolds. We would like to mention that Wahl and Pinkham kindly informed us that they have proved that the holomorphic 1-forms can extend across exceptional set, (i.e., $s^{(1)} = 0$) for 2-dimensional rational singularities (cf. [41] and [29]). The same phenomenon occurs for simple elliptic singularities (Corollary 4.9). We give various formulae for $h^{(n-1)}$ and $s^{(n-1)}$ which make them computable explicitly at least for isolated singularities with $\mathbf{C}^*$-action. In Corollary 4.2, we prove that rational singularities of arbitrary dimension with $\mathbf{C}^*$-action have $s^{(n-1)} = 0$.

In section 5 we prove the universal formula for curve singularities which is a generalization of Milnor's formula for plane curve singularities to arbitrary curve singularities and a Noether's formula for arbitrary strongly pseudo-convex manifolds. In the forthcoming paper, we will describe $h^{p,q}(M)$ for strongly pseudo-convex manifolds in more detail.

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2. Characterization of various sheaves of germs of holomorphic forms. In this section, we shall study the relationship between the sheaves of germs of holomorphic forms introduced by Grauert-Grothendieck, Noether and Ferrari. Let us first recall some lemmas in [11]. Let Y be an analytic set in an open set U in $\mathbf{C}^m$. Let $Y^0 = Y$, $Y^1 = Y_{\text{sing}}$ = set of singular points of Y, $Y^2 = Y^1_{\text{sing}}$, etc. Following Ferrari, we make the following definitions

(1) $\mathcal{H}^p = \{\omega \in \Omega^p_U : \omega/(Y \backslash Y_{\text{sing}}) = 0\}$.

(2) Let $\mathcal{H}_1^p$ be the subsheaf of Ω_U^p consisting of the germs of differential forms ω satisfying $\omega/Y_{\text{reg}}^\nu = 0$ for all $\nu \geq 0$ where Y_{reg}^ν is the regular part of Y^ν.

(3) Let $\mathcal{H}_2^p$ be the subsheaf of Ω_U^p consisting of the germs of holomorphic forms ω, for which, for any complex manifold W and any holomorphic map $\varphi: W \to U$, such that $\varphi(W) \subset Y$, one has $\varphi^*(\omega) = 0$; where $\varphi^*: \Omega_U^p \to \Omega_W^p$ is the induced map.

LEMMA 2.1. (see [11]). $\mathcal{H}_1^p = \mathcal{H}_2^p = \mathcal{H}^p$.

Definition 2.2. Let $\mathcal{K}^p = \{f\alpha + dg \wedge \beta : \alpha \in \Omega_U^p; \beta \in \Omega_U^{p-1}; f, g \in \mathcal{I}\}$ where $\mathcal{I}$ is the ideal of Y in $\mathcal{O}_U$. Let $\pi: M \to Y$ be a resolution of singularities of Y. Then Noether sheaf of germs of holomorphic p-forms on Y, $\bar{\Omega}_Y^p := R^0\pi_*\Omega_M^p$. Grauert-Grothendieck sheaf of germs of holomorphic p-forms on $\Omega_Y^p = \Omega_U^p/\mathcal{K}^p$. Ferrari sheaf of germs of holomorphic p-forms on Y, $\tilde{\Omega}_Y^p := \Omega_U^p/\mathcal{H}^p$.

LEMMA 2.3. *There are two short exact sequences*

$$0 \to K^p \to \Omega_Y^p \to \tilde{\Omega}_Y^p \to 0 \tag{2.1}$$

$$0 \to \tilde{\Omega}_Y^p \to \bar{\Omega}_Y^p \to H^p \to 0 \tag{2.2}$$

where both H^p and K^p are coherent sheaves supported on the singular points of Y.

Proof. It is easy.

The following two lemmas were proved before (c.f. [11], [43]).

LEMMA 2.4. *Let $\mathfrak{F}$ be a coherent analytic sheaf on Y. Let s be a section of $\mathfrak{F}$ over X. If* supp $s = X$ *is a nowhere dense proper subvariety of Y then s is a section of the torsion subsheaf of $\mathfrak{F}$.*

LEMMA 2.5. *Let Y be a reduced complex analytic space. Let K^p be defined as in* (2.1). *Then K^p = torsion subsheaf of Ω_Y^p.*

In response to a problem raised by Serre [34, p. 373–374], Siu [36] had the following beautiful solution. Before stating it, let us recall some notations: If $\mathfrak{F}$ is a coherent analytic sheaf on a complex analytic space X, then $S_k(\mathfrak{F})$ denotes the analytic subvariety $\{x \in X : \text{codh}\ \mathfrak{F}_x \leq k\}$. If D is an open subset of X, then $\bar{S}_k(\mathfrak{F}/D)$ denotes the topological closure of

$S_k(\mathfrak{F}/D)$ in X. If V is an analytic subvariety of X, then $\mathcal{H}_V^k(\mathfrak{F})$ denotes the sheaf defined by the presheaf $U \mapsto H_V^k(U, \mathfrak{F})$, where $H_V^k(U, \mathfrak{F})$ is the k-dimensional cohomology group of U with coefficients in $\mathfrak{F}$ and supports in V.

THEOREM (SIU). *Suppose V is an analytic subvariety of a complex analytic space $(X, \mathcal{O})$, q is a nonnegative integer, and $\mathfrak{F}$ is a coherent analytic sheaf on X. Let $\theta: X\backslash V \to X$ be the inclusion map. Then the following three statements are equivalent:*

(i) $\theta_0(\mathfrak{F}/X\backslash V), \ldots, \theta_q(\mathfrak{F}/X\backslash V)$ *(or equivalently* $\mathcal{H}_V^0(\mathfrak{F}), \ldots, \mathcal{H}_V^{q+1}(\mathfrak{F})$*) are coherent on X.*
(ii) *For every $x \in V$, $\theta_0(\mathfrak{F}/X\backslash V)_x, \ldots, \theta_q(\mathfrak{F}/X\backslash V)_x$ (or equivalently $\mathcal{H}_V^0(\mathfrak{F})_x, \ldots, \mathcal{H}_V^{q+1}(\mathfrak{F})_x$) are finitely generated over $\mathcal{O}_x$.*
(iii) dim $V \cap \bar{S}_{k+q+1}(\mathfrak{F}/X\backslash V) < k$ *for every* $k \geq 0$

where $\theta_q(\mathfrak{F})$ is the q-th direct image of $\mathfrak{F}$ under θ.

Let us introduce one more notion of sheaf of germs of differential forms on complex space. Suppose Y is a complex analytic space of dimension n with x as its only singularity. Let $\theta: Y\backslash\{x\} \to Y$ be the inclusion map. Then the 0-th direct image sheaf $\theta_* \Omega^i_{Y\backslash\{x\}} = \bar{\bar{\Omega}}_Y$ is coherent by Siu's Theorem. It is clear that we have an inclusion $\bar{\Omega}_Y^i \hookrightarrow \bar{\bar{\Omega}}_Y^i$. Define J^i by the following exact sequence

$$0 \to \bar{\Omega}_Y^i \to \bar{\bar{\Omega}}_Y^i \to J^i \to 0 \tag{2.3}$$

Remark. In case Y is a normal complex space, then the dualizing sheaf ω_Y of Grothendieck is actually the sheaf $\bar{\bar{\Omega}}_Y^n$, where n is the dimension of Y.

From now on we assume that all singularities whenever they exist are isolated.

Definition 2.6. Let Y be a complex analytic space with $x \in Y$ as an isolated singularity. Let H^i, K^i, and J^i be defined as in (2.1), (2.2), and (2.3). Then the invariants $g^{(i)}$, $m^{(i)}$, and $s^{(i)}$ at x are defined to be $\dim(H^i)_x$, $\dim(K^i)_x$, and $\dim(J^i)_x$ respectively. (See Definition 4.1 for an alternative definition of $s^{(i)}$.)

The following Lemma is obvious.

LEMMA 2.7. *Let n = dimension of Y at x and N = embedding dimension of Y at x. Then $g^{(i)} = 0$ for $i > n$ and $m^{(i)} = 0$ for $i = 0$ and $i > N$. Moreover, if Y is a Stein analytic space with x as its only singular point, let $\pi: M \to Y$ be a resolution of the singularity, then*

$$g^{(i)} = \dim \Gamma(M, \Omega^i)/\pi^*\Gamma(Y, \Omega^i)$$

and

$$m^{(i)} = \dim \operatorname{Ker}(\pi^*: \Gamma(Y, \Omega^i) \to \Gamma(M, \Omega^i)).$$

For normal singularities, $g^{(0)} = 0$. If x is a curve singularity, then $g^{(0)}$ coincides with the analytic invariant δ introduced by Serre.

Remark. Let us recall the definition of δ. Let $\mathcal{O}_{\tilde{Y},x}$ be the integral closure of $\mathcal{O}_{Y,x}$, the local ring at x. Then $\delta = \dim(\mathcal{O}_{\tilde{Y},x}/\mathcal{O}_{Y,x})$. It turns out that δ is exactly the correction term which appears in the general Plucker's formula for singular curve (p. 73–74, [34]).

THEOREM 2.8. *Let $x \in V \subseteq \mathbf{C}^{n+1}$ be an isolated hypersurface singularity, given by $f = 0$. Write*

$$\sigma: \Omega_V^n \to \omega_V = (\Omega_V^n)^{**}$$

for the canonical map; then there is a canonical exact sequence of $\mathcal{O}$-modules

$$0 \to \ker \sigma \to N \xrightarrow{f} N \to \operatorname{coker} \sigma \to 0$$

where $N = \Omega_{\mathbf{C}^{n+1}}^{n+1}/\Omega_{\mathbf{C}^{n+1}}^{n+1} \wedge df$.

In particular

$$\operatorname{coker} \sigma = N/fN = \Omega_{V,x}^{n+1}, \qquad \text{i.e.,} \quad g^{(n)} + h^{(n-1)} = \tau.$$

Here $h^{(n-1)} = \dim H^{n-1}(M, \mathcal{O})$ where $\pi: M \to V$ is a resolution of the singularity of V, and $\tau = \dim \mathbf{C}[[z_i]]/(f, (\partial f/\partial z_i))$ assuming that x is the origin.

Proof. Consider the following exact commutative diagram.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & \Omega^{n+1}_{\mathbf{C}^{n+1}} \oplus \Omega^{n}_{\mathbf{C}^{n+1}} & \xrightarrow{(0,\wedge df)} & \Omega^{n+1}_{\mathbf{C}^{n+1}} & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (\wedge df, f)} & & {\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow & & \\
\Omega^{n-1}_{\mathbf{C}^{n+1}} & \xrightarrow{\wedge df} & \Omega^{n}_{\mathbf{C}^{n+1}} & \xrightarrow{\wedge df} & \Omega^{n+1}_{\mathbf{C}^{n+1}} & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & {\scriptstyle \mathrm{Res}\left(\frac{\cdot}{f}\right)}\downarrow & & & & \\
 & & \Omega^{n}_{V} & \xrightarrow{\sigma} & \omega_V & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

The first vertical column is the definition of

$$\Omega^n_V = \Omega^n_{\mathbf{C}^{n+1}} / \Omega^{n-1}_{\mathbf{C}^{n+1}} \wedge df + f\Omega^n_{\mathbf{C}^{n+1}}.$$

The second vertical column is the Poincare residue mapping

$$\mathrm{Res}: \Omega^{n+1}(V) \otimes \mathcal{O}_V \to \omega_V.$$

The middle row is the Koszul complex of df—this is exact until the last term because $(\partial f/\partial z_i) \in m \subset \mathcal{O}$ form a regular sequence. The theorem can now be read off the diagram using the snake lemma and the fact that $\dim \omega_V/\pi_*\omega_M = \dim R^{n-1}\pi_*\mathcal{O}_M$. (cf. Theorem A of [47]).

The following Theorem gives a characterization of four different kinds of sheaves of germs of holomorphic p-forms on a complex hypersurface.

THEOREM 2.9. *Let $f(z_0, z_1, \ldots, z_n)$ be holomorphic in N, a Stein neighborhood of the origin in $\mathbf{C}^{n+1}$ with $f(0, 0, \ldots, 0) = 0$. Let $V = \{(z_0, z_1, \ldots, z_n) \in N: f(z_0, z_1, \ldots, z_n) = 0\}$ has the origin as its only singular point. Let $\pi: M \to V$ be a resolution of the singularity of V. Let*

$$\tau = \dim \mathbf{C}[[z_0, z_1, \ldots, z_n]] \Big/ \left(f, \frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)$$

and $h^{(n-1)} = \dim H^{n-1}(M, \mathcal{O})$. *Then*

(a) $\Omega_V^i = \tilde{\Omega}_V^i = \bar{\Omega}_V^i = \bar{\bar{\Omega}}_V^i$ *for* $0 \leq i \leq n-2$ *and* $\Omega_V^{n-1} = \tilde{\Omega}_V^{n-1} \subset \bar{\Omega}_V^{n-1} \subset \bar{\bar{\Omega}}_V^{n+1}$.

(b) $g^{(n)} = \tau - h^{(n-1)}$ *and* $m^{(n)} = \dim \mathbf{C}[[z_0, z_1, \ldots, z_n]]/(f, (\partial f/\partial z_0), (\partial f/\partial z_1), \ldots, (\partial f/\partial z_n))$.

(c) $\tilde{\Omega}_V^{n+1} = 0 = \bar{\Omega}_V^{n+1} = \bar{\bar{\Omega}}_V^{n+1}$, Ω_V^{n+1} *is supported on the origin and* $m^{(n+1)} = \dim \mathbf{C}[[z_0, z_1, \ldots, z_n]]/(f, (\partial f/\partial z_0), \ldots, (\partial f/\partial z_n))$.

Remark 2.10. (a) $\Omega_V^i \cong \tilde{\Omega}_V^i (0 \leq i \leq n-1)$ can be found in ([11], Proposition 2.1; [43], Theorem 2.5 and Theorem 2.7. See also [48], Proposition 1.11(i)) for the relative case

(b) $m^{(n)} = \dim_{\mathbf{C}} \mathbf{C}[z_0, z_1, \ldots, z_n]/(f, (\partial f/\partial z_0), \ldots, (\partial f/\partial z_n))$ is due to Greuel ([48], Proposition 1.11(iii)).

3. Computation for the invariant $g^{(n-1)}$ and Noether's formula for rank 2 bundle on strongly pseudo-convex 2-dimensional manifold. For the case of n-dimensional isolated hypersurface singularities, we computed all the $m^{(i)}$ and $g^{(i)}$ except $g^{(n-1)}$ in the previous section. Now we would like to investigate the more subtle invariant $g^{(n-1)}$. Then we restrict ourself to surface singularities and give a Noether formula for rank 2 bundle Ω^1.

The following lemma which was stated in p. 84 of [43] has some misprints in the last line of that lemma although it does not affect any result in that paper. For the sake of convenience to the reader, we correct the misprints here.

LEMMA 3.1. *Let*

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & A_1 & \xrightarrow{\varphi_1} & A_2' \oplus A_2'' & \xrightarrow{\varphi_2} & A_3 & \xrightarrow{\varphi_3} & A_4 & \xrightarrow{\varphi_4} & A_5' \oplus A_5'' & \xrightarrow{\varphi_5} & A_6 \\
 & & \uparrow{\scriptstyle \pi_1} & & \uparrow{\scriptstyle \pi_2'} \quad \uparrow{\scriptstyle \pi_2''} & & \uparrow{\scriptstyle \pi_3} & & \uparrow{\scriptstyle \pi_4} & & \uparrow{\scriptstyle \pi_5'} \quad \uparrow{\scriptstyle \pi_5''} & & \uparrow{\scriptstyle \pi_6} \\
0 & \longrightarrow & B_1 & \xrightarrow{\psi_1} & B_2' \oplus B_2'' & \xrightarrow{\psi_2} & B_3 & \xrightarrow{\psi_3} & B_4 & \xrightarrow{\psi_4} & B_5' \oplus B_5'' & \xrightarrow{\psi_5} & B_6
\end{array}$$

$$\begin{array}{cccccccc}
\xrightarrow{\varphi_6} & A_7 & \xrightarrow{\varphi_7} & A_8' \oplus A_8'' & \xrightarrow{\varphi_8} & A_9 & \longrightarrow & \cdots\cdots \\
 & \uparrow{\scriptstyle \pi_7} & & \uparrow{\scriptstyle \pi_8'} \quad \uparrow{\scriptstyle \pi_8''} & & \uparrow{\scriptstyle \pi_9} & & \\
\xrightarrow{\psi_6} & B_7 & \xrightarrow{\psi_7} & B_8' \oplus B_8'' & \xrightarrow{\psi_8} & B_9 & \longrightarrow & \cdots\cdots
\end{array}$$

$$\begin{array}{ccccccccc}
\longrightarrow A_{3n-2} & \xrightarrow{\varphi_{3n-2}} & A'_{3n-1} \oplus A''_{3n-1} & \xrightarrow{\varphi_{3n-1}} & A_{3n} & \xrightarrow{\varphi_{3n}} & A_{3n+1} & \xrightarrow{\varphi_{3n+1}} & 0 \\
\uparrow {\scriptstyle \pi_{3n-2}} & & \uparrow {\scriptstyle \pi'_{3n-1}} \quad \uparrow {\scriptstyle \pi''_{3n-1}} & & \uparrow {\scriptstyle \pi_{3n}} & & \uparrow {\scriptstyle \pi_{3n+1}} & & \\
\longrightarrow B_{3n-2} & \xrightarrow{\psi_{3n-2}} & B'_{3n-1} \oplus B''_{3n-1} & \xrightarrow{\psi_{3n-1}} & B_{3n} & \xrightarrow{\psi_{3n}} & B_{3n+1} & \xrightarrow{\psi_{3n+1}} & 0
\end{array}$$

be a commutative diagram with exact rows. Suppose π''_{3i-1}, π_{3i}, $1 \leq i \leq n$ *are isomorphism and all the vector spaces are finite dimensional except possibly* A'_2, B'_2, A''_{3i-1}, A_{3i}, B''_{3i-1}, B_{3i}, $1 \leq i \leq n$. *Suppose also that* $B'_{3i+2} = 0$ *for* $1 \leq i \leq n-1$. *Then*

$$\sum_{i=0}^{n} (-1)^i \dim B_{3i+1} + \sum_{i=1}^{n-1} (-1)^i \dim A'_{3i+2}$$

$$= \sum_{i=0}^{n} (-1)^i \dim A_{3i+1} + \dim \ker \pi'_2 - \dim \operatorname{coker} \pi'_2$$

The following proposition is a consequence of the Mayer-Vietoris sequence [1, p. 236] and the above Lemma.

PROPOSITION 3.2. *Let* $\bar{V} \subseteq \mathbf{P}^m$ *be a projective variety of dimension* n *with* x *as its only singularity. Let* $\pi : \overline{M} \to \bar{V}$ *be a resolution of the singularity of* V. *Let*

$$\chi^p(\overline{M}) = \sum_{q=0}^{n} (-1)^q \dim H^p(\overline{M}, \Omega^p)$$

and

$$\chi^p(\bar{V}) = \sum_{q=0}^{n} (-1)^q \dim H^q(\bar{V}, \Omega^p_V).$$

Then

$$\chi^p(\overline{M}) - \chi^p(\bar{V}) = \sum_{i=1}^{n-1} (-1)^i \dim H^i(M, \mathrm{Q}^p) - m^{(p)} + g^{(p)}$$

where M *is the strictly pseudoconvex neighborhood of* $\pi^{-1}(x)$.

If x is a hypersurface singularity, then

$$\chi^p(\overline{M}) = \chi^p(\bar{V}) + \sum_{i=1}^{n-1} (-1)^i \dim H^i(M, \Omega^p) \quad \text{for} \quad 0 \leq p \leq n-2$$

and

$$\chi^{n-1}(\overline{M}) = \chi^{n-1}(\bar{V}) + \sum_{i=1}^{n-1} (-1)^i \dim H^i(M, \Omega^{n-1}) + g^{(n-1)}.$$

Remark. One can deduce Proposition 3.2 also from Leray spectral sequence.

THEOREM 3.3. *Let $f(z_0, z_1, \ldots, z_n)$ be holomorphic in $N \subseteq \mathbf{C}^{n+1}$, $n \geq 2$, a Stein neighborhood of $(0, 0, \ldots, 0)$ with $f(0, 0, \ldots, 0) = 0$. Let $V = N \cap f^{-1}(0)$ have $(0, 0, \ldots, 0)$ as its only singular point. Let μ be the Milnor number of V at the origin. Let $\pi: M \to V$ be a resolution of V with A as the exceptional set in M. Then*

$$(3.1) \quad g^{(n-1)} = \mu + (-1)^n + (-1)^{n+1}\chi_T(A) + \sum_{p=1}^{n-1} (-1)^{n+p}\chi(\Omega^p) - 2h^{(n-1)}$$

where

$$\chi(\Omega^p) = \sum_{i=1}^{n} (-1)^i \dim H^i(M, \Omega^p),$$

$\chi_T(A) =$ *topological Euler characteristic of the exceptional set A*

$h^{(n-1)} = \dim H^{n-1}(M, \mathcal{O})$.

Proof. The proof is similar to those given in [43].

In order to make the readers feel more comfortable, let us restrict ourself to two dimensional singularities for a moment. Let us recall the Riemann-Roch formula for rank 2 bundle Ω^1 over a compact Kahler surface X. It says that

$$\dim H^0(X, \Omega^1) - \dim H^1(X, \Omega^1) + \dim H^2(X, \Omega^1) = \frac{1}{6}(c_1^2 - 5c_2)$$

where c_1 and c_2 are first and second Chern class of X respectively. In the formula (3.2) below, $\chi_T(A)$ and $K \cdot K$ should be treated as c_2 and c_1^2 respectively. The change of the sign is expected by our definition of $\chi(M, \Omega^1)$.

THEOREM 3.4. *Let $f(x, y, z)$ be holomorphic in N, a Stein neighborhood of $(0, 0, 0)$ with $f(0, 0, 0) = 0$. Let $V = \{(x, y, z) \in N : f(x, y, z) = 0\}$ has $(0, 0, 0)$ as its only singular point. Let*

$$\mu = \dim \mathbf{C}[[x, y, z]]/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

and

$$\tau = \dim \mathbf{C}[[x, y, z]]/\left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

Let $\pi: M \to V$ *be a resolution of* V *and* $A = \pi^{-1}(0, 0, 0)$. *Then*

$$\begin{aligned}(3.2)\qquad \chi(M, \Omega^1) &= \dim \Gamma(M\backslash A, \Omega^1)/\Gamma(M, \Omega^1) + \dim H^1(M, \Omega^1)\\ &= \tau - (\mu + 1) + \chi_T(A) + 2\dim H^1(M, \mathcal{O})\\ &= -\frac{1}{6}(K \cdot K - 5\chi_T(A)) + \tau - \frac{5}{6}(1 + \mu)\end{aligned}$$

where $\chi_T(A)$ is the topological Euler characteristic of A and K is the canonical divisor on M.

Proof. We apply (3.1) for $n = 2$ and get

$$\begin{aligned}(3.3)\qquad & g^{(1)} = \mu + 1 - \chi_T(A) - \chi(\Omega^1) - 2h^{(1)}\\ &\Rightarrow \dim \Gamma(M, \Omega^1)/\pi^*\Gamma(V, \Omega^1)\\ &\qquad = \mu + 1 - \chi_T(A) + \dim H^1(M, \Omega^1)\\ &\qquad\qquad - 2\dim H^1(M, \mathcal{O}).\end{aligned}$$

By the exact sequence

$$\begin{aligned}0 \to \Gamma(M, \Omega^1)/\pi^*\Gamma(V, \Omega^1) &\to \Gamma(M\backslash A, \Omega^1)/\pi^*\Gamma(V, \Omega^1)\\ &\to \Gamma(M\backslash A, \Omega^1)/\Gamma(M, \Omega^1) \to 0\end{aligned}$$

we have

$$\begin{aligned}(3.4)\qquad \dim \Gamma(M, \Omega^1)/\pi^*\Gamma(V, \Omega^1) &= \dim \Gamma(M\backslash A, \Omega^1)/\pi^*\Gamma(V, \Omega^1)\\ &\quad - \dim \Gamma M\backslash A, \Omega^1)/\Gamma(M, \Omega^1).\end{aligned}$$

The following diagram is commutative with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \Gamma(V\backslash\{0\}, \Omega^1) & \xrightarrow{\sim} & \Gamma(M\backslash A, \Omega^1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Gamma(V, K^1) & \longrightarrow & \Gamma(V, \Omega^1) & \longrightarrow & \pi^*\Gamma(V, \Omega^1) & \longrightarrow & 0
\end{array}$$

where K^1 is defined as in Lemma 2.3. By snake lemma

$$\text{(3.5)} \qquad \dim \Gamma(V\backslash\{0\}, \Omega^1)/\Gamma(V, \Omega^1) = \dim \Gamma(M\backslash A, \Omega^1)\pi^*\Gamma(V, \Omega^1).$$

Look at the local cohomology exact sequence

$$0 \to H^0_{\{0\}}(V, \Omega^1) \to H^0(V, \Omega^1) \to H^0(V\backslash\{0\}, \Omega^1)$$

$$\to H^1_{\{0\}}(V, \Omega^1) \to H^1(V, \Omega^1) \to \cdots$$

$H^1(V, \Omega^1) = 0$ because V is Stein. Hence

$$\text{(3.6)} \qquad \dim H^0(V\backslash\{0\}, \Omega^1)/H^0(V, \Omega^1) = \dim H^1_{\{0\}}(V, \Omega^1)$$

$$= \dim \operatorname{Ext}^2_{\mathcal{O}_{\mathbf{C}^3,0}}(\Omega^1_{V,0}, \mathcal{O}_{\mathbf{C}^3,0}).$$

In p. 91, (2.4) of [43], we gave an explicit resolution of $\Omega_{V,0}$ as follows.

$$0 \to \Omega^0_{\mathbf{C}^3} \xrightarrow{\tau_1} \Omega^1_{\mathbf{C}^3} \oplus \Omega^0_{\mathbf{C}^3} \xrightarrow{\tau_0} \Omega^1_{\mathbf{C}^3} \to \Omega^1_{V,0} \to 0$$

is exact at 0 in $\mathbf{C}^3$ where

$$\tau_1(\alpha) = (df \wedge \alpha, -f\alpha) \qquad \alpha \in \Omega^0_{\mathbf{C}^3}$$

and

$$\tau_0(\alpha, \beta) = f\alpha + df \wedge \beta \qquad (\alpha, \beta) \in \Omega^1_{\mathbf{C}^3} \oplus \Omega^0_{\mathbf{C}^3}$$

are $\mathcal{O}$-linear. It follows easily that

$$\dim \operatorname{Ext}^2_{\mathcal{O}_{\mathbf{C}^3,0}}(\Omega^1_{V,0}, \mathcal{O}_{\mathbf{C}^3,0}) = \dim \mathbf{C}[[x, y, z]]/\left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \tau.$$

Therefore from (3.4), (3.5) and (3.6) we have

$$(3.7)\qquad \dim \Gamma(M,\, \Omega^1)/\pi^*\Gamma(V,\, \Omega^1) = \tau - \dim \Gamma(M\backslash A,\, \Omega^1)/\Gamma(M,\, \Omega^1).$$

Put (3.7) into (3.3),

$$(3.8)\quad \dim\Gamma(M\backslash A,\Omega^1)\Gamma(M,\Omega^1) + \dim H^1(M,\Omega^1)$$

$$= \tau - (\mu + 1) + \chi_T(A) + 2\dim H^1(M,\mathcal{O})$$

$$= \tau - (\mu + 1) + \chi_T(A) - \frac{1}{6}(K^2 + \chi_T(A)) + \frac{1}{6}(1 + \mu) \quad \text{(by (3) of [22])}$$

$$= -\frac{1}{6}(K^2 - 5\chi_T(A)) + \tau - \frac{5}{6}(1 + \mu) \qquad \text{Q.E.D.}$$

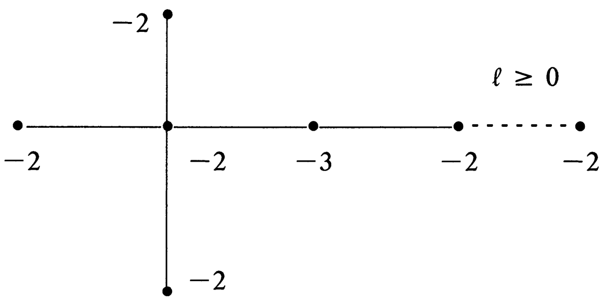
We observe that $\chi(M,\, \Omega^1)$ is not a birational invariant. So it is a good biholomorphic invariant of strongly pseudo-convex 2-dimensional manifold. The right-hand side of (3.2) is explicitly computable. Let us give one example. Before we do that, let us make a corollary first.

COROLLARY 3.5. *Let $g^{(1)}$, μ, $\chi_T(A)$, and K be as in Theorem* 3.4. *Then*

$$(3.9)\qquad g^{(1)} = \frac{5}{6}(1 + \mu) - \frac{5}{6}\chi_T(A) + \frac{1}{6}K^2 + \dim H^1(M,\Omega^1).$$

Proof. This is an easy consequence of (3.3) and (3) of [22].

Example 3.5. Let V be the locus in $\mathbf{C}^3$ of $z^2 = y^3 + x^{9+6\ell}$. Then the dual weighted graph for the exceptional set of the minimal resolution M is



This is a weakly elliptic singularity and the length of the elliptic sequence [45] is equal to $\ell + 1$. It can be calculated that $\mu = 16 + 12\ell$. By Theorem 3.7 of [45] one calculates easily that $K^2 = -(\ell + 1)$. $\chi_T(A) = \ell + 5$. Therefore

$$\chi(M, \mathcal{O}) = \frac{-1}{12}(K^2 + \chi_T(A)) + \frac{1}{12}(1 + \mu)$$

$$= \frac{-1}{12}(-\ell - 1 + \ell + 6) + \frac{1}{12}(1 + 16 + 12\ell) = \ell + 1$$

and

$$\chi(M, \Omega^1) = -\frac{1}{6}(K^2 - 5\chi_T(A)) + \tau - \frac{5}{6}(1 + \mu)$$

$$= -\frac{1}{6}(-\ell - 1 = 5\ell - 30) + 16 + 12\ell - \frac{5}{6}(1 + 16 + 12\ell)$$

$$= 3\ell + 7.$$

4. Some computations for $s^{(n-1)}$, $h^{(n-1)}$ and $h^{p,q}(M)$.

Definition 4.1. Let M be a strongly pseudo-convex manifold of dimension $n \geq 2$. Let A be the maximal compact analytic set in M. Suppose that A is connected i.e. M can be blown down to a Stein space V with x as its only singularity. We define $s^{(i)}$, $0 \leq i \leq n$, of the singularity x to be $\dim \Gamma(M \backslash A, \Omega^i)/\Gamma(M, \Omega^i)$. Recall the definition of $h^{p,q}(M)$ and $h^{(i)}$ in section 1.

THEOREM 4.2 ([14a], [36a] and [47]). *$h^{n,q} = 0$ for $1 \leq q \leq n$; $h^{p,n} = 0$ for $0 \leq p \leq n$; $s^{(n)} = h^{(n-1)}$.*

PROPOSITION 4.3. *Let M, V, A, x, $s^{(i)}$ and $h^{(i)}$ be as above. Then $s^{(i)}$ and $h^{(i)}$ are independent of the choice of M in the following sense. If U is a strongly pseudo-convex neighborhood of the exceptional set A in M, then $s^{(i)}(M) = s^{(i)}(U)$ and $h^{(i)}(M) = h^{(i)}(U)$ for $1 \leq i \leq n - 1$ where $s^{(i)}(M) =$ $\dim \Gamma(M \backslash A, \Omega^i)/\Gamma(M, \Omega^i)$ and $s^{(i)}(U) = \dim \Gamma(U \backslash A, \Omega^i)/\Gamma(U, \Omega^i)$. Moreover if V_1 is any Stein neighborhood of x in V and M_1 is any resolution of V_1, then $s^{(i)}(M) = s^{(i)}(M_1)$ and $h^{(i)}(M) = h^{(i)}(M_1)$, $1 \leq i \leq n - 1$. In any case, $s^{(0)} = 0$.*

Remark. Proposition 4.3 says that $s^{(i)}(0 \leq i \leq n)$ and $h^{(i)}(1 \leq i \leq n-1)$ are actually a numerical invariants associated to the singularity x. Actually from the proof below, it is easy to see that $s^{(i)}$ is finite for $1 \leq i \leq n$.

Proof of Proposition 4.3. Obviously the sequence

$$0 \to \bar{\Omega}_V^i \to \bar{\bar{\Omega}}_V^i \to \bar{\bar{\Omega}}_V^i/\bar{\Omega}_V^i \to 0$$

is exact. $\bar{\bar{\Omega}}_V/\bar{\Omega}_V$ is coherent and supported on $\{x\}$. Since $\Gamma(V, \bar{\Omega}_V^i) = \Gamma(M, \Omega^i)$ and $\Gamma(V, \bar{\bar{\Omega}}_V^i) = \Gamma(M \backslash A, \Omega^i)$, our claim follows from the fact that $H^1(V, \bar{\Omega}_V^i) = 0$ (Cartan Theorem B).

Recall that in [28], Narasimhan proved that given a finitely generated abelian group G and integers $k \geq 1$, $n \geq k+3$, there is a Runge domain D in $\mathbf{C}^n$ with $H_k(D, \mathbf{Z}) \approx G$. However for the strongly pseudo-convex 2-dimensional manifold we have the following.

THEOREM 4.4. *Let M be a two dimensional strongly pseudo-convex manifold in which the exceptional set may admit arbitrary singularities. Then $h^{1,1}(M) = \dim H^1(M, \Omega^1) \geq b_2$ where b_2 is the second betti number of the tubular neighborhood of the exceptional set A of M.*

LEMMA 4.5. *Let B_ϵ be an open ball in $\mathbf{C}^2$ with radius ϵ. Let $\pi: M \to B_\epsilon$ be the quadratic transformation at the origin. Then $H^1(M, \Omega^1) = \mathbf{C}$.*

Proof. Easy.

Proof of Theorem 4.4. We are going to prove that both $h^{1,1}(M)$ and b_2 will increase by one if we apply a quadratic transformation ρ at $x \in A$. The statement for b_2 is obvious. By the Mayer-Vietoris sequence [1, p. 236] argument or Leray spectral sequence,

$$\dim H^1(\tilde{M}, \Omega^1) = \dim H^1(M, \Omega^1) + \dim H^1(\rho^{-1}(D), \Omega^1)$$

where D is a Stein open neighborhood of x

$$\Rightarrow \dim H^1(\tilde{M}, \Omega^1) = \dim H^1(M, \Omega^1) + 1 \quad \text{by Lemma 4.5.}$$

This proves our claim.

By applying finite number of quadratic transformations, we may assume that A has normal crossings. We only need to prove the theorem in this case. Now one can check that the following sequence is exact

$$0 \to \Omega^1_M(\log A)(-A) \to \Omega^1_M \to \bigoplus_1^{b_2} \Omega^1_{A_i} \to 0.$$

Hence

$$H^1(M, \Omega^1_M) \to \bigoplus_1^{b_2} H^1(A_i, \Omega^1_{A_i}) \to H^2(M, \Omega^1_M(\log A)(-A))$$

is exact. By Siu theorem, $H^2(M, \Omega^1_M(\log A)(-A)) = 0$. It follows that

$$h^{1,1}(M) \geq \sum_{i=1}^{b_2} \dim H^1(A_i, \Omega^1_{A_i})$$

$$= b_2 \quad \text{since } A_i\text{'s are nonsingular.}$$ Q.E.D.

The following theorem is the first attempt to compute $h^{(1,1)}(M)$. We shall give a more general formula later.

THEOREM 4.6. *Let M be a two dimensional strongly pseudo-convex manifold M with a nonsingular Riemann surface A of genus g as its maximal compact analytic set. Then*

$$\mathrm{h}^{1,1}(M) = \dim H^1(M, \Omega^1)$$

$$= \dim \Gamma(M, \Omega^1_M \otimes \mathcal{O}_{n_0 A}(n_0 A))$$

where

$$n_0 + 1 = \max\left\{\left[\frac{2-2g}{A \cdot A}\right] + 1, 2\right\}.$$

If $((2 - 2g)/(A \cdot A)) < 1$,*then* $h^{1,1}(M) = \dim H^1(M, \Omega^1) = 1$. *In particular if A is a rational curve or an elliptic curve, then* $h^{1,1}(M) = 1$.

Before we prove Theorem 4.6, let us first recall Wahl's lemma.

LEMMA 4.7 (p. 352, [40a]). *Let* $\mathfrak{F}$ *be any local free sheaf on M. Then* $H^1_A(M, \mathfrak{F}) = \varinjlim_Z H^0(M, \mathfrak{F} \otimes \mathcal{O}_Z(Z))$ *where the limit is taken over effective divisors supported on A.*

Proof of Theorem 4.6. Recall the following exact sequence

$$(4.1) \qquad 0 \to \mathcal{O}_A(-A) \to \Omega^1_M \otimes \mathcal{O}_A \to \Omega^1_A \to 0.$$

Tensor (4.1) with $\mathcal{O}(nA)$, one therefore has an exact sequence

$$0 \to \mathcal{O}_A((n-1)A) \to \Omega_M^1 \otimes \mathcal{O}_A(nA) \to \Omega_A^1(nA) \to 0. \tag{4.2}$$

Apply the long cohomology exact sequence, we get that the sequence

$$0 \to H^0(M, \mathcal{O}_A((n-1)A)) \to H^0(M, \Omega_M^1 \otimes \mathcal{O}_A(nA)) \to H^0(M, \Omega_A^1(nA))$$

is exact. Therefore

$$H^0(M, \Omega_M^1 \otimes \mathcal{O}_A(nA)) = 0 \quad \text{for} \quad n \geq \max\left\{\left[\frac{2-2g}{A \cdot A}\right] + 1, 2\right\} \tag{4.3}$$

since $c_1(\mathcal{O}_A((n-1)A)) < 0$ and $c_1(\Omega_A^1(nA)) < 0$ by hypothesis.

The sequence

$$0 \to \mathcal{O}(-A)/\mathcal{O}(-(n+1)A) \to \mathcal{O}_{(n+1)A} \to \mathcal{O}_A \to 0 \tag{4.4}$$

is exact. Tensor (4.4) with $\mathcal{O}((n+1)A)$, we get exact sequence

$$0 \to \mathcal{O}_{nA}(nA) \to \mathcal{O}_{(n+1)A}((n+1)A) \to \mathcal{O}_A((n+1)A) \to 0 \tag{4.5}$$

Tensor (4.5) with Ω_M^1, we obtain exact sequence

$$\begin{aligned} 0 \to \Omega_M^1 \otimes \mathcal{O}_{nA}(nA) &\to \Omega_M^1 \otimes \mathcal{O}_{(n+1)A}((n+1)A) \\ &\to \Omega_M^1 \otimes \mathcal{O}_A((n+1)A) \to 0 \end{aligned} \tag{4.6}$$

By (4.3) and (4.6),

$$H^0(M, \Omega_M^1 \otimes \mathcal{O}_{nA}(nA)) \cong H^0(M, \Omega_M^1 \otimes \mathcal{O}_{(n+1)A}((n+1)A).$$

Therefore by Lemma 4.7

$$\begin{aligned} \dim H^1(M, \Omega_M^1) &= \dim H_A^1(M, \Omega_M^1) \\ &= \dim \varinjlim_{nA} H^0(M, \Omega_M^1 \otimes \mathcal{O}_{nA}(nA)) \\ &\qquad \text{(by Serre duality and [1]).} \\ &= \dim H^0(M, \Omega_M^1 \otimes \mathcal{O}_{n_0A}(n_0A)). \end{aligned} \tag{4.7}$$

If $((2-2g)/(A\cdot A)) < 1$, then $n_0 = 1$.

Set $n = 1$ in (4.2), we have

$$0 \to \mathcal{O}_A \to \Omega^1_M \otimes \mathcal{O}_A(A) \to \Omega^1_A(A) \to 0. \tag{4.8}$$

Since $c_1(\Omega^1_A(A)) = 2g - 2 + A\cdot A$ which is less than zero by our assumption $((2-2g)/(A\cdot A)) < 1$, hence $H^0(M, \Omega^1_A(A)) = 0$. By (4.7) and (4.8), we have

$$\begin{aligned} h^{1,1}(M) &= \dim H^0(M, \Omega^1_M \otimes \mathcal{O}_A(A)) \\ &= \dim H^0(M, \mathcal{O}_A) = 1. \end{aligned}$$

Q.E.D.

We come to study $s^{(1)}$ for surface singularities which are obtained by blowing down the zero section of a negative line bundle, e.g. cone singularities. Proposition 4.8 was also proved independently by Pinkham [51].

PROPOSITION 4.8. *Let N be a negative line bundle over a nonsingular compact Riemann surface A. Then*

$$\Gamma(N\backslash A, \Omega^1) = \Gamma(N, \Omega^1) \oplus \bigoplus_{n=1}^{\infty} \Gamma(A, K_A N^n) \tag{4.9}$$

$$\begin{aligned} s^{(1)} &= \dim \Gamma(N\backslash A, \Omega^1)/\Gamma(N, \Omega^1) \\ &= \begin{cases} 0 & \text{if} \quad g \le 1 \\ \sum_{n=1}^{n_0} \dim \Gamma(A, K_A N^n) & \text{if} \quad g \ge 2 \end{cases} \end{aligned}$$

where g is the genus of A, n_0 is the least integer $\geq ((2-2g)/(A\cdot A))$ and K_A is the canonical line bundle of A.

Remark. We identify the zero section of N with the compact Riemann surface A here. It should be clear that on the left-hand side of (4.9), N represents the total space of the line bundle which corresponds to M in our previous notation, while on the right-hand side, N means the line bundle itself.

Proof of Proposition 4.8. Choose an open cover $\mathfrak{U}$ of A which consists of coordinate charts (U_α, z_α) of A such that N/U_α is trivial. Let w_α

be the fiber coordinate of N/U_α. Let us denote $f_{\alpha\beta}$ and $g_{\alpha\beta}$ be the coordinate transition function for the manifold A and the transition function for the line bundle N respectively, i.e.,

$$z_\alpha = f_{\alpha\beta}(z_\beta) \quad \text{and} \quad w_\alpha = g_{\alpha\beta} w_\beta \tag{4.10}$$

(z_α, w_α) is a coordinate patch on N. A one form on N is of the form $\{\xi_\alpha(z_\alpha, w_\alpha)dz_\alpha + \eta_\alpha(z_\alpha, w_\alpha)dw_\alpha\}$. Under change of coordinate, we have

$$\xi_\beta = \xi_\alpha \frac{df_{\alpha\beta}}{dz_\beta} + \eta_\alpha \frac{d\log g_{\alpha\beta}}{dz_\beta} w_\alpha \tag{4.11}$$

$$\eta_\beta = \eta_\alpha g_{\alpha\beta}. \tag{4.12}$$

Take a Laurent series expansion of $\{\eta_\alpha\}$ along the fiber.

$$\eta_\alpha = \sum_{n=-\infty}^{\infty} \eta_{\alpha,n}(z_\alpha)w_\alpha^n, \qquad \eta_\beta = \sum_{n=-\infty}^{\infty} \eta_{\beta,n}(z_\beta)w_\beta^n.$$

Therefore (4.12) implies

$$\Rightarrow \{\eta_{\alpha,n}\} \in \Gamma(A, N^{-n-1}).$$

Since $c(N^{-n-1}) = (-n-1)c(N)$ and $c(N)$ is negative, where $c(N)$ denotes the Chern class of the bundle N we have

$$n \leq -2 \Rightarrow \{\eta_{\alpha,n}\} = 0$$

$$n = -1 \Rightarrow \{\eta_{\alpha,-1}\} \in \Gamma(A, \mathcal{O}_A)$$

i.e. $\{\eta_{\alpha,-1}\} = a$ which is a constant function on A

$$\therefore \{\eta_\alpha\} = \left\{\left(\frac{a}{w_\alpha} + \sum_{n=0}^{\infty} \eta_{\alpha,n}(z_\alpha)w_\alpha^n\right)\right\}. \tag{4.13}$$

Take the Laurent series expansion of ξ_α along the fiber

$$\xi_\alpha = \sum_{n=-\infty}^{\infty} \xi_{\alpha,n}(z_\beta)w_\alpha^n, \qquad \xi_\beta = \sum_{n=-\infty}^{\infty} \xi_{\beta,n}(z_\beta)w_\beta^n$$

From (4.11) we have

$$\begin{cases} \xi_{\beta,n}(z_\beta)g_{\alpha\beta}^{-n} = \xi_{\alpha,n}(z_\alpha)\dfrac{df_{\alpha\beta}}{dz_\beta} & \text{if } n \leq -1 \\ \xi_{\beta,0}(z_\beta) = \xi_{\alpha,0}(z_\alpha)\dfrac{df_{\alpha\beta}}{dz_\beta} + a\dfrac{d\log g_{\alpha\beta}}{dz_\beta} & \text{if } n = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \{\xi_{\alpha,n}\} \in \Gamma(A, K_A N^{-n}) \quad n \leq -1 & (4.14) \\ \text{The 1-cocycle } \left\{a \cdot \dfrac{1}{2\pi i}\dfrac{d\log g_{\alpha\beta}}{dz_\beta}dz_\beta\right\} \text{ is actually a 1-coboundary.} & (4.15) \end{cases}$$

The class $\{(1/2\pi i)((d \log g_{\alpha\beta})/dz_\beta)dz_\beta\}$ represents the Chern class of N. Since the Chern class of N is negative, the cohomology class of $\{a \cdot (1/2\pi i)((d \log g_{\alpha\beta})/dz_\beta)dz_\beta\}$ cannot be zero unless $a = 0$. By (4.15) we conclude that $a = 0$, i.e. η_α is actually holomorphic. On the other hand, if $c(K_A N^{-n}) = 2g - nA \cdot A < 0$, then $\Gamma(A, K_A N^{-n}) = 0$. Therefore our theorem follows easily from (4.14). Q.E.D.

We remark that the above computation can be generalized to compact complex manifolds of higher dimension. Recall that a surface singularity (V, x) is said to be simple elliptic if the exceptional set $\pi^{-1}(x)$ in the minimal resolution $\pi: M \to V$ is a nonsingular elliptic curve.

The following corollary tells us how to compute the invariant $s^{(1)}$ for simple elliptic singularities.

COROLLARY 4.9. *Let M be a two dimensional strongly pseudoconvex manifold M with a nonsingular elliptic curve A as its maximal compact analytic set. Then* $s^{(1)} = \dim \Gamma(M\backslash A, \Omega^1)/\Gamma(M, \Omega^1) = 0$.

Proof. By Grauert [12], a neighborhood of A in M is biholomorphic to a neighborhood of the zero set of the normal bundle of A. So by Proposition 3.4, Lemma 4.4 and Theorem 4.8, our corollary follows easily.

The following proposition is an easy consequence of [12].

PROPOSITION 4.10. *Let N be a negative line bundle over a nonsingular compact Riemann surface A. Then*

$$h^{(1)} = \dim H^1(N, \mathcal{O}) = \begin{cases} 0 & \text{if } g = 0 \\ 1 & \text{if } g = 1 \\ \sum_{n=1}^{n_0} \dim \Gamma(A, K_A N^{n+1}) & \text{if } g \geq 2 \end{cases} \tag{4.16}$$

where g is the genus of A, n_0 is the least integer $\geq ((2-2g)/A\cdot A)-1$ and K_A is the canonical line bundle of A.

There are similar formula for higher dimensional singularities. The formula below is well-known. It is also a direct consequence of Leray spectral sequence.

COROLLARY 4.11. *Let N be a negative line bundle over a $(n-1)$-dimensional compact complex manifold A (i.e. the zero section of N can be blown down.) Then*

$$h^{(n-1)} = \dim H^{n-1}(N, \mathcal{O}) = \sum_{i=-\infty}^{-1} \dim \Gamma(A, K_A N^{-i-1}) \tag{4.18}$$

where K_A is the canonical bundle on A.

COROLLARY 4.12. *Let $f(x, y, z)$ be a homogeneous polynomial which defines a nonsingular curve A of genus g in $\mathbf{CP}^2$. Let M be the dual of the hyperplane bundle restricted to A. M is a resolution of the variety V defined by $\{f=0\}$ in $\mathbf{C}^3$. Let μ, be the Milnor number of the cone singularity. Then*

$$h^{(1,1)}(M) = -\frac{1}{12}\left(\frac{2g-2}{A\cdot A}-1\right)^2(A\cdot A)+\frac{11}{12}-\frac{5}{6}g+\frac{\mu}{12} \tag{4.19}$$

$$s^{(1)} = h^{(1)} - g = -\frac{1}{12}\left(\frac{2g-2}{A\cdot A}-1\right)^2(A\cdot A)-\frac{1}{12}-\frac{5}{6}g+\frac{\mu}{12} \tag{4.20}$$

and

$$g^{(1)} = \frac{11}{12}\mu + \frac{1}{12}\left(\frac{2g-2}{A\cdot A}-1\right)^2(A\cdot A)+\frac{5}{6}g+\frac{1}{12}. \tag{4.21}$$

Proof. By (3.4)

$$\begin{aligned} s^{(1)} + h^{(1,1)}(M) &= (\tau-\mu) - 1 + \chi_T(A) + 2h^{(1)} \\ &= 1 - 2g + 2h^{(1)} \quad (f \text{ is homogeneous}) \\ \Rightarrow h^{(1,1)}(M) &= 1 - 2g + h^{(1)} + h^{(1)} - s^{(1)}. \end{aligned} \tag{4.22}$$

Put (4.9), (4.16) and (3) of [22] in (4.22), we get (4.19). Put (4.19) in (4.22) we obtain (4.20). (4.21) follows from Theorem 3.3. Q.E.D.

Example 4.13. Let $f(x, y, z) = x^k + y^k + z^k$. The singularity of $f^{-1}(0)$ may be resolved by blowing up the origin in $\mathbf{C}^3$. Let M be the resolution. The exceptional set A is a single curve of genus $\frac{1}{2}(k - 1)(k - 2)$. $h^{(1)} = \frac{1}{6}k(k - 1)(k - 2)$. By Corollary 4.12, we have

$$h^{(1,1)}(M) = 1 + \frac{1}{6}k(k-1)(k-2) - \frac{1}{2}(k-1)(k-2)$$

$$= \frac{k[(k-3)^2+2]}{6}$$

$$s^{(1)} = \frac{1}{6}k(k-1)(k-2) - \frac{1}{2}(k-1)(k-2)$$

$$= \frac{k[(k-3)^2+2]}{6} - 1.$$

If $k \geq 4$, then $h^{(1,1)}(M) > b_2 = 1$ and $s^{(1)} > 0$.

$$g^{(1)} = \frac{k(5k-7)(k-1)}{6}.$$

Definition 4.14. Let V be a n-dimensional complex analytic space with $x \in V$ as an isolated singularity. We say that $x \in V$ admits a $\mathbf{C}^*$-action if there exists an open neighborhood U of x in V and an embedding $j:(U, x) \to (\mathbf{C}^m, 0)$ for some m such that $j(U)$ is closed in $\mathbf{C}^m$ and is invariant under the $\mathbf{C}^*$-action $\tilde{\sigma}$ where $\tilde{\sigma}:\mathbf{C}^* \times \mathbf{C}^m \to \mathbf{C}^m$ is defined by

$$\tilde{\sigma}(t, (z_1, \ldots, z_m)) = (t^{q_1}z_1, \ldots, t^{q_m}z_m) \qquad q_i \text{ are positive integers.}$$

Our original proof of Theorem 4.15 is quite complicated. The following simplified proof of the theorem was suggested to us by Greuel.

THEOREM 4.15. *Let V be a Stein analytic space of dimension one with $x \in V$ an isolated singularity which admits a $\mathbf{C}^*$-action. Let N be the dimension of the Zariski tangent space of V at x. Let $g^{(i)}$ and $m^{(i)}$ be the invariants defined as before* (*cf.* section 2). *Then*

(a) $\Sigma_{i=0}^{N} (-1)^i m^{(i)} = 0$
(b) $g^{(1)} - g^{(0)} = -r + 1$

where r is the number of branches of V at x.

Before proving our theorem, let us recall a useful theorem due to Reiffen and Ferrari. The proof given here is communicated to us by Greuel.

THEOREM 4.16. *Under the assumption of Theorem* 4.15 *the complexes* $(\Omega^*_{V,x}, d)$, (K^*, d) *and* $(\tilde{\Omega}_{V,x}, d)$ *are exact where* K^* *is the torsion of* $\Omega^*_{V,x}$.

Proof. Let ξ denote the generating vector field of the $\mathbf{C}^*$-action, i_ξ the inner multiplication and $L_\xi = i_\xi d + d i_\xi$ the Lie derivation.

Let $\omega \in \Omega^p_{V,x}$ be a quasi-homogeneous element of (quasi-homogeneous) degree $q > 0$ then $L_\xi \omega = q\omega$. Hence $d\omega = 0$ implies $q\omega = L_\xi \omega = d(i_\xi \omega)$. It follows that $(\Omega^*_{V,x}, d)$ and (K^*, d) and therefore also $(\tilde{\Omega}^*_{V,x}, d)$ are exact.

Proof of Theorem 4.15. By Theorem 4.16, we have

$$\sum_{i=0}^{N} (-1)^i m^{(i)} = \sum_{i=0}^{N} (-1)^i \dim K^{(i)} = 0.$$

Since $H^1(\tilde{\Omega}^*) = \tilde{\Omega}^1/d\mathcal{O} = 0$, we have $\tilde{\Omega}^1 = d\mathcal{O}$. Statement (b) follows from the exact sequence

$$0 \to \mathbf{C}^r/\mathbf{C} \to \bar{\mathcal{O}}/\mathcal{O} \xrightarrow{d} \bar{\Omega}^1/d\mathcal{O} \to 0$$

and the fact that $g^{(1)} = \dim_{\mathbf{C}}(\bar{\Omega}^1/\tilde{\Omega}^1) = \dim_{\mathbf{C}}(\bar{\Omega}^1/d\mathcal{O})$ and $g^{(0)} = \dim_{\mathbf{C}}(\bar{\mathcal{O}}/\mathcal{O})$. Here $\bar{\mathcal{O}}$ denotes the integral closure of $\mathcal{O}$.

COROLLARY 4.17 (Milnor [26]). *Let V be a plane curve with the origin as its singularity which admits* $\mathbf{C}^*$-action. Let μ *be the Milnor number of V at* 0 *and r be the number of branches of V at* 0. *Then*

$$\mu = 2\delta - r + 1 \tag{4.23}$$

where $\delta = \dim(\tilde{\mathcal{O}}/\mathcal{O})$, $\tilde{\mathcal{O}}$ *the integral closure of* $\mathcal{O}$.

PROPOSITION 4.18 (Steenbrink [37]). *Let X be a complex manifold of dimension n, and* Γ *a property discontinuous group of automorphisms of X. Then the invariants* $s^{(i)}$, $0 \le i \le n$, *of the singularities in* $V = X/\Gamma$ *are equal to zero. In particular the* $h^{(n-1)}$ *is equal to zero.*

THEOREM 4.19. *Suppose* $V \subseteq \mathbf{C}^m$ *is an analytic variety of dimension two with the origin as its only isolated singularity. Suppose* σ *is a* $\mathbf{C}^*$*-action leaving V invariant, defined by*

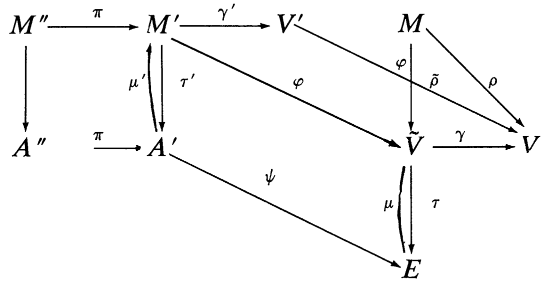
$$\sigma(t, (z_1, \ldots, z_m)) = (t^{q_1} z_1, \ldots, t^{q_m} z_m) \qquad q_i\text{'s are positive integers.}$$

Let $\varphi:\mathbf{C}^m \to \mathbf{C}^m$ be defined by $\varphi(z_1, \ldots, z_m) = (z_1^{q_1}, \ldots, z_m^{q_m})$ and let $V' = \varphi^{-1}(V)$ be the cone over V. Then V' has a natural $\mathbf{C}^*$-action defined by $\sigma'(t, (z_1, \ldots, z_m)) = (tz_1, \ldots, tz_m)$ and the induced map $\varphi: V' \to V$ commutes with the $\mathbf{C}^*$-action. Let $A' = (V' \backslash \{0\})/\mathbf{C}^* \subset \mathbf{P}^{m-1}$. Let M' be the universal subbundle (i.e. dual of the hyperplane bundle) of $\mathbf{P}^{m-1}$ restricted to A'. Identify $\mathbf{Z}_{q_i}$ with the group of q_i^{th} roots of 1. $G = \mathbf{Z}_{q_i} \oplus \cdots \oplus \mathbf{Z}_{q_m}$ acts on V' by coordinatewise multiplication. G also acts on A' and M'. Let $\pi: A'' \to A'$ be the normalization and $M'' = \pi^*(M')$, the pull back of M' by π. Then $s^{(1)}$ of the singularity in V is computed by the following formula

$$s^{(1)} = \begin{cases} 0 & \text{if } g'' \leq 1 \\ \sum_{n=1}^{n_0} \dim \Gamma(A'', K_{A''}M''^n)^G & \text{if } g'' \geq 1 \end{cases} \tag{4.24}$$

where g'' is the genus of A'', n_0 is the least integer $\geq ((2 - 2g'')/(A'' \cdot A''))$ and $K_{A''}$ is the canonical line bundle of A''. ($\Gamma(A'', K_{A''}M''^n)^G$ denotes the G-invariant sections.)

Proof. We shall use Orlik-Wagreich description of the canonical equivariant resolution of V. First observe that φ is unramified off the coordinate planes and V is the quotient of V' by G. Let $\eta':(V'\backslash\{0\}) \to A'$ be the quotient map. There is a well-known way of adding a zero section to this $\mathbf{C}^*$ bundle to get a $\mathbf{C}$ bundle. Let $\Gamma_{\eta'} \subset (V'\backslash\{0\}) \times A'$ be the graph of η', let M' be the closure of $\Gamma_{\eta'}$ in $V' \times A'$, and let $\tau': M' \to A'$ be induced by the projection on the second factor. The induced map $\gamma': M' \to V'$ is just the monoidal transform with center $0 \in V'$, and (τ', M') is the dual of the hyperplane bundle on $A' \subset \mathbf{P}^m$. Clearly $\mu': A' \to M'$ given by $\mu'(x') = (0, x')$ defines the zero section of (τ', M'). The action of $\mathbf{C}^*$ and G on V' commute, hence G acts on A' and define $E = A'/G$ we see that $E = V\backslash\{0\}/\mathbf{C}^*$. Let $\eta:(V\backslash\{0\}) \to E$ be the quotient map. As above, we would like to add a zero section to this map to get a map with fibers $\mathbf{C}$. The action of G extends to M' and we define $\tilde{V} = M'/G$. Then $\tilde{V}$ is just the closure of Γ_η in $V \times E$. Let $\gamma: \tilde{V} \to V$ be the natural map. Recalling the notation in the statement of the theorem, the map $\varphi: M' \to \tilde{V}$ is ramified only along a finite number of fibers of τ'. Hence there is a Zariski open subset $U \subset E$ so that $\tau^{-1}(U)$ is non-singular. But $\tilde{V}\backslash\mu(E)$ is non-singular since 0 is an isolated singular point of V, hence $\tilde{V}$ has only a finite number of singularities $\{q_1, \ldots, q_n\}$. These singularities are quotient singularities, hence they are rational singularities and therefore they can be resolved by a sequence of monoidal transforms with centers at isolated singular points. Let $\tilde{\rho}: M \to \tilde{V}$ be a minimal resolution of the singularities of $\tilde{V}$. Let $B_i = \tilde{\rho}^{-1}(q_i)$ $1 \leq i \leq n$. The composite map $\rho = \gamma\tilde{\rho}: M \to V$ is called the canonical equivariant resolution of V. We have a commutative diagram



Let $A = \bigcup_{i=1}^{n} B_i \cup E$ be the exceptional fiber of $\rho: M \to V$. We claim that

$$(4.25) \qquad \Gamma(M \backslash A, \Omega^1)/\Gamma(M, \Omega^1) \cong \Gamma(M'' \backslash A'', \Omega^1)^G/\Gamma(M'', \Gamma^1)^G.$$

Any form $\alpha \in \Gamma(M \backslash A, \Omega^1)$ can be considered as a form in $\Gamma(\tilde{V} \backslash E, \Omega^1)$ and hence in $\Gamma(M'' \backslash A'', \Omega^1)^G$. So we get a map $f: \Gamma(M \backslash A, \Omega^1) \to \Gamma(M'' \backslash A'', \Omega^1)^G$. Suppose $\alpha \in \Gamma(M, \Omega^1)$. Then α defines a holomorphic 1-form on $\tilde{V} \backslash \{q_1, \ldots, q_n\}$. Pulling back this 1-form to M'' by π, we get a holomorphic 1-form on M'' by Hartog extension theorem. It is a G invariant form because it comes from a form on M. We have proved that f maps $\Gamma(M, \Omega^1)$ into $\Gamma(M'', \Omega^1)^G$. This induces a natural map

$$\bar{f}: \Gamma(M \backslash A, \Omega^1)/\Gamma(M, \Omega^1) \to \Gamma(M'' \backslash A'', \Omega^1)^G/\Gamma(M'', \Omega^1)^G.$$

Let $\alpha \in \Gamma(M \backslash A, \Omega^1)$ such that $f(\alpha) \in \Gamma(M'', \Omega^1)^G$. Then $(\varphi \circ \pi)_*(f(\alpha))$ is a holomorphic form on $\tilde{V} \backslash \{q_1, \ldots, q_n\}$. By Theorem 4.18, $\tilde{\rho}^*((1/|G|)\varphi_*(f(\alpha)))$ is a holomorphic form on M which coincides with α on $M \backslash A$. This proves that $\bar{f}$ is injective. On the other hand pick any $\beta \in \Gamma(M'' \backslash A'', \Omega^1)^G$. Consider the holomorphic form $(\varphi \circ \pi)_*(\beta)$ on $\tilde{V} \backslash E$. Then $\tilde{\rho}^*((1/|G|)\varphi_*(\beta)) = \beta_1$ is holomorphic on $M \backslash A$ and $f(\beta_1) = \beta$. Hence $\bar{f}$ is surjective.

By the proof of Theorem 4.8, we have the following decomposition

$$\Gamma(M'' \backslash A'', \Omega^1) = \Gamma(M'', \Omega^1) \oplus \bigoplus_{n=-n_0}^{-1} \Gamma(A'', K_{A''} M''^{-n}).$$

Hence

$$\Gamma(M''\backslash A'', \Omega^1)^G/\Gamma(M'', \Omega^1)^G = \bigoplus_{n=-n_0}^{-1} \Gamma(A'', K_{A''}M''^{-n})^G$$

and

$$s^{(1)} = \dim \Gamma(M\backslash A, \Omega^1)/\Gamma(M, \Omega^1) = \sum_{n=-n_0}^{-1} \dim \Gamma(A'', K_{A''}M''^{-n})^G \quad \text{Q.E.D.}$$

The proof of Theorem 4.10 and the proof of Theorem 4.19 yield the following formula (4.26). This is proved independently by Pinkham [52].

THEOREM 4.20. *Suppose $V \subseteq \mathbf{C}^m$ is an analytic variety of dimension two which admits a $\mathbf{C}^*$-action with the origin as its only isolated singularity. Let G, A'', M'', K_A'' and g'' be defined as in Theorem 4.19. Then the Hironaka number $h^{(1)}$ of the singularity in V is given by the following formula*

$$h^{(1)} = \begin{cases} 0 & g'' = 0 \\ \dim \Gamma(A'', K_{A''})^G & g'' = 1 \\ \sum_{n=1}^{n_0} \dim \Gamma(A'', K_{A''}M''^{n+1})^G & \text{if} \quad g'' \geq 2 \end{cases} \tag{4.26}$$

where n_0 is the least integer $\geq ((2 - 2g'')/(A'' \cdot A'')) - 1$.

Remark. Theorem 4.20 is true for any dimension ≥ 2.

COROLLARY 4.21. *Suppose $V \subseteq \mathbf{C}^3$ is an analytic variety of dimension two which admits a $\mathbf{C}^*$-action with the origin as an isolated singularity. Let G, A'', M'', $K_{A''}$ be defined as in Theorem 4.19. Let $\rho: M \to V$ be the canonical equivariant resolution of V in the sense of Orlik and Wagreich with the maximal compact analytic set A in M as an exceptional set. Then*

$$h^{(1,1)}(M) = \chi_T(A) - 1 + h^{(1)} + \dim \Gamma(A'', K_{A''})^G$$

$$g^{(1)} = \mu + \dim \Gamma(A'', K_{A''})^G - h^{(1)}$$

and

$$s^{(1)} = h^{(1)} - \dim \Gamma(A'', K_{A''})^G$$

where $\chi_T(A)$ is the topological Euler characteristic of A.

Proof. Let f be the weighted homogeneous polynomial which defines V. Let

$$\mu = \dim \mathbf{C}[[x, y, z]] / \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

and

$$\tau = \dim \mathbf{C}[[x, y, z]] / \left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right).$$

Since f is weighted homogeneous $\mu = \tau$. By (3.4),

$$s^{(1)} + h^{(1,1)}(M) = \tau - (\mu + 1) + \chi_T(A) + 2h^{(1)} \tag{4.27}$$

$$= \chi_T(A) - 1 + 2h^{(1)}$$

$$\Rightarrow h^{(1,1)}(M) = \chi_T(A) - 1 + h^{(1)} + h^{(1)} - s^{(1)}. \tag{4.28}$$

Put (4.24) and (4.26) in (4.28), we get

$$h^{(1,1)}(M) = \chi_T(A) - 1 + h^{(1)} + \dim \Gamma(A'', K_{A''})^G \tag{4.29}$$

Corollary 4.22. *Suppose $V \subseteq \mathbf{C}^m$ is an analytic variety of dimension two which admits a $\mathbf{C}^*$-action. Let G, A'' and $K_{A''}$ be defined as in Theorem* 4.19. *Then*

$$h^{(1)} = s^{(1)} + \dim \Gamma(A'', K_{A''})^G \tag{4.30}$$

In particular,

$$h^{(1)} \geq s^{(1)} \tag{4.31}$$

Therefore $s^{(1)}$ is equal to zero for any rational singularity with $\mathbf{C}^*$-action.

The following Lemma is well-known.

Lemma 4.23. *Let $f(x, y, z)$ be a homogeneous polynomial of degree d. Let A' be a curve in $\mathbf{CP}^2$ defined by f. Let $(\mathfrak{U}_x, y_1, z_1)$ be the coordinate patch given by $x \neq 0$ and $f_1(y_1, z_1) = f(1, y_1, z_1)$. Let $\pi: A'' \to A'$ be the normalization of A'. Then any holomorphic 1-forms on A'' are of the form*:

$$\frac{P(y_1, z_1)}{\frac{\partial f_1}{\partial z_1}} dy_1$$

where P is a polynomial of degree $\leq d - 3$.

Example 4.23. Let $V = \{z^2 = y(x^4 + y^6)\} \subseteq \mathbf{C}^3$ as in Example 3.4. V admits a $\mathbf{C}^*$-action σ

$$\sigma : \mathbf{C}^* \times V \to V$$

$$(t, (x, y, z)) \to (t^3 x, t^2 y, t^7 z).$$

Then $V'' = \{x', y'z'): x'^{12} y'^2 + y'^{14} - z'^{14} = 0\}$. Identify $\mathbf{Z}_3$, $\mathbf{Z}_2$ and $\mathbf{Z}_7$ with the groups of 3^{th} roots, 2^{th} roots and 7^{th} roots of 1 respectively. $G = \mathbf{Z}_3 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_7$ acts on V' by coordinate multiplication.

An easy computation shows

$$\Gamma(A'', K_{A''})^G = \left\{ a \frac{y_1^3 dy_1}{14 z_1^7} : a \in \mathbf{C} \right\}.$$

By Corollary 4.21, we have

$$\begin{aligned}
\dim H^1(M, \Omega^1) &= \chi_T(A) - 1 + h^{(1)} + \dim \Gamma(A'', K_{A''})^G \\
&= (3 + 1 - 2) - 1 + 2 + 1 \\
&= 4
\end{aligned}$$

$$\begin{aligned}
\dim \Gamma(M \backslash A, \Omega^1)/\Gamma(M, \Omega^1) &= h^{(1)} - \dim \Gamma(A'', K_{A''})^G \\
&= 2 - 1 \\
&= 1.
\end{aligned}$$

$$\begin{aligned}
g^{(1)} &= \mu + 1 - \chi_T(A) - \chi(\Omega^1) - 2h^{(1)} \\
&= 22 + 1 - 2 + 4 - 2 \cdot 2 = 21.
\end{aligned}$$

5. Noether's formula for strongly pseudo-convex manifolds of dimension ≥ 2 and universal formula for curve singularities. In order to prove Noether's formula for arbitrary strongly pseudo-convex manifolds or a universal formula for arbitrary curve singularities, we need to introduce one more concept which has arisen naturally from the Poincaré complex. This was studied by Bloom and Herrera [8], Reiffen [30] and Saito [33].

Definition 5.1. Let X be a complex analytic space and $x_0 \in X$ an isolated singular point. Let

$$0 \longrightarrow \mathbf{C} \xrightarrow{d^{-1}} \mathcal{O}_{X,x_0} \xrightarrow{d^0} \Omega^1_{X,x_0} \xrightarrow{d^1} \Omega^2_{X,x_0} \xrightarrow{d^2} \cdots$$

be the Poincaré complex at x_0 where d^{-1} is the inclusion map. Then the Poincaré numbers of X at x_0 are defined as follows.

$$p^{(i)} = \dim \operatorname{Ker} d^i/\operatorname{Im} d^{i-1} \qquad i \geq 0.$$

We remark that $p^{(i)} = 0$ for $i > N$ where N is the embedding dimension of X at x_0. By Bloom and Herrera, all these numbers are finite.

Now let (X, x_0) be an isolated hypersurface singularity, $\dim(X, x_0) = n$. In [9], Brieskorn proved that $p^{(i)} = 0$ if $i \leq n - 2$. Later Sebastiani [49] proved that $p^{(n-1)}$ is also equal to zero. In [33], Saito proved that $p^{(n)} = 0$ if and only if (X, x_0) is quasi-homogeneous. The proofs of Brieskorn, Sebastiani and Saito are purely local (the global argument of Brieskorn in his coherence theorem can be avoided by using the main theorem of Kiehl-Verdier). In [48], section 4.4, Greuel generalized the above result to complete intersection (X, x_0).

The following universal formula for curve singularities (Theorem 5.3), is the best formula one can obtain in the sense that no condition is imposed on the singularities. Milnor's formula [26] for plane curve singularities is a particular case of our formula. The original proof of our formula is quite complicated. The proof given here is suggested to us by Greuel.

LEMMA 5.2. *Let X be a complex analytic space and $x_0 \in X$ an isolated singular point. Let*

$$0 \to \mathbf{C} \to \mathcal{O}_X \xrightarrow{d^0} \Omega^1_X \xrightarrow{d^1} \Omega^2_X \to \cdots$$

be the Poincaré complex. Let J^i be the ker *of d^i. Then there exists a Stein open neighborhood V of x_0 in X such that*

(5.1) $H^p(V, d\Omega^i) = 0$ *and* $H^p(V, J^i) = 0$ *for all* $p \geq 1$ *and* $i \geq 0$.

This V can be chosen to be arbitrary small.

Proof. By Milnor [26], we can choose arbitrary small Stein open neighborhood V of x_0 in X such that V is contractible to x_0. This implies that $H^p(V, \mathbf{C}) = 0$ for $p \geq 1$. As V is Stein, $H^p(V, \Omega^i) = 0$ for $p \geq 1$. By considering the long cohomology exact sequences associated to the following sheaf exact sequences.

$$0 \longrightarrow \mathbf{C} \longrightarrow J^0 \longrightarrow J^0/\mathbf{C} \longrightarrow 0$$

$$0 \longrightarrow J^0 \longrightarrow \mathcal{O}_V \longrightarrow d\mathcal{O}_V \longrightarrow 0$$

$$0 \longrightarrow d\Omega_V^{i-1} \longrightarrow J^i \longrightarrow J^i/d\Omega_V^{i-1} \longrightarrow 0$$

$$0 \longrightarrow J^i \longrightarrow \Omega^i \longrightarrow d\Omega^i \longrightarrow 0$$

one sees inductively that $H^p(V, d\Omega^i) = 0$ and $H^p(V, J^i)$ for all $p \geq 1$ and $i \geq 0$ by observing that $J^0/\mathbf{C}$ and $J^i/d\Omega_V^{i-1}$ $i \geq 0$ are sheaves on V with supports only on x_0. Q.E.D.

THEOREM 5.3. *Let V be a complex analytic space of dimension one with x as an isolated singularity. Let* $g^{(i)}$, $m^{(i)}$ *and* $p^{(i)}$ *be the invariants of V at x as defined before. Let N be the dimension of the Zariski tangent space of V at x. Then*

$$(5.2) \qquad \sum_{i=0}^{N} (-1)^i p^{(i)} + g^{(0)} - g^{(1)} - \sum_{i=0}^{N} (-1)^i m^{(i)} = r - 1$$

where r is the number of branches of V at x.

Proof. Since the Euler characteristic of a complex is equal to the Euler characteristic of its cohomology, we get from the sequence

$$0 \to H^1(\Omega^*) \to \Omega^1/d\mathcal{O} \to \Omega^2 \to \cdots \to \Omega^N \to 0$$

the following formula

$$p^{(1)} - \dim(\Omega^1/d\mathcal{O}) + \sum_{i=2}^{N} (-1)^i m^{(i)} = \sum_{i=2}^{N} (-1)^i p^{(i)}.$$

From the exact sequence

$$0 \to K^1 \to \Omega^1/d\mathcal{O} \to \bar{\Omega}^1/d\mathcal{O} \to \bar{\Omega}^1/\Omega^1 \to 0$$

and from $\dim(\bar{\Omega}^1/d\mathcal{O}) = \delta - r + 1$ (cf. (2.2) of [5]), the result follows. Q.E.D.

Remark. Formula (5.2) is a generalization of Milnor's formula (cf. Theorem 10.5 of [26]) for plane curve singularities to arbitrary curve singularities.

Definition 5.4. Let $\mathfrak{F}$ be a locally free sheaf over M, a strongly pseudo-convex manifold. Then $\chi(\mathfrak{F})$ and $\chi(M, \mathfrak{F})$ are defined as follows.

$$\chi(\mathfrak{F}) = \sum_{q=1}^{n} (-1)^q \dim H^q(M, \mathfrak{F})$$

$$\chi(M, \mathfrak{F}) = \dim \Gamma(M \backslash A, \mathfrak{F})/\Gamma(M, \mathfrak{F}) - \chi(\mathfrak{F})$$

where A is the maximal compact analytic set in M.

THEOREM 5.5. *Let M be a n-dimensional strongly pseudo-convex manifold of dimension n. Suppose M can be blown down to a Stein analytic space V with x as its only singularity. Let $g^{(i)}$, $m^{(i)}$, $p^{(i)}$ and $s^{(i)}$ be the invariants as defined before. Let N be the dimension of the Zariski tangent space of V at x.*

(a) *If x is a surface singularity, then*

$$(5.3) \quad \dim H^1(M, \mathcal{O}) - \dim H^1(M, \Omega^1) = \sum_{i=0}^{N} (-1)^i p^{(i)} - \sum_{i=0}^{N} (-1)^i m^{(i)} + g^{(0)} - g^{(1)} + g^{(2)} - \chi_T(A) + 1$$

(b) *If x is a higher dimensional singularity i.e. $n \geq 3$, then*

$$(5.4) \quad \sum_{i=0}^{n-1} (-1)^i \chi(M, \Omega^i) = \sum_{i=0}^{N} (-1)^i p^{(i)} - \sum_{i=0}^{N} (-1)^i m^{(i)} + \sum_{i=0}^{n} (-1)^i g^{(i)} + \sum_{i=1}^{n-1} (-1)^i s^{(i)} - \chi_T(A) + 1$$

where $\chi_T(A)$ *in* (a) *and* (b) *is the topological Euler characteristic of the exceptional set* A.

Proof. (5.3) and (5.4) are completely local. We may assume [26] that V is topological contractible to x. Let $J^i = \ker(d^i:\Omega_V^i \to \Omega_V^{i+1})$. Consider the following commutative diagram with exact rows.

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathbf{C} & \longrightarrow & \mathbf{C} & \longrightarrow & 0 & \longrightarrow & 0 & \text{on} & M \\ & & \uparrow & & \uparrow & & \uparrow & & & & \\ 0 & \longrightarrow & \mathbf{C} & \longrightarrow & J^0 & \longrightarrow & J^0/\mathbf{C} & \longrightarrow & 0 & \text{on} & V \end{array}$$

Since $H^1(V, \mathbf{C}) = 0$, the rows of the following commutative diagram are exact

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(M, \mathbf{C}) & \longrightarrow & \Gamma(M, \mathbf{C}) & \longrightarrow & 0 & \longrightarrow & 0 \\ & & {\scriptstyle \pi^*}\uparrow & & {\scriptstyle \pi^*}\uparrow & & {\scriptstyle \pi^*}\uparrow & & \\ 0 & \longrightarrow & \Gamma(V, \mathbf{C}) & \longrightarrow & \Gamma(V, J^0) & \longrightarrow & \Gamma(V, J^0/\mathbf{C}) & \longrightarrow & 0 \end{array}$$

By snake lemma,

$$\dim \ker(\pi^*:\Gamma(V, J^0) \to \Gamma(M, \mathbf{C})) - p^{(0)} = 0. \qquad (1, \text{a})$$

As $H^1(V, J^0) = 0$ by Lemma 5.2, the commutative diagram with exact rows

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathbf{C} & \longrightarrow & \mathcal{O} & \longrightarrow & d\mathcal{O} & \longrightarrow & 0 & \text{on} & M \\ & & \uparrow & & \uparrow & & \uparrow & & & & \\ 0 & \longrightarrow & J^0 & \longrightarrow & \mathcal{O} & \longrightarrow & d\mathcal{O} & \longrightarrow & 0 & \text{on} & V \end{array}$$

gives the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(M, \mathbf{C}) & \longrightarrow & \Gamma(M, \mathcal{O}) & \xrightarrow{d} & d\Gamma(M, \mathcal{O}) & \longrightarrow & 0 \\ & & {\scriptstyle \pi^*}\uparrow & & {\scriptstyle \pi^*}\uparrow & & {\scriptstyle \pi^*}\uparrow & & \\ 0 & \longrightarrow & \Gamma(V, J^0) & \longrightarrow & \Gamma(V, \mathcal{O}) & \xrightarrow{d} & \Gamma(V, d\mathcal{O}) & \longrightarrow & 0 \end{array}$$

By snake lemma

$$-\dim \ker(\pi^*:\Gamma(V, J^0) \to \Gamma(M, \mathbf{C})) + m^{(0)}$$
$$- \dim \ker(\pi^*:\Gamma(V, d\mathcal{O})$$
$$\to d\Gamma(M, \mathcal{O})) + 0 - g^{(0)}$$
$$+ \dim d\Gamma(M, \mathcal{O})/\Gamma(V, d\mathcal{O}) = 0 \qquad (1, \mathrm{b})$$

$$-\dim d\Gamma(M, \mathcal{O})/\pi^*\Gamma(V, d\mathcal{O}) - \dim \Gamma(M, d\mathcal{O})/d\Gamma(M, \mathcal{O})$$
$$+ \dim \Gamma(M, d\mathcal{O})/\pi^*\Gamma(V, d\mathcal{O}) = 0. \qquad (1, \mathrm{c})$$

From the long cohomology exact sequence,

$$0 \to \Gamma(M, \mathbf{C}) \to \Gamma(M, \mathcal{O}) \to \Gamma(M, d\mathcal{O}) \to H^1(M, \mathbf{C})$$
$$\to H^1(M, \mathcal{O}) \to H^1(M, d\mathcal{O}) \to \cdots$$

we assert that

$$\dim \Gamma(M, d\mathcal{O})/d\Gamma(M, \mathcal{O}) + \chi_T(A) - 1 - \chi(\mathcal{O}) + \chi(d\mathcal{O}) = 0 \qquad (1, \mathrm{d})$$

where

$$\chi(d\Omega^i) = \sum_{q=1}^{n} (-1)^q \dim H^q(M, d\Omega^i).$$

Consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & d\Omega^i & \longrightarrow & d\Omega^i & \longrightarrow & 0 & \longrightarrow & 0 \quad \text{on} \quad M \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & d\Omega^i & \longrightarrow & J^{i+1} & \longrightarrow & J^{i+1}/d\Omega^i & \longrightarrow & 0 \quad \text{on} \quad V
\end{array}$$

Since $H^1(V, d\Omega^i) = 0$, we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(M, d\Omega^i) & \longrightarrow & \Gamma(M, d\Omega^i) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Gamma(V, d\Omega^i) & \longrightarrow & \Gamma(V, J^{i+1}) & \longrightarrow & \Gamma(V, J^{i+1}/d\Omega^i) & \longrightarrow & 0
\end{array}$$

By snake lemma

$$(-1)^i \dim \ker(\pi^*:\Gamma(V, d\Omega^i)$$

$$\to \Gamma(M, d\Omega^i)) + (-1)^{i+1} \dim \ker(\pi^*:\Gamma(V, J^{i+1})$$

$$\to \Gamma(M, d\Omega^i)) + (-1)^i p^{(i+1)} + (-1)^{i+1} \dim \Gamma(M, d\Omega^i)/\pi^*\Gamma(V, d\Omega^i)$$

$$+ (-1)^i \dim \Gamma(M, d\Omega^i)/\pi^*\Gamma(V, J^{i+1}) = 0 \qquad (i+1, \text{a})$$

$$= 0 \qquad (i+1, \text{a})$$

As $H^1(V, J^{i+1}) = 0$ by Lemma 6.2, the commutative diagram with exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & d\Omega^i & \longrightarrow & \Omega^{i+1} & \longrightarrow & d\Omega^{i+1} & \longrightarrow & 0 \quad \text{on} \quad M \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & J^{i+1} & \longrightarrow & \Omega^{i+1} & \longrightarrow & d\Omega^{i+1} & \longrightarrow & 0 \quad \text{on} \quad V
\end{array}$$

gives the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(M, d\Omega^i) & \longrightarrow & \Gamma(M, \Omega^{i+1}) & \longrightarrow & d\Gamma(M, \Omega^{i+1}) & \longrightarrow & 0 \\
 & & \pi^* \uparrow & & \pi^* \uparrow & & \pi^* \uparrow & & \\
0 & \longrightarrow & \Gamma(V, J^{i+1}) & \longrightarrow & \Gamma(V, \Omega^{i+1}) & \longrightarrow & \Gamma(V, d\Omega^{i+1}) & \longrightarrow & 0
\end{array}$$

By snake lemma

$$(-1)^i \dim \ker(\pi^*:\Gamma(V, J^{i+1})$$

$$\to \Gamma(M, d\Omega^i)) + (-1)^{i+1} m^{(i+1)} + (-1)^i \dim \ker(\pi^*:\Gamma(V, d\Omega^{i+1})$$

$$\to d\Gamma(M, \Omega^{i+1})) + (-1)^{i+1} \dim \Gamma(M, d\Omega^i)/\pi^*\Gamma(V, J^{i+1}) + (-1)^i g^{(i+1)}$$

$$+ (-1)^{i+1} d\Gamma(M, \Omega^{i+1})/\pi^*\Gamma(V, d\Omega^{i+1}) = 0 \qquad (i+1, \text{b})$$

$$(-1)^i \dim d\Gamma(M, \Omega^{i+1})/\pi^*\Gamma(V, d\Omega^{i+1})$$

$$+ (-1)^i \dim \Gamma(M, d\Omega^{i+1})/d\Gamma(M, \Omega^{i+1})$$

$$+ (-1)^{i+1} \dim \Gamma(M, d\Omega^{i+1})/\pi^*\Gamma(V, d\Omega^{i+1}) = 0 \qquad (i+1, \text{c})$$

On the other hand from the long cohomology exact sequence

$$0 \to \Gamma(M, d\Omega^i) \to \Gamma(M, \Omega^{i+1}) \to \Gamma(M, d\Omega^{i+1}) \to H^1(M, d\Omega^i)$$

$$\to H^1(M, \Omega^{i+1}) \to H^1(M, d\Omega^{i+1}) \to \cdots$$

we get

$$(-1)^{i+1} \dim \Gamma(M, d\Omega^{i+1})/d\Gamma(M, \Omega^{i+1}) + (-1)^{i+1}\chi(d\Omega^i)$$

$$+ (-1)^{i+2}\chi(\Omega^{i+1}) + (-1)^{i+1}\chi(d\Omega^{i+1}) = 0 \qquad (i+1, \text{d})$$

Summing (1, a), (1, b), (1, c), (1, d), $(i+1, \text{a})$, $(i+1, \text{b})$, $(i+1, \text{c})$ and $(i+1, \text{d})$ for $0 \le i \le n-1$, we obtain

$$(5.5) \quad -\sum_{i=0}^{n} (-1)^i p^{(i)} + \sum_{i=0}^{n} (-1)^i m^{(i)} - \sum_{i=0}^{n} (-1)^i g^{(i)} + \chi_T(A) - 1$$

$$- \chi(\mathcal{O}) + \chi(\Omega^1) + \cdots + (-1)^n \chi(\Omega^{n-1}) + (-1)^{n+1} \dim \Gamma(V, d\Omega^n) = 0.$$

Since $H^1(V, d\Omega^n) = 0$, the sheaf exact sequence

$$0 \to d\Omega^n \to J^{n+1} \to J^{n+1}/d\Omega^n \to 0$$

on V tells us that

$$(5.6) \qquad \dim \Gamma(V, d\Omega^n) = \dim \Gamma(V, J^{n+1}) - p^{(n+1)}.$$

On the other hand, the exact sequence

$$0 \to J^{n+1} \to \Omega^{n+1} \to d\Omega^{n+1} \to 0$$

gives

$$\dim \Gamma(V, J^{n+1}) = m^{(n+1)} - \dim \Gamma(V, d\Omega^{n+1}). \tag{5.7}$$

Put (5.7) in (5.6),

$$\dim \Gamma(V, d\Omega^{n}) = m^{(n+1)} - p^{(n+1)} - \dim \Gamma(V, d\Omega^{n+1}).$$

Continuing this process, we obtain

$$\dim \Gamma(V, d\Omega^{n}) = (m^{(n+1)} - p^{(n+1)} - (m^{(n+2)} - p^{(n+2)}) + \cdots + (-1)^{N-n-1}(m^{(N)} - p^{(N)}) \tag{5.8}$$

Put (5.8) in (5.5), (5.3) and (5.4) follows easily. Q.E.D.

The following Theorem is due to Brieskorn [9] if one uses Sebastiani's result.

COROLLARY 5.6. *Let V be the Stein analytic space of dimension n with x as its isolated hypersurface singularity. Suppose in a neighborhood of x, V is isomorphic to $\{f = 0\}$ where f is holomorphic in a neighborhood of the origin in $\mathbf{C}^{n+1}$. Let*

$$\mu = \dim \mathbf{C}[[z_0, z_1, \ldots, z_n]]\Big/\left(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}\right),$$

$$\tau = \dim \mathbf{C}[[z_0, z_1, \ldots, z_n]]\Big/\left(f, \frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)$$

and $p^{(i)}$ be the Poincaré numbers of V at x. Then the Poincaré characteristic

$$\sum_{i=0}^{n+1} (-1)^i p^{(i)} = (-1)^n(\mu - \tau).$$

In particular $p^{(n)} = \mu - \tau$.

Remark. Corollary 5.6 follows also from $\mu = \dim_{\mathbf{C}} \Omega^n/d\Omega^{n-1}$ (cf. [48], Proposition 5.1) and the exact sequence

$$0 \to H^n(\Omega^*) \to \Omega^n/d\Omega^{n-1} \xrightarrow{d} \Omega^{n+1} \to 0.$$

Compare the more general statement in [48], Proposition 5.7(iii) for complete intersections.

Added to the proof. This paper has been circulating for a while. Since then, we have made two applications of our theory developed in this paper (cf. [53], [54]). Unfortunately this paper will appear in the journal form later than the above two papers.

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