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Characterization of isolated homogeneous hypersurface singularities in $\mathbb{C}^4$

Dedicated to Professor Sheng GONG on the occasion of his 75th birthday

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Abstract Let V be a hypersurface with an isolated singularity at the origin in $\mathbb{C}^{n+1}$. It is a natural question to ask when V is defined by weighted homogeneous polynomial or homogeneous polynomial up to biholomorphic change of coordinates. In 1971, a beautiful theorem of Saito gives a necessary and sufficient condition for V to be defined by a weighted homogeneous polynomial. For a two-dimensional isolated hypersurface singularity V, Xu and Yau found a coordinate free characterization for V to be defined by a homogeneous polynomial. Recently Lin and Yau gave necessary and sufficient conditions for a 3-dimensional isolated hypersurface singularity with geometric genus bigger than zero to be defined by a homogeneous polynomial. The purpose of this paper is to prove that Lin-Yau's theorem remains true for singularities with geometric genus equal to zero.

Keywords: homogeneous polynomials, hypersurface singularity, weighted homogeneous polynomial, geometric genus.

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1 Introduction

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be the germ of a complex analytic function with an isolated critical point at the origin. Let $V = \{z \in \mathbb{C}^{n+1} : f(z) = 0\}$. It is a natural question to ask when V is defined by weighted homogeneous polynomial or homogeneous polynomial up to biholomorphic change of coordinates. Recall that the Milnor number μ and the Tjurina number τ of the singularity $(V, 0)$ are defined respectively by

$$\mu = \dim \mathbb{C}\{z_0, z_1, \dots, z_n\}/(f_{z_0}, \dots, f_{z_n}),$$
$$\tau = \dim \mathbb{C}\{z_0, z_1, \dots, z_n\}/(f, f_{z_0}, \dots, f_{z_n}).$$

In 1971, Saito proved the following theorem which gives a necessary and sufficient condition for V to be defined by a weighted homogeneous polynomial.

Theorem 1.1[1]. f is a weighted homogeneous polynomial after a biholomorphic change of coordinates if and only if $\mu = \tau$.

Let $\pi : (M, A) \to (V, 0)$ be a resolution of singularity with exceptional set $A = \pi^{-1}(0)$. The geometric genus p_g of the singularity $(V, 0)$ is the dimension of $H^{n-1}(M, \mathcal{O})$ and is independent of the resolution M. In 1993, Xu and Yau[2] gave necessary and sufficient conditions for a 2-dimensional V to be defined by a homogeneous polynomial.

Theorem 1.2[2]. Let $(V, 0)$ be a 2-dimensional isolated hypersurface singularity defined by a holomorphic function $f(z_0, z_1, z_2) = 0$. Let μ be the Milnor number, τ be the Tyurina number, p_g be the geometric genus, and ν be the multiplicity of the singularity. Then f is a homogeneous polynomial after a biholomorphic change of variables if and only if $\mu = \tau$ and $\mu - \nu + 1 = 6p_g$.

The purpose of this paper is to prove the following theorem which was proved by Lin-Yau[3] in the case of $p_g > 0$.

Main Theorem. Let $(V, 0)$ be a 3-dimensional isolated hypersurface singularity defined by a weighted homogeneous polynomial $f(x, y, z, w) = 0$. Let μ be the Milnor number, p_g be the geometric genus, and ν be the multiplicity of the singularity. Then

$$\mu - (2\nu^3 - 5\nu^2 + 2\nu + 1) \geqslant 4!p_g \tag{1.1}$$

and equality in (1.1) holds if and only if f is a homogeneous polynomial.

Corollary 1.1. With the notation as in the Main Theorem, let τ be the Tyurina number of the singularity. Then f is a homogeneous polynomial after a biholomorphic change of coordinate if and only if $\mu = \tau$ and $\mu - (2\nu^3 - 5\nu^2 + 2\nu + 1) = 4!p_g$.

Since the Main Theorem is proved by Lin-Yau[3] for $p_g > 0$, we only need to consider the case of rational singularities (i.e. $p_g = 0$). It was proved in ref. [4] that the multiplicity of rational hypersurface singularity is either 2 or 3. If $(V, 0)$ is a rational singularity with $\nu = 2$, then $2\nu^3 - 5\nu^2 + 2\nu + 1 = 1$ and the Main Theorem is obviously true. In fact in this case the equality in (1.1) holds if and only if $f = x^2 + y^2 + z^2 + w^2$ after biholomorphic change of coordinates. The crucial point of this paper is to treat the case of rational singularities with multiplicity 3. The following is another Corollary of the Main Theorem.

Corollary 1.2. Let $(V, 0)$ be a 3-dimensional isolated hypersurface singularity defined by a weighted homogeneous polynomial $f(x, y, z, w) = 0$ with multiplicity 3. Let μ be the Milnor number. If $p_g = 0$, then $\mu \geqslant 16$, and $\mu = 16$ if and only if f is a homogeneous polynomial.

In sec. 2, we recall the necessary materials which are needed to prove the Main Theorem. In sec. 3, we prove the Main Theorem.

2 Preliminary

In this section, we recall some known results which are needed to prove the Main Theorem. Let $f(z_0, \ldots, z_n)$ be a germ of an analytic function at the origin such that

$f(0) = 0$. Suppose f has an isolated critical point at the origin. f can be developed in a convergent Taylor series $f(z_0, \ldots, z_n) = \sum a_\lambda z^\lambda$, where $z^\lambda = z_0^{\lambda_0} \cdots z_n^{\lambda_n}$. Recall that Newton boundary $\Gamma(f)$ is the union of compact faces of $\Gamma_+(f)$, where $\Gamma_+(f)$ is the convex hull of the union of subsets $\{\lambda + \mathbb{R}_+^{n+1}\}$ for λ such that $a_\lambda \neq 0$. Finally, let $\Gamma_-(f)$, the Newton polyhedron of f, be the cone over $\Gamma(f)$ with cone point at 0. For any closed face Δ of $\Gamma(f)$, we associate the polynomial $f_\Delta(z) = \sum_{\lambda \in \Delta} a_\lambda z^\lambda$. We say that f is nondegenerate if f_Δ has no critical point in $(\mathbb{C}^*)^{n+1}$ for any $\Delta \in \Gamma(f)$ where $\mathbb{C}^* = \mathbb{C} - \{0\}$. We say that a point p of the integral lattice $\mathbb{Z}^{n+1}$ in $\mathbb{R}^{n+1}$ is positive if all coordinates of p are positive. The following beautiful theorem is due to Merle-Teissier[5].

Theorem 2.1[5]. Let $(V, 0)$ be an isolated hypersurface singularity defined by a nondegenerate holomorphic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. Then the geometric genus $p_g = \#\{p \in \mathbb{Z}^{n+1} \cap \Gamma_-(f) : p \text{ is positive}\}$.

A polynomial $f(z_0, \ldots, z_n)$ is weighted homogeneous of type $(w_0, w_1, \ldots, w_n)$, where $w_0, \ldots, w_n$ are fixed positive rational numbers, if it can be expressed as a linear combination of monomials $z_0^{i_0} z_1^{i_1} \cdots z_n^{i_n}$ for which $i_0/w_0 + i_1/w_1 + \cdots + i_n/w_n = 1$. As a consequence of the theorem of Merle-Teissier, for isolated singularity defined by a weighted homogeneous polynomial, computing the geometric genus is equivalent to counting the number of positive integral points in the tetrahedron defined by $x_0/w_0 + \cdots + x_n/w_n \leqslant 1$, $x_0 \geqslant 0, \ldots, x_n \geqslant 0$. We also need the following result.

Theorem 2.2[6]. Let $f(z_0, z_1, \ldots, z_n)$ be a weighted homogeneous polynomial of type $(w_0, w_1, \ldots, w_n)$ with isolated singularity at the origin. Then the Milnor number $\mu = (w_0 - 1)(w_1 - 1) \cdots (w_n - 1)$.

The following theorems are proved by Lin-Yau[3].

Theorem 2.3[3]. Let $a \geqslant b \geqslant c \geqslant d \geqslant 2$ be real numbers. Let $P_4 = \#\{(x, y, z, w) \in \mathbb{Z}_+^4 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leqslant 1\}$. Suppose d is not an integer and $d = [d] + \beta$ where β is either $\frac{d}{c}$ or $\frac{d}{b}$. Define $\mu = (a-1)(b-1)(c-1)(d-1)$. Then

$$\begin{aligned} 24\,P_4 &< \mu - (2\nu^3 - 5\nu^2 + 2\nu + 1)\big|_{\nu = d - \beta + 1} \\ &= abcd - (abc + abd + acd + bcd) + (ab + ac + ad + bc + bd + cd) - (a + b + c) \\ &\quad - (2d^3 + d^2 - d - 1) + 2\beta^3 - \beta^2(6d + 1) + \beta(6d^2 + 2d - 2). \end{aligned} \tag{2.1}$$

Theorem 2.4[3]. Let $(V, 0)$ be a 3-dimensional isolated singularity defined by a weighted homogeneous polynomial $f(x, y, z, w) = 0$. Let a, b, c and d be the weights of x, y, z and w respectively and d be an integer. Suppose $a \geqslant b \geqslant c \geqslant d \geqslant 3$ or $a \geqslant b \geqslant c \geqslant d = 2$. Let p_g be the geometric genus and μ be the Milnor number of the singularity. Then

$$\mu \geqslant 24\,p_g + 2d^3 - 5d^2 + 2d + 1 \tag{2.2}$$

and the equality in (2.2) holds if and only if $a = b = c = d =$ integer, i.e. $f(x, y, z, w)$ is a homogeneous polynomial.

Lemma 2.1[3]. Let w_0, w_1, w_2, and w_3 be the weights of a weighted homogeneous polynomial $f(z_0, z_1, z_2, z_3)$ with an isolated critical point at the origin. Suppose $w_{i_0} \geqslant w_{i_1} \geqslant w_{i_2} \geqslant w_{i_3}$ and w_{i_3} is not an integer, where $\{i_0, i_1, i_2, i_3\} = \{0, 1, 2, 3\}$. Let

$w_{i_3} = [w_{i_3}] + \beta$ with $0 < \beta < 1$. Then β is either $\frac{w_{i_3}}{w_{i_0}}$ or $\frac{w_{i_3}}{w_{i_1}}$ or $\frac{w_{i_3}}{w_{i_2}}$.

Theorem 2.5[3]**.** Let $(V, 0)$ be a 3-dimensional isolated singularity defined by a weighted homogeneous polynomial $f(x, y, z, w) = 0$. Let p_g be the geometric genus, ν be the multiplicity and μ be the Milnor number of the singularity. Suppose $p_g > 0$. Then

$$\mu \geqslant 24\,p_g + 2\nu^3 - 5\nu^2 + 2\nu + 1 \tag{2.3}$$

and equality in (2.3) holds if and only if f is a homogeneous polynomial.

3 Proof of Main Theorem

Let a, b, c, and d be the weights of x, y, z, and w respectively so that $f(x, y, z, w)$ is a weighted homogeneous polynomial. We may assume without loss of generality that $a \geqslant b \geqslant c \geqslant d$. Note that p_g is precisely the number of positive integral solutions of $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{u}{d} \leqslant 1$, i.e. $p_g = \#\{(x, y, z, w) \in Z_+^4 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leqslant 1\}$. On the other hand $\mu = (a-1)(b-1)(c-1)(d-1)$. Recall that $\nu = \inf\{n \in Z_+ : n \geqslant \inf(a, b, c, d)\}$. If d is an integer, then the Main Theorem follows from Theorem 2.4.

If d is not an integer, then write $d = [d] + \beta$ with $0 < \beta < 1$. It follows that the multiplicity $\nu = [d] + 1 = d - \beta + 1$. By Lemma 2.1, β is either $\frac{d}{c}$ or $\frac{d}{b}$ or $\frac{d}{a}$. Suppose $a \geqslant b \geqslant c \geqslant d \geqslant 3$. The Main Theorem with a strict inequality in (1.1) follows from Theorem 2.5. For the case $a \geqslant b \geqslant c \geqslant d \geqslant 2$, in view of Theorem 2.5, the Main Theorem with a strict inequality in (1.1) is true for $p_g > 0$; and by Theorem 2.3, the Main Theorem with a strict inequality in (1.1) is true for $p_g \geqslant 0$ if $\beta = \frac{d}{b}$ or $\beta = \frac{d}{c}$. Therefore we only need to prove the Main Theorem with a strict inequality in (1.1) for the case $a \geqslant b \geqslant c \geqslant d = 2 + \beta = 2 + \frac{d}{a}$ and $p_g = 0$. In the following proof, $w = (w_0, w_1, w_2, w_3)$ is the weights of x, y, z, and w, but we do not assume any ordering for w_0, w_1, w_2, and w_3. For the case we consider now, $\nu = d - \beta + 1 = 3$, $a\beta = 2 + \beta$, $a = 1 + \frac{2}{\beta} > 3$, and $2\nu^3 - 5\nu^2 + 2\nu + 1 = 16$. Because $p_g = 0$ for this case we only need to show that $\nu = (a-1)(b-1)(c-1)(d-1) > 16$ for all the possible combination of a, b, c, and d. It is well known that any weighted homogeneous polynomial with an isolated singularity at the origin can be deformed into one of the following 19 classes of weighted homogeneous singularities with weights preserved (see ref. [7] and footnote 1) for example). Since the multiplicity remains constant in the constant weights family, the rest of the proof can be reduced to case by case analysis as follows:

Class 1. $f(z_0, z_1, z_2, z_3) = z_0^{a_0} + z_1^{a_1} + z_2^{a_2} + z_3^{a_3}$.

Since the weights are integers, we do not need to consider this case.

Class 2. $f(z_0, z_1, z_2, z_3) = z_0^{a_0} + z_1^{a_1} + z_2^{a_2} + z_2 z_3^{a_3}$, $w = (a_0, a_1, a_2, \frac{a_2 a_3}{a_2 - 1})$.

We only need to consider when $d = \frac{a_2 a_3}{a_2 - 1} = 2 + \frac{d}{a} = a_3 + \frac{a_3}{a_2 - 1}$. In particular, we have $a_2 \geqslant \frac{a_2 a_3}{a_2 - 1}$ which implies $1 \geqslant \frac{a_3}{a_2 - 1}$. Since d is not an integer $1 > \frac{a_3}{a_2 - 1} = \beta$. Hence $a = a_2$, and $a_3 = 2$. Without loss of generality we can assume $a_0 \geqslant a_1$. Hence $(a, b, c, d) = (a_2, a_0, a_1, \frac{a_2 a_3}{a_2 - 1})$ and $a = a_2, b = a_0, c = a_1$ are integer. Note that $a > 3$

1) Yau S S T, Yu Y. Algebraic classification of rational CR-structures on topological 5-spheres with transversal holomorphic S^1-action in $\mathbb{C}^4$. Math Nachrichten, (to appear)

and $b \geqslant c \geqslant d > 2$. So $a \geqslant 4$, $b \geqslant 3$, and $c \geqslant 3$. Therefore

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1) = (a-1)(b-1)(c-1)\Big(1+\frac{2}{a-1}\Big)\\ &= (b-1)(c-1)(a+1) \geqslant (2)(2)(5) = 20 > 16.\end{aligned}$$

Class 3. $f(z_0, z_1, z_2, z_3) = z_0^{a_0} + z_1^{a_1} + z_2^{a_2}z_3 + z_2z_3^{a_3}$, $w = (a_0, a_1, \frac{a_2a_3-1}{a_3-1}, \frac{a_2a_3-1}{a_2-1})$.

Without loss of generality we can let $d = \frac{a_2a_3-1}{a_2-1} = a_3 + \frac{a_3-1}{a_2-1}$. In particular, $\frac{a_2a_3-1}{a_3-1} \geqslant \frac{a_2a_3-1}{a_2-1}$ which implies $1 \geqslant \frac{a_3-1}{a_2-1}$. Since d is not an integer, $1 > \frac{a_3-1}{a_2-1} = \beta$. Hence $a_3 = 2$, $a = \frac{a_2a_3-1}{a_3-1} = (2a_2-1)$, $\beta = \frac{d}{a} = \frac{a_3-1}{a_2-1} = \frac{1}{a_2-1}$. Without loss of generality we can assume $(a,b,c,d) = (2a_2-1, a_0, a_1, 2+\frac{2}{a_2-1})$, because interchanging the value of b, and c does not change the value of μ. Observe that a, b, and c are integer because a_0, a_1, and a_2 are positive integers. So $b = a_0 \geqslant 3$, $c = a_1 \geqslant 3$, and $a_2 \geqslant 3$. Therefore

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1) = 2(a_2-1)(a_0-1)(a_1-1)\Big(\frac{a_2+1}{a_2-1}\Big)\\ &= 2(a_0-1)(a_1-1)(a_2+1) \geqslant (2)(2)(2)(4) = 32 > 16.\end{aligned}$$

Class 4. $f(z_0, z_1, z_2, z_3) = z_0^{a_0} + z_0z_1^{a_1} + z_2^{a_2} + z_2z_3^{a_3}$, $w = (a_0, \frac{a_0a_1}{a_0-1}, a_2, \frac{a_2a_3}{a_2-1})$.

Without loss of generality we can let $d = \frac{a_2a_3}{a_2-1} = a_3 + \frac{a_3}{a_2-1}$. Since d is not an integer , $a_2 > \frac{a_2a_3}{a_2-1}$ which implies $a_2 - 1 > a_3$. Hence $a_3 = 2$, $a = a_2 \geqslant 4$, $\beta = \frac{d}{a} = \frac{a_3}{a_2-1} = \frac{2}{a_2-1}$. Without loss of generality we can assume $(a,b,c,d) = (a_2, a_0, \frac{a_0a_1}{a_0-1}, 2+\frac{2}{a_2-1})$, because μ is symmetric with respect to b and c, and we don't use the condition $b \geqslant c$ in the proof except the fact that $b \geqslant d$. Observe that $b = a_0 \geqslant 3$, because $a_0 > d > 2$ and $a = a_2 \geqslant 4$. If $a_1 = 1$ then $\frac{a_0a_1}{a_0-1} < 2$ which is a contradiction. Hence $a_1 \geqslant 2$. It follows that

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1) = (a_2-1)(a_0-1)\Big(\frac{a_0a_1-a_0+1}{a_0-1}\Big)\Big(\frac{a_2+1}{a_2-1}\Big)\\ &= (a_0a_1-a_0+1)(a_2+1) = [a_0(a_1-1)+1](a_2+1)\\ &\geqslant [3(1)+1](5) = 20 > 16.\end{aligned}$$

Class 5. $f(z_0, z_1, z_2, z_3) = z_0^{a_0}z_1 + z_0z_1^{a_1} + z_2^{a_2} + z_2z_3^{a_3}$, $w = (\frac{a_0a_1-1}{a_1-1}, \frac{a_0a_1-1}{a_0-1}, a_2, \frac{a_2a_3}{a_2-1})$.

If $d = \frac{a_2a_3}{a_2-1}$, then $d = a_3 + \frac{a_3}{a_2-1}$. Since d is not an integer, $a_2 > \frac{a_2a_3}{a_2-1}$ which implies $a_2 - 1 > a_3$. Hence $a = a_2 \geqslant 4$ and $a_3 = 2$. Without loss of generality we can assume $(a,b,c,d) = (a_2, \frac{a_0a_1-1}{a_1-1}, \frac{a_0a_1-1}{a_0-1}, 2+\frac{2}{a_2-1})$. Similar to class 4

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1) = (a_2-1)\Big(\frac{a_0a_1-a_1}{a_1-1}\Big)\Big(\frac{a_0a_1-a_0}{a_0-1}\Big)\Big(\frac{a_2+1}{a_2-1}\Big)\\ &= a_0a_1(a_2+1) \geqslant (2)(2)(5) = 20 > 16.\end{aligned}$$

Without loss of generality we can consider $d = \frac{a_0a_1-1}{a_0-1}$, then $d = a_1 + \frac{a_1-1}{a_0-1}$. Since d is not an integer, $\frac{a_0a_1-1}{a_1-1} > \frac{a_0a_1-1}{a_0-1}$ which implies $a_0 > a_1$. Hence $a = \frac{a_0a_1-1}{a_1-1}$ and $a_1 = 2$. Without loss of generality we can assume $(a,b,c,d) = (\frac{a_0a_1-1}{a_1-1}, a_2, \frac{a_2a_3}{a_2-1}, 2+\frac{1}{a_0-1})$. Because $a_0 > a_1 = 2$, $a_0 \geqslant 3$. We also have $a_2 = b \geqslant 3$, because $a_2 > d > 2$. Hence

$$\mu = (a-1)(b-1)(c-1)(d-1) = a_1a_0(a_2+1) \geqslant (2)(3)(4) = 24 > 16.$$

Class 6. $f(z_0, z_1, z_2, z_3) = z_0^{a_0}z_1 + z_0z_1^{a_1} + z_2^{a_2}z_3 + z_2z_3^{a_3}$, $w = (\frac{a_0a_1-1}{a_1-1}, \frac{a_0a_1-1}{a_0-1}, \frac{a_2a_3-1}{a_3-1}, \frac{a_2a_3-1}{a_2-1})$.

Without loss of generality we can let $d = \frac{a_2a_3-1}{a_2-1} = a_3 + \frac{a_3-1}{a_2-1}$. Since d is not an integer, $\frac{a_2a_3-1}{a_3-1} > \frac{a_2a_3-1}{a_2-1}$, which implies $a_2 > a_3$. Hence $a_3 = 2$, $a = \frac{a_2a_3-1}{a_3-1} =$

$a_2 + \frac{a_2-1}{a_3-1} > 3$, $\beta = \frac{d}{a} = \frac{a_3-1}{a_2-1} = \frac{1}{a_2-1}$. Without loss of generality we can assume $(a,b,c,d) = (\frac{a_2a_3-1}{a_3-1}, \frac{a_0a_1-1}{a_1-1}, \frac{a_0a_1-1}{a_0-1}, 2+\frac{1}{a_2-1})$. Since $a_2 > a_3 = 2$, $a_2 \geqslant 3$. We also have $a_1 \geqslant 2$, because a_1 is positive integer and if $a_1 = 1$ then $\frac{a_0a_1-1}{a_0-1} = 1 < 2$, which would be a contradiction. Similarly we can prove $a_0 \geqslant 2$. Therefore

$$\mu = (a-1)(b-1)(c-1)(d-1) = \Big(\frac{a_3(a_2-1)}{a_3-1}\Big)\Big(\frac{a_1(a_0-1)}{a_1-1}\Big)\Big(\frac{a_0(a_1-1)}{a_0-1}\Big)\Big(\frac{a_2}{a_2-1}\Big)$$
$$= 2a_1a_0a_2 \geqslant 2(2)(2)(3) = 24 > 16.$$

Class 7. $f(z_0,z_1,z_2,z_3) = z_0^{a_0} + z_1^{a_1} + z_1z_2^{a_2} + z_2z_3^{a_3}$, weights $w = (a_0, a_1, \frac{a_1a_2}{a_1-1}, \frac{a_1a_2a_3}{a_1(a_2-1)+1})$, $a_1 \geqslant 2$.

If $d = \frac{a_1a_2}{a_1-1}$, then $d = a_2 + \frac{a_2}{a_1-1}$. Since d is not an integer, $a_1 > \frac{a_1a_2}{a_1-1}$ which implies $a_1-1 > a_2$. Hence $a = a_1 \geqslant 4$ and $a_2 = 2$. Observe that $\frac{a_1a_2a_3}{a_1(a_2-1)+1} = a_3 + \frac{(a_1-1)a_3}{a_1(a_2-1)+1} = a_3 + \frac{(a_1-1)a_3}{a_1+1}$. If $a_3 = 1$, then $a_3 + \frac{(a_1-1)a_3}{a_1+1} = 1 + \frac{a_1-1}{a_1+1} < 2$. Hence $a_3 \geqslant 2$. Without loss of generality we can assume $(a,b,c,d) = (a_1, a_0, \frac{a_1a_2a_3}{a_1(a_2-1)+1}, a_2 + \frac{a_2}{a_1-1})$. Observe that $a_0 > d > 2$. Therefore

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1)\\ &= (a_1-1)(a_0-1)\Big(\frac{a_1a_2a_3 - a_1a_2 + a_1 - 1}{a_1a_2 - a_1 + 1}\Big)\Big(\frac{a_1+1}{a_1-1}\Big)\\ &= (a_0-1)(2a_1a_3 - a_1 - 1) \geqslant (a_0-1)(3a_1-1) \geqslant (2)(11) > 16.\end{aligned}$$

If $d = \frac{a_1a_2a_3}{a_1(a_2-1)+1} = a_3 + \frac{(a_1-1)a_3}{a_1(a_2-1)+1}$, then $\frac{a_1a_2}{a_1-1} \geqslant \frac{a_1a_2a_3}{a_1(a_2-1)+1}$ which implies $1 \geqslant \frac{(a_1-1)a_3}{a_1(a_2-1)+1}$. Since d is not an integer, $1 > \frac{(a_1-1)a_3}{a_1(a_2-1)+1}$, and $d = 2 + \frac{(a_1-1)2}{a_1(a_2-1)+1}$ where $a_3 = 2$, and $a = \frac{a_1a_2}{a_1-1}$. From $a = \frac{a_1a_2}{a_1-1} > 3$ we get $a_1a_2 > 3a_1 - 3$. Without loss of generality we can assume $(a,b,c,d) = (\frac{a_1a_2}{a_1-1}, a_0, a_1, 2 + \frac{(a_1-1)2}{a_1(a_2-1)+1})$. We also have $a_0 \geqslant 3$, $a_1 \geqslant 3$ because a_0, a_1 are integers greater than or equal to $d > 2$. Therefore

$$\mu = (a-1)(b-1)(c-1)(d-1) = \Big(\frac{a_1a_2 - a_1 + 1}{a_1-1}\Big)(a_0-1)(a_1-1)\Big(\frac{a_1a_2 + a_1 - 1}{a_1a_2 - a_1 + 1}\Big)$$
$$= (a_0-1)(a_1a_2 + a_1 - 1) > (a_0-1)(4(a_1-1)) \geqslant (2)(8) = 16.$$

Class 8. $f(z_0,z_1,z_2,z_3) = z_0^{a_0} + z_1^{a_1} + z_1z_2^{a_2} + z_1z_3^{a_3} + z_2^pz_3^q$ with the condition $\frac{p(a_1-1)}{a_1a_2} + \frac{q(a_1-1)}{a_1a_3} = 1$, $a_1 \geqslant 2$, $w = (a_0, a_1, \frac{a_1a_2}{a_1-1}, \frac{a_1a_3}{a_1-1})$.

Without loss of generality, we only need to consider when $d = \frac{a_1a_3}{a_1-1}$. Since d is not an integer, $a_1 > \frac{a_1a_3}{a_1-1}$ which implies $a_1 - 1 > a_3$. Therefore $d = \frac{a_1a_3}{a_1-1} = a_3 + \frac{a_3}{a_1-1} = 2 + \frac{2}{a_1-1}$, where $a_3 = 2$ and $a = a_1$. Hence $a = a_1 \geqslant 4$, and $a_2 \geqslant a_3 = 2$. Without loss of generality we can assume $(a,b,c,d) = (a_1, a_0, \frac{a_1a_2}{a_1-1}, 2 + \frac{2}{a_1-1})$. Observe that $a_0 \geqslant 3$ and from $\frac{a_1a_2}{a_1-1} > 2$ $a_1a_2 > 2a_1 - 2$. If $a_2 = 2$, $c = d$ and note that d is not an integer. The condition $\frac{p(a_1-1)}{a_1a_2} + \frac{q(a_1-1)}{a_1a_3} = 1$ now becomes $\frac{p}{c} + \frac{q}{d} = \frac{1}{d}(p+q) = 1$ which is a contradiction because $p+q$ is an integer. Hence $a_2 \geqslant 3$. Therefore

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1) = (a_1-1)(a_0-1)\Big(\frac{a_1a_2 - a_1 + 1}{a_1-1}\Big)\Big(\frac{a_1+1}{a_1-1}\Big)\\ &= (a_0-1)\Big(\frac{a_1a_2 - a_1 + 1}{a_1-1}\Big)(a_1+1) = (a_0-1)\Big(\frac{a_1a_2}{a_1-1} - 1\Big)(a_1+1)\\ &> (a_0-1)(a_2-1)(a_1+1) \geqslant (2)(2)(5) > 16.\end{aligned}$$

Class 9. $f(z_0,z_1,z_2,z_3) = z_0^{a_0} + z_1^{a_1}z_3 + z_2^{a_2}z_3 + z_1z_3^{a_3} + z_1^pz_2^q$ with the condition $\frac{p(a_3-1)}{(a_1a_3-1)} + \frac{qa_1(a_3-1)}{a_2(a_1a_3-1)} = 1$, $w = (a_0, \frac{a_1a_3-1}{a_3-1}, \frac{a_2(a_1a_3-1)}{a_1(a_3-1)}, \frac{a_1a_3-1}{a_1-1})$.

Without loss of generality we only need to consider $d = \frac{a_1a_3-1}{a_1-1}$ and $d = \frac{a_2(a_1a_3-1)}{a_1(a_3-1)}$. We consider first when $d = \frac{a_1a_3-1}{a_1-1} = a_3 + \frac{a_3-1}{a_1-1}$. Then $\frac{a_1a_3-1}{a_3-1} \geqslant \frac{a_1a_3-1}{a_1-1}$ which implies $1 \geqslant \frac{a_3-1}{a_1-1}$. Since d is not an integer, $1 > \frac{a_3-1}{a_1-1}$ which implies $a_1 > a_3$. Therefore $d = 2 + \frac{1}{a_1-1}$, where $a_3 = 2$ and $a = \frac{a_1a_3-1}{a_3-1} = 2a_1 - 1$. Hence $a_1 \geqslant 3$ because $a_1 > a_3 = 2$. Without loss of generality we can assume $(a,b,c,d) = (\frac{a_1a_3-1}{a_3-1}, a_0, \frac{a_2(a_1a_3-1)}{a_1(a_3-1)}, 2+\frac{1}{a_1-1})$. Observe that if $a_2 = 1$, then $\frac{a_2(a_1a_3-1)}{a_1(a_3-1)} = \frac{2a_1-1}{a_1} < 2$. Hence $a_2 \geqslant 2$. $a_0 \geqslant 3$ because $a_0 \geqslant d > 2$. Therefore

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1) = (2(a_1-1))(a_0-1)\Big(\frac{2a_1a_2-a_2-a_1}{a_1}\Big)\Big(\frac{a_1}{a_1-1}\Big)\\ &= 2(a_0-1)(2a_1a_2-a_2-a_1) = 2(a_0-1)[a_1(a_2-1)+a_2(a_1-1)]\\ &> (2)(2)(3+4) = 28 > 16.\end{aligned}$$

If $d = \frac{a_2(a_1a_3-1)}{a_1(a_3-1)} = a_2 + \frac{a_2(a_1-1)}{a_1(a_3-1)}$, then $\frac{a_1a_3-1}{a_1-1} \geqslant \frac{a_2(a_1a_3-1)}{a_1(a_3-1)}$ which implies $1 \geqslant \frac{a_2(a_1-1)}{a_1(a_3-1)}$. Since d is not an integer, $1 > \frac{a_2(a_1-1)}{a_1(a_3-1)}$. Hence $d = 2 + \frac{2(a_1-1)}{a_1(a_3-1)}$, where $a_2 = 2$ and $a = \frac{a_1a_3-1}{a_1-1}$. From $1 > \frac{a_2(a_1-1)}{a_1(a_3-1)}$ and $a_2 = 2$, $a_1a_3 > 3a_1 - 2$. It follows that $a = \frac{a_1a_3-1}{a_1-1} > \frac{3a_1-3}{a_1-1} = 3$. This implies $a_3 \geqslant 3$, otherwise we cannot get $a > 3$. Without loss of generality we can assume $(a,b,c,d) = (\frac{a_1a_3-1}{a_1-1}, a_0, \frac{a_1a_3-1}{a_3-1}, 2 + \frac{2(a_1-1)}{a_1(a_3-1)})$. Hence $a_0 \geqslant 3$. If $a_1 = 1$ then $\frac{a_1a_3-1}{a_3-1} = 1$, hence $a_1 \geqslant 2$. If $a_1 = 2$ then $c = d = 2 + \frac{1}{a_3-1}$. Hence the condition $\frac{p(a_3-1)}{(a_1a_3-1)} + \frac{qa_1(a_3-1)}{a_2(a_1a_3-1)} = 1$ becomes $\frac{p}{c} + \frac{q}{d} = \frac{1}{d}(p+q) = 1$ which is a contradiction since $p+q$ is an integer. Therefore $a_1 \geqslant 3$. Observe also that $\frac{a_1a_3+a_1-2}{a_3-1} \geqslant \frac{3a_3+1}{a_3-1} > 3$. Therefore

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1) = \Big(\frac{a_1a_3-a_1}{a_1-1}\Big)(a_0-1)\Big(\frac{a_1a_3-a_3}{a_3-1}\Big)\Big(\frac{a_1a_3+a_1-2}{a_1(a_3-1)}\Big)\\ &= (a_0-1)a_3\Big(\frac{a_1a_3+a_1-2}{a_3-1}\Big) > (2)(3)(3) > 16.\end{aligned}$$

Class 10. $f(z_0,z_1,z_2,z_3) = z_0^{a_0} + z_1^{a_1}z_2 + z_2^{a_2}z_3 + z_1z_3^{a_3}$, $w = (a_0, \frac{a_1a_2a_3+1}{a_3(a_2-1)+1}, \frac{a_1a_2a_3+1}{a_1(a_3-1)+1}, \frac{a_1a_2a_3+1}{a_2(a_1-1)+1})$.

Without loss of generality, we only need to consider when $d = \frac{a_1a_2a_3+1}{a_2(a_1-1)+1} = a_3 + \frac{a_3(a_2-1)+1}{a_2(a_1-1)+1}$. Since d is not an integer, $\frac{a_1a_2a_3+1}{a_3(a_2-1)+1} > \frac{a_1a_2a_3+1}{a_2(a_1-1)+1}$ which implies $1 > \frac{a_3(a_2-1)+1}{a_2(a_1-1)+1}$. Hence $d = 2 + \frac{2(a_2-1)+1}{a_2(a_1-1)+1}$, where $a = \frac{a_1a_2a_3+1}{a_3(a_2-1)+1} > 3$, and $a_3 = 2$. Without loss of generality we can assume $(a,b,c,d) = (\frac{a_1a_2a_3+1}{a_3(a_2-1)+1}, a_0, \frac{2a_1a_2+1}{a_1+1}, 2 + \frac{2(a_2-1)+1}{a_2(a_1-1)+1})$. Observe that $a_0 \geqslant 3$. From $\frac{2a_1a_2+1}{a_1+1} > 2$ $a_2 \geqslant 2$. From $\frac{2a_1a_2+1}{2a_2-1} > 3$ we can show $a_1 \geqslant 2$. $1 > \frac{2(a_2-1)+1}{a_2(a_1-1)+1}$ implies either $a_1 \geqslant 2$, $a_2 > 2$ or $a_1 > 2$, $a_2 \geqslant 2$. Therefore

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1)\\ &= \Big(\frac{2a_1a_2-2a_2+2}{2a_2-1}\Big)(a_0-1)\Big(\frac{2a_1a_2-a_1}{a_1+1}\Big)\Big(\frac{a_1a_2+a_2}{a_1a_2-a_2+1}\Big)\\ &= 2(a_0-1)a_1a_2 > 2(2)(2)(2) = 16.\end{aligned}$$

Class 11. $f(z_0,z_1,z_2,z_3) = z_0^{a_0} + z_0z_1^{a_1} + z_1z_2^{a_2} + z_2z_3^{a_3}$, $w = (a_0, \frac{a_0a_1}{a_0-1}, \frac{a_0a_1a_2}{a_0(a_1-1)+1}, \frac{a_0a_1a_2a_3}{a_0a_1(a_2-1)+(a_0-1)})$.

If $d = \frac{a_0a_1}{a_0-1} = a_1 + \frac{a_1}{a_0-1}$: Since d is not an integer, $a_0 > \frac{a_0a_1}{a_0-1}$ which implies $1 > \frac{a_1}{a_0-1}$. Hence $d = 2 + \frac{2}{a_0-1}$, and $a_1 = 2$. It follows that $a = a_0 \geqslant 4$. Without loss of generality we can assume $(a,b,c,d) = (a_0, \frac{a_0a_1a_2}{a_0(a_1-1)+1}, \frac{a_0a_1a_2a_3}{a_0a_1(a_2-1)+(a_0-1)}, 2 + \frac{2}{a_0-1})$. If $a_2 = 1$, then $\frac{a_0a_1a_2}{a_0(a_1-1)+1} = \frac{2a_0}{a_0+1} < 2$. Hence $a_2 \geqslant 2$. If $a_3 = 1$, then $\frac{a_0a_1a_2a_3}{a_0a_1(a_2-1)+(a_0-1)} = \frac{2a_0a_2}{2a_0a_2-a_0-1} = 1 + \frac{a_0+1}{2a_0a_2-a_0-1} < 1 + \frac{a_0+1}{4a_0-a_0-1} < 2$. Hence $a_3 \geqslant 2$. Therefore

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1)\\ &= (a_0-1)\Big(\frac{a_0a_1a_2-a_0a_1+a_0-1}{a_0a_1-a_0+1}\Big)\Big(\frac{a_0a_1a_2a_3-a_0a_1a_2+a_0a_1-a_0+1}{a_0a_1(a_2-1)+(a_0-1)}\Big)\Big(\frac{a_0+1}{a_0-1}\Big)\\ &= \Big(\frac{2a_0a_2-a_0-1}{a_0+1}\Big)\Big(\frac{2a_0a_2a_3-2a_0a_2+a_0+1}{2a_0a_2-a_0-1}\Big)(a_0+1)\\ &= 2a_0a_2(a_3-1)+a_0+1 \geqslant 2(4)(2)(1)+4+1 = 21 > 16.\end{aligned}$$

If $d = \frac{a_0a_1a_2}{a_0(a_1-1)+1} = a_2 + \frac{(a_0-1)a_2}{a_0(a_1-1)+1}$: Since d is not an integer, $\frac{a_0a_1}{a_0-1} > \frac{a_0a_1a_2}{a_0(a_1-1)+1}$ which implies $1 > \frac{(a_0-1)a_2}{a_0(a_1-1)+1}$. Hence $d = 2 + \frac{2(a_0-1)}{a_0(a_1-1)+1}$, $a = \frac{a_0a_1}{a_0-1}$ and $a_2 = 2$. Without loss of generality we can assume $(a,b,c,d) = (\frac{a_0a_1}{a_0-1}, a_0, \frac{a_0a_1a_2a_3}{a_0a_1(a_2-1)+(a_0-1)}, 2 + \frac{2(a_0-1)}{a_0(a_1-1)+1})$. Hence $a_0 \geqslant 3$. From $\frac{a_0a_1}{a_0-1} > 3$, $a_0a_1 > 3(a_0-1)$, which is equivalent to $a_1 > 3 - \frac{3}{a_0}$. Since $a_0 \geqslant 3$, $a_1 > 3-1 = 2$ i.e. $a_1 \geqslant 3$. If $a_3 = 1$, then $\frac{a_0a_1a_2a_3}{a_0a_1(a_2-1)+(a_0-1)} = \frac{2a_0a_1}{a_0a_1+a_0-1} < \frac{2a_0a_1+2a_0-2}{a_0a_1+a_0-1} = 2$, hence $a_3 \geqslant 2$. Therefore

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1)\\ &= \Big(\frac{a_0a_1-a_0+1}{a_0-1}\Big)(a_0-1)\Big(\frac{2a_0a_1a_3-a_0a_1-a_0+1}{a_0a_1+a_0-1}\Big)\Big(\frac{a_0a_1+a_0-1}{a_0a_1-a_0+1}\Big)\\ &= 2a_0a_1a_3-a_0a_1-a_0+1 = a_0a_1(a_3-1)+a_0(a_1a_3-1)+1\\ &> 9+3(5)+1 = 25 > 16.\end{aligned}$$

If $d = \frac{a_0a_1a_2a_3}{a_0a_1(a_2-1)+(a_0-1)} = a_3 + \frac{a_0a_1a_3-a_0a_3+a_3}{a_0a_1(a_2-1)+(a_0-1)}$: Since d is not an integer, $\frac{a_0a_1a_2}{a_0(a_1-1)+1} > \frac{a_0a_1a_2a_3}{a_0a_1(a_2-1)+(a_0-1)}$ which implies $1 > \frac{a_0a_1a_3-a_0a_3+a_3}{a_0a_1(a_2-1)+(a_0-1)}$. Hence we get $a_3 = 2$ and $a = \frac{a_0a_1a_2}{a_0(a_1-1)+1}$. Without loss of generality we can assume $(a,b,c,d) = (\frac{a_0a_1a_2}{a_0(a_1-1)+1}, a_0, \frac{a_0a_1}{a_0-1}, 2 + \frac{2a_0a_1-2a_0+2}{a_0a_1(a_2-1)+(a_0-1)})$. If $a_1 = 1$, then $\frac{a_0a_1}{a_0-1} \leqslant 2$. This contradicts the fact that $\frac{a_0a_1}{a_0-1} \geqslant d > 2$. Hence $a_1 \geqslant 2$. Clearly $a_0 \geqslant 3$. If $a_2 = 1$, then $\frac{a_0a_1a_2}{a_0a_1-a_0-1} = 1 + \frac{a_0-1}{a_0a_1-a_0+1} < 2$. Hence $a_2 \geqslant 2$. Observe that $1 > \frac{2a_0a_1-2a_0+2}{a_0a_1(a_2-1)+(a_0-1)}$ implies $a_0a_1a_2 > 3a_0a_1 - 3a_0 + 3$. Therefore

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1)\\ &= \Big(\frac{a_0a_1a_2-a_0a_1+a_0-1}{a_0a_1-a_0+1}\Big)(a_0-1)\Big(\frac{a_0a_1-a_0+1}{a_0-1}\Big)\Big(\frac{a_0a_1a_2+a_0a_1-a_0+1}{a_0a_1(a_2-1)+a_0-1}\Big)\\ &= a_0a_1a_2+a_0(a_1-1)+1 > 3a_0a_1-3a_0+3+a_0a_1-a_0+1\\ &= 4a_0a_1-4a_0+4 \geqslant 4a_0+4 \geqslant 16.\end{aligned}$$

Class 12. $f(z_0,z_1,z_2,z_3) = z_0^{a_0} + z_0z_1^{a_1} + z_0z_2^{a_2} + z_2z_3^{a_3} + z_1^pz_2^q$ with the condition $\frac{p(a_0-1)}{a_0a_1} + \frac{q(a_0-1)}{a_0a_2} = 1$, $w = (a_0, \frac{a_0a_1}{a_0-1}, \frac{a_0a_2}{a_0-1}, \frac{a_0a_1a_3}{a_0(a_1-1)+1})$.

Without loss of generality we only need to consider $d = \frac{a_0a_2}{a_0-1}$ and $d = \frac{a_0a_1a_3}{a_0(a_1-1)+1}$. We consider first when $d = \frac{a_0a_2}{a_0-1} = a_2 + \frac{a_2}{a_0-1}$. Since d is not an integer, $a_0 > \frac{a_0a_2}{a_0-1}$ which implies $1 > \frac{a_2}{a_0-1}$. Hence $d = 2 + \frac{a_2}{a_0-1}$, $a = a_0$ and $a_2 = 2$. It follows that $a_0 \geqslant 4$. Without loss of generality we can assume $(a,b,c,d) = (a_0, \frac{a_0a_1}{a_0-1}, \frac{a_0a_1a_3}{a_0(a_1-1)+1}, \frac{a_0a_2}{a_0-1})$. Hence $a_1 \geqslant a_2$. Observe that if $a_1 = 1$, then $\frac{a_0a_1}{a_0-1} = \frac{a_0}{a_0-1} < 2$. This contradicts the fact that $\frac{a_0a_1}{a_0-1} \geqslant d > 2$. Hence $a_1 \geqslant 2$. If $a_3 = 1$, then $\frac{a_0a_1a_3}{a_0(a_1-1)+1} = \frac{a_0a_1}{a_0(a_1-1)+1} < 2$. Hence $a_3 \geqslant 2$. If $a_1 = 2$, then $b = d = 2 + \frac{2}{a_0-1}$. This implies $\frac{p}{b} + \frac{q}{d} = \frac{1}{d}(p+q) = 1$

which contradicts the fact that d is not an integer. Hence $a_1 \geqslant 3$. Therefore

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1)\\ &= (a_0-1)\Big(\frac{a_0(a_1-1)+1}{a_0-1}\Big)\Big(\frac{a_0a_1a_3-a_0a_1+a_0-1}{a_0a_1-a_0+1}\Big)\Big(\frac{a_0+1}{a_0-1}\Big)\\ &= [a_0a_1(a_3-1)+a_0-1]\Big(\frac{a_0+1}{a_0-1}\Big) = \Big[\frac{a_0a_1(a_3-1)}{a_0-1}+1\Big](a_0+1)\\ &> [a_1(a_3-1)+1](a_0+1) \geqslant 20 > 16.\end{aligned}$$

If $d = \frac{a_0a_1a_3}{a_0(a_1-1)+1} = a_3 + \frac{(a_0-1)a_3}{a_0(a_1-1)+1}$: Since d is not an integer, $\frac{a_0a_1}{a_0-1} > \frac{a_0a_1a_3}{a_0(a_1-1)+1}$ which implies $1 > \frac{(a_0-1)a_3}{a_0(a_1-1)+1}$. Hence $d = 2 + \frac{2(a_0-1)}{a_0a_1-a_0+1}$, $a_3 = 2$ and $a = \frac{a_0a_1}{a_0-1} > 3$. Without loss of generality we can assume $(a,b,c,d) = (\frac{a_0a_1}{a_0-1}, a_0, \frac{a_0a_2}{a_0-1}, 2+\frac{2(a_0-1)}{a_0a_1-a_0+1})$. If $a_2 = 1$, then $\frac{a_0a_2}{a_0-1} = \frac{a_0}{a_0-1} \leqslant 2$, which contradicts to the fact that $\frac{a_0a_2}{a_0-1} \geqslant d > 2$. Hence $a_2 \geqslant 2$. Therefore

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1)\\ &= \Big(\frac{a_0a_1-a_0+1}{a_0-1}\Big)(a_0-1)\Big(\frac{a_0a_2-a_0+1}{a_0-1}\Big)\Big(\frac{a_0a_1+a_0-1}{a_0a_1-a_0+1}\Big)\\ &= [a_0(a_2-1)+1](\tfrac{a_0a_1}{a_0-1}+1) > [3+1](4) = 16.\end{aligned}$$

Class 13. $f(z_0,z_1,z_2,z_3) = z_0^{a_0} + z_0z_1^{a_1} + z_1z_2^{a_2} + z_1z_3^{a_3} + z_2^pz_3^q$ with the condition $\frac{p(a_0a_1-a_0+1)}{a_0a_1a_2} + \frac{q(a_0a_1-a_0+1)}{a_0a_1a_3} = 1$, $w = (a_0, \frac{a_0a_1}{a_0-1}, \frac{a_0a_1a_2}{a_0(a_1-1)+1}, \frac{a_0a_1a_3}{a_0(a_1-1)+1})$.

Without loss of generality we only need to consider $d = \frac{a_0a_1}{a_0-1}$ and $d = \frac{a_0a_1a_3}{a_0(a_1-1)+1}$. We consider first when $d = \frac{a_0a_1}{a_0-1} = a_1 + \frac{a_1}{a_0-1}$. Since d is not an integer, $a_0 > \frac{a_0a_1}{a_0-1}$ which implies $1 > \frac{a_1}{a_0-1}$. Hence $d = 2 + \frac{2}{a_0-1}$, $a_1 = 2$ and $a = a_0 \geqslant 4$. Without loss of generality we can assume $(a,b,c,d) = (a_0, \frac{a_0a_1a_2}{a_0(a_1-1)+1}, \frac{a_0a_1a_3}{a_0(a_1-1)+1}, 2+\frac{2}{a_0-1})$. Observe that if $a_2 = 1$, then $\frac{a_0a_1a_2}{a_0(a_1-1)+1} = \frac{2a_0}{a_0+1} < 2$. Hence $a_2 \geqslant 2$. Similar argument will show that $a_3 \geqslant 2$. Therefore

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1) = (a_0-1)\Big(\frac{2a_0a_2}{a_0+1}-1\Big)\Big(\frac{2a_0a_3}{a_0+1}-1\Big)\Big(\frac{a_0+1}{a_0-1}\Big)\\ &= \frac{(2a_0a_2-a_0-1)(2a_0a_3-a_0-1)}{a_0+1} = \frac{[a_0(2a_2-1)-1][a_0(2a_3-1)-1]}{a_0+1}\\ &\geqslant \frac{(3a_0-1)(3a_0-1)}{a_0+1} > 16 \quad \text{for } a_0 \geqslant 4.\end{aligned}$$

If $d = \frac{a_0a_1a_3}{a_0(a_1-1)+1} = a_3 + \frac{(a_0-1)a_3}{a_0(a_1-1)+1}$: Since d is not an integer, $\frac{a_0a_1}{a_0-1} > \frac{a_0a_1a_3}{a_0(a_1-1)+1}$ which implies $1 > \frac{(a_0-1)a_3}{a_0(a_1-1)+1}$. Hence $d = 2 + \frac{2(a_0-1)}{a_0a_1-a_0+1}$, $a_3 = 2$ and $a = \frac{a_0a_1}{a_0-1} > 3$. Without loss of generality we can assume $(a,b,c,d) = (\frac{a_0a_1}{a_0-1}, a_0, \frac{a_0a_1a_2}{a_0(a_1-1)+1}, 2+\frac{2(a_0-1)}{a_0a_1-a_0+1})$. Observe that $a_0 \geqslant 3$ because $a_0 \geqslant d > 2$. If $a_1 = 1$, then $\frac{a_0a_1}{a_0-1} = \frac{a_0}{a_0-1} < 2$. Hence $a_1 \geqslant 2$. If $a_1 = 2$, then $\frac{a_0a_1}{a_0-1} = \frac{2a_0}{a_0-1} \leqslant 3$. Hence $a_1 \geqslant 3$. If $a_2 = 1$, then $\frac{a_0a_1a_2}{a_0(a_1-1)+1} = \frac{a_0a_1}{a_0(a_1-1)+1} < 2$. Hence $a_2 \geqslant 2$. If $a_2 = 2$, then $c = d = 2 + \frac{2(a_0-1)}{a_0a_1-a_0+1}$. Hence the condition $\frac{p(a_0a_1-a_0+1)}{a_0a_1a_2} + \frac{q(a_0a_1-a_0+1)}{a_0a_1a_3} = 1$ becomes $\frac{p}{c} + \frac{q}{d} = \frac{1}{d}(p+q) = 1$ which contradicts the fact that d is not an integer. Hence $a_2 \geqslant 3$. It follows that

$(\frac{a_0a_1a_2}{a_0(a_1-1)+1}-1) \geqslant \frac{3a_0a_1}{a_0a_1-a_0+1}-1 = 2+\frac{3a_0-3}{a_0a_1-a_0+1} \geqslant 2$. Therefore

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1)\\ &= \Big(\frac{a_0a_1-a_0+1}{a_0-1}\Big)(a_0-1)\Big(\frac{a_0a_1a_2-a_0a_1+a_0-1}{a_0a_1-a_0+1}\Big)\Big(\frac{a_0a_1+a_0-1}{a_0a_1-a_0+1}\Big)\\ &= \Big(\frac{a_0a_1a_2-a_0a_1+a_0-1}{a_0a_1-a_0+1}\Big)(a_0a_1+a_0-1) = \Big(\frac{a_0a_1a_2}{a_0(a_1-1)+1}-1\Big)(a_0a_1+a_0-1)\\ &\geqslant (2)(9+3-1) = 22 > 16.\end{aligned}$$

Class 14. $f(z_0,z_1,z_2,z_3) = z_0^{a_0}+z_0z_1^{a_1}+z_0z_2^{a_2}+z_0z_3^{a_3}+z_1^pz_2^q+z_2^rz_3^s$ with the condition $\frac{p(a_0-1)}{a_0a_1}+\frac{q(a_0-1)}{a_0a_2}=1$ and $\frac{r(a_0-1)}{a_0a_2}+\frac{s(a_0-1)}{a_0a_3}=1$, $w=(a_0,\frac{a_0a_1}{a_0-1},\frac{a_0a_2}{a_0-1},\frac{a_0a_3}{a_0-1})$.

Without loss of generality we only need to consider $d=\frac{a_0a_3}{a_0-1}$ or $d=\frac{a_0a_2}{a_0-1}$. If $d=\frac{a_0a_3}{a_0-1}=a_3+\frac{a_3}{a_0-1}$: Since d is not an integer, $a_0>\frac{a_0a_3}{a_0-1}$ which implies $1>\frac{a_3}{a_0-1}$. Hence $d=2+\frac{2}{a_0-1}$, $a_3=2$ and $a=a_0\geqslant 4$. Consider first the case when $(a,b,c,d)=(a_0,\frac{a_0a_1}{a_0-1},\frac{a_0a_2}{a_0-1},2+\frac{2}{a_0-1})$. If $a_1=1$, then $\frac{a_0a_1}{a_0-1}<2$, which contradicts to the fact that $\frac{a_0a_1}{a_0-1}\geqslant d>2$. Hence $a_1\geqslant 2$. If $a_1=2$, then $b=2+\frac{2}{a_0-1}=d$, which implies $b=c=d$. Hence $\frac{p(a_0-1)}{a_0a_1}+\frac{q(a_0-1)}{a_0a_2}=\frac{p}{b}+\frac{q}{c}=\frac{1}{d}(p+q)=1$, which contradicts d is not an integer. Hence $a_1\geqslant 3$. If $a_2=1$, then $\frac{a_0a_2}{a_0-1}<2$, which contradicts $\frac{a_0a_2}{a_0-1}\geqslant d>2$. Hence $a_2\geqslant 2$. If $a_2=2$, then $c=d=2+\frac{2}{a_0-1}$. Hence the condition $\frac{r(a_0-1)}{a_0a_2}+\frac{s(a_0-1)}{a_0a_3}=1$ becomes $\frac{r}{c}+\frac{s}{d}=\frac{1}{d}(r+s)=1$, which contradicts d is not an integer. Hence $a_2\geqslant 3$. If $(a,b,c,d)=(a_0,\frac{a_0a_2}{a_0-1},\frac{a_0a_1}{a_0-1},2+\frac{2}{a_0-1})$ we can similarly prove $a_1\geqslant 3$ and $a_2\geqslant 3$. Therefore

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1)=(a_0-1)\Big(\frac{a_0a_1-a_0+1}{a_0-1}\Big)\Big(\frac{a_0a_2-a_0+1}{a_0-1}\Big)\Big(\frac{a_0+1}{a_0-1}\Big)\\ &= \Big(\frac{a_0a_1}{a_0-1}-1\Big)\Big(\frac{a_0a_2}{a_0-1}-1\Big)(a_0+1) > (2)(2)(5) = 20 > 16.\end{aligned}$$

We next consider $d=\frac{a_0a_2}{a_0-1}=a_2+\frac{a_2}{a_0-1}$. Since d is not an integer, $a_0>\frac{a_0a_2}{a_0-1}$ which implies $1>\frac{a_2}{a_0-1}$. Hence $d=2+\frac{2}{a_0-1}$, $a_2=2$ and $a=a_0\geqslant 4$. Consider first the case when $(a,b,c,d)=(a_0,\frac{a_0a_1}{a_0-1},\frac{a_0a_3}{a_0-1},2+\frac{2}{a_0-1})$. If $a_1=1$, then $\frac{a_0a_1}{a_0-1}<2$, which contradicts to the fact that $\frac{a_0a_1}{a_0-1}\geqslant d>2$. Hence $a_1\geqslant 2$. If $a_1=2$, then $b=2+\frac{2}{a_0-1}=d$, which implies $b=c=d$. Hence $\frac{p(a_0-1)}{a_0a_1}+\frac{q(a_0-1)}{a_0a_2}=\frac{p}{b}+\frac{q}{d}=\frac{1}{d}(p+q)=1$, which contradicts d is not an integer. Hence $a_1\geqslant 3$. If $a_3=1$, then $\frac{a_0a_3}{a_0-1}<2$, which contradicts $\frac{a_0a_3}{a_0-1}\geqslant d>2$. Hence $a_3\geqslant 2$. If $a_3=2$, then $c=d=2+\frac{2}{a_0-1}$. Hence the condition $\frac{r(a_0-1)}{a_0a_2}+\frac{s(a_0-1)}{a_0a_3}=1$ becomes $\frac{r}{c}+\frac{s}{d}=\frac{1}{d}(r+s)=1$, which contradicts d is not an integer. Hence $a_3\geqslant 3$. If $(a,b,c,d)=(a_0,\frac{a_0a_3}{a_0-1},\frac{a_0a_1}{a_0-1},2+\frac{2}{a_0-1})$ we can similarly prove $a_1\geqslant 3$ and $a_3\geqslant 3$. Therefore

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1)=(a_0-1)\Big(\frac{a_0a_1-a_0+1}{a_0-1}\Big)\Big(\frac{a_0a_3-a_0+1}{a_0-1}\Big)\Big(\frac{a_0+1}{a_0-1}\Big)\\ &= \Big(\frac{a_0a_1}{a_0-1}-1\Big)\Big(\frac{a_0a_3}{a_0-1}-1\Big)(a_0+1) > (2)(2)(5) = 20 > 16.\end{aligned}$$

Class 15. $f(z_0,z_1,z_2,z_3) = z_0^{a_0}z_1+z_0z_1^{a_1}+z_0z_2^{a_2}+z_2z_3^{a_3}+z_1^pz_2^q$ with the condition $\frac{p(a_0-1)}{a_0a_1-1}+\frac{qa_1(a_0-1)}{a_2(a_0a_1-1)}=1$, $w=(\frac{a_0a_1-1}{a_1-1},\frac{a_0a_1-1}{a_0-1},\frac{a_0a_1a_2-a_2}{a_0a_1-a_1},\frac{a_2a_3(a_0a_1-1)}{a_2(a_0a_1-1)-a_1(a_0-1)})$.

If $d=\frac{a_0a_1-1}{a_0-1}=a_1+\frac{a_1-1}{a_0-1}$: Since d is not an integer, $\frac{a_0a_1-1}{a_1-1}>\frac{a_0a_1-1}{a_0-1}$ which implies $1>\frac{a_1-1}{a_0-1}$. Hence $d=2+\frac{1}{a_0-1}$, $a=\frac{a_0a_1-1}{a_1-1}$ and $a_1=2$. Observe that $a=\frac{a_0a_1-1}{a_1-1}=2a_0-1>3$, which implies $a_0\geqslant 3$. If $a_2=1$ then $\frac{a_0a_1a_2-a_2}{a_0a_1-a_1}=\frac{2a_0-1}{2a_0-2}<2$. Hence $a_2\geqslant 2$. If $a_2=2$, then $\frac{p+q}{d}=1$ which implies d is an integer. Therefore $a_2\geqslant 3$.

If $a_3 = 1$ then $\frac{a_2a_3(a_0a_1-1)}{a_2(a_0a_1-1)-a_1(a_0-1)} = \frac{2a_0a_2-a_2}{2a_0a_2-a_2-2a_0+2} = 1 + \frac{2a_0-2}{2a_0a_2-a_2-2a_0+2}$. From $(2a_0-1)(a_2-2)+2 > 0$, $2a_0a_2 - a_2 - 4a_0 + 4 > 0$ i.e. $2a_0 - 2 < 2a_0a_2 - a_2 - 2a_0 + 2$. Therefore $1 + \frac{2a_0-2}{2a_0a_2-a_2-2a_0+2} < 2$. Hence $a_3 \geqslant 2$. Without loss of generality we can assume $(a,b,c,d) = (\frac{a_0a_1-1}{a_1-1}, \frac{a_0a_1a_2-a_2}{a_0a_1-a_1}, \frac{a_2a_3(a_0a_1-1)}{a_2(a_0a_1-1)-a_1(a_0-1)}, 2+\frac{1}{a_0-1})$. Therefore

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1)\\ &= 2(a_0-1)\Big(\frac{2a_0a_2-a_2-2a_0+2}{2a_0-2}\Big)\Big(\frac{2a_0a_2a_3-a_2a_3-2a_0a_2+a_2+2a_0-2}{2a_0a_2-a_2-2a_0+2}\Big)\Big(\frac{a_0}{a_0-1}\Big)\\ &= (2a_0a_2a_3-a_2a_3-2a_0a_2+a_2+2a_0-2)\Big(\frac{a_0}{a_0-1}\Big)\\ &= [a_2a_3(a_0-1)+a_0a_2(a_3-1)-a_2(a_0-1)+2(a_0-1)]\Big(\frac{a_0}{a_0-1}\Big)\\ &= \Big[a_2a_3+\frac{a_0a_2(a_3-1)}{a_0-1}-a_2+2\Big]a_0 \geqslant [(2)(3)+2]3 = 24 > 16.\end{aligned}$$

If $d = \frac{a_0a_1-1}{a_1-1} = a_0 + \frac{a_0-1}{a_1-1}$: Since d is not an integer, $\frac{a_0a_1-1}{a_0-1} > \frac{a_0a_1-1}{a_1-1}$ which implies $1 > \frac{a_0-1}{a_1-1}$. Hence $d = 2+\frac{1}{a_1-1}$, $a = \frac{a_0a_1-1}{a_0-1}$ and $a_0 = 2$. Observe that $a = \frac{a_0a_1-1}{a_0-1} = a_1 + \frac{a_1-1}{a_0-1} = 2a_1-1 > 3$, which implies $a_1 \geqslant 3$. If $a_2 = 1$ then $\frac{a_0a_1a_2-a_2}{a_0a_1-a_1} = \frac{2a_1-1}{a_1} < 2$. Hence $a_2 \geqslant 2$. If $a_3 = 1$ then $\frac{a_2a_3(a_0a_1-1)}{a_2(a_0a_1-1)-a_1(a_0-1)} = \frac{2a_1a_2-a_2}{2a_1a_2-a_2-a_1} < 2$. Hence $a_3 \geqslant 2$. Without loss of generality we can assume $(a,b,c,d) = (\frac{a_0a_1-1}{a_0-1}, \frac{a_0a_1a_2-a_2}{a_0a_1-a_1}, \frac{a_2a_3(a_0a_1-1)}{a_2(a_0a_1-1)-a_1(a_0-1)}, 2+\frac{1}{a_1-1})$. Therefore

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1)\\ &= 2(a_1-1)\Big(\frac{2a_1a_2-a_2-a_1}{a_1}\Big)\Big(\frac{2a_1a_2a_3-a_2a_3-2a_1a_2+a_2+a_1}{2a_1a_2-a_2-a_1}\Big)\Big(\frac{a_1}{a_1-1}\Big)\\ &= 2[a_2a_3(a_1-1)+a_1a_2(a_3-2)+a_2+a_1]\\ &\geqslant 2[(2)(2)(2)+2+3] = 26 > 16.\end{aligned}$$

If $d = \frac{a_0a_1a_2-a_2}{a_0a_1-a_1} = a_2 + \frac{a_2(a_1-1)}{a_1(a_0-1)}$: Since d is not an integer, $\frac{a_0a_1-1}{a_1-1} > \frac{a_0a_1a_2-a_2}{a_0a_1-a_1}$ which implies $1 > \frac{a_2(a_1-1)}{a_1(a_0-1)}$. Hence $d = 2 + \frac{2(a_1-1)}{a_1(a_0-1)}$, $a = \frac{a_0a_1-1}{a_1-1}$ and $a_2 = 2$. From $\frac{a_0a_1-1}{a_1-1} \geqslant \frac{a_0a_1-1}{,}a_0 - 1$ $a_0 \geqslant a_1$. If $a_1 = 1$ then $\frac{a_0a_1-1}{a_0-1} = 1 < 2$. Hence $a_1 \geqslant 2$. Observe that $\frac{a_0a_1-1}{a_1-1} = a_0 + \frac{a_0-1}{a_1-1} > 3$. Hence $a_0 \geqslant 3$. If $a_3 = 1$ then $\frac{a_2a_3(a_0a_1-1)}{a_2(a_0a_1-1)-a_1(a_0-1)} = \frac{a_2(a_0a_1-1)}{a_2(a_0a_1-1)-a_1(a_0-1)} = 1+\frac{a_0a_1-a_1}{a_2(a_0a_1-1)-a_1(a_0-1)} = 1+\frac{a_0a_1-a_1}{a_0a_1+a_1-2} < 2$, hence $a_3 \geqslant 2$. Without loss of generality we can assume $(a,b,c,d) = (\frac{a_0a_1-1}{a_1-1}, \frac{a_0a_1-1}{a_0-1}, \frac{a_2a_3(a_0a_1-1)}{a_2(a_0a_1-1)-a_1(a_0-1)}, 2+\frac{2(a_1-1)}{a_1(a_0-1)})$. Therefore

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1)\\ &= \Big(\frac{a_0a_1-1}{a_1-1}-1\Big)\Big(\frac{a_0a_1-1}{a_0-1}-1\Big)\Big(\frac{2a_0a_1a_3-2a_3-2a_0a_1+2+a_0a_1-a_1}{2a_0a_1-2-a_0a_1+a_1}\Big)\Big(\frac{a_0a_1+a_1-2}{a_0a_1-a_1}\Big)\\ &= \Big(\frac{a_0}{a_0-1}\Big)(2a_0a_1a_3-a_0a_1-2a_3-a_1+2) = a_0\Big[\frac{(a_0a_1-1)(2a_3-2)}{a_0-1}+a_1\Big]\\ &> (a_0a_1-1)(2a_3-2)+a_0a_1 \geqslant (5)(2)+6 = 16.\end{aligned}$$

If $d = \frac{a_2a_3(a_0a_1-1)}{a_2(a_0a_1-1)-a_1(a_0-1)} = a_3 + \frac{a_1a_3(a_0-1)}{a_2(a_0a_1-1)-a_1(a_0-1)}$: Since d is not an integer, $\frac{a_0a_1a_2-a_2}{a_0a_1-a_1} > \frac{a_2a_3(a_0a_1-1)}{a_2(a_0a_1-1)-a_1(a_0-1)}$ which implies $1 > \frac{a_1a_3(a_0-1)}{a_2(a_0a_1-1)-a_1(a_0-1)}$. Hence $d = 2 + \frac{2a_1(a_0-1)}{a_2(a_0a_1-1)-a_1(a_0-1)}$, $a = \frac{a_0a_1a_2-a_2}{a_0a_1-a_1} > 3$ and $a_3 = 2$. If $a_0 = 1$, then $\frac{a_0a_1-1}{a_1-1} = \frac{a_1-1}{a_1-1} = 1$. Hence $a_0 \geqslant 2$. If $a_1 = 1$, then $\frac{a_0a_1-1}{a_0-1} = 1$. Hence $a_1 \geqslant 2$. If $a_2 = 1$, then $\frac{a_0a_1a_2-a_2}{a_0a_1-a_1} = \frac{a_0a_1-1}{a_0a_1-a_1} = 1+\frac{a_1-1}{a_0a_1-a_1} < 1+\frac{a_1}{a_1(a_0-1)} \leqslant 2$. Hence $a_2 \geqslant 2$. $1 > \frac{2a_1(a_0-1)}{a_2(a_0a_1-1)-a_1(a_0-1)}$ implies $\frac{a_0a_1a_2-a_2}{a_0a_1-a_1} > 3$. Without loss of generality we can assume $(a,b,c,d) = (\frac{a_0a_1a_2-a_2}{a_0a_1-a_1}, \frac{a_0a_1-1}{a_1-1}, \frac{a_0a_1-1}{a_0-1}, 2+\frac{2a_1(a_0-1)}{a_2(a_0a_1-1)-a_1(a_0-1)})$. Therefore

$$\begin{aligned}
\mu &= (a-1)(b-1)(c-1)(d-1)\\
&= \Big(\frac{a_0a_1a_2-a_2-a_0a_1+a_1}{a_0a_1-a_1}\Big)\Big(\frac{a_0a_1-a_1}{a_1-1}\Big)\Big(\frac{a_0a_1-a_0}{a_0-1}\Big)\Big(\frac{a_0a_1a_2-a_2+a_0a_1-a_1}{a_0a_1(a_2-1)+a_1-a_2}\Big)\\
&= \Big(\frac{a_0}{a_0-1}\Big)(a_0a_1a_2-a_2+a_0a_1-a_1)\\
&= \Big(\frac{a_0}{a_0-1}\Big)[a_2(a_0a_1-1)+a_1(a_0-1)]\\
&= \frac{a_0a_2(a_0a_1-1)}{a_0-1}+a_0a_1 = a_0a_1\Big[\frac{a_2(a_0a_1-1)}{a_1(a_0-1)}+1\Big] > a_0a_1(4) \geqslant 16.
\end{aligned}$$

Class 16. $f(z_0,z_1,z_2,z_3) = z_0^{a_0}z_1 + z_0z_1^{a_1} + z_0z_2^{a_2} + z_0z_3^{a_3} + z_1^pz_2^q + z_2^rz_3^s$ with the condition $\frac{p(a_0-1)}{a_0a_1-1}+\frac{qa_1(a_0-1)}{a_2(a_0a_1-1)} = 1$ and $\frac{ra_1(a_0-1)}{a_2(a_0a_1-1)}+\frac{sa_1(a_0-1)}{a_3(a_0a_1-1)} = 1$, $w = (\frac{a_0a_1-1}{a_1-1}, \frac{a_0a_1-1}{a_0-1}, \frac{a_2(a_0a_1-1)}{a_1(a_0-1)}, \frac{a_3(a_0a_1-1)}{a_1(a_0-1)})$.

If $d = \frac{a_0a_1-1}{a_0-1} = a_1+\frac{a_1-1}{a_0-1}$: Since d is not an integer, $\frac{a_0a_1-1}{a_1-1} > \frac{a_0a_1-1}{a_0-1}$, which implies $1 > \frac{a_1-1}{a_0-1}$. Hence $d = 2+\frac{1}{a_0-1}$, $a = \frac{a_0a_1-1}{a_1-1}$ and $a_1 = 2$. Observe that $a = \frac{a_0a_1-1}{a_1-1} = 2a_0-1 > 3$. Hence $a_0 \geqslant 3$. If $a_2 = 1$ then $\frac{a_0a_1a_2-a_2}{a_0a_1-a_1} = \frac{2a_0-1}{2a_0-2} < 2$, hence $a_2 \geqslant 2$. If $a_2 = 2$ then $\frac{a_2(a_0a_1-1)}{a_1(a_0-1)} = 2+\frac{1}{a_0-1}$. Hence $\frac{p+q}{d} = 1$ which implies d is an integer. Hence $a_2 \geqslant 3$. If $a_3 = 1$, then $\frac{a_3(a_0a_1-1)}{a_1(a_0-1)} = 1+\frac{a_1-1}{a_1(a_0-1)} < 2$. Hence $a_3 \geqslant 2$. Without loss of generality we can assume $(a,b,c,d) = (\frac{a_0a_1-1}{a_1-1}, \frac{a_2(a_0a_1-1)}{a_1(a_0-1)}, \frac{a_3(a_0a_1-1)}{a_1(a_0-1)}, 2+\frac{1}{a_0-1})$. Therefore

$$\begin{aligned}
\mu &= (a-1)(b-1)(c-1)(d-1)\\
&= \Big(\frac{a_1a_0-a_1}{a_1-1}\Big)\Big(\frac{a_0a_1a_2-a_2-a_1a_0+a_1}{a_1a_0-a_1}\Big)\Big(\frac{a_0a_1a_3-a_3-a_1a_0+a_1}{a_1a_0-a_1}\Big)\Big(\frac{a_0}{a_0-1}\Big)\\
&= \Big[\frac{(a_0a_1-1)(a_2-1)+(a_1-1)}{a_1-1}\Big]\Big[\frac{a_3(a_0a_1-1)-a_1(a_0-1)}{a_1(a_0-1)}\Big]\Big(\frac{a_0}{a_0-1}\Big)\\
&= \Big[\Big(\frac{a_0a_1-1}{a_1-1}\Big)(a_2-1)+1\Big]\Big(\frac{a_3(a_0a_1-1)}{a_1(a_0-1)}-1\Big)\Big(\frac{a_0}{a_0-1}\Big)\\
&= [(2a_0-1)(a_2-1)+1]\Big[a_3+\frac{a_3}{2(a_0-1)}-1\Big]\Big(\frac{a_0}{a_0-1}\Big).
\end{aligned}$$

We will show that if $a_3 = 2$, $a_2 = 3$, then $a_0 = 4$. If $a_3 = 2$, $a_2 = 3$, the condition $\frac{ra_1(a_0-1)}{a_2(a_0a_1-1)}+\frac{sa_1(a_0-1)}{a_3(a_0a_1-1)} = (\frac{2a_0-2}{2a_0-1})(\frac{r}{3}+\frac{s}{2}) = 1$. Hence $(\frac{r}{3}+\frac{s}{2}) = \frac{2a_0-1}{2a_0-2} > 1$. The sequence $(\frac{2a_0-1}{2a_0-2})\ |_{a_0\geqslant 3}$ is a decreasing sequence of a_0 and converges to 1 from one side. The maximum of the sequence is at $a_0 = 3$. Hence we have $(\frac{2a_0-1}{2a_0-2})\ |_{a_0=3} = \frac{5}{4} \geqslant \frac{r}{3}+\frac{s}{2} > 1$. $(r,s) \neq (1,1)$ because $\frac{r}{3}+\frac{s}{2} > 1$. If $(r,s) = (2,1)$ then $\frac{r}{3}+\frac{s}{2} = \frac{7}{6}$ and $a_0 = 4$. If $(r,s) = (1,2)$ then $\frac{r}{3}+\frac{s}{2} = \frac{4}{3}$ which is larger than $\frac{5}{4}$. For all other choices of r and s, $\frac{r}{3}+\frac{s}{2}$ is larger than $\frac{5}{4}$. Therefore we have $a_3 = 2$, $a_2 = 3$, $a_0 = 4$, and $[(2a_0-1)(a_2-1)+1][a_3+\frac{a_3}{2(a_0-1)}-1](\frac{a_0}{a_0-1}) = [(7)(2)+1][1+\frac{1}{3}](\frac{4}{3}) = (15)(\frac{16}{9}) > 16$. If $a_3 = 2$, $a_2 \geqslant 4$, then $[(2a_0-1)(a_2-1)+1][a_3+\frac{a_3}{2(a_0-1)}-1](\frac{a_0}{a_0-1}) > [(5)(3)+1](\frac{a_0}{a_0-1}) > 16$. If $a_3 \geqslant 3$, then $[(2a_0-1)(a_2-1)+1][a_3+\frac{a_3}{2(a_0-1)}-1](\frac{a_0}{a_0-1}) > [(5)(2)+1][2] = 22 > 16$. Hence we get $\mu > 16$.

If $d = \frac{a_0a_1-1}{a_1-1} = a_0 + \frac{a_0-1}{a_1-1}$: Since d is not an integer, $\frac{a_0a_1-1}{a_0-1} > \frac{a_0a_1-1}{a_1-1}$, which implies $1 > \frac{a_0-1}{a_1-1}$. Hence $d = 2 + \frac{1}{a_1-1}$, $a = \frac{a_0a_1-1}{a_0-1}$ and $a_0 = 2$. Observe that $a = \frac{a_0a_1-1}{a_0-1} = a_1 + \frac{a_1-1}{a_0-1} = 2a_1 - 1 > 3$. Hence $a_1 \geqslant 3$. If $a_2 = 1$ then $\frac{a_2(a_0a_1-1)}{a_1(a_0-1)} = \frac{2a_1-1}{a_1} < 2$. Hence $a_2 \geqslant 2$. If $a_3 = 1$ then $\frac{a_3(a_0a_1-1)}{a_1(a_0-1)} = \frac{2a_1-1}{a_1} < 2$. Hence $a_3 \geqslant 2$. Observe that by $(\frac{a_2}{a_1})(\frac{a_0a_1-1}{a_0-1}) \leqslant \frac{a_0a_1-1}{a_0-1}$, $\frac{a_2}{a_1} < 1$. Without loss of generality we can assume $(a,b,c,d) = (2a_1-1, \frac{a_2(2a_1-1)}{a_1}, \frac{a_3(2a_1-1)}{a_1}, 2+\frac{1}{a_1-1})$. Therefore

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1)\\ &= 2(a_1-1)\Big(\frac{2a_1a_2-a_2-a_1}{a_1}\Big)\Big(\frac{2a_1a_3-a_3-a_1}{a_1}\Big)\Big(\frac{a_1}{a_1-1}\Big)\\ &\geqslant 2(2a_2-2)(2a_1a_3-a_3-a_1)\\ &\geqslant 2\cdot 2[(a_1-1)a_3 + a_1(a_3-1)] = 4(2\cdot 2+3) = 28 > 16.\end{aligned}$$

If $d = \frac{a_3(a_0a_1-1)}{a_1(a_0-1)} = a_3 + \frac{a_3(a_1-1)}{a_1(a_0-1)}$: Since d is not an integer, $\frac{a_0a_1-1}{a_1-1} > \frac{a_3(a_0a_1-1)}{a_1(a_0-1)}$, which implies $1 > \frac{a_3(a_1-1)}{a_1(a_0-1)}$. Hence $d = 2 + \frac{2(a_1-1)}{a_1(a_0-1)}$, $a = \frac{a_0a_1-1}{a_1-1} > 3$ and $a_3 = 2$. If $a_1 = 1$, then $\frac{a_0a_1-1}{a_0-1} = 1 < 2$. Hence $a_1 \geqslant 2$. If $a_0 = 1$, then $\frac{a_0a_1-1}{a_1-1} = 1 < 3$. Hence $a_0 \geqslant 2$. If $a_0 = 2$ then $\frac{a_0a_1-1}{a_1-1} = 2 + \frac{1}{a_1-1} \leqslant 3$. Hence $a_0 \geqslant 3$. From $(\frac{a_2}{a_1})(\frac{a_0a_1-1}{a_0-1}) \geqslant (\frac{a_3}{a_1})\frac{a_0a_1-1}{a_0-1}$, $a_2 \geqslant a_3 = 2$. If $a_2 = 2$, then $\frac{ra_1(a_0-1)}{a_2(a_0a_1-1)} + \frac{sa_1(a_0-1)}{a_3(a_0a_1-1)} = 1$ becomes $\frac{1}{d}(r+s) = 1$ which contradicts the fact that d is not an integer. Hence $a_2 \geqslant 3$. Without loss of generality we can assume $(a,b,c,d) = (\frac{a_0a_1-1}{a_1-1}, \frac{a_0a_1-1}{a_0-1}, \frac{a_2(a_0a_1-1)}{a_1(a_0-1)}, 2+\frac{2(a_1-1)}{a_1(a_0-1)})$. Therefore

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1)\\ &= \Big(\frac{a_0a_1-a_1}{a_1-1}\Big)\Big(\frac{a_0a_1-a_0}{a_0-1}\Big)\Big(\frac{a_0a_1a_2-a_2-a_1a_0+a_1}{a_0a_1-a_1}\Big)\Big(\frac{a_0a_1+a_1-2}{a_0a_1-a_1}\Big)\\ &= \Big[\frac{a_2(a_0a_1-1)}{a_1(a_0-1)} - 1\Big]a_0\Big[\frac{a_1(a_0-1)+2(a_1-1)}{a_0-1}\Big]\\ &= \Big[a_2 + \frac{a_2(a_1-1)}{a_1(a_0-1)} - 1\Big]\Big[a_0a_1 + 2(a_1-1)\Big(\frac{a_0}{a_0-1}\Big)\Big]\\ &\geqslant \Big(2 + \frac{3(a_1-1)}{a_1(a_0-1)}\Big)\Big[a_0a_1 + 2(a_1-1)\Big(\frac{a_0}{a_0-1}\Big)\Big]\\ &> 2a_0a_1 + 3(a_1-1) + 4(a_1-1) \geqslant (2)(3)(2) + (3)(1) + (4)(1) = 19 > 16.\end{aligned}$$

If $d = \frac{a_2(a_0a_1-1)}{a_1(a_0-1)} = a_2 + \frac{a_2(a_1-1)}{a_1(a_0-1)}$: Since d is not an integer, $\frac{a_0a_1-1}{a_1-1} > \frac{a_2(a_0a_1-1)}{a_1(a_0-1)}$, which implies $1 > \frac{a_2(a_1-1)}{a_1(a_0-1)}$. Hence $d = 2 + \frac{2(a_1-1)}{a_1(a_0-1)}$, $a = \frac{a_0a_1-1}{a_1-1} > 3$ and $a_2 = 2$. If $a_1 = 1$, then $\frac{a_0a_1-1}{a_0-1} = 1 < 2$. Hence $a_1 \geqslant 2$. If $a_0 = 1$, then $\frac{a_0a_1-1}{a_1-1} = 1 < 3$. Hence $a_0 \geqslant 2$. If $a_0 = 2$, then $\frac{a_0a_1-1}{a_1-1} = 2 + \frac{1}{a_1-1} \leqslant 3$. Hence $a_0 \geqslant 3$. From $(\frac{a_2}{a_1})(\frac{a_0a_1-1}{a_0-1}) \leqslant (\frac{a_3}{a_1})\frac{a_0a_1-1}{a_0-1}$ we have $a_3 \geqslant a_2 = 2$. If $a_3 = 2$, then $\frac{ra_1(a_0-1)}{a_2(a_0a_1-1)} + \frac{sa_1(a_0-1)}{a_3(a_0a_1-1)} = 1$ becomes $\frac{1}{d}(r+s) = 1$ which contradicts d is not an integer. Hence $a_3 \geqslant 3$. Without loss of generality we can assume $(a,b,c,d) = (\frac{a_0a_1-1}{a_1-1}, \frac{a_0a_1-1}{a_0-1}, \frac{a_3(a_0a_1-1)}{a_1(a_0-1)}, 2+\frac{2(a_1-1)}{a_1(a_0-1)})$. Therefore

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1)\\ &= \Big(\frac{a_0a_1-a_1}{a_1-1}\Big)\Big(\frac{a_0a_1-a_0}{a_0-1}\Big)\Big(\frac{a_0a_1a_3-a_3-a_1a_0+a_1}{a_0a_1-a_1}\Big)\Big(\frac{a_0a_1+a_1-2}{a_0a_1-a_1}\Big)\\ &= \Big[\frac{a_3(a_0a_1-1)}{a_1(a_0-1)}-1\Big]a_0\Big[\frac{a_1(a_0-1)+2(a_1-1)}{a_0-1}\Big]\\ &= \Big[a_3+\frac{a_3(a_1-1)}{a_1(a_0-1)}-1\Big]\Big[a_0a_1+2(a_1-1)\Big(\frac{a_0}{a_0-1}\Big)\Big]\\ &\geqslant \Big(2+\frac{3(a_1-1)}{a_1(a_0-1)}\Big)\Big[a_0a_1+2(a_1-1)\Big(\frac{a_0}{a_0-1}\Big)\Big]\\ &> 2a_0a_1+3(a_1-1)+4(a_1-1)\geqslant(2)(3)(2)+(3)(1)+(4)(1)=19>16.\end{aligned}$$

Class 17. $f(z_0,z_1,z_2,z_3)=z_0^{a_0}z_1+z_0z_1^{a_1}+z_1z_2^{a_2}+z_0z_3^{a_3}+z_1^pz_3^q+z_0^rz_2^s$ with the condition $\frac{p(a_0-1)}{a_0a_1-1}+\frac{qa_1(a_0-1)}{a_3(a_0a_1-1)}=1$ and $\frac{r(a_1-1)}{(a_0a_1-1)}+\frac{sa_0(a_1-1)}{a_2(a_0a_1-1)}=1$, $w=(\frac{a_0a_1-1}{a_1-1},\frac{a_0a_1-1}{a_0-1},\frac{a_2(a_0a_1-1)}{a_0(a_1-1)},\frac{a_3(a_0a_1-1)}{a_1(a_0-1)})$.

If $d=\frac{a_0a_1-1}{a_0-1}=a_1+\frac{a_1-1}{a_0-1}$: Since d is not an integer, $\frac{a_0a_1-1}{a_1-1}>\frac{a_0a_1-1}{a_0-1}$, which implies $1>\frac{a_1-1}{a_0-1}$. Hence $d=2+\frac{1}{a_0-1}$, $a=\frac{a_0a_1-1}{a_1-1}$ and $a_1=2$. Observe that $a=\frac{a_0a_1-1}{a_1-1}=a_0+\frac{a_0-1}{a_1-1}=2a_0-1>3$. Hence $a_0\geqslant 3$. If $a_2=1$, then $\frac{a_0a_1a_2-a_2}{a_0a_1-a_0}=\frac{2a_0-1}{a_0}<2$. Hence $a_2\geqslant 2$. From $\frac{a_3(a_0a_1-1)}{a_1(a_0-1)}\geqslant\frac{a_0a_1-1}{a_0-1}$, we have $a_3\geqslant a_1=2$. If $a_3=2$, then $\frac{a_3(a_0a_1-1)}{a_1(a_0-1)}=\frac{(a_0a_1-1)}{(a_0-1)}=d$. Hence the condition $\frac{p(a_0-1)}{a_0a_1-1}+\frac{qa_1(a_0-1)}{a_3(a_0a_1-1)}=1$ becomes $\frac{1}{d}(p+q)=1$ which contradicts the fact that d is not an integer. Hence $a_3\geqslant 3$. Without loss of generality we can assume $(a,b,c,d)=(2a_0-1,\frac{a_2(2a_0-1)}{a_0},\frac{a_3(2a_0-1)}{2(a_0-1)},2+\frac{1}{a_0-1})$. Therefore

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1)\\ &= (2a_0-2)\Big(\frac{2a_0a_2-a_2-a_0}{a_0}\Big)\Big(\frac{2a_0a_3-a_3-2a_0+2}{2a_0-2}\Big)\Big(\frac{a_0}{a_0-1}\Big)\\ &= \Big[\frac{(a_0-1)(2a_2-1)+(a_2-1)}{a_0-1}\Big][(2a_0-1)(a_3-1)+1]\\ &> (2a_2-1)[(2a_0-1)(a_3-1)+1]\geqslant(3)[(5)(2)+1]>16.\end{aligned}$$

If $d=\frac{a_0a_1-1}{a_1-1}=a_0+\frac{a_0-1}{a_1-1}$: Since d is not an integer, $\frac{a_0a_1-1}{a_0-1}>\frac{a_0a_1-1}{a_1-1}$, which implies $1>\frac{a_0-1}{a_1-1}$. Hence $d=2+\frac{1}{a_1-1}$, $a=\frac{a_0a_1-1}{a_0-1}$ and $a_0=2$. Observe that $a=\frac{a_0a_1-1}{a_0-1}=a_1+\frac{a_1-1}{a_0-1}=2a_1-1>3$. Hence $a_1\geqslant 3$. If $a_2=1$, then $\frac{a_0a_1a_2-a_2}{a_0a_1-a_0}=\frac{2a_1-1}{2a_1-2}<2$. Hence $a_2\geqslant 2$. If $a_2=2$, then $\frac{a_2(a_0a_1-1)}{a_0(a_1-1)}=d$. Hence the condition $\frac{r(a_1-1)}{(a_0a_1-1)}+\frac{sa_0(a_1-1)}{a_2(a_0a_1-1)}=1$ becomes $\frac{1}{d}(r+s)=1$ which contradicts d is not an integer. Hence $a_2\geqslant 3$. If $a_3=1$, then $\frac{a_3(a_0a_1-1)}{a_1(a_0-1)}=\frac{2a_1-1}{a_1}<2$. Hence $a_3\geqslant 2$. Without loss of generality we can assume $(a,b,c,d)=(2a_1-1,\frac{a_2(2a_1-1)}{2(a_1-1)},\frac{a_3(2a_1-1)}{a_1},2+\frac{1}{a_1-1})$. Therefore

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1)\\ &= 2(a_1-1)\Big(\frac{2a_1a_2-a_2-2a_1+2}{2(a_1-1)}\Big)\Big(\frac{2a_1a_3-a_3-a_1}{a_1}\Big)\Big(\frac{a_1}{a_1-1}\Big)\\ &> [(2a_1-1)(a_2-1)]\Big[\frac{(a_1-1)(2a_3-1)+a_3-1}{a_1-1}\Big]\geqslant[(5)(2)][3]>16.\end{aligned}$$

If $d=\frac{a_3(a_0a_1-1)}{a_1(a_0-1)}=a_3+\frac{a_3(a_1-1)}{a_1(a_0-1)}$: Since d is not an integer, $\frac{a_0a_1-1}{a_1-1}>\frac{a_3(a_0a_1-1)}{a_1(a_0-1)}$, which implies $1>\frac{a_3(a_1-1)}{a_1(a_0-1)}$. Hence $d=2+\frac{2(a_1-1)}{a_1(a_0-1)}$, $a=\frac{a_0a_1-1}{a_1-1}>3$ and $a_3=2$.

If $a_0 = 1$, then $\frac{a_0a_1-1}{a_1-1} = 1 < 3$. Hence $a_0 \geqslant 2$. If $a_1 = 1$, then $\frac{a_0a_1-1}{a_0-1} = 1 < 2$. Hence $a_1 \geqslant 2$. If $a_0 = 2$, then $\frac{a_0a_1-1}{a_1-1} = 2 + \frac{1}{a_1-1} \leqslant 3$ Hence $a_0 \geqslant 3$. If $a_1 = 2$, then $\frac{a_0a_1-1}{a_0-1} = 2 + \frac{1}{a_0-1} = d$. Hence the condition $\frac{p(a_0-1)}{a_0a_1-1} + \frac{qa_1(a_0-1)}{a_3(a_0a_1-1)} = 1$ becomes $\frac{1}{d}(p+q) = 1$ which contradicts the fact that d is not an integer. Hence $a_1 \geqslant 3$. If $a_2 = 1$, then $\frac{a_2(a_0a_1-1)}{a_0(a_1-1)} = 1 + \frac{a_0-1}{a_0(a_1-1)} < 2$. Hence $a_2 \geqslant 2$. From $\frac{a_0a_1-1}{a_1-1} \geqslant \frac{a_0a_1-1}{a_0-1}$ we have $\frac{a_0-1}{a_1-1} \geqslant 1$. Hence $\frac{a_0a_1-1}{a_1-1} = a_0 + \frac{a_0-1}{a_1-1} \geqslant 4$. $\frac{a_0a_1-1}{a_0-1} = a_1 + \frac{a_1-1}{a_0-1} > 3$. Without loss of generality we can assume $(a,b,c,d) = (\frac{a_0a_1-1}{a_1-1}, \frac{a_0a_1-1}{a_0-1}, \frac{a_2(a_0a_1-1)}{a_0(a_1-1)}, 2 + \frac{2(a_1-1)}{a_1(a_0-1)})$. Therefore

$$\begin{aligned}
\mu &= (a-1)(b-1)(c-1)(d-1) \\
&= \Big(\frac{a_0a_1 - a_1}{a_1-1}\Big)\Big(\frac{a_0a_1 - a_0}{a_0-1}\Big)\Big(\frac{a_0a_1a_2 - a_2 - a_1a_0 + a_0}{a_0a_1 - a_0}\Big)\Big(\frac{a_0a_1 + a_1 - 2}{a_0a_1 - a_1}\Big) \\
&= \Big[\frac{(a_0a_1-1)(a_2-1) + (a_0-1)}{a_0-1}\Big]\Big[\frac{a_0a_1-1}{a_1-1} + 1\Big] \\
&= \Big[\Big(\frac{a_0a_1-1}{a_0-1}\Big)(a_2-1) + 1\Big]\Big[\frac{a_0a_1-1}{a_1-1} + 1\Big] > [3+1][4+1] > 16.
\end{aligned}$$

If $d = \frac{a_2(a_0a_1-1)}{a_0(a_1-1)} = a_2 + \frac{a_2(a_0-1)}{a_0(a_1-1)}$: Since d is not an integer, $\frac{a_0a_1-1}{a_0-1} > \frac{a_2(a_0a_1-1)}{a_0(a_1-1)}$, which implies $1 > \frac{a_2(a_0-1)}{a_0(a_1-1)}$. Hence $d = 2 + \frac{2(a_0-1)}{a_0(a_1-1)}$, $a = \frac{a_0a_1-1}{a_0-1} > 3$ and $a_2 = 2$. If $a_0 = 1$, then $\frac{a_0a_1-1}{a_1-1} = 1 < 2$. Hence $a_0 \geqslant 2$. If $a_0 = 2$, then $\frac{a_0a_1-1}{a_1-1} = 2 + \frac{1}{a_1-1} = d$. Hence $\frac{r(a_1-1)}{(a_0a_1-1)} + \frac{sa_0(a_1-1)}{a_2(a_0a_1-1)} = 1$ becomes $\frac{1}{d}(r+s) = 1$ which contradicts d is not an integer. Hence $a_0 \geqslant 3$. If $a_1 = 1$, then $\frac{a_0a_1-1}{a_0-1} = 1 < 3$. Hence $a_1 \geqslant 2$. If $a_1 = 2$, then $a = \frac{a_0a_1-1}{a_0-1} = \frac{2a_0-1}{a_0-1} \leqslant 3$. Hence $a_1 \geqslant 3$. From $\frac{a_0a_1-1}{a_0-1} \geqslant \frac{a_0a_1-1}{a_1-1}$ we have $\frac{a_1-1}{a_0-1} \geqslant 1$. Hence $\frac{a_0a_1-1}{a_0-1} = a_1 + \frac{a_1-1}{a_0-1} \geqslant 4$. $\frac{a_0a_1-1}{a_1-1} = a_0 + \frac{a_0-1}{a_1-1} \geqslant 3$. Since $a_0 \geqslant 3$, $\frac{a_1-1}{a_0a_1-a_1} < 1$. If $a_3 = 1$, then $\frac{a_3(a_0a_1-1)}{a_1(a_0-1)} = 1 + \frac{a_1-1}{a_0a_1-a_1} < 2$. Hence $a_3 \geqslant 2$. Without loss of generality we can assume $(a,b,c,d) = (\frac{a_0a_1-1}{a_0-1}, \frac{a_0a_1-1}{a_1-1}, \frac{a_3(a_0a_1-1)}{a_1(a_0-1)}, 2 + \frac{2(a_0-1)}{a_0(a_1-1)})$. Therefore

$$\begin{aligned}
\mu &= (a-1)(b-1)(c-1)(d-1) \\
&= \Big(\frac{a_0a_1 - a_0}{a_0-1}\Big)\Big(\frac{a_0a_1 - a_1}{a_1-1}\Big)\Big(\frac{a_0a_1a_3 - a_3 - a_1a_0 + a_1}{a_0a_1 - a_1}\Big)\Big(\frac{a_0a_1 + a_0 - 2}{a_0a_1 - a_0}\Big) \\
&= \Big[\frac{(a_0a_1-1)(a_3-1) + a_1 - 1}{a_1-1}\Big]\Big[\frac{a_0a_1 - 1 + a_0 - 1}{a_0-1}\Big] \\
&= \Big[\Big(\frac{a_0a_1-1}{a_1-1}\Big)(a_3-1) + 1\Big]\Big[\frac{a_0a_1-1}{a_0-1} + 1\Big] > [4][5] > 16.
\end{aligned}$$

Class 18. $f(z_0,z_1,z_2,z_3) = z_0^{a_0}z_2 + z_0z_1^{a_1} + z_1z_2^{a_2} + z_1z_3^{a_3} + z_2^pz_3^q$ with the condition $\frac{p(a_0a_1-a_0+1)}{a_0a_1a_2+1} + \frac{qa_2(a_0a_1-a_0+1)}{a_3(a_0a_1a_2+1)} = 1$, $w = (\frac{a_0a_1a_2+1}{a_1(a_2-1)+1}, \frac{a_0a_1a_2+1}{a_2(a_0-1)+1}, \frac{a_0a_1a_2+1}{a_0(a_1-1)+1}, \frac{a_3(a_0a_1a_2+1)}{a_2(a_0(a_1-1)+1)})$.

We consider first when $d = \frac{a_0a_1a_2+1}{a_0(a_1-1)+1} = a_2 + \frac{a_2(a_0-1)+1}{a_0(a_1-1)+1}$. Since d is not an integer, $\frac{a_0a_1a_2+1}{a_2(a_0-1)+1} > \frac{a_0a_1a_2+1}{a_0(a_1-1)+1}$, which implies $1 > \frac{a_2(a_0-1)+1}{a_0(a_1-1)+1}$. Hence $d = 2 + \frac{2(a_0-1)+1}{a_0(a_1-1)+1}$, $a = \frac{a_0a_1a_2+1}{a_2(a_0-1)+1}$ and $a_2 = 2$. Without loss of generality we can assume $(a,b,c,d) = (\frac{2a_0a_1+1}{2a_0-1}, \frac{2a_0a_1+1}{a_1+1}, \frac{a_3(2a_0a_1+1)}{2(a_0(a_1-1)+1)}, 2+\frac{2(a_0-1)+1}{a_0(a_1-1)+1})$. Observe that if $a_1 = 1$, then $\frac{a_0a_1a_2+1}{a_2(a_0-1)+1} = \frac{2a_0a_1+1}{2a_0-1} = \frac{2a_0+1}{2a_0-1} < 2$. Hence $a_1 \geqslant 2$. If $a_0 = 1$, then $\frac{a_0a_1a_2+1}{a_1(a_2-1)+1} = \frac{2a_0a_1+1}{a_1+1} = \frac{2a_1+1}{a_1+1} < 2$. Hence $a_0 \geqslant 2$. If $a_1 = 2$, then $a = \frac{a_0a_1a_2+1}{a_2(a_0-1)+1} = \frac{2a_0a_1+1}{2a_0-1} = \frac{4a_0+1}{2a_0-1} = 2 + \frac{3}{2a_0-1} \leqslant 3$. Since $a > 3$ $a_1 \geqslant 3$. If $a_3 = 1$, then $\frac{a_3(2a_0a_1+1)}{2(a_0(a_1-1)+1)} = \frac{2a_0a_1+1}{2(a_0(a_1-1)+1)} < \frac{2a_0a_1+1}{2a_0a_1} < 2$. Hence

$a_3 \geqslant 2$. Therefore

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1)\\ &= \Big(\frac{2a_0a_1-2a_0+2}{2a_0-1}\Big)\Big(\frac{2a_0a_1-a_1}{a_1+1}\Big)\Big(\frac{2a_0a_1a_3+a_3-2a_0a_1+2a_0-2}{2a_0a_1-2a_0+2}\Big)\Big(\frac{a_0a_1+a_0}{a_0a_1-a_0+1}\Big)\\ &= \Big(\frac{a_0a_1}{a_0a_1-a_0+1}\Big)(2a_0a_1a_3+a_3-2a_0a_1+2a_0-2)\\ &= a_0a_1\Big(\frac{2a_0a_1a_3+a_3}{a_0a_1-a_0+1}-2\Big).\end{aligned}$$

It is obvious that $a_0a_1(\frac{2a_0a_1a_3+a_3}{a_0a_1-a_0+1}-2)$ is an increasing function of a_3. Since $a_3 \geqslant 2$, the minimum of $a_0a_1(\frac{2a_0a_1a_3+a_3}{a_0a_1-a_0+1}-2)$ is at $a_3 = 2$. $a_0a_1(\frac{2a_0a_1a_3+a_3}{a_0a_1-a_0+1}-2)\,|_{a_3=2}= a_0a_1(\frac{2a_0a_1+2a_0}{a_0a_1-a_0+1}) > 2a_0a_1+2a_0 \geqslant (2)(2)(3)+(2)(2) = 16$. Hence we have $\mu = (a-1)(b-1)(c-1)(d-1) > 16$.

Similarly we can prove the cases for $d = \frac{a_0a_1a_2+1}{a_1(a_2-1)+1}$, and $d = \frac{a_0a_1a_2+1}{a_2(a_0-1)+1}$.

If $d = \frac{a_3(a_0a_1a_2+1)}{a_2(a_0(a_1-1)+1)} = a_3 + \frac{a_2a_3(a_0-1)+a_3}{a_2(a_0(a_1-1)+1)}$: Since d is not an integer, $\frac{a_0a_1a_2+1}{a_2(a_0-1)+1} > \frac{a_3(a_0a_1a_2+1)}{a_2(a_0(a_1-1)+1)}$, which implies $1 > \frac{a_2a_3(a_0-1)+a_3}{a_2(a_0(a_1-1)+1)}$. Hence $d = 2 + \frac{2(a_0a_2-a_2+1)}{a_2(a_0(a_1-1)+1)}$, where $a = \frac{a_0a_1a_2+1}{a_2(a_0-1)+1}$ and $a_3 = 2$. Without loss of generality we can assume $(a,b,c,d) = (\frac{a_0a_1a_2+1}{a_2(a_0-1)+1}, \frac{a_0a_1a_2+1}{a_1(a_2-1)+1}, \frac{a_0a_1a_2+1}{a_0(a_1-1)+1}, 2+\frac{2(a_0a_2-a_2+1)}{a_2(a_0(a_1-1)+1)})$. We will show that $a_0 \geqslant 2$, $a_1 \geqslant 2$, and $a_2 \geqslant 2$. If $a_0 = a_1 = a_2 = 1$, then $a = b = c = 2$ which is a contradiction. If $(a_0,a_1,a_2) = (1,1,2)$ then $\frac{a_0a_1a_2+1}{a_1(a_2-1)+1} = \frac{3}{2} < 2$ which is a contradiction. Similarly we can prove (a_0,a_1,a_2) cannot be $(2,1,1)$ and $(1,2,1)$. If $(a_0,a_1,a_2) = (1,2,2)$ then $\frac{a_0a_1a_2+1}{a_1(a_2-1)+1} = \frac{5}{3} < 2$ which is a contradiction. Similarly we can prove (a_0,a_1,a_2) cannot be $(2,1,2)$ and $(2,2,1)$. Hence we get $a_0 \geqslant 2$, $a_1 \geqslant 2$, and $a_2 \geqslant 2$. If $a_0 = a_1 = a_2 = 2$, then $a = \frac{a_0a_1a_2+1}{a_2(a_0-1)+1} = 3$. Hence at least one of a_0, a_1, and a_2 must be greater than 2. Therefore

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1)\\ &= \Big(\frac{a_0a_1a_2-a_0a_2+a_2}{a_0a_2-a_2+1}\Big)\Big(\frac{a_0a_1a_2-a_1a_2+a_1}{a_1a_2-a_1+1}\Big)\Big(\frac{a_0a_1a_2-a_0a_1+a_0}{a_0a_1-a_0+1}\Big)\\ &\quad\cdot\Big(\frac{a_0a_1a_2+a_0a_2-a_2+2}{a_2(a_0a_1-a_0+1)}\Big)\\ &= \Big(\frac{a_0a_1}{a_0a_1-a_0+1}\Big)(a_0a_1a_2+a_0a_2-a_2+2)\\ &= a_0a_1\Big(a_2+\frac{2(a_0a_2-a_2+1)}{a_0a_1-a_0+1}\Big) > a_0a_1a_2+2(a_0a_2-a_2+1).\end{aligned}$$

It is obvious that $a_0a_1a_2+2(a_0a_2-a_2+1)$ is an increasing function of a_0, a_1, and a_2. Note also at least one of them is greater than or equal to 3. If $a_0 = 3$, $a_0a_1a_2+2(a_0a_2-a_2+1) \geqslant 3a_1a_2+2(2a_2+1) \geqslant 12+10 = 22 > 16$. If $a_1 = 3$, $a_0a_1a_2+2(a_0a_2-a_2+1) \geqslant 3a_0a_2+2(a_2(a_0-1)+1) \geqslant 12+6 = 18 > 16$. If $a_2 = 3$, $a_0a_1a_2+2(a_0a_2-a_2+1) \geqslant 3a_0a_1+2(a_2(a_0-1)+1) \geqslant 12+8 = 20 > 16$.

Class 19. $f(z_0,z_1,z_2,z_3) = z_0^{a_0}z_2+z_0z_1^{a_1}+z_3z_2^{a_2}+z_1z_3^{a_3}$, $w = (\frac{a_0a_1a_2a_3-1}{a_1(a_3(a_2-1)+1)-1}, \frac{a_0a_1a_2a_3-1}{a_3(a_2(a_0-1)+1)-1}, \frac{a_0a_1a_2a_3-1}{a_0(a_1(a_3-1)+1)-1}, \frac{a_0a_1a_2a_3-1}{a_2(a_0(a_1-1)+1)-1})$.

By the property of symmetry we consider only the case when $d = \frac{a_0a_1a_2a_3-1}{a_2(a_0(a_1-1)+1)-1} = a_3 + \frac{a_3(a_0a_2-a_2+1)-1}{a_2(a_0(a_1-1)+1)-1}$. Since d is not an integer, $\frac{a_0a_1a_2a_3-1}{a_3(a_2(a_0-1)+1)-1} > \frac{a_0a_1a_2a_3-1}{a_2(a_0(a_1-1)+1)-1}$, which implies $1 > \frac{a_3(a_0a_2-a_2+1)-1}{a_2(a_0(a_1-1)+1)-1}$. Hence $d = 2+\frac{a_3(a_0a_2-a_2+1)-1}{a_2(a_0(a_1-1)+1)-1}$, $a = \frac{a_0a_1a_2a_3-1}{a_3(a_2(a_0-1)+1)-1}$

and $a_3 = 2$. We will show that $a_0 \geqslant 2$, $a_1 \geqslant 2$, and $a_2 \geqslant 2$. If $(a_0, a_1, a_2) = (1, 1, 1)$, then $\frac{a_0a_1a_2a_3-1}{a_0(a_1(a_3-1)+1)-1} = 1 < 2$ which is a contradiction. If $(a_0, a_1, a_2) = (2, 1, 1)$, then $\frac{a_0a_1a_2a_3-1}{a_3(a_2(a_0-1)+1)-1} = 1 < 2$ which is a contradiction. If $(a_0, a_1, a_2) = (1, 2, 1)$ or $(a_0, a_1, a_2) = (1, 1, 2)$, then $a = \frac{a_0a_1a_2a_3-1}{a_3(a_2(a_0-1)+1)-1} = 3$ which is a contradiction. If $(a_0, a_1, a_2) = (1, 2, 2)$ then $\frac{a_0a_1a_2a_3-1}{a_1(a_3(a_2-1)+1)-1} = \frac{7}{5} < 2$ which is a contradiction. Similarly we can prove that (a_0, a_1, a_2) cannot be $(2, 1, 2)$ and $(2, 2, 1)$. But if $a_0 = a_1 = a_2 = a_3 = 2$, then $a = \frac{a_0a_1a_2a_3-1}{a_3(a_2(a_0-1)+1)-1} = 3$. Hence one of a_0, a_1, a_2 must be greater than or equal to 3. Therefore

$$\begin{aligned}\mu &= (a-1)(b-1)(c-1)(d-1) = a_0a_1a_2a_3 = 2a_0a_1a_2 \\ &\geqslant 2(2)(2)(3) = 24 > 16.\end{aligned}$$

The proof of the main theorem is completed.

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