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On the GLY Conjecture of upper estimate of positive integral points in real right-angled simplices ☆

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Abstract

The GLY (Granville–Lin–Yau) Conjecture is a generalization of Lin, Xu and Yau's results. An important application of GLY is its use in characterizing an affine hypersurface in C^n as a cone over a nonsingular projective variety. In addition, the Rough Upper Estimate Conjecture in GLY, recently proved by Yau and Zhang, implies the Durfee Conjecture in singularity theory. This paper develops a unified approach to prove the Sharp Upper Estimate Conjecture for general n. Using this unified approach, we prove that the Sharp Upper Estimate Conjecture is true for $n = 4, 5, 6$. After giving a counter-example to show that the Sharp Upper Estimate Conjecture is not true for $n = 7$, we propose a Modified GLY Conjecture. For each fixed n, our unified approach can be used to prove this Modified GLY Conjecture.

1. Introduction

An n-dimensional right-angled simplex Δ is defined by $x_1 \geqslant 0, \ldots, x_n \geqslant 0$ and

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leqslant 1, \tag{1.1}$$

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where $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant 1$. Define $Q_n(a_1, a_2, \ldots, a_n)$ and $P_n(a_1, a_2, \ldots, a_n)$ to be the number of nonnegative and positive integral solutions of the right-angled simplices, respectively. Calculating Q_n and P_n has been a research topic for several decades. Q_n and P_n are related by the following formula:

$$P_n(a_1, a_2, \ldots, a_n) = Q_n\big(a_1(1-a), a_2(1-a), \ldots, a_n(1-a)\big),$$

where $a = \frac{1}{a_1} + \cdots + \frac{1}{a_n}$. So the study of Q_n and of P_n is equivalent. This study can be broken into three areas, depending on whether $a_1, a_2, \ldots, a_n$ are integers, rational, or real numbers.

There are exact formulas to compute Q_n in the case of integral right-angled simplices where $a_1, a_2, \ldots, a_n$ are positive integers. As early as in 1899, Pick [31] gave the famous formula for Q_2:

$$Q_2 = \text{area}(\Delta) + \frac{|\partial\Delta \cap Z^n|}{2} + 1,$$

where $\partial\Delta$ is the boundary of the right-angled simplex. This formula tells us that the number of lattice points inside a 2-dimensional right-angled simplex is determined by the area of the right-angled simplex and the number of integer points on the boundary. In 1951, Mordell [29] gave a formula for Q_3, using Dedekind sums under the condition that no two of a_1, a_2, a_3 have a common factor.

A major result was obtained by Ehrhart [13] in 1967. Let $Q_n(k)$ be the number of nonnegative solutions of the right-angled simplex dilated from (1.1) by a factor k, where k is a positive integer. Ehrhart proved that $Q_n(k)$ is a polynomial in k of degree n:

$$Q_n(k) = b_n k^n + b_{n-1} k^{n-1} + \cdots + b_0. \tag{1.2}$$

He also showed that b_n is the volume of Δ bounded by (1.1), and b_{n-1} is one half of the surface area of (1.1) measured in $n-1$ space. The constant term is 1. The polynomial on the right-hand side of (1.2) is also called the *Ehrhart polynomial*. The Ehrhart polynomial is very important since we can compute Q_n easily if all its coefficients can be determined [2]. The major contribution related to the computation of these coefficients is from Danilov's work. For each right-angled simplex Δ, there is an associated n-dimensional toric variety X_Δ, which has a naturally defined Todd class [14]. Let Γ be the face of Δ, $V(\Gamma)$ be the closed subvariety of X_Δ corresponding to Γ and $[V(\Gamma)]$ be the class in the group of rational equivalence classes $(A_* X_\Delta)_{\mathbf{Q}}$. In 1989, Danilov [10] showed that if the Todd class of X_Δ has the form

$$T\,dX_\Delta = \sum r_\Gamma\big[V(\Gamma)\big], \tag{1.3}$$

where $r_\Gamma \in Q$, then the coefficient b_k in the Ehrhart polynomial is determined by

$$b_k = \sum_{\dim \Gamma = k} r_\Gamma \operatorname{Vol}(\Gamma),$$

Morelli [30] later showed that (1.3) is indeed true. Using this result, Pommersheim [32] gave the coefficient for b_1 when $n = 3$. Using a similar method, Kantor and Khovanskii [19] computed

b_{n-2} for general n. In 1994, Cappell and Shaneson [8] announced an explicit formula to compute all the coefficients b_i in (1.2). Their results on the behavior of some important algebraic and topological invariants under morphisms of projective algebraic varieties allowed them to calculate the Todd class of the toric variety. Diaz and Robins [11] later obtained another formula for the coefficients of the Ehrhart polynomial by means of Fourier analysis. Most recently Beck [5] has used the residue theorem to determine the coefficients of Ehrhart polynomial. However all these results are under the condition that $a_1, a_2, \dots, a_n$ must be integers.

A rational right-angled simplex is a right-angled simplex whose vertices, $(a_1, 0, \dots, 0), \dots, (0, \dots, 0, a_n)$, are rational. Ehrhart's result remains true for rational right-angled simplices if we replace "polynomial" by "quasi-polynomial" in the statement. This means the number of lattice points in a dilated rational right-angled simplex can also be expressed as a quasi-polynomial of k. In 1983, using valuation theory, McMullen [25,26] showed that Q_n can be expressed as a linear combination of the volumes of the polytope faces F, with the coefficients determined by a function ϕ over the rational cone:

$$Q_n = \sum_F \operatorname{vol}(F)\phi\big(\operatorname{cone}(\Delta, F)\big),$$

where ϕ is invariant under lattice translations. However, finding a computable function ϕ for rational polytopes remains an unsolved problem [3].

Brion and Vergne [7] gave a generating function for Q_n. The most recent result is from Beck–Diaz–Robins [4,6]. Applying the residue theorem to the case of rational right-angled simplices, they found formulas involving generalized Dedekind sums.

The third category is real right-angled simplex, where $a_1, a_2, \dots, a_n$ are real numbers. Due to its connection to Diophantine approximation problems, Hardy and Littlewood discussed the case $n = 2$ in their three famous papers [16] in 1920 and [17,18] in 1922, which have some applications to problems in Diophantine approximation. For a 2-dimensional right-angled simplex like (1.1), they showed that

$$Q_2 = \frac{1}{2}a_1a_2 - \frac{1}{2}(a_1 + a_2) + \frac{1}{2}\beta_1 + \frac{1}{2}\theta\beta_1(1 - \beta_1) + \sum_{\nu=1}^{[a_1]}\{\nu\theta - \beta_2\},$$

where $\theta = a_2/a_1$, $\beta_1 = a_1 - [a_1]$, $\beta_2 = a_2 - [a_2]$ and $\{x\} = x - [x] - 1/2$.

Later in 1939, Rosser [34] got a lower bound for Q_n in the general case. In 1940, Lehmer [20] constructed two polynomials $l(a_1, \dots, a_n)$ and $L(a_1, \dots, a_n)$ to approximate Q_n from below and above, respectively. A more general approximation of Q_n was obtained by Spencer [35,36] in 1942 via complex function-theoretic methods:

$$P_n = (-1)^n \zeta_n\left(0, 1 \,\middle|\, \frac{1}{a_1}, \dots, \frac{1}{a_n}\right) + T_0^{(n)}(1),$$

where ζ_n is the ζ-function of Barnes [1] and $T_0^{(n)}(1)$ is the sum of terms resulting from integral residues, which in general is of small order. Unlike integral or rational right-angled simplices, so far there is no exact formula to compute P_n in the general case. However, when $a = a_1 = a_2 = \cdots = a_n$, we can get Q_n very easily [20]:

$$Q_n = \binom{[a]+n}{n}.$$

An estimate R_n of Q_n is said to be *sharp* if

$$R_n|_{a_1=\cdots=a_n=a=\text{integer}} = \binom{[a]+n}{n}.$$

By this standard all the above estimates are far from sharp.

Finding a sharp estimate of P_n for real right-angled simplices is related to many other mathematical problems. In number theory, consider a set of primes $p_1 < p_2 < \cdots < p_n \leqslant y$. Given a positive number $u > 2$, consider the set

$$S = \{\alpha \mid \alpha = p_1^{l_1} \cdots p_n^{l_n} \leqslant y^u,\ l_i \geqslant 0 \text{ is integer}\}.$$

Counting the size of set S is equivalent to counting the number of nonnegative solutions of

$$\frac{l_1}{a_1} + \frac{l_2}{a_2} + \cdots + \frac{l_n}{a_n} \leqslant 1, \quad \text{where } a_i = \frac{\log y^u}{\log p_i} \geqslant u > 2, \tag{1.4}$$

which is an n-dimensional real right-angled simplex. Granville [15] pointed out that the estimate of Q_n for (1.4) has many applications in number theory to finding large gaps between primes, to Waring's problem, to primality testing and factoring algorithms, and to getting bounds for the least prime kth power residues and nonresidues (mod n).

In geometry and singularity theory, estimating P_n for real right-angled simplices is connected with the Durfee Conjecture. Let $f:(\mathbf{C}^n,0) \to (\mathbf{C},0)$ be a germ of a complex analytic function with an isolated critical point at the origin. Let $V = \{(z_1,\ldots,z_n) \in \mathbf{C}^n:\ f(z_1,\ldots,z_n) = 0\}$. The Milnor number of the singularity $(V,0)$ is defined as

$$\mu = \dim \mathbf{C}\{z_1,\ldots,z_n\}/(f_{z_1},\ldots,f_{z_n}).$$

The geometric genus p_g of $(V,0)$ is defined as

$$p_g = \dim H^{n-2}(M,\mathfrak{O}),$$

where M is a resolution of V. In 1978, Durfee [12] made the following conjecture:

Durfee Conjecture. *$n!p_g \leqslant \mu$ with equality only when $\mu = 0$.*

If $f(z_1,\ldots,z_n)$ is a weighted homogeneous polynomial of type $(a_1,\ldots,a_n)$ with an isolated singularity at the origin, Milnor and Orlik [28] proved that $\mu = (a_1-1)\cdots(a_n-1)$. On the other hand, Merle and Teissier [27] showed that $p_g = P_n$, where P_n is the number of positive integral solutions of (1.1). So finding a good estimate of P_n will eventually lead to a resolution of the Durfee Conjecture.

Starting from early 90's, Yau, Xu and Lin [22,37,39] tried to get a sharp upper estimates of P_n when a_i are just positive real numbers. They were able to obtain it under certain conditions, specifically when $n = 3, 4$, and 5. Surprising enough, these sharp estimates are all polynomials of a_i:

$$3!P_3 \leqslant f_3 = a_1a_2a_3 - (a_1a_2 + a_1a_3 + a_2a_3) + a_1 + a_2,$$

$$4!P_4 \leqslant f_4 = a_1a_2a_3a_4 - \frac{2}{3}(a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4) + \frac{11}{3}(a_1a_2 + a_1a_3 + a_2a_3) - 2(a_1 + a_2 + a_3),$$

$$5!P_5 \leqslant f_5 = a_1a_2a_3a_4a_5 - 2(a_1a_2a_3a_4 + a_1a_2a_3a_5 + a_1a_2a_4a_5 + a_1a_3a_4a_5 + a_2a_3a_4a_4) + \frac{35}{4}(a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4) - \frac{50}{6}(a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4) + 6(a_1 + a_2 + a_3 + a_4).$$

These estimates are considered sharp because the equality holds true if and only if all a_i take the same integer.

Inspired by the similarity of these estimates, the general form of the upper estimate was conjectured.

GLY (Granville–Lin–Yau) Conjecture. *Let* $P_n = \#\{(x_1, x_2, \ldots, x_n) \in \mathbf{Z}_+^n;\ \frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leqslant 1\}$. *Let* $n \geqslant 3$,

(1) *Sharp estimate*: *If* $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant n-1$, *then*

$$n!P_n \leqslant f_n := A_0^n + \frac{s(n, n-1)}{n}A_1^n + \sum_{l=1}^{n-2}\frac{s(n, n-1-l)}{\binom{n-1}{l}}A_l^{n-1} \tag{1.5}$$

and $n!P_n = f_n$ *if and only if* $a_1 = a_2 = \cdots = a_n = \text{integer}$,

(2) *Rough estimate*: *If* $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n > 1$, *then*

$$n!P_n < q_n := \prod_{i=1}^{n}(a_i - 1), \tag{1.6}$$

where A_0^n, A_1^n *and* A_l^{n-1} *are polynomials of* $a_1, a_2, \ldots, a_n$ *defined by* (2.1), $s(n,k)$ *is the Stirling number of the first kind defined by* (2.2).

When $n = 3, 4$ and 5, this conjecture is true [22,23,37,39]. The sharp estimate conjecture was first formulated in [24]. In private communication, to the second author, Granville formulated this sharp estimate conjecture independently after reading [23]. Notice that the sharp estimate conjecture is for n-dimensional real right-angled simplices with $a_n \geqslant n-1$. When we use induction to prove the sharp estimate conjecture by dissecting the n-dimensional right-angled simplex along the x_n-axis into several $(n-1)$-dimensional right-angled simplices, we must face the difficulty that we cannot apply the lower-dimensional sharp estimate conjecture in every level. Therefore the lower-dimensional rough estimate conjecture must be used.

The importance of this Upper Estimate Conjecture is twofold. First the Durfee Conjecture in singularity theory becomes a special case. And second, more importantly, it is the first main step to prove the following conjecture made by Yau in 1995:

Conjecture 1. *Let $f:(C^{n+1},0)\to(C,0)$ be a germ of a weighted homogeneous polynomial with isolated critical points at the origin. Then*

$$\mu - h(\nu) \geqslant (n+1)!p_g$$

with equality if and only if f is a homogeneous polynomial, where $h(\nu)$ is a polynomial function of the multiplicity ν with the properties $h(\nu)\geqslant 0$ and $h(\nu)=0$ if and only if $\nu=1$. In fact $h(\nu)=(\nu-1)^{n+1}-\nu(\nu-1)\cdots(\nu-n)$.

The above conjecture was proven for the case $n=3$ in [38] and for the case $n=4$ in [21]. It leads to the following numerical characterization of an affine variety in C^{n+1} as a cone over nonsingular projective variety in CP^n.

Conjecture 2. *Let V be an affine hypersurface in C^{n+1}. Then V is a cone over nonsingular hypersurface in CP^n if and only if V has only isolated singularity at the origin, $\mu=\tau$ and $\mu-(\nu-1)^{n+1}+\nu(\nu-1)\cdots(\nu-n)=(n+1)!p_g$.*

The main task of this paper is to prove the following theorem:

Main Theorem. *The Upper Estimate Conjecture is true for $n=4,5,6$, and there is a counter-example to the sharp estimate of the Sharp Upper Estimate Conjecture for $n=7$.*

In view of the above theorem, we must modify GLY Conjecture as follows:

Modified GLY Conjecture. *There exists an integer α which depends only on n such that the sharp estimate* (1.5) *holds when $a_1\geqslant a_2\geqslant\cdots\geqslant a_n\geqslant\alpha$.*

While we follow similar idea in [22,39] for $n=4,5$ to prove the Main Theorem, we treat the problem uniformly in the general case. Thus a unified approach is developed to prove the Main Theorem. In order to get the estimate of P_n, we first partition the n-dimensional right-angled simplex into several $(n-1)$-dimensional right-angled simplices by assigning $x_n=k$, $k=1,2,\ldots,[a_n]$. Since we know the Upper Estimate Conjecture is true for $n=3$, we can assume that the Main Theorem is true for each $(n-1)$-dimensional right-angled simplex. By summing up the upper estimate for each $(n-1)$-dimensional right-angled simplex, we can get the upper estimate for an n-dimensional right-angled simplex. Comparing this summed upper estimate to f_n and q_n, we show that the Main Theorem is true for n-dimensional right-angled simplices. In this paper, we develop a formula for the comparison for general n. So starting from $n=4$, all $n\geqslant 4$ are treated in the same way inductively.

The difference between the summed upper estimate and f_n (respectively q_n) is actually a polynomial of $a_1,a_2,\ldots,a_n$. The major breakthrough of this paper is that we are able to understand this polynomial via Lemmas 4.1, 4.2 and 4.3 which are true for any dimension. We use a special method to determine the sign of this polynomial. Lemmas 3.1 and 3.2 are the main tools of this special method. In order to facilitate the computation, we use Maple 7 to do the major computation.

The first version of this paper was submitted for publication more than two years ago. In the summer of 2005 the second author and Letian Zhang gave a simple proof of the rough estimate conjecture:

Theorem *(Yau–Zhang).* [40] *Let* $P_n = \#\{(x_1, x_2, \ldots, x_n) \in \mathbf{Z}_+^n;\ \frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leqslant 1\}$, *where* $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n > 1$ *are real numbers. If* $n \geqslant 3$, *then*

$$n!P_n \leqslant (a_1 - 1)(a_2 - 1)\cdots(a_n - 1),$$

equality holds if and only if $a_n = 1$.

2. Notation and preliminary results

Let us first introduce some notations.

Definition 2.1 *(Polynomial of* a_i*).*

$$A_k^n = \left(\prod_{i=1}^n a_i\right)\left(\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant n} \frac{1}{a_{i_1}a_{i_2}\cdots a_{i_k}}\right), \tag{2.1}$$

where $k = 1, 2, \ldots, n-1$. In particular, we have

$$A_0^n = a_1a_2\cdots a_n, \qquad A_n^n = 1.$$

Notice when $a = a_1 = a_2 = \cdots = a_n$, we have

$$A_k^n = \binom{n}{k} a^{n-k}.$$

The recursion formula for A_k^n is

$$A_k^n = a_n A_k^{n-1} + A_{k-1}^{n-1}.$$

Definition 2.2 *(Stirling number of the first kind* $s(n,k)$*).* [9] $s(n,k)$ is defined by generating function:

$$x(x-1)\cdots(x-n+1) = \sum_{k=0}^n s(n,k)x^k. \tag{2.2}$$

Let

$$b_k^n = \sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant n} i_1i_2\cdots i_k, \tag{2.3}$$

where $k = 1, 2, \ldots, n$. Then we have

$$s(n,k) = (-1)^{n-k} b_{n-k}^{n-1}.$$

Some special value of $s(n,m)$ are:

$$s(n,n)=1,$$
$$s(n,n-1)=-\binom{n}{2},$$
$$s(n,1)=(-1)^{n-1}(n-1)!,$$
$$s(n,0)=0,\quad n>0,$$
$$s(n,k)=0,\quad k>n.$$

For any real number x, define

$$\binom{x}{n}=\frac{x(x-1)\cdots(x-n+1)}{n!}.$$

Then (2.2) can be rewritten as

$$\sum_{k=0}^{n} s(n,k)x^k=n!\binom{x}{n}.$$

Definition 2.3 *(Bernoulli number B_k).* [33] Bernoulli number B_k is defined by recursion formula:

$$B_k=\sum_{i=0}^{k}\binom{k}{i}B_i,\quad \text{with } B_0=1,\ B_1=-\frac{1}{2},\ B_2=\frac{1}{6}.$$

The most important property of Bernoulli number is

$$B_{2k+1}=0\quad \text{for } k\geqslant 1. \tag{2.4}$$

Definition 2.4 *(Bernoulli polynomial $B_k[x]$).* [33] Bernoulli polynomial $B_k[x]$ is defined as

$$B_k[x]=\sum_{i=0}^{k}\binom{k}{i}B_i x^{k-i}.$$

We will use the following equalities about Bernoulli number and Bernoulli polynomial:

$$\sum_{k=1}^{m}k^n=\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}(m+1)^{n+1-k}B_k\quad \text{for } n\geqslant 1, \tag{2.5}$$

$$B_n[1-x]=(-1)^n B_n[x]. \tag{2.6}$$

We shall recall the Upper Estimate Conjecture of sharp upper estimate of P_n for an n-dimensional right-angled simplex. When $a_1=a_2=\cdots=a_n$, P_n can be computed very easily. The next theorem is an extension of the result from Lehmer [20].

Theorem 2.1. *Let P_n be the number of positive integer solutions of*

$$x_1 + x_2 + \cdots + x_n \leqslant \lambda, \tag{2.7}$$

where $\lambda \geqslant n$. Then

$$P_n = \binom{[\lambda]}{n}.$$

The following example shows that the sharp estimate cannot apply to $n = 7$.

Counter-Example to the Upper Estimate Conjecture for $n = 7$. Let $a = a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 2000$ and $a_7 = 6.09$. Then consider the following 7-dimensional right-angled simplex: $x_i > 0$, $1 \leqslant i \leqslant 7$:

$$\frac{x_1}{2000} + \frac{x_2}{2000} + \frac{x_3}{2000} + \frac{x_4}{2000} + \frac{x_5}{2000} + \frac{x_6}{2000} + \frac{x_7}{6.09} \leqslant 1. \tag{2.8}$$

Now we need to find P_7 for (2.8). The possible number for x_7 to take is $i = 1, 2, 3, 4, 5, 6$. Plugging in each i into x_7 of (2.8), we have

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leqslant \left(1 - \frac{i}{6.09}\right)2000. \tag{2.9}$$

Let

$$\lambda_i = \left(1 - \frac{i}{6.09}\right)2000.$$

Consider tetrahedra: $x_i > 0,\ i = 1, 2, 3, 4, 5, 6$,

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leqslant \lambda_i. \tag{2.10}$$

By Theorem 2.1, we have

$$P_6(i) = \binom{[\lambda_i]}{6}.$$

So

$$P_7 = \sum_{i=1}^{6} \binom{[\lambda_i]}{6} = 39\,656\,226\,290\,532\,420.$$

Now we compute the sharp estimate f_7 when $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 2000$ and $a_7 = 6.09$.

$$f_7 = A_0^7 + A_1^7 \frac{s(7,6)}{7} + \sum_{l=1}^{5} A_l^6 \frac{s(7,6-l)}{\binom{6}{l}}$$
$$= 199\,840\,412\,984\,945\,440\,000.$$

So we have

$$\begin{aligned} f_7 - 7!P_7 &= 199\,840\,412\,984\,945\,440\,000 - 7! \times 39\,656\,226\,290\,532\,420 \\ &= -26\,967\,519\,337\,956\,800 \\ &< 0. \end{aligned}$$

This shows that the sharp estimate of the Upper Estimate Conjecture fails in this case. On the other hand, the rough estimate is true for this case:

$$\begin{aligned} q_7 - 7!P_7 &= \prod_{i=1}^{7}(a_i - 1) - 7! \times 39\,656\,226\,290\,532\,420 \\ &= 324\,783\,940\,785\,905\,338\,925.09 - 7! \times 39\,656\,226\,290\,532\,420 \\ &= 124\,916\,560\,281\,621\,942\,125.09 \\ &> 0. \end{aligned}$$

However, in case that $a_1 = a_2 = \cdots = a_n$, the Upper Estimate Conjecture is true for all $n \geqslant 1$.

Theorem 2.2. *Given the n-dimensional right-angled simplex defined by* (1.1), *if $a_1 = a_2 = \cdots = a_n$, then*

$$n!P_n \leqslant f_n,$$

where P_n and f_n are defined in the Upper Estimate Conjecture. The equality is true when $a_1 = a_2 = \cdots = a_n =$ integer.

Proof. Let $a = a_i$. From Theorem 2.1, we have

$$P_n = \binom{[a]}{n}. \tag{2.11}$$

On the other hand,

$$f_n = \sum_{k=0}^{n} s(n,k)a^k = n!\binom{a}{n}.$$

By (2.11), we have

$$n!P_n = n!\binom{[a]}{n} \leqslant n!\binom{a}{n} = f_n.$$

The equality is true if a is an integer. □

3. Two useful lemmas

The following two lemmas give an easy way to check the sign of the polynomials with some special properties. They are used extensively in the proof of the Main Theorem. The first lemma allows us to determine the sign of the polynomial by checking the summation of the coefficients.

Lemma 3.1. *Let $f(\beta)$ be a polynomial defined by*

$$f(\beta)=\sum_{i=0}^{n} c_i\beta^i, \quad \text{where } \beta\in(0,1).$$

If for any $k=0,1,\dots,n$

$$\sum_{i=0}^{k} c_i \geqslant 0,$$

then $f(\beta)\geqslant 0$ for $\beta\in(0,1)$.

Proof. We will prove this lemma by mathematical induction. It is obviously true when $n=0$. Let assume that the lemma is true for $n=m-1$. In case $n=m$, we have

$$f(\beta)=\sum_{i=0}^{m-2} c_i\beta^i + c_{m-1}\beta^{m-1}+c_m\beta^m.$$

By assumption, we have $\sum_{i=0}^{m-1} c_i\beta^i\geqslant 0$ and $\sum_{i=0}^{m-2} c_i\beta^i\geqslant 0$.

If $c_m\geqslant 0$, then $f(\beta)\geqslant 0$ and the lemma is true. If $c_m<0$, we have

$$f(\beta)=\sum_{i=0}^{m-2} c_i\beta^i + (c_{m-1}+c_m\beta)\beta^{m-1}.$$

Since $\beta\in(0,1)$, $c_m\beta\geqslant c_m$. So

$$f(\beta)\geqslant\sum_{i=0}^{m-2} c_i\beta^i + (c_{m-1}+c_m)\beta^{m-1}.$$

Since $\sum_{i=0}^{m-2} c_i+(c_{m-1}+c_m)=\sum_{i=0}^{m} c_i\geqslant 0$, by induction for $n=m-1$, we have $f(\beta)\geqslant 0$. This finishes the proof. □

The second lemma allows us to use the initial value of all partial derivatives to determine the sign of the polynomial. For $1\leqslant i_1\leqslant i_2\leqslant\dots\leqslant i_k\leqslant n-1$, $k=1,2,\dots,n-1$, we use the following notation:

$$f^{(k)}(i_1,i_2,\dots,i_k)=\frac{\partial^k f}{\partial a_{i_1}\partial a_{i_2}\cdots\partial a_{i_k}}.$$

Lemma 3.2. *Let $f(a_1, a_2, \dots, a_n, \beta)$ be a polynomial of a_i, $1 \leqslant i \leqslant n$, and β, where the degree of variable a_i, $i = 1, 2, \dots, n-1$, is 1 and $\beta \in (0, 1)$. If*

(1) $f(a_n, a_n, \dots, a_n, \beta) \geqslant 0$, *for* $a_n \geqslant \alpha$ *and* $\beta \in (0, 1)$;
(2) $f^{(k)}(i_1, i_2, \dots, i_k)|_{(a_n, a_n, \dots, a_n, \beta)} \geqslant 0$, *for* $a_n \geqslant \alpha$ *and* $\beta \in (0, 1)$, *and for all* $1 \leqslant i_1 \leqslant i_2 \leqslant \dots \leqslant i_k \leqslant n-1$, $k = 1, 2, \dots, n-1$,

then $f(a_1, a_2, \dots, a_n, \beta) \geqslant 0$ *for* $a_1 \geqslant a_2 \geqslant \dots \geqslant a_n \geqslant \alpha$ *and* $\beta \in (0, 1)$.

Proof. Regarding a_n, β as coefficient parameters, we can treat $f(a_1, a_2, \dots, a_n, \beta)$ as polynomial of $a_1, a_2, \dots, a_{n-1}$. Notice that for $1 \leqslant i_1 \leqslant i_2 \leqslant \dots \leqslant i_{n-1} \leqslant n-1$

$$f^{(n-1)}(i_1, i_2, \dots, i_{n-1}) = \begin{cases} f^{(n-1)}(1, 2, \dots, n-1), & i_1 = 1,\ i_2 = 2, \dots,\ i_{n-1} = n-1, \\ 0, & \text{otherwise}, \end{cases}$$

since the maximum degree of a_i in f is 1. Observe that $f^{(n-1)}(1, 2, \dots, n-1)$ only contains variables a_n and β. It follows that

$$f^{(n-1)}(i_1, i_2, \dots, i_{n-1}) \geqslant 0 \quad \text{for } a_1 \geqslant \dots \geqslant a_n \geqslant \alpha \text{ and } \beta \in (0, 1). \tag{3.1}$$

By (3.1) and condition (2), it follows that for $1 \leqslant i_1 \leqslant i_2 \leqslant \dots \leqslant i_{n-2} \leqslant n-1$

$$f^{(n-2)}(i_1, i_2, \dots, i_{n-2}) \geqslant 0 \quad \text{for } a_1 \geqslant a_2 \geqslant \dots \geqslant a_{n-1} \geqslant a_n \geqslant \alpha \text{ and } \beta \in (0, 1).$$

By applying the same argument to the kth partial derivative inductively for $k = n-3, n-4, \dots, 1$, all the first partial derivatives of f are nonnegative for $a_1 \geqslant a_2 \geqslant \dots \geqslant a_{n-1} \geqslant a_n \geqslant \alpha$ and $\beta \in (0, 1)$. By condition (1), the lemma follows immediately. □

When the first t number of variables of $a_1, a_2, \dots, a_n$ are the same, say $a_i = a$, $1 \leqslant i \leqslant t$, we have the following corollary immediately from Lemma 3.2.

Corollary 3.1. *For $1 \leqslant t \leqslant n$, let $f(a, a_{t+1}, \dots, a_n, \beta)$ be a polynomial of a, a_i, $t+1 \leqslant i \leqslant n$, and β, where the degree of variable a is t and the degree of a_i, $i = t+1, \dots, n-1$, is 1 and $\beta \in (0, 1)$. If*

(1) $f(a_n, a_n, \dots, a_n, \beta) \geqslant 0$ *for* $a_n \geqslant \alpha$ *and* $\beta \in (0, 1)$;
(2) $\frac{\partial^s f}{\partial a^s}|_{(a_n, a_n, \dots, a_n, \beta)} \geqslant 0$ *for* $1 \leqslant s \leqslant t$ *and* $a_n \geqslant \alpha$ *and* $\beta \in (0, 1)$;
(3) *for* $0 \leqslant s \leqslant t$ *and* $1 \leqslant k \leqslant n-1-t$,

$$\frac{\partial^k}{\partial a_{i_{t+1}} \partial a_{i_{t+2}} \cdots \partial a_{i_{t+k}}} \left(\frac{\partial^s f}{\partial a^s} \right) \bigg|_{(a_n, a_n, \dots, a_n, \beta)} \geqslant 0 \quad \textit{for } a_n \geqslant \alpha \textit{ and } \beta \in (0, 1),$$

where $t+1 \leqslant i_{t+1} \leqslant i_{t+2} \leqslant \dots \leqslant i_{t+k} \leqslant n-1$,

then $f(a, a_{t+1}, \dots, a_n, \beta) \geqslant 0$ *for* $a \geqslant a_{t+1} \geqslant \dots \geqslant a_n \geqslant \alpha$ *and* $\beta \in (0, 1)$.

When $t = 1$, this corollary is the same as Lemma 3.2. When $t = n$, condition (3) is not needed.

4. Sharp estimate analysis

Since the conjecture has been proved true for case $n = 3, 4, 5$, we can use induction method to prove case $n = 4, 5, 6$ starting from the result of $n = 3$. The basic approach is to partition the higher dimension right-angled simplices into several lower dimension right-angled simplices. Then we can apply the proved results on lower dimension cases. We shall first take a look at how to partition a right-angled simplices in general.

Let k be the possible integer such that $1 \leqslant k \leqslant [a_n]$, where $[a_n]$ is the biggest integer $\leqslant a_n$. For each k, we have an $n - 1$ dimension right-angled simplices:

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{k}{a_n} \leqslant 1. \tag{4.1}$$

Using a simple computation, we can change the above form into:

$$\frac{x_1}{a_1(1 - \frac{k}{a_n})} + \frac{x_2}{a_2(1 - \frac{k}{a_n})} + \cdots + \frac{x_{n-1}}{a_{n-1}(1 - \frac{k}{a_n})} \leqslant 1. \tag{4.2}$$

Let $P^{(k)}_{n-1}$ be the number of positive integral solution of (4.2). Then

$$P_n = \sum_{k=1}^{[a_n]} P^{(k)}_{n-1}. \tag{4.3}$$

Assume $a_{n-1}(1 - \frac{k}{a_n}) \geqslant n - 2$ for all $1 \leqslant k \leqslant [a_n]$. We can apply sharp estimate from the induction assumption on this $n - 1$ dimension right-angled simplices. Let

$$\mu_k = \left(1 - \frac{k}{a_n}\right),$$

we have sharp estimate $f_{(n-1)}(k)$:

$$(n-1)!P^{(k)}_{n-1} \leqslant f_{(n-1)}(k),$$
$$f_{(n-1)}(k) = \overline{A}^{n-1}_0 + \frac{s(n-1, n-2)}{n-1}\overline{A}^{n-1}_1 + \sum_{l=1}^{n-3} \frac{s(n-1, n-2-l)}{\binom{n-2}{l}}\overline{A}^{n-2}_l,$$

where

$$\overline{A}^{n-1}_t = \left(\prod_{i=1}^{n-1} a_i \mu_k\right)\left(\sum_{1 \leqslant i_1 < i_2 < \cdots < i_t \leqslant n-1} \frac{1}{a_{i_1} a_{i_2} \cdots a_{i_t}} \frac{1}{\mu^t_k}\right) = A^{n-1}_t \mu^{n-1-t}_k. \tag{4.4}$$

So

$$f_{(n-1)}(k) = A_0^{n-1}\left(1-\frac{k}{a_n}\right)^{n-1} + \frac{s(n-1,n-2)}{n-1}A_1^{n-1}\left(1-\frac{k}{a_n}\right)^{n-2} + \sum_{l=1}^{n-3}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}A_l^{n-2}\left(1-\frac{k}{a_n}\right)^{n-2-l}.$$

By (4.3), we have

$$n!P_n = n\sum_{k=1}^{[a_n]}(n-1)!P_{n-1}^{(k)} \leqslant n\sum_{k=1}^{[a_n]} f_{(n-1)}(k).$$

In order to prove $n!P_n \leqslant f_n$, it is sufficient to prove

$$a_n^{n-1} f_n - n\sum_{k=1}^{[a_n]} a_n^{n-1} f_{(n-1)}(k) \geqslant 0. \tag{4.5}$$

In general, we may not be able to use this sharp estimate for all k, since they need to satisfy the condition:

$$a_{n-1}\left(1-\frac{k}{a_n}\right) \geqslant n-2. \tag{4.6}$$

However, if $k = m'$ satisfy this condition, then all $1 \leqslant k < m'$ must satisfy this condition. This is true since

$$a_{n-1}\left(1-\frac{k}{a_n}\right) > a_{n-1}\left(1-\frac{m'}{a_n}\right) \geqslant n-2 \quad \text{for } 1 \leqslant k < m'.$$

So we can sum up k from 1 to m'.

The left-hand side of (4.5) is a polynomial of $a_1, a_2, \ldots, a_n$. It is time consuming and hard to compute this polynomial manually for each n. In order to use Maple to do the computation and simplify the expression of (4.5), we will transform the left-hand side satisfying the following two requirements:

(1) The lower and upper limits of the summation are only determined by n.
(2) The number of summation symbols in one term is minimized.

The next lemma addresses the first requirement.

Lemma 4.1. *Let*

$$G(m') = \sum_{k=1}^{m'} f_{(n-1)}(k), \tag{4.7}$$

then $G(m')$ can be expressed by the summation whose limit is determined only by n:

$$\begin{aligned} G(m') &= \frac{1}{n} A_0^{n-1} \sum_{i=0}^{n-2} \binom{n}{i} \left(-\frac{1}{a_n}\right)^{n-1-i} \sum_{k=0}^{n-1-i} \binom{n-i}{k} (m'+1)^{n-i-k} B_k \\ &\quad + \frac{1}{n-1} \frac{s(n-1,n-2)}{n-1} A_1^{n-1} \\ &\quad \times \sum_{i=0}^{n-3} \binom{n-1}{i} \left(-\frac{1}{a_n}\right)^{n-2-i} \sum_{k=0}^{n-2-i} \binom{n-1-i}{k} (m'+1)^{n-1-i-k} B_k \\ &\quad + \sum_{l=1}^{n-3} \frac{1}{n-1-l} \frac{s(n-1,n-2-l)}{\binom{n-2}{l}} A_l^{n-2} \sum_{i=0}^{n-3-l} \binom{n-1-l}{i} \left(-\frac{1}{a_n}\right)^{n-2-l-i} \\ &\quad \times \sum_{k=0}^{n-2-l-i} \binom{n-1-l-i}{k} (m'+1)^{n-1-l-i-k} B_k + T(m'), \end{aligned} \tag{4.8}$$

where

$$T(m') = m' \left[A_0^{n-1} + \frac{s(n-1,n-2)}{n-1} A_1^{n-1} + \sum_{l=1}^{n-3} \frac{s(n-1,n-2-l)}{\binom{n-2}{l}} A_l^{n-2} \right] \tag{4.9}$$

and B_k is the Bernoulli number.

Proof. Easy exercise. □

Notice that the maximum number of summation symbols in one term in the expression of $G(m')$ is three. Let

$$\beta = a_n - [a_n], \tag{4.10}$$

$$k = [a_n] - h = a_n - \beta - h, \quad \text{where } h = 0, 1, 2, \ldots, [a_n] - 1, \tag{4.11}$$

$$m' = a_n - \beta - m,$$

then

$$G(a_n - \beta - m) = \sum_{h=m}^{a_n-\beta-1} f_{(n-1)}(a_n - \beta - h). \tag{4.12}$$

Using this notation, we can further simplify $G(m')$ by reducing the number of summation symbols.

Lemma 4.2. *Let*

$$g(m') = na_n^{n-1}G(m'), \tag{4.13}$$

then

$$\begin{aligned}
g(a_n-\beta-m) &= na_n^{n-1}\sum_{h=m}^{a_n-\beta-1} f_{(n-1)}(a_n-\beta-h) \\
&= A_0^{n-1}\left[-na_n^{n-1}-\sum_{s=0}^{n}(-1)^{s-1}B_s\binom{n}{s}a_n^{n-s}+(-1)^{n-1}B_n(1-\beta-m)\right] \\
&\quad+\frac{n}{n-1}\frac{s(n-1,n-2)}{n-1}A_1^{n-1} \\
&\quad\times\left[-(n-1)a_n^{n-1}-\sum_{s=0}^{n-1}(-1)^{s-1}B_s\binom{n-1}{s}a_n^{n-s}\right. \\
&\quad\left.+(-1)^{n-2}a_nB_{n-1}(1-\beta-m)\right] \\
&\quad+\sum_{l=1}^{n-3}\frac{n}{n-1-l}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}A_l^{n-2} \\
&\quad\times\left[-(n-1-l)a_n^{n-1}-\sum_{s=0}^{n-1-l}(-1)^{s-1}B_s\binom{n-1-l}{s}a_n^{n-s}\right. \\
&\quad\left.+(-1)^{n-2-l}a_n^{l+1}B_{n-1-l}(1-\beta-m)\right].
\end{aligned} \tag{4.14}$$

Proof. The proof is similar to those in Lemma 4.3. □

Now we can study the difference between the sharp estimate f_n and the sum of the lower dimension sharp estimates $g(a_n-\beta-m)$. The next lemma plays a crucial rule in our later computation.

Lemma 4.3. *Let*

$$\Delta_0(a_n-\beta-m) = a_n^{n-1}f_n - g(a_n-\beta-m), \tag{4.15}$$

then

$$\Delta_0(a_n-\beta-m) = \sum_{i=n}^{2n-2}T_i + T_{n-1}(m) + \Phi(m,\beta), \tag{4.16}$$

where T_i, $n \leqslant i \leqslant 2n-2$, are polynomials of $a_1, a_2, \ldots, a_n$ with coefficients depending only on n. Each term in T_i has degree of i. The expressions of T_i are:

$$T_{2n-2}=\frac{1}{2}A_0^{n-1}a_n^{n-1}-\frac{1}{2}\frac{1}{n-1}A_1^{n-1}a_n^n,$$

$$T_{2n-3}=\frac{s(n,n-2)}{\binom{n-1}{1}}A_1^{n-1}a_n^{n-1}-\binom{n}{2}B_2A_0^{n-1}a_n^{n-2}+\frac{1}{2}\frac{n}{n-1}s(n-1,n-2)A_1^{n-1}a_n^{n-1}$$
$$-\frac{n}{n-2}\frac{s(n-1,n-3)}{\binom{n-2}{1}}A_1^{n-2}a_n^n,$$

$$T_{2n-4}=-\frac{n}{n-1}\binom{n-1}{2}\frac{s(n-1,n-2)}{n-1}B_2A_1^{n-1}a_n^{n-2}+\frac{s(n,n-3)}{\binom{n-1}{2}}A_2^{n-1}a_n^{n-1}$$
$$-\frac{n}{n-3}\frac{s(n-1,n-4)}{\binom{n-2}{2}}A_2^{n-2}a_n^n+\frac{1}{2}\frac{n}{n-2}s(n-1,n-3)A_1^{n-2}a_n^{n-1}.$$

For $i=n+1,\dots,2n-5$, we have

$$T_i=\frac{s(n,i-n+1)}{\binom{n-1}{2n-2-i}}A_{2n-2-i}^{n-1}a_n^{n-1}+(-1)^i\binom{n}{2n-1-i}B_{2n-1-i}A_0^{n-1}a_n^{i-n+1}$$
$$-\frac{n}{n-1}\frac{s(n-1,n-2)}{n-1}(-1)^i\binom{n-1}{2n-2-i}B_{2n-2-i}A_1^{n-1}a_n^{i-n+2}$$
$$-\frac{n}{i-n+1}\frac{s(n-1,i-n)}{\binom{n-2}{2n-2-i}}A_{2n-2-i}^{n-2}a_n^n+\frac{1}{2}n\frac{s(n-1,i-n+1)}{\binom{n-2}{2n-3-i}}A_{2n-3-i}^{n-2}a_n^{n-1}$$
$$+(-1)^i\sum_{s=1}^{2n-4-i}\frac{(-1)^{1+s}n}{n-1-s}\binom{n-1-s}{2n-2-i-s}\frac{s(n-1,n-2-s)}{\binom{n-2}{s}}B_{2n-2-i-s}A_s^{n-2}a_n^{i+s-n+2}$$

and

$$T_n=\frac{s(n,1)}{\binom{n-1}{n-2}}A_{n-2}^{n-1}a_n^{n-1}+(-1)^{n-2}\binom{n}{n-1}B_{n-1}A_0^{n-1}a_n$$
$$+(-1)^{n-1}\frac{n}{n-1}s(n-1,n-2)B_{n-2}A_1^{n-1}a_n^2+\frac{1}{2}\frac{n}{n-2}s(n-1,1)A_{n-3}^{n-2}a_n^{n-1}$$
$$+\sum_{l=1}^{n-4}(-1)^{n-1-l}\frac{n}{\binom{n-2}{l}}s(n-1,n-2-l)B_{n-2-l}A_l^{n-2}a_n^{l+2}.$$

$T_{n-1}(m)$ is a polynomial of $a_1,a_2,\dots,a_n$ with coefficients depending only on n and m. Each term in $T_{n-1}(m)$ has degree of $n-1$. The expression of $T_{n-1}(m)$ is

$$T_{n-1}(m)=(-1)^{n-2}\big[B_n[1-m]-B_n\big]A_0^{n-1}$$
$$+(-1)^{n-1}\frac{n}{n-1}\frac{s(n-1,n-2)}{n-1}\big[B_{n-1}[1-m]-B_{n-1}\big]A_1^{n-1}a_n$$
$$+(-1)^{n-2}\sum_{l=1}^{n-3}\frac{n(-1)^{1+l}}{n-1-l}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}\big[B_{n-1-l}[1-m]-B_{n-1-l}\big]A_l^{n-2}a_n^{l+1}.$$

$\Phi(m,\beta)$ is the polynomial of $a_1, a_2, \ldots, a_n$ with coefficients depending on n, m and β. $\Phi(m,0)=0$. Each term in $\Phi(m,\beta)$ has degree of $n-1$ and

$$\Phi(m,\beta) = (-1)^{n-2} A_0^{n-1} \Psi(n,m,\beta) + (-1)^{n-1} \frac{n}{n-1} \frac{s(n-1,n-2)}{n-1} A_1^{n-1} a_n \Psi(n-1,m,\beta)$$
$$- \sum_{l=1}^{n-3} (-1)^{n-1} \frac{n(-1)^{1+l}}{n-1-l} \frac{s(n-1,n-2-l)}{\binom{n-2}{l}} A_l^{n-2} a_n^{l+1} \Psi(n-1-l,m,\beta),$$

where

$$\Psi(n,m,\beta) = (-1)^n \sum_{s=0}^{n-1} \binom{n}{s} B_s[m] \beta^{n-s}. \tag{4.17}$$

Proof. Notice that

$$a_n^{n-1} f_n = A_0^n a_n^{n-1} + \frac{s(n,n-1)}{n} A_1^n a_n^{n-1} + \sum_{l=1}^{n-2} \frac{s(n,n-1-l)}{\binom{n-1}{l}} A_l^{n-1} a_n^{n-1}$$

and

$$\begin{aligned} A_0^n a_n^{n-1} &= A_0^{n-1} a_n^n, \\ A_1^n a_n^{n-1} &= \left(a_n A_1^{n-1} + A_0^{n-1}\right) a_n^{n-1} \\ &= A_1^{n-1} a_n^n + A_0^{n-1} a_n^{n-1}. \end{aligned}$$

Then by Lemma 4.2

$$\begin{aligned} \Delta_0 = {} & A_0^{n-1} a_n^n (1-1) + A_0^{n-1} a_n^{n-1} \left(\frac{s(n,n-1)}{n} + n + nB_1 \right) \\ & + A_1^{n-1} a_n^n \left(\frac{s(n,n-1)}{n} - \frac{n}{n-1} \frac{s(n-1,n-2)}{n-1} \right) \\ & + \sum_{l=1}^{n-2} \frac{s(n,n-1-l)}{\binom{n-1}{l}} A_l^{n-1} a_n^{n-1} \\ & - A_0^{n-1} \left[- \sum_{s=2}^{n} (-1)^{s-1} B_s \binom{n}{s} a_n^{n-s} + (-1)^{n-1} B_n[1-\beta-m] \right] \\ & - \frac{n}{n-1} \frac{s(n-1,n-2)}{n-1} A_1^{n-1} \\ & \times \left[-\frac{1}{2}(n-1) a_n^{n-1} - \sum_{s=2}^{n-1} (-1)^{s-1} B_s \binom{n-1}{s} a_n^{n-s} + (-1)^{n-2} B_{n-1}[1-\beta-m] a_n \right] \end{aligned}$$

$$
\begin{aligned}
&-\sum_{l=1}^{n-3}\frac{n}{n-1-l}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}A_l^{n-2}\\
&\times\Bigg[a_n^n-\frac{1}{2}(n-1-l)a_n^{n-1}-\sum_{s=2}^{n-1-l}(-1)^{s-1}B_s\binom{n-1-l}{s}a_n^{n-s}\\
&+(-1)^{n-2-l}B_{n-1-l}[1-\beta-m]a_n^{l+1}\Bigg]\\
={}&\frac{1}{2}A_0^{n-1}a_n^{n-1}+\left(-\frac{1}{2}\right)\frac{1}{n-1}A_1^{n-1}a_n^n+\sum_{l=1}^{n-2}\frac{s(n,n-1-l)}{\binom{n-1}{l}}A_l^{n-1}a_n^{n-1}\\
&-A_0^{n-1}\Bigg[-\sum_{s=2}^{n}(-1)^{s-1}B_s\binom{n}{s}a_n^{n-s}+(-1)^{n-1}B_n[1-\beta-m]\Bigg]\\
&-\frac{n}{n-1}\frac{s(n-1,n-2)}{n-1}A_1^{n-1}\\
&\times\Bigg[-\frac{1}{2}(n-1)a_n^{n-1}-\sum_{s=2}^{n-1}(-1)^{s-1}B_s\binom{n-1}{s}a_n^{n-s}+(-1)^{n-2}B_{n-1}[1-\beta-m]a_n\Bigg]\\
&-\sum_{l=1}^{n-3}\frac{n}{n-1-l}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}A_l^{n-2}\\
&\times\Bigg[a_n^n-\frac{1}{2}(n-1-l)a_n^{n-1}-\sum_{s=2}^{n-1-l}(-1)^{s-1}B_s\binom{n-1-l}{s}a_n^{n-s}\\
&+(-1)^{n-2-l}B_{n-1-l}[1-\beta-m]a_n^{l+1}\Bigg].
\end{aligned}
$$

In the last term, define

$$
\begin{aligned}
\overline{\Delta}&=-\sum_{l=1}^{n-3}\frac{n}{n-1-l}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}A_l^{n-2}(-1)\sum_{s=2}^{n-1-l}(-1)^{s-1}B_s\binom{n-1-l}{s}a_n^{n-s}\\
&=\sum_{l=1}^{n-3}\frac{n}{n-1-l}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}\sum_{s=2}^{n-1-l}(-1)^{s-1}B_s\binom{n-1-l}{s}A_l^{n-2}a_n^{n-s}.
\end{aligned}
$$

Define the new index $i=s+l$, we have the

$$
\begin{aligned}
\overline{\Delta}&=\sum_{l=1}^{n-3}\frac{n}{n-1-l}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}\sum_{i=2+l}^{n-1}(-1)^{s-1}B_s\binom{n-1-l}{s}A_l^{n-2}a_n^{n-s}\\
&=\sum_{i=3}^{n-1}\sum_{l=1}^{i-2}\frac{n}{n-1-l}\frac{s(n-1,n-2-l)}{\binom{n-2}{l}}(-1)^{s-1}B_s\binom{n-1-l}{s}A_l^{n-2}a_n^{n-s}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{l=1}^{n-3} (-1)^{n+l} \frac{n}{n-1-l} \frac{s(n-1, n-2-l)}{\binom{n-2}{l}} B_{n-1-l} A_l^{n-2} a_n^{1+l} \\
&\quad + \sum_{l=1}^{n-4} (-1)^{n-1+l} n \frac{s(n-1, n-2-l)}{\binom{n-2}{l}} B_{n-2-l} A_l^{n-2} a_n^{2+l} \\
&\quad + \sum_{i=3}^{n-3} (-1)^i \sum_{l=1}^{i-2} \frac{n(-1)^{1+l}}{n-1-i} \frac{s(n-1, n-2-l)}{\binom{n-2}{l}} B_{i-l} A_l^{n-2} a_n^{n-i+l}.
\end{aligned}
$$

Also notice that,

$$
\begin{aligned}
B_n[1-\beta-m] &= \sum_{k=0}^{n-1} \binom{n}{k} (1-\beta-m)^{n-k} B_k + B_n \\
&= \sum_{k=0}^{n-1} \binom{n}{k} B_k (1-m)^{n-k} + B_n + \sum_{k=0}^{n-1} \binom{n}{k} B_k \sum_{t=0}^{n-k-1} \binom{n-k}{t} (1-m)^t (-\beta)^{n-k-t} \\
&= B_n[1-m] + \sum_{k=0}^{n-1} \binom{n}{k} B_k \sum_{t=0}^{n-k-1} \binom{n-k}{t} (1-m)^t (-\beta)^{n-k-t}.
\end{aligned}
$$

Similarly,

$$
B_{n-1}[1-\beta-m] = B_{n-1}[1-m] + \sum_{k=0}^{n-2} \binom{n-1}{k} B_k \sum_{t=0}^{n-k-2} \binom{n-k-1}{t} (1-m)^t (-\beta)^{n-1-k-t}
$$

$$
\begin{aligned}
B_{n-1-l}[1-\beta-m] &= B_{n-1-l}[1-m] \\
&\quad + \sum_{k=0}^{n-2-l} \binom{n-1-l}{k} B_k \sum_{t=0}^{n-k-2-l} \binom{n-k-1-l}{t} (1-m)^t (-\beta)^{n-1-k-t-l}.
\end{aligned}
$$

Let

$$
\Psi(n, m, \beta) = \sum_{k=0}^{n-1} \binom{n}{k} B_k \sum_{t=0}^{n-k-1} \binom{n-k}{t} (1-m)^t (-\beta)^{n-k-t}.
$$

Define the new index $s = k + t$. We have

$$
\begin{aligned}
\Psi(n, m, \beta) &= \sum_{k=0}^{n-1} \binom{n}{k} B_k \sum_{s=k}^{n-1} \binom{n-k}{s-k} (1-m)^{s-k} (-\beta)^{n-s} \\
&= \sum_{s=0}^{n-1} \sum_{k=0}^{s} \binom{n}{k} B_k \binom{n-k}{s-k} (1-m)^{s-k} (-\beta)^{n-s}
\end{aligned}
$$

$$= \sum_{s=0}^{n-1} (-\beta)^{n-s} \binom{n}{s} \sum_{k=0}^{s} \binom{s}{k} (1-m)^{s-k} B_k$$

$$= \sum_{s=0}^{n-1} \binom{n}{s} (-\beta)^{n-s} B_s[1-m].$$

Notice that $B_n[1-x] = (-1)^n B_n[x]$. So

$$\Psi(n, m, \beta) = (-1)^n \sum_{s=0}^{n-1} \binom{n}{s} B_s[m] \beta^{n-s}.$$

Then

$$B_n[1-\beta-m] = B_n[1-m] + \Psi(n, m, \beta),$$
$$B_{n-1}[1-\beta-m] = B_{n-1}[1-m] + \Psi(n-1, m, \beta),$$
$$B_{n-1-l}[1-\beta-m] = B_{n-1-l}[1-m] + \Psi(n-1-l, m, \beta).$$

Using the above results and collecting terms with the same degree, we can get (4.16) as the expression of Δ_0. □

Remark 1. From the definition of A_k^n, it can be easily seen that the degree of each a_j, $1 \leqslant j \leqslant n-1$, in T_i is 1. We will use this property later.

5. Unified proof for sharp upper estimate

We know the Main Theorem is true for $n = 3, 4, 5$ [22,37,39]. In order to get the unified proof for all $4 \leqslant n \leqslant 6$, we assume the Main Theorem is true for $(n-1)$-dimensional tetrahedron. Then we shall prove the n-dimensional case. We should dissect the n-dimensional tetrahedron along x_n-axis into several $(n-1)$-dimensional tetrahedron. Inequality (4.1) is the kth level of such $(n-1)$-dimensional tetrahedron. Using the notation $k = a_n - \beta - h$ in (4.11), we can transform the kth tetrahedron (4.1) into the following form:

$$\frac{x_1}{\frac{a_1}{a_n}(\beta+h)} + \frac{x_2}{\frac{a_2}{a_n}(\beta+h)} + \cdots + \frac{x_{n-1}}{\frac{a_{n-1}}{a_n}(\beta+h)} \leqslant 1, \tag{5.1}$$

where $h = 0, 1, 2, \ldots, a_n - \beta - 1$.

Let $P_{n-1}(h)$ be the number of positive integer solution of (5.1). Then we have

$$P_n = \sum_{h=0}^{[a_n]-1} P_{n-1}(h).$$

We also use the notation $q_{n-1}(a_n - \beta - h)$ and $f_{n-1}(a_n - \beta - h)$ to denote the rough and sharp estimate defined in the Upper Estimate Conjecture for (5.1).

For each $P_{n-1}(h)$, there are three cases regarding its upper estimate:

(a) $P_{n-1}(h) = 0$, then we do not need to consider the tetrahedron on this level. Also if $P_{n-1}(h') = 0$, then $P_{n-1}(h) = 0$ for all $h \leqslant h'$.
(b) $P_{n-1}(h) > 0$, and $\frac{a_{n-1}}{a_n}(\beta + h) < n - 2$. We know $\frac{a_{n-1}}{a_n}(\beta + h) > 1$. Then we can apply the rough estimate:

$$(n-1)!P_{n-1}(h) \leqslant q_{n-1}(a_n - \beta - h).$$

(c) $P_{n-1}(h) > 0$, and $\frac{a_{n-1}}{a_n}(\beta + h) \geqslant n - 2$. Then we can apply the sharp estimate:

$$(n-1)!P_{n-1}(h) \leqslant f_{n-1}(a_n - \beta - h).$$

So we have

$$\begin{aligned} n!P_n &= n \sum_{h=h_0}^{[a_n]-1} (n-1)!P_{n-1}(h) \\ &\leqslant n \sum_{h=h_0}^{m-1} q_{n-1}(a_n - \beta - h) + n \sum_{h=m}^{[a_n]-1} f_{n-1}(a_n - \beta - h), \end{aligned}$$

where m is the smallest integer for which the sharp estimate condition $\frac{a_{n-1}}{a_n}(\beta + m) \geqslant n - 2$ is true. h_0 is the smallest integer for which $P_{n-1}(h_0) > 0$.

In order to show $n!P_n \leqslant f_n$, we only need to show that

$$f_n \geqslant n \sum_{h=h_0}^{m-1} q_{n-1}(a_n - \beta - h) + n \sum_{h=m}^{[a_n]-1} f_{n-1}(a_n - \beta - h).$$

Now define

$$\Delta = a_n^{n-1} f_n - n a_n^{n-1} \sum_{h=m}^{[a_n]-1} f_{n-1}(a_n - \beta - h) - n a_n^{n-1} \sum_{h=h_0}^{m-1} q_{n-1}(a_n - \beta - h).$$

Using the definition in (4.15) and (4.14), we have

$$\begin{aligned} \Delta &= a_n^{n-1} f_n - g(a_n - \beta - m) - n a_n^{n-1} \sum_{h=h_0}^{m-1} q_{n-1}(a_n - \beta - h) \\ &= \Delta_0(a_n - \beta - m) - n a_n^{n-1} \sum_{h=h_0}^{m-1} q_{n-1}(a_n - \beta - h). \end{aligned} \tag{5.2}$$

Notice that Δ is the polynomial of $a_1, a_2, \ldots, a_n, \beta$. Our task here is to show that $\Delta \geqslant 0$ for $a_1 \geqslant a_2 \geqslant a_3 \geqslant \cdots \geqslant a_n \geqslant n - 1$ and the equality holds when $a_1 = a_2 = a_3 = \cdots = a_n =$ integer. Please notice that we have already proved the sharp estimate is true for all $n \geqslant 2$ when $a_1 = a_2 = \cdots = a_n$ in Theorem 2.2. In (5.2), for each fixed n, if we can determine m and h_0, then we can use Lemmas 3.1 and 3.2 to determine the sign of (5.2). By definition of m, $m \leqslant n - 2$.

This means the maximum number of $(n-1)$-dimensional tetrahedron on which we have to apply rough estimate is $n-2$. We need to study the sign of Δ in each situations of $P_{n-1}(k)=0$, where $k=0,1,\dots,n-2$. For this reason, we shall study Δ in $(n-1)\times n$ subcases determined by

$$a_1=a_2=\cdots=a_{n-i}\geqslant a_{n-i+1}\geqslant\cdots\geqslant a_{n-1},\quad \text{where } 1\leqslant i\leqslant n-1,$$
$$P_{n-1}(n-2-j)=0,\quad P_{n-1}(n-1-j)>0,\quad \text{where } 0\leqslant j\leqslant n-1.$$

By (5.1), it is obvious that $P_{n-1}(h)=0$ imply that $P_{n-1}(1)=P_{n-1}(2)=\cdots=P_{n-1}(h-1)=0$.

Case 1. $i=1$ implies $a_1=a_2=\cdots=a_{n-1}\geqslant a_n$. Let $a=a_1=a_2=\cdots=a_{n-1}$. We study the following subcases.

Subcase 1.0. $j=0$ implies $P_{n-1}(1)=P_{n-1}(2)=\cdots=P_{n-1}(n-3)=P_{n-1}(n-2)=0$.

Since $\frac{a_{n-1}}{a_n}(\beta+h)\geqslant n-2$ for all $h>n-2$, we can apply sharp estimate to $P_{n-1}(h)$ for $h\geqslant n-1$. Then $m=n-1$ and $h_0=n-1$ and (5.2) becomes

$$\Delta=\Delta_0\big(a_n-\beta-(n-1)\big). \tag{5.3}$$

Let

$$\Delta_{1.0}=\Delta|_{a_1=a_2=\cdots=a_{n-1}=a}.$$

By Remark 1 in Section 4, $\Delta_{1.0}$ is actually the polynomial of a, a_n and β such that

$$\frac{\partial^n\Delta_{1.0}}{\partial a^n}=0.$$

For $n\leqslant 6$, by using Lemma 3.1, we can verify that

$$\left.\frac{\partial^k\Delta_{10}}{\partial a^k}\right|_{a=a_n}\geqslant 0\quad \text{for } a_n\geqslant n-1 \text{ and } 0<\beta<1,\ 0\leqslant k\leqslant n-1. \tag{5.4}$$

Then by Corollary 3.1, we have

$$\Delta=\Delta_{1.0}>0\quad \text{for all } a\geqslant a_n\geqslant n-1 \text{ and } 0<\beta<1.$$

Subcase 1.1. $P_{n-1}(0)=P_{n-1}(1)=\cdots=P_{n-1}(n-3)=0$, $P_{n-1}(n-2)>0$.

Since $\frac{a_{n-1}}{a_n}(\beta+h)\geqslant n-2$ for all $h\geqslant n-2$, we can apply sharp estimate to $P_{n-1}(h)$. So $m=n-2$ and $h_0=n-2$. We have

$$\Delta=\Delta_0\big(a_n-\beta-(n-2)\big). \tag{5.5}$$

Let

$$\Delta_{1.1}=\Delta|_{a_1=a_2=\cdots=a_{n-1}=a}.$$

Similar to previous case, $\Delta_{1.1}$ is actually the polynomial of a, a_n, β such that

$$\frac{\partial^n \Delta_{1.1}}{\partial a^n} = 0.$$

Since $P_{n-1}(n-2) > 0$, we have

$$\frac{1}{\frac{a_1}{a_n}(\beta+n-2)} + \frac{1}{\frac{a_2}{a_n}(\beta+n-2)} + \cdots + \frac{1}{\frac{a_{n-1}}{a_n}(\beta+n-2)} \leqslant 1.$$

This is equivalent to

$$\frac{a}{a_n}(\beta+n-2) \geqslant n-1.$$

We have the minimum value for a:

$$a \geqslant a_0 := \frac{n-1}{\beta+n-2} a_n. \tag{5.6}$$

Obviously, $a_0 > a_n$. Since $\Delta_{1.1}$ does not have value for $a \in [a_n, a_0)$, we extends the definition of $\Delta_{1.1}$ to interval $[a_n, a_0]$ by assigning

$$\Delta_{1.1}(a, a_n, \beta) = \Delta_{1.1}(a_0, a_n, \beta) \quad \text{for } a \in [a_n, a_0). \tag{5.7}$$

So we can check the derivative of $\Delta_{1.1}$ at $a = a_0$ instead of $a = a_n$. For $k = 0, 1, 2, \ldots, n-1$ by Lemma 3.1, we can verify that

$$\left.\frac{\partial^k \Delta_{1.1}}{\partial a^k}\right|_{a=a_0} \geqslant 0 \quad \text{for } a_n \geqslant n-1 \text{ and } 0 < \beta < 1. \tag{5.8}$$

Recall we have extended the definition of $\Delta_{1.1}$ in (5.7). So a_0 can be replaced by a_n in (5.8). By Corollary 3.1, we have

$$\Delta = \Delta_{1.1} \geqslant 0 \quad \text{for } a \geqslant a_n \geqslant n-1 \text{ and } 0 < \beta < 1.$$

In general we have Subcase $1.j$, where $j = 0, 1, 2, \ldots, n-2$.

Subcase $1.j$. $P_{n-1}(1) = P_{n-1}(2) = \cdots = P_{n-1}(n-2-j) = 0$, $P_{n-1}(n-1-j) > 0$. In this case, $m = n-2$ and $h_0 = n-1-j$. Then

$$\Delta = \Delta_0\big(a_n - \beta - (n-2)\big) - n a_n^{n-1} \sum_{h=n-1-j}^{n-3} q_{n-1}(a_n - \beta - h). \tag{5.9}$$

Let

$$\Delta_{1.j} = \Delta|_{a_1=a_2=\cdots=a_{n-1}=a}.$$

Since $\Delta_{1.j}$ is the polynomial of a, a_n, β, we also use $\Delta_{1.j}(a, a_n, \beta)$ for $\Delta_{1.j}$. From $P_{n-1}(n-1-j) > 0$, we can get

$$a \geqslant a_0 := \begin{cases} a_n & \text{for } j = 0, \\ \frac{n-1}{\beta+n-2} a_n & \text{for } j = 1, \\ \frac{n-1}{n-j} a_n & \text{for } j \geqslant 2. \end{cases} \tag{5.10}$$

The definition of $\Delta_{1.j}$ is extended to $[a_n, a_0]$ by assigning the value of $\Delta_{1.j}$ at a_0. Using the same technique as in Case 1.1, we compute and verify that for $0 \leqslant k \leqslant n$

$$\left.\frac{\partial^k \Delta_{1j}}{\partial a^k}\right|_{a=a_n} \geqslant 0 \quad \text{for } a_n \geqslant n-1 \text{ and } 0 < \beta < 1.$$

By Corollary 3.1, we have

$$\Delta = \Delta_{1.j}(a, a_n) \geqslant 0 \quad \text{for } a \geqslant a_n \geqslant n-1 \text{ and } 0 < \beta < 1.$$

In general, we can have Case i, where $2 \leqslant i \leqslant n-1$.

Case i. $a_1 = a_2 = \cdots = a_{n-i} \geqslant a_{n-i+1} \geqslant \cdots \geqslant a_{n-1} \geqslant a_n$.

We also use $P_{n-1}(h)$ to divide this case into $n-1$ subcases: Subcase $i.j$, $0 \leqslant j \leqslant n-2$, where $P_{n-1}(1) = P_{n-1}(1) = P_{n-1}(n-2-j) = 0$, $P_{n-1}(n-1-j) > 0$. For each Subcase $i.j$, we can compute Δ by (5.9). Let $a = a_1 = a_2 = \cdots = a_{n-i}$. Define

$$\Delta_{i.j} = \Delta|_{a_1=a_2=\cdots=a_{n-i}=a}.$$

$\Delta_{i.j}$ is polynomial in $a, a_{n-i+1}, \ldots, a_n, \beta$. We write $\Delta_{i.j}$ as

$$\Delta_{i.j} = \Delta_{i.j}(a, a_{n-i+1}, \ldots, a_n).$$

Notice that

$$\Delta_{i.j}(a, a, a_{n-i+2}, \ldots, a_n) = \Delta_{(i-1).j}(a, a_{n-i+2}, \ldots, a_n).$$

Since we already proved in Subcase $(i-1).j$ that

$$\Delta_{(i-1).j}(a, a_{n-i+2}, \ldots, a_n) \geqslant 0 \quad \text{for } a \geqslant a_{n-i+2} \geqslant \cdots \geqslant a_n \geqslant n-1 \text{ and } 0 < \beta < 1.$$

It follows that for $a \geqslant a_{n-i+2} \geqslant \cdots \geqslant a_n \geqslant n-1$ and $0 < \beta < 1$

$$\Delta_{i.j}|_{a=a_{n-i+1}} = \Delta_{i.j}(a, a, a_{n-i+2}, \ldots, a_n) \geqslant 0. \tag{5.11}$$

Now we need to check the sign of $\frac{\partial \Delta_{i.j}}{\partial a}$. Notice that in Subcase $i.j$, $\frac{\partial \Delta_{i.j}}{\partial a}$ is a polynomial of $a, a_{n-i+1}, \ldots, a_{n-1}, a_n$ and β. The degree of a is $n-1-i$, the degree of a_s, $n-i+1 \leqslant s \leqslant n-1$, is 1. Using the same technique as in Case 1, we can show that for $a_n \geqslant \alpha$ and $\beta \in (0, 1)$

$$\left.\frac{\partial^s \Delta_{i.j}}{\partial a^s}\right|_{a=a_{n-i+1}=\cdots=a_{n-1}=a_n} \geqslant 0,$$

$$\left.\frac{\partial^k}{\partial a_{i_{n-i+1}} \partial a_{i_{n-i+2}} \cdots \partial a_{i_{n-i+k}}} \left(\frac{\partial^s \Delta_{i.j}}{\partial a^s}\right)\right|_{a=a_{n-i+1}=\cdots=a_{n-1}=a_n} \geqslant 0,$$

where $0 \leqslant s \leqslant n-i$, $n-i+1 \leqslant i_{n-i+1} \leqslant i_{n-i+2} \leqslant \cdots \leqslant i_{n-i+k} \leqslant n-1$ and $1 \leqslant k \leqslant i-1$. By Corollary 3.1, we have

$$\frac{\partial \Delta_{i.j}}{\partial a} \geqslant 0 \quad \text{for all } a \geqslant a_{n-i+1} \geqslant \cdots \geqslant a_{n-1} \geqslant a_n.$$

By (5.11), we have

$$\Delta_{i.j}(a, a_{n-i+1}, \ldots, a_n) \geqslant 0 \quad \text{for all } a \geqslant a_{n-i+1} \geqslant \cdots \geqslant a_{n-1} \geqslant a_n.$$

At the case $i = n-1$, we have for all $a = a_1 \geqslant a_2 \geqslant \cdots \geqslant a_{n-1} \geqslant a_n \geqslant n-1$

$$\Delta_{(n-1).j}(a_1, a_2, \ldots, a_{n-1}, a_n) \geqslant 0.$$

This finishes the proof.

6. Conclusion

In this paper we present a unified proof for the GLY (Granville–Lin–Yau) conjecture for $n = 4, 5, 6$. An example shows that the sharp upper estimate of the Upper Estimate Conjecture is not true for $n = 7$. The lower bound for a_n has to be further investigated. The $a_n \geqslant n-1$ used in the Main Theorem is too loose for tetrahedron of $n \geqslant 7$.

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