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The diffeomorphic types of the complements of arrangements in $\mathbb{CP}^3$ II

Dedicated to Professor LU QiKeng on the occasion of his 80th birthday

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Abstract For any arrangement of hyperplanes in $\mathbb{CP}^3$, we introduce the soul of this arrangement. The soul, which is a pseudo-complex, is determined by the combinatorics of the arrangement of hyperplanes. In this paper, we give a sufficient combinatoric condition for two arrangements of hyperplanes to be diffeomorphic to each other. In particular we have found sufficient conditions on combinatorics for the arrangement of hyperplanes whose moduli space is connected. This generalizes our previous result on hyperplane point arrangements in $\mathbb{CP}^3$.

Keywords: **arrangement of hyperplanes, soul of arrangement, combinational geometry of arrangement, differential and topological structure of complement of arrangement**

MSC(2000): 32S22, 57R50, 57R52, 05B35, 14N20

1 Introduction

An arrangement of hyperplanes $\mathcal{A}^*$ in $\mathbb{CP}^n$ is a finite collection of hyperplanes of dimension $n-1$ in $\mathbb{CP}^n$. Associated with $\mathcal{A}^*$ is an open real $2n$-manifold, the complement $M(\mathcal{A}^*) = \mathbb{CP}^n - \bigcup_{H^*\in\mathcal{A}^*} H^*$. One of the central problems in this area is to decide to what extent the topology or differentiable structure of $M(\mathcal{A}^*)$ is determined by the combinatorial geometry of $\mathcal{A}^*$ and vice versa. It is well-known that the combinatorial data of $\mathcal{A}^*$ are coded by $L(\mathcal{A}^*)$ which is the set of all intersections of elements of $\mathcal{A}^*$ partially ordered by reverse inclusion. In a series of papers[1−3], Falk studied the question whether $L(\mathcal{A}^*)$ is a homotopic invariant. In [3], Falk constructed two arrangements of hyperplanes in $\mathbb{CP}^2$, each of which has two triple points and nine double points, but their combinatorial data are different. The homotopic equivalence of their complements was shown in [3]. Therefore $L(\mathcal{A}^*)$ is not a homotopic invariant. In 1993, Jiang and Yau [4, 5] proved that $L(\mathcal{A}^*)$ is indeed a topological invariant if $\mathcal{A}^*$ is an arrangement of hyperplanes in $\mathbb{CP}^2$. In their proof, they made use of some deep results of Waldhausen on three-manifolds. Indeed $L(\mathcal{A}^*)$ is no longer a topological invariant for arrangement of hyperplanes $\mathcal{A}^*$ in $\mathbb{CP}^n$, $n \geqslant 3$ (cf. [6]).

The difficult and still unsolved problem is whether the topological or diffeomorphic type

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of complement $M(\mathcal{A}^*)$ of an arrangement is combinatorial in nature. In a famous preprint, Rybnikov[7] announced the existence of two line arrangements $\mathcal{A}_1^*$ and $\mathcal{A}_2^*$ in $\mathbb{CP}^2$ which have the same combinatorics but whose complements $M(\mathcal{A}_1^*)$ and $M(\mathcal{A}_2^*)$ are not homeomorphic. Unfortunately there is no detail proof of the above result. Recently Bartolo, Ruber, Agustin and Buzunariz[8] prove the existence of complexified real arrangements with same combinatorics but different topology for complements of arrangements. The first step towards finding such pairs of arrangements involves finding combinatorics whose moduli space is not connected. On the other hand, if an arrangement $\mathcal{A}^*$ whose moduli space is connected, then Randell's lattice-isotopic theorem[9] implies that there is only one differentiable structure for any arrangement lying in this moduli space. For a central arrangement of hyperplanes $\mathcal{A}$ in $\mathbb{C}^{n+1}$, one can define the underlying matroid $\mathcal{G}(\mathcal{A})$ of $\mathcal{A}$ (see for example [10]). Recall that the moduli space of arrangements is the same as the realization space of the underlying matroid (cf. [10]). In view of the result of Randell[9], the moduli space of Rybnikov arrangements[7] and the moduli space of Bartolo, Ruber, Agustin and Buzunariz[8] arrangements are nonconnected. Therefore there is enormous interest of finding combinatorics for which the moduli space is connected. In 1994, Jiang and Yau[11] first successfully described a large class of line arrangements in $\mathbb{CP}^2$ whose moduli spaces are connected. In 2005 we[12] have described a much larger class of line arrangements in $\mathbb{CP}^2$ whose moduli spaces are still connected. Recently we[13] introduced the concept of point arrangements of hyperplanes in $\mathbb{CP}^3$ and proved the following theorem.

Theorem 1.1. *Let $\mathcal{A}^*$ be a nice point arrangement of hyperplanes in $\mathbb{CP}^3$. The moduli space of $\mathcal{A}^*$ with fixed combinatorics $L(\mathcal{A}^*)$ is connected.*

In this paper we generalize the above theorem to a class of nice arrangement of hyperplanes in $\mathbb{CP}^3$. In general, for any arrangement $\mathcal{A}^*$ of hyperplanes in $\mathbb{CP}^3$, we introduce a soul $\mathcal{G}(\mathcal{A}^*)$ which is a pseudo-complex completely determined by the combinatoric data of the arrangement. If the soul consists of $\mathcal{G}(1)$ (a set of lines or 1-simplices) and $\mathcal{G}(2)$ (a set of planes or 2-simplices), then the arrangement is called a line arrangement. A line arrangement is called a nice arrangement if after removing disjoint stars of $\mathcal{G}$, the remaining pseudo-complex contains no loop (cf. Definition 2.7). We prove that the theorem above still holds for the nice line arrangements in $\mathbb{CP}^3$.

Theorem A. *Let $\mathcal{A}_0^*$ and $\mathcal{A}_1^*$ be two nice arrangements of hyperplanes in $\mathbb{CP}^3$. If $L(\mathcal{A}_0^*)$ and $L(\mathcal{A}_1^*)$ are isomorphic, then $M(\mathcal{A}_0^*)$ and $M(\mathcal{A}_1^*)$ are diffeomorphic to each other.*

In the course of proving Theorem A, we have proved the following Theorem.

Theorem B. *Let $\mathcal{A}^*$ be a nice arrangement of hyperplanes in $\mathbb{CP}^3$. The moduli space of $\mathcal{A}^*$ with fixed combinatorics $L(\mathcal{A}^*)$ is connected.*

Our paper is organized as follows. In Section 2, for any arrangement $\mathcal{A}^*$ in $\mathbb{CP}^3$, we introduce a pseudo-complex $\mathcal{G}(\mathcal{A}^*)$ which is called the soul of $\mathcal{A}^*$. $\mathcal{G}(\mathcal{A}^*)$ is determined by the combinatorial data $L(\mathcal{A}^*)$. We also introduce the definition of the nice arrangement of hyperplanes. In Section 3, we prove Theorem A and Theorem B for nice line arrangements. In the final section, we prove Theorem A and Theorem B for nice arrangements of hyperplanes in $\mathbb{CP}^3$.

2 Nice arrangements of hyperplanes in $\mathbb{CP}^3$

In this paper we denote $\mathcal{A}^*$ arrangement of hyperplanes in $\mathbb{CP}^3$. Let $L(\mathcal{A}^*)$ be the set of all

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intersections of subsets of $\mathcal{A}^*$, partially ordered by reverse inclusion.

We give some definitions and examples of nice arrangements of hyperplanes in $\mathbb{CP}^3$ for the following sections.

Definition 2.1. *A point* p *in* $\mathbb{CP}^3$ *is of multiplicity* k*, denoted by* $m(p)$*, in* $\mathcal{A}^*$ *if* p *is the intersection of exactly* k *hyperplanes in* $\mathcal{A}^*$*. A line* l *in* $\mathbb{CP}^3$ *is of multiplicity* k*, denoted by* $m(l)$*, in* $\mathcal{A}^*$ *if* l *is the intersection of exactly* k *hyperplanes in* $\mathcal{A}^*$*.*

To study the combinatorial properties of $\mathcal{A}^*$, we need to consider all intersections (lines and points) of $\mathcal{A}^*$ in $\mathbb{CP}^3$. For an arrangement in $\mathbb{CP}^3$, any two planes must meet at a line. We only need to consider those intersection lines whose multiplicity is not less than 3. For any plane and line, if the line does not lie on the plane, they must intersect at a point with multiplicity 3 in the arrangement. We also know that a point may be an intersection of two lines. So, we need to consider those intersection points whose multiplicity is not less than 4. To get rid of the trivial situation that a point has multiplicity at least 4 which is obtained by a plan and a line with multiplicity at least 3, we need to add a condition for the intersection points: there are four planes passing through this point in the arrangement $\mathcal{A}^*$ such that every three of them are in general position. Now we can give the following definition naturally.

Definition 2.2. *Let* $p_k(\mathcal{A}^*)$ *be the number of points of multiplicity* $k\,(\geqslant 4)$ *each of which has the property that there are four planes passing through this point in the arrangement* $\mathcal{A}^*$ *such that every three of them are in general position. Let* $l_k(\mathcal{A}^*)$ *be the number of lines of multiplicity* $k\,(\geqslant 3)$ *in the arrangement* $\mathcal{A}^*$*. Then the complexity* $c(\mathcal{A}^*)$ *of* $\mathcal{A}^*$ *is defined to be* $\sum_{k\geqslant 4}(k-3)p_k(\mathcal{A}^*)+\sum_{k\geqslant 3}(k-2)l_k(\mathcal{A}^*)$*.*

Definition 2.3. *A soul* $\mathcal{G}$ *of an arrangement* $\mathcal{A}^*$ *of hyperplanes in* $\mathbb{CP}^3$ *is a pseudo-complex which is defined as follows:*

Let $\mathcal{G}(0)$ *be the set of* 0*-simplices of* $\mathcal{G}$ *defined by* $\{p\in L(\mathcal{A}^*)$ *is a point* $|m(p)\geqslant 4$ *and there are four planes passing through* p *in* $\mathcal{A}^*$ *from which any three of them are in general position*$\}$*. An element of* $\mathcal{G}(0)$ *is called a point.*

Let $\mathcal{G}(1)$ *be the set of* 1*-simplices of* $\mathcal{G}$ *which is the set of lines of* $L(\mathcal{A}^*)$ *with multiplicity* $m(l)\geqslant 3$*. An element of* $\mathcal{G}(1)$ *is called a line.*

Let $\mathcal{G}(2)$ *be the set of* 2*-simplices of* $\mathcal{G}$*. Each element of* $\mathcal{G}(2)$ *is a hyperplane of* $\mathcal{A}^*$ *that passes through an element of* $\mathcal{G}(0)\cup\mathcal{G}(1)$*. This means that it contains a point or line of* $\mathcal{G}(0)\cup\mathcal{G}(1)$*. An element of* $\mathcal{G}(2)$ *is called a plane.*

We say that two different simplices of $\mathcal{G}$ intersect to each other in $\mathcal{G}$ if and only if they contain a same element of $\mathcal{G}(0)\cup\mathcal{G}(1)$ (See Example 2.8 below).

A path in $\mathcal{G}$ is defined to be a finite sequence of simplices $a_0,h_1,a_1,h_2,\dots,a_{k-1},h_k,a_k$ $(k>0)$ of $\mathcal{G}$ where a_i and a_{i+1} are distinct elements in $\mathcal{G}(0)\cup\mathcal{G}(1)$, $h_{i+1}\in\mathcal{G}(2)$, which contains both a_i and a_{i+1} for $i=0,1,\dots,k-1$ and h_j are distinct for $j=1,\dots,k$. k is called the length of the path from a_0 to a_k. When $a_0=a_k, k\geqslant 3$, we call this path a loop.

For two elements a_1 and $a_2\in\mathcal{G}(0)\cup\mathcal{G}(1)$, the distance from a_1 to a_2 is the minimum length of the path among all paths from a_1 to a_2.

Say a_1 to be a k-element of a_2 if the distance from a_1 to a_2 is k. If a_1 is a point, we call a_1 as a k-point of a_2. If a_1 is a line, we call a_1 as a k-line of a_2.

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Remark 2.4. From the discussion and definitions above, we know that in $\mathbb{CP}^3$, each two planes must meet at a line and each plane and a line must intersect at a point. Hence we do not need to consider these trivial cases in our definition of the pseudo-complex soul $\mathcal{G}$. Thus, it is easy to see that for two souls $\mathcal{G}_1$ and $\mathcal{G}_2$, if $\mathcal{G}_1$ is isomorphic to $\mathcal{G}_2$ and $|\mathcal{A}_1^*| = |\mathcal{A}_2^*|$, then $\mathcal{A}_1^*$ is isomorphic to $\mathcal{A}_2^*$.

Definition 2.5. *For an arbitrary $u \in \mathcal{G}(0) \cup \mathcal{G}(1)$, a star $St(u)$ of u is $\{u\} \cup \{$2-simplices of $\mathcal{G}$ which contain $u\}$.*

A point $v \in \mathcal{G}(0)$ $(\neq u)$ is called an end point of the star $St(u)$ if $St(u)$ has a 2-simplex which contains v.

A line $l \in \mathcal{G}(1)(\neq u)$ is called an end line of the star $St(u)$ if $St(u)$ has a 2-simplex which contains l.

The end points and end lines of the star $St(u)$ are all called the end elements of the star $St(u)$.

For the stars $St(u_1), \ldots, St(u_m)$ in $\mathcal{G}$ $(m > 0)$, let $\mathcal{G}' = \mathcal{G} - \{St(u_1) \cup \cdots \cup St(u_m)\}$. $St(u_1), \ldots, St(u_m)$ are said to be simple joint in $\mathcal{G}$ if

(1) any end element of $St(u_1), \ldots, St(u_m)$ can connect to at most one another end element by a path in $\mathcal{G}'$,

(2) any two end elements of $St(u_1), \ldots, St(u_m)$ can be connected by at most one path in $\mathcal{G}'$.

Definition 2.6. *An arrangement $\mathcal{A}^*$ of hyperplanes in $\mathbb{CP}^3$ is said to be nice if the soul $\mathcal{G}$ from $\mathcal{A}^*$ has the following properties:*

(1) $\mathcal{G}(0)$ and $\mathcal{G}(1)$ are disjoint, i.e. for any $p \in \mathcal{G}(0)$ and any $q \in \mathcal{G}(1)$, p is not contained in q;

(2) all lines $\in \mathcal{G}(1)$ lie on one plane;

(3) $\mathcal{G}$ has no loop; or

(4) there are simple joint stars $St(u_1), \ldots, St(u_m)$ which are pairwise disjoint in $\mathcal{G}$ such that $\mathcal{G}' = \mathcal{G} - \{St(u_1) \cup \cdots \cup St(u_m)\}$ contains no loop where $u_1, \ldots, u_m$ in $\mathcal{G}(0) \cup \mathcal{G}(1)$.

Definition 2.7. *An arrangement $\mathcal{A}^*$ of hyperplanes in $\mathbb{CP}^3$ is called a point arrangement of hyperplanes if the $\mathcal{G}(1)$ of $\mathcal{A}^*$ is empty. This means that $\mathcal{G}$ consists of the set of the points (0-simplices) and the set of the planes (2-simplices).*

If a point arrangement is nice it is called a nice point arrangement.

An arrangement $\mathcal{A}^*$ of hyperplanes in $\mathbb{CP}^3$ is called a line arrangement of hyperplanes if the $\mathcal{G}(0)$ of $\mathcal{A}^*$ is empty. This means that $\mathcal{G}$ consists of the set of the lines (1-simplices) and the set of the planes (2-simplices).

If a line arrangement is nice it is called a nice line arrangement.

In the following we give some examples to show the nice line arrangement and the nice point arrangement in $\mathbb{CP}^3$.

Example 2.8. Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{C}^4$ consisting of the elements

$$H_1 : \{(x,y,z,w) \in C^4 : x = 0\}, \quad H_2 : \{(x,y,z,w) \in C^4 : y = 0\},$$
$$H_3 : \{(x,y,z,w) \in C^4 : z = 0\}, \quad H_4 : \{(x,y,z,w) \in C^4 : w = 0\},$$
$$H_5 : \{(x,y,z,w) \in C^4 : x = y\}.$$

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The corresponding projective arrangement $\mathcal{A}^*$ is a nice arrangement in $\mathbb{CP}^3$. As shown in Figure 1, the pseudo-complex soul $\mathcal{G}$ of $\mathcal{A}^*$ consists of five 2-simplices ABD, AED, ACD, ABC and DBC, and one 1-simplices AD. We can see that AD incidents with ABD, AED and ACD. Also, we can see, two 2-simplices ABD and ADC intersect at a 1-simplex AD. Notice, there is no 0-simplices because no point in the Figure 1 satisfies the condition that any three of planes are in general position in Definition 2.3. $\mathcal{G}$ contains no loop. Hence, it is a nice line arrangement.

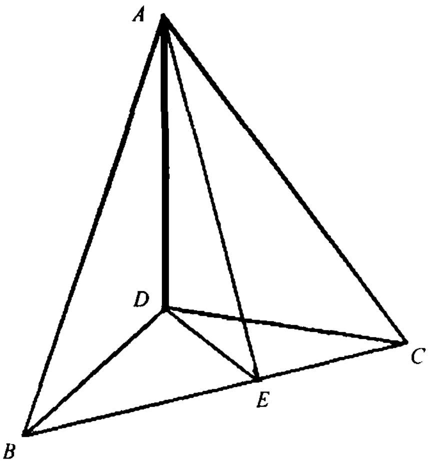

Figure 1. A nice line arrangement in $\mathbb{CP}^3$

Example 2.8 is an example of a line arrangement and it is a nice arrangement. We give another example of nice arrangement as follows:

Example 2.9. Let $\mathcal{A}$ be a central arrangement of hyperplanes in $\mathbb{C}^4$ and $\mathcal{A}^*$ the associated arrangement in $\mathbb{CP}^3$ which is obtained by adding four more planes HJI, HLK, MNO and MQP to the projective arrangement $\mathcal{A}^*$ in Example 2.8. The two new planes, HJI and HLK, pass through H and the other two new planes, MNO and MQP, pass through M (see Figure 2). There is a loop: $AD, ABD, H, ABC, M, ADC, AD$. This is also a nice arrangement since deleting $St(H)$ gives a sub-pseudo-complex with no loop, where $St(H)$ has no any end element. So, it is simple joint.

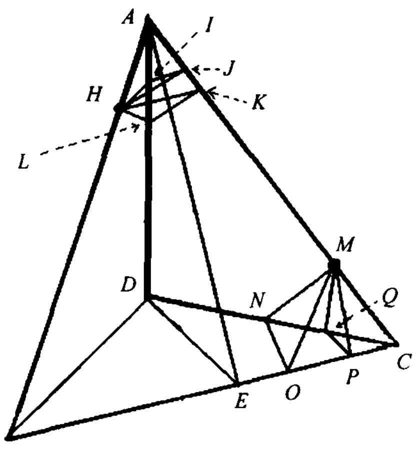

Figure 2. A nice arrangement in $\mathbb{CP}^3$

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Here,
$\mathcal{G}(0)=\{H,M\}$, $\mathcal{G}(1)=\{AD\}$,
$\mathcal{G}(2)=\{ABD,AED,ACD,ABC,DBC,HJI,HLK,MNO,MQP\}$,
$St(H)=\{H,ABD,ABC,HIJ,HKL\}$ (see Figure 3),
$\mathcal{G}-St(H)=\{AED,ACD,DBC,MNO,MQP,AD,M\}$ (see Figure 4).

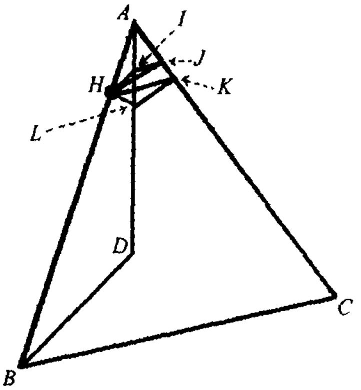

Figure 3. The pseudo-complexes $St(H)$ in $\mathbb{CP}^3$

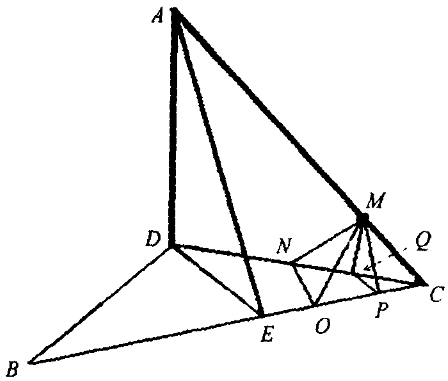

Figure 4. The pseudo-complexes $\mathcal{G}-St(H)$ in $\mathbb{CP}^3$

3 Diffeomorphic types of nice line arrangement in $\mathbb{CP}^3$

In this section, we shall state and prove the theorem about nice line arrangement of hyperplanes. First, we give some definitions and lemmas as follows:

Definition 3.1. *Let $(x_i:y_i)$, $(x_j:y_j)$, $(x_k:y_k)\in(\mathbb{CP}^1)^3$. $(x_i:y_i)$ is called irregular for the following equation*

$$ay_ix_jx_k+bx_iy_jx_k+cx_ix_jy_k+dx_iy_jy_k+ey_ix_jy_k+fy_iy_jx_k=0, \tag{3.1}$$

where $abcdef\neq 0$. if $(ay_i)x_jx_k+(bx_i+fy_i)y_jx_k+(cx_i+ey_i)x_jy_k+(dx_i)y_jy_k$ is a reducible polynomial of the other two variables $(x_j:y_j)$ and $(x_k:y_k)$. Otherwise we call $(x_i:y_i)$ regular for the equation (3.1).

Definition 3.2. *Let $\{F_1,F_2,\ldots,F_n\}$ be the planes in $\mathbb{CP}^3$ where F_i may be represented by an equation $f_{i1}x+f_{i2}y+f_{i3}z+f_{i4}w=0$, $i=1,2,\ldots,n$. This equation is determined by its coefficients, which is a tetrad of homogeneous plane-coordinates. We call $(f_{i1},f_{i2},f_{i3},f_{i4})$ a normal vector of the plane. Since the plane and its normal vector are one-to-one correspondence, we will identify the plane with its normal vector and write $F_i=(f_{i1},f_{i2},f_{i3},f_{i4})$, $i=1,2,\ldots,n$.*

Let $\widehat{\mathbb{CP}^3}$ denote the space of all hyperplanes in $\mathbb{CP}^3$. Under the above identification, $\widehat{\mathbb{CP}^3}$ is naturally isomorphic to $\mathbb{CP}^3$.

If the equation

$$a_1F_1+a_2F_2+\cdots+a_nF_n=0,\qquad a_i\in\mathbb{C},\quad i=1,2,\ldots,n$$

has only the zero solution $a_i=0$, $i=1,2,\ldots,n$, then $\{F_1,F_2,\ldots,F_n\}$ are called independent.

Lemma 3.3. *Assume $((x_1:y_1),(x_2:y_2),(x_3:y_3))\in(\mathbb{CP}^1)^3$ is a solution of* (3.1). *If $(x_1:y_1)$ is irregular, then either $(x_2:y_2)$ or $(x_3:y_3)$ is irregular for* (3.1). *If $(x_1:y_1)$ is regular, then $(x_2:y_2)$ and $(x_3:y_3)$ are either both regular or both irregular for* (3.1).

Proof. When $y_1=0$, (3.1) becomes

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$$by_2x_3 + cx_2y_3 + dy_2y_3 = 0, \tag{3.2}$$

which is irreducible. Hence, if $(x_1 : y_1)$ is irregular, then $x_1 \neq 0$ and $y_1 \neq 0$.

Write (3.1) as polynomial of $(x_2 : y_2)$ and $(x_3 : y_3)$

$$(ay_1)x_2x_3 + (bx_1 + fy_1)y_2x_3 + (cx_1 + ey_1)x_2y_3 + (dx_1)y_2y_3 = 0. \tag{3.3}$$

It is reducible if and only if

$$(bx_1 + fy_1)(cx_1 + ey_1) = adx_1y_1 \tag{3.4}$$

or

$$bcx_1^2 + (be + fc - ad)x_1y_1 + efy_1^2 = 0, \tag{3.5}$$

which has at most two roots of $(x_1 : y_1)$. When $(x_1 : y_1)$ is a root of the equation above, from (3.4) we get

$$\frac{bx_1 + fy_1}{ay_1} = \frac{dx_1}{cx_1 + ey_1}.$$

Thus, (3.3) becomes

$$[(ay_1)x_3 + (cx_1 + ey_1)y_3]\left[x_2 + \frac{dx_1}{cx_1 + ey_1}y_2\right] = 0 \tag{3.6}$$

from which we have the solution either $(x_2 : y_2) = (-dx_1 : cx_1 + ey_1)$ or $(x_3 : y_3) = (-(cx_1 + ey_1) : ay_1)$.

In the first case, we have $\frac{x_1}{y_1} = -\frac{ex_2}{cx_2+dy_2}$.

Putting these into (3.5) yields

$$bc\frac{e^2x_2^2}{(cx_2 + dy_2)^2} - (be + fc - da)\frac{ex_2}{(cx_2 + dy_2)} + ef = 0. \tag{3.7}$$

Combining the like terms we get

$$\begin{aligned} &bcex_2^2 - (be + fc - da)(cx_2^2 + dx_2y_2) + (fc^2x_2^2 + 2cdfx_2y_2 + fd^2y_2^2) = 0, \\ &cax_2^2 + (ad + fc - be)x_2y_2 + fdy_2^2 = 0. \end{aligned} \tag{3.8}$$

The last equation (3.8) is a necessary and sufficient condition for $(x_2 : y_2)$ being irregular of (3.1).

For the second case, we have the same conclusion for $(x_3 : y_3)$.

From the argument above we also have

Lemma 3.4. *Assume* $((x_i : y_i), (x_j : y_j), (x_k : y_k)) \in (\mathbb{CP}^1)^3$ *is a solution of* (3.1). *For each* $m = i, j, k$, *there are at most two irregular* $(x_m : y_m)$ *of* (3.1). *Therefore, the set of irregular of* (3.1) *is finite.* $(0 : 1)$ *and* $(1 : 0)$ *are regular of* (3.1).

Proof. Assume $i = 1$. From the proof above, the necessary and sufficient conditions that $(x_1 : y_1)$ is irregular of (3.1) is that equation (3.5) holds, which have at most two solutions.

Similarly, we can consider $(x_i : y_i)$, $(x_j : y_j)$, $(x_k : y_k)$ and $(x_l : y_l)$.

It is clear that $(0 : 1)$ and $(1 : 0)$ do not satisfy (3.5). Hence, $(0 : 1)$ and $(1 : 0)$ are regular of (3.1).

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Lemma 3.5. *For each fixed regular* $(x_1 : y_1)$ *of* (3.1), *the following relation produces an automorphism of* $\mathbb{CP}^1$

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = K\begin{pmatrix} -ey_1 - cx_1 & -dx_1 \\ ay_1 & bx_1 + fy_1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \quad K \in \mathbb{C}^*, \tag{3.9}$$

which sends regular values to regular values of (3.1). *In particular* $(x_1 : y_1) = (x_2 : y_2) = (0 : 1)$ (*respectively,* $(1 : 0)$) *corresponds to* $(x_3 : y_3) = (0 : 1)$ (*respectively,* $(1 : 0)$).

Proof. Consider

$$\begin{vmatrix} -ey_1 - cx_1 & -dx_1 \\ ay_1 & bx_1 + fy_1 \end{vmatrix} = -bcx_1^2 - (be + fc - da)x_1y_1 - fey_1^2.$$

Since $(x_1 : y_1)$ is a regular value, the above expression is nonzero by (3.5). Hence (3.9) is an automorphism of $\mathbb{CP}^1$. Clearly (3.9) satisfies equation (3.1). By Lemma 3.3, the mapping (3.9) sends regular values of (3.1) to regular values of (3.1). The last statement of the lemma is obvious.

Remark 3.6. Equation (3.9) is equivalent to equation (3.1).

If we write (3.1) as $(ay_1x_2 + bx_1y_2 + fy_1y_2)x_3 + (dx_1y_2 + ey_1x_2 + cx_1x_2)y_3 = 0$, then $(x_3, y_3) = K(-dx_1y_2 - ey_1x_2 - cx_1x_2, ay_1x_2 + bx_1y_2 + fy_1y_2)$ which is (3.9). Hence, if $(x_1 : y_1)$ and $(x_2 : y_2)$ are regular of (3.1), then there is a unique regular $(x_3 : y_3)$ solved in terms of $(x_1 : y_1)$ and $(x_2 : y_2)$. We call such procedure "fixing two variables to solve the other" and call $(x_1 : y_1)$, $(x_2 : y_2)$, $(x_3 : y_3)$ "solved variables".

Lemma 3.7. *Let* F_1, F_2, F_3, F *be planes in* $\mathbb{CP}^3$. *If* F *is not a linear combination of* F_1, F_2, F_3, *then* F_1, F_2, F_3, F *are independent if and only if* F_1, F_2, F_3 *are independent.*

In particular, $\det(F_1, F_2, F_3, F) = 0$ *if and only if* $\operatorname{rank}(F_1, F_2, F_3) < 3$.

Proof. Assume that F_1, F_2, F_3, F are independent, and it implies directly that F_1, F_2, F_3 are independent.

On the other hand, assume that F_1, F_2, F_3 are independent. If $a_1F_1 + a_2F_2 + a_3F_3 + aF = 0$, because F is not a linear combination of F_1, F_2, F_3, we have a=0 and it implies $a_1F_1 + a_2F_2 + a_3F_3 = 0$. Hence, a_1=a_2=a_3=0 because F_1, F_2, F_3 are independent. It implies that F_1, F_2, F_3, F are independent.

Lemma 3.8. *Let* $G_1, G_2, G_3 \in \mathbb{CP}^3$ *pass through a line* L_1, $H_1, H_2, H_3 \in \mathbb{CP}^3$ *pass through a line* L_2. *If* L_1 *and* L_2 *lie in a plane, then, the dimension of space of all* F_1, F_2, F_3 *in* $\widehat{\mathbb{CP}^3}$ *is less than* 3, *where* $F_i = x_iG_i + y_iH_i$ *and* $(x_i : y_i)$ *varies in* $\mathbb{CP}^1$ *for* $i = 1, 2, 3$.

Proof. Let P_1 and P_2 be the 2-dimension planes passing through the origin in $\mathbb{C}^4$ which correspond to the projective lines L_1 and L_2 in $\mathbb{CP}^3$ respectively. Clearly $1 \leqslant \dim(P_1 \cap P_2) \leqslant 2$.

If $\dim(P_1 \cap P_2) = 2$, i.e. $P_1 = P_2$, $L_1 = L_2$, let $F = a_1F_1 + a_2F_2 + a_3F_3$. Then

$$\begin{aligned} F &= a_1(x_1G_1 + y_1H_1) + a_2(x_2G_2 + y_2H_2) + a_3(x_3G_3 + y_3H_3) \\ &= (a_1x_1G_1 + a_2x_2G_2 + a_3x_3G_3) + (a_1y_1H_1 + a_2y_2H_2 + a_3y_3H_3). \end{aligned}$$

The first part of the above expression is a hyperplane containing P_1 while the second part is a hyperplane containing P_2. Therefore F is a hyperplane containing a plane P_1. Hence the linear combination of F_1, F_2, F_3 has a dimension less than 3.

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If $\dim(P_1 \cap P_2) = 1$, i.e. P_1 and P_2 intersect at a line L in $\mathbb{C}^4$, and L_1 and L_2 intersect at a point Q in $\mathbb{CP}^3$, then we choose the coordinates such that the line L is x-axis, P_1 is xy-plane and P_2 is xz-plane. Then the planes G_i passing through L_1 can be written as $z + \theta_i w = 0$, with normal vector $(0, 0, 1, \theta_i)$, $i = 1, 2, 3$, and the planes H_i passing through L_2 can be written as $y + \phi_i w = 0$, with normal vector $(0, 1, 0, \phi_i)$, $i = 1, 2, 3$. Hence,

$$\begin{aligned} F_i &= x_i G_i + y_i H_i = x_i(0,0,1,\theta_i) + y_i(0,1,0,\phi_i) = (0, y_i, x_i, x_i\theta_i + y_i\phi_i) \\ &= y_i\left(0, 1, \frac{x_i}{y_i}, \frac{x_i}{y_i}\theta_i + \phi_i\right) = y_i(0, 1, t_i, t_i\theta_i + \phi_i), \quad \text{where } t_i = \frac{x_i}{y_i} \in \mathbb{C}, \quad i = 1,2,3. \end{aligned}$$

So, the space of all F_1, F_2, F_3 has dimension less than 3 in $\mathbb{CP}^3$.

Lemma 3.9. *Let $\mathcal{A}^*$ be a line arrangement of hyperplanes. Then for each two lines of $\mathcal{G} = \mathcal{G}(\mathcal{A}^*)$, there is at most one plane containing both of them.*

Proof. It is obvious because these two lines decide at most one plane in $\mathbb{CP}^3$.

Lemma 3.10. *Let $\mathcal{G}$ be a pseudo-complex, $St(L_1), \ldots, St(L_m)$ be simple joint stars of $\mathcal{G}$ and $\mathcal{G}' = \mathcal{G} - \bigcup_{i=1}^m St(L_i)$. If L is a line of $\mathcal{G}'$, then L cannot connect to more than two end lines of $St(L_1), \ldots, St(L_m)$ by path in $\mathcal{G}'$. If L connects two end lines of $St(L_1), \ldots, St(L_m)$ by two paths in $\mathcal{G}'$ respectively, then the two paths are unique.*

Proof. Assume L connects to three end lines w_1, w_2 and w_3 of $St(L_1), \ldots, St(L_m)$ by paths in $\mathcal{G}'$. Then w_1 connects to other two end points w_2 and w_3 through L. It is a contradiction because $St(L_1), \ldots, St(L_m)$ are simple joint.

If L connects two end lines of $St(L_1), \ldots, St(L_m)$ by more than two paths in $\mathcal{G}'$, say, $\mathcal{P}_1$ and $\mathcal{P}_2$ connect L to an end line w_1, $\mathcal{P}_3$ connects L to another end line w_2, then there are two paths:

$$w_1, \mathcal{P}_1, L, \mathcal{P}_3, w_2 \quad \text{and} \quad w_1, \mathcal{P}_2, L, \mathcal{P}_3, w_2,$$

which connect w_1 and w_2. It is also a contradiction because $St(L_1), \ldots, St(L_m)$ are simple joint.

Corollary 3.11. *Let $\mathcal{G}$ be a pseudo-complex, $St(v_1), \ldots, St(v_m)$ be simple joint stars of $\mathcal{G}$ and $\mathcal{G}' = \mathcal{G} - \bigcup_{i=1}^m St(L_i)$. If L is a line in $\mathcal{G}'$ connecting to $St(L_1), \ldots, St(L_m)$, then only one of the following cases occurs:*

(1) *L connects to only one end line of $St(L_1), \ldots, St(L_m)$ in $\mathcal{G}'$.*

(2) *L connects to two end lines w_1 and w_2 of $St(L_1), \ldots, St(L_m)$. Moreover, the path in $\mathcal{G}'$ from L to w_i, $i = 1, 2$ is unique.*

Proof. It is obvious from Lemma 3.10.

Lemma 3.12. *Let $\mathcal{G}$ be a pseudo-complex, L_1, L_2 and L_3 be three lines of $\mathcal{G}$. If L_1, L_2 and L_3 are pairwise connected to each other by the paths, each of which does not contain all three lines, then there is a loop in $\mathcal{G}$.*

Proof. Assume that L_1 and L_2 are connected by the path $\mathcal{P}_1$, L_2 and L_3 are connected by the path $\mathcal{P}_2$, and L_3 and L_1 are connected by the path $\mathcal{P}_3$. Then there is a loop: $L_1, \mathcal{P}_1, L_2, \mathcal{P}_2, L_3, \mathcal{P}_3, L_1$.

Corollary 3.13. *Let $\mathcal{G}$ be a pseudo-complex. If $\mathcal{G}$ has no loop, then any three lines in $\mathcal{G}$ can not be pairwise connected each other by the paths, each of which does not contain all three lines.*

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Proof. It is obvious from Lemma 3.12.

Lemma 3.14 (Lattice-Isotopy Theorem)[9]. *If two arrangements are connected by a one-parameter family of arrangements $\{\mathcal{A}(t)\}$ which have the same $L(\mathcal{A})$, then the complements are diffeomorphic, hence of the same homotopy type.*

Theorem 3.15. *Let $\mathcal{A}_0^*$ and $\mathcal{A}_1^*$ be two nice line projective arrangements in $\mathbb{CP}^3$. If $L(\mathcal{A}_0^*)$ and $L(\mathcal{A}_1^*)$ are isomorphic, then the complements $M(\mathcal{A}_0^*)$ and $M(\mathcal{A}_1^*)$ in $\mathbb{CP}^3$ are diffeomorphic to each other.*

Proof. Similar to the proof of Theorem A[13], we represent the two arrangements as $\mathcal{A}_1^* = \{H_1, H_2, \dots, H_n\}$ and $\mathcal{A}_0^* = \{G_1, G_2, \dots, G_n\}$ where $H_i = (h_{i1}, h_{i2}, h_{i3}, h_{i4})$ and $G_i = (g_{i1}, g_{i2}, g_{i3}, g_{i4})$ are in $\mathbb{CP}^3$. We shall construct a one-parameter family of arrangements $\mathcal{A}^*(t)$ such that $\mathcal{A}^*(0) = \mathcal{A}_0^*$, $\mathcal{A}^*(1) = \mathcal{A}_1^*$ and $L(\mathcal{A}(t)) \equiv L(\mathcal{A}_0)$ for all $t \in [0,1]$.

Let $\mathcal{A}^* = \{F_1, F_2, \dots, F_n\}$ where $F_i = x_iG_i + y_iH_i$ for some $x_i, y_i \in \mathbb{C}$, $i = 1, 2, \dots, n$. Let $I = \{(i,j,k) : 1 \leqslant i < j < k \leqslant n\}$. So $|I| = \binom{n}{3}$. Consider any triple $\{F_i, F_j, F_k\}$, $(i,j,k) \in I$. Denote the matrix

$$\begin{pmatrix} x_ig_{i1}+y_ih_{i1} & x_ig_{i2}+y_ih_{i2} & x_ig_{i3}+y_ih_{i3} & x_ig_{i4}+y_ih_{i4} \\ x_jg_{j1}+y_jh_{j1} & x_jg_{j2}+y_jh_{j2} & x_jg_{j3}+y_jh_{j3} & x_jg_{j4}+y_jh_{j4} \\ x_kg_{k1}+y_kh_{k1} & x_kg_{k2}+y_kh_{k2} & x_kg_{k3}+y_kh_{k3} & x_kg_{k4}+y_kh_{k4} \end{pmatrix}$$

by (F_i, F_j, F_k). Since each of the two planes in $\mathbb{CP}^3$ meets on one line, to get $L(\mathcal{A}) \equiv L(\mathcal{A}_0)$, it is sufficient to have the following conclusion. For any $(i,j,k) \in I$, $\operatorname{rank}(F_i, F_j, F_k) < 3$ if and only if $\operatorname{rank}(G_i, G_j, G_k) < 3$.

Since $\mathcal{A}_0^*$ is a nice line arrangement all lines $\in \mathcal{G}(1)$ lie on one plane. Similarly, all lines $\in \mathcal{G}(1)$ of $\mathcal{A}_1^*$ also lie on one plane. Without loss of generality we assume that all those lines $\in \mathcal{G}(1)$ lie in a same plane. By Lemma 3.8, for each $(i,j,k) \in I$, the dimension of the space of all F_i, F_j, F_k is less than 3 in $\mathbb{CP}^3$. Hence, there is a plane F which is not a linear combination of F_i, F_j and F_k. Moreover, notice that there are finite planes in $\mathcal{A}_0^*$ and $\mathcal{A}_1^*$, we can choose the plane F such that F is not the linear combinations of each three planes of F_i, F_j, F_k for each $(i,j,k) \in I$. Furthermore, we can change the coordinates such that F is represented as $(0,0,0,1)$.

By Lemma 3.7, $\operatorname{rank}(F_i, F_j, F_k) < 3$ if and only if $\det(F_i, F_j, F_k, F) = 0$. Here

$$\det(F_iF_jF_kF) = \begin{vmatrix} x_ig_{i1}+y_ih_{i1} & x_ig_{i2}+y_ih_{i2} & x_ig_{i3}+y_ih_{i3} & x_ig_{i4}+y_ih_{i4} \\ x_jg_{j1}+y_jh_{j1} & x_jg_{j2}+y_jh_{j2} & x_jg_{j3}+y_jh_{j3} & x_jg_{j4}+y_jh_{j4} \\ x_kg_{k1}+y_kh_{k1} & x_kg_{k2}+y_kh_{k2} & x_kg_{k3}+y_kh_{k3} & x_kg_{k4}+y_kh_{k4} \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x_ig_{i1}+y_ih_{i1} & x_ig_{i2}+y_ih_{i2} & x_ig_{i3}+y_ih_{i3} \\ x_jg_{j1}+y_jh_{j1} & x_jg_{j2}+y_jh_{j2} & x_jg_{j3}+y_jh_{j3} \\ x_kg_{k1}+y_kh_{k1} & x_kg_{k2}+y_kh_{k2} & x_kg_{k3}+y_kh_{k3} \end{vmatrix}$$

$$= D_3(G_iG_jG_k)x_ix_jx_k + D_3(H_iG_jG_k)y_ix_jx_k + D_3(G_iH_jG_k)x_iy_jx_k + D_3(G_iG_jH_k)x_ix_jy_k + D_3(G_iH_jH_k)x_iy_jy_k + D_3(H_iG_jH_k)y_ix_jy_k$$

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$$+ D_3(H_iH_jG_k)y_iy_jx_k + D_3(H_iH_jH_k)y_iy_jy_k, \tag{3.10}$$

where $D_3(G_iG_jG_k)$ is the left upper 3-subdeterminate of (G_i, G_j, G_k), and so on.

Hence, to get $L(\mathcal{A}) \equiv L(\mathcal{A}_0)$, it is sufficient to have the following: for any $(i, j, k) \in I$,

$$\det(F_iF_jF_kF) = 0, \text{ if and only if } D_3(G_iG_jG_k) = 0. \tag{3.11}$$

Let $l = \sum_{j \geqslant 3} \binom{j}{3} t_j(\mathcal{A}_0^*)$. To prove (3.11), we need to consider l equations and $\binom{n}{3} - l$ inequalities

$$P_1 = 0, \dots, P_l = 0, \tag{3.12}$$

$$Q_1 \neq 0, \dots, Q_{\binom{n}{3}-l} \neq 0. \tag{3.13}$$

Both P_i and Q_j have the forms like (3.10). But for P_i, the first term and last term are zero since $D_3(G_iG_jG_k) = D_3(H_iH_jH_k) = 0$ by (3.11). Among $P_1, \dots, P_l$ at most $c(\mathcal{A}_0^*) = \sum_{j \geqslant 3}(j-2)t_j(\mathcal{A}_0^*)$ of them are independent. To see this, we consider a j-tuple line v $(j \geqslant 3)$. Let $F_1, \dots, F_j$ be the planes of $\mathcal{A}^*$ containing v. We have $\binom{j}{3}$ equations $D_3(F_iF_jF_k) = 0, \dots$, etc. Since $\{F_1, \dots, F_j\}$ can be linearly generated by F_1 and F_2, the $\binom{j}{3}$ equations are reduced equivalently to $j-2$ equations $D_3(F_1F_2F_k) = 0$ for $i = 3, \dots, j$. Now consider all j-tuple lines $(j \geqslant 3)$. We have a system of $c(\mathcal{A}_0^*)$ equations, say $\{P_1 = 0, \dots, P_{c(\mathcal{A}_0^*)} = 0\}$ which is equivalent to $\{P_1 = 0, \dots, P_l = 0\}$.

As we observed before, each P_r can be written as

$$\begin{aligned} P_r = {} & a_r y_{i_r} x_{j_r} x_{k_r} + b_r x_{i_r} y_{j_r} x_{k_r} + c_r x_{i_r} x_{j_r} y_{k_r} + d_r x_{i_r} y_{j_r} y_{k_r} \\ & + e_r y_{i_r} x_{j_r} y_{k_r} + f_r y_{i_r} y_{j_r} x_{k_r} = 0, \end{aligned} \tag{3.14}$$

where $a_r = D_3(H_{i_r}G_{j_r}G_{k_r})$ etc. Replaying $\mathcal{A}^*$ by $\phi(\mathcal{A}^*)$ if necessary where $\phi : \mathbb{CP}^3 \to \mathbb{CP}^3$ is a complex analytic automorphism, we assume without loss of generality that any one (two) plane(s) in $\mathcal{A}_0^*$ and any two (one) plane(s) in $\mathcal{A}_1^*$ do not intersect at a line. This means that $a_rb_rc_rd_re_rf_r \neq 0$ for all $r = 1, \dots, c(\mathcal{A}_0^*)$.

Note that P_r is viewed as a polynomial in $((x_1 : y_1), \dots, (x_n : y_n)) \in (\mathbb{CP}^1)^n$. For each r, indices i_r, j_r, k_r are pairwise distinct and $(i_r, j_r, k_r) \neq (i_s, j_s, k_s)$ for $r \neq s$ where $1 \leqslant i_r, j_r, k_r, i_s, j_s, k_s \leqslant n$ and $1 \leqslant r, s \leqslant c(\mathcal{A}_0^*)$.

Since $\mathcal{A}_0^*$ is a nice line arrangement of hyperplane in $\mathbb{CP}^3$, if $\mathcal{G}$ has no loop, it is clear that we can solve all variables. Hence, we consider that there are simple joint stars, say $St(L_1), \dots, St(L_m)$ in $\mathcal{G}$ such that they are disjoint and $\mathcal{G}' = \mathcal{G} - \bigcup_{i=1}^m St(L_i)$ has no loop.

We shall prove that all variables can be solved in terms of some variables (in the sense of Remark 3.6) without ambiguity. Here we shall use the notation in Definition 2.3.

Case 1. $s = 1$ and L_1 is a line of multiplicity k in $\mathcal{A}_0^*$.

Since $k \geqslant 3$ by the definition of $\mathcal{G}$, there are k variables appearing in $k-2$ equations of (3.14). Without loss of generality we suppose that these variables are $(x_1 : y_1), \dots, (x_k : y_k)$ and $(x_1 : y_1)$ and $(x_2 : y_2)$ appear in each of these $k-2$ equations. Thus, we can fix$(x_1 : y_1)$ and $(x_2 : y_2)$ to solve $(x_3 : y_3), \dots, (x_k : y_k)$. Hence, we can solve all variables about the star $St(L_1)$.

We know that at each line there are k variables appearing in $k-2$ equations of (3.14). If at most two variables are solved at this point, then we can use these two variables to solve all

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others. Hence, in the following discussion, we only need to show that at most two variables are solved at each line.

The rest of the unsolved variables and equations in (3.14) correspond to the pseudo-complex $\mathcal{G}'$ which has no loop and is a set of some stars.

We use induction on the distance from the lines to L_1.

First, we consider the end lines of $St(L_1)$, they are 1-lines of L_1. Then we consider 2-lines of L_1, and so on.

Case 1.1. If each of two end lines of $St(L_1)$ is not connected by the path in $\mathcal{G}' = \mathcal{G} - St(v_1)$, we can pick each end line of $St(L_1)$ separately. Assume we first pick an end line of $St(L_1)$, $u_{1,1}$. By Lemma 3.9 there is only one plane in the star $St(L_1)$ containing $u_{1,1}$, which means that only one variable corresponding to the plane is solved. We choose another variable corresponding to another plane containing $u_{1,1}$, and then fix these two variables and solve all other variables at $u_{1,1}$. Next we pick another end line, $u_{1,2}$, which does not connect to $u_{1,1}$ by a path in $\mathcal{G}'$. Hence we can solve all variables at $u_{1,2}$ by using same method of solving variables at $u_{1,1}$. Continuing this procedure, we can solve all variables at all end lines of $St(L_1)$ which are 1-lines of L_1.

Case 1.2. If there are two end lines of $St(L_1)$ which are connected by a path in $\mathcal{G}'$, we can choose an end line of $St(L_1)$, say $u_{1,1}$, such that $u_{1,1}$ connects to one end line $u_{1,2}$ of $St(v_1)$. By Lemma 3.9 there is only one plane, say P_1, in the star $St(L_1)$ containing $u_{1,1}$. We can use the variable corresponding to P_1 and choose another variable, and then solve all variables at $u_{1,1}$. Since $St(L_1)$ is simple joint, there is only one path which connects $u_{1,1}$ and $u_{1,2}$. Assume the plane containing $u_{1.2}$ in the path is P_1. By Lemma 3.9 there is only one plane, say P_2, in the star $St(L_1)$ containing $u_{1,2}$. Then we can fix these two variables corresponding to P_1 and P_2, and solve other variables at $u_{1,2}$. Then consider another end line $u_{1,3}$. Similarly, since $St(L_1)$ is simple joint, only one of $u_{1,1}$ and $u_{1,2}$ can connect to $u_{1,3}$ by a path in $\mathcal{G}'$. Hence at most two variables at $u_{1,3}$ are solved. Using these two variables we can solve all other variables at $u_{1,3}$. Continuing this procedure, we can solve all variables at all end lines of $St(L_1)$ which are 1-lines of L_1.

Assume we can solve all variables at the (k-1)-lines of L_1 $u_{k-1,1}, \dots, u_{k-1,m}$. Then consider the k-lines of L_1. Without loss of the generality we assume that k-line $u_{k,1}$ is an end line of $St(u_{k-1,1})$ which connects to an end line $u_{1,1}$ of $St(L_1)$. From induction assumption, all variables at $u_{k-1,1}$ are solved. For $u_{k,1}$, there is only one plane containing $u_{k,1}$ and $u_{k-1,1}$ by Lemma 3.9, $u_{k,1}$ cannot connect to another line that connects to $u_{1,1}$ by Corollary 3.13, and $u_{k,1}$ cannot connect to other two j-point ($j < k$) by the path in $\mathcal{G}'$ by Corollary 3.11. Hence at most two variables are solved at $u_{k,1}$. Thus, we can solve all variables at $u_{k,1}$. Similarly, using this procedure, we can solve other variables at all k-lines.

By induction, we can solve all variables at all lines of $\mathcal{G}$.

Case 2. $s = 2$. By the same procedure above we can solve all variables at L_1 and L_2. If $St(L_1)$ and $St(L_2)$ are not connected by a path in $\mathcal{G}'$, we can solve all variables from them separately. Hence, we only need to consider the case when they are connected.

We choose an end line of $St(L_1)$, say $u_{1,1}$. It is 1-line of L_1. By Lemma 3.9 there is only one plane in $St(L_1)$ passing through $u_{1,1}$. Hence, we can solve the all variables at $u_{1,1}$. For the other end lines of $St(L_1)$, we can solve the variables by the same method as in Case 1.

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Now we consider an end line of $St(L_2)$, say $w_{1,1}$, which connects to an end line of $St(L_1)$, say $u_{1,1}$. We know from Definition 2.6 that $w_{1,1}$ only connects to $u_{1,1}$ by one path. Assume the plane containing $w_{1,1}$ in the path is P_1. Also, by Lemma 3.9, there is only one plane, say P_2, in $St(L_2)$ which containing $w_{1,1}$. Then we use these two solved variables corresponding to P_1 and P_2 and solve other variables at $w_{1,1}$.

Next, we pick another end line, say $w_{1,2}$. Because $w_{1,2}$ connects to at most one end line of $St(L_1)$ or $St(L_2)$ by Definition 2.6, and at most one plane in $St(L_2)$ passes L_2 and its end line by Lemma 3.9, we know that there are at most two solved variables at $w_{1,2}$. Hence we can use these two solved variables to solve other variables at $w_{1,2}$. Continuing the same procedure, we can solve all variables at the end lines of $St(L_1)$ or $St(L_2)$.

Since $\mathcal{G}'$ has no loop any three lines can not connected pairwise by Corollary 3.13 and any line can connect to only one end line or connect to two end lines of $St(L_1)$ and $St(L_2)$ by two unique paths in $\mathcal{G}'$ from Corollary 3.11, we can continue this procedure and solve all variables without ambiguity.

Similarly, we can consider the case of $s > 2$.

Thus we can solve all variables in terms of some variables without ambiguity since $\mathcal{G}'$ has no loop.

Now, all variables are presented as $((x_1 : y_1), \ldots, (x_n : y_n)) = f((x_1 : y_1), \ldots, (x_p : y_p))$, where each component of f is a composition by some maps as (3.9). So they are homogeneous polynomial of $(x_1 : y_1), \ldots, (x_p : y_p)$.

Let $U := (\mathbb{CP}^1)^p - \{((x_1 : y_1), \ldots, (x_p : y_p)) : \text{for some } 1 \leqslant i \leqslant p, (x_i : y_i) \text{ is irregular of some equation of (3.14)}\}$. By Lemma 3.4, U is an open connected set of $(\mathbb{CP}^1)^p$. By Lemma 3.5, f defines an embedding from $U \subset (\mathbb{CP}^1)^p$ to $(\mathbb{CP}^1)^n$. Since U is irreducible, so is $f(U)$ irreducible. Observe that $(0:1)^n = ((0:1), \ldots, (0:1))$ and $(1:0)^n = ((1:0), \ldots, (1:0))$ are contained in $f(U)$. We deduce that $(0:1)^n$ and $(1:0)^n$ are in the same irreducible component of $\{P_1 = 0, \ldots, P_{c(\mathcal{A}_0^*)} = 0\}$. In fact, put $(1:0)^n$ ($(0:1)^n$, respectively) to (3.14), and we can see by (3.11), $P_r = 0$ for all $r = 1, \ldots, c(\mathcal{A}_0^*)$, and $Q_s = D_3(G_{si}G_{sj}G_{sk})$ ($D_3(H_{si}H_{sj}H_{sk})$, respectively) $\neq 0$ for all $s = 1, \ldots, \binom{n}{4} - c(\mathcal{A}_0^*)$.

Recall that irreducible variety minus a subvariety is still a connected set. Therefore, the irreducible component of $\{P_1 = 0, \ldots, P_{c(\mathcal{A}_0^*)} = 0\} - \bigcup_{s=1}^{\binom{n}{4}-c(\mathcal{A}_0^*)}\{Q_s = 0\}$ is a connected set which contains $((1:0), \ldots, (1:0))$ and $((0:1), \ldots, (0:1))$. So there is a curve from $((1:0), \ldots, (1:0))$ to $((0:1), \ldots, (0:1))$ such that(3.12) and (3.13) are satisfied for any point lying in the curve. This means that we have constructed a one-parameter family of arrangements $\mathcal{A}^*(t)$ such that $\mathcal{A}^*(0) = \mathcal{A}_0^*$, $\mathcal{A}^*(1) = \mathcal{A}_1^*$ and $L(\mathcal{A}(t)) \equiv L(\mathcal{A}_0)$ for all $t \in [0, 1]$.

Now we can apply Lemma 3.14 and finish the proof of Theorem 1.1.

In the course of proving Theorem 3.15, we have proved the following theorem.

Theorem 3.16. *Let* $\mathcal{A}^*$ *be a nice line arrangement of hyperplanes in* $\mathbb{CP}^3$. *The moduli space of* $\mathcal{A}^*$ *with fixed combinatorics* $L(\mathcal{A}^*)$ *is connected.*

Proof. For given two nice line arrangements $\mathcal{A}_0^*$ and $\mathcal{A}_1^*$ of hyperplanes in $\mathbb{CP}^3$ with fixed combinatorics $L(\mathcal{A}^*)$, in the proof of Theorem 3.15, we have constructed a one-parameter family $\mathcal{A}^*(t)$ of hyperplanes in $\mathbb{CP}^3$ with fixed combinatorics $L(\mathcal{A}^*)$ connecting $\mathcal{A}_0^*$ and $\mathcal{A}_1^*$. Therefore the moduli space of $\mathcal{A}^*$ with fixed combinatorics $L(\mathcal{A}^*)$ is connected.

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Theorem 3.17. *The homotopy groups of the complement $M(\mathcal{A}^*)$ of a nice line arrangement of hyperplanes in $\mathbb{CP}^3$ depend only on $L(\mathcal{A}^*)$ (or the lattice $L(\mathcal{A})$).*

Proof. Since the topology of $M(\mathcal{A}^*)$ is determined by $L(\mathcal{A}^*)$, by Theorem 3.15, the homotopy groups of the complement $M(\mathcal{A}^*)$ are determined by $L(\mathcal{A}^*)$.

4 Main theorem

We have proved

Theorem A[13]**.** *Let $\mathcal{A}_0^*$ and $\mathcal{A}_1^*$ be two nice point arrangements of hyperplanes in $\mathbb{CP}^3$. If $L(\mathcal{A}_0^*)$ and $L(\mathcal{A}_1^*)$ are isomorphic, then the complements $M(\mathcal{A}_0^*)$ and $M(\mathcal{A}_1^*)$ in $\mathbb{CP}^3$ are diffeomorphic to each other.*

In Section 3 we proved Theorem 3.15, which states that Theorem A still holds for nice line arrangements of hyperplanes in $\mathbb{CP}^3$. Now, we will prove the main Theorem, which means that for nice arrangements of hyperplanes in $\mathbb{CP}^3$, Theorem A still holds. We give some lemmas first.

Lemma 4.1. *Let $\mathcal{A}^*$ be a nice arrangement of hyperplanes in $\mathbb{CP}^3$. Then*

(1) for each two points of $\mathcal{G} = \mathcal{G}(\mathcal{A}^*)$, there are at most two planes passing through both of them;

(2) for each two lines of $\mathcal{G} = \mathcal{G}(\mathcal{A}^*)$, there is at most one plane containing both of them;

(3) for each point and line of $\mathcal{G} = \mathcal{G}(\mathcal{A}^*)$, there is at most one plane containing both of them.

Proof. Because $\mathcal{A}^*$ is a nice arrangement, $\mathcal{G}(0)$ and $\mathcal{G}(1)$ are disjoint from Definition 2.6.

If two points Q_1 and Q_2 are connected by three planes P_1, P_2 and P_3, then the line L passing through Q_1 and Q_2 is an intersection of three planes P_1, P_2 and P_3. It implies that L has multiplicity great than 3 and $L \in \mathcal{A}^*$. But Q_1 and Q_2 are on the line L, which is a contradiction to that $\mathcal{G}(0)$ and $\mathcal{G}(1)$ are disjoint. Hence (1) holds.

It is obvious that the two lines at most decide one plane. For a point Q and a line L in $\mathcal{G}$, the point Q can not be on the line L. Otherwise, $\mathcal{G}(0)$ and $\mathcal{G}(1)$ are not disjoint. We also know that a point and a line only determine a plane. Hence, for each two lines or each point and line of $\mathcal{G}$, there is at most one plane containing both of them.

Lemma 4.2. *Let $\mathcal{G}$ be a pseudo-complex, $St(U_1), \dots, St(U_m)$ be simple joint stars of $\mathcal{G}$ and $\mathcal{G}' = \mathcal{G} - \bigcup_{i=1}^m St(U_i)$. If U be a point or a line of $\mathcal{G}'$, then U cannot connect to more than two end elements of $St(U_1), \dots, St(U_m)$ by path in $\mathcal{G}'$. If U connects two end elements of $St(U_1), \dots, St(U_m)$ by two paths in $\mathcal{G}'$ respectively, then the two paths are unique.*

Proof. Assume U connects to three end elements w_1, w_2 and w_3 of $St(U_1), \dots, St(U_m)$ by paths in $\mathcal{G}'$. Then w_1 connects to other two end elements w_2 and w_3 through U. It is a contradiction because $St(U_1), \dots, St(U_m)$ are simple joint.

If U connects two end elements of $St(U_1), \dots, St(U_m)$ by more than two paths in $\mathcal{G}'$, say $\mathcal{P}_1$ and $\mathcal{P}_2$ connect U to an end element w_1, $\mathcal{P}_3$ connects U to another end element w_2, then there are two paths:

$$w_1, \mathcal{P}_1, U, \mathcal{P}_3, w_2, \quad \text{and} \quad w_1, \mathcal{P}_2, U, \mathcal{P}_3, w_2,$$

which connect w_1 and w_2. It is also a contradiction because $St(U_1), \dots, St(U_m)$ are simple joint.

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Corollary 4.3. *Let $\mathcal{G}$ be a pseudo-complex, $St(U_1), \ldots, St(U_m)$ be simple joint stars of $\mathcal{G}$ and $\mathcal{G}' = \mathcal{G} - \bigcup_{i=1}^{m} St(U_i)$. If U is a point or a line in $\mathcal{G}'$ connecting to $St(U_1), \ldots, St(U_m)$, then only one of the following cases occurs:*

(1) U connects to only one end element of $St(U_1), \ldots, St(U_m)$ in $\mathcal{G}'$;

(2) U connects to two end elements w_1 and w_2 of $St(U_1), \ldots, St(U_m)$. Moreover, the path in $\mathcal{G}'$ from U to w_i, $i = 1, 2$ is unique.

Proof. It is obvious from Lemma 4.2.

Lemma 4.4. *Let $\mathcal{A}^*$ be a nice arrangement in $\mathbb{CP}^3$ and $\mathcal{G}$ be a pseudo-complex, U_1, U_2 and U_3 be three elements of $\mathcal{G}$. If U_1, U_2 and U_3 are pairwise connected each other by the paths, each of which does not pass all three U_1, U_2 and U_3, then there is a loop in $\mathcal{G}$.*

Proof. Assume that U_1 and U_2 are connected by the path $\mathcal{P}_1$, U_2 and U_3 are connected by the path $\mathcal{P}_2$, and U_3 and U_1 are connected by the path $\mathcal{P}_3$. Then there is a loop: $U_1, \mathcal{P}_1, U_2, \mathcal{P}_2, U_3, \mathcal{P}_3, U_1$.

Corollary 4.5. *Let $\mathcal{A}^*$ be a nice arrangement in $\mathbb{CP}^3$ and $\mathcal{G}$ be a pseudo-complex. If $\mathcal{G}$ has no loop, then any three elements in $\mathcal{G}$ cannot be pairwise connected each other by the paths, each of which does not pass all three elements.*

Proof. It is obvious from Lemma 4.4.

Now we can state and prove our main Theorem 4.6.

Theorem 4.6. *Let $\mathcal{A}_0^*$ and $\mathcal{A}_1^*$ be two nice projective arrangements of hyperplanes in $\mathbb{CP}^3$. If $L(\mathcal{A}_0^*)$ and $L(\mathcal{A}_1^*)$ are isomorphic, then the complements $M(\mathcal{A}_0^*)$ and $M(\mathcal{A}_1^*)$ in $\mathbb{CP}^3$ are diffeomorphic to each other.*

Proof. Similar to the proof of Theorem A[13] and Theorem 3.15, we represent the two arrangements as $\mathcal{A}_1^* = \{H_1, H_2, \ldots, H_n\}$ and $\mathcal{A}_0^* = \{G_1, G_2, \ldots, G_n\}$ where $H_i = (h_{i1}, h_{i2}, h_{i3}, h_{i4})$ and $G_i = (g_{i1}, g_{i2}, g_{i3}, g_{i4})$ are in $\mathbb{CP}^3$. We shall construct a one-parameter family of arrangements $\mathcal{A}^*(t)$ such that $\mathcal{A}^*(0) = \mathcal{A}_0^*$, $\mathcal{A}^*(1) = \mathcal{A}_1^*$ and $L(\mathcal{A}(t)) \equiv L(\mathcal{A}_0)$ for all $t \in [0, 1]$.

Let $\mathcal{A}^* = \{F_1, F_2, \ldots, F_n\}$ where $F_i = x_i G_i + y_i H_i$ for some $x_i, y_i \in \mathbb{C}$ such that F_i is in $\mathbb{CP}^3$, $i = 1, 2, \ldots, n$.

The point of $\mathcal{A}_0^*$ is an intersection of $\{G_i, G_j, G_k, G_l\}$. Similar to the proof of Theorem A[13], we need to consider quaternion $\{F_i, F_j, F_k, F_l\}$ and solve the variables of the following equations:

$$\begin{aligned} &a_r y_{i_r} x_{j_r} x_{k_r} x_{l_r} + b_r x_{i_r} y_{j_r} x_{k_r} x_{l_r} + c_r x_{i_r} x_{j_r} y_{k_r} x_{l_r} + d_r x_{i_r} x_{j_r} x_{k_r} y_{l_r} + A_r x_{i_r} x_{j_r} y_{k_r} y_{l_r} \\ &\quad + B_r x_{i_r} y_{j_r} x_{k_r} y_{l_r} + C_r x_{i_r} y_{j_r} y_{k_r} x_{l_r} + D_r y_{i_r} x_{j_r} x_{k_r} y_{l_r} + E_r y_{i_r} x_{j_r} y_{k_r} x_{l_r} + F_r y_{i_r} y_{j_r} x_{k_r} x_{l_r} \\ &\quad + e_r x_{i_r} y_{j_r} y_{k_r} y_{l_r} + f_r y_{i_r} x_{j_r} y_{k_r} y_{l_r} + g_r y_{i_r} y_{j_r} x_{k_r} y_{l_r} + h_r y_{i_r} y_{j_r} y_{k_r} x_{l_r} = 0, \end{aligned} \tag{4.1}$$

where $a_r = |H_{i_r} G_{j_r} G_{k_r} G_{l_r}|$, $b_r = |G_{i_r} H_{j_r} G_{k_r} G_{l_r}|$, etc. and $a_r b_r c_r d_r A_r B_r C_r D_r E_r F_r e_r f_r g_r h_r \neq 0$ for all $r = 1, \ldots, c(\mathcal{A}_0^*)$.

For the line of $\mathcal{A}_0^*$, which is an intersection of $\{G_i, G_j, G_k\}$. Same as the proof of Theorem 3.15 we need to consider triple $\{F_i, F_j, F_k\}$ and solve the variables of the following equations:

$$a_r y_{i_r} x_{j_r} x_{k_r} + b_r x_{i_r} y_{j_r} x_{k_r} + c_r x_{i_r} x_{j_r} y_{k_r} + d_r x_{i_r} y_{j_r} y_{k_r} + e_r y_{i_r} x_{j_r} y_{k_r} + f_r y_{i_r} y_{j_r} x_{k_r} = 0, \tag{4.2}$$

where $a_r = D_3(H_{i_r} G_{j_r} G_{k_r})$ etc. and $a_r b_r c_r d_r e_r f_r \neq 0$ for all $r = 1, \ldots, c(\mathcal{A}_0^*)$.

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If we can prove that all variables of (4.1) and (4.2) about the points and lines of $\mathcal{G}$ can be solved in terms of some variables, similar to the proof of Theorem A[13] and Theorem 3.15, we can show the Main Theorem is true.

Since $\mathcal{A}_0^*$ is a nice arrangement in $\mathbb{CP}^3$, if $\mathcal{G}$ has no loop, it is clear that we are able to solve all variables of (4.1) and (4.2). Without loss of the generality, we assume that there are simple joint stars, say $St(U_1), \ldots, St(U_s)$ in $\mathcal{G}$ such that they are disjoint and $\mathcal{G}' = \mathcal{G} - \bigcup_{i=1}^{s} St(U_i)$ has no loop, where all $U_i \in \mathcal{G}(0) \cup \mathcal{G}(1)$.

Similar to the proof of Theorem A[13] and Theorem 3.15, we can solve all variables at $U_1, \ldots, U_s$.

Case 1. $s = 1$, this means that there is only one star $St(U_1)$.

We use induction on the distance from the elements to U_1. We consider the end elements of the stars $St(U_1), \ldots, St(U_s)$, which are 1-elements of U_1. Then we consider 2-elements of U_1, and so on.

Case 1.1. Let U_1 be a point. If we pick one end point $u_{1,1}$ of $St(U_1)$. By Lemma 4.1, at most two planes in $St(U_1)$ pass through U_1 and $u_{1,1}$. By Definition 2.6, $u_{1,1}$ can connect to at most one more end element of $St(U_1)$. This means that at most three variables at $u_{1,1}$ are solved. Hence we can use these three variables and solve all others at $u_{1,1}$. If we pick one end line $u_{1,2}$ of $St(U_1)$, by Lemma 4.1, there is at most one plane in $St(U_1)$ containing U_1 and $u_{1,2}$. By Definition 2.6, $u_{1,2}$ can connect to at most one more end element of $St(U_1)$. This means that at most two variables at $u_{1,2}$ are solved. Hence we can use these two variables and solve all others at $u_{1,2}$. Continuing this procedure, we can solve all variables at all end elements of $St(U_1)$ which are 1-elements of U_1.

Case 1.2. Let U_1 be a line. If we pick one end point $u_{1,1}$ of $St(U_1)$. By Lemma 4.1, at most one plane in $St(U_1)$ contains U_1 and $u_{1,1}$. By Definition 2.6, $u_{1,1}$ can connect to at most one more end element of $St(U_1)$. This means that at most two variables at $u_{1,1}$ are solved. Hence we can use these two variables and solve all others at u_{11}. If we pick one end line $u_{1,2}$ of $St(U_1)$, by Lemma 4.1, there is at most one plane in $St(U_1)$ containing U_1 and $u_{1,2}$. By Definition 2.6, $u_{1,2}$ can connect to at most one more end element of $St(U_1)$. This means that at most two variables at $u_{1,2}$ are solved. Hence we can use these two variables and solve all others at $u_{1,2}$. Continuing this procedure, we can solve all variables at all end elements of $St(U_1)$ which are 1-elements of U_1.

Assume we can solve all variables at the $(k-1)$-elements of U_1 $u_{k-1,1}, \ldots, u_{k-1,m}$. Then consider the k-elements of U_1. Without loss of the generality we assume that k-element $u_{k,1}$ is an end element of $St(u_{k-1,1})$ which connects to an end element $u_{1,1}$ of $St(U_1)$. From induction assumption, all variables at $u_{k-1,1}$ are solved. For $u_{k,1}$, if $u_{k,1}$ is a point, there is at most two planes containing $u_{k,1}$ and $u_{k-1,1}$ by Lemma 4.1. If $u_{k,1}$ is a line, there is only one plane containing $u_{k,1}$ and $u_{k-1,1}$ by Lemma 4.1. $u_{k,1}$ cannot connect to another point or line that connects to $u_{1,1}$ by Corollary 4.5, and $u_{k,1}$ cannot connect to other two j-point $(j < k)$ by the path in $\mathcal{G}'$ by Corollary 4.3. Hence at most three variables are solved at $u_{k,1}$ if $u_{k,1}$ is a point or at most two variables are solved at $u_{k,1}$ if $u_{k,1}$ is a line. Thus, we can solve all variables at $u_{k,1}$. Similarly, using this procedure, we can solve other variables at all k-elements of U_1.

By induction, we can solve all variables at all elements of $\mathcal{G}$.

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Case 2. $s = 2$. By the same procedure above we can solve all variables at U_1 and U_2. If $St(U_1)$ and $St(U_2)$ are not connected by a path in $\mathcal{G}'$, we can solve all variables from them separately. Hence, we only need to consider the case that they are connected.

We choose an end element of $St(U_1)$, say $u_{1,1}$. It is a 1-element of $u_{1,1}$.

Case 2.1. If U_1 is a line, by Lemma 4.1 there is only one plane in $St(U_1)$ containing $u_{1,1}$. Since $\mathcal{A}_0^*$ is a nice arrangement of hyperplanes, $St(U_1)$ and $St(U_2)$ are simple joint, $u_{1,1}$ can connect to at most one end element, hence, at most two variables at $u_{1,1}$ are solved. We use these two solved variables to solve all variables at $u_{1,1}$.

Case 2.2. If U_1 is a point and $u_{1,1}$ is a line, by Lemma 4.1 there is only one plane in $St(U_1)$ containing $u_{1,1}$. If U_1 is a point and $u_{1,1}$ is also a point, by Lemma 4.1 there are only two planes in $St(U_1)$ passing through $u_{1,1}$. Similar to Case 2.1, $u_{1,1}$ can connect to at most one end element. Hence, if $u_{1,1}$ is a point, then at most three variables at $u_{1,1}$ are solved. If $u_{1,1}$ is a line, then at most two variables at $u_{1,1}$ are solved. For both cases, we can solve the all variables at $u_{1,1}$.

For other end elements of $St(U_1)$, we can solve the variables by the same method as in Case 1.

Now we consider an end element of $St(U_2)$, say $w_{1,1}$, which connects to an end element of $St(U_1)$, say $u_{1,1}$. We know from Definition 2.6 that $w_{1,1}$ only connects to $u_{1,1}$ by one path. Assume the plane containing $w_{1,1}$ in the path is P_1. If $w_{1,1}$ is a line, by Lemma 4.1, there is only one plane, say P_2, in $St(U_2)$ which contains $w_{1,1}$. If $w_{1,1}$ is a point, by Lemma 4.1, there are at most two planes, say P_3 and P_4, in $St(U_2)$ which passes through $w_{1,1}$. Then for both cases, we use these two solved variables corresponding to P_1 and P_2 or these three variables corresponding to P_1, P_3 and P_4 and solve other variables at $w_{1,1}$.

Next, we pick another end element, say $w_{1,2}$. Because $w_{1,2}$ connects to at most one end element of $St(U_1)$ or $St(U_2)$ from Definition 2.6, and at most one plane in $St(U_2)$ contains U_2 and $w_{1,2}$ if $w_{1,2}$ is an end line or at most two planes in $St(U_2)$ pass U_2 and $w_{1,2}$ if $w_{1,2}$ is an end point by Lemma 4.1, we know that there are at most two or three solved variables at $w_{1,2}$. Hence we can use these solved variables to solve other variables at $w_{1,2}$. Continuing the same procedure, we can solve all variables at the end elements of $St(U_1)$ or $St(U_2)$.

Since $\mathcal{G}'$ has no loop any three elements can not connected pairwise by Corollary 4.5 and any point or element can connect to only one end element or connect to two end elements of $St(U_1)$ and $St(U_2)$ by two unique paths in $\mathcal{G}'$ from Corollary 4.3, we can continue this procedure and solve all variables without ambiguity.

Similarly, we can consider the case of $s > 2$.

We can solve all variables in terms of some variables without ambiguity since $\mathcal{G}'$ has no loop.

Similar to the proof of Theorem A[13] and Theorem 3.15, we can apply Lemma 4.13 and finish the proof of the Main Theorem.

In the course of proving Theorem 4.6, we have proved the following Theorems.

Theorem B. *Let* $\mathcal{A}^*$ *be a nice arrangement of hyperplanes in* $\mathbb{CP}^3$. *The moduli space of* $\mathcal{A}^*$ *with fixed combinatorics* $L(\mathcal{A}^*)$ *is connected.*

Proof. For given two nice arrangements $\mathcal{A}_0^*$ and $\mathcal{A}_1^*$ of hyperplanes in $\mathbb{CP}^3$ with fixed combi-

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natorics $L(\mathcal{A}^*)$, in the proof of Theorem 4.6, we have constructed a one-parameter family $\mathcal{A}^*(t)$ of hyperplanes in $\mathbb{CP}^3$ with fixed combinatorics $L(\mathcal{A}^*)$ connecting $\mathcal{A}_0^*$ and $\mathcal{A}_1^*$. Therefore the moduli space of $\mathcal{A}^*$ with fixed combinatorics $L(\mathcal{A}^*)$ is connected.

Theorem C. *The homotopy groups of the complement $M(\mathcal{A}^*)$ of a nice arrangement of hyperplanes in $\mathbb{CP}^3$ depend only on $L(\mathcal{A}^*)$ (or the lattice $L(\mathcal{A})$).*

Proof. Since the topology of $M(\mathcal{A}^*)$ is determined by $L(\mathcal{A}^*)$, by Theorem 4.6, the homotopy groups of the complement $M(\mathcal{A}^*)$ are determined by $L(\mathcal{A}^*)$.

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