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On formulas for Dedekind sums and the number of lattice points in tetrahedra

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ABSTRACT

This paper explores a simple yet powerful relationship between the problem of counting lattice points and the computation of Dedekind sums. We begin by constructing and proving a sharp upper estimate for the number of lattice points in tetrahedra with some irrational coordinates for the vertices. Besides providing a sharper estimate, this upper bound (Theorem 1.1) becomes an equality (i.e. gives the exact number of lattice points) in a tetrahedron where the lengths of the edges divide each other. This equality condition can then be applied to the explicit computation of the classical Dedekind sums, a topic that is the central focus in the second half of our paper. In this half of the paper, we come up with a number of interesting results related to Dedekind sums, based on our upper estimate (Theorem 1.1). Among these findings, Theorem 1.9 and Theorem 1.10 deserve special attention, for they successfully generalize two of Apostol's formulas in [T.M. Apostol, Modular Functions and Dirichlet Series in Number Theory, Springer-Verlag, New York, 1997], and also directly imply the famous Reciprocity Law of Dedekind sums.

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1. Introduction

1.1. Number of lattice points in a tetrahedron

Let $\Delta(a_1, a_2, \ldots, a_n)$ be an n-dimensional tetrahedron described by

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leqslant 1, \quad x_1, x_2, \ldots, x_n \geqslant 0, \tag{1.1}$$

where $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant 1$ are positive real numbers. Define $P_{(a_1,a_2,\ldots,a_n)}$ and $Q_{(a_1,a_2,\ldots,a_n)}$ to be the number of positive and nonnegative integral solutions of (1.1), respectively (i.e. the number of positive and nonnegative integral points in tetrahedron $\Delta(a_1, a_2, \ldots, a_n)$). If we let $a = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$, then $P_{(a_1,a_2,\ldots,a_n)}$ and $Q_{(a_1,a_2,\ldots,a_n)}$ are related by the following formulas [Li-Ya 1]:

$$Q_{(a_1,a_2,\ldots,a_n)} = P_{(a_1(1+a),a_2(1+a),\ldots,a_n(1+a))}, \tag{1.2}$$

$$P_{(a_1,a_2,\ldots,a_n)} = Q_{(a_1(1-a),a_2(1-a),\ldots,a_n(1-a))}. \tag{1.3}$$

Hence, the study of $P_{(a_1,a_2,\ldots,a_n)}$ and the study of $Q_{(a_1,a_2,\ldots,a_n)}$ are equivalent.

The computation of $P_{(a_1,a_2,\ldots,a_n)}$ and $Q_{(a_1,a_2,\ldots,a_n)}$ has received attention from many distinguished mathematicians. Hardy and Littlewood wrote several papers on the subject that have applications to problems of Diophantine approximation [Ha-Li 1], [Ha-Li 2], [Ha-Li 3], [Ha-Li 4]. D.C. Spencer followed up the efforts of Hardy and Littlewood and wrote two papers on the estimation of $Q_{(a_1,a_2,\ldots,a_n)}$ as well [Sp 1], [Sp 2]. In 1951, Mordell gave a formula for $Q_{(a_1,a_2,a_3)}$, expressed in terms of three Dedekind sums, in the case that a_1, a_2, a_3 are pairwise relatively prime [Mo]. Using toric varieties, Pommersheim in 1993 gave a formula for $Q_{(a_1,a_2,a_3)}$ for arbitrary positive integers a_1, a_2 and a_3 [Po]. More generally, let Δ be a polytope of dimension n in the lattice $\mathbb{Z}^n$, and denote $l(k)$ to be the number of lattice points in Δ dilated by a factor of k, where k is a positive integer:

$$l_\Delta(k) := \#\big(k\Delta \cap \mathbb{Z}^n\big), \quad k \in \mathbb{Z}_+. \tag{1.4}$$

Ehrhart proved that $l_\Delta(k)$ is a polynomial in k of degree n,

$$l_\Delta(k) = b_n k^n + b_{n-1} k^{n-1} + \cdots + b_0, \tag{1.5}$$

where $b_n =$ volume of Δ, $b_{n-1} =$ half the sum of the volumes of $(n-1)$-dimensional faces of Δ. In 1993, Kantor and Khovänskii [Ka-Kh] succeeded in computing b_{n-2}. In fact they gave a general formula for the number of integral points in any integral polytope in R_4. In 1994, Cappell and Shaneson [Ca-Sh] announced a fantastic result with which they can compute all of the coefficients b_i in (1.5). Brion and Vergne [Br-Ve 2] and Diaz and Robins [Di-Ro] have also done beautiful works in finding the exact formula for $Q_{(a_1,a_2,\ldots,a_n)}$.

However, an explicit formula for $P_{(a_1,a_2,\ldots,a_n)}$ or $Q_{(a_1,a_2,\ldots,a_n)}$ in terms of $a_1, a_2, \ldots, a_n$ remains elusive, although many explicit upper bounds have been formulated. In a series of papers, Yau, Xu, Lin, and Wang have proved that the following Sharp Polynomial Upper Estimate Conjecture formulated in [Li-Ya 3] is true for $3 \leqslant n \leqslant 6$ ([Xu-Ya 1], [Xu-Ya 2] for $n = 3, 4$, [Li-Ya 2] for $n = 5$), and Wang ([Wa-Ya] for $n = 6$). Let

$$S_k^{n-1} = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n-1} i_1 i_2 \cdots i_k, \qquad S_0^{n-1} = 1, \qquad S_{n-1}^{n-1} = (n-1)!, \tag{1.6}$$

where $i_1, i_2, \ldots, i_k$ are integers, and

$$A_{n-k}^n = \left(\prod_{i=1}^n a_i\right) \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} \frac{1}{a_{i_1} a_{i_2} \cdots a_{i_k}},$$
$$A_n^n = a_1 a_2 \cdots a_n, \qquad A_0^n = 1. \tag{1.7}$$

Observe that A_{n-k}^n is a polynomial in $a_1, \ldots, a_n$ of degree $n-k$.

Sharp Polynomial Upper Estimate Conjecture. *(See [Li-Ya 3].) Denote $P_{(a_1,a_2,\ldots,a_n)}$ to be the number of positive integral points in an n-dimensional real tetrahedron, where $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant n-1$ are positive real numbers. If $n \geqslant 3$, then*

$$\begin{aligned} n! P_{(a_1,a_2,\ldots,a_n)} \leqslant{} & A_n^n + (-1)\frac{S_1^{n-1}}{n} A_{n-1}^n + (-1)^2 \frac{S_2^{n-1}}{\binom{n-1}{1}} A_{n-2}^{n-1} + (-1)^3 \frac{S_3^{n-1}}{\binom{n-1}{2}} A_{n-3}^{n-1} \\ & + (-1)^4 \frac{S_4^{n-1}}{\binom{n-1}{3}} A_{n-4}^{n-1} + \cdots + (-1)^{k+1} \frac{S_{k+1}^{n-1}}{\binom{n-1}{k}} A_{n-k-1}^{n-1} \\ & + \cdots + (-1)^{n-1} \frac{S_{n-1}^{n-1}}{\binom{n-1}{n-2}} A_1^{n-1}, \end{aligned} \tag{1.8}$$

and the above equality holds if and only if $a_1 = a_2 = \cdots = a_n =$ integer.

In [Wa-Ya], Wang and Yau found that the above conjecture must be modified. The statement "$a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant n-1$" has to be replaced by the statement "$a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant \alpha(n)$", where $\alpha(n)$ is a positive integer depending on n. Because the Sharp Polynomial Upper Estimate begins to lose its sharpness when $a_1 \gg a_n$, Benson made the following conjecture [Ben] in 1997.

Benson's Conjecture. *Denote $P_{(a_1,a_2,a_3)}$ to be the number of positive integral points in a 3-dimensional tetrahedron, where $a_1 \geqslant a_2 \geqslant a_3 \geqslant 0$ are positive integers. Then*

$$6P_{(a_1,a_2,a_3)} \leqslant a_1 a_2 a_3 - \frac{3a_1(a_2+a_3)}{2} + \frac{a_1(a_2+3a_3)}{2a_3}. \tag{1.9}$$

If a_3 divides a_2 and a_2 divides a_1, then (1.9) becomes an equality.

For $n=3$, one can show that (1.9) is strictly sharper than (1.8), except when equality holds in both equations. Therefore Benson's conjecture would give us a better upper bound when dealing with integers, if it can be proved. However unfortunately, we have recently found a few counterexamples to Benson's conjecture.

Counterexample. *Let $a_1 = 100$, $a_2 = 4$, and $a_3 = 3$, then $6P_{a_1,a_2,a_3} = 65$ while the* R.H.S. *of* (1.10) *is* 61.11. *Hence in this case Benson's conjecture fails to serve as an upper bound.*

In general, there is a possibility that Benson's conjecture may fail when the difference between a_2 and a_3 is small. Fortunately, a slight modification of the conditions of Benson's conjecture solves the problem.

Theorem 1.1. *Let $P_{(a_1,a_2,a_3)}$ be the number of positive integral points in a 3-dimensional tetrahedron, where $a_2 \geqslant a_3 \geqslant 0$ are positive integers and a_1 is a positive real number. If $a_3 \mid a_2$ (i.e. a_2 is an integral multiple of a_3), then*

$$6P_{(a_1,a_2,a_3)} \leqslant a_1 a_2 a_3 - \frac{3a_1(a_2+a_3)}{2} + \frac{a_1(a_2+3a_3)}{2a_3}. \tag{1.10}$$

Equality holds if and only if

(i) $a_3 = 1$,
(ii) $a_2 = a_3 = 2$, *or*
(iii) a_1 *is an integer and* $a_2 \mid a_1$.

Theorem 1.1 differs from Benson's conjecture in the additional restriction on the values of a_2 and a_3 ($a_3 \mid a_2$). On the other hand, Theorem 1.1 somewhat compensates this extra constraint by allowing the value of a_1 to be any positive real number. Moreover, it retains the sharpness of Benson's conjecture, as well as the equality condition.

For the sake of its applications later in the paper, Theorem 1.1 is also presented in terms of $Q_{(a_1,a_2,a_3)}$.

Corollary 1.1. *Let* $Q_{(a_1,a_2,a_3)}$ *be the number of nonnegative integral points in a 3-dimensional tetrahedron, where* $a_2 \geqslant a_3 \geqslant 0$ *are positive integers and* a_1 *is a positive real number. If* $a_3 \mid a_2$ *(i.e.* a_2 *is an integral multiple of* a_3*), then*

$$6Q_{(a_1,a_2,a_3)} \leqslant a_1a_2a_3 + \frac{3a_1(a_2+a_3)}{2} + \frac{a_1(a_2+3a_3)}{2a_3} + 3a_2(a_3+1) + 6a_3 + 6. \tag{1.11}$$

Equality holds if and only if $a_2 \mid a_1$.

Although Theorem 1.1 applies only to 3-dimensional tetrahedra, it can easily be modified to suit higher dimensions. For readers' convenience, we list modified versions of Theorem 1.1 for 4-dimensional and 5-dimensional tetrahedra. The derivation of higher dimensional cases is straightforward once the proof of Theorem 1.1 is given.

Theorem 1.2. *Let* $Q_{(a_1,a_2,a_3,a_4)}$ *be the number of nonnegative integral points in a 4-dimensional tetrahedron, where* $a_2 \geqslant a_3 \geqslant a_4 \geqslant 0$ *are positive integers and* a_1 *is a positive real number. If* $a_4 \mid a_3$ *and* $a_3 \mid a_2$*, then*

$$\begin{aligned}24Q_{(a_1,a_2,a_3,a_4)} \leqslant{} & a_1a_2a_3a_4 + 2(a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + 2a_2a_3a_4) \\ & + 3(a_1a_2 + a_1a_3 + a_1a_4 + 2a_2a_3 + 2a_2a_4 + 4a_3a_4) + 3(a_1 + 2a_2 + 4a_3 + 8a_4) \\ & + \frac{a_1a_2(a_4+1)}{a_3} + \frac{a_1a_2a_3 + a_1a_2 + a_1a_3 + 2a_2a_3}{a_4} + 24.\end{aligned} \tag{1.12}$$

Equality holds if and only if $a_2 \mid a_1$.

Theorem 1.3. *Let* $Q_{(a_1,a_2,\dots,a_5)}$ *be the number of nonnegative integral points in a 5-dimensional tetrahedron, where* $a_2 \geqslant a_3 \geqslant a_4 \geqslant a_5 \geqslant 0$ *are positive integers and* a_1 *is a positive real number. If* $a_5 \mid a_4$ *and* $a_4 \mid a_3$ *and* $a_3 \mid a_2$*, then*

$$\begin{aligned}120Q_{(a_1,a_2,\dots,a_5)} \leqslant{} & a_1a_2a_3a_4a_5 + \frac{5}{2}(a_1a_2a_3a_4 + a_1a_2a_3a_5 + a_1a_2a_4a_5 + a_1a_3a_4a_5 + 2a_2a_3a_4a_5) \\ & + 5(a_1a_2a_3 + a_1a_2a_4 + a_1a_2a_5 + a_1a_3a_4 + a_1a_3a_5 + a_1a_4a_5) \\ & + 10(a_2a_3a_4 + a_2a_3a_5 + a_2a_4a_5 + 2a_3a_4a_5) + \frac{15}{2}(a_1a_2 + a_1a_3 + a_1a_4 + a_1a_5) \\ & + 15(a_2a_3 + a_2a_4 + a_2a_5 + 2a_3a_4 + 2a_3a_5 + 4a_4a_5) \\ & + \frac{15}{2}(a_1 + 2a_2 + 4a_3 + 8a_4 + 16a_5) + \frac{5a_1a_2}{2a_3}(a_4 + a_5 + 1)\end{aligned}$$

$$+\frac{5}{2a_4}(a_1a_2+a_1a_3+a_1a_2a_3+a_1a_2a_5+a_1a_3a_5+2a_2a_3+2a_2a_3a_5)$$
$$+\frac{5}{2a_5}(a_1a_2+a_1a_3+a_1a_4+a_1a_2a_3+a_1a_2a_4+a_1a_3a_4)$$
$$+\frac{5}{a_5}(a_2a_3+a_2a_4+a_2a_3a_4+2a_3a_4)+\frac{5a_1a_2}{3}\left(\frac{a_3a_4}{a_5}+\frac{a_3a_5}{a_4}+\frac{a_4a_5}{a_3}\right)$$
$$+\frac{5a_1a_2}{6a_5}\left(\frac{a_3}{a_4}+\frac{a_4}{a_3}\right)-\frac{a_1a_2a_3a_4}{6a_5^3}+120. \tag{1.13}$$

Equality holds if and only if $a_2 \mid a_1$.

A few simple variations on the proof of Theorem 1.1 also leads us to a lower bound for the special semi-integral tetrahedron. However unfortunately, this lower bound does not guarantee us equality when the lengths of the edges divide each other, as Theorem 1.1 has done. Hence it has less mathematical significance than Theorem 1.1. Nonetheless, we present it here for its applications to Dedekind sums.

Theorem 1.4. *Denote $Q_{(a_1,a_2,a_3)}$ to be the number of nonnegative integral points in a 3-dimensional tetrahedron, where $a_2 \geqslant a_3 \geqslant 0$ are positive integers and a_1 is a positive real number. If $a_3 \mid a_2$ (i.e. a_2 is an integral multiple of a_3), then*

$$6Q_3 > a_1a_2a_3+\frac{3a_1(a_2+a_3)}{2}+\frac{a_1(a_2+3a_3)}{2a_3}+6a_3+6. \tag{1.14}$$

1.2. Dedekind sums

The second half of our paper is dedicated to the discussion of the classical Dedekind sums, which play important role in both geometry and topology (see [H-Z-G]). Mathematicians have known that the classical Dedekind sum is somewhat related to the enumeration of lattice points, ever since Mordell in 1951 constructed a formula for $Q_{(a_1,a_2,a_3)}$ in terms of Dedekind sums [Mo]. However, relatively little work has been done that further explores this connection. Moreover, among these few works, most are using Dedekind sums to compute $Q_{(a_1,a_2,a_3)}$ [Mo], [Po]. The purpose of this paper is to explore this connection in the other direction, by applying the estimates on $Q_{(a_1,a_2,a_3)}$ (Theorem 1.1) to the computation of Dedekind sums.

Given two relatively prime positive integers, a and b, the Dedekind sum is defined by

$$s(b,a) \equiv \sum_{i=1}^{a}\left(\left(\frac{i}{a}\right)\right)\left(\left(\frac{ib}{a}\right)\right), \tag{1.15}$$

where

$$((m)) \equiv \begin{cases} m-\lfloor m\rfloor-\frac{1}{2}, & \text{if } m\notin\mathbb{Z},\\ 0, & \text{if } m\in\mathbb{Z}.\end{cases} \tag{1.16}$$

First introduced by Dedekind in the nineteenth century, Dedekind sums play the fundamental role in elliptic modular functions. Using the upper estimate of Corollary 1.1 and the lower estimate of Theorem 1.4, we obtain the following upper and lower estimates for $S(b,a)$ where a and b are relatively prime positive integers,

$$s(b,a) \geqslant \frac{1}{12}\left(\frac{1}{ab}+\frac{a}{b}+\frac{b}{a}\right)-\frac{(b+1)^2}{4b}+\frac{3}{4b}, \tag{1.17}$$

$$s(b,a) < \frac{1}{12}\left(\frac{1}{ab}+\frac{a}{b}+\frac{b}{a}\right)+\frac{(b+1)^2}{4b}-\frac{2b-1}{4b}. \tag{1.18}$$

(1.17) becomes an equality if and only if $b=1$.

Based on the Reciprocity Law, Beck suggests the following upper and lower estimates for Dedekind sums, which are better than (1.17) and (1.18).

Theorem 1.5 *(Beck).*

$$s(b,a) \geqslant \frac{1}{12}\left(\frac{a}{b}+\frac{b}{a}+\frac{1}{ab}\right)-\frac{1}{6b}-\frac{b}{12}, \tag{1.19}$$

$$s(b,a) \leqslant \frac{1}{12}\left(\frac{a}{b}+\frac{b}{a}+\frac{1}{ab}\right)+\frac{1}{6b}+\frac{b}{12}-\frac{1}{2}. \tag{1.20}$$

Mathematicians have known that the classical Dedekind sum is somewhat related to the enumeration of lattice points, ever since Mordell in 1951 constructed a formula for $Q_{(a_1,a_2,a_3)}$ in terms of Dedekind sums [Mo]. However, relatively little work has been done that further explores this connection. Moreover, among these few works, most are using Dedekind sums to compute $Q_{(a_1,a_2,a_3)}$ [Mo], [Po]. The purpose of this paper is to explore this connection in the other direction, by applying the estimates on $Q_{(a_1,a_2,a_3)}$ (Theorem 1.1) to the computation of Dedekind sums.

First, we introduce the following relationships between $Q_{(a_1,a_2,a_3)}$ and $s(b,a)$. They are derived from Pommersheim's results (see Theorem 5 of p. 17 in [Po]), while incorporating Corollary 1.1 and Theorem 1.4.

Theorem 1.6. *Let E_1 be the error of Corollary* 1.1 (*i.e.* $E_1 = \text{RHS}$ *of* (1.11) $- Q_3$) *and E_2 be the error of Theorem* 1.4 (*i.e.* $E_2 = \text{RHS}$ *of* (1.14) $- Q_3$) *under the condition $a_2 = a_3$ and* $\gcd(a_1,a_2)=1$,

$$s(b,a) = \textit{lower estimate}\,(1.17) + \frac{E_1}{b}, \tag{1.21}$$

$$s(b,a) = \textit{upper estimate}\,(1.18) - \frac{E_2}{b}. \tag{1.22}$$

Theorem 1.6 successfully transforms the calculation of Dedekind sums into counting the number of lattice points, and it will be the conceptual basis of our discussion on Dedekind sums. Such transformation gives us the opportunity to examine the Dedekind sums from a different angle and to use a different approach that may produce interesting new results. We arrived at the following formulas.

Theorem 1.7. *Let a and b be relatively prime positive integers, then*

$$s(b,a) = \frac{1}{12}\left(\frac{1}{ab}+\frac{a}{b}+\frac{b}{a}\right)-\frac{4a-3}{6b}-\frac{b(4a+3)}{12}-\frac{2a+1}{2}+\frac{1}{b}\sum_{k=0}^{b}\left(\sum_{n=0}^{b-k}\left\lceil\frac{a(k+n)}{b}\right\rceil\right), \tag{1.23}$$

where $\lceil x\rceil$ denotes the least integer greater than or equal to x.

Theorem 1.8. *Let a and b be relatively prime positive integers, then*

$$s(b,a)=\frac{1}{12}\left(\frac{1}{ab}+\frac{a}{b}+\frac{b}{a}\right)+\frac{(2a-3)(b+3)}{12}+\frac{4a+3b+6}{12b}-\frac{1}{b}\sum_{k=0}^{b}\left(\sum_{n=0}^{k}\left\lfloor\frac{a(k-n)}{b}\right\rfloor\right), \quad (1.24)$$

where $\lfloor x\rfloor$ *denotes the greatest integer less than or equal to x.*

(1.15) gives the following property [Kn]:

$$s(b,a)=s(b\pm ka,a), \quad (1.25)$$

where k is an integer.

Although mathematicians have not yet found a formula that explicitly evaluates the classical Dedekind sums, there exists a well-known property of Dedekind sums, often referred to as the Reciprocity Law, which makes the calculation of Dedekind sums much easier [Be], [Ca], [De], [Di], [Ha], [Ra-Gr], [Si]. The Reciprocity Law states:

$$s(b,a)+s(a,b)=\frac{-1}{4}+\frac{1}{12}\left(\frac{a}{b}+\frac{b}{a}+\frac{1}{ab}\right). \quad (1.26)$$

Combining (1.25) and the Reciprocity Law, Apostol has done beautiful work to come up with the following result: let $a\equiv r \pmod b$, then

$$12ab\cdot s(b,a)=a^2-\left[12\cdot s(r,a)+3\right]ab+b^2+1. \quad (1.27)$$

(1.27) is a useful formula, for it reduces the calculation of $s(b,a)$ to that of $s(r,a)$.

However, if r is a large value, we still have the burden of computing $s(r,a)$. Hence, while (1.27) can efficiently calculate Dedekind sums with small r values, the evaluation of larger r values becomes tedious.

Based on Theorem 1.6, we have successfully generalized (1.27). Our generalized version of (1.27), Theorem 1.9 below, can determine $s(b,a)$ without knowning $s(r,a)$. Hence Theorem 1.9 greatly reduces the time in calculating Dedekind sums with large r values.

Theorem 1.9. *Given two relatively prime positive integers a and b, if* $a\equiv r \pmod b$ *and* $b\equiv t \pmod r$ *and* $w_t\equiv mt \pmod r$, *where* $b-1\geqslant r\geqslant 1$ *and* $r-1\geqslant t\geqslant 1$ *and* $r-1\geqslant w_m\geqslant 1$, *then*

$$12ab\cdot s(b,a)=a^2+b^2+1-\frac{b^2+3r(r-1)b+(r^2+1)}{r}a+\frac{12ab}{r^2}\sum_{m=1}^{r-1}mw_m. \quad (1.28)$$

Theorem 1.9 implies (1.27):

Corollary 1.2 *(Apostol). Let* $a\equiv r \pmod b$, *then*

$$12ab\cdot s(b,a)=a^2-\left[12\cdot s(r,a)+3\right]ab+b^2+1. \quad (1.29)$$

Theorem 1.9 also gives us the following properties of Dedekind sums:

Corollary 1.3. *Given four positive integers* a_1, b_1, a_2, *and* b_2 *where* $\gcd(a_1,b_1)=1$ *and* $\gcd(a_2,b_2)=1$, *let* $a_1\equiv r \pmod{b_1}$ *where* $b_1\equiv t \pmod r$ *and* $a_2\equiv r \pmod{b_2}$ *where* $b_2\equiv -t \pmod r$, *if* $b_1-1\geqslant r\geqslant 1$, $b_2-1\geqslant r\geqslant 1$, *and* $r-1\geqslant t\geqslant 1$, *then*

$$s(b_1,a_1)+s(b_2,a_2)=\frac{1}{12}\left(\frac{b_2}{a_2}+\frac{a_2}{b_2}+\frac{1}{a_2b_2}+\frac{b_1}{a_1}+\frac{a_1}{b_1}+\frac{1}{a_1b_1}\right)$$
$$-\frac{r^2+1}{12r}\left(\frac{1}{b_1}+\frac{1}{b_2}\right)-\frac{b_1+b_2}{12r}. \tag{1.30}$$

Corollary 1.4 *(Apostol). (See p. 73 of [Ap].) Given two relatively prime positive integers a and b, if* $a\equiv r \pmod b$ *and* $b\equiv t \pmod r$ *where* $b-1\geqslant r\geqslant 1$ *and* $t=\pm 1$*, then*

$$12ab\cdot s(b,a)=a^2+b^2+1-\frac{b^2-t(r-1)(r-2)b+r^2+1}{r}a. \tag{1.31}$$

Notice the restriction on t in Corollary 1.4. In general, Corollary 1.4 gives explicit formulas when $a\equiv r \pmod b$, where $4\geqslant r\geqslant -4$ or $r=\pm 6$.

Observe that in Theorem 1.9, we can explicitly express $\frac{12ab}{r^2}\sum_{m=1}^{r-1} m w_m$ once numerical values of r and t are given. Hence Theorem 1.9 becomes an explicit formula for Dedekind sums, if r and t are known (see Example 4.1 and Example 4.2 in Section 4). Theorem 1.9 is useful when $a \pmod b$ is known, and now we present Theorem 1.10, which evaluates $s(b,a)$ when we are given $b \pmod a$.

Theorem 1.10. *Given two relatively prime positive integers a and b, if* $b\equiv u \pmod a$ *and* $a\equiv f \pmod u$ *and* $z_m\equiv mf \pmod u$*, where* $a-1\geqslant u\geqslant 1$ *and* $u-1\geqslant f\geqslant 1$ *and* $u-1\geqslant z_m\geqslant 1$*, then*

$$12ab\cdot s(b,a)=\frac{a^2+u^2+1}{u}b+3ab(u-2)-\frac{12ab}{u^2}\sum_{m=1}^{u-1} m z_m. \tag{1.32}$$

Theorem 1.10 also leads us to the following corollary, which provides explicit formulas when $b\equiv u \pmod a$, where $4\geqslant u\geqslant -4$ or $u=\pm 6$.

Corollary 1.5. *Given two relatively prime positive integers a and b, if* $b\equiv u \pmod a$ *and* $a\equiv f \pmod u$ *where* $a-1\geqslant u\geqslant 1$ *and* $f=\pm 1$*, then*

$$12ab\cdot s(b,a)=\frac{a^2+u^2+1}{u}b+abf\left(\frac{2}{u}+u-3\right)-3ab. \tag{1.33}$$

As mentioned above, the Reciprocity Law is one of the most important properties of the classical Dedekind sums. Since Dedekind [De] first introduced it in 1953, many mathematicians have written different proofs of it (see [Be], [Ca], [Di], [Ha], [Ra-Gr], [Si]). As we will show in Section 4, the Reciprocity Law is a direct consequence of Theorem 1.9 and Theorem 1.10. Hence this paper gives another proof to this important property.

Corollary 1.6 *(Reciprocity Law). (See [Be], [Ca], [Di], [De], [Ha], [Ra-Gr], [Si].) Given two relatively prime positive integers a and b, then*

$$12ab\cdot s(b,a)+12ab\cdot s(a,b)=a^2+b^2+1-3ab. \tag{1.34}$$

In Section 2, we shall prove Corollary 1.1, Theorem 1.2, and Theorem 1.4. Since the proofs of Theorem 1.1 and Theorem 1.3 can be easily derived, based on the proofs of Corollary 1.1 and Theorem 1.2, we omit these two proofs. In Section 3, we make a comparison between our upper bound and the existing upper bound, the Sharp Polynomial Upper Estimate. In Section 4, we apply Corollary 1.1 to give explicit computation of Dedekind sums. We will close out the discussion by constructing an alternative proof of the famous Reciprocity Law of Dedekind sums.

2. Sharp rational function upper estimate for integral tetrahedra

The purpose of this section is to prove Corollary 1.1, Theorem 1.2, and Theorem 1.4 above. The following Lemma 2.1 is a modification of Preposition 2.1 of [Xu-Ya 1].

Lemma 2.1. *Let $Q_{(r,s)}$ be the number of nonnegative integral solutions of*

$$\frac{x_1}{r}+\frac{x_2}{s}\leqslant 1, \tag{2.1}$$

(*i.e.* $Q_{(r,s)}$ *is the number of nonnegative lattice points in a 2-dimensional tetrahedron*), *where r is a positive real number and s is a positive integer, then*

$$Q_{(r,s)}\leqslant\frac{(r+2)(s+1)}{2}. \tag{2.2}$$

The above inequality becomes an equality if and only if $s \mid r$, i.e. s divides r.

Proof. We sum the nonnegative integral solutions of (2.1) line by line horizontally. In view of (2.1), we have $x_1\leqslant\frac{r(s-x_2)}{s}+1$.

$$\begin{aligned}Q_{(r,s)}&\leqslant\sum_{n=0}^{s-1}\left(\frac{r(s-n)}{s}+1\right)+1\\&=\frac{r}{s}\left[s^2-\frac{(s-1)s}{2}\right]+s+1\\&=\frac{rs}{2}+\frac{r}{2}+s+1\\&=\frac{(r+2)(s+1)}{2}.\end{aligned}$$

The equality above holds if and only if $\frac{r(s-n)}{s}$, for all n where $0\leqslant n\leqslant s-1$, are integers. This is true if and only if r is an integral multiple of s. □

Proof of Corollary 1.1. We shall prove Corollary 1.1 by slicing the three-dimensional tetrahedron along the x_3-axis and reduce the three-dimensional tetrahedron into a_3 numbers of two-dimensional tetrahedra. Then we sum up the number of lattice points in all two-dimensional tetrahedra. Specifically, the two-dimensional tetrahedron at $x_3=k$ is given by

$$\frac{x_1}{a_1}+\frac{x_2}{a_2}+\frac{k}{a_3}\leqslant 1\quad\Longrightarrow\quad\frac{x_1}{\frac{a_1(a_3-k)}{a_3}}+\frac{x_2}{\frac{a_2(a_3-k)}{a_3}}\leqslant 1. \tag{2.3}$$

Let $r=\frac{a_1(a_3-k)}{a_3}$ and $s=\frac{a_2(a_3-k)}{a_3}$, where $0\leqslant k\leqslant a_3-1$. $Q_{(r,s)}(k)$ is the number of nonnegative integral points in (2.3). Observe that since $a_3\mid a_2$, s is an integer and we can apply lemma 2.1 to get

$$\begin{aligned}Q_{(a_1,a_2,a_3)}&=\sum_{k=0}^{a_3-1}Q_{(r,s)}(k)+1\\&\leqslant\frac{1}{2}\sum_{k=0}^{a_3-1}\left[\left(\frac{a_1}{a_3}(a_3-k)+2\right)\left(\frac{a_2}{a_3}(a_3-k)+1\right)\right]+1\end{aligned}$$

$$= \frac{a_1a_2a_3}{2} + a_2a_3 + \frac{a_1a_3}{2} + a_3 - \left(\frac{a_1a_2(a_3-1)}{2} + \frac{a_2(a_3-1)}{2} + \frac{a_1(a_3-1)}{4}\right)$$
$$+ \left(\frac{a_1a_2a_3}{6} - \frac{a_1a_2}{4} + \frac{a_1a_2}{12a_3}\right) + 1$$
$$= \frac{a_1a_2a_3}{6} + \frac{a_1(a_2+a_3)}{4} + \frac{a_1(a_2+3a_3)}{12a_3} + \frac{a_2(a_3+1)}{2} + a_3 + 1.$$

From Lemma 2.1, equality holds if and only if $s \mid r$, which is equivalent to $a_2 \mid a_1$. □

Proof of Theorem 1.2. Following essentially the same idea, we shall prove Theorem 1.2 by slicing the four-dimensional tetrahedron along the x_4-axis and reduce the four-dimensional tetrahedron into $a_4 - 1$ numbers of three-dimensional tetrahedra. The number of nonnegative lattice points in the three-dimensional tetrahedra can be estimated by Corollary 1.1. Then we sum up the number of lattice points in all three-dimensional tetrahedra. The three-dimensional tetrahedron at $x_4 = m$ is given by

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{m}{a_4} \leqslant 1 \quad \Longrightarrow \quad \frac{x_1}{\frac{a_1(a_4-m)}{a_4}} + \frac{x_2}{\frac{a_2(a_4-m)}{a_4}} + \frac{x_3}{\frac{a_3(a_4-m)}{a_4}} \leqslant 1. \tag{2.4}$$

By the hypothesis of Theorem 1.2; $a_3 \mid a_4$ and $a_2 \mid a_3$, $\frac{a_2}{a_4}(a_4 - m)$ and $\frac{a_3}{a_4}(a_4 - m)$ are integers and $\frac{a_3}{a_4}(a_4 - m)$ divides $\frac{a_2}{a_4}(a_4 - m)$. Hence we can apply Corollary 1.1 to get

$$Q_3(m) \leqslant \frac{a_1a_2a_3}{a_4^3}(a_4-m)^3 - \frac{3}{2}\frac{a_1}{a_4}(a_4-m)\left[\frac{a_2}{a_4}(a_4-m) + \frac{a_3}{a_4}(a_4-m)\right]$$
$$+ \frac{a_1}{2a_3}\left[\frac{a_2}{a_4}(a_4-m) + \frac{a_3}{a_4}(a_4-m)\right], \tag{2.5}$$

where $Q_3(m)$ is the number of nonnegative integral points in (2.4).

$$Q_{(a_1,a_2,a_3,a_4)} = \sum_{m=0}^{a_4-1} Q_3(m) + 1$$
$$\leqslant \frac{1}{6}\sum_{m=0}^{a_4-1}\left(\frac{a_1a_2a_3(a_4-m)^3}{a_4^3}\right) + \frac{1}{4}\sum_{m=0}^{a_4-1}\left[\frac{a_1(a_4-m)}{a_4}\left(\frac{(a_2+a_3)(a_4-m)}{a_4}\right)\right]$$
$$+ \frac{1}{12}\sum_{m=0}^{a_4-1}\left[\frac{a_1}{a_3}\left(\frac{(a_2+3a_3)(a_4-m)}{a_4}\right)\right]$$
$$+ \frac{1}{2}\sum_{m=0}^{a_4-1}\left[\frac{a_2(a_4-m)}{a_4}\left(\frac{a_3(a_4-m)}{a_4} + 1\right)\right] + \sum_{m=0}^{a_4-1}\left(\frac{a_3(a_4-m)}{a_4}\right) + a_4 + 1$$
$$= \frac{a_1a_2a_3a_4}{24} + \frac{(a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + 2a_2a_3a_4)}{12}$$
$$+ \frac{(a_1a_2 + a_1a_3 + a_1a_4 + 2a_2a_3 + 2a_2a_4 + 4a_3a_4)}{8} + \frac{(a_1 + 2a_2 + 4a_3 + 8a_4)}{8}$$
$$+ \frac{a_1a_2(a_4+1)}{24a_3} + \frac{a_1a_2a_3 + a_1a_2 + a_1a_3 + 2a_2a_3}{24a_4} + 1.$$

From Corollary 1.1, the above inequality becomes an equality if and only if $a_2 \mid a_1$. □

Lemma 2.2. *Let $Q_{(r,s)}$ be the number of nonnegative integral solutions of*

$$\frac{x_1}{r} + \frac{x_2}{s} \leqslant 1, \tag{2.6}$$

(i.e. $Q_{(r,s)}$ is the number of nonnegative lattice points in a 2-dimensional tetrahedron), where r is a positive real number and s is a positive integer, then

$$Q_{(r,s)} > \frac{r(s+1)}{2} + 1. \tag{2.7}$$

Proof. Again, we sum the nonnegative integral solutions of (2.6) horizontally. In view of (2.6), we have $x_1 > \frac{r(s-x_2)}{s}$. It follows that

$$Q_{(r,s)} > \sum_{k=0}^{s-1} \left(\frac{r(s-k)}{s} \right) + 1 = \frac{r(s+1)}{2} + 1. \qquad \square$$

Proof of Theorem 1.4. Again, based on the proof of Corollary 1.1, we slice the three-dimensional tetrahedron along the x_3 axis and the two-dimensional tetrahedron at $x_3 = k$ is given by

$$\frac{x_1}{\frac{a_1(a_3-k)}{a_3}} + \frac{x_2}{\frac{a_2(a_3-k)}{a_3}} \leqslant 1. \tag{2.8}$$

Let $r = \frac{a_1(a_3-k)}{a_3}$ and $s = \frac{a_2(a_3-k)}{a_3}$, where $1 \leqslant k \leqslant a_3 - 1$. $Q_{(r,s)}(k)$ is the number of nonnegative integral points in (2.8). Observe that since $a_3 \mid a_2$, s is an integer and we can apply Lemma 2.2 to get

$$\begin{aligned} Q_{(a_1,a_2,a_3)} &= \sum_{k=0}^{a_3-1} Q_{(r,s)}(k) + 1 \\ &> \frac{1}{2} \sum_{k=0}^{a_3-1} \frac{a_1}{a_3}(a_3-k)\left[\frac{a_2}{a_3}(a_3-k)+1\right] + a_3 + 1 \\ &= \frac{1}{6}\left[a_1a_2a_3 + \frac{3a_1(a_2+a_3)}{2} + \frac{a_1(a_2+3a_3)}{2a_3} + 6a_3 + 6\right]. \qquad \square \end{aligned}$$

3. A comparison between Theorem 1.1 and the existing upper bound

The sharpest existing upper bound for three-dimensional real tetrahedra is by Xu and Yau [Xu-Ya 1]:

Sharp Polynomial Upper Estimate. *If $a_1 \geqslant a_2 \geqslant a_3 \geqslant 2$, then*

$$6P_{(a_1,a_2,a_3)} \leqslant (a_1-1)(a_2-1)(a_3-1) - a_3 + 1. \tag{3.1}$$

While the Sharp Polynomial Upper Estimate is the sharpest existing upper bound for real tetrahedra, Theorem 1.1 gives even a sharper estimate for special semi-integral tetrahedra. In fact, Theorem 1.1 is strictly sharper than Xu–Yau's upper bound unless $a_1 = a_2 = a_3$, in which case the equality holds for both bounds.

Theorem 3.1. *Theorem 1.1 is strictly sharper than Xu–Yau's upper bound unless $a_1 = a_2 = a_3$. More precisely, if $a_1 \geqslant a_2 \geqslant a_3 \geqslant 2$, where a_2 and a_3 are integers and $a_3 \mid a_2$, then*

$$a_1a_2a_3 - \frac{3a_1(a_2+a_3)}{2} + \frac{a_1(a_2+3a_3)}{2a_3} \leqslant (a_1-1)(a_2-1)(a_3-1) - a_3 + 1. \tag{3.2}$$

Equality holds if and only if $a_1 = a_2 = a_3$.

While this paper covers only the case of $n = 3$, higher dimensional bounds (Theorems 1.2 and 1.3) also have the same level of sharpness as Theorem 1.1, in comparison to the Sharp Polynomial Upper Estimate.

Proof of Theorem 3.1. We show that the following is always nonnegative under the assumption that $a_1 \geqslant a_2 \geqslant a_3 \geqslant 2$, where a_2 and a_3 are integers and $a_3 \mid a_2$.

$$\begin{aligned}&\big[(a_1-1)(a_2-1)(a_3-1) - a_3 + 1\big] - \left(a_1a_2a_3 - \frac{3a_1(a_2+a_3)}{2} + \frac{a_1(a_2+3a_3)}{2a_3}\right)\\ &= \frac{a_1a_2}{2} + \frac{a_1a_3}{2} + a_2 - \frac{a_1}{2} - \frac{a_1a_2}{2a_3} - a_2a_3.\end{aligned} \tag{3.3}$$

There are two cases to consider: (i) $a_1 \geqslant 2a_2$ and (ii) $a_1 < 2a_2$.

Case 1. $a_1 \geqslant 2a_2$.

We shall show that (3.3) is strictly increasing with respect to a_3.

$$\frac{d}{da_3}\left(\frac{a_1a_2}{2} + \frac{a_1a_3}{2} + a_2 - \frac{a_1}{2} - \frac{a_1a_2}{2a_3} - a_2a_3\right) = \frac{a_1}{2} - a_2 + \frac{a_1a_2}{2a_3^2} = \frac{a_3^2(a_1-2a_2) + a_1a_2}{2a_3^2}. \tag{3.4}$$

Clearly, when $a_1 \geqslant 2a_2$,

$$\frac{a_3^2(a_1-2a_2) + a_1a_2}{2a_3^2} > 0.$$

Thus we have proved that given $a_1 \geqslant 2a_2$, Theorem 1.1 is strictly sharper.

Case 2. $a_1 < 2a_2$.

We further divide this case into two subcases.

Subcase 1. $a_2 \neq a_3$.

$$\begin{aligned}\frac{a_1a_2}{2} + \frac{a_1a_3}{2} + a_2 - \frac{a_1}{2} - \frac{a_1a_2}{2a_3} - a_2a_3 &> \frac{a_1a_2}{2} + \frac{a_1a_3}{2} - \frac{a_1a_2}{2a_3} - a_2a_3\\ &= \frac{1}{2a_3}\big(a_1a_3^2 - 2a_2a_3^2 - a_1a_2 + a_1a_2a_3\big)\\ &\geqslant \frac{1}{2a_3}\big(-a_1a_2 - a_2a_3^2 + a_1a_2a_3\big)\\ &= \frac{a_2}{2a_3}\big[a_1(a_3-1) - a_3^2\big].\end{aligned}$$

Table 1
Comparison between Theorem 1.1 and Xu–Yau's bound.

a_1, a_2, a_3	Actual values of P_3	Values from Theorem 1.1	Xu–Yau's bound
3, 3, 3	1	1	1
10, 9, 3	18	20	23.66
13, 6, 3	14	15	16
15, 10, 5	70	75	84
23, 4, 4	21	23	33
30, 18, 6	360	375	410
35, 20, 10	885	918.75	967.5
45, 30, 15	2835	2887.5	2975
60, 20, 10	1575	1575	1680
99, 3, 3	33	33	65
99, 99, 3	2673	2673	3201
100, 4, 4	100	100	148.5
50.5, 20, 10	1290	1325.625	1410.75
17.8, 3, 3	5	5.93	11.2

In our assumption $a_3 \mid a_2$ and $a_2 \neq a_3$, so we have $a_2 \geqslant 2a_3$. It follows that

$$\begin{aligned} a_1(a_3-1)-a_3^2 &\geqslant 2a_3(a_3-1)-a_3^2 \\ &= a_3^2-2a_3 \\ &\geqslant 0. \end{aligned}$$

Subcase 2. $a_2 = a_3 \leqslant a_1$.

$$\begin{aligned} \frac{a_1a_2}{2}+\frac{a_1a_3}{2}+a_2-\frac{a_1}{2}-\frac{a_1a_2}{2a_3}-a_2a_3 &= \frac{a_1a_2}{2}+\frac{a_1a_2}{2}+a_2-\frac{a_1}{2}-\frac{a_1a_2}{2a_2}-a_2^2 \\ &= a_1(a_2-1)-a_2(a_2-1) \\ &\geqslant 0. \end{aligned}$$

Observe that the inequality above becomes an equality if and only if $a_1 = a_2$. □

Table 1 compares Theorem 1.1 to the Sharp Polynomial Upper Estimate.

4. Proofs and examples on the classical Dedekind sums

Let $Q_3 = Q(a_1, a_2, a_3)$ be the number of nonnegative integral points satisfying

$$\frac{x_1}{a_1}+\frac{x_2}{a_2}+\frac{x_3}{a_3} \leqslant 1, \tag{4.1}$$

where a_1, a_2, and a_3 are positive integers.

As we mentioned in the introduction, Pommersheim [Po] in 1993 derived the three-dimensional Ehrhart polynomial using toric variety. Specifically, for a tetrahedron with vertices at $((a,0,0),(0,b,0),(0,0,c))$, where $\gcd(a,b,c)=1$, the number of nonnegative lattice points in such a tetrahedron dilated by a factor of k is

$$l_\Delta(k) = \frac{abc}{6}k^3 + \frac{(ab+ac+bc+d)}{4}k^2 + \frac{1}{12}\left(\frac{ac}{b} + \frac{bc}{a} + \frac{ab}{c} + \frac{d^2}{abc}\right)k + \frac{(a+b+c+A+B+C)}{4}k - A\cdot s\left(\frac{bc}{d}, \frac{aA}{d}\right)\cdot k - B\cdot s\left(\frac{ac}{d}, \frac{bB}{d}\right)\cdot k - C\cdot s\left(\frac{ab}{d}, \frac{cC}{d}\right)\cdot k + 1, \tag{4.2}$$

where $A = \gcd(b,c)$, $B = \gcd(a,c)$, $C = \gcd(a,b)$, $d = ABC$.

Throughout the rest of this section we let a_1 and a_2 be relatively prime and $a_2 = a_3$. Thus (4.1) becomes

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_2} \leqslant 1, \tag{4.3}$$

where a_1 and a_2 are relatively prime positive integers.

Proof of Theorem 1.6 and Theorem 1.5. Theorem 1.6 follows directly from (4.2). Theorem 1.5 comes from the Reciprocity Law of Dedekind sums, which we shall prove later.

$$s(b,a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right) - s(a,b), \tag{4.4}$$

and

$$-s(1,b) \leqslant s(a,b) \leqslant s(1,b). \tag{4.5}$$

Thus we obtain

$$-\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right) - s(1,b) \leqslant s(b,a) \leqslant -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right) + s(1,b). \tag{4.6}$$

Once we insert the identity $s(1,b) = -\frac{1}{4} + \frac{1}{6b} + \frac{b}{12}$, we have arrived at the desired result. □

Proof of Theorem 1.7. The proof of Lemma 2.1 implies that the number of nonnegative integral solutions in (2.1) is given by

$$Q_2 = \sum_{n=0}^{s}\left\lfloor \frac{r(s-n)}{s} + 1 \right\rfloor, \tag{4.7}$$

where $\lfloor x \rfloor$ represents the smallest integer less than or equal to x. After dissecting the three-dimensional tetrahedron in (4.3), the two-dimensional tetrahedron at $x_3 = k$ is given by

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{k}{a_2} \leqslant 1 \quad\Longrightarrow\quad \frac{x_1}{\frac{a_1(a_2-k)}{a_2}} + \frac{x_2}{(a_2-k)} \leqslant 1. \tag{4.8}$$

Let $r = \frac{a_1(a_2-k)}{a_2}$ and $s = a_2 - k$, we apply (4.14) to get

$$Q_3 = \sum_{k=0}^{a_2}\left(\sum_{n=0}^{a_2-k}\left\lfloor \frac{a_1(a_2-k-n)}{a_2} + 1 \right\rfloor\right). \tag{4.9}$$

Now we can calculate the error of Corollary 1.1 in estimating the actual Q_3 value under the condition $a_2 = a_3$ and $\gcd(a_1, a_2) = 1$,

$$E_1 = \frac{a_1 a_2^2}{6} + \frac{a_1 a_2}{2} + \frac{a_1}{3} + \frac{a_2(a_2+1)}{2} + a_2 + 1 - \sum_{k=0}^{a_2}\left(\sum_{n=0}^{a_2-k}\left\lfloor \frac{a_1(a_2-k-n)}{a_2} + 1\right\rfloor\right). \tag{4.10}$$

Then we apply Theorem 1.6

$$\begin{aligned}
s(a_2, a_1) &= \frac{1}{a_2}\left(\frac{a_1 a_2^2}{6} + \frac{a_1 a_2}{2} + \frac{a_1}{3} + \frac{a_2(a_2+1)}{2} + a_2 + 1\right) \\
&\quad - \frac{1}{a_2}\sum_{k=0}^{a_2}\left[\sum_{n=0}^{a_2-k}\left(a_1 - \left\lceil \frac{a_1(k+n)}{a_2}\right\rceil + 1\right)\right] \\
&\quad + \frac{1}{12}\left(\frac{1}{a_1 a_2} + \frac{a_1}{a_2} + \frac{a_2}{a_1}\right) - \frac{(a_2+1)^2}{4a_2} + \frac{3}{4a_2} \\
&= \frac{a_1 a_2}{6} + \frac{a_1}{2} + \frac{a_1}{3a_2} + \frac{a_2}{2} + \frac{1}{2} + 1 + \frac{1}{a_2} \\
&\quad - \frac{1}{a_2}\left[\frac{a_1 a_2^2}{2} + \frac{3a_1 a_2}{2} + a_1 + \frac{a_2^2}{2} + \frac{3a_2}{2} + 1 - \sum_{k=0}^{a_2}\left(\sum_{n=0}^{a_2-k}\left\lceil \frac{a_1(k+n)}{a_2}\right\rceil\right)\right] \\
&\quad + \frac{1}{12}\left(\frac{1}{a_1 a_2} + \frac{a_1}{a_2} + \frac{a_2}{a_1}\right) - \frac{1}{2} - \frac{1}{4a_2} - \frac{a_2}{4} + \frac{3}{4a_2} \\
&= \frac{1}{12}\left(\frac{1}{a_1 a_2} + \frac{a_1}{a_2} + \frac{a_2}{a_1}\right) - \frac{a_1 a_2}{3} - a_1 - \frac{2a_1}{3a_2} - \frac{a_2}{4} + \frac{1}{2a_2} - \frac{1}{2} \\
&\quad + \frac{1}{a_2}\sum_{k=0}^{a_2}\left(\sum_{n=0}^{a_2-k}\left\lceil \frac{a_1(k+n)}{a_2}\right\rceil\right) \\
&= \frac{1}{12}\left(\frac{1}{a_1 a_2} + \frac{a_1}{a_2} + \frac{a_2}{a_1}\right) - \frac{4a_1 - 3}{6a_2} - \frac{a_2(4a_1+3)}{12} - \frac{2a_1+1}{2} \\
&\quad + \frac{1}{a_2}\sum_{k=0}^{a_2}\left(\sum_{n=0}^{a_2-k}\left\lceil \frac{a_1(k+n)}{a_2}\right\rceil\right). \qquad \square
\end{aligned}$$

Proof of Theorem 1.8. The proof of Theorem 1.8 is almost identical to that of Theorem 1.7. Instead of dissecting the three-dimensional tetrahedron at $x_3 = k$, we dissect it at $x_3 = a_2 - k$.

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{a_2 - k}{a_2} \leqslant 1 \quad \Longrightarrow \quad \frac{x_1}{\frac{a_1 k}{a_2}} + \frac{x_2}{k} \leqslant 1. \tag{4.11}$$

Let $r = \frac{a_1 k}{a_2}$ and $s = k$ and applying (4.7), we have

$$Q_3 = \sum_{k=0}^{a_2}\left(\sum_{n=0}^{k}\left\lfloor \frac{a_1(k-n)}{a_2} + 1\right\rfloor\right). \tag{4.12}$$

In this case, the error of Corollary 1.1 when a_1 and a_2 are relatively prime integers and $a_2 = a_3$ can be expressed as

$$E_1 = \frac{a_1 a_2^2}{6} + \frac{a_1 a_2}{2} + \frac{a_1}{3} + \frac{a_2(a_2+1)}{2} + a_2 + 1 - \sum_{k=0}^{a_2}\left(\sum_{n=0}^{k}\left\lfloor \frac{a_1(k-n)}{a_2} + 1\right\rfloor\right), \qquad (4.13)$$

and Theorem 1.6 gives us the following:

$$\begin{aligned}
s(a_2, a_1) &= \frac{1}{a_2}\left(\frac{a_1 a_2^2}{6} + \frac{a_1 a_2}{2} + \frac{a_1}{3} + \frac{a_2(a_2+1)}{2} + a_2 + 1\right) \\
&\quad - \frac{1}{a_2}\sum_{k=0}^{a_2}\left[\sum_{n=0}^{k}\left(\left\lfloor \frac{a_1(k-n)}{a_2}\right\rfloor + 1\right)\right] \\
&\quad + \frac{1}{12}\left(\frac{1}{a_1 a_2} + \frac{a_1}{a_2} + \frac{a_2}{a_1}\right) - \frac{(a_2+1)^2}{4a_2} + \frac{3}{4a_2} \\
&= \frac{a_1 a_2}{6} + \frac{a_1}{2} + \frac{a_1}{3a_2} + \frac{a_2}{2} + \frac{1}{2} + 1 + \frac{1}{a_2} \\
&\quad - \frac{1}{a_2}\left[\frac{a_2^2}{2} + \frac{3a_2}{2} + 1 + \sum_{k=0}^{a_2}\left(\sum_{n=0}^{k}\left\lfloor \frac{a_1(k-n)}{a_2}\right\rfloor\right)\right] \\
&\quad + \frac{1}{12}\left(\frac{1}{a_1 a_2} + \frac{a_1}{a_2} + \frac{a_2}{a_1}\right) - \frac{1}{2} - \frac{1}{4a_2} - \frac{a_2}{4} + \frac{3}{4a_2} \\
&= \frac{a_1 a_2}{6} + \frac{a_1}{2} + \frac{a_1}{3a_2} - \frac{1}{a_2}\sum_{k=0}^{a_2}\left(\sum_{n=0}^{k}\left\lfloor \frac{a_1(k-n)}{a_2}\right\rfloor\right) \\
&\quad + \frac{1}{12}\left(\frac{1}{a_1 a_2} + \frac{a_1}{a_2} + \frac{a_2}{a_1}\right) - \frac{1}{2} - \frac{1}{4a_2} - \frac{a_2}{4} + \frac{3}{4a_2} \\
&= \frac{1}{12}\left(\frac{1}{ab} + \frac{a_1}{a_2} + \frac{a_2}{a_1}\right) + \frac{(2a_1-3)(a_2+3)}{12} + \frac{4a_1 + 3a_2 + 6}{12a_2} \\
&\quad - \frac{1}{a_2}\sum_{k=0}^{a_2}\left(\sum_{n=0}^{k}\left\lfloor \frac{a_1}{a_2}(k-n)\right\rfloor\right). \qquad \square
\end{aligned}$$

Proof of Theorem 1.9. The proof of Lemma 2.1 implies that the number of nonnegative integral solutions in (2.1) is given by

$$Q_{(r,s)} = \sum_{n=0}^{s}\left\lfloor \frac{r(s-n)}{s} + 1\right\rfloor, \qquad (4.14)$$

where $\lfloor x \rfloor$ represents the smallest integer less than or equal to x. After slicing the three-dimensional tetrahedron in (4.3), the two-dimensional tetrahedron at $x_3 = k$ is given by

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{k}{a_2} \leqslant 1 \quad \Longrightarrow \quad \frac{x_1}{\frac{a_1(a_2-k)}{a_2}} + \frac{x_2}{(a_2-k)} \leqslant 1. \qquad (4.15)$$

Let $r = \frac{a_1(a_2-k)}{a_2}$ and $s = a_2 - k$, we apply (4.14) to get

$$Q_3 = \sum_{k=0}^{a_2}\left(\sum_{n=0}^{a_2-k}\left\lfloor \frac{a_1(a_2-k-n)}{a_2} + 1\right\rfloor\right). \tag{4.16}$$

Let $a_1 \equiv r \pmod{a_2}$, $a_2 \equiv t \pmod{r}$, and $w_t \equiv mt \pmod{r}$, where $a_2 - 1 \geqslant r \geqslant 1$, $r - 1 \geqslant t \geqslant 1$, and $r - 1 \geqslant w_m \geqslant 1$ (notice that the r here is different from the r in (4.14)), from (4.16), we have

$$\begin{aligned} Q_3 &= \sum_{k=0}^{a_2}\left(\sum_{n=0}^{a_2-k}\left\lfloor \frac{a_1(a_2-k-n)}{a_2} + 1\right\rfloor\right) \\ &= \sum_{k=0}^{a_2}\left(\sum_{n=0}^{a_2-k}\left\lfloor \frac{(a_1-r)}{a_2}(a_2-k-n) + 1\right\rfloor\right) \\ &\quad + \sum_{k=0}^{a_2}\left(\sum_{n=0}^{a_2-k}\left\lfloor \frac{r}{a_2}(a_2-k-n)\right\rfloor\right). \end{aligned} \tag{4.17}$$

Since $a_2 \mid (a_1 - r)$, $\frac{a_1-r}{a_2}(a_2-k-n)$ is an integer and

$$\sum_{k=0}^{a_2}\left(\sum_{n=0}^{a_2-k}\left\lfloor \frac{(a_1-r)}{a_2}(a_2-k-n) + 1\right\rfloor\right) = \sum_{k=0}^{a_2}\left(\sum_{n=0}^{a_2-k} \frac{(a_1-r)}{a_2}(a_2-k-n) + 1\right),$$

$\sum_{k=0}^{a_2}(\sum_{n=0}^{a_2-k} \frac{(a_1-r)}{a_2}(a_2-k-n)+1)$ can be evaluated by Corollary 1.1, which becomes an exact formula when the lengths of the edges of the tetrahedron divide each other. In our case, we have

$$\frac{x_1}{a_1 - r} + \frac{x_2}{a_2} + \frac{x_3}{a_2} \leqslant 1, \tag{4.18}$$

where a_1 and a_2 are relatively prime positive integers and $a \equiv r \pmod{b}$. From Corollary 1.1, (4.17) becomes the following:

$$\begin{aligned} Q_3 = {} & \frac{(a_1-r)a_2^2}{6} + \frac{(a_1-r)a_2}{2} + \frac{(a_1-r)}{3} + \frac{a_2(a_2+1)}{2} + a_2 + 1 \\ & + \sum_{k=0}^{a_2}\left(\sum_{n=0}^{a_2-k}\left\lfloor \frac{r}{a_2}(a_2-k-n)\right\rfloor\right). \end{aligned} \tag{4.19}$$

However, we are still interested in finding $\sum_{k=0}^{a_2}(\sum_{n=0}^{a_2-k}\lfloor \frac{r}{a_2}(a_2-k-n)\rfloor)$, which is always greater than or equal to zero but less than or equal to r. We let

$$\#\left\{(k,n) \in \mathbb{Z}_{\geqslant 0}\colon\ m_1 > \left\lfloor \frac{r}{a_2}(a_2-k-n)\right\rfloor \geqslant m_2\right\}$$

be the number of combinations of k and n satisfying $m_1 > \lfloor \frac{r}{a_2}(a_2-k-n)\rfloor \geqslant m_2$, then

$$\sum_{k=0}^{a_2}\left(\sum_{n=0}^{a_2-k}\left\lfloor\frac{r}{a_2}(a_2-k-n)\right\rfloor\right)=\left(\#\left\{(k,n)\in\mathbb{Z}_{\geqslant 0}\colon\ 2>\left\lfloor\frac{r}{a_2}(a_2-k-n)\right\rfloor\geqslant 1\right\}\right)$$
$$+2\left(\#\left\{(k,n)\in\mathbb{Z}_{\geqslant 0}\colon\ 3>\left\lfloor\frac{r}{a_2}(a_2-k-n)\right\rfloor\geqslant 2\right\}\right)$$
$$+\cdots+r\left(\#\left\{(k,n)\in\mathbb{Z}_{\geqslant 0}\colon\ \left\lfloor\frac{r}{a_2}(a_2-k-n)\right\rfloor=r\right\}\right)$$
$$=\left(\#\left\{(k,n)\in\mathbb{Z}_{\geqslant 0}\colon\ \left\lfloor\frac{r}{a_2}(a_2-k-n)\right\rfloor\geqslant 1\right\}\right)$$
$$+\left(\#\left\{(k,n)\in\mathbb{Z}_{\geqslant 0}\colon\ \left\lfloor\frac{r}{a_2}(a_2-k-n)\right\rfloor\geqslant 2\right\}\right)$$
$$+\cdots+\left(\#\left\{(k,n)\in\mathbb{Z}_{\geqslant 0}\colon\ \left\lfloor\frac{r}{a_2}(a_2-k-n)\right\rfloor=r\right\}\right).$$

Given two integers k and n where $a_2\geqslant k\geqslant 0$ and $(a_2-k)\geqslant n\geqslant 0$, we will first find the number of combinations of k and n satisfying $\lfloor\frac{r}{a_2}(a_2-k-n)\rfloor\geqslant 1$. Clearly $\lfloor\frac{r}{a_2}(a_2-k-n)\rfloor\geqslant 1$ if and only if $k+n\leqslant\frac{r-1}{r}a_2$. However, since $k+n$ is always an integer, what we really want is the integral part of $\frac{r-1}{r}a_2$.

$$\begin{aligned}\left\lfloor\frac{r-1}{r}a_2\right\rfloor&=a_2-\left\lceil\frac{a_2}{r}\right\rceil\\&=a_2-\left(\frac{a_2-w_1}{r}+1\right)\\&=\frac{r-1}{r}a_2-\frac{r-w_1}{r}.\end{aligned}\tag{4.20}$$

In finding the number of k and n satisfying $k+n\leqslant(\frac{r-1}{r}a_2-\frac{r-w_1}{r})$, we have the following:

If $k=0$, then $0\leqslant n\leqslant(\frac{r-1}{r}a_2-\frac{r-w_1}{r})$.
If $k=1$, then $0\leqslant n\leqslant(\frac{r-1}{r}a_2-\frac{r-w_1}{r}-1)$.
$\vdots$
If $k=(\frac{r-1}{r}a_2-\frac{r-w_1}{r}-1)$, then $0\leqslant n\leqslant 1$.
If $k=(\frac{r-1}{r}a_2-\frac{r-w_1}{r})$, then $n=0$.

As the value of k increases, the range of n decreases accordingly. Specifically, as k increases from zero to $(\frac{r-1}{r}a_2-\frac{r-w_1}{r})$, the range of values for n decreases from $(\frac{r-1}{r}a_2-\frac{r-w_1}{r}+1)$ to 1. Therefore, there are total of $\frac{1}{2}(\frac{r-1}{r}a_2-\frac{r-w_1}{r}+2)(\frac{r-1}{r}a_2-\frac{r-w_1}{r}+1)$ combinations of k and n satisfying $\lfloor\frac{r}{a_2}(a_2-k-n)\rfloor\geqslant 1$.

In finding the number of combinations of k and n satisfying $\lfloor\frac{r}{a_2}(a_2-k-n)\rfloor\geqslant 2$, we notice that $\lfloor\frac{r}{a_2}(a_2-k-n)\rfloor\geqslant 2$ if and only if $k+n\leqslant\frac{r-2}{r}a_2$. It comes naturally from (4.20) that $\lfloor\frac{r-2}{r}a_2\rfloor=(\frac{r-2}{r}a_2-\frac{r-w_2}{r})$. By using the same reasoning as in the previous case, we determine the number of combinations of k and n satisfying $\lfloor\frac{r}{a_2}(a_2-k-n)\rfloor\geqslant 2$ to be $\frac{1}{2}(\frac{r-2}{r}a_2-\frac{r-w_2}{r}+2)(\frac{r-2}{r}a_2-\frac{r-w_2}{r}+1)$.

In general, let $m \leqslant r-1$, then $\lfloor \frac{r}{a_2}(a_2-k-n)\rfloor \geqslant m$ if and only if $k+n \leqslant (\frac{r-m}{r}a_2 - \frac{r-w_m}{r})$. The number of combinations of k and n satisfying $\lfloor \frac{r}{a_2}(a_2-k-n)\rfloor \geqslant m$, where $m \leqslant r-1$, is $\frac{1}{2}(\frac{r-m}{r}a_2 - \frac{r-w_m}{r}+2)(\frac{r-m}{r}a_2 - \frac{r-w_m}{r}+1)$.

Notice that $\lfloor \frac{r}{a_2}(a_2-k-n)\rfloor = r$ if and only if $k=n=0$. Adding up all the combinations of k and n, we have

$$\sum_{k=0}^{a_2}\left(\sum_{n=0}^{a_2-k}\left\lfloor \frac{r}{a_2}(a_2-k-n)\right\rfloor\right) = \sum_{m=1}^{r-1}\left[\frac{1}{2}\left(\frac{r-m}{r}a_2 - \frac{r-w_m}{r}+2\right)\left(\frac{r-m}{r}a_2 - \frac{r-w_m}{r}+1\right)\right]+1,$$

and

$$\begin{aligned} Q_3 = {} & \frac{(a_1-r)a_2^2}{6} + \frac{(a_1-r)a_2}{2} + \frac{(a_1-r)}{3} + \frac{a_2(a_2+1)}{2} + a_2 + 1 \\ & + \sum_{m=1}^{r-1}\frac{1}{2}\left(\frac{r-m}{r}a_2 - \frac{r-w_m}{r}+2\right)\left(\frac{r-m}{r}a_2 - \frac{r-w_m}{r}+1\right)+1. \end{aligned}$$

Although we cannot calculate each w_m distinctively, we can evaluate the sum. Since $\gcd(a_1,a_2)=1$ and $\gcd(t,r)=1$, t is a generator of $\mathbb{Z}_r$, i.e. the cyclic group of r. For $1 \leqslant m \leqslant r-1$, $mt \pmod r$ becomes the set $\{1,2,3,\ldots,r-2,r-1\}$. In other words,

$$\{w_1, w_2, w_3, \ldots, w_{r-1}\} = \{1,2,3,\ldots,r-2,r-1\}. \tag{4.21}$$

The error of Corollary 1.1 is given by

$$\begin{aligned} E_1 = {} & \frac{a_1a_2^2}{6} + \frac{a_1a_2}{2} + \frac{a_1}{3} + \frac{a_2(a_2+1)}{2} + a_2 + 1 \\ & - \Bigg[\frac{(a_1-r)a_2^2}{6} + \frac{(a_1-r)a_2}{2} + \frac{(a_1-r)}{3} + \frac{a_2(a_2+1)}{2} + a_2 + 1 \\ & + \sum_{m=1}^{r-1}\frac{1}{2}\left(\frac{r-m}{r}a_2 - \frac{r-w_m}{r}+2\right)\left(\frac{r-m}{r}a_2 - \frac{r-w_m}{r}+1\right)+1\Bigg] \\ = {} & \frac{a_2^2r}{6} + \frac{a_2r}{2} + \frac{r}{3} - \sum_{m=1}^{r-1}\left(\frac{a_2^2m^2}{2r^2} - \frac{a_2^2m}{r} + \frac{a_2^2}{2} - \frac{a_2m}{2r} + \frac{a_2}{2}\right) \\ & - \sum_{m=1}^{r-1}\left(\frac{w_m^2}{2r^2} + \frac{a_2w_m}{r} + \frac{w_m}{2r}\right) + \sum_{m=1}^{r-1}\frac{a_2mw_m}{r^2} - 1 \\ = {} & \frac{a_2^2r}{6} + \frac{a_2r}{2} + \frac{r}{3} - \left(\frac{a_2^2}{2r^2}\cdot\frac{(r-1)r(2r-1)}{6} - \frac{a_2(r-1)r}{4r} + \frac{a_2(r-1)}{2}\right) \\ & - \left(\frac{(r-1)r(2r-1)}{12r^2} + \frac{a_2(r-1)}{2} + \frac{r-1}{4}\right) + \frac{a_2}{r^2}\sum_{m=1}^{r-1}mw_m - 1. \end{aligned}$$

If we apply Theorem 1.6, it becomes

$$\begin{aligned} s(a_2,a_1) &= \frac{a_2 r}{6} + \frac{r}{2} + \frac{r}{3a_2} - \left(\frac{a_2}{2r}\cdot\frac{(r-1)(2r-1)}{6} - \frac{r-1}{4} + \frac{r-1}{2}\right) \\ &\quad - \left(\frac{(r-1)(2r-1)}{12a_2 r} + \frac{r-1}{2} + \frac{r-1}{4a_2}\right) + \frac{1}{r^2}\sum_{m=1}^{r-1} m w_m - \frac{1}{a_2} \\ &\quad + \frac{1}{12}\left(\frac{1}{a_1 a_2} + \frac{a_1}{a_2} + \frac{a_2}{a_1}\right) - \frac{(a_2+1)^2}{4a_2} + \frac{3}{4a_2} \\ &= \frac{1}{12a_1a_2} + \frac{a_1}{12a_2} + \frac{a_2}{12a_1} - \frac{a_2}{12r} - \frac{r}{12a_2} - \frac{1}{12a_2 r} - \frac{r}{4} + \frac{1}{4} + \frac{1}{r^2}\sum_{m=1}^{r-1} m w_m. \end{aligned}$$

It follows that

$$\begin{aligned} 12a_1a_2\cdot s(a_2,a_1) &= a_1^2 + a_2^2 + 1 - \left(\frac{a_1 a_2^2}{r} + 3a_1a_2 r - 3a_1a_2 + a_1 r + \frac{a_1}{r}\right) + \frac{12a_1a_2}{r^2}\sum_{m=1}^{r-1} m w_m \\ &= a_1^2 + a_2^2 + 1 - \frac{a_2^2 + 3r(r-1)a_2 + r^2 + 1}{r} a_1 + \frac{12a_1a_2}{r^2}\sum_{m=1}^{r-1} m w_m. \qquad \Box \end{aligned}$$

Below are some special cases of Theorem 1.9. More special cases, including when $6 \geqslant r \geqslant -4$, are listed in [Ap].

Example 4.1. Given two relatively prime positive integers a and b:

If $a \equiv 7 \pmod b$ and $b \equiv t \pmod 7$ where $t = \pm 1$, ± 2, or ± 3, then

$$12ab\cdot s(b,a) = a^2 + b^2 + 1 - \frac{b^2 - t[(t^2-7)^2 - 6]b + 50}{7} a. \tag{4.22}$$

If $a \equiv 8 \pmod b$ and $b \equiv t \pmod 8$ where $t = \pm 1$ or ± 3, then

$$12ab\cdot s(b,a) = a^2 + b^2 + 1 - \frac{b^2 + t(5t^2 - 47)b + 65}{8} a. \tag{4.23}$$

If $a \equiv 9 \pmod b$ and $b \equiv t \pmod 9$ where $t = \pm 1$, ± 2, or ± 4, then

$$12ab\cdot s(b,a) = a^2 + b^2 + 1 - \frac{b^2 - t(t^4 - 21t^2 + 76)b + 82}{9} a. \tag{4.24}$$

If $a \equiv 10 \pmod b$ and $b \equiv t \pmod{10}$ where $t = \pm 1$ or ± 3, then

$$12ab\cdot s(b,a) = a^2 + b^2 + 1 - \frac{b^2 + 9t(t-3)(t+3)b + 101}{10} a. \tag{4.25}$$

Example 4.2. Given two relatively prime positive integers a and b:

If $a \equiv -5 \pmod b$ and $b \equiv t \pmod 5$ where $t = \pm 1$ or ± 2, then

$$12ab\cdot s(b,a) = a^2 + b^2 + 1 - \frac{b^2 + (4t^3 - 16t - 30)b + 26}{-5} a. \tag{4.26}$$

If $a \equiv -7 \pmod{b}$ and $b \equiv t \pmod{7}$ where $t = \pm 1$, $t = \pm 2$, or ± 3, then

$$12ab \cdot s(b,a) = a^2 + b^2 + 1 - \frac{b^2 - (t^5 - 14t^3 + 43t + 42)b + 50}{-7}a. \tag{4.27}$$

If $a \equiv -8 \pmod{b}$ and $b \equiv t \pmod{8}$ where $t = \pm 1$ or ± 3, then

$$12ab \cdot s(b,a) = a^2 + b^2 + 1 - \frac{b^2 + (5t^3 - 47t - 48)b + 65}{-8}a. \tag{4.28}$$

If $a \equiv -9 \pmod{b}$ and $b \equiv t \pmod{9}$ where $t = \pm 1$, $t = \pm 2$, or ± 4, then

$$12ab \cdot s(b,a) = a^2 + b^2 + 1 - \frac{b^2 - (t^5 - 21t^3 + 76t + 54)b + 82}{-9}a. \tag{4.29}$$

If $a \equiv -10 \pmod{b}$ and $b \equiv t \pmod{10}$ where $t = \pm 1$ or $t = \pm 3$, then

$$12ab \cdot s(b,a) = a^2 + b^2 + 1 - \frac{b^2 + (9t^3 - 81t - 60)b + 101}{-10}a. \tag{4.30}$$

Proof of Corollary 1.3. Given four positive integers a_1, b_1, a_2, and b_2 where $\gcd(a_1, b_1) = 1$ and $\gcd(a_2, b_2) = 1$, let $a_1 \equiv r \pmod{b_1}$ where $b_1 \equiv t \pmod{r}$, $a_2 \equiv r \pmod{b_2}$ where $b_2 \equiv -t \pmod{r}$, and $w_t \equiv mt \pmod{r}$, if $b_1 - 1 \geqslant r \geqslant 1$, $b_2 - 1 \geqslant r \geqslant 1$, $r - 1 \geqslant t \geqslant 1$, and $r - 1 \geqslant w_m \geqslant 1$, then from Theorem 1.9

$$12a_1b_1 \cdot s(b_1, a_1) = a_1^2 + b_1^2 + 1 - \frac{b_1^2 + 3r(r-1)b_1 + (r^2+1)}{r}a_1 + \frac{12a_1b_1}{r^2}\sum_{m=1}^{r-1} m w_m, \tag{4.31}$$

and

$$\begin{aligned}
12a_2b_2 \cdot s(b_2, a_2) &= a_2^2 + b_2^2 + 1 - \frac{b_2^2 + 3r(r-1)b_2 + (r^2+1)}{r}a_2 + \frac{12a_2b_2}{r^2}\sum_{m=1}^{r-1} m(r - w_m) \\
&= a_2^2 + b_2^2 + 1 - \frac{b_2^2 + 3r(r-1)b_2 + (r^2+1)}{r}a_2 + 6a_2b_2(r-1) \\
&\quad - \frac{12a_2b_2}{r^2}\sum_{m=1}^{r-1} m w_m \\
&= a_2^2 + b_2^2 + 1 - \frac{b_2^2 - 3r(r-1)b_2 + (r^2+1)}{r}a_2 - \frac{12a_2b_2}{r^2}\sum_{m=1}^{r-1} m w_m.
\end{aligned} \tag{4.32}$$

(4.31) becomes

$$-3r(r-1) + \frac{12}{r}\sum_{m=1}^{r-1} m w_m = \frac{r}{a_1b_1}\left(12a_1b_1 \cdot s(b_1, a_1) - a_1^2 - b_1^2 - 1\right) + \frac{r^2}{b_1} + \frac{1}{b_1} + b_1, \tag{4.33}$$

and (4.32) becomes

$$3r(r-1)-\frac{12}{r}\sum_{m=1}^{r-1}mw_m=\frac{r}{a_2b_2}\bigl(12a_2b_2\cdot s(b_2,a_2)-a_2^2-b_2^2-1\bigr)+\frac{r^2}{b_2}+\frac{1}{b_2}+b_2. \tag{4.34}$$

We add up (4.33) and (4.34),

$$\begin{aligned}0&=\frac{r}{a_1b_1}\bigl(12a_1b_1\cdot s(b_1,a_1)-a_1^2-b_1^2-1\bigr)+r^2+1+b_1^2\\&\quad+\frac{r}{a_2b_2}\bigl(12a_2b_2\cdot s(b_2,a_2)-a_2^2-b_2^2-1\bigr)+r^2+1+b_2^2\\0&=12\cdot s(b_1,a_1)-\frac{a_1}{b_1}-\frac{b_1}{a_1}-\frac{1}{a_1b_1}+12\cdot s(b_2,a_2)-\frac{a_2}{b_2}-\frac{b_2}{a_2}-\frac{1}{a_2b_2}\\&\quad+\frac{1}{r}\left(\frac{r^2}{b_1}+\frac{1}{b_1}+b_1+\frac{r^2}{b_2}+\frac{1}{b_2}+b_2\right).\end{aligned}$$

And we have the desired result,

$$\begin{aligned}s(b_1,a_1)+(b_2,a_2)&=\frac{1}{12}\left(\frac{b_2}{a_2}+\frac{a_2}{b_2}+\frac{1}{a_2b_2}+\frac{b_1}{a_1}+\frac{a_1}{b_1}+\frac{1}{a_1b_1}\right)\\&\quad-\frac{r^2+1}{12r}\left(\frac{1}{b_1}+\frac{1}{b_2}\right)-\frac{b_1+b_2}{12r}.\qquad\square\end{aligned}$$

Proof of Theorem 1.10. Given two relatively prime positive integers a and b, let $b\equiv u \pmod a$, $b\equiv f \pmod u$, and $z_m\equiv mf \pmod u$, where $a-1\geqslant u\geqslant 1$, $u-1\geqslant f\geqslant 1$, and $u-1\geqslant z_m\geqslant 1$, we can prove Theorem 1.10 in the same fashion as we proved Theorem 1.9, by means of lattice points enumeration. However in creating such a proof we need to modify Corollary 1.1. To make our case simpler, we construct a alternative proof, based on the following definition of the Dedekind sum:

$$s(b,a)=\sum_{q=1}^{a-1}\frac{q}{a}\left(\frac{bq}{a}-\left\lfloor\frac{bq}{a}\right\rfloor-\frac{1}{2}\right). \tag{4.35}$$

(4.35) is simply another way of defining the Dedekind sum.

$$\begin{aligned}s(b,a)&=\frac{b}{a^2}\sum_{q=1}^{a-1}q^2-\frac{1}{a}\sum_{q=1}^{a-1}\left(q\left\lfloor\frac{(b-u)}{a}q+\frac{u}{a}q\right\rfloor\right)-\frac{1}{2a}\sum_{q=1}^{a-1}q\\&=\frac{b(a-1)(2a-1)}{6a}-\frac{1}{a}\sum_{q=1}^{a-1}\left(\frac{b-u}{a}q^2\right)-\frac{1}{a}\sum_{q=1}^{a}\left(q\left\lfloor\frac{u}{a}q\right\rfloor\right)\\&\quad+u-\frac{a-1}{4}.\end{aligned} \tag{4.36}$$

The last equality comes from the fact that $a\mid(b-u)$ and $\lfloor\frac{b-u}{a}q^2\rfloor=\frac{b-u}{a}q^2$. In solving $\sum_{q=1}^{a}(q\lfloor\frac{u}{a}q\rfloor)$, we first find all the q values where $2>\lfloor\frac{u}{a}q\rfloor\geqslant 1$. We notice $\lfloor\frac{u}{a}q\rfloor\geqslant 1$ if and only if $q\geqslant\frac{a}{u}$. However since q is an integer but $a\nmid u$, it will be more convenient for us to write it as $\lfloor\frac{u}{a}q\rfloor\geqslant 1$ if and only if $q\geqslant\lfloor\frac{a}{u}\rfloor+1$. Similarly, since $a\nmid 2u$, we obtain $\lfloor\frac{u}{a}q\rfloor<2$ if and only if $q<\lfloor\frac{2a}{u}\rfloor$.

Let m be an arbitrary positive integer and $(u-2)\geqslant m\geqslant 1$, then $m+1>\lfloor\frac{u}{a}q\rfloor\geqslant m$ if and only if $\frac{(m+1)a}{u}>q\geqslant\frac{ma}{u}$. Since $u\nmid(m+1)a$ and $u\nmid ma$, we can replace $\frac{(m+1)a}{u}$ and $\frac{ma}{u}$ with $\lfloor\frac{(m+1)a}{u}\rfloor$ and

$\lfloor \frac{ma}{u} \rfloor + 1$, respectively. Hence we conclude that $m+1 > \lfloor \frac{u}{a}q \rfloor \geqslant m$ if and only if $\lfloor \frac{(m+1)a}{u} \rfloor > q \geqslant \lfloor \frac{ma}{u} \rfloor + 1$. However, we still need to consider the case where $u > \lfloor \frac{u}{a}q \rfloor \geqslant u-1$. Clearly $\lfloor \frac{u}{a}q \rfloor \geqslant u-1$ if and only if $q \geqslant \frac{(u-1)a}{u}$. Since $u \nmid (u-1)a$, we can safely state that $\lfloor \frac{u}{a}q \rfloor \geqslant u-1$ if and only if $q \geqslant \lfloor \frac{(u-1)a}{u} \rfloor + 1$. Notice that $\lfloor \frac{u}{a}q \rfloor = u$ if and only if $q = a$.

$$\begin{aligned}
\sum_{q=1}^{a}\left(q\left\lfloor \frac{u}{a}q \right\rfloor\right) &= \sum_{q=\lfloor \frac{a}{u} \rfloor+1}^{\lfloor \frac{2a}{u} \rfloor} q + \sum_{q=\lfloor \frac{2a}{u} \rfloor+1}^{\lfloor \frac{3a}{u} \rfloor} 2q + \sum_{q=\lfloor \frac{3a}{u} \rfloor+1}^{\lfloor \frac{4a}{u} \rfloor} 3q + \cdots + \sum_{q=\lfloor \frac{(u-1)a}{u} \rfloor+1}^{a-1} (u-1)q + au \\
&= \sum_{q=\lfloor \frac{a}{u} \rfloor+1}^{a} q + \sum_{q=\lfloor \frac{2a}{u} \rfloor+1}^{a} q + \cdots + \sum_{q=\lfloor \frac{(u-1)a}{u} \rfloor+1}^{a} q + a \\
&= \sum_{m=1}^{u-1}\left(\sum_{q=\lfloor \frac{am}{u} \rfloor+1}^{a} q\right) + a \\
&= \frac{1}{2}\sum_{m=1}^{u-1}\left[\left(a - \left\lfloor \frac{am}{u} \right\rfloor\right)\left(a + \left\lfloor \frac{am}{u} \right\rfloor + 1\right)\right] + a \\
&= \frac{a(a+1)(u-1)}{2} - \frac{1}{2}\sum_{m=1}^{u-1}\left(\frac{am}{u} - \frac{z_m}{u}\right) - \frac{1}{2}\sum_{m=1}^{u-1}\left(\frac{am}{u} - \frac{z_m}{u}\right)^2 + a.
\end{aligned}$$

The last equality comes from the fact that $\lfloor \frac{am}{u} \rfloor = \frac{am}{u} - \frac{z_m}{u}$. As we did to the sum of w_m in the proof of Theorem 1.9, the sum of z_m can also be computed in a similar manner. In fact, f is a generator of $\mathbb{Z}_u$, i.e. the cyclic group of u. For $1 \leqslant m \leqslant u-1$, $mf \pmod u$ becomes the set $\{1, 2, 3, \ldots, u-2, u-1\}$. In other words,

$$\{z_1, z_2, z_3, \ldots, z_{u-1}\} = \{1, 2, 3, \ldots, u-2, u-1\}, \tag{4.37}$$

$$\begin{aligned}
\sum_{q=1}^{a}\left(q\left\lfloor \frac{u}{a}q \right\rfloor\right) = \frac{a(a+1)(u-1)}{2} - \frac{a(u-1)}{4} + \frac{u-1}{4} - \frac{a^2(2u-1)(u-1)}{12u} \\
- \frac{(u-1)(2u-1)}{12u} + \frac{a}{u^2}\sum_{m=1}^{u-1} m z_m + a.
\end{aligned} \tag{4.38}$$

We plug (4.38) back into (4.36) and have

$$\begin{aligned}
s(b,a) = \frac{b(a-1)(2a-1)}{6a} - \frac{b-u}{a}\cdot\frac{(2a-1)(a-1)}{6} - \frac{(a+1)(u-1)}{2} + \frac{u-1}{4} - \frac{u-1}{4a} \\
+ \frac{a(2u-1)(u-1)}{12u} + \frac{(u-1)(2u-1)}{12au} - \frac{1}{u^2}\sum_{m=1}^{u-1} m z_m - 1 + u - \frac{a-1}{4} \\
= \frac{a}{12u} + \frac{u}{12a} + \frac{1}{12au} + \frac{u}{4} - \frac{1}{2} - \frac{1}{u^2}\sum_{m=1}^{u-1} m z_m.
\end{aligned}$$

Then we have the desired result,

$$12ab\cdot s(b,a)=\frac{a^2+u^2+1}{u}b+3ab(u-2)-\frac{12ab}{u^2}\sum_{m=1}^{u-1}mz_m. \qquad \square$$

Proof of Corollary 1.5. We have two relatively prime positive integers a and b where $b\equiv r \pmod a$, $b\equiv f \pmod u$, and $z_m\equiv mf \pmod u$.

If $f=1$, then $z_m\equiv m \pmod u$. Since in our case m ranges from one to $u-1$, it is clear that $z_m=m$. From Theorem 1.10, we have

$$\begin{aligned}12ab\cdot s(b,a)&=\frac{a^2+u^2+1}{u}b+3ab(u-2)-\frac{2ab(u-1)(2u-1)}{u}\\&=\frac{a^2+u^2+1}{u}b-ab\left(\frac{2}{u}+u-3\right)-3ab.\end{aligned}$$

Similarly, if $f=-1$, then $z_m=u-m$ and it follows from Theorem 1.10,

$$\begin{aligned}12ab\cdot s(b,a)&=\frac{a^2+u^2+1}{u}b+3ab(u-2)-\frac{2ab(u-1)(u+1)}{u}\\&=\frac{a^2+u^2+1}{u}b+ab\left(\frac{2}{u}+u-3\right)-3ab. \qquad \square\end{aligned}$$

Another Proof of the Reciprocity Law. We introduce another proof of the Reciprocity Law that comes directly from the results of Theorem 1.9 and Theorem 1.10.

Let $a\equiv r \pmod b$, $b\equiv t \pmod r$, and $w_m\equiv mt \pmod r$, Theorem 1.9 states

$$12ab\cdot s(b,a)=a^2+b^2+1-\frac{b^2+3r(r-1)b+(r^2+1)}{r}a+\frac{12ab}{r^2}\sum_{m=1}^{r-1}mw_m. \tag{4.39}$$

Similarly, in Theorem 1.10, if we exchange the positions of the two relatively prime positive integers a and b and let $a\equiv u \pmod b$ and $b\equiv f \pmod u$ and $z_m\equiv mf \pmod u$, then

$$12ab\cdot s(a,b)=\frac{b^2+u^2+1}{u}a+3ab(u-2)-\frac{12ab}{u^2}\sum_{m=1}^{u-1}mz_m. \tag{4.40}$$

We combine (4.39) and (4.40) to get

$$\begin{aligned}12ab\cdot s(b,a)+12ab\cdot s(a,b)={}&a^2+b^2+1-\frac{b^2+3r(r-1)b+(r^2+1)}{r}a\\&+\frac{12ab}{r^2}\sum_{m=1}^{r-1}mw_m+\frac{b^2+u^2+1}{u}a+3ab(u-2)\\&-\frac{12ab}{u^2}\sum_{m=1}^{u-1}mz_m.\end{aligned} \tag{4.41}$$

Since in this case $r = u$, $t = f$, and $w_m = z_m$, (4.41) becomes

$$12ab \cdot s(b,a) + 12ab \cdot s(a,b) = a^2 + b^2 + 1 - 3ab. \qquad \square$$

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