Science in China Series A: Mathematics
Dec., 2009, Vol. 52, No. 12, 2792–2828
www.scichina.com math.scichina.com

Classification of gradient space of dimension 8 by a reducible $s\ell(2, \mathrm{C})$ action

Dedicated to Professor ZHONG TongDe on the occasion of his 80th birthday

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Abstract This paper deals with a reducible $s\ell(2, \mathbf{C})$ action on the formal power series ring. The purpose of this paper is to confirm a special case of the Yau conjecture: Suppose that $s\ell(2, \mathbf{C})$ acts on the formal power series ring via (1.1). Then $I(f) = (\ell_{i_1}) \oplus (\ell_{i_2}) \oplus \cdots \oplus (\ell_{i_s})$ modulo some one dimensional $s\ell(2, \mathbf{C})$ representations where (ℓ_i) is an irreducible $s\ell(2, \mathbf{C})$ representation of ℓ_i dimension and $\{\ell_{i_1}, \ell_{i_2}, \ldots, \ell_{i_s}\} \subseteq \{\ell_1, \ell_2, \ldots, \ell_r\}$. Unlike classical invariant theory which deals only with irreducible action and 1-dimensional representations, we treat the reducible action and higher dimensional representations successively.

Keywords: **principal ideal ring, Steinberg group, intermediate subgroups**

MSC(2000): **17B10 (14B05, 17B20)**

1 Introduction

In [1, 2], the first author developed a new theory which connects isolated singularities on the one hand, and finite dimensional Lie algebras on the other hand. These Lie algebras are called Yau algebras in [3–5]. They are very useful in studying isolated hypersurface singularities. For example, in [6] Seeley and the first author showed that one can construct a continuous numerical invariant from Yau algebras. Xu and the first author showed that Yau algebras can also be used to detect the quasi-homogeneity of the original singularities. Yau algebras are not arbitrary finite dimensional Lie algebra. It was shown in [2] that these algebras are solvable Lie algebras. Since every Lie algebra is a semidirect product of semi-simple Lie algebra and a solvable Lie algebra, in proving that his Lie algebras are solvable, the first author only needs to show that his Lie algebras do not contain $s\ell(2, \mathbf{C})$. This leads him to study $s\ell(2, \mathbf{C})$ action via derivations preserving m-adic filtration on the formal power series ring. In [7], the first author classifies all these actions.

Theorem (Yau). *Let $L = s\ell(2, \mathbf{C})$ act on the formal power series ring via derivations pre-*

Received April 29, 2008; accepted October 19, 2008; published online September 29, 2009
DOI: 10.1007/s11425-009-0047-1
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This work was supported by National Natural Science Foundation of China (Grant No. 10731030) and PSSCS of Shanghai

Citation: Yau S S T, Yu Y, Zuo H Q. Classification of gradient space of dimension 8 by a reducible $s\ell(2, \mathrm{C})$ action. Sci China Ser A, 2009, 52(12): 2792–2828, DOI: 10.1007/s11425-009-0047-1

serving m-adic filtration where m is the maximal ideal (i.e., $L(m^k)\subseteq m^k$). Then there exists a coordinate $x_1, x_2, \dots, x_{\ell_1}, x_{\ell_1+1}, \dots, x_{\ell_1+\ell_2}, \dots, x_{\ell_1+\cdots+\ell_{r-1}+1}, \dots, x_{\ell_1+\cdots+\ell_r}, x_{\ell_1+\cdots+\ell_r+1}, \dots, x_n$ such that the action of L is given by

$$\tau = D_{\tau,1}+\cdots+D_{\tau,r},\quad X_+ = D_{X_+,1}+\cdots+D_{X_+,r},\quad X_- = D_{X_-,1}+\cdots+D_{X_-,r}, \tag{1.1}$$

where

$$\begin{aligned}
D_{\tau,1} &= (\ell_i-1)x_{\ell_1+\cdots+\ell_{i-1}+1}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_{i-1}+1}} + (\ell_i-3)x_{\ell_1+\cdots+\ell_{i-1}+2}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_{i-1}+2}}\\
&\quad + \cdots + (-(\ell_i-3))x_{\ell_1+\cdots+\ell_i-1}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_i-1}} + (-(\ell_i-1))x_{\ell_1+\cdots+\ell_i}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_i}},\\
D_{X_+,i} &= (\ell_i-1)x_{\ell_1+\cdots+\ell_{i-1}+1}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_{i-1}+2}}\\
&\quad + \cdots + (j(\ell_i-j))x_{\ell_1+\cdots+\ell_{i-1}+j}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_{i-1}+j+1}}\\
&\quad + \cdots + (\ell_i-1)x_{\ell_1+\cdots+\ell_i-1}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_i}},\\
D_{X_-,i} &= x_{\ell_1+\cdots+\ell_{i-1}+2}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_{i-1}+1}} + \cdots + x_{\ell_1+\cdots+\ell_{i-1}+j}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_{i-1}+j-1}}\\
&\quad + \cdots + x_{\ell_1+\cdots+\ell_i}\frac{\partial}{\partial x_{\ell_1+\cdots+\ell_i-1}}.
\end{aligned}$$

Let f be a homogeneous polynomial of degree $k+1\geqslant 3$ in n variables, and let $I(f)$ be the vector space spanned by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}$. In 1985, the first author gave the following conjecture about the structure of $I(f)$ if $I(f)$ is an $s\ell(2,\mathbf{C})$ module.

Yau's conjecture. Suppose that $s\ell(2,\mathbf{C})$ acts on the formal power series ring. The purpose of this paper is to confirm a special case of Yau conjecture: suppose that $s\ell(2,\mathbf{C})$ acts on the formal power series ring via (1.1). Then $I(f) = (\ell_{i_1})\oplus(\ell_{i_2})\oplus\cdots\oplus(\ell_{i_s})$ modulo some one-dimensional $s\ell(2,\mathbf{C})$ representations where (ℓ_i) is an irreducible $s\ell(2,\mathbf{C})$ representation of ℓ_i dimension and $\{\ell_{i_1},\ell_{i_2},\dots,\ell_{i_s}\}\subseteq\{\ell_1,\ell_2,\dots,\ell_r\}$.

If the $s\ell(2,\mathbf{C})$ action is irreducible, i.e., $\ell_1=n$ in the above theorem of Yau, then Yau conjecture was confirmed by Sampson-Yau-Yu[8]. Actually, they proved that f must be an invariant polynomial if $I(f)$ is an $s\ell(2,\mathbf{C})$ module. This special statement was proved independently by Kempf[9]. In fact he proved that this special statement is true also for other semisimple Lie algebras as well. In [10], Yau's conjecture was proved for any $s\ell(2,\mathbf{C})$-action for $n\leqslant 5$. In a special case of $n=6$, the Yau's conjecture was confirmed by the second author[3]. The purpose of this paper is to confirm this conjecture for a special case of $n=8$.

Main theorem. *Let $s\ell(2,\mathbf{C})$ act on the formal power series ring in 8 variables via*

$$\begin{aligned}
\tau &= 3x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} - x_3\frac{\partial}{\partial x_3} - 3x_4\frac{\partial}{\partial x_4},\\
X_+ &= 3x_1\frac{\partial}{\partial x_2} + 4x_2\frac{\partial}{\partial x_3} + 3x_3\frac{\partial}{\partial x_4},\\
X_- &= x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3}.
\end{aligned}$$

Let f be a homogeneous polynomial of degree $k+l$ in 8 variables where $k \geqslant 2$. If $I(f) = \langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_8} \rangle$ is an $s\ell(2,\mathbf{C})$ submodule and $\dim I = 8$, then f is an $s\ell(2,\mathbf{C})$ invariant polynomial in $x_1, x_2, \dots, x_8$ variables and $I = (4) \oplus (1) \oplus (1) \oplus (1) \oplus (1)$. Moreover we have $X_- \frac{\partial f}{\partial x_1} = 0$, $X_- \frac{\partial f}{\partial x_2} = -\frac{\partial f}{\partial x_1}$, $X_- \frac{\partial f}{\partial x_3} = -\frac{\partial f}{\partial x_2}$, $X_- \frac{\partial f}{\partial x_4} = -\frac{\partial f}{\partial x_3}$, $X_- \frac{\partial f}{\partial x_5} = X_- \frac{\partial f}{\partial x_6} = X_- \frac{\partial f}{\partial x_7} = X_- \frac{\partial f}{\partial x_8} = 0$ and $X_+ \frac{\partial f}{\partial x_1} = -3\frac{\partial f}{\partial x_2}$, $X_+ \frac{\partial f}{\partial x_2} = -4\frac{\partial f}{\partial x_3}$, $X_+ \frac{\partial f}{\partial x_3} = -3\frac{\partial f}{\partial x_4}$, $X_+ \frac{\partial f}{\partial x_4} = 0$, $X_+ \frac{\partial f}{\partial x_5} = X_+ \frac{\partial f}{\partial x_6} = X_+ \frac{\partial f}{\partial x_7} = X_+ \frac{\partial f}{\partial x_8} = 0$.

2 Some lemmas

Lemma 1. *Let f be a polynomial in $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ and x_8 variables. Suppose $\frac{\partial f}{\partial x_6} = r_1 \frac{\partial f}{\partial x_5}$, $\frac{\partial f}{\partial x_7} = r_2 \frac{\partial f}{\partial x_5}$ and $\frac{\partial f}{\partial x_8} = r_3 \frac{\partial f}{\partial x_5}$. Then there exists a polynomial $g(y_1, y_2, y_3, y_4, y_5)$ such that $f(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = g(x_1, x_2, x_3, x_4, x_5 + r_1 x_6 + r_2 x_7 + r_3 x_8)$.*

Proof. Let

$$\begin{aligned} &y_1 = x_1, \quad y_2 = x_2, \quad y_3 = x_3, \quad y_4 = x_4, \\ &y_5 = x_5 + r_1 x_6 + r_2 x_7 + r_3 x_8, \quad y_6 = x_6, \quad y_7 = x_7, \quad y_8 = x_8. \end{aligned}$$

Set $g(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) = f(y_1, y_2, y_3, y_4, y_5 - r_1 y_6 - r_2 y_7 - r_3 y_8, y_6, y_7, y_8)$. We claim that g depends only on y_1, y_2, y_3, y_4 and y_5 variables.

$$\begin{aligned} \frac{\partial g}{\partial y_6} &= \frac{\partial f}{\partial x_5}(y_1, y_2, y_3, y_4, y_5 - r_1 y_6 - r_2 y_7 - r_3 y_8, y_6, y_7, y_8) \frac{\partial x_5}{\partial y_6} \\ &\quad + \frac{\partial f}{\partial x_6}(y_1, y_2, y_3, y_4, y_5 - r_1 y_6 - r_2 y_7 - r_3 y_8, y_6, y_7, y_8) \frac{\partial x_6}{\partial y_6} \\ &= -r_1 \frac{\partial f}{\partial x_5}(y_1, y_2, y_3, y_4, y_5 - r_1 y_6 - r_2 y_7 - r_3 y_8, y_6, y_7, y_8) \\ &\quad + \frac{\partial f}{\partial x_6}(y_1, y_2, y_3, y_4, y_5 - r_1 y_6 - r_2 y_7 - r_3 y_8, y_6, y_7, y_8) \\ &= 0. \end{aligned}$$

Similarly $\frac{\partial g}{\partial y_7} = 0$ and $\frac{\partial g}{\partial y_8} = 0$.

Thus g depends only on y_1, y_2, y_3, y_4 and y_5 variables.

Lemma 2. *Let $f^m_{k+1}(x_1, x_2, x_3, x_5, x_6, x_7, x_8)$ be a nonzero homogeneous polynomial of degree $k+1 (k \geqslant 2)$, weight m ($m \geqslant 0$) and in $x_1, x_2, x_3, x_5, x_6, x_7, x_8$ variables.* $\mathrm{wt}(x_1) = 3$, $\mathrm{wt}(x_2) = 1$, $\mathrm{wt}(x_3) = -1$, $\mathrm{wt}(x_i) = 0$, $5 \leqslant i \leqslant 8$ *and* $X_- = x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3}$. *Write $f^m_{k+1} = \sum_{\beta \geqslant 0} g^{m+\beta}_{k+1-\beta}(x_1, x_2, x_5, x_6, x_7, x_8) x_3^\beta$ where $g^{m+\beta}_{k+1-\beta}(x_1, x_2, x_5, x_6, x_7, x_8)$ is a homogeneous polynomial of degree $k+1-\beta$, weight $m+\beta$ and in $x_1, x_2, x_5, x_6, x_7, x_8$ variables. Let β_0 be the largest integer such that $g^{m+\beta_0}_{k+1-\beta_0}(x_1, x_2, x_5, x_6, x_7, x_8) \neq 0$. If $\beta_0 = 0$ and $m = 0$, then $f^m_{k+1}(x_1, x_2, x_3, x_5, x_6, x_7, x_8) = f^0_{k+1}(x_5, x_6, x_7, x_8)$, otherwise either $X_-^{\beta_0} \frac{\partial f^m_{k+1}}{\partial x_1}$ or $X_-^{\beta_0} \frac{\partial f^m_{k+1}}{\partial x_2}$ is not zero.*

Proof. If $\beta_0 \neq 0$ or $m \neq 0$, since $\mathrm{wt}(x_3) = -1$, $\mathrm{wt}(x_1) = 3$, $\mathrm{wt}(x_2) = 1$, $\mathrm{wt}(x_j) = 0$, $5 \leqslant j \leqslant 8$, then either $\frac{\partial g^{m+\beta_0}_{k+1-\beta_0}}{\partial x_1}$ or $\frac{\partial g^{m+\beta_0}_{k+1-\beta_0}}{\partial x_2}$ is nonzero.

Without loss of generality, we suppose $\frac{\partial g^{m+\beta_0}_{k+1-\beta_0}}{\partial x_1} \neq 0$. $\frac{\partial f^m_{k+1}}{\partial x_1} = \sum_{0 \leqslant \beta \leqslant \beta_0} \frac{\partial g^{m+\beta}_{k+1-\beta}}{\partial x_1} x_3^\beta$.

$$X_-^{\beta_0}\left(\frac{\partial f^m_{k+1}}{\partial x_1}\right) = \sum_{0 \leqslant \beta \leqslant \beta_0 - 1} X_-^{\beta_0}\left(\frac{\partial g^{m+\beta}_{k+1-\beta}}{\partial x_1} \cdot x_3^\beta\right) + X_-^{\beta_0}\left(\frac{\partial g^{m+\beta_0}_{k+1-\beta_0}}{\partial x_1} \cdot x_3^{\beta_0}\right)$$

$$= \sum_{0\leqslant\beta\leqslant\beta_0-1}\left[\sum_{i=0}^{\beta_0}\binom{\beta_0}{i}X_-^{\beta_0-i}\left(\frac{\partial g_{k+1-\beta}^{m+\beta}}{\partial x_1}\right)X_-^i(x_3^\beta)\right]$$
$$+\sum_{i=0}^{\beta_0}\binom{\beta_0}{i}X_-^{\beta_0-i}\left(\frac{\partial g_{k+1-\beta_0}^{m+\beta_0}}{\partial x_1}\right)X_-^i(x_3^{\beta_0})$$
$$= \sum_{0\leqslant\beta\leqslant\beta_0-1}\left[\sum_{i=0}^{\beta}\binom{\beta_0}{i}X_-^{\beta_0-i}\left(\frac{\partial g_{k+1-\beta}^{m+\beta}}{\partial x_1}\right)X_-^i(x_3^\beta)\right]$$
$$+\sum_{i=0}^{\beta_0}\binom{\beta_0}{i}\beta_0(\beta_0-1)\cdots(\beta_0-i+1)X_-^{\beta_0-i}\left(\frac{\partial g_{k+1-\beta_0}^{m+\beta_0}}{\partial x_1}\right)x_3^{\beta_0-i}x_4^i$$
$$= \sum_{0\leqslant\beta\leqslant\beta_0-1}\sum_{i=0}^{\beta}\binom{\beta_0}{i}\beta(\beta-1)\cdots(\beta-i+1)X_-^{\beta_0-i}\left(\frac{\partial g_{k+1-\beta}^{m+\beta}}{\partial x_1}\right)x_3^{\beta-i}x_4^i$$
$$+\sum_{i=0}^{\beta_0-1}\binom{\beta_0}{i}\beta_0(\beta_0-1)\cdots(\beta_0-i+1)X_-^{\beta_0-i}\left(\frac{\partial g_{k+1-\beta_0}^{m+\beta_0}}{\partial x_1}\right)x_3^{\beta_0-i}x_4^i$$
$$+\beta_0!\frac{\partial g_{k+1-\beta_0}^{m+\beta_0}}{\partial x_1}x_4^{\beta_0}.$$

It is easy to see that the first two terms above do not have factor $x_4^{\beta_0}$, and since $\frac{\partial g_{k+1-\beta_0}^{m+\beta_0}}{\partial x_1}\neq 0$, $X_-^{\beta_0}(\frac{\partial f_{k+1}^m}{\partial x_1})\neq 0$.

If $\beta_0=0$, $m=0$, then it is easy to see that $f_{k+1}^m(x_1,x_2,x_3,x_5,x_6,x_7,x_8)=f_{k+1}^0(x_5,x_6,x_7,x_8)$.

Corollary 3. *Let $f_{k+1}^m(x_1,x_2,x_3)$ be a nonzero homogeneous polynomial of degree $k+1$ $(k\geqslant 2)$, weight m $(m\geqslant 0)$ and in x_1,x_2,x_3 variables.* $\mathrm{wt}(x_1)=3$, $\mathrm{wt}(x_2)=1$, $\mathrm{wt}(x_3)=-1$, *and $X_-=x_2\frac{\partial}{\partial x_1}+x_3\frac{\partial}{\partial x_2}+x_4\frac{\partial}{\partial x_3}$. Write $f_{k+1}^m=\sum_{\beta\geqslant 0}b_\beta x_1^{\frac{m-k-1+2\beta}{2}}\cdot x_2^{\frac{3k+3-4\beta-m}{2}}\cdot x_3^\beta$. Let β_0 be the largest integer such that $b_{\beta_0}\neq 0$, then either $X_-^{\beta_0}\frac{\partial f_{k+1}^m}{\partial x_1}$ or $X_-^{\beta_0}\frac{\partial f_{k+1}^m}{\partial x_2}$ is not zero.*

Proof. This is an immediate consequence of Lemma 2.

3 Proof of main theorem

Theorem 4. *Suppose that $s\ell(2,\mathbf{C})$ acts on the space of homogeneous polynomials of degree $k\geqslant 2$ in $x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8$ via*

$$\tau=3x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}-x_3\frac{\partial}{\partial x_3}-3x_4\frac{\partial}{\partial x_4},\quad X_+=3x_1\frac{\partial}{\partial x_2}+4x_2\frac{\partial}{\partial x_3}+3x_3\frac{\partial}{\partial x_4},$$
$$X_-=x_2\frac{\partial}{\partial x_1}+x_3\frac{\partial}{\partial x_2}+x_4\frac{\partial}{\partial x_3}.$$

Suppose that the weight of x_i is given by the corresponding coefficient in the expression of τ above, i.e.,

$$\mathrm{wt}(x_1)=3,\quad \mathrm{wt}(x_2)=1,\quad \mathrm{wt}(x_3)=-1,\quad \mathrm{wt}(x_4)=-3,\quad \mathrm{wt}(x_j)=0,\quad j=5,6,7,8.$$

Let I be the complex vector space spanned by $\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},\frac{\partial f}{\partial x_3},\frac{\partial f}{\partial x_4},\frac{\partial f}{\partial x_5},\frac{\partial f}{\partial x_6},\frac{\partial f}{\partial x_7}$ and $\frac{\partial f}{\partial x_8}$, where f is a homogeneous polynomial of degree $k+1$. If I is an $s\ell(2,\mathbf{C})$-submodule and $\dim I=8$, *then f is an $s\ell(2,\mathbf{C})$ invariant polynomial in $x_1,x_2,x_3,x_4,x_5,x_6,x_7$ and x_8 variables and $I=$*

$(4) \oplus (1) \oplus (1) \oplus (1) \oplus (1)$. *Moreover we have*

$$X_- \frac{\partial f}{\partial x_1} = 0, \quad X_- \frac{\partial f}{\partial x_2} = -\frac{\partial f}{\partial x_1}, \quad X_- \frac{\partial f}{\partial x_3} = -\frac{\partial f}{\partial x_2}, \quad X_- \frac{\partial f}{\partial x_4} = -\frac{\partial f}{\partial x_3},$$
$$X_- \frac{\partial f}{\partial x_5} = X_- \frac{\partial f}{\partial x_6} = X_- \frac{\partial f}{\partial x_7} = X_- \frac{\partial f}{\partial x_8} = 0$$

and

$$X_+ \frac{\partial f}{\partial x_1} = -3\frac{\partial f}{\partial x_2}, \quad X_+ \frac{\partial f}{\partial x_2} = -4\frac{\partial f}{\partial x_3}, \quad X_+ \frac{\partial f}{\partial x_3} = -3\frac{\partial f}{\partial x_4}, \quad X_+ \frac{\partial f}{\partial x_4} = 0,$$
$$X_+ \frac{\partial f}{\partial x_5} = X_+ \frac{\partial f}{\partial x_6} = X_+ \frac{\partial f}{\partial x_7} = X_+ \frac{\partial f}{\partial x_8} = 0.$$

Proof. **Case 1.** $I = (8)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $7, 5, 3, 1, -1, -3, -5, -7$. Write $f = \sum_{i=-\infty}^{\infty} f_{k+1}^i$, where f_{k+1}^i is a homogeneous polynomial of degree $k+1$ and weight i.

For i being an odd integer, $\mathrm{wt}\frac{\partial f_{k+1}^i}{\partial x_j}$ is an even integer for all $1 \leqslant j \leqslant 4$. Hence $\frac{\partial f_{k+1}^i}{\partial x_j} = 0$ for all $1 \leqslant j \leqslant 4$. It follows that f_{k+1}^i depends only on x_5, x_6, x_7, x_8 variables. Since weights of x_5, x_6, x_7, x_8 are zero, we conclude that $f_{k+1}^i = 0$. Thus we have $f = \sum_{i=-\infty}^{\infty} f_{k+1}^{2i}$.

Since $\mathrm{wt}\frac{\partial f_{k+1}^{2i}}{\partial x_j} = 2i$ is an even integer for all $5 \leqslant j \leqslant 8$, hence $\frac{\partial f_{k+1}^{2i}}{\partial x_j} = 0$. It follows that $\frac{\partial f}{\partial x_j} = 0$ for all $5 \leqslant j \leqslant 8$. So $\dim I \leqslant 4$. This contradicts our hypothesis $I = (8)$. We conclude that Case 1 cannot occur.

Case 2. $I = (7) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $6, 4, 2, 0, -2, -4, -6$.

For i is an even integer, $\mathrm{wt}\frac{\partial f_{k+1}^i}{\partial x_j}$ is an odd integer for all $1 \leqslant j \leqslant 4$. It follows that f_{k+1}^i depends only on x_5, x_6, x_7, x_8 variables. Since $\mathrm{wt}(x_j) = 0$ for all $5 \leqslant j \leqslant 8$, $f_{k+1}^i = 0$ for $i \neq 0$ and $f_{k+1}^0 = f_{k+1}^0(x_5, x_6, x_7, x_8)$. Since $\frac{\partial f_{k+1}^0}{\partial x_j} \in (1)$ for all $5 \leqslant j \leqslant 8$, in view of Lemma 1, there exist constants r_1, r_2, r_3, r_4 such that $f_{k+1}^0 = (r_1 x_5 + r_2 x_6 + r_3 x_7 + r_4 x_8)^{k+1}$.

For $|i| \geqslant 11$ and i is odd,

$$\left| \mathrm{wt}\frac{\partial f_{k+1}^i}{\partial x_j} \right| \geqslant 8 \Rightarrow \frac{\partial f_{k+1}^i}{\partial x_j} = 0 \Rightarrow f_{k+1}^i = 0, \quad 1 \leqslant j \leqslant 8.$$

For $i = \pm 9, \pm 7, \pm 5, \pm 3, \pm 1$, $\mathrm{wt}\frac{\partial f_{k+1}^i}{\partial x_j}$ is an odd integer for all $5 \leqslant j \leqslant 8$. Therefore f_{k+1}^i depends only on x_1, x_2, x_3, x_4 variables. Let $g(x_2, x_2, x_3, x_4) = \sum_{i=-5}^{4} f_{k+1}^{2i+1}$, then

$$f = \sum_{i=-5}^{4} f_{k+1}^{2i+1} + f_{k+1}^0(x_5, x_6, x_7, x_8) = g(x_1, x_2, x_3, x_4) + (r_1 x_5 + r_2 x_6 + r_3 x_7 + r_4 x_8)^{k+1}.$$

Since $\langle \frac{\partial f}{\partial x_j}, j = 5, 6, 7, 8 \rangle = \langle (r_1 x_5 + r_2 x_6 + r_3 x_7 + r_4 x_8)^k \rangle$ is at most dimension one, $\dim I \leqslant 5$. This contradicts our hypothesis $I = (7) \oplus (1)$. We conclude that Case 2 cannot occur.

Case 3. $I = (6) \oplus (2)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $5, 3, 1, -1, -3, -5$.

This case cannot occur by the same argument as Case 1.

Case 4. $I = (6)\oplus(1)\oplus(1)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $5,3,1,-1,-3,-5,0$.

For $|i|\geqslant 9$,

$$\left|\mathrm{wt}\frac{\partial f^i_{k+1}}{\partial x_j}\right|\geqslant 6 \Rightarrow \frac{\partial f^i_{k+1}}{\partial x_j}=0\Rightarrow f^i_{k+1}=0, 1\leqslant j\leqslant 8.$$

For $i=8$,

$$\left|\mathrm{wt}\frac{\partial f^8_{k+1}}{\partial x_j}\right|\geqslant 7, 2\leqslant j\leqslant 8\Rightarrow f^8_{k+1}\,\text{depends only on } x_1 \text{ variable}\Rightarrow f^8_{k+1}=0\,\text{since wt}(x_1)=3.$$

Similarly, we can prove that $f^{-8}_{k+1}=0$.

For $i=7$,

$$\mathrm{wt}\frac{\partial f^7_{k+1}}{\partial x_j}\text{ is nonzero even for }1\leqslant j\leqslant 4, \mathrm{wt}\frac{\partial f^7_{k+1}}{\partial x_j}=7\,\text{for }5\leqslant j\leqslant 8\Rightarrow f^7_{k+1}=0.$$

Similarly we can prove that $f^{-7}_{k+1}=0$.

For $i=6$,

$$\begin{aligned}\mathrm{wt}\frac{\partial f^6_{k+1}}{\partial x_j}\geqslant 6, 3\leqslant j\leqslant 8 &\Rightarrow f^6_{k+1}\,\text{depends only on } x_1,x_2 \text{ variables}\\ &\Rightarrow f^6_{k+1}=cx_1x_2^3 \text{ or } f^6_{k+1}=cx_2^6 \text{ for some constant } c.\end{aligned}$$

If $f^6_{k+1}=cx_1x_2^3$ and $c\neq 0$, then $x_2^3=\frac{1}{c}\frac{\partial f^6_{k+1}}{\partial x_1}$ is in I. It follows that $\frac{1}{162}X_+^3(x_2^3)=x_1^3$ is in I. Thus $\langle x_1^3, X_-^i(x_1^3), 1\leqslant i\leqslant 9\rangle$ is a subspace of dimension 10 in I which contradicts that dim $I=8$.

If $f^6_{k+1}=cx_2^6$ and $c\neq 0$, then the same argument above shows that this is impossible. Thus $f^6_{k+1}=0$.

Similarly we can prove that $f^{-6}_{k+1}=0$.

For $i=5$,

$$\begin{aligned}&\mathrm{wt}\frac{\partial f^5_{k+1}}{\partial x_j}\text{is nonzero even for }1\leqslant j\leqslant 4,\ \mathrm{wt}\tfrac{\partial f^5_{k+1}}{\partial x_j}=5\text{ for }5\leqslant j\leqslant 8\\ &\Rightarrow f^5_{k+1}\,\text{depends only on } x_5,x_6,x_7,x_8 \text{ variables}\\ &\Rightarrow f^5_{k+1}=0 \text{ since wt}(x_j)=0 \text{ for } 5\leqslant j\leqslant 8.\end{aligned}$$

Similarly we can prove that $f^{-5}_{k+1}=0$.

For $i=4$,

$$\mathrm{wt}\frac{\partial f^4_{k+1}}{\partial x_4}=7, \mathrm{wt}\frac{\partial f^4_{k+1}}{\partial x_j}=4\text{ for }5\leqslant j\leqslant 8\Rightarrow f^4_{k+1}\,\text{depends only on }x_1,x_2,x_3\text{ variables.}$$

Write $f^4_{k+1}=\sum_{\beta\geqslant 0}b_\beta x_1^{\frac{2\beta-k+3}{2}}x_2^{\frac{-4\beta+3k-1}{2}}x_3^\beta$.

Suppose $f^4_{k+1}\neq 0$. Let β_0 be the largest integer such that $b_{\beta_0}\neq 0$. By Corollary 3, either $x_-^{\beta_0}\frac{\partial f^4_{k+1}}{\partial x_1}$ or $x_-^{\beta_0}\frac{\partial f^4_{k+1}}{\partial x_2}$ is nonzero. Since

$$\begin{aligned}&\mathrm{wt}X_-^{\beta_0}\frac{\partial f^4_{k+1}}{\partial x_1}=1-2\beta_0, \mathrm{wt}X_-^{\beta_0}\frac{\partial f^4_{k+1}}{\partial x_2}=3-2\beta_0 \text{ and } X_-^{\beta_0}\frac{\partial f^4_{k+1}}{\partial x_i}\in I, i=1,2\\ &\Rightarrow 1-2\beta_0\geqslant -5 \text{ or } 3-2\beta_0\geqslant -5\Rightarrow \beta_0\leqslant 3 \text{ or } \beta_0\leqslant 4,\end{aligned}$$

without loss of generality, we suppose $\beta_0\leqslant 4$.

$\beta_0=4\Rightarrow b_4\neq 0$,

$$\begin{aligned}k&=7, &f^4_8&=b_2x_2^6x_3^2+b_3x_1x_2^4x_3^3+b_4x_1^2x_2^2x_3^4,\\ k&=9, &f^4_{10}&=b_3x_2^7x_3^3+b_4x_1x_2^5x_3^4,\\ k&=11, &f^4_{12}&=b_4x_2^8x_3^4.\end{aligned}$$

$\beta_0=3\Rightarrow b_3\neq 0$,

$$\begin{aligned}k&=5, &f^4_6&=b_1x_2^5x_3+b_2x_1x_2^3x_3^2+b_3x_1^2x_2x_3^3,\\ k&=7, &f^4_8&=b_2x_2^6x_3^2+b_3x_1x_2^4x_3^3,\\ k&=9, &f^4_{10}&=b_3x_2^7x_3^3.\end{aligned}$$

$\beta_0=2\Rightarrow b_2\neq 0$,

$$\begin{aligned}k&=3, &f^4_4&=b_0x_2^4+b_1x_1x_2^2x_3+b_2x_1^2x_3^2,\\ k&=5, &f^4_6&=b_1x_2^5x_3+b_2x_1x_2^3x_3^2,\\ k&=7, &f^4_8&=b_2x_2^6x_3^2.\end{aligned}$$

$\beta_0=1\Rightarrow b_1\neq 0$, $k=3, f^4_4=b_0x_2^4+b_1x_1x_2^2x_3, k=5, f^4_6=b_1x_2^5x_3$.

$\beta_0=0\Rightarrow b_0\neq 0$, $k=3, f^4_4=b_0x_2^4$.

If $f^4_8=b_2x_2^6x_3^2+b_3x_1x_2^4x_3^3+b_4x_1^2x_2^2x_3^4$ where $b_4\neq 0$,

$$\begin{aligned}&\mathrm{wt}\left(X_+\frac{\partial f^4_8}{\partial x_3}\right)=7\\ &\Rightarrow 0=X_+\frac{\partial f^4_8}{\partial x_3}=(36b_2+24b_3)x_1x_2^5x_3+24b_4x_1^3x_2x_3^3+(48b_4+35b_3)x_1^2x_2^3x_3^2+8b_2x_2^7\\ &\Rightarrow b_2=b_3=b_4=0.\end{aligned}$$

This contradicts that $b_4\neq 0$.

If $f^4_{10}=b_3x_2^7x_3^3+b_4x_1x_2^5x_3^4$ where $b_4\neq 0$,

$$\begin{aligned}\mathrm{wt}\left(X_+\frac{\partial f^4_{10}}{\partial x_3}\right)=7&\Rightarrow 0=X_+\frac{\partial f^4_{10}}{\partial x_3}=(36b_3+48b_4)x_1x_2^6x_3^2+24b_3x_2^8x_3+60b_4x_1^2x_2^4x_3^3\\ &\Rightarrow b_3=b_4=0.\end{aligned}$$

This contradicts that $b_4\neq 0$.

If $f^4_{12}=b_4x_2^8x_3^4$ where $b_4\neq 0$,

$$\mathrm{wt}\left(X_+\frac{\partial f^4_{12}}{\partial x_3}\right)=7\Rightarrow 0=X_+\frac{\partial f^4_{12}}{\partial x_3}=96b_4x_1x_2^7x_3^3+48b_4x_2^9x_3^2\Rightarrow b_4=0.$$

This contradicts that $b_4 \neq 0$.

Therefore $f^4_{k+1} = 0$.

Similarly, we can prove that $f^4_{k+1} = 0$ for $\beta_0 = 3, 2, 1, 0$.

Similarly, we can prove that $f^{-4}_{k+1} = 0$.

For $i = 3$,

$$\mathrm{wt}\frac{\partial f^3_{k+1}}{\partial x_j}\text{is nonzero even for } 2 \leqslant j \leqslant 4, \mathrm{wt}\frac{\partial f^3_{k+1}}{\partial x_j} = 3 \text{ for } 5 \leqslant j \leqslant 8$$
$$\Rightarrow f^3_{k+1} \text{ depends only on } x_1, x_5, x_6, x_7, x_8 \text{ variables.}$$

In view of Lemma 1, there are constants r_1, r_2, r_3, r_4 such that $f^3_{k+1} = x_1(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^k$. Suppose $f^3_{k+1} \neq 0$. We shall assume without loss of generality that $r_1 \neq 0$,

$$\frac{\partial f^3_{k+1}}{\partial x_5} = kr_1x_1(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1}.$$

It is easy to see that

$$(4) = \left\langle \frac{\partial f^3_{k+1}}{\partial x_5}, X^j_- \frac{\partial f^3_{k+1}}{\partial x_5}, 1 \leqslant j \leqslant 3 \right\rangle \subseteq I.$$

This contradicts that $I = (6) \oplus (1) \oplus (1)$. Thus $f^3_{k+1} = 0$.

Similarly, we can prove that $f^{-3}_{k+1} = 0$.

For $i = \pm 2$,

$$\mathrm{wt}\frac{\partial f^{-2}_{k+1}}{\partial x_j} = -2, \mathrm{wt}\frac{\partial f^2_{k+1}}{\partial x_j} = 2 \text{ for } 5 \leqslant j \leqslant 8$$
$$\Rightarrow f^i_{k+1} \text{ depends only on } x_1, x_2, x_3, x_4 \text{ variables.}$$

For $i = 1$,

$$\mathrm{wt}\frac{\partial f^1_{k+1}}{\partial x_j} \text{ is nonzero even for } j = 1, 3, 4, \mathrm{wt}\frac{\partial f^1_{k+1}}{\partial x_j} = 1 \text{ for } 5 \leqslant j \leqslant 8$$
$$\Rightarrow f^1_{k+1} \text{ depends only on } x_2, x_5, x_6, x_7, x_8 \text{ variables.}$$

In view of Lemma 1, there are constants r_1, r_2, r_3, r_4 such that $f^1_{k+1} = x_2(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^k$. Suppose that $f^1_{k+1} \neq 0$. We shall assume, without loss of generality, that $r_1 \neq 0$,

$$\frac{\partial f^1_{k+1}}{\partial x_5} = kr_1x_2(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^k.$$

It is easy to see that

$$(4) = \left\langle \frac{\partial f^1_{k+1}}{\partial x_5}, X_+ \frac{\partial f^1_{k+1}}{\partial x_5}, X^j_- \frac{\partial f^1_{k+1}}{\partial x_5}, j = 1, 2 \right\rangle \subseteq I.$$

This contradicts that $I = (6) \oplus (1) \oplus (1)$. Thus $f^1_{k+1} = 0$.

Similarly, we can prove that $f^{-1}_{k+1} = 0$.

So $f = f^{-2}_{k+1}(x_1, x_2, x_3, x_4) + f^0_{k+1}(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) + f^2_{k+1}(x_1, x_2, x_3, x_4)$, $\frac{\partial f}{\partial x_j} = \frac{\partial f^0_{k+1}}{\partial x_j}$ for $5 \leqslant j \leqslant 8$. Since $\mathrm{wt}\frac{\partial f^0_{k+1}}{\partial x_j} = 0$ and $I = (6) \oplus (1) \oplus (1)$, dim $\langle \frac{\partial f}{\partial x_5}, \frac{\partial f}{\partial x_6}, \frac{\partial f}{\partial x_7}, \frac{\partial f}{\partial x_8} \rangle \leqslant 2$. This implies that dim $I \leqslant 6$ contradicts dim $I = 8$. We conclude that Case 4 cannot occur.

Case 5. $I = (5) \oplus (3)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $4, 2, 0, -2, -4$.

For $|i| \geqslant 8$,

$$\left|\mathrm{wt}\frac{\partial f_{k+1}^i}{\partial x_j}\right| \geqslant 5 \Rightarrow \frac{\partial f_{k+1}^i}{\partial x_j} = 0 \Rightarrow f_{k+1}^i = 0, 1 \leqslant j \leqslant 8.$$

For $i = \pm 6, \pm 4, \pm 2, 0$,

$$\mathrm{wt}\frac{\partial f_{k+1}^i}{\partial x_j} \text{ is odd for } 1 \leqslant j \leqslant 4,\ \mathrm{wt}\frac{\partial f_{k+1}^i}{\partial x_j} = i \text{ for } 5 \leqslant j \leqslant 8$$
$$\Rightarrow f_{k+1}^i = 0 \,\mathrm{for}\, i = \pm 6, \pm 4, \pm 2,$$

and f_{k+1}^0 depends only on x_5, x_6, x_7, x_8 variables since $\mathrm{wt}(x_j) = 0$ for $5 \leqslant j \leqslant 8$.

If one of $\frac{\partial f_{k+1}^0}{\partial x_j} \neq 0$ for $5 \leqslant j \leqslant 8$, say $\frac{\partial f_{k+1}^0}{\partial x_5} \neq 0$, then $(1) = \langle \frac{\partial f_{k+1}^0}{\partial x_5} \rangle \subseteq I$. This contradicts that $I = (5) \oplus (3)$. Thus $f_{k+1}^0 = 0$.

For $i = \pm 7, \pm 5, \pm 3, \pm 1$,

$$\mathrm{wt}\frac{\partial f_{k+1}^i}{\partial x_j} \text{ is odd for } 5 \leqslant j \leqslant 8 \Rightarrow f_{k+1}^i \,\text{depends only on}\, x_1, x_2, x_3, x_4 \text{ variables.}$$

Thus $f = \sum_{i=-4}^{3} f_{k+1}^{2i+1}$ depends only on x_1, x_2, x_3, x_4 variables $\Rightarrow \dim I \leqslant 4$. This contradicts that $\dim I = 8$. We conclude that Case 5 cannot occur.

Case 6. $I = (5) \oplus (2) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $4, 2, 0, -2, -4, 1, -1$.

For $|i| \geqslant 8$,

$$\left|\mathrm{wt}\frac{\partial f_{k+1}^i}{\partial x_j}\right| \geqslant 5 \Rightarrow \frac{\partial f_{k+1}^i}{\partial x_j} = 0 \Rightarrow f_{k+1}^i = 0, 1 \leqslant j \leqslant 8.$$

For $i = 7$,

$$\mathrm{wt}\frac{\partial f_{k+1}^7}{\partial x_j} \geqslant 6, \ \ 2 \leqslant j \leqslant 8$$
$$\Rightarrow f_{k+1}^7 \,\text{depends only on}\, x_1 \text{ variable}$$
$$\Rightarrow f_{k+1}^7 = 0 \text{ since } \mathrm{wt}(x_1) = 3.$$

Similarly, we can prove that $f_{k+1}^{-7} = 0$.

For $i = 6$, $\mathrm{wt}\frac{\partial f_{k+1}^6}{\partial x_1} = 3, \mathrm{wt}\frac{\partial f_{k+1}^6}{\partial x_j} \geqslant 5, 2 \leqslant j \leqslant 8 \Rightarrow f_{k+1}^6 = 0$. Similarly we can prove that $f_{k+1}^{-6} = 0$.

For $i = 5$,

$$\mathrm{wt}\frac{\partial f_{k+1}^5}{\partial x_j} \geqslant 6, 3 \leqslant j \leqslant 8 \Rightarrow f_{k+1}^5 \,\text{depends only on}\, x_1, x_2 \text{ variables}$$
$$\Rightarrow f_{k+1}^5 = c x_1 x_2^2 \,\text{or}\, c x_2^5 \text{ for some constant } c.$$

If $f_{k+1}^5 = c x_1 x_2^2$, then $\mathrm{wt}(X_+^2 \frac{\partial f_{k+1}^5}{\partial x_1}) = 6 \Rightarrow 0 = X_+^2 \frac{\partial f_{k+1}^5}{\partial x_1} = 18 c x_1^2 \Rightarrow c = 0 \Rightarrow f_{k+1}^5 = 0$.

If $f_{k+1}^5 = cx_2^5$, then $\mathrm{wt}(X_+\frac{\partial f_{k+1}^5}{\partial x_2}) = 6 \Rightarrow 0 = X_+\frac{\partial f_{k+1}^5}{\partial x_2} = 60cx_1x_2^3 \Rightarrow c = 0 \Rightarrow f_{k+1}^5 = 0$.

Similarly, we can prove that $f_{k+1}^{-5} = 0$.

For $i = 4$,

$$\mathrm{wt}\frac{\partial f_{k+1}^4}{\partial x_2} = 3, \quad \mathrm{wt}\frac{\partial f_{k+1}^4}{\partial x_3} = 5, \quad \mathrm{wt}\frac{\partial f_{k+1}^4}{\partial x_4} = 7$$
$$\Rightarrow f_{k+1}^4 \text{ depends only on } x_1, x_5, x_6, x_7, x_8 \text{ variables}$$
$$\Rightarrow f_{k+1}^4 = 0 \text{ since } \mathrm{wt}(x_1) = 3, \mathrm{wt}(x_j) = 0, 5 \leqslant j \leqslant 8.$$

Similarly we can prove that $f_{k+1}^{-4} = 0$.

For $i = 3$,

$$\mathrm{wt}\frac{\partial f_{k+1}^3}{\partial x_4} = 6, \quad \mathrm{wt}\frac{\partial f_{k+1}^3}{\partial x_j} = 3, 5 \leqslant j \leqslant 8 \Rightarrow f_{k+1}^3 \text{ depends only on } x_1, x_2, x_3 \text{ variables.}$$

Write $f_{k+1}^3 = \sum_{\beta \geqslant 0} b_\beta x_1^{\frac{2\beta-k+2}{2}} x_2^{\frac{-4\beta+3k}{2}} x_3^\beta$. Suppose that $f_{k+1}^3 \neq 0$. Let β_0 be the largest integer such that $b_{\beta_0} \neq 0$. By Corollary 3, either $X_-^{\beta_0}\frac{\partial f_{k+1}^3}{\partial x_1}$ or $X_-^{\beta_0}\frac{\partial f_{k+1}^3}{\partial x_2}$ is nonzero. Since $\mathrm{wt}X_-^{\beta_0}\frac{\partial f_{k+1}^3}{\partial x_1} = -2\beta_0$, $\mathrm{wt}X_-^{\beta_0}\frac{\partial f_{k+1}^3}{\partial x_2} = 2 - 2\beta_0$ and $X_-^{\beta_0}\frac{\partial f_{k+1}^3}{\partial x_i} \in I$, $i = 1, 2$, $-2\beta_0 \geqslant -4$ or $2 - 2\beta_0 \geqslant -4$, thus $\beta_0 \leqslant 2$ or $\beta_0 \leqslant 3$. Without loss of generality, we suppose that $\beta_0 \leqslant 3$.

$\beta_0 = 3 \Rightarrow b_3 \neq 0$,

$$\begin{aligned} k &= 4, \qquad f_5^3 = b_1x_2^4x_3 + b_2x_1x_2^2x_3^2 + b_3x_1^2x_3^3, \\ k &= 6, \qquad f_7^3 = b_2x_2^5x_3^2 + b_3x_1x_2^3x_3^3, \\ k &= 8, \qquad f_9^3 = b_3x_2^6x_3^3. \end{aligned}$$

$\beta_0 = 2 \Rightarrow b_2 \neq 0$, $k = 4, f_5^3 = b_1x_2^4x_3 + b_2x_1x_2^2x_3^2$, $k = 6, f_7^3 = b_2x_2^5x_3^2$.

$\beta_0 = 1 \Rightarrow b_1 \neq 0$, $k = 2, f_3^3 = b_0x_2^3 + b_1x_1x_2x_3$.

$\beta_0 = 0 \Rightarrow b_0 \neq 0$, $k = 2, f_3^3 = b_0x_2^3$. If $f_5^3 = b_1x_2^4x_3 + b_2x_1x_2^2x_3^2 + b_3x_1^2x_3^3$ where $b_3 \neq 0$, then

$$\mathrm{wt}\left(X_+\frac{\partial f_5^3}{\partial x_3}\right) = 6 \Rightarrow 0 = X_+\frac{\partial f_5^3}{\partial x_3} = (12b_1 + 8b_2)x_1x_2^3 + (12b_2 + 24b_3)x_1^2x_2x_3$$
$$\Rightarrow b_2 = -2b_3, \quad b_1 = \frac{4}{3}b_3.$$

Thus

$$\begin{aligned} f_5^3 &= \frac{4}{3}b_3x_2^4x_3 - 2b_3x_1x_2^2x_3^2 + b_3x_1^2x_3^3 \\ &= \frac{b_3}{3}(4x_2^4x_3 - 6x_1x_2^2x_3^2 + 3x_1^2x_3^3) \\ &= c(4x_2^4x_3 - 6x_1x_2^2x_3^2 + 3x_1^2x_3^3) \text{ for some constant } c \neq 0. \end{aligned}$$

Since

$$\frac{\partial f_5^3}{\partial x_2} = c(16x_2^3x_3 - 12x_1x_2x_3^2), \quad \mathrm{wt}\frac{\partial f_5^3}{\partial x_2} = 2,$$
$$X_-\frac{\partial f_5^3}{\partial x_3} = c(4x_2^3x_3 - 6x_1x_2x_3^2 + 18x_1^2x_3x_4 - 12x_1x_2^2x_4) \text{ and } \mathrm{wt}\left(X_-\frac{\partial f_5^3}{\partial x_3}\right) = 2.$$

thus

$$X_-\frac{\partial f_5^3}{\partial x_3} = \widetilde{c}\frac{\partial f_5^3}{\partial x_3} \text{ for some constant } \widetilde{c}$$
$$\Rightarrow c(4x_2^3x_3 - 6x_1x_2x_3^2 + 18x_1^2x_3x_4 - 12x_1x_2^2x_4) = c\widetilde{c}(16x_2^3x_3 - 12x_1x_2x_3^2)$$
$$\Rightarrow c = 0.$$

If $f_7^3 = b_2x_2^5x_3^2 + b_3x_1x_2^3x_3^3$ where $b_3 \neq 0$, then

$$\mathrm{wt}\left(X_+\frac{\partial f_7^3}{\partial x_3}\right) = 6 \Rightarrow 0 = X_+\frac{\partial f_7^3}{\partial x_3} = 30b_2x_1x_2^4x_3 + 8b_2x_2^6 + 27b_3x_1^2x_2^2x_3^2 + 24b_3x_1x_2^4x_3$$
$$\Rightarrow b_2 = b_3 = 0.$$

If $f_9^3 = b_3x_2^6x_3^3$ where $b_3 \neq 0$, then

$$\mathrm{wt}\left(X_+\frac{\partial f_9^3}{\partial x_3}\right) = 6 \Rightarrow 0 = X_+\frac{\partial f_9^3}{\partial x_3} = 54b_3x_1x_2^5x_3^2 + 24b_3x_2^7x_3 \Rightarrow b_3 = 0.$$

Therefore $f_{k+1}^3 = 0$.

Similarly, we can prove that $f_{k+1}^3 = 0$ for $\beta_0 = 2, 1, 0$.

Similarly, we can prove that $f_{k+1}^{-3} = 0$.

For $i = 2$,

$$\mathrm{wt}\frac{\partial f_{k+1}^2}{\partial x_3} = 3, \quad \mathrm{wt}\frac{\partial f_{k+1}^2}{\partial x_4} = 5, \quad \mathrm{wt}\frac{\partial f_{k+1}^2}{\partial x_j} = 2, \quad 5 \leqslant j \leqslant 8$$
$$\Rightarrow f_{k+1}^2 \text{ depends only on } x_1, x_2, x_5, x_6, x_7, x_8 \text{ variables.}$$

In view of Lemma 1, there are constants r_1, r_2, r_3, r_4 such that $f_{k+1}^2 = x_2^2(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1}$. Suppose there $f_{k+1}^2 \neq 0$. We shall assume without loss of generality that $r_1 \neq 0$.

$$\frac{\partial f_{k+1}^2}{\partial x_5} = r_1(k-1)x_2^2(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-2}.$$
$$\mathrm{wt}\left(X_+^2\frac{\partial f_{k+1}^2}{\partial x_5}\right) = 6 \Rightarrow 0 = X_+^2\frac{\partial f_{k+1}^2}{\partial x_5} = 18r_1(k-1)x_1^2(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-2}$$
$$\Rightarrow r_1 = 0.$$

Thus $f_{k+1}^2 = 0$.

Similarly, we can prove that $f_{k+1}^{-2} = 0$.

$$\mathrm{wt}\frac{\partial f_{k+1}^{-1}}{\partial x_1} = -4, \quad \mathrm{wt}\frac{\partial f_{k+1}^{-1}}{\partial x_2} = -2, \quad \mathrm{wt}\frac{\partial f_{k+1}^{-1}}{\partial x_3} = 0, \quad \mathrm{wt}\frac{\partial f_{k+1}^{-1}}{\partial x_4} = 2, \quad \mathrm{wt}\frac{\partial f_{k+1}^{-1}}{\partial x_j} = -1,$$
$$\mathrm{wt}\frac{\partial f_{k+1}^{0}}{\partial x_1} = -3, \quad \mathrm{wt}\frac{\partial f_{k+1}^{0}}{\partial x_2} = -1, \quad \mathrm{wt}\frac{\partial f_{k+1}^{0}}{\partial x_3} = 1, \quad \mathrm{wt}\frac{\partial f_{k+1}^{0}}{\partial x_4} = 3, \quad \mathrm{wt}\frac{\partial f_{k+1}^{0}}{\partial x_j} = 0,$$
$$\mathrm{wt}\frac{\partial f_{k+1}^{1}}{\partial x_1} = -2, \quad \mathrm{wt}\frac{\partial f_{k+1}^{1}}{\partial x_2} = 0, \quad \mathrm{wt}\frac{\partial f_{k+1}^{1}}{\partial x_3} = 2, \quad \mathrm{wt}\frac{\partial f_{k+1}^{1}}{\partial x_4} = 4, \quad \mathrm{wt}\frac{\partial f_{k+1}^{1}}{\partial x_j} = 1,$$

$5 \leqslant j \leqslant 8$. Thus $f = f_{k+1}^{-1}(x_1, x_2, \ldots, x_8) + f_{k+1}^0(x_2, x_3, x_5, x_6, x_7, x_8) + f_{k+1}^1(x_1, x_2, \ldots, x_8)$.

If $f_{k+1}^0 = 0$ then $\langle \frac{\partial f}{\partial x_j}, 5 \leqslant j \leqslant 8\rangle = \langle \frac{\partial f_{k+1}^{-1}}{\partial x_j} + \frac{\partial f_{k+1}^1}{\partial x_j}, 5 \leqslant j \leqslant 8\rangle \subseteq \langle \frac{\partial f_{k+1}^{-1}}{\partial x_j}, \frac{\partial f_{k+1}^1}{\partial x_j}, 5 \leqslant j \leqslant 8\rangle \subseteq (2) \Rightarrow \dim\langle \frac{\partial f}{\partial x_j}, 5 \leqslant j \leqslant 8\rangle \leqslant 2 \Rightarrow \dim\ I \leqslant 6$.

This contradicts that dim $I = 8$.

Thus $f_{k+1}^0 \neq 0$.

Since f_{k+1}^0 must be either dependent on both x_2 and x_3 variables or independent on both x_2 and x_3 variables by weight consideration,

$$f_{k+1}^0 = f_{k+1}^0(x_5, x_6, x_7, x_8)$$

or

$$f_{k+1}^0 = \sum_{\beta \geqslant 1} x_2^\beta x_3^\beta g_{k+1-2\beta}(x_5, x_6, x_7, x_8)$$

for some $g_{k+1-2\beta} \neq 0$.

If $f_{k+1}^0 = f_{k+1}^0(x_5, x_6, x_7, x_8)$, then $\langle \frac{\partial f_{k+1}^0}{\partial x_j}, 5 \leqslant j \leqslant 8 \rangle \subseteq (1)$.

Thus $\langle \frac{\partial f}{\partial x_j} \rangle = \langle \frac{\partial f_{k+1}^{-1}}{\partial x_j} + \frac{\partial f_{k+1}^0}{\partial x_j} + \frac{\partial f_{k+1}^1}{\partial x_j} \rangle \subseteq (2) \oplus (1) \Rightarrow \dim \langle \frac{\partial f}{\partial x_j} \rangle \leqslant 3 \Rightarrow \dim I \leqslant 7$ for $5 \leqslant j \leqslant 8$. This contradicts that dim $I = 8$.

If $f_{k+1}^0 = \sum_{\beta \geqslant 1} x_2^\beta x_3^\beta g_{k+1-2\beta}(x_5, x_6, x_7, x_8)$. Let β_0 be the largest integer such that $g_{k+1-2\beta_0} \neq 0$. By Lemma 2, $X_-^{\beta_0} \frac{\partial f_{k+1}^0}{\partial x_2} \neq 0$.

$$\begin{aligned} &\mathrm{wt} X_-^{\beta_0} \frac{\partial f_{k+1}^0}{\partial x_2} = -1 - 2\beta_0 \text{ and } X_-^{\beta_0} \frac{\partial f_{k+1}^0}{\partial x_2} \in I \\ &\quad \Rightarrow -1 - 2\beta_0 \geqslant -4 \Rightarrow \beta_0 \leqslant 1 \quad \text{since } \beta_0 \geqslant 1 \Rightarrow \beta_0 = 1. \end{aligned}$$

Therefore $f_{k+1}^0 = x_2 x_3 g_{k-1}(x_5, x_6, x_7, x_8)$ where $g_{k-1}(x_5, x_6, x_7, x_8) \neq 0$.

It is easy to see that $x_3 g_{k-1}(x_5, x_6, x_7, x_8) = \frac{\partial f_{k+1}^0}{\partial x_2} \in I$.

Therefore

$$(4) = \langle x_1 g_{k-1}, x_2 g_{k-1}, x_3 g_{k-1}, x_4 g_{k-1} \rangle \subseteq I.$$

This contradicts that $I = (5) \oplus (2) \oplus (1)$.

We conclude that Case 6 cannot occur.

Case 7. $I = (5) \oplus (1) \oplus (1) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $4, 2, 0, -2, -4$.

For $|i| \geqslant 8$,

$$\left| \mathrm{wt} \frac{\partial f_{k+1}^i}{\partial x_j} \right| \geqslant 5 \Rightarrow \frac{\partial f_{k+1}^i}{\partial x_j} = 0 \Rightarrow f_{k+1}^i = 0, \quad 1 \leqslant j \leqslant 8.$$

For $i = \pm 6, \pm 4, \pm 2, 0$,

$$\begin{aligned} &\mathrm{wt} \frac{\partial f_{k+1}^i}{\partial x_j} \text{ is odd for } 1 \leqslant j \leqslant 4, \mathrm{wt} \frac{\partial f_{k+1}^i}{\partial x_j} = i \ \text{ for } 5 \leqslant j \leqslant 8 \\ &\quad \Rightarrow f_{k+1}^i = 0 \text{ for } i = \pm 6, \pm 5, \pm 2 \end{aligned}$$

and f_{k+1}^0depends only on x_5, x_6, x_7, x_8 variables since $\mathrm{wt}(x_j) = 0$ for $5 \leqslant j \leqslant 8$.

For $i = \pm 7, \pm 5, \pm 3, \pm 1$,

$$\mathrm{wt} \frac{\partial f_{k+1}^i}{\partial x_j} \text{ is odd for } 5 \leqslant j \leqslant 8 \Rightarrow f_{k+1}^i \text{ depends only on } x_1, x_2, x_3, x_4 \text{ variables.}$$

Let $g(x_1,x_2,x_3,x_4) = \sum_{i=-4}^{3} f_{k+1}^{2i+1}$. Then $f = \sum_{i=-4}^{3} f_{k+1}^{2i+1} + f_{k+1}^0 = g(x_1,x_2,x_3,x_4) + f_{k+1}^0(x_5,x_6,x_7,x_8)$. Since $\frac{\partial f_{k+1}^0}{\partial x_j}$ is invariant for $5 \leqslant j \leqslant 8$, $\langle \frac{\partial f_{k+1}^0}{\partial x_j} \rangle \subseteq (1) \oplus (1) \oplus (1) \Rightarrow \dim \langle \frac{\partial f}{\partial x_j} \rangle = \dim \langle \frac{\partial f_{k+1}^0}{\partial x_j} \rangle \leqslant 3 \Rightarrow \dim I \leqslant 7$. This contradicts that $\dim I = 8$. We conclude that Case 7 cannot occur.

Case 8. $I = (4) \oplus (4)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $3, 1, -1, -3$.

For $|i| \geqslant 7$,

$$\left|\mathrm{wt}\frac{\partial f_{k+1}^i}{\partial x_j}\right| \geqslant 4 \Rightarrow \frac{\partial f_{k+1}^i}{\partial x_j} = 0 \Rightarrow f_{k+1}^i = 0, 1 \leqslant j \leqslant 8.$$

For $i = 6$,

$$\mathrm{wt}\frac{\partial f_{k+1}^6}{\partial x_j} \geqslant 5, \quad 2 \leqslant j \leqslant 8 \Rightarrow f_{k+1}^6 \text{ depends only on } x_1 \text{ variable}$$
$$\Rightarrow f_{k+1}^6 = cx_1^2 \Rightarrow f_{k+1}^6 = 0 \text{ because } k \geqslant 2.$$

Similarly, we can prove that $f_{k+1}^{-6} = 0$.

For $i = \pm 5$, $\mathrm{wt}\frac{\partial f_{k+1}^i}{\partial x_j}$ is an even integer, $1 \leqslant j \leqslant 4$ and

$$\left|\mathrm{wt}\frac{\partial f_{k+1}^i}{\partial x_j}\right| = 5, 5 \leqslant j \leqslant 8 \Rightarrow \frac{f_{k+1}^i}{\partial x_j} = 0, 1 \leqslant j \leqslant 8 \Rightarrow f_{k+1}^i = 0.$$

For $i = 4$,

$$\begin{aligned}&\mathrm{wt}\frac{\partial f_{k+1}^4}{\partial x_j} \geqslant 4, 3 \leqslant j \leqslant 8\\ &\Rightarrow f_{k+1}^4 \text{ depends only on } x_1 \text{ and } x_2 \text{ variables}\\ &\Rightarrow f_{k+1}^4 = cx_1x_2 + dx_2^4 \Rightarrow f_{k+1}^4 = dx_2^4 \text{ because } k \geqslant 2.\end{aligned}$$

If $d \neq 0$, then $x_2^3 = \frac{1}{4d}\frac{\partial f_{k+1}^4}{\partial x_2} \in I$. Thus, $x_1^3 = \frac{1}{162}X_+^3(x_2^3) \in I$. Then $\langle x_1^3, X_-^i(x_1^3), 1 \leqslant i \leqslant 9\rangle$ is a subspace of dimension 10 in I which contradicts that $\dim I = 8$. Thus $f_{k+1}^4 = 0$. Similarly, we can prove that $f_{k+1}^{-4} = 0$.

For $i = 3$,

$$\begin{aligned}&\mathrm{wt}\frac{\partial f_{k+1}^3}{\partial x_j} \text{ is even, } 1 \leqslant j \leqslant 4\\ &\Rightarrow f_{k+1}^3 \text{ depends only on } x_5, x_6, x_7, x_8 \text{ variables}\\ &\Rightarrow f_{k+1}^3 = 0 \text{ because } \mathrm{wt}(x_j) = 0, 5 \leqslant j \leqslant 8.\end{aligned}$$

Similarly, we can prove that $f_{k+1}^{-3} = 0$.

The same argument as above shows that $f_{k+1}^1 = f_{k+1}^{-1} = 0$. Thus we have $f = f_{k+1}^{-2} + f_{k+1}^0 + f_{k+1}^2$.

$$\mathrm{wt}\frac{\partial f_{k+1}^2}{\partial x_4} = 5, \quad \mathrm{wt}\frac{\partial f_{k+1}^2}{\partial x_j} = 2, \quad \mathrm{wt}\frac{\partial f_{k+1}^0}{\partial x_j} = 0, \quad \mathrm{wt}\frac{\partial f_{k+1}^{-2}}{\partial x_1} = -5, \quad \mathrm{wt}\frac{\partial f_{k+1}^{-2}}{\partial x_j} = -2, 5 \leqslant j \leqslant 8$$

$$\Rightarrow f = f_{k+1}^{-2}(x_2,x_3,x_4) + f_{k+1}^{0}(x_1,x_2,x_3,x_4) + f_{k+1}^{2}(x_1,x_2,x_3)$$
$$\Rightarrow \frac{\partial f}{\partial x_j} = 0, 5 \leqslant j \leqslant 8 \Rightarrow \dim I \leqslant 4.$$

This contradicts that $\dim I = 8$. We conclude that Case 8 cannot occur.

Case 9. $I = (4) \oplus (3) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $3, 1, -1, -3, 2, 0, -2$.

By the same argument as Case 8, we have $f_{k+1}^{i} = 0$ for $|i| \geqslant 6$.

For $i = 5$,

$$\mathrm{wt}\frac{\partial f_{k+1}^{5}}{\partial x_j} \geqslant 4, 2 \leqslant j \leqslant 8 \Rightarrow f_{k+1}^{5} \text{ depends only on } x_1 \text{ variable}$$
$$\Rightarrow f_{k+1}^{5} = 0 \text{ because } \mathrm{wt}(x_1) = 3.$$

Similarly, we can prove that $f_{k+1}^{-5} = 0$.

For $i = 4$,

$$\mathrm{wt}\frac{\partial f_{k+1}^{4}}{\partial x_j} \geqslant 4, 3 \leqslant j \leqslant 8 \Rightarrow f_{k+1}^{4} \text{ depends only on } x_1, x_2 \text{ variables}$$
$$\Rightarrow f_{k+1}^{4} = cx_2^4 \text{ for some constant } c.$$

If $c \neq 0$ then $x_2^3 = \frac{1}{4c}\frac{\partial f_{k+1}^4}{\partial x_2} \in I$, $x_1^3 = \frac{1}{162}X_+^3(x_2^3) \in I$. Thus $\langle x_1^3, X_-^i(x_1^3), 1 \leqslant i \leqslant 9\rangle$ is a subspace of dimension 10 in I which contradicts that $\dim I = 8$. Thus $f_{k+1}^4 = 0$.

Similarly, we can prove that $f_{k+1}^{-4} = 0$.

For $i = 3$,

$$\mathrm{wt}\frac{\partial f_{k+1}^{3}}{\partial x_j} \geqslant 4, \quad j = 3,4 \Rightarrow f_{k+1}^{3} \text{ depends only on } x_1, x_2, x_5, x_6, x_7, x_8 \text{ variables}$$
$$\Rightarrow \mathrm{wt}\frac{\partial f_{k+1}^{3}}{\partial x_j} = 3, 5 \leqslant j \leqslant 8.$$

In view of Lemma 1, there are constants r_1, r_2, r_3, r_4 and c such that

$$f_{k+1}^3 = x_1(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^k \quad \text{or} \quad cx_2^3(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-2}.$$

If $f_{k+1}^3 = cx_2^3(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-2}$, then

$$\mathrm{wt}\left(X_+^2\frac{\partial f_{k+1}^3}{\partial x_2}\right) = 6 \Rightarrow 0 = X_+^2\frac{\partial f_{k+1}^3}{\partial x_2} = 54cx_1^2(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-2}$$
$$\Rightarrow c = 0 \Rightarrow f_{k+1}^3 = 0.$$

Thus $f_{k+1}^3 = x_1(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^k$.

Similarly, we can prove that $f_{k+1}^{-3} = x_4(r_5x_5 + r_6x_6 + r_7x_7 + r_8x_8)^k$ for some constants r_5, r_6, r_7, r_8.

If one of f_{k+1}^3 and f_{k+1}^{-3} is not zero, we may assume without loss of generality that $f_{k+1}^3 \neq 0$ and $r_1 \neq 0$. Then

$$(4) = \langle x_1(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1}, x_2(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1},$$

$$x_3(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1}, x_4(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1}\rangle,$$

$$(1)=\left\langle\frac{\partial f^3_{k+1}}{\partial x_1}\right\rangle=\langle(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1}\rangle.$$

Since $\frac{\partial f^{-3}_{k+1}}{\partial x_4}=(r_5x_5+r_6x_6+r_7x_7+r_8x_8)^k\in(1)$, $\frac{\partial f^{-3}_{k+1}}{\partial x_4}=d_1(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^k$ for some constant d_1. Thus

$$\begin{aligned}
&f^{-3}_{k+1}=d_1x_4(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^k,\\
&\mathrm{wt}\frac{\partial f^2_{k+1}}{\partial x_1}=-1\Rightarrow\frac{\partial f^2_{k+1}}{\partial x_1}=c_1x_3(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1},\\
&\mathrm{wt}\frac{\partial f^2_{k+1}}{\partial x_2}=1\Rightarrow\frac{\partial f^2_{k+1}}{\partial x_2}=c_2x_2(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1},\\
&\mathrm{wt}\frac{\partial f^2_{k+1}}{\partial x_3}=3\Rightarrow\frac{\partial f^2_{k+1}}{\partial x_3}=c_3x_1(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1},
\end{aligned}$$

for some constants c_1,c_2,c_3.

$$\begin{aligned}
&\mathrm{wt}\frac{\partial f^2_{k+1}}{\partial x_4}=5\Rightarrow f^2_{k+1}\text{ does not depend on }x_4\text{ variable.}\\
&f^2_{k+1}=c_1x_1x_3(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1}+g_1(x_2,x_3,x_5,x_6,x_7,x_8).\\
&\frac{\partial f^2_{k+1}}{\partial x_2}=\frac{\partial}{\partial x_2}g_1(x_2,x_3,x_5,x_6,x_7,x_8)=c_2x_2(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1}\\
&\Rightarrow g_1=\frac{c_2}{2}x_2^2(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1}+g_2(x_3,x_5,x_6,x_7,x_8).
\end{aligned}$$

Since $\mathrm{wt}(g_2)=2$, $\mathrm{wt}(x_3)=-1$, $\mathrm{wt}(x_j)=0$, $5\leqslant j\leqslant 8$, thus $g_2=0$.

Thus

$$\begin{aligned}
f^2_{k+1}&=\left(c_1x_1x_3+\frac{c_2}{2}x_2^2\right)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1}.\\
\frac{\partial f^2_{k+1}}{\partial x_5}&=(k-1)r_1\left(c_1x_1x_3+\frac{c_2}{2}x_2^2\right)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-2}.
\end{aligned}$$

Since $\mathrm{wt}X_+\frac{\partial f^2_{k+1}}{\partial x_5}=4$, $0=X_+\frac{\partial f^2_{k+1}}{\partial x_5}=(k-1)r_1(4c_1+3c_2)x_1x_2(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-2}$.

Thus $4c_1+3c_2=0\Rightarrow c_2=-\frac{4}{3}c_1\Rightarrow$

$$\begin{aligned}
f^2_{k+1}&=\frac{1}{3}c_1(3x_1x_3-2x_2^2)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1}\\
&=d_2(3x_1x_3-2x_2^2)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1}
\end{aligned}$$

for some constant d_2.

Similarly, we can prove that $f^{-2}_{k+1}=d_3(2x_2x_4-x_3^2)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1}$ for some constant d_3.

If $f^2_{k+1}\neq 2$ or $f^{-2}_{k+1}\neq 0$, i.e., $d_2\neq 0$ or $d_3\neq 0$ then

$$\begin{aligned}
(3)=\langle&(3x_1x_3-2x_2^2)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-2},\\
&(3x_1x_4-x_2x_3)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-2},\\
&(2x_2x_4-x_3^2)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-2}\rangle,
\end{aligned}$$

$$\mathrm{wt}\frac{\partial f^1_{k+1}}{\partial x_1} = -2 \Rightarrow \frac{\partial f^1_{k+1}}{\partial x_1} = c_4(2x_2x_4 - x_3^2)(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-2},$$
$$\mathrm{wt}\frac{\partial f^1_{k+1}}{\partial x_2} = 0 \Rightarrow \frac{\partial f^1_{k+1}}{\partial x_2} = c_5(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k} + c_6(3x_1x_4 - x_2x_3)(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-2},$$
$$\mathrm{wt}\frac{\partial f^1_{k+1}}{\partial x_3} = 2 \Rightarrow \frac{\partial f^1_{k+1}}{\partial x_3} = c_7(3x_1x_3 - 2x_2^2)(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-2},$$
$$\mathrm{wt}\frac{\partial f^1_{k+1}}{\partial x_4} = 4 \Rightarrow f^1_{k+1} \text{ does not depend on } x_4 \text{ variable},$$
$$\mathrm{wt}\frac{\partial f^1_{k+1}}{\partial x_j} = 1 \Rightarrow \frac{\partial f^1_{k+1}}{\partial x_j} = e_jx_2(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1},\ 5 \leqslant j \leqslant 8,$$

for some constants c_i, $4 \leqslant i \leqslant 7$, e_j, $5 \leqslant j \leqslant 8$.

$$f^1_{k+1} = c_4x_1(2x_2x_4 - x_3^2)(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-2} + g_3(x_2, x_3, x_5, x_6, x_7, x_8),$$
$$\frac{\partial f^1_{k+1}}{\partial x_3} = -2c_4x_1x_3(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-2} + \frac{\partial}{\partial x_3}g_3(x_2, x_3, x_5, x_6, x_7, x_8)$$
$$= c_7(3x_1x_3 - 2x_x^2)(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-2}$$
$$\Rightarrow -2c_4 = 3c_7 \Rightarrow c_7 = -\frac{2}{3}c_4,$$
$$\frac{\partial g_3}{\partial x_3} = -2c_7x_2^2(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-2}$$
$$\Rightarrow g_3 = -2c_7x_2^2x_3(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-2} + g_4(x_2, x_5, x_6, x_7, x_8).$$
$$f^1_{k+1} = \left[c_4x_1(2x_2x_4 - x_3^2) + \frac{4}{3}c_4x_2^2x_3\right](r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-2} + g_4(x_2, x_5, x_6, x_7, x_8),$$
$$\frac{\partial f^1_{k+1}}{\partial x_2} = \left(2c_4x_1x_4 + \frac{8}{3}c_4x_2x_3\right)(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-2} + \frac{\partial}{\partial x_2}g_4(x_2, x_5, x_6, x_7, x_8)$$
$$= c_5(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k} + c_6(3x_1x_4 - x_2x_3)(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-2}$$
$$\Rightarrow 2c_4 = 3c_6 \text{ and } \frac{8}{3}c_4 = -c_6 \Rightarrow c_4 = c_6 = 0,$$
$$\frac{\partial g_4}{\partial x_2} = c_5(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k}$$
$$\Rightarrow g_4 = c_5x_2(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k} + g_5(x_5, x_6, x_7, x_8).$$

Since $\mathrm{wt}(g_5) = 1$, $\mathrm{wt}(x_j) = 0$, $5 \leqslant j \leqslant 8$, $g_5 = 0$.

Thus $f^1_{k+1} = d_4x_2(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^k$ for some constant d_4.

Similarly, we can prove that $f^{-1}_{k+1} = d_5x_3(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^k$ for some constant d_5.

$$\mathrm{wt}\frac{\partial f^0_{k+1}}{\partial x_1} = -3 \Rightarrow \frac{\partial f^0_{k+1}}{\partial x_1} = c_8x_4(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1},$$
$$\mathrm{wt}\frac{\partial f^0_{k+1}}{\partial x_2} = -1 \Rightarrow \frac{\partial f^0_{k+1}}{\partial x_2} = c_9x_3(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1},$$
$$\mathrm{wt}\frac{\partial f^0_{k+1}}{\partial x_3} = 1 \Rightarrow \frac{\partial f^0_{k+1}}{\partial x_3} = c_{10}x_2(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1},$$
$$\mathrm{wt}\frac{\partial f^0_{k+1}}{\partial x_4} = 3 \Rightarrow \frac{\partial f^0_{k+1}}{\partial x_4} = c_{11}x_1(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1},$$

$$\mathrm{wt}\frac{\partial f_{k+1}^0}{\partial x_j}=0\Rightarrow\frac{\partial f_{k+1}^0}{\partial x_j}=\widetilde{c}_j(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^k$$
$$+\widetilde{e}_j(3x_1x_4-x_2x_3)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-2},\quad 5\leqslant j\leqslant 8,$$

for some constants c_i, $8\leqslant j\leqslant 11$, $\widetilde{c}_j,\widetilde{e}_j$, $5\leqslant j\leqslant 8$.

$$\begin{aligned}
f_{k+1}^0&=c_8x_1x_4(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1}+g_6(x_2,x_3,x_4,x_5,x_6,x_7,x_8),\\
\frac{\partial f_{k+1}^0}{\partial x_2}&=\frac{\partial}{\partial x_2}g_6(x_2,x_3,x_4,x_5,x_6,x_7,x_8)=c_9x_3(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1}\\
&\Rightarrow g_6=c_9x_2x_3(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1}+g_7(x_3,x_4,x_5,x_6,x_7,x_8),\\
f_{k+1}^0&=(c_8x_1x_4+c_9x_2x_3)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1}+g_7(x_3,x_4,x_5,x_6,x_7,x_8),\\
\frac{\partial f_{k+1}^0}{\partial x_3}&=c_9x_2(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1}+\frac{\partial}{\partial x_3}g_7(x_3,x_4,x_5,x_6,x_7,x_8)\\
&=c_{10}x_2(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1}\\
&\Rightarrow c_9=c_{10},\\
\frac{\partial g_7}{\partial x_3}&=0\Rightarrow g_7=g_7(x_4,x_5,x_6,x_7,x_8).\\
f_{k+1}^0&=(c_8x_1x_4+c_9x_2x_3)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1}+g_7(x_4,x_5,x_6,x_7,x_8),\\
\frac{\partial f_{k+1}^0}{\partial x_4}&=c_8x_1(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1}+\frac{\partial}{\partial x_4}g_7(x_4,x_5,x_6,x_7,x_8)\\
&=c_{11}x_1(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1}\\
&\Rightarrow c_8=c_{11},\\
\frac{\partial g_7}{\partial x_4}&=0\Rightarrow g_7=g_7(x_5,x_6,x_7,x_8).\\
f_{k+1}^0&=(c_8x_1x_4+c_9x_2x_3)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1}+g_7(x_5,x_6,x_7,x_8),\\
\frac{\partial f_{k+1}^0}{\partial x_j}&=(k-1)r_j(c_8x_1x_4+c_9x_2x_3)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-2}\\
&\quad+\frac{\partial}{\partial x_j}g_7(x_5,x_6,x_7,x_8)=\widetilde{c}_j(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^k\\
&\quad+\widetilde{e}_j(3x_1x_4-x_2x_3)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-2},5\leqslant j\leqslant 8\\
&\Rightarrow (k-1)r_jc_8=3\widetilde{e}_j\text{ and }(k-1)r_jc_9=-\widetilde{e}_j,\\
\frac{\partial g_7}{\partial x_j}&=\widetilde{c}_j(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^k\\
&\Rightarrow c_8=-3c_9,g_7=d_7(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k+1}\text{ for some constant }d_7,\\
f_{k+1}^0&=(-3c_9x_1x_4+c_9x_2x_3)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1}\\
&\quad+d_7(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k+1}\\
&=d_6(3x_1x_4-x_2x_3)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1}\\
&\quad+d_7(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k+1}\text{ for some constant }d_6.
\end{aligned}$$

So

$$\begin{aligned}
f&=f_{k+1}^{-3}+f_{k+1}^{-2}+f_{k+1}^{-1}+f_{k+1}^0+f_{k+1}^1+f_{k+1}^2+f_{k+1}^3\\
&=d_1x_4(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^k
\end{aligned}$$

$$
\begin{aligned}
&+ d_3(2x_2x_4 - x_3^2)(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1}\\
&+ d_5x_3(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k}\\
&+ d_6(3x_1x_4 - x_2x_3)(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1}\\
&+ d_7(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k+1}\\
&+ d_4x_2(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k}\\
&+ d_2(3x_1x_3 - 2x_2^2)(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k+1}\\
&+ x_1(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k}\\
\Rightarrow\ & \frac{\partial f}{\partial x_6} = \frac{r_2}{r_1}\frac{\partial f}{\partial x_5}, \quad \frac{\partial f}{\partial x_7} = \frac{r_3}{r_1}\frac{\partial f}{\partial x_5}, \quad \frac{\partial f}{\partial x_8} = \frac{r_4}{r_1}\frac{\partial f}{\partial x_5}\\
\Rightarrow\ & \dim I \leqslant 5.
\end{aligned}
$$

This contradicts that $\dim I = 8$. Thus $f_{k+1}^{2} = 0$ and $f_{k+1}^{-2} = 0$. So

$$
\begin{aligned}
f &= f_{k+1}^{-3} + f_{k+1}^{-1} + f_{k+1}^{0} + f_{k+1}^{1} + f_{k+1}^{3},\\
f_{k+1}^{-3} &= d_1x_4(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k},\\
f_{k+1}^{3} &= x_1(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k}, r_1 \neq 0.\\
(4) &= \langle x_1(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1}, x_2(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1},\\
&\quad x_3(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1}, x_4(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1}\rangle,\\
(1) &= \left\langle \frac{\partial f_{k+1}^{3}}{\partial x_1} \right\rangle = \langle (r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k}\rangle,\\
f_{k+1}^{0} &= (c_8x_1x_4 + c_9x_2x_3)(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1} + g_7(x_5, x_6, x_7, x_8).\\
f_{k+1}^{1} &= f_{k+1}^{1}(x_1, x_2, x_3, x_5, x_6, x_7, x_8),\\
f_{k+1}^{-1} &= f_{k+1}^{-1}(x_2, x_3, x_4, x_5, x_6, x_7, x_8),\\
\mathrm{wt}\frac{\partial f_{k+1}^{1}}{\partial x_j} &= 1, 5 \leqslant j \leqslant 8 \Rightarrow \frac{\partial f_{k+1}^{1}}{\partial x_j} = a_jx_2(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1}.
\end{aligned}
$$

for some constant a_j, $5 \leqslant j \leqslant 8 \Rightarrow f_{k+1}^{1} = d_8x_2(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k} + g_8(x_1, x_2, x_3)$ for some constant d_8.

$$
\mathrm{wt}\frac{\partial f_{k+1}^{-1}}{\partial x_j} = -1, 5 \leqslant j \leqslant 8 \Rightarrow \frac{\partial f_{k+1}^{-1}}{\partial x_j} = \widetilde{a}_jx_3(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1}
$$

for some constant $\widetilde{a}_j$, $5 \leqslant j \leqslant 8 \Rightarrow f_{k+1}^{-1} = d_9x_3(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k} + g_9(x_2, x_3, x_4)$ for some constant d_9.

Write

$$
g_8(x_1, x_2, x_3) = \sum_{\beta \geqslant 0} b_\beta x_1^{\beta - \frac{k}{2}} x_2^{-2\beta + \frac{3}{2}k + 1} x_3^{\beta}.
$$

If $g_8 \neq 0$, let β_0 be the largest integer such that $b_{\beta_0} \neq 0$. By Corollary 3, either $X_-^{\beta_0}\frac{\partial g_8}{\partial x_1}$ or $X_-^{\beta_0}\frac{\partial g_8}{\partial x_2}$ is nonzero. So, $\mathrm{wt}X_-^{\beta_0}\frac{\partial g_8}{\partial x_1} = -2-2\beta_0$, $\mathrm{wt}X_-^{\beta_0}\frac{\partial g_8}{\partial x_2} = -2\beta_0$ and $X_-^{\beta_0}\frac{\partial g_8}{\partial x_i} = X_-^{\beta_0}\frac{\partial f_{k+1}^{1}}{\partial x_i} \in I$, $i = 1, 2$, thus $-2 - 2\beta_0 \geqslant -3$, or $-2\beta_0 \geqslant -3 \Rightarrow \beta_0 \leqslant 0$ or $\beta_0 \leqslant 1$. Without loss of generality, we suppose $\beta_0 \leqslant 1 \Rightarrow g_8 = b_0x_2^2x_3$, then $\frac{\partial g_8}{\partial x_3} = b_0x_2^2$, $X_+\frac{\partial g_8}{\partial x_3} = 6b_0x_1x_2 = 0$ since $\mathrm{wt}(X_+\frac{\partial g_8}{\partial x_3}) = 4 \Rightarrow b_0 = 0 \Rightarrow g_8 = 0$.

Similarly, we can prove that $g_9=0$.

These imply that no elements in I are of weight $2,-2$. This contradicts that $I=(4)\oplus(3)\oplus(1)$.

Therefore $f_{k+1}^{-3}=0$ and $f_{k+1}^{3}=0$, so $f=f_{k+1}^{-2}+f_{k+1}^{-1}+f_{k+1}^{0}+f_{k+1}^{1}+f_{k+1}^{2}$.

For $i=2$,wt$\frac{\partial f_{k+1}^2}{\partial x_4}=5\Rightarrow f_{k+1}^2$ does not depend on x_4 variable.

Write $f_{k+1}^2=\sum_{\beta\geqslant 0}h_{k+1-\beta}^{2+\beta}(x_1,x_2,x_5,x_6,x_7,x_8)x_3^{\beta}$ where $h_{k+1-\beta}^{2+\beta}(x_1,x_2,x_5,x_6,x_7,x_8)$ is a homogeneous polynomial of degree $k+1-\beta$, weight $2+\beta$ and in x_1,x_2,x_5,x_6,x_7,x_8 variables.

If $f_{k+1}^1\neq 0$, let β_0 be the largest integer such that $h_{k+1-\beta_0}^{2+\beta_0}\neq 0$. By Lemma 2. either $X_-^{\beta_0}\frac{\partial f_{k+1}^2}{\partial x_1}$ or $X_-^{\beta_0}\frac{\partial f_{k+1}^2}{\partial x_2}$ is nonzero.

Since wt$X_-^{\beta_0}\frac{\partial f_{k+1}^2}{\partial x_1}=-1-2\beta_0$, $X_-^{\beta_0}\frac{\partial f_{k+1}^2}{\partial x_2}=1-2\beta_0$ and $X_-^{\beta_0}\frac{\partial f_{k+1}^2}{\partial x_i}\in I$, $i=1,2$, thus $-1-2\beta_0\geqslant -3$ or $1-2\beta_0\geqslant -3$, this implies $\beta_0\leqslant 1$ or $\beta_0\leqslant 2$. Without loss of generality, suppose $\beta_0\leqslant 2$. Since wt$(\frac{\partial f_{k+1}^2}{\partial x_j})=2$, $5\leqslant j\leqslant 8$, in view of Lemma 1, there are constants $c_1,c_2,c_3,c_4,c_5,r_1,r_2,r_3,r_4$ such that

$$\begin{aligned}
f_{k+1}^2&=(c_1x_2^2+c_2x_1x_3)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1}\\
&\quad+(c_3x_2^3x_3+c_4x_1x_2x_3^2)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-3}\\
&\quad+c_5x_2^4x_3^2(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-5},\quad \text{for}\, k\geqslant 6;\\
f_{k+1}^2&=(c_1x_2^2+c_2x_1x_3)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{4}\\
&\quad+(c_3x_2^3x_3+c_4x_1x_2x_3^2)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{2}\\
&\quad+c_5x_2^4x_3^2,\quad \text{for}\ \ k=5;\\
f_{k+1}^2&=(c_1x_2^2+c_2x_1x_3)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{3}\\
&\quad+(c_3x_2^3x_3+c_4x_1x_2x_3^2)(r_1x_5+r_2x_6+r_3x_7+r_4x_8),\quad \text{for}\ \ k=4;\\
f_{k+1}^2&=(c_1x_2^2+c_2x_1x_3)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{2}+c_3x_2^3x_3+c_4x_1x_2x_3^2,\quad \text{for}\ \ k=3;\\
f_{k+1}^2&=(c_1x_2^2+c_2x_1x_3)(r_1x_2x_6+r_3x_7+r_4x_8);\quad \text{for}\, k=2.
\end{aligned}$$

Since wt$(X_+\frac{\partial f_{k+1}^2}{\partial x_j})=4$, $X_+\frac{\partial f_{k+1}^2}{\partial x_j}=0$, $5\leqslant j\leqslant 8$.

$$\begin{aligned}
\frac{\partial f_{k+1}^2}{\partial x_j}&=(c_1x_2^2+c_2x_1x_3)(k-1)r_j(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-2}\\
&\quad+(c_3x_2^3x_3+c_4x_1x_2x_3^2)(k-3)r_j(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-4}\\
&\quad+c_5x_2^4x_3^2(k-5)r_j(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-6},\quad 5\leqslant j\leqslant 8,\quad \text{for}\ \ k\geqslant 6;\\
\frac{\partial f_{k+1}^2}{\partial x_j}&=4(c_1x_2^2+c_2x_1x_3)r_j(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{3}\\
&\quad+2(c_3x_2^3x_3+c_4x_1x_2x_3^2)r_j(r_1x_5+r_2x_6+r_3x_7+r_4x_8),\quad \text{for}\ \ k=5;\\
\frac{\partial f_{k+1}^2}{\partial x_j}&=3(c_1x_2^2+c_2x_1x_3)r_j(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{2}+(c_3x_2^3x_3+c_4x_1x_2x_3^2)r_j,\ \text{for}\ k=4;\\
\frac{\partial f_{k+1}^2}{\partial x_j}&=2(c_1x_2^2+c_2x_1x_3)r_j(r_1x_5+r_2x_6+r_3x_7+r_4x_8),\quad \text{for}\ \ k=3;\\
f_{k+1}^2&=(c_1x_2^2+c_2x_1x_3)r_j,\quad \text{for}\ \ k=2.
\end{aligned}$$

So

$$X_+\left(\frac{\partial f_{k+1}^2}{\partial x_j}\right)=(6c_1+4c_2)x_1x_2(k-1)r_j(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-2}$$

$$+ [(9c_3+8c_4)x_1x_2^2x_3+4c_3x_2^4+3c_4x_1^2x_3^2](k-3)r_j(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-4}$$
$$+ c_5(14x_1x_2^3x_3^2+8x_2^5x_3)(k-5)r_j(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-6}=0,\quad 5\leqslant j\leqslant 8$$
$$\Rightarrow 6c_1+4c_2=0,\quad c_3=c_4=c_5=0 \Rightarrow c_1=-\frac{2}{3}c_2,\quad \text{for } k\geqslant 6;$$

$$X_+\left(\frac{\partial f_{k+1}^2}{\partial x_j}\right)=4(6c_1+4c_2)x_1x_2r_j(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^3$$
$$+ 2[(9c_3+8c_4)x_1x_2^2x_3+4c_3x_2^4+3c_4x_1^2x_3^2]$$
$$\cdot r_j(r_1x_5+r_2x_6+r_3x_7+r_4x_8)=0,\quad 5\leqslant j\leqslant 8$$
$$\Rightarrow 6c_1+4c_2=0,\ c_3=c_4=0 \Rightarrow c_1=-\frac{2}{3}c_2,\quad \text{for } k=5;$$

$$X_+\left(\frac{\partial f_{k+1}^2}{\partial x_j}\right)=3(6c_1+4c_2)x_1x_2r_j(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^2$$
$$+ r_j[(9c_3+8c_4)x_1x_2^2x_3+4c_3x_2^4+3c_4x_1^2x_3^2]=0,\quad 5\leqslant j\leqslant 8$$
$$\Rightarrow 6c_1+4c_2=0,\ c_3=c_4=0 \Rightarrow c_1=-\frac{2}{3}c_2,\quad \text{for } k=4;$$

$$X_+\left(\frac{\partial f_{k+1}^2}{\partial x_j}\right)=2(6c_1+4c_2)x_1x_2r_j(r_1x_5+r_2x_6+r_3x_7+r_4x_8)=0,\quad 5\leqslant j\leqslant 8$$
$$\Rightarrow 6c_1+4c_2=0 \Rightarrow c_1=-\frac{2}{3}c_2,\quad \text{for } k=3;$$

$$X_+\left(\frac{\partial f_{k+1}^2}{\partial x_j}\right)=(6c_1+4c_2)x_1x_2r_j=0,\quad 5\leqslant j\leqslant 8$$
$$\Rightarrow 6c_1+4c_2=0 \Rightarrow c_1=-\frac{2}{3}c_2,\quad \text{for}\, k=2.$$

Thus

$$f_{k+1}^2=\frac{c_2}{3}(3x_1x_3-2x_2^2)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1},\quad \text{for } k\geqslant 6;$$
$$f_{k+1}^2=\frac{c_2}{3}(3x_1x_3-2x_2^2)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^4+c_5x_2^4x_3^2$$
$$\Rightarrow X_+\frac{\partial f_{k+1}^2}{\partial x_3}=24c_5x_1x_2^3x_3+8c_5x_2^5=0 \text{ since } \mathrm{wt}X_+\frac{\partial f_{k+1}^2}{\partial x_3}=5$$
$$\Rightarrow c_5=0.$$

Thus

$$f_{k+1}^2=\frac{c_2}{3}(3x_1x_3-2x_2^2)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^4,\quad \text{for}\, k=5;$$
$$f_{k+1}^2=\frac{c_2}{3}(3x_1x_3-2x_2^2)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^3,\quad \text{for}\, k=4;$$
$$f_{k+1}^2=\frac{c_2}{3}(3x_1x_3-2x_2^2)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^2+c_3x_2^3x_3+c_4x_1x_2x_3^2$$
$$\Rightarrow X_+\frac{\partial f_{k+1}^2}{\partial x_3}=9c_3x_1x_2^2+8c_4x_1x_2^2+6c_4x_1^2x_3=0 \ \text{ since }\ \mathrm{wt}X_+\frac{\partial f_{k+1}^2}{\partial x_3}=5$$
$$\Rightarrow c_3=c_4=0.$$

Thus

$$f_{k+1}^2=\frac{c_2}{3}(3x_1x_3-2x_2^2)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^2,\quad \text{for } k=3;$$

$$f_{k+1}^2 = \frac{c_2}{3}(3x_1x_3 - 2x_2^2)(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8), \quad \text{for } k = 2.$$

Without loss of generality, we can suppose $f_{k+1}^2 = (3x_1x_3 - 2x_2^2)(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1}$ and assume $r_1 \neq 0$.

$$\begin{aligned}
(4) &= \left\langle \frac{\partial f_{k+1}^2}{\partial x_3}, \frac{\partial f_{k+1}^2}{\partial x_2}, \frac{\partial f_{k+1}^2}{\partial x_1}, X_-\left(\frac{\partial f_{k+1}^2}{\partial x_1}\right) \right\rangle \\
&= \langle x_1(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1}, x_2(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1}, \\
&\quad x_3(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1}, x_4(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1}\rangle, \\
(3) &= \left\langle \frac{\partial f_{k+1}^2}{\partial x_5}, X_-\left(\frac{\partial f_{k+1}^2}{\partial x_5}\right), X_-^2\left(\frac{\partial f_{k+1}^2}{\partial x_5}\right) \right\rangle \\
&= \langle (3x_1x_3 - 2x_2^2)(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-2}, \\
&\qquad (3x_1x_4 - x_2x_3)(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-2}, \\
&\qquad (2x_2x_4 - x_3^2)(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-2}\rangle.
\end{aligned}$$

As before we can prove that

$$\begin{aligned}
f_{k+1}^{-2} &= d_1(2x_2x_4 - x_3^2)(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1}, \\
f_{k+1}^{1} &= d_2x_2(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k}, \\
f_{k+1}^{-1} &= d_3x_3(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k},
\end{aligned}$$

for some constants d_1, d_2, d_3.

Suppose $d_2 = 0$ and $d_3 = 0$.

Let $(1) = \langle h \rangle$ for some $h \in I$. Then as before we have

$$f_{k+1}^0 = (c_4x_1x_4 + c_5x_2x_3)(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1} + g(x_5, x_6, x_7, x_8)$$

for some constants c_4, c_5 and g is a homogeneous polynomial of degree $k+1$ with weight 0 in x_5, x_6, x_7, x_8 variables.

$$\mathrm{wt}\frac{\partial f_{k+1}^0}{\partial x_j} = 0, 5 \leqslant j \leqslant 8 \Rightarrow \frac{\partial f_{k+1}^0}{\partial x_j} = \widetilde{c}_j h + \widetilde{e}_j(3x_1x_4 - x_2x_3)(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-2}$$

for some constants $\widetilde{c}_j, \widetilde{e}_j$, $5 \leqslant j \leqslant 8$,

$$\begin{aligned}
&\Rightarrow (k-1)r_j(c_4x_1x_4 + c_5x_2x_3)(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-2} + \frac{\partial g}{\partial x_j} \\
&\quad = \widetilde{c}_j h + \widetilde{e}_j(3x_1x_4 - x_2x_3)(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-2} \\
&\Rightarrow (k-1)r_jc_4 = 3\widetilde{e}_j, \quad (k-1)r_jc_5 = -\widetilde{e}_j, \quad \frac{\partial g}{\partial x_j} = \widetilde{c}_j h \\
&\Rightarrow c_4 = -3c_5, \quad g(x_5, x_6, x_7, x_8) = (r_5x_5 + r_6x_6 + r_7x_7 + r_8x_8)^{k+1}
\end{aligned}$$

for some constants r_5, r_6, r_7, r_8, for $\frac{\partial g}{\partial x_j} \in (1)$ and Lemma 1.

If $g = 0$ then no elements of I in (1), so $g \neq 0$. We may assume that $r_5 \neq 0$ and $(1) = \langle (r_5x_5 + r_6x_6 + r_7x_7 + r_8x_8)^k \rangle$.

Thus

$$f_{k+1}^0 = d_4(3x_1x_4 - x_2x_3)(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1} + (r_5x_5 + r_6x_6 + r_7x_7 + r_8x_8)^{k+1}$$

for some constant d_4.

Now $f = f_{k+1}^{-2} + f_{k+1}^{0} + f_{k+1}^{2}$. Then $\dim\langle \frac{\partial f}{\partial x_j}, 5 \leqslant j \leqslant 8\rangle \leqslant 2$.

Since

$$\operatorname{rank}\begin{pmatrix} (k-1)r_1 & d_4(k-1)r_1 & d_1(k-1)r_1 & (k+1)r_5 \\ (k-1)r_2 & d_4(k-1)r_2 & d_1(k-1)r_2 & (k+1)r_6 \\ (k-1)r_3 & d_4(k-1)r_3 & d_1(k-1)r_3 & (k+1)r_7 \\ (k-1)r_4 & d_4(k-1)r_4 & d_1(k-1)r_4 & (k-1)r_8 \end{pmatrix} \leqslant 2 \Rightarrow \dim I \leqslant 6,$$

this contradicts that $\dim I = 8$. Thus $d_2 \neq 0$ or $d_3 \neq 0$. Then $(1) = \langle (r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^k\rangle$. As before we can prove that

$$f_{k+1}^0 = d_5(3x_1x_4 - x_2x_3)(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1} + d_6(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k+1}$$

for some constants d_5, d_6.

Now

$$\begin{aligned} f &= f_{k+1}^{-2} + f_{k+1}^{-1} + f_{k+1}^{0} + f_{k+1}^{1} + f_{k+1}^{2} \\ &\Rightarrow \frac{\partial f}{\partial x_6} = \frac{r_2}{r_1}\frac{\partial f}{\partial x_5}, \frac{\partial f}{\partial x_7} = \frac{r_3}{r_1}\frac{\partial f}{\partial x_5}, \frac{\partial f}{\partial x_8} = \frac{r_4}{r_1}\frac{\partial f}{\partial x_5} \\ &\Rightarrow \dim I \leqslant 5. \end{aligned}$$

This contradicts that $\dim I = 8$. Therefore, $f_{k+1}^2 = 0$.

Similarly, we can prove that $f_{k+1}^{-2} = 0$.

For $i = 1$, if $f_{k+1}^1 \neq 0$, then as before (for $i = 2$) we can prove that

$$f_{k+1}^1 = c_1x_2(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^k + c_2x_2^2x_3(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-2}$$

for some constant $c_1, c_2, r_1, r_2, r_3, r_4$. Since $\mathrm{wt}(X_+\frac{\partial f_{k+1}^1}{\partial x_3}) = 4$, $0 = X_+\frac{\partial f_{k+1}^1}{\partial x_3} = X_+[c_2x_2^2(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-2}] = 6c_2x_1x_2(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-2} \Rightarrow c_2 = 0$. Thus $f_{k+1}^1 = c_1x_2(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^k$, $c_1 \neq 0$ and assume $r_1 \neq 0$. So

$$\begin{aligned} (4) = \langle & x_1(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1}, x_2(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1}, \\ & x_3(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1}, x_4(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1}\rangle. \\ (1) = \langle & (r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^k\rangle. \end{aligned}$$

As before we can prove that $f_{k+1}^{-1} = c_3x_3(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^k$ for some constant c_3. Then no elements in I are of weights $2, -2$. This contradicts that $I = (4) \oplus (3) \oplus (1)$. Thus $f_{k+1}^1 = 0$.

Similarly we can prove that $f_{k+1}^{-1} = 0$.

Therefore $f = f_{k+1}^0$. Then no elements in I are of weights $2, -2$. This contradicts that $I = (4) \oplus (3) \oplus (1)$. We conclude that Case 9 cannot occur.

Case 10. $I = (4) \oplus (2) \oplus (2)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $3, 1, -1, -3$.

For i is an odd integer, $\frac{\partial f_{k+1}^i}{\partial x_j}$ is an even integer for all $1 \leqslant j \leqslant 4$. It follows that f_{k+1}^i depends only on x_5, x_6, x_7, x_8 variables. Since $\mathrm{wt}(x_j) = 0$ for all $5 \leqslant j \leqslant 8$, $f_{k+1}^i = 0$.

For $|i| \geqslant 8$ and i is an even integer,

$$\left|\mathrm{wt}\frac{\partial f_{k+1}^i}{\partial x_j}\right| \geqslant 5 \Rightarrow \frac{\partial f_{k+1}^i}{\partial x_j} = 0, \Rightarrow f_{k+1}^i = 0, 1 \leqslant j \leqslant 8.$$

For $i = 6$,

$$\begin{aligned} \mathrm{wt}\frac{f_{k+1}^6}{\partial x_j} \geqslant 5 \Rightarrow & \frac{f_{k+1}^6}{\partial x_j} = 0, \quad 2 \leqslant j \leqslant 8 \\ \Rightarrow & f_{k+1}^6 \text{ depends only on } x_1 \text{ variable} \\ \Rightarrow & f_{k+1}^6 = c x_1^2 \text{ where } c \text{ is a constant} \\ \Rightarrow & f_{k+1}^6 = 0 \text{ because } k \geqslant 2. \end{aligned}$$

Similarly, we can prove that $f_{k+1}^{-6} = 0$.

For $i = 4$,

$$\begin{aligned} \mathrm{wt}\frac{\partial f_{k+1}^4}{\partial x_j} \geqslant 4 \Rightarrow & \frac{\partial f_{k+1}^4}{\partial x_j} = 0 \quad \text{for } 3 \leqslant j \leqslant 8 \\ \Rightarrow & f_{k+1}^4 \text{ depends only on } x_1, x_2 \text{ variables} \\ \Rightarrow & f_{k+1}^4 = c x_1 x_2 \text{ or } d x_2^4 \text{ for some constants } c, d \\ \Rightarrow & f_{k+1}^4 = d x_2^4 \text{ because } k \geqslant 2. \end{aligned}$$

If $d \neq 0$, then $\frac{1}{4d}\frac{\partial f_{k+1}^4}{\partial x_2} = x_2^3 \in I$. Thus $\frac{1}{162}X_+^3(x_2^3) = x_1^3 \in I$. So $\langle x_1^3, X_-^i(x_1^3), 1 \leqslant i \leqslant 9\rangle$ is a subspace of dimension 10 in I, which contradicts that $\dim I = 8$. Thus $f_{k+1}^4 = 0$.

Similarly, we can prove that $f_{k+1}^{-4} = 0$.

For $i = 2$,

$$\begin{aligned} \mathrm{wt}\frac{\partial f_{k+1}^2}{\partial x_j} = 5, \quad \mathrm{wt}\frac{\partial f_{k+1}^2}{\partial x_j} = 2 \Rightarrow & \frac{\partial f_{k+1}^2}{\partial x_4} = \frac{\partial f_{k+1}^2}{\partial x_j} = 0, \quad 5 \leqslant j \leqslant 8 \\ \Rightarrow & f_{k+1}^2 = f_{k+1}^2(x_1, x_2, x_3). \end{aligned}$$

Similarly, we can prove that $f_{k+1}^{-2} = f_{k+1}^{-2}(x_2, x_3, x_4)$.

For $i = 0$,

$$\mathrm{wt}\frac{\partial f_{k+1}^0}{\partial x_j} = 0 \Rightarrow \frac{\partial f_{k+1}^0}{\partial x_j} = 0 \Rightarrow f_{k+1}^0 = f_{k+1}^0(x_1, x_2, x_3, x_4), \quad 5 \leqslant j \leqslant 8.$$

We can write

$$\begin{aligned} f = & f_{k+1}^{-2}(x_2, x_3, x_4) + f_{k+1}^0(x_1, x_2, x_3, x_4) + f_{k+1}^2(x_1, x_2, x_3) \\ & \Rightarrow \frac{\partial f}{\partial x_5} = \frac{\partial f}{\partial x_6} = \frac{\partial f}{\partial x_7} = \frac{\partial f}{\partial x_8} = 0 \\ & \Rightarrow \dim I \leqslant 4 \text{ which contradicts that } \dim I = 8. \end{aligned}$$

Therefore this case cannot occur.

Case 11. $I = (4) \oplus (2) \oplus (1) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $3, 1, -1, -3, 0$.

For i is an odd integer and $|i| \geqslant 5$, $\mathrm{wt}\frac{\partial f^i_{k+1}}{\partial x_j}$ is an even integer and nonzero for all $1 \leqslant j \leqslant 4$. It follows that f^i_{k+1} depends only on x_5, x_6, x_7, x_8 variables. Since $\mathrm{wt}(x_j) = 0$ for all $5 \leqslant j \leqslant 8$, $f^i_{k+1} = 0$.

For $i = 3$,

$$\begin{aligned}
&\mathrm{wt}\frac{\partial f^3_{k+1}}{\partial x_2} = 2, \mathrm{wt}\frac{\partial f^3_{k+1}}{\partial x_3} = 4, \mathrm{wt}\frac{\partial f^3_{k+1}}{\partial x_4} = 6\\
&\quad \Rightarrow \frac{\partial f^3_{k+1}}{\partial x_2} = \frac{\partial f^3_{k+1}}{\partial x_3} = \frac{\partial f^3_{k+1}}{\partial x_4} = 0\\
&\quad \Rightarrow f^3_{k+1} \text{ depends only on } x_1, x_5, x_6, x_7, x_8 \text{ variables}\\
&\quad \Rightarrow f^3_{k+1} = x_1 g_k(x_5, x_6, x_7, x_8),
\end{aligned}$$

where $g_k(x_5, x_6, x_7, x_8)$ is a homogeneous polynomial of degree k in x_5, x_6, x_7, x_8 variables.

Similarly, we can prove that $f^{-3}_{k+1} = x_4 g'_k(x_5, x_6, x_7, x_8)$, where $g'_k(x_5, x_6, x_7, x_8)$ is a homogeneous polynomial of degree k in x_5, x_6, x_7, x_8 variables.

For $i = 1$,

$$\begin{aligned}
&\mathrm{wt}\frac{\partial f^1_{k+1}}{\partial x_1} = -2, \mathrm{wt}\frac{\partial f^1_{k+1}}{\partial x_3} = 2, \mathrm{wt}\frac{\partial f^1_{k+1}}{\partial x_4} = 4\\
&\quad \Rightarrow \frac{\partial f^1_{k+1}}{\partial x_1} = \frac{\partial f^1_{k+1}}{\partial x_3} = \frac{\partial f^1_{k+1}}{\partial x_4} = 0\\
&\quad \Rightarrow f^1_{k+1} \text{ depends only on } x_2, x_5, x_6, x_7, x_8 \text{ variables}\\
&\quad \Rightarrow f^1_{k+1} = x_2 h_k(x_5, x_6, x_7, x_8),
\end{aligned}$$

where $h_k(x_5, x_6, x_7, x_8)$ is a homogeneous polynomial of degree k in x_5, x_6, x_7, x_8 variables.

Similarly, we can prove that $f^{-1}_{k+1} = x_3 h'_k(x_5, x_6, x_7, x_8)$, where $h'_k(x_5, x_6, x_7, x_8)$ is a homogeneous polynomial of degree k in x_5, x_6, x_7, x_8 variables. With the same argument as Case 10, we can prove that i is an even integer and $|i| \geqslant 4$

$$f^i_{k+1} = 0 \quad \text{and} \quad f^2_{k+1} = f^2_{k+1}(x_1, x_2, x_3), \quad f^{-2}_{k+1} = f^{-2}_{k+1}(x_2, x_3, x_4).$$

Suppose on the contrary that $f^2_{k+1} \neq 0$, write $f^2_{k+1} = \sum_{\beta \geqslant 0} x_1^{\frac{2\beta-k+1}{2}} x_2^{\frac{3k+1-4\beta}{2}} x_3^{\beta}$.

Let β_0 be the largest integer such that $b_{\beta_0} \neq 0$. By Corollary 3, either $X_-^{\beta_0}\frac{\partial f^2_{k+1}}{\partial x_1}$ or $X_-^{\beta_0}\frac{\partial f^2_{k+1}}{\partial x_2}$ is nonzero. Since

$$\begin{aligned}
&\mathrm{wt}\left(X_-^{\beta_0}\frac{\partial f^2_{k+1}}{\partial x_1}\right) = -1 - 2\beta_0, \quad \mathrm{wt}\left(X_-^{\beta_0}\frac{\partial f^2_{k+1}}{\partial x_2}\right) = 1 - 2\beta_0 \text{ and } X_-^{\beta_0}\frac{\partial f^2_{k+1}}{\partial x_i} \in I,\ i = 1, 2\\
&\quad \Rightarrow -1 - 2\beta_0 \geqslant 3 \text{ or } 1 - 2\beta_0 \geqslant -3\\
&\quad \Rightarrow \beta_0 \leqslant 1 \text{ or } \beta_0 \leqslant 2,
\end{aligned}$$

without loss of generality, we suppose that $\beta_0 \leqslant 2$.

$\beta_0 = 2 \Rightarrow b_2 \neq 0$, $k = 3$, $f^2_4 = b_1 x_2^3 x_3 + b_2 x_1 x_2 x_3^2$, $k = 5$, $f^2_6 = b_2 x_2^4 x_3^2$.

$\beta_0 = 1 \Rightarrow b_1 \neq 0$, $k = 3, f_4^2 = b_1 x_2^3 x_3$. If $f_4^2 = b_1 x_2^3 x_3 + b_2 x_1 x_2 x_3^2$ where $b_2 \neq 0$,

$$\mathrm{wt}\left(X_+ \frac{\partial f_4^2}{\partial x_3}\right) = 5 \Rightarrow 0 = X_+ \frac{\partial f_4^2}{\partial x_3} = (9b_1 + 8b_2) x_1 x_2^2 + 6b_2 x_1^2 x_3 \Rightarrow b_1 = b_2 = 0.$$

This contradicts that $b_2 \neq 0$.

If $f_6^2 = b_2 x_2^4 x_3^2$ where $b_2 \neq 0$,

$$\mathrm{wt}\left(X_+ \frac{\partial f_4^2}{\partial x_3}\right) = 5 \Rightarrow 0 = X_+ \frac{\partial f_4^2}{\partial x_3} = 24 b_2 x_1 x_2^3 x_3 + 8 b_2 x_2^5 \Rightarrow b_2 = 0.$$

This contradicts that $b_2 \neq 0$.

If $f_4^2 = b_1 x_2^3 x_3$ where $b_1 \neq 0$,

$$\mathrm{wt}\left(X_+ \frac{\partial f_4^2}{\partial x_3}\right) = 5 \Rightarrow 0 = X_+ \frac{\partial f_4^2}{\partial x_3} = 9 b_1 x_1 x_2^2 \Rightarrow b_1 = 0.$$

This contradicts that $b_1 \neq 0$. Therefore $f_{k+1}^2 = 0$.

Similarly, we can prove that $f_{k+1}^{-2} = 0$. Thus we can write

$$\begin{aligned} f &= f_{k+1}^{-3} + f_{k+1}^{-1} + f_{k+1}^{0} + f_{k+1}^{1} + f_{k+1}^{3} \\ &= x_4 g_k'(x_5, x_6, x_7, x_8) + x_3 h_k'(x_5, x_6, x_7, x_8) + f_{k+1}^0(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \\ &\quad + x_2 h_k(x_5, x_6, x_7, x_8) + x_1 g_k(x_5, x_6, x_7, x_8). \end{aligned}$$

Again since $\mathrm{wt}\frac{\partial f_{k+1}^3}{\partial x_j} = 3$, $5 \leqslant j \leqslant 8$, in view of Lemma 1 there are constants r_1, r_2, r_3 and r_4 such that $f_{k+1}^3 = x_1(r_1 x_5 + r_2 x_6 + r_3 x_7 + r_4 x_8)^k$. Similarly, we can prove that $f_{k+1}^{-3} = x_4(r_5 x_5 + r_6 x_6 + r_7 x_7 + r_8 x_8)^k$ for some constants r_5, r_6, r_7 and r_8.

If one of f_{k+1}^3 and f_{k+1}^{-3} is not zero, we may assume without loss of generality that $f_{k+1}^3 \neq 0$ and $r_1 \neq 0$. Then

$$\begin{aligned} (4) = \langle & x_1(r_1 x_5 + r_2 x_6 + r_3 x_7 + r_4 x_8)^{k-1}, x_2(r_1 x_5 + r_2 x_6 + r_3 x_7 + r_4 x_8)^{k-1}, \\ & x_3(r_1 x_5 + r_2 x_6 + r_3 x_7 + r_4 x_8)^{k-1}, x_4(r_1 x_5 + r_2 x_6 + r_3 x_7 + r_4 x_8)^{k-1} \rangle. \end{aligned}$$

If $f_{k+1}^{-3} = x_4(r_5 x_5 + r_6 x_6 + r_7 x_7 + r_8 x_8)^k \neq 0$, then there exist $5 \leqslant t \leqslant 8$ such that $\frac{\partial f_{k+1}^{-3}}{\partial x_t} = k r_t x_4 (r_5 x_5 + r_6 x_6 + r_7 x_7 + r_8 x_8)^{k-1} \neq 0$ since $\mathrm{wt}\frac{\partial f_{k+1}^{-3}}{\partial x_t} = -3$. So $\frac{\partial f_{k+1}^{-3}}{\partial x_t} = k r_t x_4 (r_5 x_5 + r_6 x_6 + r_7 x_7 + r_8 x_8)^{k-1} = c x_4 (r_1 x_5 + r_2 x_6 + r_3 x_7 + r_4 x_8)^{k-1}$ for some constant c.

Thus $\frac{\partial f_{k+1}^{-3}}{\partial x_t} = d x_4 (r_5 x_5 + r_6 x_6 + r_7 x_7 + r_8 x_8)^{k-1}$ for some nonzero constant d. This means $f_{k+1}^{-3} = e x_4 (r_1 x_5 + r_2 x_6 + r_3 x_7 + r_4 x_8)^k$ for some nonzero constant e. Since

$$\begin{aligned} \mathrm{wt}\frac{\partial f_{k+1}^0}{\partial x_1} = -3 &\Rightarrow \frac{\partial f_{k+1}^0}{\partial x_1} = c_1 x_4 (r_1 x_5 + r_2 x_6 + r_3 x_7 + r_4 x_8)^{k-1}, \\ \mathrm{wt}\frac{\partial f_{k+1}^0}{\partial x_1} = 3 &\Rightarrow \frac{\partial f_{k+1}^0}{\partial x_4} = c_2 x_1 (r_1 x_5 + r_2 x_6 + r_3 x_7 + r_4 x_8)^{k-1} \end{aligned}$$

for some constants c_1 and c_2,

$$\begin{aligned} f_{k+1}^0 &= c_1 x_1 x_4 (r_1 x_5 + r_2 x_6 + r_3 x_7 + r_4 x_8)^{k-1} + g_{k+1}^0(x_2, x_3, x_4, x_5, x_6, x_7, x_8) \\ &\Rightarrow \frac{\partial f_{k+1}^0}{\partial x_4} = c_1 x_1 (r_1 x_5 + r_2 x_6 + r_3 x_7 + r_4 x_8)^{k-1} + \frac{\partial g_{k+1}^0}{\partial x_4} \end{aligned}$$

$$= c_2 x_1 (r_1 x_5 + r_2 x_6 + r_3 x_7 + r_4 x_8)^{k-1}$$
$$\Rightarrow c_1 = c_2 \text{ and } \frac{\partial g^0_{k+1}}{\partial x_4} = 0$$
$$\Rightarrow f^0_{k+1} = c_1 x_1 x_4 (r_1 x_5 + r_2 x_6 + r_3 x_7 + r_4 x_8)^{k-1} + g^0_{k+1}(x_2, x_3, x_5, x_6, x_7, x_8),$$

where $g^0_{k+1}(x_2, x_3, x_5, x_6, x_7, x_8)$ is a homogeneous polynomial of degree $k+1$, weight 0 and in $x_2, x_3, x_5, x_6, x_7, x_8$ variables.

Write $g^0_{k+1}(x_2, x_3, x_5, x_6, x_7, x_8) = \sum_{\beta_0 \geqslant 0} h^0_{k+1-2\beta}(x_5, x_6, x_7, x_8) x_2^\beta x_3^\beta$ where $h^0_{k+1-2\beta}(x_5, x_6, x_7, x_8)$ is a homogeneous polynomial of degree $k+1-2\beta$, weight 0 and in x_5, x_6, x_7, x_8 variables.

If $g^0_{k+1} \neq 0$, then let β_0 be the largest integer such that $h^0_{k+1-2\beta_0} \neq 0$. By Lemma 2

(1) $\beta_0 \neq 0 \Rightarrow X_-^{\beta_0} \frac{\partial f^0_{k+1}}{\partial x_2} = X_-^{\beta_0} \frac{\partial g^0_{k+1}}{\partial x_2} \neq 0 \Rightarrow -1 - 2\beta_0 \geqslant -3 \Rightarrow \beta_0 \leqslant 1 \Rightarrow \beta_0 = 1.$

$$\beta_0 = 1 \Rightarrow h^0_{k-1}(x_5, x_6, x_7, x_8) \neq 0, g^0_{k+1} = h^0_{k+1}(x_5, x_6, x_7, x_8) + h^0_{k-1}(x_5, x_6, x_7, x_8) x_2 x_3.$$

(2) $\beta_0 = 0 \Rightarrow g^0_{k+1} = h^0_{k+1}(x_5, x_6, x_7, x_8)$.

If $g^0_{k+1} = h^0_{k+1}(x_5, x_6, x_7, x_8) + h^0_{k-1}(x_5, x_6, x_7, x_8) x_2 x_3$ where $h^0_{k-1} \neq 0$, since $h^0_{k-1} \neq 0 \Rightarrow \exists\, 5 \leqslant t \leqslant 8$, then $\frac{\partial h^0_{k-1}}{\partial x_t} \neq 0$.

$$\text{wt} X_+ \frac{\partial f^0_{k+1}}{\partial x_t} = 2 \Rightarrow 0 = X_+ \frac{\partial f^0_{k+1}}{\partial x_t} = 3c_1(k-1) r_t x_1 x_3 (r_1 x_5 + r_2 x_6 + r_3 x_7 + r_4 x_8)^{k-2} + 3\frac{\partial h^0_{k-1}}{\partial x_t} x_1 x_3 + 4 \frac{\partial h^0_{k-1}}{\partial x_t} x_2^2 \Rightarrow \frac{\partial h^0_{k-1}}{\partial x_t} = 0.$$

This contradicts that $\frac{\partial h^0_{k-1}}{\partial x_t} \neq 0$.

Therefore $g^0_{k+1} = h^0_{k+1}(x_5, x_6, x_7, x_8)$. Thus $f^0_{k+1} = c_1 x_1 x_4 (r_1 x_5 + r_2 x_6 + r_3 x_7 + r_4 x_8)^{k-1} + h^0_{k+1}(x_5, x_6, x_7, x_8)$. Now $\text{wt} \frac{\partial f^0_{k+1}}{\partial x_5} = 0 \Rightarrow X_- \frac{\partial f^0_{k+1}}{\partial x_5} = 0$

$$\Rightarrow r_1 c_1 (k-1)(r_1 x_5 + r_2 x_6 + r_3 x_7 + r_4 x_8)^{k-2} = 0$$
$$\Rightarrow c_1 = 0 \text{ since } r_1 \neq 0 \Rightarrow f^0_{k+1} = h^0_{k+1}(x_5, x_6, x_7, x_8).$$

We can write

$$f = e x_4 (r_1 x_5 + r_2 x_6 + r_3 x_7 + r_4 x_8)^{k-1} + x_3 h'_k(x_5, x_6, x_7, x_8) + h^0_{k+1}(x_5, x_6, x_7, x_8) + x_2 h_k(x_5, x_6, x_7, x_8) + x_1 (r_1 x_5 + r_2 x_6 + r_3 x_7 + r_4 x_8)^k$$
$$\Rightarrow \frac{\partial f}{\partial x_4} = e \frac{\partial f}{\partial x_1}$$
$$\Rightarrow \dim I \leqslant 7, \text{ which contradicts that } \dim I = 8.$$

If $g^0_{k+1}(x_2, x_3, x_5, x_6, x_7, x_8) = 0$, then $f^0_{k+1} = c_1 x_1 x_4 (r_1 x_5 + r_2 x_6 + r_3 x_7 + r_4 x_8)^{k-1}$. Similarly, we can prove $c_1 = 0$ i.e., $f^0_{k+1} = 0$.

We can write

$$f = e x_4 (r_1 x_5 + r_2 x_6 + r_3 x_7 + r_4 x_8)^{k-1} + x_3 h'_k(x_5, x_6, x_7, x_8) + x_2 h_k(x_5, x_6, x_7, x_8) + x_1 (r_1 x_5 + r_2 x_6 + r_3 x_7 + r_4 x_8)^k$$
$$\Rightarrow \frac{\partial f}{\partial x_4} = e \frac{\partial f}{\partial x_1}$$

$$\Rightarrow \dim I \leqslant 7, \text{ which contradicts that } \dim I = 8.$$

If $f_{k+1}^{-3} = x_4(r_5x_5 + r_6x_6 + r_7x_7 + r_8x_8)^k = 0$, with the same argument as before, we can prove $f_{k+1}^0 = c_1x_1x_4(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^{k-1} + g_{k+1}^0(x_2, x_3, x_5, x_6, x_7, x_8)$.

If $g_{k+1}^0 \neq 0$, with the same argument as before, we can prove $f_{k+1}^0 = h_{k+1}^0(x_5, x_6, x_7, x_8)$. Write

$$\begin{aligned} f &= f_{k+1}^{-1} + f_{k+1}^0 + f_{k+1}^1 + f_{k+1}^3 \\ &= x_3h'_k(x_5, x_6, x_7, x_8) + h_{k+1}^0(x_5, x_6, x_7, x_8) \\ &\quad + x_2h_k(x_5, x_6, x_7, x_8) + x_1(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^k. \end{aligned}$$

Thus $\frac{\partial f}{\partial x_4} = 0 \Rightarrow \dim I \leqslant 7$, which contradicts that $\dim I = 8$.

If $g_{k+1}^0 = 0$, with the same argument as before, we can prove that $f_{k+1}^0 = 0$. Write $f = f_{k+1}^{-1} + f_{k+1}^1 + f_{k+1}^3 = x_3h'_k(x_5, x_6, x_7, x_8) + x_2h_k(x_5, x_6, x_7, x_8) + x_1(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8)^k \Rightarrow \frac{\partial f}{\partial x_4} = 0 \Rightarrow \dim I \leqslant 7$, which contradicts that $\dim I = 8$.

Therefore $f_{k+1}^3 = 0$ and $f_{k+1}^{-3} = 0$, so

$$f = f_{k+1}^{-1} + f_{k+1}^0 + f_{k+1}^1 = x_3h'_k(x_5, x_6, x_7, x_8) + f_{k+1}^0 + x_2h_k(x_5, x_6, x_7, x_8).$$

If one of the f_{k+1}^1 and f_{k+1}^{-1} is not zero, without lost of generality, we may assume that $f_{k+1}^1 \neq 0$, and $\frac{\partial h_k}{\partial x_5} \neq 0$. Then

$$\begin{aligned} &(4) = \left\langle x_1\frac{\partial h_k}{\partial x_5}, x_2\frac{\partial h_k}{\partial x_5}, x_3\frac{\partial h_k}{\partial x_5}, x_4\frac{\partial h_k}{\partial x_5} \right\rangle \\ &\mathrm{wt}\frac{\partial f_{k+1}^0}{\partial x_1} = -3 \Rightarrow \frac{\partial f_{k+1}^0}{\partial x_1} = c_3x_4\frac{\partial h_k}{\partial x_5}, \\ &\mathrm{wt}\frac{\partial f_{k+1}^0}{\partial x_4} = 3 \Rightarrow \frac{\partial f_{k+1}^0}{\partial x_4} = c_4x_1\frac{\partial h_k}{\partial x_5}, \end{aligned}$$

for some constants c_3, c_4,

$$\begin{aligned} f_{k+1}^0 &= c_3x_1x_4\frac{\partial h_k}{\partial x_5} + h_{k+1}^0(x_2, x_3, x_4, x_5, x_6, x_7, x_8) \\ &\Rightarrow \frac{\partial f_{k+1}^0}{\partial x_4} = c_3x_1\frac{\partial h_k}{\partial x_5} + \frac{\partial h_{k+1}^0}{\partial x_4} = c_4x_1\frac{\partial h_k}{\partial x_5} \\ &\Rightarrow c_3 = c_4, \quad \frac{\partial h_{k+1}^0}{\partial x_4} = 0 \\ &\Rightarrow f_{k+1}^0 = c_3x_1x_4\frac{\partial h_k}{\partial x_5} + h_{k+1}^0(x_2, x_3, x_5, x_6, x_7, x_8), \end{aligned}$$

where h_{k+1}^0 is a homogeneous polynomial of degree $k+1$, weight 0 and in $x_2, x_3, x_5, x_6, x_7, x_8$ variables. Write $h_{k+1}^0 = \sum_{\beta \geqslant 0} h_{k+1-2\beta}^0(x_5, x_6, x_7, x_8)x_2^\beta x_3^\beta$ where $h_{k+1-2\beta}^0(x_5, x_6, x_7, x_8)$ is a homogeneous polynomial of degree $k+1-2\beta$, weight 0 and in x_5, x_6, x_7, x_8 variables for any integer β.

If $h_{k+1}^0 \neq 0$, then let β_0 be the largest integer such that $h_{k+1-2\beta_0}^0 \neq 0$, i.e., there exists $5 \leqslant t \leqslant 8$, such that $\frac{\partial h_{k+1-2\beta_0}^0}{\partial x_t} \neq 0$,

$$\frac{\partial f_{k+1}^0}{\partial x_t} = c_3x_1x_4\frac{\partial^2 h_k}{\partial x_t \partial x_5} + \sum_{\beta \geqslant 0}\frac{\partial h_{k+1-2\beta}^0}{\partial x_t}x_2^\beta x_3^\beta \neq 0 \in I.$$

$$\mathrm{wt}\frac{\partial f_{k+1}^0}{\partial x_t} = 0 \Rightarrow \mathrm{wt}\left(X_-\frac{\partial f_{k+1}^0}{\partial x_t}\right) = -2 \Rightarrow X_-\left(\frac{\partial f_{k+1}^0}{\partial x_t}\right) = 0$$
$$\Rightarrow c_3x_2x_4\frac{\partial^2 h_k}{\partial x_t\partial x_5} + \sum_{\beta\geqslant 0}\frac{\partial h_{k+1-2\beta}^0}{\partial x_t}(\beta x_2^{\beta-1}x_3^{\beta+1} + \beta x_2^\beta x_3^{\beta-1}x_4) = 0$$
$$\Rightarrow \beta_0 = 0.$$

Thus $f_{k+1}^0 = c_3x_1x_4\frac{\partial h_k}{\partial x_5} + h_{k+1}^0(x_5,x_6,x_7,x_8)$. If $c_3 \neq 0$ and since $\frac{\partial h_k}{\partial x_5} \neq 0$ and $k \geqslant 2$, there exists $5 \leqslant t \leqslant 8$ such that

$$\frac{\partial^2 h_k}{\partial x_t\partial x_5} \neq 0 \Rightarrow \frac{\partial f_{k+1}^0}{\partial x_t} = c_3x_1x_4\frac{\partial^2 h_k}{\partial x_t\partial x_5} + \frac{\partial h_{k+1}^0}{\partial x_t} \neq 0 \in I,$$
$$\mathrm{wt}\frac{\partial f_{k+1}^0}{\partial x_t} = 0 \Rightarrow \mathrm{wt}\left(X_-\frac{\partial f_{k+1}^0}{\partial x_t}\right) = -2 \Rightarrow X_-\left(\frac{\partial f_{k+1}^0}{\partial x_t}\right) = 0$$
$$\Rightarrow c_3x_2x_4\frac{\partial^2 h_k}{\partial x_t\partial x_5} = 0, \text{ which contradicts that } c_3 \neq 0$$
$$\Rightarrow f_{k+1}^0 = h_{k+1}^0(x_5,x_6,x_7,x_8).$$

We can write

$$f = x_3h_k'(x_5,x_6,x_7,x_8) + h_{k+1}^0(x_5,x_6,x_7,x_8) + x_2h_k(x_5,x_6,x_7,x_8)$$
$$\Rightarrow \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_4} = 0 \Rightarrow \dim I \leqslant 6, \text{ which contradicts that } \dim I = 8.$$

If $h_{k+1}^0 = 0$, then $f_{k+1}^0 = c_3x_1x_4\frac{\partial h_k}{\partial x_5}$.

With the same argument as before, we can prove that $c_3 = 0$, i.e., $f_{k+1}^0 = 0$. We can write

$$f = x_3h_k'(x_5,x_6,x_7,x_8) + x_2h_k(x_5,x_6,x_7,x_8)$$
$$\Rightarrow \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_4} = 0 \Rightarrow \dim I \leqslant 6, \text{ which contradicts that } \dim I = 8.$$

Therefore $f_{k+1}^1 = 0$ and $f_{k+1}^{-1} = 0$. We write $f = f_{k+1}^0(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)$. Since $\mathrm{wt}\frac{\partial f_{k+1}^0}{\partial x_j} = 0$, $5 \leqslant j \leqslant 8$,

$$\left\langle\frac{\partial f_{k+1}^0}{\partial x_5},\frac{\partial f_{k+1}^0}{\partial x_6},\frac{\partial f_{k+1}^0}{\partial x_7},\frac{\partial f_{k+1}^0}{\partial x_8}\right\rangle \subseteq (1)\oplus(1)$$
$$\Rightarrow \dim I = \langle\tfrac{\partial f}{\partial x_1},\tfrac{\partial f}{\partial x_2},\ldots,\tfrac{\partial f}{\partial x_8}\rangle = \dim\left\langle\frac{\partial f_{k+1}^0}{\partial x_1},\frac{\partial f_{k+1}^0}{\partial x_2},\ldots,\frac{\partial f_{k+1}^0}{\partial x_8}\right\rangle \leqslant 6,$$

which contradicts that $\dim I = 8$.

Therefore this case cannot happen.

Case 12. $I = (4)\oplus(1)\oplus(1)\oplus(1)\oplus(1)$.

With the same argument as Case 11, we can write $f = f_{k+1}^0(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)$. Let us first observe that

$$\left[\frac{\partial}{\partial x_1},X_-\right] = 0,\quad \left[\frac{\partial}{\partial x_2},X_-\right] = \frac{\partial}{\partial x_1},\quad \left[\frac{\partial}{\partial x_3},X_-\right] = \frac{\partial}{\partial x_2},$$
$$\left[\frac{\partial}{\partial x_4},X_-\right] = \frac{\partial}{\partial x_3},\quad \left[\frac{\partial}{\partial x_i},X_-\right] = 0,\quad 5 \leqslant i \leqslant 8.$$

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $-3,-1,1,3$ and 0. Since f is of weight 0, we have

$$\begin{aligned}
&\mathrm{wt}\frac{\partial f}{\partial x_1}=-3,\quad \mathrm{wt}\frac{\partial f}{\partial x_2}=-1,\quad \mathrm{wt}\frac{\partial f}{\partial x_3}=1,\quad \mathrm{wt}\frac{\partial f}{\partial x_4}=3,\quad \mathrm{wt}\frac{\partial f}{\partial x_i}=0,\quad 5\leqslant i\leqslant 8\\
&\Rightarrow \mathrm{wt}X_-\frac{\partial f}{\partial x_1}=-5,\quad \mathrm{wt}X_-\frac{\partial f}{\partial x_2}=-3,\quad \mathrm{wt}X_-\frac{\partial f}{\partial x_3}=-1,\\
&\qquad \mathrm{wt}X_-\frac{\partial f}{\partial x_4}=1,\quad \mathrm{wt}X_-\frac{\partial f}{\partial x_i}=-2,\quad 5\leqslant i\leqslant 8\\
&\Rightarrow X_-\frac{\partial f}{\partial x_1}=0,\quad X_-\frac{\partial f}{\partial x_2}=c_1\frac{\partial f}{\partial x_1},\quad X_-\frac{\partial f}{\partial x_3}=c_2\frac{\partial f}{\partial x_2},\\
&\qquad X_-\frac{\partial f}{\partial x_4}=c_3\frac{\partial f}{\partial x_3},\quad X_-\frac{\partial f}{\partial x_i}=0, 5\leqslant i\leqslant 8
\end{aligned}$$

$$\Rightarrow \left.\begin{array}{l}
\dfrac{\partial}{\partial x_1}(X_-f)=X_-\left(\dfrac{\partial f}{\partial x_1}\right)=0,\\
\dfrac{\partial}{\partial x_2}(X_-f)=X_-\left(\dfrac{\partial f}{\partial x_2}\right)+\dfrac{\partial f}{\partial x_1}=(c_1+1)\dfrac{\partial f}{\partial x_1},\\
\dfrac{\partial}{\partial x_3}(X_-f)=X_-\left(\dfrac{\partial f}{\partial x_3}\right)+\dfrac{\partial f}{\partial x_2}=(c_2+1)\dfrac{\partial f}{\partial x_2},\\
\dfrac{\partial}{\partial x_4}(X_-f)=X_-\left(\dfrac{\partial f}{\partial x_4}\right)+\dfrac{\partial f}{\partial x_3}=(c_3+1)\dfrac{\partial f}{\partial x_3},\\
\dfrac{\partial}{\partial x_i}(X_-f)=X_-\left(\dfrac{\partial f}{\partial x_i}\right)=0,\quad 5\leqslant i\leqslant 8
\end{array}\right\}\qquad (*)$$

$$\begin{aligned}
&\Rightarrow \frac{\partial}{\partial x_1}(X_-^2f)=X_-\left(\frac{\partial}{\partial x_1}(X_-f)\right)=0,\\
&\quad \frac{\partial}{\partial x_2}(X_-^2f)=X_-\left(\frac{\partial}{\partial x_2}(X_-f)\right)+\frac{\partial}{\partial x_1}X_-f=0,\\
&\quad \frac{\partial}{\partial x_3}(X_-^2f)=X_-\left(\frac{\partial}{\partial x_3}(X_-f)\right)+\frac{\partial}{\partial x_2}X_-f=(c_1c_2+2c_1+1)\frac{\partial f}{\partial x_1},\\
&\quad \frac{\partial}{\partial x_4}(X_-^2f)=X_-\left(\frac{\partial}{\partial x_4}(X_-f)\right)+\frac{\partial}{\partial x_3}X_-f=(c_2c_3+2c_2+1)\frac{\partial f}{\partial x_2},\\
&\quad \frac{\partial}{\partial x_i}(X_-^2f)=X_-\left(\frac{\partial}{\partial x_i}(X_-f)\right)=0\\
&\Rightarrow X_-^2f \text{ is a homogeneous polynomial in } x_3,x_4 \text{ variable of weight } -4\\
&\Rightarrow X_-^2f=d_1x_3^4+d_2x_3x_4.
\end{aligned}$$

Since degree of X_-^2f is $k+1\geqslant 3$. We have $X_-^2f=d_1x_3^4$. If $d_1\neq 0$ then $x_3^3=\frac{1}{4d_1}\frac{\partial}{\partial x_3}X_-^2f$ is an element in I. It follows that $\langle x_1^3, x_1^2x_2, 3x_1^2x_3+8x_1x_2^2, 9x_1x_2x_3+4x_2^2, 3x_1x_3^2+8x_2^2x_3, x_2x_3^2, x_3^2, x_3^2x_4, x_3x_4^2, x_4^3\rangle$ is a ten dimensional subspace in I. This contradicts that dim $I=8$.

Thus, we have $X_-^2f=0$, consequently we have $c_1c_2+2c_1+1=0=c_2c_3+2c_2+1$. Therefore either c_1,c_2 and c_3 are -1 or c_1,c_2 and c_3 are not -1. We claim the latter case cannot occur. Suppose on the contrary that $c_1\neq -1$, $c_2\neq -1$ and $c_3\neq -1$. From $(*)$, we know that $\frac{\partial f}{\partial x_1}$, $\frac{\partial f}{\partial x_2}$ and $\frac{\partial f}{\partial x_3}$ are polynomials in x_2,x_3 and x_4 variables. Hence, we have

$$\frac{\partial^2 f}{\partial x_1^2}=\frac{\partial^2 f}{\partial x_i\partial x_1}=\frac{\partial^2 f}{\partial x_1\partial x_2}=\frac{\partial^2 f}{\partial x_i\partial x_2}=\frac{\partial^2 f}{\partial x_1\partial x_3}=\frac{\partial^2 f}{\partial x_i\partial x_3}=0,\quad 5\leqslant i\leqslant 8.$$

Consequently, f does not involve $x_1^2, x_1x_i, x_1x_2, x_1x_3, x_2x_i$ and x_3x_i, $5 \leqslant i \leqslant 8$. It follows easily that

$$f = c_4x_1x_4 + \Phi^0_{k+1}(x_2, x_3, x_4, x_5, x_6, x_7, x_8),$$

where $\Phi^0_{k+1}(x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ is a homogeneous polynomial of degree $k+1$ and weight 0. Since f is a homogeneous polynomial of degree $k+1 \geqslant 3$. We have $c_4 = 0$ and $f = \Phi^0_{k+1}(x_2, x_3, x_4, x_5, x_6, x_7, x_8)$. Thus we have $\frac{\partial f}{\partial x_1} = 0$ and $\dim I \leqslant 7$. This contradict that $\dim I = 8$.

In conclusion, we have $c_1 = -1 = c_2 = c_3$ and $\frac{\partial}{\partial x_i}(X_-f) = 0$ for all $1 \leqslant i \leqslant 8$. So X_-f is necessary zero and we have

$$X_-\frac{\partial f}{\partial x_1} = 0, \quad X_-\frac{\partial f}{\partial x_2} = -\frac{\partial f}{\partial x_1}, \quad X_-\frac{\partial f}{\partial x_3} = -\frac{\partial f}{\partial x_2},$$
$$X_-\frac{\partial f}{\partial x_4} = -\frac{\partial f}{\partial x_3}, \quad X_-\frac{\partial f}{\partial x_i} = 0, \quad 5 \leqslant i \leqslant 8.$$

Similarly, we can prove that X_+f is a zero and we have

$$X_+\frac{\partial f}{\partial x_1} = -3\frac{\partial f}{\partial x_2}, \quad X_+\frac{\partial f}{\partial x_2} = -4\frac{\partial f}{\partial x_3}, \quad X_+\frac{\partial f}{\partial x_3} = -3\frac{\partial f}{\partial x_4},$$
$$X_+\frac{\partial f}{\partial x_4} = 0, \quad X_+\frac{\partial f}{\partial x_i} = 0, \quad 5 \leqslant i \leqslant 8.$$

Therefore f is an $s\ell(2, \mathbf{C})$ invariant homogeneous polynomial.

Case 13. $I = (3) \oplus (3) \oplus (2)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $2, 0, -2, 1$ and -1.

For $|i| \geqslant 6$,

$$\left|\mathrm{wt}\frac{\partial f^i_{k+1}}{\partial x_j}\right| \geqslant 3 \Rightarrow \frac{\partial f^i_{k+1}}{\partial x_j} = 0 \Rightarrow f^i_{k+1} = 0, \quad 1 \leqslant j \leqslant 8.$$

For $i = 4, 5$,

$$\begin{aligned} \mathrm{wt}\frac{\partial f^i_{k+1}}{\partial x_j} \geqslant 3 \Rightarrow & \frac{\partial f^i_{k+1}}{\partial x_j} = 0, \quad 2 \leqslant j \leqslant 8 \\ \Rightarrow & f^i_{k+1} \text{ depends only on } x_1 \text{ variable} \\ \Rightarrow & f^4_{k+1} = f^5_{k+1} = 0 \quad \text{because } \mathrm{wt}(x_1) = 3. \end{aligned}$$

Similarly, we can prove that $f^{-4}_{k+1} = f^{-5}_{k+1} = 0$.

For $i = 3$,

$$\begin{aligned} & \mathrm{wt}\frac{\partial f^3_{k+1}}{\partial x_3} = 4, \quad \mathrm{wt}\frac{\partial f^3_{k+1}}{\partial x_4} = 6, \quad \mathrm{wt}\frac{\partial f^3_{k+1}}{\partial x_j} = 3, \quad 5 \leqslant j \leqslant 8 \\ & \Rightarrow \frac{\partial f^3_{k+1}}{\partial x_j} = 0, \quad 3 \leqslant j \leqslant 8 \\ & \Rightarrow f^3_{k+1} \text{ depends only on } x_1, x_2 \text{ variables} \\ & \Rightarrow f^3_{k+1} = cx_1 \text{ or } dx_2^3 \text{ for some cnostants } c, d \\ & \Rightarrow f^3_{k+1} = dx_2^3 \quad \text{because } k \geqslant 2. \end{aligned}$$

Since $\mathrm{wt}X_+\frac{\partial f_{k+1}^3}{\partial x_2}=4$, $0=X_+\frac{\partial f_{k+1}^3}{\partial x_2}=18dx_1x_2\Rightarrow d=0\Rightarrow f_{k+1}^3=0$.

Similarly we can prove that $f_{k+1}^{-3}=0$.

For $i=2$,

$$\begin{aligned}\mathrm{wt}\frac{\partial f_{k+1}^2}{\partial x_3}=3,\mathrm{wt}\frac{\partial f_{k+1}^2}{\partial x_4}=5\Rightarrow&\frac{\partial f_{k+1}^2}{\partial x_3}=\frac{\partial f_{k+1}^2}{\partial x_4}=0\\ \Rightarrow& f_{k+1}^2 \text{ depends only on } x_1,x_2,x_5,x_6,x_7,x_8 \text{ variables}\\ \Rightarrow& f_{k+1}^2=x_2^2g_{k-1}(x_5,x_6,x_7,x_8)\end{aligned}$$

where $g_{k-1}(x_5,x_6,x_7,x_8)$ is a homogeneous polynomial of degree $k-1$ in x_5,x_6,x_7,x_8 variables.

If $f_{k+1}^2\neq 0$, then since $k\geqslant 2$, $\frac{\partial g_{k-1}}{\partial x_t}\neq 0$ for some $5\leqslant t\leqslant 8$.

Since $\mathrm{wt}X_+\frac{\partial f_{k+1}^2}{\partial x_t}=4$, $0=X_+\frac{\partial f_{k+1}^2}{\partial x_t}=6x_1x_2\frac{\partial g_{k-1}}{\partial x_t}\Rightarrow\frac{\partial g_{k-1}}{\partial x_t}=0$. Therefore $f_{k+1}^2=0$.

Similarly, we can prove that $f_{k+1}^{-2}=0$.

For $i=1$,

$$\mathrm{wt}\frac{\partial f_{k+1}^1}{\partial x_4}=4\Rightarrow\frac{\partial f_{k+1}^1}{\partial x_4}=0\Rightarrow f_{k+1}^1 \text{ is independent on } x_4 \text{ variables.}$$

Write $f_{k+1}^1=\sum_{\beta\geqslant 0}h_{k+1-\beta}^{1+\beta}(x_1,x_2,x_5,x_6,x_7,x_8)x_3^\beta$ where $h_{k+1-\beta}^{1+\beta}(x_1,x_2,x_5,x_6,x_7,x_8)$ is a homogeneous polynomial of degree $k+1-\beta$ and weight $1+\beta$ in x_1,x_2,x_5,x_6,x_7,x_8 variables. If $f_{k+1}^1\neq 0$, let β_0 be the largest integer such that $h_{k+1-\beta_0}^{1+\beta_0}\neq 0$. By Lemma 2, either $X_-^{\beta_0}\frac{\partial f_{k+1}^1}{\partial x_1}$ or $X_-^{\beta_0}\frac{\partial f_{k+1}^1}{\partial x_2}$ is nonzero.

$$\begin{aligned}&\mathrm{wt}X_-^{\beta_0}\frac{\partial f_{k+1}^1}{\partial x_1}=-2-2\beta_0,\ \mathrm{wt}X_-^{\beta_0}\frac{\partial f_{k+1}^1}{\partial x_2}=-2\beta_0,\ \mathrm{wt}X_-^{\beta_0}\frac{\partial f_{k+1}^1}{\partial x_i}\in I,\ i=1,2\\ &\Rightarrow -2-2\beta_0\geqslant -2 \text{ or } -2\beta_0\geqslant -2\\ &\Rightarrow \beta_0\leqslant 0 \text{ or } \beta_0\leqslant 1.\end{aligned}$$

Without loss of generality, we suppose $\beta_0\leqslant 1$. Since $\frac{\partial f_{k+1}^1}{\partial x_j}=1$, $5\leqslant j\leqslant 8$, in view of Lemma 1, there are constant c_1,c_2,r_1,r_2,r_3,r_4 such that

$$\begin{aligned}&f_{k+1}^1=c_1x_2(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^k+c_2x_2^2x_3(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-2},\ \text{for } k\geqslant 3;\\ &f_{k+1}^1=c_1x_2(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^2+c_2x_2^2x_3,\quad \text{for } k=2;\end{aligned}$$

Since $\mathrm{wt}(X_+\frac{\partial f_{k+1}^1}{\partial x_j})=3$, so $X_+\frac{\partial f_{k+1}^1}{\partial x_j}=0$, $5\leqslant j\leqslant 8$.

$$\begin{aligned}\frac{\partial f_{k+1}^1}{\partial x_j}=&c_1kr_jx_2(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1}\\ &+c_2(k-2)r_jx_2^2x_3(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-3},\quad 5\leqslant j\leqslant 8,\ \text{for } k\geqslant 3;\\ \frac{\partial f_{k+1}^1}{\partial x_j}=&2c_1r_jx_2(r_1x_5+r_2x_6+r_3x_7+r_4x_8),\quad 5\leqslant j\leqslant 8,\ \text{for } k=2;\end{aligned}$$

So

$$\begin{aligned}X_+\frac{\partial f_{k+1}^1}{\partial x_j}=&3c_1kr_jx_1(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-1}\\ &+c_2(k-2)r_j(6x_1x_2x_3+4x_2^3)(r_1x_5+r_2x_6+r_3x_7+r_4x_8)^{k-3}=0,\quad 5\leqslant j\leqslant 8\end{aligned}$$

$$\Rightarrow c_1 = c_2 = 0 \Rightarrow f^1_{k+1} = 0, \text{ for } k \geqslant 3;$$

$$X_+\frac{\partial f^1_{k+1}}{\partial x_j} = 6c_1 r_j x_1(r_1x_5 + r_2x_6 + r_3x_7 + r_4x_8) = 0, \quad 5 \leqslant j \leqslant 8$$

$$\Rightarrow c_1 = 0 \Rightarrow f^1_{k+1} = 0, \text{ for } k = 2.$$

Therefore $f^1_{k+1} = 0$.

Similarly, we can prove $f^{-1}_{k+1} = 0$.

For $i = 0$,

$$\text{wt}\frac{\partial f^0_{k+1}}{\partial x_1} = -3, \text{wt}\frac{\partial f^0_{k+1}}{\partial x_4} = 3 \Rightarrow \frac{\partial f^0_{k+1}}{\partial x_1} = \frac{\partial f^0_{k+1}}{\partial x_4} = 0$$
$$\Rightarrow f^0_{k+1} \text{depends only on } x_2, x_3, x_5, x_6, x_7, x_8 \text{ variables.}$$

Write $f^0_{k+1} = \sum_{m\geqslant 0} t_{k+1-2m}(x_5, x_6, x_7, x_8)x_2^m x_3^m$ where t_{k+1-2m} is a homogeneous polynomial of degree $k+1-2m$, weight 0 and in x_5, x_6, x_7, x_8 variable.

If $f^0_{k+1} \neq 0$, let m_0 be the largest integer such that $t_{k+1-2m_0} \neq 0$. Thus, there exist $5 \leqslant t \leqslant 8$, such that $\frac{\partial t_{k+1-2m_0}}{\partial x_t} \neq 0$. So $\frac{\partial f^0_{k+1}}{\partial x_t} = \sum_{m\geqslant 0} \frac{\partial t_{k-1-2m}}{\partial x_t} x_2^m x_3^m \neq 0$. If $m_0 \geqslant 2$, since

$$\frac{\partial f^0_{k+1}}{\partial x_t} \neq 0 \in I \text{ and } \text{wt}\frac{\partial f^0_{k+1}}{\partial x_t} = 0 \Rightarrow \text{wt}X_-^2\left(\frac{\partial f^0_{k+1}}{\partial x_t}\right) = -4 \Rightarrow X_-^2\left(\frac{\partial f^0_{k+1}}{\partial x_t}\right) = 0,$$

$$X_-\left(\frac{\partial f^0_{k+1}}{\partial x_t}\right) = \sum_{m\geqslant 0}\frac{\partial t_{k+1-2m}}{\partial x_t}(mx_2^{m-1}x_3^{m+1} + mx_2^m x_3^{m-1}x_4),$$

$$X_-^2\left(\frac{\partial f^0_{k+1}}{\partial x_t}\right) = \sum_{m\geqslant 0}\frac{\partial t_{k+1-2m}}{\partial x_t}(m(m-1)x_2^{m-2}x_3^{m+2} + m(m+1)x_2^{m-1}x_3^m x_4 + m^2 x_2^{m-1}x_3^m x_4 + m(m-1)x_2^m x_3^{m-2}x_4^2) = 0.$$

The coefficient of $x_3^{m_0+2}$ is $m_0(m_0-1)x_2^{m_0-2}\frac{\partial t_{k+1-3m_0}}{\partial x_t}$. Since $\frac{\partial t_{k+1-2m_0}}{\partial x_t} \neq 0$, therefore $m_0(m_0-1)x_2^{m_0-2}\frac{\partial t_{k+1-2m_0}}{\partial x_t} = 0 \longleftrightarrow m_0(m_0-1) = 0$ which contradicts that $m_0 \geqslant 2$, therefore $m_0 \leqslant 1$.

If $m_0 = 1$, then $f^0_{k+1} = t_{k-1}(x_5, x_6, x_7, x_8)x_2x_3 + t_{k+1}(x_5, x_6, x_7, x_8)$ where

$$\frac{\partial t_{k-1}(x_5, x_6, x_7, x_8)}{\partial x_t} \neq 0 \Rightarrow \frac{\partial f^0_{k+1}}{\partial x_t} = \frac{\partial t_{k-1}}{\partial x_t}x_2x_3 + \frac{\partial t_{k+1}}{\partial x_t}$$
$$\Rightarrow X_-\left(\frac{\partial f^0_{k+1}}{\partial x_t}\right) = \frac{\partial t_{k-1}}{\partial x_t}(x_3^2 + x_2x_4)$$
$$\Rightarrow X_-^2\left(\frac{\partial f^0_{k+1}}{\partial x_t}\right) = \frac{\partial t_{k-1}}{\partial x_t}(3x_3x_4) \neq 0,$$

which contradicts that $X_-^2(\frac{\partial f^0_{k+1}}{\partial x_t}) = 0$. Thus $m_0 = 0 \Rightarrow f^0_{k+1} = f^0_{k+1}(x_5, x_6, x_7, x_8)$. We can write $f = f^0_{k+1}(x_5, x_6, x_7, x_8)$. Since $\frac{\partial f}{\partial x_i} = \frac{\partial f^0_{k+1}}{\partial x_i} = 0$, $1 \leqslant i \leqslant 4 \Rightarrow \dim I \leqslant 4$ which contradicts that $\dim I = 8$, therefore this case cannot occur.

Case 14. $I = (3) \oplus (3) \oplus (1) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $2, 0$ and -2.

For i and even integer, $\text{wt}\frac{\partial f^i_{k+1}}{\partial x_j}$ is an odd integer for all $1 \leqslant i \leqslant 4$.

It follows that f_{k+1}^i depends only on x_5, x_6, x_7, x_8 variables, since $\mathrm{wt}(x_j) = 0$ for all $5 \leqslant j \leqslant 8$. So, $f_{k+1}^i = 0$ for $i \neq 0$ and $f_{k+1}^0 = f_{k+1}^0(x_5, x_6, x_7, x_8)$.

For $|i| \geqslant 7$ and i is an odd integer,

$$\left|\mathrm{wt}\frac{\partial f_{k+1}^i}{\partial x_j}\right| \geqslant 4 \Rightarrow \frac{\partial f_{k+1}^i}{\partial x_j} = 0 \Rightarrow f_{k+1}^i = 0, \quad 1 \leqslant j \leqslant 8.$$

For $i = 5$,

$$\begin{aligned}
&\mathrm{wt}\frac{\partial f_{k+1}^5}{\partial x_j} = 5, \quad 5 \leqslant j \leqslant 8, \\
&\mathrm{wt}\frac{\partial f_{k+1}^5}{\partial x_1} = 2, \quad \mathrm{wt}\frac{\partial f_{k+1}^5}{\partial x_2} = 4, \quad \mathrm{wt}\frac{\partial f_{k+1}^5}{\partial x_3} = 6, \quad \mathrm{wt}\frac{\partial f_{k+1}^5}{\partial x_4} = 8 \\
&\quad \Rightarrow \frac{\partial f_{k+1}^5}{\partial x_j} = 0, \quad 2 \leqslant j \leqslant 8 \\
&\quad \Rightarrow f_{k+1}^5 \text{ depends only on } x_1 \text{ variables} \\
&\quad \Rightarrow f_{k+1}^5 = 0 \quad \text{because } \mathrm{wt}(x_1) = 3.
\end{aligned}$$

Similarly, we can prove that $f_{k+1}^{-5} = 0$.

For $i = 3$,

$$\begin{aligned}
&\mathrm{wt}\frac{\partial f_{k+1}^3}{\partial x_1} = 0,\ \mathrm{wt}\frac{\partial f_{k+1}^3}{\partial x_2} = 2,\ \mathrm{wt}\frac{\partial f_{k+1}^3}{\partial x_3} = 4,\ \mathrm{wt}\frac{\partial f_{k+1}^3}{\partial x_4} = 6,\ \mathrm{wt}\frac{\partial f_{k+1}^3}{\partial x_j} = 3,\ 5 \leqslant j \leqslant 8 \\
&\Rightarrow \frac{\partial f_{k+1}^3}{\partial x_3} = \frac{\partial f_{k+1}^3}{\partial x_4} = \frac{\partial f_{k+1}^3}{\partial x_j} = 0, \quad 5 \leqslant j \leqslant 8 \\
&\Rightarrow f_{k+1}^3 \text{ depends only on } x_1, x_2 \text{ variables} \\
&\Rightarrow f_{k+1}^3 = ax_1 \text{ or } bx_2^3 \text{ for some constants } a, b \\
&\Rightarrow f_{k+1}^3 = bx_2^3 \text{ because } k \geqslant 2.
\end{aligned}$$

Since $\mathrm{wt} X_+ \frac{\partial f_{k+1}^3}{\partial x_2} = 4$, $0 = X_+ \frac{\partial f_{k+1}^2}{\partial x_2} = 18bx_1x_2 \Rightarrow b = 0$. Thus $f_{k+1}^3 = 0$.

Similarly, we can prove that $f_{k+1}^{-3} = 0$.

For $i = 1$.

$$\begin{aligned}
&\mathrm{wt}\frac{\partial f_{k+1}^1}{\partial x_1} = -2,\ \mathrm{wt}\frac{\partial f_{k+1}^1}{\partial x_2} = 0,\ \mathrm{wt}\frac{\partial f_{k+1}^1}{\partial x_3} = 2,\ \mathrm{wt}\frac{\partial f_{k+1}^1}{\partial x_4} = 4,\ \mathrm{wt}\frac{\partial f_{k+1}^1}{\partial x_j} = 1 \\
&\quad \Rightarrow \frac{\partial f_{k+1}^1}{\partial x_4} = \frac{\partial f_{k+1}^1}{\partial x_j} = 0 \Rightarrow f_{k+1}^1 = f_{k+1}^1(x_1, x_2, x_3), \quad 5 \leqslant j \leqslant 8.
\end{aligned}$$

Similarly, we can prove that $f_{k+1}^{-1} = f_{k+1}^{-1}(x_2, x_3, x_4)$.

We can write $f = f_{k+1}^{-1}(x_2, x_3, x_4) + f_{k+1}^0(x_5, x_6, x_7, x_8) + f_{k+1}^1(x_1, x_2, x_3)$. So

$$\begin{aligned}
I(f) &= \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_8} \right\rangle \\
&= \left\langle \frac{\partial f_{k+1}^1}{\partial x_1}, \frac{\partial f_{k+1}^{-1}}{\partial x_2} + \frac{\partial f_{k+1}^1}{\partial x_2}, \frac{\partial f_{k+1}^{-1}}{\partial x_3} + \frac{\partial f_{k+1}^1}{\partial x_3}, \frac{\partial f_{k+1}^{-1}}{\partial x_4}, \frac{\partial f_{k+1}^0}{\partial x_5}, \frac{\partial f_{k+1}^0}{\partial x_6}, \frac{\partial f_{k+1}^0}{\partial x_7}, \frac{\partial f_{k+1}^0}{\partial x_8} \right\rangle \\
&= \left\langle \frac{\partial f_{k+1}^0}{\partial x_1}, \frac{\partial f_{k+1}^{-1}}{\partial x_2} + \frac{\partial f_{k+1}^1}{\partial x_2}, \frac{\partial f_{k+1}^{-1}}{\partial x_3} + \frac{\partial f_{k+1}^1}{\partial x_3}, \frac{\partial f_{k+1}^{-1}}{\partial x_4} \right\rangle
\end{aligned}$$

$$\oplus\left\langle\frac{\partial f^0_{k+1}}{\partial x_5}\right\rangle\oplus\left\langle\frac{\partial f^0_{k+1}}{\partial x_6}\right\rangle\oplus\left\langle\frac{\partial f^0_{k+1}}{\partial x_7}\right\rangle\oplus\left\langle\frac{\partial f^0_{k+1}}{\partial x_8}\right\rangle$$

which contradicts that $I=(3)\oplus(3)\oplus(1)\oplus(1)$. Therefore this case cannot occur.

Case 15. $I=(3)\oplus(2)\oplus(2)\oplus(1)$.

With the same argument as case 13, we can write

$$f=f^{-1}_{k+1}(x_2,x_3,x_4,x_5,x_6,x_7,x_8)+f^0_{k+1}(x_5,x_6,x_7,x_8)+f^1_{k+1}(x_1,x_2,x_3,x_5,x_6,x_7,x_8).$$

Write $f^1_{k+1}=\sum_{\beta\geqslant 0}h^{1+\beta}_{k+1-\beta}(x_1,x_2,x_5,x_6,x_7,x_8)x_3^\beta$ where $h^{1+\beta}_{k+1-\beta}(x_1,x_2,x_5,x_6,x_7,x_8)$ is a homogeneous polynomial of degree $k+1-\beta$, weight $1+\beta$ and in x_1,x_2,x_5,x_6,x_7,x_8 variables. If $f^1_{k+1}\neq 0$, let β_0 be the largest integer such that $h^{1+\beta_0}_{k+1-\beta_0}\neq 0$. By Lemma 2, either $X_-^{\beta_0}\frac{\partial f^1_{k+1}}{\partial x_1}$ or $X_-^{\beta_0}\frac{\partial f^1_{k+1}}{\partial x_2}$ is nonzero.

$$\begin{aligned}&\mathrm{wt}X_-^{\beta_0}\frac{\partial f^1_{k+1}}{\partial x_1}=-2-2\beta_0,\ \mathrm{wt}X_-^{\beta_0}\frac{\partial f^1_{k+1}}{\partial x_2}=-2\beta_0\ \text{ and }\ X_-^{\beta_0}\frac{\partial f^1_{k+1}}{\partial x_i}\in I,\ i=1,2\\ &\Rightarrow -2-\beta_0\geqslant -2,\text{ or }-2\beta_0\geqslant -2\ \Rightarrow \beta_0\leqslant 0\,\text{or}\,\beta_0\leqslant 1.\end{aligned}$$

Without loss of generality, we suppose $\beta_0\leqslant 1$,

$$\begin{aligned}&\beta_0=1\Rightarrow\ h^0_{k-2}(x_5,x_6,x_7,x_8)\neq 0,\quad f^1_{k+1}=x_2h^0_k(x_5,x_6,x_7,x_8)+x_2^2x_3h^0_{k-2}(x_5,x_6,x_7,x_8),\\ &\beta_0=0\Rightarrow\ h^0_k(x_5,x_6,x_7,x_8)\neq 0,\quad f^1_{k+1}=x_2h^0_k(x_5,x_6,x_7,x_8).\end{aligned}$$

If $f^1_{k+1}=x_2h^0_k(x_5,x_6,x_7,x_8)+x_2^2x_3h^0_{k-2}(x_5,x_6,x_7,x_8)$ where $h^0_{k-2}(x_5,x_6,x_7,x_8)\neq 0$, since $\mathrm{wt}X_+\frac{\partial f^1_{k+1}}{\partial x_3}=4\Rightarrow 0=X_+\frac{\partial f^1_{k+1}}{\partial x_3}=6x_1x_2h^0_{k-2}(x_5,x_6,x_7,x_8)\Rightarrow h^0_{k-2}(x_5,x_6,x_7,x_8)=0$, This contradicts that $h^0_{k-2}(x_5,x_6,x_7,x_8)\neq 0$.

If $f^1_{k+1}=x_2h^0_k(x_5,x_6,x_7,x_8)$, where $h^0_k\neq 0$ since $h^0_k\neq 0\Rightarrow\exists 5\leqslant t\leqslant 8$, thus $\frac{\partial h^0_k}{\partial x_t}\neq 0\Rightarrow\frac{\partial f^1_{k+1}}{\partial x_t}=x_2\frac{\partial h^0_k}{\partial x_t}\neq 0$.

Since $\mathrm{wt}X_+\frac{\partial f^1_{k+1}}{\partial x_t}=3\Rightarrow 0=X_+\frac{\partial f^1_{k+1}}{\partial x_t}=3x_1\frac{\partial h^0_k}{\partial x_t}\Rightarrow\frac{\partial h^0_k}{\partial x_t}=0$, this contradicts that $\frac{\partial h^0_k}{\partial x_t}\neq 0$.

Therefore $f^1_{k+1}=0$.

Similarly we can prove that $f^{-1}_{k+1}=0$.

We can write $f=f^0_{k+1}(x_5,x_6,x_7,x_8)$. Since $\frac{\partial f}{\partial x_i}=\frac{\partial f^0_{k+1}}{\partial x_i}=0,\ 1\leqslant i\leqslant 4\Rightarrow\dim I\leqslant 4$ which contradicts that $\dim I=8$, this case cannot occur.

Case 16. $I=(3)\oplus(2)\oplus(1)\oplus(1)\oplus(1)$.

This case cannot occur by the same argument as Case 13.

Case 17. $I=(3)\oplus(1)\oplus(1)\oplus(1)\oplus(1)\oplus(1)$.

With the same argument as Case 14, we can write $f=f^{-1}_{k+1}(x_2,x_3,x_4)+f^0_{k+1}(x_5,x_6,x_7,x_8)+f^1_{k+1}(x_1,x_2,x_3)$. If $f^1_{k+1}(x_1,x_2,x_3)=0$ then $\frac{\partial f}{\partial x_1}=0$. Thus $\dim I\leqslant 7$ which contradicts that $\dim I=8$. Therefore $f^1_{k+1}(x_1,x_2,x_3)\neq 0$. Write

$$f^1_{k+1}(x_1,x_2,x_3)=\sum_{b\geqslant 0}b_\beta x_1^{\frac{2\beta-k}{2}}x_2^{\frac{3k+2-4\beta}{2}}x_3^\beta.$$

Let β_0 be the largest integer such that $b_{\beta_0}\neq 0$. By Corollary 3, either $X_-^{\beta_0}\frac{\partial f^1_{k+1}}{\partial x_1}$ or $X_-^{\beta_0}\frac{\partial f^1_{k+1}}{\partial x_2}$ is nonzero. Since

$$\mathrm{wt}\left(X_-^{\beta_0}\frac{\partial f^1_{k+1}}{\partial x_1}\right)=-2-2\beta_0,\ \mathrm{wt}\left(X_-^{\beta_0}\frac{\partial f^1_{k+1}}{\partial x_2}\right)=-2\beta_0,\ \text{and}\ X_-^{\beta_0}\frac{\partial f^1_{k+1}}{\partial x_2}\in I,\ i=1,2$$

$$\Rightarrow -2-2\beta_0 \geqslant -2 \text{ or } -2\beta_0 \geqslant -2 \Rightarrow \beta_0 \leqslant 0 \text{ or } \beta_0 \leqslant 1,$$

Without loss of generality, we suppose $\beta_0 \leqslant 1 \Rightarrow f_{k+1}^1 = b_0 x_2^2 x_3$ where $b_0 \neq 0$.

$$\text{wt}\left(X_+ \frac{\partial f_{k+1}^1}{\partial x_3}\right) = 4 \Rightarrow 0 = X_+ \frac{\partial f_{k+1}^1}{\partial x_3} = 6b_0 x_1 x_2 \Rightarrow b_0 = 0.$$

This contradicts that $b_0 \neq 0$. Thus $f_{k+1}^1 = 0$ which contradicts that hypothesis $f_{k+1}^1 \neq 0$. Therefore, this case cannot occur.

Case 18. $I = (2) \oplus (2) \oplus (2) \oplus (2)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $1, -1$.

For i is an odd integer, $\text{wt}\frac{\partial f_{k+1}^i}{\partial x_j}$ is an even integer for all $1 \leqslant j \leqslant 4$. It follows that f_{k+1}^i depends only on x_5, x_6, x_7, x_8 variables. Since $\text{wt}(x_j) = 0$ for all $5 \leqslant j \leqslant 8$, $f_{k+1}^i = 0$.

For i is an even integer and $|i| \geqslant 6$,

$$\left|\text{wt}\frac{\partial f_{k+1}^i}{\partial x_j}\right| \geqslant 3 \Rightarrow \frac{\partial f_{k+1}^i}{\partial x_j} = 0 \Rightarrow f_{k+1}^i = 0, \quad 1 \leqslant j \leqslant 8.$$

For $i = 4$,

$$\begin{aligned}
&\text{wt}\frac{\partial f_{k+1}^4}{\partial x_1} = 1, \quad \text{wt}\frac{\partial f_{k+1}^4}{\partial x_2} = 3, \quad \text{wt}\frac{\partial f_{k+1}^4}{\partial x_3} = 5,\\
&\text{wt}\frac{\partial f_{k+1}^4}{\partial x_4} = 7, \quad \text{wt}\frac{\partial f_{k+1}^4}{\partial x_j} = 4, \quad 5 \leqslant j \leqslant 8\\
&\quad \Rightarrow \frac{\partial f_{k+1}^4}{\partial x_j} = 0,\ 2 \leqslant j \leqslant 8 \Rightarrow f_{k+1}^4 \text{ depends only on } x_1 \text{ variable}\\
&\quad \Rightarrow f_{k+1}^4 = 0 \text{ because } \text{wt}(x_1) = 3.
\end{aligned}$$

Similarly, we can prove that $f_{k+1}^{-4} = 0$. So we write $f = f_{k+1}^{-2} + f_{k+1}^0 + f_{k+1}^2$. Since

$$\begin{aligned}
&\text{wt}\frac{\partial f^{-2}}{\partial x_j} = -2, \text{wt}\frac{\partial f_{k+1}^0}{\partial x_j} = 0, \frac{\partial f_{k+1}^2}{\partial x_j} = 2, \quad 5 \leqslant j \leqslant 8\\
&\quad \Rightarrow \frac{\partial f_{k+1}^{-2}}{\partial x_j} = \frac{\partial f^0}{\partial x_j} = \frac{\partial f_{k+1}^2}{\partial x_j} = 0, \quad 5 \leqslant j \leqslant 8\\
&\quad \Rightarrow \frac{\partial f}{\partial x_j} = 0, \quad 5 \leqslant j \leqslant 8\\
&\quad \Rightarrow \dim I \leqslant 4 \text{ which contradicts that } \dim I = 8,
\end{aligned}$$

this case cannot occur.

Case 19. $I = (2) \oplus (2) \oplus (2) \oplus (1) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $1, -1, 0$.

For $|i| \geqslant 5$,

$$\left|\text{wt}\frac{\partial f_{k+1}^i}{\partial x_j}\right| \geqslant 2 \Rightarrow \frac{\partial f_{k+1}^i}{\partial x_j} = 0 \Rightarrow f_{k+1}^i = 0, \quad 1 \leqslant j \leqslant 8.$$

For $i=4$,

$$\mathrm{wt}\frac{\partial f^4_{k+1}}{\partial x_j} =\geqslant 3 \Rightarrow \frac{\partial f^4_{k+1}}{\partial x_j}=0, \quad 2\leqslant j\leqslant 8$$
$$\Rightarrow f^4_{k+1} \text{ depends only on } x_1 \text{ variable} \Rightarrow f^4_{k+1}=0 \text{ because } \mathrm{wt}(x_1)=3.$$

Similarly, we can prove that $f^{-4}_{k+1}=0$.

For $i=3$,

$$\mathrm{wt}\frac{\partial f^3_{k+1}}{\partial x_j}=2,\ 2\leqslant j\leqslant 8 \Rightarrow \frac{\partial f^3_{k+1}}{\partial x_j}=0,\ 2\leqslant j\leqslant 8$$
$$\Rightarrow f^3_{k+1} \text{ depends only on } x_1 \text{ variable} \Rightarrow f^3_{k+1}=cx_1 \text{ for some constant } c$$
$$\Rightarrow f^3_{k+1}=0 \text{ because } k\geqslant 2.$$

Similarly, we can prove that $f^{-3}_{k+1}=0$.

For $i=2$,

$$\mathrm{wt}\frac{\partial f^2_{k+1}}{\partial x_1}=-1,\ \mathrm{wt}\frac{\partial f^2_{k+1}}{\partial x_2}=1,\ \mathrm{wt}\frac{\partial f^2_{k+1}}{\partial x_3}=3,\ \mathrm{wt}\frac{\partial f^2_{k+1}}{\partial x_4}=5,\ \mathrm{wt}\frac{\partial f^2_{k+1}}{\partial x_j}=2, \quad 5\leqslant j\leqslant 8$$
$$\Rightarrow \frac{\partial f^2_{k+1}}{\partial x_j}=0, \quad 3\leqslant j\leqslant 8 \quad \Rightarrow f^2_{k+1} \text{ depends only on } x_1,x_2 \text{ variables}$$
$$\Rightarrow f^2_{k+1}=cx_2^2 \text{ for some constant } c \quad \Rightarrow f^2_{k+1}=0 \text{ because } k\geqslant 2.$$

Similarly, we can prove that $f^{-2}_{k+1}=0$.

For $i=1$,

$$\mathrm{wt}\frac{\partial f^1_{k+1}}{\partial x_1}=-2,\ \mathrm{wt}\frac{\partial f^1_{k+1}}{\partial x_3}=2,\ \mathrm{wt}\frac{\partial f^1_{k+1}}{\partial x_4}=4 \Rightarrow \frac{\partial f^1_{k+1}}{\partial x_1}=\frac{\partial f^1_{k+1}}{\partial x_3}=\frac{\partial f^1_{k+1}}{\partial x_4}=0$$
$$\Rightarrow f^1_{k+1}=f^1_{k+1}(x_2,x_5,x_6,x_7,x_8).$$

Similarly, we can prove that $f^{-2}_{k+1}=f^{-1}_{k+1}(x_3,x_5,x_6,x_7,x_8)$.

For $i=0$,

$$\mathrm{wt}\frac{\partial f^0_{k+1}}{\partial x_1}=-3, \mathrm{wt}\frac{\partial f^0_{k+1}}{\partial x_4}=3 \Rightarrow \frac{\partial f^0_{k+1}}{\partial x_1}=\frac{\partial f^0_{k+1}}{\partial x_4}=0 \Rightarrow f^0_{k+1}=f^0_{k+1}(x_2,x_3,x_5,x_6,x_7,x_8).$$

Write $f=f^{-1}_{k+1}(x_3,x_5,x_6,x_7,x_8)+f^0_{k+1}(x_2,x_3,x_5,x_6,x_7,x_8)+f^1_{k+1}(x_2,x_5,x_6,x_7,x_8)$. Thus

$$\frac{\partial f}{\partial x_1}=\frac{\partial f}{\partial x_4}=0 \ \Rightarrow \dim I\leqslant 6 \text{ which contradicts that } \dim I=8.$$

Therefore, this case cannot occur.

Case 20. $I=(2)\oplus(2)\oplus(1)\oplus(1)\oplus(1)\oplus(1)$.

This case cannot occur by the same argument as Case 19.

Case 21. $I=(2)\oplus(1)\oplus(1)\oplus(1)\oplus(1)\oplus(1)\oplus(1)$.

This case cannot occur by the same argument as Case 19.

Case 22. $I=(1)\oplus(1)\oplus(1)\oplus(1)\oplus(1)\oplus(1)\oplus(1)\oplus(1)$.

Elements of I are linear combination of homogeneous polynomials of degree k and weights 0.

For i is an even integer, $\mathrm{wt}\frac{\partial f_{k+1}^i}{\partial x_j}$ is an odd integer for all $1 \leqslant j \leqslant 4$. It follows that f_{k+1}^i depends only on x_5, x_6, x_7, x_8 variable. Since $\mathrm{wt}(x_j) = 0$ for all $5 \leqslant j \leqslant 8$, $f_{k+1}^i = 0$ for all $i \neq 0$ and $f_{k+1}^0 = f_{k+1}^0(x_5, x_6, x_7, x_8)$.

For i is an old integer and $|i| \geqslant 5$,

$$\left|\mathrm{wt}\frac{\partial f_{k+1}^i}{\partial x_j}\right| \geqslant 2 \Rightarrow \frac{\partial f_{k+1}^i}{\partial x_j} = 0 \Rightarrow f_{k+1}^i = 0, \quad 1 \leqslant j \leqslant 8.$$

For $i = 3$,

$$\mathrm{wt}\frac{\partial f_{k+1}^3}{\partial x_1} = 0,\ \mathrm{wt}\frac{\partial f_{k+1}^3}{\partial x_2} = 2,\ \mathrm{wt}\frac{\partial f_{k+1}^3}{\partial x_3} = 4,\ \mathrm{wt}\frac{\partial f_{k+1}^3}{\partial x_4} = 6,\ \mathrm{wt}\frac{\partial f_{k+1}^3}{\partial x_j} = 3,\ 5 \leqslant j \leqslant 8$$

$$\Rightarrow f_{k+1}^3 \text{ depends only on } x_1 \text{ variable } \Rightarrow f_{k+1}^3 = cx_1 \text{ for some constant } c$$

$$\Rightarrow f_{k+1}^3 = 0 \text{ because } k \geqslant 2.$$

Similarly, we can prove that $f_{k+1}^{-3} = 0$.

For $i = 1$,

$$\mathrm{wt}\frac{\partial f_{k+1}^1}{\partial x_1} = -2,\ \mathrm{wt}\frac{\partial f_{k+1}^1}{\partial x_2} = 0,\ \mathrm{wt}\frac{\partial f_{k+1}^1}{\partial x_3} = 2,\ \frac{\partial f_{k+1}^1}{\partial x_4} = 4,\ \frac{\partial f_{k+1}^1}{\partial x_j} = 1,\ 5 \leqslant j \leqslant 8$$

$$\Rightarrow f_{k+1}^1 \text{ depends only on } x_2 \text{ variable} \Rightarrow f_{k+1}^1 = cx_2 \text{ for some constant } c$$

$$\Rightarrow f_{k+1}^1 = 0 \text{ because } k \geqslant 2.$$

Similarly, $f_{k+1}^{-1} = 0$.

Write

$$f = f_{k+1}^0(x_5, x_6, x_7, x_8) \Rightarrow \frac{\partial f}{\partial x_i} = \frac{\partial f_{k+1}^0}{\partial x_i} = 0, \quad 1 \leqslant i \leqslant 4$$

$$\Rightarrow \dim I \leqslant 4 \text{ which contradicts that } \dim I = 8.$$

Therefore this case cannot occur.

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