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Diffeomorphic types of complements of nice point arrangements in $\mathbf{CP}^l$

Dedicated to Professor ZHONG TongDe on the occasion of his 80th birthday

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Abstract We use a new method to study arrangement in $\mathbf{CP}^l$, define a class of nice point arrangements and show that if two nice point arrangements have the same combinatorics, then their complements are diffeomorphic to each other. In particular, the moduli space of nice point arrangements with same combinatorics in $\mathbf{CP}^l$ is connected. It generalizes the result on point arrangements in $\mathbf{CP}^3$ to point arrangements in $\mathbf{CP}^l$ for any l.

Keywords: diffeomorphic type, hyperplane arrangement, lattice isotopy

MSC(2000): 14F05, 14H30

1 Introduction

A complex l-arrangement $\mathcal{A}$ is a finite collection of hyperplanes in a vector space $\mathbf{C}^l$. A central complex l-arrangement means that all hyperplanes of $\mathcal{A}$ contain the origin. Given a central arrangement, there is an arrangement $\mathcal{A}^*$ in $\mathbf{CP}^{l-1}$ which is the natural projection of $\mathcal{A}$. Let $M(\mathcal{A}) = \mathbf{C}^l - \bigcup_{H\in\mathcal{A}} H$ be the complement of $\mathcal{A}$. Let $M(\mathcal{A}^*) = \mathbf{CP}^{l-1} - \bigcup_{H^*\in\mathcal{A}^*} H^*$. One of the central topic in this area is to determine the relation between the topological or differential structure of $M(\mathcal{A})$ (or $M(\mathcal{A}^*)$) and the combinatorial geometry of $\mathcal{A}$ (or $\mathcal{A}^*$). Here the combinatorial geometry is referred as the geometry of $L(\mathcal{A})$ (or $L(\mathcal{A}^*)$) which is the set of intersections of elements of $\mathcal{A}$ (or $\mathcal{A}^*$) partially ordered by reverse inclusion (see [1, Definition 2.1]). When the arrangement $\mathcal{A}$ is a central arrangement, i.e. the intersection of all elements of $\mathcal{A}$ is nonempty, $L(\mathcal{A})$ is indeed a geometric lattice (see [1, Lemma 2.3]). In general, $L(\mathcal{A}^*)$ may not be a lattice. By abusing the terminology we still call $L(\mathcal{A}^*)$ the lattice of $\mathcal{A}^*$. Let $\mathcal{A}_1 = \{H_1, \dots, H_n\}$ and $\mathcal{A}_2 = \{G_1, \dots, G_n\}$ be two arrangements of hyperplanes. $L(\mathcal{A}_1)$ and $L(\mathcal{A}_2)$ are said to be isomorphic if $\bigcap_{i\in I} H_i = \bigcap_{i\in I} G_i$, for any $I \subseteq \{1, \dots, n\}$.

In 1980, Orlik and Solomon [2] computed the cohomology algebra of $M(\mathcal{A})$ in terms of $L(\mathcal{A})$. They further conjectured that various homotopy invariants of the complement depend only on the lattice of $\mathcal{A}$. In [3–5], Falk studied the question whether $L(\mathcal{A}^*)$ is a homotopy invariant. He proved that there exist pairs of central arrangements in $\mathbf{C}^3$ with different underlying lattices

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whose complements are homotopy equivalent, see [5]. However, a result of Jiang and Yau, see [6, 7], does assert that $L(\mathcal{A}^*)$ is a topological invariant for arrangements in $\mathbf{CP}^2$. In a famous preprint [12], Rybnikov showed that there exist two line arrangements in $\mathbf{CP}^2$ with same combinatorics, but the fundamental groups of the complements are not isomorphic. It is a natural question to ask when the combinatoric of an arrangement will uniquely determine the topological or diffeomorphic type of the complement. One pioneering result in this direction is the lattice-isotopy theorem, proved by Randell [9]. It asserts that if two arrangements are connected by a one-parameter family of arrangements whose lattices are isomorphic to a fixed lattice then their complements are diffeomorphic. In other words, if the moduli space of arrangements with isomorphic lattice structrue is connected then the complements of arrangements have the same differential structure.

Inspired by Randell's theorem, one naturally wants to know what kind of combinatorics of an arrangement will warrant its moduli space to be connected. More generally, one can ask when the differential structure of the complement of an arrangement $\mathcal{A}^*$ is determined by its combinatorial data $L(\mathcal{A}^*)$. Partial results have been gotten by Jiang and Yau [10, 11] for arrangements in $\mathbf{CP}^2$, and Wang and Yau for arrangements in $\mathbf{CP}^2$ in [12] and for arrangements in $\mathbf{CP}^3$ in [13, 14]. They defined a large class of arrangements called nice arrangement whose moduli are connected. To prove that the moduli space of arrangements with fixed combinatorial data is connected, we need to construct a one-parameter family of arrangements with a fixed lattice which connects the two arbitrary given arrangements in this moduli space. Given two isomorphic arrangements $\mathcal{A}_1 = \{H_1, \ldots, H_n\}$ and $\mathcal{A}_2 = \{G_1, \ldots, G_n\}$ in $\mathbf{CP}^l$ with the properties that $\bigcap_{i\in I} H_i = \bigcap_{i\in I} G_i$, it is natural to consider the family of arrangements with parameter t in $(\mathbf{CP}^1)^n$:

$$\mathcal{A}(t) = \{x_i H_i + y_i G_i : (x_i, y_i) \in \mathbf{CP}^1, i = 1, 2, \ldots, n\}.$$

The idea is to construct one-parameter subfamily of $\mathcal{A}(t)$ connecting $\mathcal{A}_1$ and $\mathcal{A}_2$ such that lattice structure remains constant within this one-parameter subfamily. This one-parameter family is easier to construct if the arrangement is in a low dimensional projective space. For instance, for any two arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ with the same lattice in $\mathbf{CP}^2$, to construct the one-parameter family connecting $\mathcal{A}_1$ and $\mathcal{A}_2$ with a fixed lattice structure, it is sufficient to keep track of all possible three lines intersecting at one point (see [10–12] for detail). Notice that three lines intersecting at one point in $\mathbf{CP}^2$ is equivalent to the fact that the determinant of the coefficient matrix of three lines is equal to zero and there exists a non-vanishing 2×2 minor of this coefficient matrix. Therefore we need to consider the quasi-projective variety in $(\mathbf{CP}^1)^n$ defined by a set of polynomial equations and non-equalities. If we can find a curve in this quasi-projective variety connecting $(1, 0)^n$ and $(0, 1)^n$ in $(\mathbf{CP}^1)^n$, then we can do so. In general this quasi-projective variety may not have positive dimension, which implies that such a one-parameter family of hyperplane arrangements with fixed lattice structure may not exist. It was shown that this quasi-projective variety has a component with positive dimension connecting $(1, 0)^n$ and $(0, 1)^n$ if the defining polynomials are all irreducible (see [10–12]).

It is an interesting and difficult problem how to construct one parameter family of some arrangements of hyperplanes in a higher dimensional projective space. In other words, given certain arrangements of hyperplanes in a higher dimensional projective space with fixed combinatorial structure, are their complements diffeomorphic to each other? What properties do the moduli spaces have? An approach has been made by Wang and Yau in [13, 14] to study nice arrangements in $\mathbf{CP}^3$. They accomplished for nice arrangements in $\mathbf{CP}^3$ that the diffeomorphic type of complements of nice arrangements in $\mathbf{CP}^3$ is determined by their combinatorial structure and their moduli spaces are connected. But as you may see from their paper, even in $\mathbf{CP}^3$, in general the lattices of arrangements are too complicated to be used to construct one-parameter family. In higher dimension, we have to face much more complicated combinatorial structure of an arrangement. Besides, unlike the $\mathbf{CP}^2$ and $\mathbf{CP}^3$ cases, we have to face a serious new difficulty of arranging that the polynomial equations we obtained are general polynomials in the sense that all the coefficients are nonzero. We are allowed to use the automorphism group of $\mathbf{CP}^l$ to change the coefficient of these polynomials. However, the difficulty is that the number of coefficients in the polynomials equations is greater than the dimension of the automorphism group of $\mathbf{CP}^l$. Except that, if one wants to use the method for higher dimension, the work will be too hard to be controlled. In this paper, we simplify the idea and method to construct one parameter family of arrangements. This enables us to study the above problem of some arrangements, which we called nice point arrangement in $\mathbf{CP}^l$ (for precise definition, see Definition 2.8 below). As a result, we prove that the moduli spaces of nice point arrangements with same lattice $L(\mathcal{A})$ in $\mathbf{CP}^n$ are connected. More precisely, we prove the following theorem:

Main theorem. *Let $\mathcal{A}_0^*$ and $\mathcal{A}_1^*$ be two nice point arrangements of hyperplanes in $\mathbf{CP}^l$. If $L(\mathcal{A}_0^*)$ and $L(\mathcal{A}_1^*)$ are isomorphic, then there exists a one-parameter family of arrangements $\mathcal{A}_t$ with $L(\mathcal{A}_t^*) \cong L(\mathcal{A}_0^*)$, which connects $\mathcal{A}_0^*$ and $\mathcal{A}_1^*$. In particular, the complements $M(\mathcal{A}_0^*)$ and $M(\mathcal{A}_1^*)$ in $\mathbf{CP}^l$ are diffeomorphic to each other.*

2 Definitions

In this paper, a projective arrangement of hyperplanes in $\mathbf{CP}^l$ will be denoted by $\mathcal{A}^*$. Denote the lattice (intersection poset) of $\mathcal{A}^*$ by $L(\mathcal{A}^*)$. We will give some definitions and example of nice point arrangements in this section.

Definition 2.1. *Given an arrangement $\mathcal{A}^*$ and lattice $L(\mathcal{A}^*)$, an element $e \in L(\mathcal{A}^*)$ is of multiplicity k, if and only if it is the intersection of exactly k hyperplanes in $\mathcal{A}^*$. Denote the multiplicity of an element $e \in L(\mathcal{A}^*)$ by $m(e)$. Denote by $p_i(\mathcal{A}^*)$ the number of elements of dimension 0 in $L(\mathcal{A}^*)$ with multiplicity i.*

Definition 2.2. *A soul $\mathcal{G}$ of an arrangement $\mathcal{A}^*$ is a pseudo-complex whose simplices are defined as follows:*

Let $\mathcal{G}(k)$, $k \leqslant l-2$, be the sets of k-simplices of $\mathcal{G}$ defined by $\{e \in L(\mathcal{A}^*) : \dim e = k, m(e) \geqslant l+1-k, \exists H_1, \dots, H_{l+1-k} \in \mathcal{A}^*$ passing through $e = \bigcap_{i=0}^{l+1-k} H_i$ such that any $l-k$ of them are in general position$\}$.

Let $\mathcal{G}(l-1)$ be the set of $(l-1)$-simplices of $\mathcal{G}$ defined by $\{H \in \mathcal{A}^* : \exists\, e \in \bigcup_{k=0}^{l-2} \mathcal{G}(k)$ such that $e \subset H\}$.

In the following, a point will be referred as an element in $\mathcal{G}(0)$.

Remark 2.3. To study the combinatorics properties of $\mathcal{A}^*$, we need to consider all intersections in $L(\mathcal{A}^*)$. Notice that any two hyperplanes in $\mathbf{CP}^l$ must intersect at an $(l-2)$-

dimensional subspace of $\mathbf{CP}^l$. So we only need to consider those $(l-2)$-dimensional intersections $e_{l-2} \in L(\mathcal{A}^*)$ of multiplicity greater than 2. Given an $(l-2)$-dimensional intersection $e_{l-2} \in L(\mathcal{A}^*)$ and $H \in \mathcal{A}^*$, if $e_{l-2} \nsubseteq H$, then $\dim e_{l-2} \cap H = l-3$. Obviously, this is a trivial way to get an $(l-3)$-dimensional subspace of $\mathbf{CP}^l$. So we can get rid of such a trivial case and only consider all elements in $\mathcal{G}(l-3)$.

In fact, given any element $e \in \mathcal{G}(k)$ and $H \in \mathcal{A}^*$, if $e \nsubseteq H$, then $\dim e \cap H = k-1$. We can also get rid of such trivial cases and only consider all $(k-1)$-dimensional elements in $\mathcal{G}(k-1)$.

Based on the above argument, we say that given two arrangements of hyperplanes $\mathcal{A}_1^*$ and $\mathcal{A}_2^*$ in $\mathbf{CP}^l$, $L(\mathcal{A}_1^*) \cong L(\mathcal{A}_2^*)$ if and only if $|\mathcal{A}_1^*| = |\mathcal{A}_2^*|$ and $\mathcal{G}_1 \cong \mathcal{G}_2$.

Definition 2.4. *A path in* $\mathcal{G}$ *is defined to be a finite sequence of simplices* $a_0, h_1, a_1, h_2, a_2, \ldots, a_{k-1}, h_k, a_k$ $(k \geqslant 1)$ *of* $\mathcal{G}$ *where* $a_0, a_1, \ldots, a_k$ *are distinct elements in* $\bigcup_{j=0}^{l-2} \mathcal{G}(j)$, *and* $h_1, \ldots, h_k$ *are distinct and* $h_{i+1} \in \mathcal{G}(l-1)$ *contains* a_i *and* a_{i+1} *for* $i = 0, 1, \ldots, k-1$. *We say that an element* a *connects an elements* a' *by path, if there is a path* $a = a_0, h_1, a_1, \ldots, a_{k-1}, h_k, a_k = a'$ $(k \geqslant 1)$. *A loop in* $\mathcal{G}$ *is a path* $a_0, h_1, a_1, h_2, a_{k-1}, \ldots, h_k, a_k$ *with* $a_0 = a_k$.

Definition 2.5. *For any* $u \in \bigcup_{k=0}^{l-2} \mathcal{G}(k)$, *a star* $St(u)$ *of* u *is* $\{u\} \cup \{H \in \mathcal{A}^* : u \subset H\}$. *A* k*-dimensional element* $v \in \mathcal{G}(k)\,(\neq u)$ *is called an end* k*-element of* u, *if* $St(v) \cap St(u) \neq \emptyset$. *An end 0-element will also be called an end point. Two stars* $St(u)$ *and* $St(w)$ *are called disjoint if* $St(u) \cap St(w) = \emptyset$.

Definition 2.6. *For the stars* $St(u_1), \ldots, St(u_m)$ *in* $\mathcal{G}$ $(m > 0)$, *Let* $\mathcal{G}' = \mathcal{G} \setminus (St(u_1) \cup \cdots \cup St(u_m))$. $St(u_1), \ldots, St(u_m)$ *are said to be simple joint in* $\mathcal{G}$, *if the following conditions are satisfied:*

(1) *Given any end element of the stars* $St(u_1), \ldots, St(u_m)$, *it can connect to at most one other end element of the stars* $St(u_1), \ldots, St(u_m)$ *by path(s) in* $\mathcal{G}'$.

(2) *Given any two end elements of the stars* $St(u_1), \ldots, St(u_m)$, *they can be connected by at most one path in* $\mathcal{G}'$.

(3) *If several stars share the same end element, the total number of hyperplanes containing the end element and the stars must be at most* l.

Lemma 2.7. *Suppose that* u *is an element of* $\mathcal{G}'$, *then only two cases will occur: either*

(1) u *connects to only one end element of the stars* $St(u_1), \ldots, St(u_m)$ *by path(s) in* $\mathcal{G}'$, *or*

(2) *If* u *connects to two end elements of the stars* $St(u_1), \ldots, St(u_m)$, *then there is a unique path in* $\mathcal{G}'$ *connecting* u *and the two end elements.*

Proof. First, we claim that u can connect to at most two end elements of the stars, $St(u_1), \ldots, St(u_m)$. If u connects to three distinct end elements, v_1, v_2, v_3, of the stars, then one of v_1, v_2, v_3 will connect to the other two end elements simultaneously; this condition contradicts the item (1) in Definition 2.6. Assume that u connects two end elements w_1 and w_2 of $St(u_1), \ldots, St(u_m)$. If u connects w_1 or w_2 by two or more paths in $\mathcal{G}'$, then there will be more than one path in $\mathcal{G}'$ connecting w_1 and w_2; this contradicts the item (2) in Definition 2.6.

Definition 2.8. *An arrangement* $\mathcal{A}^*$ *of hyperplanes in* $\mathbf{CP}^l$ *is said to be a nice point arrangement, if the soul* $\mathcal{G} = \mathcal{G}(0) \cup \mathcal{G}(l-1)$ *of* $\mathcal{A}^*$ *and either*

(1) $\mathcal{G}$ *has no loop, or*

(2) *there are simple joint stars* $St(u_1), \ldots, St(u_m)$ *which are pairwise disjoint in* $\mathcal{G}$ *such that*

$\mathcal{G}'$ *contains no loop.*

Lemma 2.9. *Let p be a point in $\mathcal{G}(0)$ and $H_1, \ldots, H_q$, $q > l$, are the hyperplanes passing through p. Then, any l of those hyperplanes is in general position.*

Proof. Let $H_{i_1}, \ldots, H_{i_l}$ be l hyperplanes passing through p. If they are not in general position, then they are linearly dependent which implies that there exsits $\bigcap_{j \in I} H_{i_j} \in \mathcal{G}(k)$ for some $1 \leqslant k \leqslant l-2$, where $I \subset \{1, 2, \ldots, l\}$. It contradicts that $\mathcal{G}(i) = \emptyset$ for any $1 \leqslant i \leqslant l-2$.

Lemma 2.10. *Let $\mathcal{A}^*$ be a nice point arrangement of hyperplanes in $\mathbf{CP}^l$. Then the soul $\mathcal{G}$ of $\mathcal{A}^*$ has the following property: for any two points in $\mathcal{G}(0)$, there are at most $l-1$ hyperplanes passing through these two points at the same time.*

Proof. Let p_1 and p_2 be in $\mathcal{G}(0)$. Let $H_1, \ldots, H_q$, $q \geqslant l$ be the hyperplanes passing through p_1 and p_2. Then the line which connects p_1 and p_2 is of multiplicity at least l. The line may be an element of $\mathcal{G}(1)$. If it is not in $\mathcal{G}(1)$, the only possibility is that it is the intersection of a hyperplane H and a plane P of multiplicity at least $l-1$. If P is not in $\mathcal{G}(2)$, P must be the intersection of a 3-dimensional subspace e_3 of multiplicity at least $l-2$ and a hyperplane in $\mathcal{A}^*$. Continuing the argument, if e_3 is not in $\mathcal{G}(3)$, finally we get an $(l-2)$-dimensional subspace of multiplicity at least 3. Since all $(l-2)$-dimensional intersections of multiplicity at least 3 are all in $\mathcal{G}(l-2)$, then $e_{l-2} \in \mathcal{G}(k)$. It contradicts that $\mathcal{G}(i) = \emptyset$ for any $1 \leqslant i \leqslant l-2$.

Example 2.11. Let $\mathcal{A}^*$ be an arrangement of $2l+2$ hyperplanes in $\mathbf{CP}^l$ and $\mathcal{G}$ be the soul of $\mathcal{A}^*$. Assume that $\mathcal{G}(0) = \{A, B, C\}$ with the multiplicities $m(A) = 2l-1$, $m(B) = m(C) = l+1$. Let $L_{AB} \in L(\mathcal{A}^*)$ be a line of multiplicity $l-1$ which connects A and B. Let $L_{AC} \in L(\mathcal{A}^*)$ be a line of multiplicity $l-1$ which connects A and C. Let $H \in \mathcal{A}^*$ be the unique hyperplane containing B and C. Figure 1 shows the generic projection of $\mathcal{A}^*$ onto $\mathbf{CP^2} \subset \mathbf{CP}^l$. Then $\mathcal{A}^*$ is a nice point arrangement of hyperplanes in $\mathbf{CP}^l$.

Firstly, one can easily verify that $\mathcal{G}(i) = \emptyset$ for all $1 \leqslant i \leqslant l-2$. Obviously, $St(A)$ is a simple joint star of the soul $\mathcal{G}$, since the two end elements B and C are connected by only one hyperplane. After removing $St(A)$, we find that the remaining part of $\mathcal{G}$ has no loop (see Figure 2). So $\mathcal{A}^*$ is a nice point arrangement.

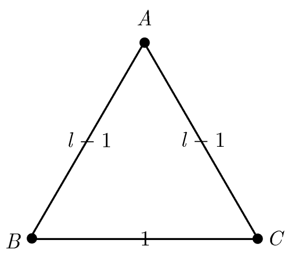

Figure 1

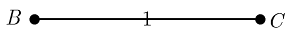

Figure 2

Example 2.12. Let $\mathcal{A}^*$ be an arrangement of $2l+1$ hyperplanes in $\mathbf{CP}^l$ and $\mathcal{G}$ be the soul of $\mathcal{A}^*$. Assume that $\mathcal{G}(0) = \{A, B, C\}$ with multiplicities $m(A) = 2l-3$, $m(B) = m(C) = l+1$. Let S_{AB} be a plane of multiplicity $l-2$ which connects A and B. Let S_{AC} be a plane of multiplicity $l-2$ which connects A and C. Moreover, assume that there are only two hyperplanes passing though both B and C. Then $\mathcal{A}^*$ is not a nice point arrangement of hyperplanes in $\mathbf{CP}^l$.

One can easily check that $\mathcal{G}(i) = \emptyset$ for all $1 \leqslant i \leqslant l-2$. But none of the three stars $St(A)$, $St(B)$ and $St(C)$ is a simple joint star of the soul $\mathcal{G}$, since the two end points of any of the stars are connected by more than one paths. Plus, any two stars are not disjoint. So there are no disjoint simple joint stars in $\mathcal{G}$ such that $\mathcal{G}'$ has no loop. Hence $\mathcal{A}^*$ is not a nice point arrangemen of hyperplanes in $\mathbf{CP}^l$.

Example 2.13. Let $\mathcal{A}^*$ be an arrangement of hyperplanes in $\mathbf{CP}^l$ and $\mathcal{G}$ be the soul of $\mathcal{A}^*$. Assume that $\mathcal{G}(0) = \{A, B, C, D\}$. Figure 3 shows the generic projection of $\mathcal{A}^*$ onto $\mathbf{CP^2} \subset \mathbf{CP}^l$. Line AB and AD are the only lines in $L(\mathcal{A}^*)$ which are both of multiplicity $l-2$. There are two hyperplanes containing B and C but not A and D. There are two hyperplanes containing D and C but not A and B.

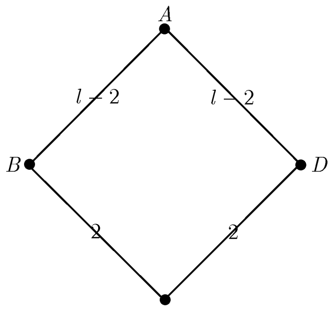

Figure 3

Figure 4

Note that we can choose $St(A)$ and $St(C)$ as simple joint stars. Because B and D are shared end points, the numbers of shared hyperplanes are both l. Moreover, $\mathcal{G}' = \mathcal{G} \setminus (St(A) \cup St(C))$ (see Figure 4) has no loop. Therefore $\mathcal{A}^*$ is also a nice point arrangement.

Example 2.14. Let $\mathcal{A}^*$ be an arrangement of hyperplanes in $\mathbf{CP}^l$ and $\mathcal{G}$ be the soul of $\mathcal{A}^*$. Assume that $\mathcal{G}(0) = \{A, B, C, D, E, F\}$. Figure 5 shows the generic projection of $\mathcal{A}^*$ on $\mathbf{CP^2} \subset \mathbf{CP}^l$.

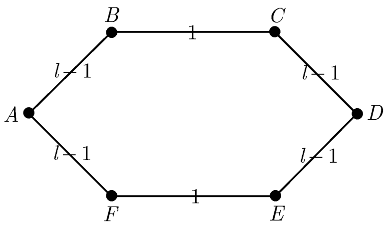

Figure 5

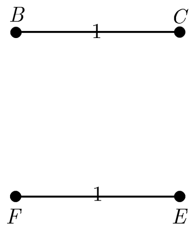

Figure 6

We can easily make $\mathcal{G}(i) = \emptyset$ for all $1 \leqslant i \leqslant n-2$ (see Figure 6) so that $\mathcal{A}^*$ is a nice point arrangement of hyperplanes in $\mathbf{CP}^l$.

3 Setup

Let $\mathcal{N} = \{0, 1, \dots, l\}$. We need to consider equations of the following form:

$$P = \sum_{\substack{I_1 \cup I_2 = \mathcal{N} \\ I_1 \cap I_2 = \emptyset \\ I_1 \neq \emptyset, I_2 \neq \emptyset}} \big(C_{(I_1, I_2)} \prod_{i \in I_1} x_i \prod_{j \in I_2} y_j\big) = 0, \tag{1}$$

where $C_{(I_1,I_2)}$ are constants and $(x_r, y_r) \in \mathbf{CP}^1$ for $r \in \mathcal{N}$.

Definition 3.1. *An $(l-1)$-tuple $((x_{r_1}, y_{r_1}), \dots, (x_{r_{l-1}}, y_{r_{l-1}})) \in (\mathbf{CP}^1)^{(l-1)}$ is called an irregular $(l-1)$-tuple for Equation (1) if and only if the homogeneous polynomial P is a reducible polynomial of the remaining two variables, where $r_i \in \mathcal{N}$, for $i = 1, \dots, l-1$ and $r_i \neq r_j$ when $i \neq j$.*

Lemma 3.2. *Let $ax_ix_j + by_ix_j + cx_iy_j + dy_iy_j$ be a homogeneous degree two polynomial. Then it is reducible if and only if*

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0.$$

Proof. Suppose that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0.$$

Then $(a, b) = k(c, d)$, where k is a constant in $\mathbf{C}^* = \mathbf{C} - \{0\}$. Thus $ax_ix_j + by_ix_j + cx_iy_j + dy_iy_j = (ax_i + by_i)(x_j + \frac{1}{k}y_j)$. So it is reducible.

Conversely, suppose that the polynomial is reducible. Then the polynomial can only be factorized into $(Ax_i + By_i)(Cx_j + Dy_j)$. So, $(Ax_i + By_i)(Cx_j + Dy_j) = ACx_ix_j + BCy_ix_j + ADx_iy_j + BDy_iy_j = ax_ix_j + by_ix_j + cx_iy_j + dy_iy_j$. By comparing the coefficients, it follows that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} AC & BC \\ AD & BD \end{vmatrix} = ABCD - ABCD = 0.$$

Corollary 3.3. *Suppose that all of the coefficients of equation* (1) *are non-zero. Then the following hold*:

(i) *The set of irregular $(l-1)$-tuples, $((x_{r_1}, y_{r_1}), \dots, (x_{r_{l-1}}, y_{r_{l-1}})) \in (\mathbf{CP}^1)^{(l-1)}$ for equation* (1), *is a hypersurface in $(\mathbf{CP}^1)^{l-1}$.*

(ii) *The set of l-tuples, $((x_{r_0}, y_{r_0}), \dots, (x_{r_{l-1}}, y_{r_{l-1}})) \in (\mathbf{CP}^1)^{(l-1)}$, which contains an irregular $(l-1)$-tuple for equation* (1), *is the union of finite hypersurfaces in $(\mathbf{CP}^1)^l$.*

(iii) *The set of $(l+1)$-tuples, $((x_{r_0}, y_{r_0}), \dots, (x_{r_l}, y_{r_l})) \in (\mathbf{CP}^1)^l$, which contains an irregular $(l-1)$-tuple for equation* (1), *is the union of finite hypersurfaces in $(\mathbf{CP}^1)^{l+1}$.*

Proof. Let $((x_{r_1}, y_{r_1}), \dots, (x_{r_{l-1}}, y_{r_{l-1}})) \in (\mathbf{CP}^1)^{(l-1)}$ be an irregular $(l-1)$-tuple for equation (1). Suppose the remaining two variables are (x_{r_0}, y_{r_0}) and (x_{r_l}, y_{r_l}). Then the equation can be written as

$$\left[\sum_{\substack{I_1\cup I_2=\mathcal{N}\\ I_1\cap I_2=\emptyset\\ \{0,l\}\subset I_1\\ I_2\neq\emptyset}} \left(C_{(I_1,I_2)} \prod_{i\in I_1\backslash\{0,l\}} x_{r_i} \prod_{j\in I_2} y_{r_j}\right)\right] x_{r_0}x_{r_l}$$

$$+ \left[\sum_{\substack{I_1\cup I_2=\mathcal{N}\\ I_1\cap I_2=\emptyset\\ l\in I_1\\ 0\in I_2}} \left(C_{(I_1,I_2)} \prod_{i\in I_1\backslash\{l\}} x_{r_i} \prod_{j\in I_2\backslash\{0\}} y_{r_j}\right)\right] y_{r_0}x_{r_l}$$

$$+\Bigg[\sum_{\substack{I_1\cup I_2=\mathcal{N}\\ I_1\cap I_2=\emptyset\\ 0\in I_1\\ l\in I_2}}\Bigg(C_{(I_1,I_2)}\prod_{i\in I_1\setminus\{0\}}x_{r_i}\prod_{j\in I_2\setminus\{l\}}y_{r_j}\Bigg)\Bigg]x_{r_0}y_{r_l}$$

$$+\Bigg[\sum_{\substack{I_1\cup I_2=\mathcal{N}\\ I_1\cap I_2=\emptyset\\ I_1\neq\emptyset\\ \{0,l\}\subset I_2}}\Bigg(C_{(I_1,I_2)}\prod_{i\in I_1}x_{r_i}\prod_{j\in I_2\setminus\{0,l\}}y_{r_j}\Bigg)\Bigg]y_{r_0}y_{r_l}=0.$$

Hence polynomial at the left-hand side of equation (1) is reducible if and only if the determinant

$$\begin{vmatrix}\Bigg[\sum\limits_{\substack{I_1\cup I_2=\mathcal{N}\\ I_1\cap I_2=\emptyset\\ \{0,l\}\subset I_1,\, I_2\neq\emptyset}}\Bigg(C_{(I_1,I_2)}\prod\limits_{i\in I_1\setminus\{0,l\}}x_{r_i}\prod\limits_{j\in I_2}y_{r_j}\Bigg)\Bigg] & \Bigg[\sum\limits_{\substack{I_1\cup I_2=\mathcal{N}\\ I_1\cap I_2=\emptyset\\ l\in I_1,\, 0\in I_2}}\Bigg(C_{(I_1,I_2)}\prod\limits_{i\in I_1\setminus\{l\}}x_{r_i}\prod\limits_{j\in I_2\setminus\{0\}}y_{r_j}\Bigg)\Bigg]\\ \Bigg[\sum\limits_{\substack{I_1\cup I_2=\mathcal{N}\\ I_1\cap I_2=\emptyset\\ 0\in I_1,\, l\in I_2}}\Bigg(C_{(I_1,I_2)}\prod\limits_{i\in I_1\setminus\{0\}}x_{r_i}\prod\limits_{j\in I_2\setminus\{l\}}y_{r_j}\Bigg)\Bigg] & \Bigg[\sum\limits_{\substack{I_1\cup I_2=\mathcal{N}\\ I_1\cap I_2=\emptyset\\ I_1\neq\emptyset,\, \{0,l\}\subset I_2}}\Bigg(C_{(I_1,I_2)}\prod\limits_{i\in I_1}x_{r_i}\prod\limits_{j\in I_2\setminus\{0,l\}}y_{r_j}\Bigg)\Bigg]\end{vmatrix} \tag{2}$$

vanishes, which defines a hypersurface in $(\mathbf{CP}^1)^{l-1}$. Hence (i) holds.

Now we prove (ii). Clearly, there are only l different forms of $(l-1)$-tuples in $((x_{r_0},y_{r_0}),\dots,(x_{r_{l-1}},y_{r_{l-1}}))\in(\mathbf{CP}^1)^l$ for equation (1). For a given form, the irregular $(l-1)$-tuples generates a polynomial equation of form (ii) which can be regard as an equation of variables $(x_{r_0},y_{r_0}),\dots,(x_{r_{l-1}},y_{r_{l-1}})$, i.e. a hypersurface in $(\mathbf{CP}^1)^l$. Then (ii) follows.

To prove (iii), we only need to point out that there are $\binom{l+1}{2}$ different forms of $(l-1)$-tuples in $((x_{r_0},y_{r_0}),\dots,(x_{r_l},y_{r_l}))\in(\mathbf{CP}^1)^{(l+1)}$ for equation (1). Immigrating the argument in the proof of (ii), one will easily get (iii).

Corollary 3.3. *Suppose that all of the coefficients of equation* (1) *are non-zero. Then the* $(l-1)$*-tuples* $((1,0))^{l-1}$ *and* $((0,1))^{l-1}$ *are both non-irregular for equation* (1).

Proof. Plug $((1,0))^{l-1}$ into the determinant (2) and simplify each entries, one can easily check that the determinant equals

$$\begin{vmatrix}0 & C_{(\{1,\dots,l\},\{0\})}\\ C_{(\{0,\dots,l-1\},\{l\})} & C_{(\{1,\dots,l-1\},\{0,l\})}\end{vmatrix}.$$

By the assumption, the coefficients $C_{(I_1,I_2)}$ in equation (1) are all non-zero. So $C_{(\{1,\dots,l\},\{0\})}$, $C_{(\{0,\dots,l-1\},\{l\})}$ and $C_{(\{1,\dots,l-1\},\{0,l\})}$ are all non-zero, which implies that the above determinant is non-zero. Hence $((1,0))^{l-1}$ is non-irregular. Similarly, we know that $((0,1))^{l-1}$ is also non-irregular.

Before we introduce the next lemma, let us give some notations at first.

$$P_{11}=\sum_{\substack{I_1\cup I_2=\mathcal{N}\\ I_1\cap I_2=\emptyset\\ \{0,l\}\subset I_1,\, I_2\neq\emptyset}}\Bigg(C_{(I_1,I_2)}\prod_{i\in I_1\setminus\{0,l\}}x_{r_i}\prod_{j\in I_2}y_{r_j}\Bigg), \tag{3}$$

$$P_{12}=\sum_{\substack{I_1\cup I_2=\mathcal{N}\\ I_1\cap I_2=\emptyset\\ l\in I_1,\, 0\in I_2}}\Bigg(C_{(I_1,I_2)}\prod_{i\in I_1\setminus\{l\}}x_{r_i}\prod_{j\in I_2\setminus\{0\}}y_{r_j}\Bigg), \tag{4}$$

$$P_{21} = \sum_{\substack{I_1 \cup I_2 = \mathcal{N} \\ I_1 \cap I_2 = \emptyset \\ 0 \in I_1,\, l \in I_2}} \left(C_{(I_1,I_2)} \prod_{i \in I_1 \setminus \{0\}} x_{r_i} \prod_{j \in I_2 \setminus \{l\}} y_{r_j} \right), \tag{5}$$

$$P_{22} = \sum_{\substack{I_1 \cup I_2 = \mathcal{N} \\ I_1 \cap I_2 = \emptyset \\ I_1 \neq \emptyset,\, \{0,l\} \subset I_2}} \left(C_{(I_1,I_2)} \prod_{i \in I_1} x_{r_i} \prod_{j \in I_2 \setminus \{0,l\}} y_{r_j} \right). \tag{6}$$

Lemma 3.5. *Suppose that all of the coefficients of equation* (1) *are non-zero. Given a non-irregular* $(l-1)$*-tuple* $((x_{r_1}, y_{r_1}), \ldots, (x_{r_{l-1}}, y_{r_{l-1}})) \in (\mathbf{CP}^1)^{(l-1)}$ *for equation* (1)*, let* P_{ij}*,* $i, j = 1, 2$ *be the polynomials defined above. Then*

$$\begin{pmatrix} x_{r_l} \\ y_{r_l} \end{pmatrix} = K \begin{pmatrix} P_{21} & P_{22} \\ -P_{11} & -P_{12} \end{pmatrix} \begin{pmatrix} x_{r_0} \\ y_{r_0} \end{pmatrix}, \quad K \in \mathbf{C}^* \tag{7}$$

is an automorphism of $\mathbf{CP}^1$*. In particular, if* $((x_{r_1}, y_{r_1}), \ldots, (x_{r_{l-1}}, y_{r_{l-1}})) = ((1,0))^{l-1}$ *, then the automorphism* (7) *sends* $(1,0)$ *to* $(1,0)$*. If* $((x_{r_1}, y_{r_1}), \ldots, (x_{r_{l-1}}, y_{r_{l-1}})) = ((0,1))^{l-1}$*, then the automorphism* (7) *sends* $(0,1)$ *to* $(0,1)$*.*

Proof. Notice that equation (1) can be written as

$$P_{11} x_{r_0} x_{r_l} + P_{12} y_{r_0} x_{r_l} + P_{21} x_{r_0} y_{r_l} + P_{22} y_{r_0} y_{r_l} = 0,$$

which implies that

$$\begin{pmatrix} x_{r_l} \\ y_{r_l} \end{pmatrix} = K \begin{pmatrix} P_{21} x_{r_0} + P_{22} y_{r_0} \\ -P_{11} x_{r_0} - P_{12} y_{r_0} \end{pmatrix} = K \begin{pmatrix} P_{21} & P_{22} \\ -P_{11} & -P_{12} \end{pmatrix} \begin{pmatrix} x_{r_0} \\ y_{r_0} \end{pmatrix}, \quad K \in \mathbf{C}^*.$$

Since $((x_{r_1}, y_{r_1}), \ldots, (x_{r_{l-1}}, y_{r_{l-1}}))$ is non-irregular, by Lemma 3.2

$$-\begin{vmatrix} P_{21} & P_{22} \\ -P_{11} & -P_{12} \end{vmatrix} = \begin{vmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{vmatrix} \neq 0.$$

Therefore the matrix

$$\begin{pmatrix} P_{11} & P_{12} \\ -P_{21} & -P_{22} \end{pmatrix}$$

is an invertible matrix which defines an automorphism of $\mathbf{CP}^1$. When $((x_{r_1}, y_{r_1}), \ldots, (x_{r_{l-1}}, y_{r_{l-1}})) = ((1,0))^{l-1}$, by Corollary 3.4 we know

$$\begin{pmatrix} P_{21} & P_{22} \\ -P_{11} & -P_{12} \end{pmatrix} = \begin{pmatrix} C_{(\{0,\ldots,l-1\},\{l\})} & C_{(\{1,\ldots,l-1\},\{0,l\})} \\ 0 & -C_{(\{1,\ldots,l\},\{0\})} \end{pmatrix}$$

is invertible. Then

$$\begin{pmatrix} P_{21} & P_{22} \\ -P_{11} & -P_{12} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} C_{(\{0,\ldots,l-1\},\{l\})} & C_{(\{1,\ldots,l-1\},\{0,l\})} \\ 0 & -C_{(\{1,\ldots,l\},\{0\})}. \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = K \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Similarly, when $((x_{r_1}, y_{r_1}), \ldots, (x_{r_{l-1}}, y_{r_{l-1}})) = ((0,1))^{l-1}$, the automorphism (7) sends $(0,1)$ to $(0,1)$.

Remark 3.6. Under the assumption and the proof of the above lemmas, we know that the automorphism (7) and equation (1) are equivalent. Fixing n variables (x_{i_0}, y_{i_0}), (x_{i_1}, y_{i_1}), $\ldots$, $(x_{i_{l-1}}, y_{i_{l-1}})$ of equation (1), if there is a non-irregular $(l-1)$-tuple, say, (x_{i_1}, y_{i_1}), $\ldots$, $(x_{i_{l-1}}, y_{i_{l-1}})$, then (x_{i_l}, y_{i_l}) will be uniquely determined by automorphism (7). We call such procedure "fixing l variables to solve the remaining one".

Theorem 3.7. (Lattice-Isotopy Theorem [9]) *If two arrangements are connected by a one-parameter family of arragements* $\{\mathcal{A}_t\}$ *which have the same lattice* $L(\mathcal{A})$*, then the complements are diffeomorphic, hence of the same homotopy type.*

4 Diffeomorphic type of complement of nice arrangement in $\mathbf{CP}^l$

In this section we will prove the main theorem. The idea of the proof is to construct a one-parameter family between $\mathcal{A}_0^*$ and $\mathcal{A}_1^*$ such that $L(\mathcal{A}_t^*) \cong L(\mathcal{A}_0^*) \cong L(\mathcal{A}_1^*)$ for any t moving on a real curve. In fact, the lattice conditions require us to solve a family of equations and some inequalities and show that the solution set has a real curve which connects the points corresponding to $\mathcal{A}_0^*$ and $\mathcal{A}_1^*$. First, we will prove that the solution set for the equations is a variety depending only on some free variables. Roughly speaking, the variety is in some sense an embedding of $(\mathbf{CP}^1)^m$ in $(\mathbf{CP}^1)^n$, i.e. there exists an open subset of $(\mathbf{CP}^1)^m$ on which the equations define an embedding of this open subset in $(\mathbf{CP}^1)^n$ and the image is a subset of the whole solution set. We will prove that there exists a such open subset which is path-connected. Furthermore, The points corresponding to $\mathcal{A}_0^*$ and $\mathcal{A}_1^*$ are contained in this open subset.

Proof of Main theorem. Suppose that $\mathcal{A}_0^* = \{G_1, G_2, \ldots, G_n\}$ and $\mathcal{A}_1^* = \{H_1, H_2, \ldots, H_n\}$ are two arrangements of hyperplanes in $\mathbf{CP}^l$, where $G_i = (g_{0i}, g_{1i}, \ldots, g_{li})^T$ and $H_i = (h_{0i}, h_{1i}, \ldots, h_{li})^T$ are the dual representations of the defining equations of the hyperplanes in $\mathbf{CP}^l$. We need to construct a one-parameter family of arrangements $\mathcal{A}_t^*$ which connects $\mathcal{A}_0^*$ and $\mathcal{A}_1^*$ and $L(\mathcal{A}_t^*) \equiv L(\mathcal{A}_0^*) \equiv L(\mathcal{A}_1^*)$ for all t on a real curve.

Let $\mathcal{A}_t^* = \{F_1, F_2, \ldots, F_n\}$ where $F_i = x_i G_i + y_i H_i$ and $(x_i, y_i) \in \mathbf{CP}^1$. In dual representation, $F_i = (x_i g_{0i} + y_i h_{0i}, x_i g_{1i} + y_i h_{1i}, \ldots, x_i g_{li} + y_i h_{li})^T$. Thus $\mathcal{A}_t^*$ is an arrangement in $\mathbf{CP}^l$.

For any point $e \in \mathcal{G}(\mathcal{A}_t^*)$, let $\mathcal{S} = \{(r_0, \ldots, r_l) : r_i \in \mathcal{N}$ and $1 \leqslant r_0 < r_1 < \ldots < r_l \leqslant n\}$. So $|\mathcal{S}| = \binom{n}{l+1}$. For any $l+1$ hyperplanes $F_{r_0}, \ldots, F_{r_l}$ passing through point e, denote by $|F_{r_0} \cdots F_{r_l}|$ the determinant of the $(l+1) \times (l+1)$ matrix $(F_{r_0} \cdots F_{r_l})$. Since $L(\mathcal{A}_t^*) \cong L(\mathcal{A}_0^*)$ if and only if $\mathcal{G}(\mathcal{A}_t^*) \cong \mathcal{G}(\mathcal{A}_0^*)$, by definition of nice point arrangement, we only need to focus on the points in the souls. Actually, we only need consider all the possible $l+1$ hyperplanes intersecting exactly at one point. Therefore, to get $L(\mathcal{A}_t^*) \cong L(\mathcal{A}_0^*)$ it is sufficient and necessary to show that $\mathrm{rank}(F_{r_0} \cdots F_{r_l}) = l$ if and only if $\mathrm{rank}(G_{r_0} \cdots G_{r_l}) = l$ for any $(r_0, \ldots, r_l) \in \mathcal{S}$. This is equivalent to

(1) $|F_{r_0} \cdots F_{r_l}| = 0$ if and only if $|G_{r_0} \cdots G_{r_l}| = 0$, and

(2) there exists one non-zero l-th order minor $D_l(F_{r_0} \cdots F_{r_l})$ if and only if there exists one non-zero l-th order minor $D_l(G_{r_0} \cdots G_{r_l})$.

For $I \subset \mathcal{N} = \{0, 1, 2, \ldots, n\}$, let $|H_{\mathcal{N} \setminus I} G_I|$ be the determinant of $(l+1) \times (l+1)$ matrix whose i-th column is G_{r_i} if $i \in I$ and H_{r_i} if $i \in \mathcal{N} \setminus I$ where $i \in I$. By the multilinear property of determinant of matrix, using the new notation, we have

$$|F_{r_0} \cdots F_{r_l}| = \sum_{I \subset \mathcal{N}} \left(\left| H_{\mathcal{N} \setminus I} G_I \right| \cdot \prod_{i \in \mathcal{N} \setminus I} x_i \prod_{j \in I} y_j \right). \tag{8}$$

Let $d = \sum_{i \geqslant l+1} \binom{i}{l+1} p_i(\mathcal{A}_0^*)$. The condition (1) is equivalent to the following two conditions:

(A) If $|G_{r_0} \cdots G_{r_l}| = 0$, then polynomial (8) vanishes and $|H_{r_0} \cdots H_{r_l}| = 0$, i.e. equation (1) holds.

(B) If $|G_{r_0} \cdots G_{r_l}| \neq 0$, then polynomial (8) does not equal zero.

Hence, to show condition (1), we have to consider a system of d equations and a system of $\binom{n}{l+1} - d$ inequalities:

$$P_1 = 0, P_2 = 0, \dots, P_d = 0;$$

$$Q_1 \neq 0, Q_2 \neq 0, \dots, Q_{\binom{n}{l+1} - d} \neq 0.$$

Here, P_i's are polynomials of equation (1) and Q_i's are polynomials of equation (8). To show condition (2), we can consider an l-th order minor $D_l(F_{r_0} \cdots F_{r_l})$ of $(F_{r_0} \cdots F_{r_l})$ such that, $D_l(F_{r_0} \cdots F_{r_l}) \neq 0$ if and only if $D_l(G_{r_0} \cdots G_{r_l}) \neq 0$. Notice that among $P_1 = 0, P_2 = 0, \dots, P_d = 0$, there are only $c(\mathcal{A}_0^*) = \sum_{j > l} (j - l) p_j(\mathcal{A}_0^*)$ of them which are independent, because at every point there are l hyperplanes in general position passing though the point in $\mathbf{CP}^l$.

To prove the theorem, we need to construct a one parameter family of arrangements $\mathcal{A}_t^*$ with isomorphic lattices. We will prove that there exists a real curve connnecting the points $((1,0), \dots, (1,0))$ and $((0,1), \dots, (0,1))$ which lies on some irreducible component of $\{P_1 = 0, \dots, P_{c(\mathcal{A}_0^*)} = 0\}$. This curve can be chosen so that it will neither intersect with $\bigcup_{i=1}^{\binom{n}{l+1} - d} \{Q_i = 0\}$ nor intersect with the intersection of all $\{D_l(F_{r_0} \cdots F_{r_l}) = 0\}$.

We shall prove that all of the coefficients of the equations $P_1 = 0, P_2 = 0, \dots, P_{c(\mathcal{A}_0^*)} = 0$ are non-zero. Fixing $\mathcal{A}_0^*$, we claim that there exists an automorphism of $\mathbf{CP}^l$ acting on $\mathcal{A}_1^*$ such that all of the coefficients of the polynomials are non-zero after renewing $\mathcal{A}_1^*$ under the action. We know that the automorphism of projective space is defined by a matrix (see [15, Example 7.1.1]):

$$\Gamma = \begin{pmatrix} \alpha_{00} & \cdots & \alpha_{0l} \\ \vdots & \ddots & \vdots \\ \alpha_{l0} & \cdots & \alpha_{ll} \end{pmatrix},$$

whose determinant is nonzero. That the determinant equals zero defines exactly a hypersurface in $\mathbf{CP}^{l^2+2l}$ by regarding the entries as variables.

For any hyperplane in $\mathcal{A}_1^*$, say $H_i = (h_{0i}, \dots, h_{li})^T$, $1 \leqslant i \leqslant q$, the action of Γ on H_i gives a new hyperplane:

$$H_i' = \left(\sum_{r=0}^{l} \alpha_{0r} h_{ri}, \dots, C \sum_{r=0}^{l} \alpha_{lr} h_{ri} \right)^T.$$

Suppose on the contrary that for any automorphism of $\mathbf{CP}^l$ acting on $\mathcal{A}_1^*$ there is a coefficient in one of the polynomials $P_1, \dots, P_{c(\mathcal{A}_0^*)}$ which remains zero. Replacing all $F_i = x_i H_i + y_i G_i$ by

$F_i' = x_iH_i' + y_iG_i$, one find that that a coefficient $|H'_{\mathcal{N}\setminus I}G_I|$ in the polynomial is zero defines a determinant-type polynomial equation of variables α_{ij}. We shall check that this determinant-type polynomial equation of variables α_{ij} is not identically zero, i.e. $|H'_{\mathcal{N}\setminus I}G_I| \not\equiv 0$, for $\emptyset \neq I \subsetneq \mathcal{N}$. Since G_{r_i}, $i \in I$, are in general position, then there exists a non-zero $|I|\times|I|$ minor. We may assume that the non-zero $|I|\times|I|$ minor sits on the last $|I|$ rows of $|G_{r_{i_1}}\cdots G_{r_{i_{|I|}}}|$, i.e.,

$$\begin{vmatrix} g_{(l-i)r_{i_1}} & \cdots & g_{(l-i)r_{i_{|I|}}} \\ \vdots & \ddots & \vdots \\ g_{lr_{i_1}} & \cdots & g_{lr_{i_{|I|}}} \end{vmatrix} \neq 0, \tag{9}$$

where $i_1 < i_2 < \ldots < i_{|I|}$ are different elements in I. We claim that the cofactor of the above non-zero minor (9) must be zero identically, i.e.

$$\begin{vmatrix} \sum_{k=0}^{l}\alpha_{0k}h_{kr_{j_1}} & \cdots & \sum_{k=0}^{l}\alpha_{0k}h_{kr_{j_{|\mathcal{N}\setminus I|}}} \\ \vdots & \ddots & \vdots \\ \sum_{k=0}^{l}\alpha_{(l-|I|)k}h_{kr_{j_1}} & \cdots & \sum_{k=0}^{l}\alpha_{(l-|I|)k}h_{kr_{j_{|\mathcal{N}\setminus I|}}} \end{vmatrix} \equiv 0. \tag{10}$$

In fact, by Laplace theorem,

$$|H'_{\mathcal{N}\setminus I}G_I| = \sum_{j_1<j_2<\ldots<j_{|I|}} \begin{vmatrix} g_{j_1r_{i_1}} & \cdots & g_{j_1r_{i_{|I|}}} \\ \vdots & \ddots & \vdots \\ g_{j_{|I|}r_{i_1}} & \cdots & g_{j_{|I|}r_{i_{|I|}}} \end{vmatrix}$$

$$\cdot(-1)^{j_1+\ldots+j_{|\mathcal{N}\setminus I|}+i_1+\ldots+i_{|I|}} \begin{vmatrix} \sum_{k=0}^{l}\alpha_{j_1'k}h_{kr_{j_1}} & \cdots & \sum_{k=0}^{l}\alpha_{j_1'k}h_{kr_{j_{|\mathcal{N}\setminus I|}}} \\ \vdots & \ddots & \vdots \\ \sum_{k=0}^{l}\alpha_{j'_{|\mathcal{N}\setminus I|}k}h_{kr_{j_1}} & \cdots & \sum_{k=0}^{l}\alpha_{j'_{|\mathcal{N}\setminus I|}k}h_{kr_{j_{|\mathcal{N}\setminus I|}}} \end{vmatrix},$$

where $\{j_1'\cdots j'_{|\mathcal{N}\setminus I|}\} = \{0,\ldots,l\}\setminus\{j_1\cdots j_{|I|}\}$, and $j_1' < \ldots < j'_{|\mathcal{N}\setminus I|}$. Note that in the above expansion any two terms have no like terms in variables α_{ij}. Then $|H'_{\mathcal{N}\setminus I}G_I| \equiv 0$ implies equation (10). By using the multilinear property of determinant, the identity (10) can be easily written as

$$\sum_{0\leqslant k_1<k_2<\cdots<k_{|\mathcal{N}\setminus I|}\leqslant l} \begin{vmatrix} h_{k_1r_{j_1}} & \cdots & h_{k_1r_{j_{|\mathcal{N}\setminus I|}}} \\ \vdots & \ddots & \vdots \\ h_{k_{|\mathcal{N}\setminus I|}r_{j_1}} & \cdots & h_{k_{|\mathcal{N}\setminus I|}r_{j_{|\mathcal{N}\setminus I|}}} \end{vmatrix} \begin{vmatrix} \alpha_{0k_1} & \cdots & \alpha_{0k_{|\mathcal{N}\setminus I|}} \\ \vdots & \ddots & \vdots \\ \alpha_{(l-|I|)k_1} & \cdots & \alpha_{(l-|I|)k_{|\mathcal{N}\setminus I|}} \end{vmatrix} \equiv 0.$$

Thus as a polynomial equation of variables α_{ij}, all the coefficients in the above equality must be zero, i.e.

$$\begin{vmatrix} h_{k_1r_{j_1}} & \cdots & h_{k_1r_{j_{|\mathcal{N}\setminus I|}}} \\ \vdots & \ddots & \vdots \\ h_{k_{|\mathcal{N}\setminus I|}r_{j_1}} & \cdots & h_{k_{|\mathcal{N}\setminus I|}r_{j_{|\mathcal{N}\setminus I|}}} \end{vmatrix} = 0$$

for $0 \leqslant k_1 < k_2 < \cdots < k_{|\mathcal{N}\setminus I|} \leqslant l$. Hence all the $|\mathcal{N}\setminus I| \times |\mathcal{N}\setminus I|$ minors of the $l \times |\mathcal{N}\setminus I|$ matrix

$$\begin{vmatrix} h_{0r_{j_1}} & \cdots & h_{0r_{j_{|\mathcal{N}\setminus I|}}} \\ \vdots & \ddots & \vdots \\ h_{lr_{j_1}} & \cdots & h_{lr_{j_{|\mathcal{N}\setminus I|}}} \end{vmatrix}$$

are zero, where $j_1 < j_2 < \cdots < j_{|\mathcal{N}\setminus I|}$ are elements in $\mathcal{N}\setminus I$. Therefore $H_{j_1}, \ldots, H_{j_{|\mathcal{N}\setminus I|}}$ are linearly dependent, which contradicts Lemma 2.9. We then prove that all the determinant-type polynomials $|H'_{\mathcal{N}\setminus I}G_I|$ of α_{ij} in $\mathbf{CP}^{l^2+2l}$ are not identically zero, i.e. $|H'_{\mathcal{N}\setminus I}G_I| = 0$ defines a hypersurface in $\mathbf{CP}^{l^2+2l}$. Let P'_i be the polynomial derived from P_i by replacing H_j by H'_j. Then that a coefficient of the polynomials $P'_1, P'_2, \ldots, P'_{c(\mathcal{A}_0^*)}$ equals zero defines a hypersurface in $\mathbf{CP}^{l^2+2l}$. These hypersurfaces together with hypersurface defined by $\det(\Gamma) = 0$ generate an arrangement $\mathcal{A}_{\mathcal{S}}$ of finite hypersurfaces in $\mathbf{CP}^{l^2+2l}$. Obviously, the union of the hypersurfaces in $\mathcal{A}_{\mathcal{S}}$ is not the whole projective space. Then there exist points in the complement of the union in $\mathbf{CP}^{l^2+2l}$, i.e. an automorphism exists such that all the coefficients of polynomials P_i, $1 \leqslant i \leqslant c(\mathcal{A}_0^*)$ are nonzero after the action of the automorphism Γ on $\mathcal{A}_1^*$. Notice that under the automorphism of the projective space, the diffeormorphic type of the complement is unchanged. So we may assume at the beginning that all the coefficients of the polynomials $P_1, P_2, \ldots, P_{c(\mathcal{A}_0^*)}$ are nonzero.

Next we will prove that the system of the polynomial equations $P_1 = 0, \ldots, P_{c(\mathcal{A}_0^*)} = 0$ is solvable, i.e. we can solve all variables in terms of some fixing variables. The key argument is to show that at each point in the soul $\mathcal{G}(\mathcal{A}_t^*)$, there are at most l fixing variables. Then we can fix l variables and solve other variables in terms of these l variables.

Case 0. Assume $\mathcal{G}(\mathcal{A}_t^*)$ has no loop, i.e. $\mathcal{G}(\mathcal{A}_t^*)$ is like a tree. Pick up a point $p \in \mathcal{G}(\mathcal{A}_t^*)$ of multiplicity $m(p)$ which we regard as the root of the tree. There are $m(p)$ variables, say, $(x_1, y_1), \ldots, (x_{m(p)}, y_{m(p)})$ appearing in $m(p) - l$ equations and $(x_1, y_1), \ldots, (x_l, y_l)$ appearing in each of these $m(p) - l$ equations. Thus we may fix $(x_1, y_1), \ldots, (x_l, y_l)$ to solve all the other variables at p. By Lemma 2.10, any end element of p connects with p by at most $l-1$ hyperplanes. Hence we can solve all variables at the end elements of p by fixing l variables in which some are variables shared with p and some are new. Note that for two points in $\mathcal{G}(\mathcal{A}_t^*)$ there are at most $l-1$ hyperplanes containing both of them by Lemma 2.10. Then chasing the points from the "root" p along the "branches" (i.e. paths) to the "treetops" (i.e. points) in $\mathcal{G}(\mathcal{A}_t^*)$, we can solve all the polynomial equations $P_1 = 0, \ldots, P_{c(\mathcal{A}_0^*)}$ at all points of $\mathcal{G}(\mathcal{A}_t^*)$. Clearly, the solution set can be obtained in terms of at least l fixing variables.

Now suppose $\mathcal{G}(\mathcal{A}_t^*)$ has loops. we will show that the solution set is also of dimension no less than l. By the definition of nice point arrangement, there exist simple joint stars, say $St(u_1), \ldots, St(u_k)$ in $\mathcal{G}(\mathcal{A}_t^*)$ such that $\mathcal{G}(\mathcal{A}_t^*)' = \mathcal{G}(\mathcal{A}_t^*) - \{St(u_1), \ldots, St(u_k)\}$ has no loop. We will show by induction that the system of polynomial equations is solvable.

Case 1. Consider that $k = 1$. This means that there is only one star $St(u_1)$. Like Case 0, we can fix arbitrary l variables at u_1 and solve all the remaining variables at u_1. Now consider the end points of u_1. For any two end points of u_1, say u_{11}, u_{12}, there are two subcases. Case

1(a): Suppose that there is no path in $\mathcal{G}(\mathcal{A}_t^*)'$ connecting them. By Lemma 2.10, there are at most $l-1$ hyperplanes connecting u_1 , u_{11} or u_1, u_{12} respectively. Fixing l variables at u_{11} (u_{12} respectively) which include the variables shared with the equations at u_1, we can solve all the remaining variables at u_{11} (u_{12} respectively). Case 1(b): Suppose that u_{11} and u_{12} are connected by path in $\mathcal{G}(\mathcal{A}_t^*)'$. By the definition of simple joint stars, there is only one path connecting u_{11} and u_{12}. By Definition 2.4, each two adjacent points a_{k-1} and a_k in a path are connected by only one hyperplane h_k in this path. If the variables at an end point u_{11} are solved, we can solve all variables at the points in the path between u_{11} and u_{12} and the path will deliver only one solved variable to the end point u_{12}. Thus there are at most l variables at u_{12} being solved and we can use the solved variables to solve the remaining ones. Then all the variables at the star and the end elements can be solved. For other points in $\mathcal{G}(\mathcal{A}_t^*)'$, since $\mathcal{G}(\mathcal{A}_t^*)'$ has no loop, like Case 0, we can also solve all variables at the points in $\mathcal{G}(\mathcal{A}_t^*)'$. Consequently, we can solve all variables at all points in finite steps.

Case 2. For $h=2$, say, there are two stars $St(u_1), St(u_2)$. First we fix arbitrary l variables at u_1 and u_2 respectively and we can solve all variables at u_1 and u_2. If there is no path in $\mathcal{G}(\mathcal{A}_t^*)'$ connected them, we can solve all variables coming form them separately like Case 1. Therefore, we only need consider the case when they are connected.

Suppose $St(u_1), St(u_2)$ are connected by paths in $\mathcal{G}(\mathcal{A}_t^*)'$. All variables at both u_1 and u_2 can be solved at first. If $St(u_1)$ and $St(u_2)$ have shared end elements, by the item (3) in Definition 2.6, at each shared end element there are at most l hyperplanes containing this end element and one of the u_1's or u_2's. Then all variables at each shared end element can be solved based on those l shared variables. Now suppose there is no shared end element. Case 1 shows that we can solve all variables at any end point of u_1 or u_2, if the end points of u_1 only connect with end points of u_1 and the end points of u_2 only connect with end points of u_2 in $\mathcal{G}(\mathcal{A}_t^*)'$. Otherwise, for an end point of u_1, say u_{11}, it can connect with at most one end point u_{21} of u_2. And there is a unique path in $\mathcal{G}(\mathcal{A}_t^*)'$ which connects u_{11} and u_{21}. The path can only deliver one solved variable from u_{11} to u_{21}. Similarly to Case 1, there are at most l solved variables together from u_2 and u_{11}. Then all variables at u_{21} can be solved. Chasing points in the tree $\mathcal{G}(\mathcal{A}_t^*)'$ by paths, we can solve all variables at points in $\mathcal{G}(\mathcal{A}_t^*)'$ as Case 0. So all variables can be solved in finite steps.

Suppose that we can solve all variables for the stars $St(u_1)$, $St(u_2)$, …, $St(u_k)$, $k>2$. For a new coming star $St(u_{k+1})$, each end element of $St(u_{k+1})$ either connects with at most one end element of $St(u_1), \dots, St(u_k)$ by a path in $\mathcal{G}(\mathcal{A}_t^*)'$ or is a shared end element. In both cases, we know that there are at most l solved variables. So all variables coming with it can be solved like Case 1 or Case 2. By induction, all of the variables in $\mathcal{G}(\mathcal{A}_t^*)$ can be solved.

Finally, we will prove that there is a connected open set which contains the points corresponding to $\mathcal{A}_0^*$ and $\mathcal{A}_1^*$.

Suppose that the number of free variables in the solution set is t. Then the solution set of equations is determined by $(\mathbf{CP}^1)^t$. By Remark 3.6, when there exists non-irregular $(l-1)$-tuple in solved variables for equation (1), we will get a unique solution for the remaining variable. We claim that there exists a connected open subset W of $(\mathbf{CP}^1)^t$ which contains $((1,0))^t$ and

$((0,1))^t$ such that all of the solutions for the equations are determined by the automorphisms in Lemma 3.5, i.e,

$$((x_1,y_1),\dots,(x_n,y_n)) = ((x_1,y_1),\dots,(x_t,y_t), f((x_1,y_1),\dots,(x_t,y_t))),$$

where each component of f is a composition of automorphisms.

To prove the existence of such a connected open subset, the idea is that we can shrink the base solution set by removing finite hypersurfaces. We shall introduce an information checking and returning algorithm. By working on Case 0, one shall see that the method works well for all the other cases.

Suppose that the soul $\mathcal{G}(\mathcal{A}_t^*)$ has no loop. Pick up a points $pt_1 \in \mathcal{G}(\mathcal{A}_t^*)$ of multiplicity j_1. Since any l of the hyperplanes passing through pt_1 is in general position, then the intersection structure of this point is determined only by $j_1 - l$ equations $P_1^1 = 0, \dots, P_{(j_1-l)}^1 = 0$. We may assume that the equations at this point have l shared variables, say, $(x_1,y_1),\dots,(x_l,y_l)$. We call them free variables for the solution of the equations. Let I_1 be a set of the points $((x_1,y_1),\dots,(x_l,y_l)) \in (\mathbf{CP}^1)^l$ such that $((x_1,y_1),\dots,(x_{(l-1)},y_{(l-1)}))$ is an irregular $(l-1)$-tuple of one of the equations. According to the proof of Corollary 3.3, we know that I_1 is the union of $j_1 - l$ hypersurfaces in $(\mathbf{CP}^1)^l$. Let U_1 be the complement $(\mathbf{CP}^1)^l - I_1$. Then any point in U_1 contains a non-irregular $(l-1)$-tuple $((x_1,y_1),\dots,(x_{(l-1)},y_{(l-1)}))$. So each point in U_1 will induce a unique solution for each equation by a embedding

$$f_1((x_1,y_1),\dots,(x_l,y_l)) = ((x_1,y_1),\dots,(x_l,y_l),\phi_1^1((x_l,y_l)),\dots,\phi_{(j_1-l)}^1((x_l,y_l))),$$

where ϕ_i^1 is an automorphism of form (7) introduced in Lemma 3.5. Clearly, $f_1(U_1) \subset (\mathbf{CP}^1)^{j_1}$ is a set of solutions for equations $P_1^1 = 0,\dots,P_{(j_1-l)}^1 = 0$. Note that each point in $f_1(U_1)$ is given by a point $((x_1,y_1),\dots,(x_l,y_l))$ in U_1 under the embedding f_1. We call $(x_1,y_1),\dots,(x_l,y_l)$ the free variables.

Now let pt_2 be an end point of multiplicity j_2 which connects with pt_1 by l_1, $l_1 \leqslant l-1$, hyperplanes. Then there are $j_2 - l$ equations $P_1^2 = 0,\dots,P_{(j_2-l)}^2 = 0$ associated to this point. Suppose that $(x_{i_1},y_{i_1}),\dots,(x_{i_{l_1}},y_{i_{l_1}})$ are the variables associated to the l_1 hyperplanes which contain both pt_1 and pt_2. Choosing $l-l_1$ variables $(x_{j_1+1},y_{j_1+1}),\dots,(x_{j_1+l-l_1},y_{j_1+l-l_1})$ which are associated to the hyperplanes passing through pt_2 but not pt_1, we may then assume that $(x_{i_1},y_{i_1}),\dots,(x_{i_{l_1}},y_{i_{l_1}})$ and $(x_{j_1+1},y_{j_1+1}),\dots,(x_{j_1+l-l_1},y_{j_1+l-l_1})$ are variables appearing in each equation $P_i^2 = 0$ and call the last $l-l_1$ variables $(x_{j_1+1},y_{j_1+1}),\dots,(x_{j_1+l-l_1},y_{j_1+l-l_1})$ new free variables. Let $p_1 = l - l_1$ and $\widetilde{U_1} := f_1(U_1) \times (\mathbf{CP}^1)^{p_1} \subset (\mathbf{CP}^1)^{(j_1+p_1)}$, where p_1 is just the number of the new free variables. Denote I_2' to be the set of points $((x_1,y_1),\dots,(x_{j_1},y_{j_1}),$ $(x_{j_1+1},y_{j_1+1}),\dots,(x_{j_1+p_1},y_{j_1+p_1}))$ in $\widetilde{U_1}$ such that $((x_{i_1},y_{i_1}),\dots,(x_{i_{l_1}},y_{i_{l_1}}),(x_{j_1+1},y_{j_1+1}),\dots,$ $(x_{j_1+p_1-1},y_{j_1+p_1-1}))$ is an irregular $(l-1)$-tuple for one of the equations at point pt_2. By Corollary 3.3, the irregular $(l-1)$-tuple $((x_{i_1},y_{i_1}),\dots,(x_{i_{l-1}},y_{i_{l-1}}))$ for an equation is a hypersurface defined by a polynomial of variables $(x_{i_1},y_{i_1}),\dots,(x_{i_{l-1}},y_{i_{l-1}})$. By regarding the polynomials as polynomials in higher dimensional product space $(\mathbf{CP}^1)^{(j_1+p_1)}$, one will find that I_2' equals the intersection of $\widetilde{U_1}$ and the union of $j_2 - l$ hypersurfaces in $(\mathbf{CP}^1)^{(j_1+p_1)}$. Note that the pull back of a hypersurface in $(\mathbf{CP}^1)^{(j_1+p_1)}$ under $f_1 \times (Id)^{p_1}$ is nothing but a hypersurface in $(\mathbf{CP}^1)^{(l+p_1)}$. Denote by I_2 the pull back of the union of the j_2-l hypersurfaces in $(\mathbf{CP}^1)^{(j_1+p_1)}$. Then I_2 is the union of j_2-l hypersurfaces in $(\mathbf{CP}^1)^{(l+p_1)}$. Let $U_2=U_1\times(\mathbf{CP}^1)^{p_1}-I_2=(\mathbf{CP}^1)^{(l+p_1)}\setminus((I_1\times(\mathbf{CP}^1)^{p_1})\cup I_2))$, then $f_1\times(Id)^{p_1}(U_2)=\widetilde{U_1}\setminus I_2'$ and each point in $f_1\times(Id)^{p_1}(U_2)$ will induce a unique solution for the equations $P_1^1=0,\ldots,P_{j_1-l}^1=0,P_1^2=0,\ldots,P_{(j_2-l)}^2=0$ by an embedding

$$\begin{aligned}&f_2'((x_1,y_1),\ldots,(x_{j_1+p_1}),y_{(j_1+p_1)}))\\&\quad=((x_1,y_1),\ldots,(x_{j_1+p_1},y_{j_1+p_1}),\phi_1^2(x_{j_1+p_1},y_{j_1+p1}),\ldots,\phi_{j_2-l}^2(x_{j_1+p_1},y_{j_1+p1})),\end{aligned}$$

where $\phi_1^2,\ldots,\phi_{j_1-l}^2$ are automorphisms of form (7). Let $f_2=f_2'(f_1\times Id^{p_1})$, then f_2 gives an embedding from $(\mathbf{CP}^1)^{l+p_1}$ to $(\mathbf{CP}^1)^{j_1+j_2-l_1}$ and $f_2(U_2)$ is a solution set of the equations $P_1^1=0,\ldots,P_{j_1-l}^1=0,P_1^2=0,\ldots,P_{(j_2-l)}^2=0$. Similarly, we can solve the equations for all other end points of pt_1 as for pt_2.

Now let pt_3 be an end point of multiplicity j_3 which connects with pt_2 by $l_2(<l)$ hyperplanes. Then there are j_3-l equations $P_1^3=0,\ldots,P_{(j_3-l)}^3=0$ attached to pt_3. Suppose that $(x_{k_1},y_{k_1}),\ldots,(x_{k_{l_2}},y_{k_{l_2}})$ are the variables associated to the $l_2(<l)$ shared hyperplanes between pt_2 and pt_3. Let $c_{12}=j_1+j_2-l_1$ and $p_2=l-l_2$. Freely, choosing p_2 variables $(x_{c_{12}+1},y_{c_{12}+1}),\ldots,(x_{c_{12}+p_2},y_{c_{12}+p_2})$ associated to p_2 hyperplanes which are passing through pt_3 but not pt_2, we may then assume that $(x_{k_1},y_{k_1}),\ldots,(x_{k_{l_2}},y_{k_{l_2}})$ and $(x_{c_{12}+1},y_{c_{12}+1}),\ldots,(x_{c_{12}+p_2},y_{c_{12}+p_2})$ are variables appearing in each equation $P_i^3=0$. We call the last p_2 variables new free variables. Let $\widetilde{U_2}=:f_2(U_2)\times(\mathbf{CP}^1)^{p_2}\subset(\mathbf{CP}^1)^{(c_{12}+p_2)}$ and I_3'' be the set of points $((x_1,y_1),\ldots,(x_{c_{12}+p_2},y_{c_{12}+p_2}))$ in $\widetilde{U_2}$ such that $((x_{k_1},y_{k_1}),\ldots,(x_{k_{l_2}},y_{k_{l_2}}),(x_{c_{12}+1},y_{c_{12}+1}),\ldots,(x_{c_{12}+p_2-1},y_{c_{12}+p_2-1}))$ is an irregular $(l-1)$-tuple for one of the equations at point pt_3. By Corollary 3.3, the irregular $(l-1)$-tuples for an equation are on a hypersurface. By regarding the defining polynomials as polynomials in higher dimensional product space, we know that I_3'' equals to the intersection of $\widetilde{U_2}$ and the union of j_3-l_2 hypersurfaces in $(\mathbf{CP}^1)^{(c_{12}+p_2)}$. Pulling back a hypersurface in $(\mathbf{CP}^1)^{(c_{12}+p_2)}$ under $f_2'\times(Id)^{p_2}$ we have nothing but a hypersurface in $(\mathbf{CP}^1)^{(j_1+p_1+p_2)}$. Denote by I_3' the pull-back of the union of the j_3-l_2 hypersurfaces in $(\mathbf{CP}^1)^{(j_1+p_1+p_2)}$. Pulling back I_3' again under the embedding $f_1\times(Id)^{p_1}\times(Id)^{p_2}$, we will get a set I_3 which is the union of j_3-l_2 hypersurfaces in $(\mathbf{CP}^1)^{(l+p_1+p_2)}$. Let $U_3=U_2\times(\mathbf{CP}^1)^{p_2}-I_3=U_1\times(\mathbf{CP}^1)^{p_1}\times(\mathbf{CP}^1)^{p_2}\setminus(I_2\times(\mathbf{CP}^1)^{p_2}\cup I_3)=(\mathbf{CP}^1)^{(l+p_1+p_2)}\setminus((I_1\times(\mathbf{CP}^1)^{p_1})\times(\mathbf{CP}^1)^{p_2}\cup I_2\times(\mathbf{CP}^1)^{p_2}\cup I_3)$, then $f_2\times(Id)^{p_2}(U_3)=\widetilde{U_2}\setminus I_3''$ and each point in $f_2\times(Id)^{p_2}(U_3)$ induces a unique solution for the equations $P_1^1=0,\ldots,P_{j_1-l}^1=0,P_1^2=0,\ldots,P_{(j_2-l)}^2=0,P_1^3=0,\ldots,P_{(j_3-l)}^3=0$ by a embedding

$$\begin{aligned}&f_3'((x_1,y_1),\ldots,(x_{c_{12}+p_2}),y_{(c_{12}+p_2)}))\\&\quad=((x_1,y_1),\ldots,(x_{c_{12}+p_2},y_{c_{12}+p_2}),\phi_1^3(x_{c_{12}+p_2},y_{c_{12}+p_2}),\ldots\phi_{j_3-l}^3(x_{c_{12}+p_2},y_{c_{12}+p_2})),\end{aligned}$$

where $\phi_1^2,\ldots,\phi_{j_3-l}^2$ are automorphisms of form (7). Let $f_3=f_3'(f_2\times Id^{p_1})$, then f_3 gives a embedding from $(\mathbf{CP}^1)^{l+p_1+p_2}$ to $(\mathbf{CP}^1)^{j_1+j_2+j_3-l_1-l_2}$ and $f_3(U_3)$ is a solution set of the equations $P_1^1=0,\ldots,P_{j_1-l}^1=0,P_1^2=0,\ldots,P_{(j_2-l)}^2=0,P_1^3=0,\ldots,P_{(j_3-l)}^3=0$.

Since there are only finite hyperplanes in the arrangements, repeating such procedure, we can solve all the equations in finite steps. Let t be the number of all free variables in the sense of above argument. By repeating the above argument, it is clear that after removing $c(\mathcal{A}_t^*)$ hypersurfaces from $(\mathbf{CP}^1)^t$, we have a subset $W \subseteq (\mathbf{CP}^1)^t$ on which there is an embedding so that the image of W under the embedding is a solution set of all equations.

Note that so far we have only considered the solutions of the equations, but lattice condition requires the restriction of the solution on the open set $V_1 = \{((x,y))^n \in (\mathbf{CP}^1)^n : Q_i(((x,y))^n) \neq 0, i = 1,2,\dots,\binom{n}{l+1} - d\}$ and $V_2 = \{((x,y))^n \in (\mathbf{CP}^1)^n : D_l(F_{r_0} \cdots F_{r_l}) \neq 0, \forall (r_0,\dots,r_l) \in \mathcal{S}\}$. Clearly, under the embedding, the pullbacks of complements of V_1 and V_2 are the union of finite hypersurfaces in $(\mathbf{CP}^1)^t$, say, C. Let $U = W \setminus C$, then U is still the complement of the union of finite hypersurfaces in $(\mathbf{CP}^1)^t$. Therefore U is path-connected. Let S be the embedding of U in $(\mathbf{CP}^1)^n$. Then S is homeomorphic to U, so is also path-connected.

It is obvious that $(0,1)^n$ and $(1,0)^n$ both satisfy the equations and inequalities. By Corollary 3.4 and Lemma 3.5, in fact they are both contained in S. Therefore there exists a continuous curve connecting $(0,1)^n$ and $(1,0)^n$ which defines a one-parameter family of arrangements $\mathcal{A}_t^*$ such that $\mathcal{A}_0^*$ corresponds to $(0,1)^n$ and $\mathcal{A}_1^*$ corresponds to $(0,1)^n$. Applying Theorem 3.7, we complete the proof.

Corollary 4.1. *The homotopy groups of the complement $M(\mathcal{A}^*)$ in $\mathbf{CP}^l$ depend only on the lattice $L(\mathcal{A}^*)$.*

Proof. By Main Theorem, the diffeomophic structure of $M(\mathcal{A}^*)$ is determined by $L(\mathcal{A}^*)$, so the homotopy groups of $M(\mathcal{A}^*)$ depend only on $L(\mathcal{A}^*)$

Theorem 4.2. *Let $\mathcal{A}^*$ be a nice point arrangement of hyperplanes in $\mathbf{CP}^n$. Then the moduli space of $\mathcal{A}^*$ with $L(\mathcal{A}^*)$ fixed is connected.*

Proof. In the course of proving Main theorem, for any two nice arrangements $\mathcal{A}_0^*$ and $\mathcal{A}_1^*$ with fixed $L(\mathcal{A}^*)$, there is a one-parameter family $L(\mathcal{A}_t^*)$ with the same $L(\mathcal{A}^*)$ connecting $\mathcal{A}_0^*$ and $\mathcal{A}_1^*$. Thus the moduli space of $\mathcal{A}^*$ with $L(\mathcal{A}^*)$ fixed is connected.

Remark 4.3. As one can see from the proof, the definition of the nice point arrangement allows us to solve all of the equations which reflect the combinatoric information of the family of the arrangements. In fact our method works for a much larger class of arrangements. Example 2.12 is an example of such an arrangement. Since at every point, there are at most l shared hyperplanes, and the number l is less than the number of variables in each equation, one can solve all variables of the equations without conflict. In other word, if the combinatorics of the arrangement is "proper", then the combinatorics will determine the diffeomorphic type of the complement. Here the word "proper" means that the combinatorics will allow us to solve all variables of the equations by chasing the points in the soul.

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