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The second pluri-genus of smoothable Gorenstein surface singularities

Dedicated to Professor Yang Lo on the Occasion of his 70th Birthday

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Abstract For complete intersection isolated surface singularities, Okuma gave a formula for the second pluri-genus relating geometry genus, Milnor number and Tjurina number. We generalize Okuma's theorem to Gorenstein smoothable surface singularities.

Keywords second pluri-genus, smoothable, Gorenstein, Milnor number, Tjurina number

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1 Introduction

Let (X,x) be a normal surface singularity over $\mathbb{C}$ and $f\colon (M,E)\to(X,x)$ the minimal good resolution of the singularity (X,x), i.e., the smallest resolution for which an exceptional divisor E consist of non-singular curves intersecting transversally, with no three through one point. Watanabe [12] introduced plurigenus $\{\delta_m(X,x)\}_{m\in\mathbb{N}}$ (for n ($\geqslant 2$)-dimensional normal isolated singularities). It is well-known that, for a normal surface singularity (X,x), $\delta_m(X,x)=0$ for any $m\in\mathbb{N}$ if and only if (X,x) is a log-terminal singularity, and $\delta_m(X,x)\leqslant 1$ for any $m\in\mathbb{N}$ if and only if (X,x) is a log-canonical singularity [4].

In a series of papers [5–7], Okuma studied the pluri-genus of surface singularities. In particular, for second pluri-genus of Gorenstein surface singularities, he proved that this invariant is determined by the geometry genus and the dual graph of the resolution in [5]. And for complete intersection singularities, he considered relations among the invariants the second pluri-genus $\delta_2(X,x)$, geometry genus $p_g(X,x)$, Milnor number $\mu(X,x)$ and Tjurina number $\tau(X,x)$ [5].

The main purpose of this paper is to generalize Okuma's theorem to Gorenstein smoothable surface singularities.

Main Theorem. *If (X,x) is a Gorenstein smoothable surface singularity with $p_g\geqslant 1$, then*

$$\delta_2(X,x)=h^1(\mathcal{S})+\mu(X,x)-\tau(X,x)-p_g(X,x)+1,$$

where $\mathcal{S}=(\Omega^1_M(\log E))^*$, $\mu(X,x)$ and $\tau(X,x)$ are the generalized Milnor number and Tjurina number of (X,x), respectively.

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2 Preliminary

Let (X,x) be a normal surface singularity over $\mathbb{C}$ and $f\colon (M,E)\to(X,x)$ the minimal good resolution of the singularity (X,x). It is well-known that there is a unique minimal good resolution. Let $E=\bigcup_{i=1}^k E_i$ be the decomposition of the exceptional set E into irreducible components. Let g_i be the genus of E_i, $g=\sum g_i$ and denote by $\tilde{E}$ the disjoint union of E_i. Let b be the number of loops in the dual graph of E.

First let us recall the definition of smoothable singularities, generalized Milnor number and Tjurina number.

Definition 2.1. *If there exists a flat morphism $\pi:\mathcal{V}\to T$ of local analytic spaces such that $\pi^{-1}(t_0)\simeq X$ and $\pi^{-1}(t)\simeq X_t$ is nonsingular for $t\neq t_0$, then the singularity (X,x) is called smoothable.*

More generally, let (X,x) be a local analytic variety with isolated singularity of pure dimension n. To any smoothing $\pi:\mathcal{V}\to T$ of X, one can attach a Milnor fibre $F:=B_\epsilon\cap\pi^{-1}(t)$, where B_ϵ is a ball in some $\mathbb{C}^N$ containing $\mathcal{V}$ and $t\in T$ [9]. F is a $2n$-manifold with boundary, with the homotopy type of a finite complex of dimension n. We define Milnor number $\mu=rkH_n(F)$.

As is well-known, any complex analytic germ (X,x) with isolated singularity admits a semi-universal deformation

$$(X,x)\hookrightarrow(\mathscr{Y},y)\xrightarrow{F}(S,s)$$

(cf. [2] and [1]). Given a semi-universal deformation of (X,x), we call an irreducible component (S',s) of (S,s) a smooth component if the general fibre of F over this component is smooth. In general however, the germ (S,s) is not smooth and indeed, it may have irreducible components of various dimensions. Any smooth component has dimension between $\dim_{\mathbb{C}}T^1_X-\dim_{\mathbb{C}}T^2_X$ and $\dim_{\mathbb{C}}T^1_X$, where T^1_X, T^2_X are respectively the first order deformation and obstruction spaces.

For a smoothing, $\pi:\mathcal{V}\to T$ of (X,x) and a smooth component (S',s) on which $\pi:\mathcal{V}\to T$ lies, Wahl had a conjecture in [11]:

$$\dim_{\mathbb{C}}(S',s)=\dim_{\mathbb{C}}\mathrm{coker}(\Theta_{\mathcal{V}/T}\otimes\mathcal{O}_X\to\Theta_X).$$

Wahl himself verified his conjecture for special cases. Later, Greuel and Looijenga [3] proved this conjecture completely.

We define $\tau(X,x)=\dim_{\mathbb{C}}T^1_X$. If (X,x) has no obstructed deformations (e.g. (X,x) is a complete intersection), then (S,s) is nonsingular and

$$\dim_{\mathbb{C}}(S,s)=\dim_{\mathbb{C}}T^1_X.$$

So $\tau(X,x)$ generalizes the usual Tjurina number.

Definition 2.2. *The geometric genus p_g and the irregularity q of the singularity are defined as follows (cf. [8, 13]) :*

$$p_g:=\dim\Gamma(M-E,\Omega^2_M)/\Gamma(M,\Omega^2_M), \tag{2.1}$$

$$q:=\dim\Gamma(M-E,\Omega^1_M)/\Gamma(M,\Omega^1_M). \tag{2.2}$$

We denote by K the canonical divisor on M.

Definition 2.3 [12]**.** *We define the pluri-genera $\{\delta_m(X,x)\}_{m\in\mathbb{N}}$ as follows:*

$$\delta_m(X,x)=\dim_{\mathbb{C}}\frac{H^0(\mathcal{O}_{M-E}(mK))}{H^0(\mathcal{O}_M(mK+(m-1)E))}.$$

Note that $\delta_1(X,x)=p_g(X,x)$.

Let $\Omega^1_M(\log E)$ be the sheaf of 1-form with logarithmic poles along E, and $\mathcal{S}$ its dual. Then there is an exact sequence

$$0\longrightarrow\Omega^1_M(\log E)(-E)\longrightarrow\Omega^1_M\longrightarrow\Omega^1_{\tilde{E}}\longrightarrow 0. \tag{2.3}$$

It follows that $\wedge^2\Omega^1_M(\log E)=\Omega^2_M(E)$, and there is an exact sequence

$$0\longrightarrow\Omega^1_M\longrightarrow\Omega^1_M(\log E)\longrightarrow\mathscr{O}^1_{\tilde{E}}\longrightarrow 0. \tag{2.4}$$

Theorem 2.4 [5]. *If (X,x) is a Gorenstein singularity with $p_g(X,x) \geqslant 1$ then*

$$\delta_2(X,x) = p_g(X,x) - \frac{1}{2}(2K+E)(K+E).$$

Theorem 2.5 [5]. *If (X,x) is a complete intersection singularity with $p_g(X,x) \geqslant 1$, then*

$$\delta_2(X,x) = h^1(\mathcal{S}) + \mu(X,x) - \tau(X,x) - p_g(X,x) + 1,$$

where $\mu(X,x)$ is the Milnor number and $\tau(X,x)$ is the Tjurina number.

We want to generalize Theorem 2.5 to Gorenstein smoothable surface singularities. Before doing that, we need to recall some notations.

The following Lemma 2.6 can be found in [8].

Lemma 2.6. (1) *The composition $H^0(\mathcal{O}_{\tilde{E}}) \to H^1(\Omega^1_M) \to H^1(\Omega^1_{\tilde{E}})$ is an isomorphism.*

(2) $H^0(\Omega^1_M) \tilde{\to} H^0(\Omega^1_M(\log E))$.

Recall that k is the number of the components of E.

Since $H^0_E(\mathcal{O}_{\tilde{E}}) \tilde{\to} H^0(\mathcal{O}_{\tilde{E}})$, the map $H^0(\mathcal{O}_{\tilde{E}}) \to H^1(\Omega^1_M)$ factors via $H^1_E(\Omega^1_M)$. Therefore, by Lemma 2.6, Steenbrink [8] defined a nonnegative integer

$$\gamma := rk(H^1_E(\Omega^1_M) \to H^1(\Omega^1_M)) - k. \tag{2.5}$$

Besides γ, Steenbrink introduced two other invariants

$$\alpha := \dim H^0(\Omega^2_M)/dH^0(\Omega^1_M(\log E)(-E)), \tag{2.6}$$

$$\beta := \dim H^0(\Omega^1_{\tilde{E}})/\mathrm{Im} H^0(\Omega^1_M). \tag{2.7}$$

Moreover, the authors found a relation between these invariants and the irregularity q in [8].

Theorem 2.7 [8]. *With the notations defined as above,*

$$q = p_g - g - b - \alpha - \beta - \gamma. \tag{2.8}$$

Theorem 2.8 [10]. *Let (X,x) be an isolated Gorenstein surface singularity. For a given smoothing, let $\mu(X,x)$ and $\tau(X,x)$ be the generalized Milnor number and Tjurina number of (X,x) respectively. Then*

$$\mu(X,x) - \tau(X,x) = 2\alpha + 2\beta + \gamma + b. \tag{2.9}$$

3 Main result

We use the same notation as in the previous sections. Let (X,x) be a Gorenstein singularity with contractible X. Let Z be a cycle such that $\mathcal{O}_M(K) \simeq \mathcal{O}_M(-Z)$.

Lemma 3.1.

$$h^1(\Omega^1_M(\log E)) = p_g - \alpha - \beta - b, \tag{3.1}$$

Proof. We have the spectral sequence

$$E_1^{p,q} = H^q(M, \Omega^p_M(\log E)) \Rightarrow \mathbf{H}^{p+q}(M, \Omega^\bullet_M(\log E)) \cong H^{p+q}(M-E,\mathbb{C}).$$

The spectral sequence induces an exact sequence of small order terms

$$\begin{aligned} 0 &\to H^1(M, \Omega^\bullet_M(\log E)) \to H^1(M-E,\mathbb{C}) \to E_2^{0,1} \\ &\to H^2(M, \Omega^\bullet_M(\log E)) \to H^2(M-E,\mathbb{C}) \to E_2^{1,1} \to 0, \end{aligned}$$

where

$$E_2^{0,1} = \ker(H^1(M,\mathcal{O}_M) \to H^1(M, \Omega^1_M(\log E))),$$

$$E_2^{1,1} = \text{coker}(H^1(M, \mathcal{O}_M) \to H^1(M, \Omega_M^1(\log E))).$$

So

$$\begin{aligned} h^1(M,\Omega_M^\bullet(\log E)) - h^1(M-E) + \dim E_2^{0,1} - h^2(M, \Omega_M^\bullet(\log E)) \\ + h^2(M-E) - \dim E_2^{1,1} = 0. \end{aligned}$$

Since

$$\dim E_2^{0,1} - \dim E_2^{1,1} = h^1(M, \mathcal{O}_M) - h^1(M, \Omega_M^1(\log E)),$$

we have

$$\begin{aligned} h^1(M,\Omega_M^\bullet(\log E)) - h^1(M-E) - h^2(M, \Omega_M^\bullet(\log E)) + h^2(M-E) \\ + h^1(M, \mathcal{O}_M) - h^1(M, \Omega_M^1(\log E)) = 0. \end{aligned}$$

Since $h^1(M-E) = h^2(M-E)$, we get

$$h^1(M, \Omega_M^\bullet(\log E)) - h^2(M, \Omega_M^\bullet(\log E)) + h^1(M, \mathcal{O}_M) - h^1(M, \Omega_M^1(\log E)) = 0.$$

From (2.4), we have

$$\begin{aligned} 0 \to & H^0(M, \Omega_M^1) \to H^0(M, \Omega_M^1(\log E)) \to H^0(M, \mathcal{O}_{\tilde{E}}) \\ \to & H^1(M, \Omega_M^1) \to H^1(M, \Omega_M^1(\log E)) \to H^1(M, \mathcal{O}_{\tilde{E}}) \to 0. \end{aligned}$$

Since $H^0(M, \Omega_M^1) \simeq H^0(M, \Omega_M^1(\log E))$ [8], we have

$$h^0(M, \mathcal{O}_{\tilde{E}}) - h^1(M, \Omega_M^1) + h^1(M, \Omega_M^1(\log E)) - h^1(M, \mathcal{O}_{\tilde{E}}) = 0.$$

From [10, (1.9.2)], we know

$$h^1(M, \Omega_M^1) = p_g - g - b - \alpha - \beta + k, \tag{3.2}$$

so

$$h^1(M, \Omega_M^1(\log E)) = h^1(M, \mathcal{O}_{\tilde{E}}) + h^1(M, \Omega_M^1) - h^0(M, \mathcal{O}_{\tilde{E}}) = p_g - \alpha - \beta - b. \qquad \square$$

Before we begin to prove Main Theorem, we need the following lemma.

Lemma 3.2 [10]**.** *On the minimal good resolution of a Gorenstein singularity (X, x), let $\mathcal{O}_M(K) \simeq \mathcal{O}_M(-Z)$ and $Z = \sum r_i E_i$. Excluding the rational double point case, all $r_i \geqslant 1$, and $r_i = 1$ implies either*

a) *E_i is rational with at most 2 intersection points with other curves, or*

b) *$Z = E$ is one non-singular elliptic curve, and (X, x) is simply elliptic.*

Theorem 3.3. *If (X, x) is a Gorenstein smoothable surface singularity with $p_g \geqslant 1$, then*

$$\delta_2(X, x) = h^1(\mathcal{S}) + \mu(X, x) - \tau(X, x) - p_g(X, x) + 1,$$

where $\mathcal{S} = (\Omega_M^1(\log E))^*$, $\mu(X, x)$ and $\tau(X, x)$ are the generalized Milnor number and Tjurina number of (X, x), respectively.

Proof. Let $\mathcal{O}_M(K) \simeq \mathcal{O}_M(-Z)$, then $\mathcal{S} = (\Omega_M^1(\log E))^* \simeq \Omega_M^1(\log E)(Z - E)$. From Lemma 3.2 and the fact that simply elliptic singularities are hypersurface singularities, we can suppose $Z > E$.

We start from the exact sequence

$$0 \to \Omega_M^1(\log E) \to \Omega_M^1(\log E)(Z - E) \to \Omega_M^1(\log E) \otimes \mathcal{O}_{Z-E}(Z - E) \to 0.$$

Since $H^0(M, \Omega_M^1(\log E)(Z - E)) = H^0(M - E, \Omega_M^1)$ [10, (3.5.3)] and $H^0(M, \Omega_M^1(\log E)) = H^0(M, \Omega_M^1)$ [8], we can get

$$q - \chi(\Omega_M^1(\log E) \otimes \mathcal{O}_{Z-E}(Z - E)) + h^1(\Omega_M^1(\log E)) - h^1(\mathcal{S}) = 0. \tag{3.3}$$

Using Riemann-Roch theorem on $Z - E$, we get

$$\chi(\Omega_M^1(\log E) \otimes \mathcal{O}_{Z-E}(Z - E))$$

$$
\begin{aligned}
&= (Z-E)c_1(\Omega^1_M(\log E)\otimes \mathcal{O}_{Z-E}(Z-E)) - (Z-E)(Z-E+K) \\
&= (Z-E)(-Z+E+2(Z-E)) - (Z-E)(Z-E-Z) \\
&= Z(Z-E). \qquad (3.4)
\end{aligned}
$$

So by (3.3), (3.4), Theorem 2.8, Lemma 3.1, Theorem 2.7 and Theorem 2.4, we have

$$
\begin{aligned}
&h^1(\mathcal{S}) + \mu(X,x) - \tau(X,x) - p_g(X,x) + 1 \\
&\quad = h^1(\Omega^1_M(\log E)) + q - Z(Z-E) + 2\alpha + 2\beta + \gamma + b - p_g(X,x) + 1 \\
&\quad = p_g(X,x) - b - g + 1 - Z(Z-E) \\
&\quad = p_g(X,x) + \chi(\mathcal{O}_E) - Z(Z-E) \\
&\quad = p_g(X,x) - \frac{E(-Z+E)}{2} - Z(Z-E) \\
&\quad = \delta_2(X,x).
\end{aligned}
$$

□

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