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Explicit construction of moduli space of bounded complete Reinhardt domains in $\mathbb{C}^n$

Rong Du, Yun Gao and Stephen S. T. Yau

One of the most fundamental problems in complex geometry is to determine when two bounded domains in $\mathbb{C}^n$ are biholomorphically equivalent. Even for complete Reinhardt domains, this fundamental problem remains unsolved completely for many years. Using the Bergmann function theory, we construct an infinite family of numerical invariants from the Bergman functions for complete Reinhardt domains in $\mathbb{C}^n$. These infinite family of numerical invariants are actually a complete set of invariants if the domains are pseudoconvex with C^1 boundaries. For bounded complete Reinhardt domains with real analytic boundaries, the complete set of numerical invariants can be reduced dramatically although the set is still infinite. As a consequence, we have constructed the natural moduli spaces for these domains for the first time.

1. Introduction

One of the basic problems in complex geometry is to find a reasonable object which parametrizes all non-isomorphic complex manifolds. This is the well-known moduli problem. Let D_1 and D_2 be two domains in $\mathbb{C}^n$. One of the most fundamental problems in complex geometry is to find necessary and sufficient conditions which will imply that D_1 and D_2 are biholomorphically equivalent. For $n = 1$, the celebrated Riemann mapping theorem states that any simply connected domains in $\mathbb{C}$ are biholomorphically equivalent. For $n \geqslant 2$, there are many domains which are topologically equivalent to the ball but not biholomorphically equivalent to the ball (see [14]). Poincaré studied the invariance properties of the CR manifolds, which are real hypersurfaces in $\mathbb{C}^n$, with respect to biholomorphic transformations. The systematic study of such properties for real hypersurface was made by Cartan [2] and later by Chern and Moser [3]. A main result of the theory is the existence of a complete system of local differential invariants for CR-structures on real hypersurface. In 1974, Fefferman [5] proved that a biholomorphic mapping between two strongly pseudoconvex domains is smooth up to the boundaries and the induced boundary mapping is a CR-equivalence on the boundary. Thus, one can use Chern–Moser invariants to study the biholomorphically equivalent problem of two strongly pseudoconvex domains. Using the Chern–Moser theory, Webster [18] gave a complete characterization when two ellipsoids in $\mathbb{C}^n$ are biholomorphically equivalent. In 1978, Burns *et al.* [1] showed that the "number of moduli" of a "moduli space" of a strongly pseudoconvex bounded domain has to be infinite. Thus the moduli problem of open manifolds is really a very difficult one. Lempert [10] made significant progress in the subject. He was able to construct the moduli space of bounded strictly convex domains of $\mathbb{C}^n$ with marking at the origin. Although the theory established by Lempert is beautiful, the computation of his invariants is a hard problem. For more details on the global and local equivalence problem for real sub-manifolds in $\mathbb{C}^n$, we refer the readers to the survey paper by Huang [8].

Recall that an open subset $D \subseteq \mathbb{C}^n$ is called a complete Reinhardt domain if, whenever $(z_1, \ldots, z_n) \in D$, then $(\zeta_1 z_1, \ldots, \zeta_n z_n) \in D$ for all complex numbers ζ_j with $|\zeta_j| \leqslant 1$. There is a beautiful theorem of Sunada [17] which relates two such domains by a permutation map. For constructing the biholomorphic moduli space of bounded complete pseudoconvex Reinhardt domains with C^1 boundary in $\mathbb{C}^2$, by using a result of Sunada one may identify this moduli space with the quotient space of curves in $\mathbb{R}^2_+$ given in polar coordinates by $r = f(\theta)$ where $f : [0, \frac{\pi}{2}] \to \mathbb{R}_+$ is C^1 satisfying

$$f(0) = 1 = f\left(\frac{\pi}{2}\right), \quad f'(0) = 0 = f'\left(\frac{\pi}{2}\right)$$
$$\frac{f'(\theta) + \cot(\theta) \cdot f(\theta)}{f'(\theta) - \tan(\theta) \cdot f(\theta)} \quad \text{increasing,}$$

modulo identification of each curve with the companion curve obtained by switching the roles of the x_1- and x_2-axis; equivalently, the desired moduli space can be identified with the space of functions f satisfying the above conditions modulo identification of $f(\theta)$ with $f(\frac{\pi}{2} - \theta)$. Apparently, this result cannot be regarded as a solution from algebraic geometry point of view because of lack of coordinates on the quotient space of functions. In the spirit of our methods, one can try to remedy this by providing a complete set of invariants:

$$\alpha(\theta) = f(\theta) + f\left(\frac{\pi}{2}\right),$$
$$\beta(\theta) = f(\theta) \cdot f\left(\frac{\pi}{2}\right),$$

$$\gamma(\theta,\psi) = \left(f(\theta) - f\left(\frac{\pi}{2} - \theta\right)\right) \cdot \left(f(\psi) - f\left(\frac{\pi}{2} - \psi\right)\right).$$

To get a countable complete set of invariants, one can restrict to a countable dense subset of θ's and ψ's. However, there are three major drawbacks for this approach. The first one is that the embedding constructed in this way is not canonical. The other one is that it lacks the uniform solution to all dimensions. Thirdly, it is difficult if not impossible to identify the image of the embedding. The results of this paper will solve these problems.

Recently Du and Yau [4] studied the moduli problem of complete Reinhardt domains in $\mathbb{C}^2$ from a different point of view. The main tool to solve this moduli problem with geometry information is the new biholomorphic invariant Bergman function defined by Yau [20] on pseudoconvex domains in a variety with only isolated singularity. Du and Yau discovered that the moduli problem of complete Reinhardt domains in $\mathbb{C}^2$ is equivalent to the moduli problem of complete Reinhardt domains in A_n-variety $\{(x,y,z)\colon xy = z^{n+1}\}$, which is the quotient of $\mathbb{C}^2$ by a cyclic group of order $n+1$. The great advantage of working on the moduli problem of domains in A_n-variety is that the biholomorphic maps between these domains are dramatically smaller. This is because these biholomorphic maps not only have to send the boundary of one domain to the boundary of another domain, but also have to leave the A_n-variety invariant. In addition, these biholomorphic maps need to preserve the Bergman functions which are positive functions. Du and Yau used these facts to show that all the biholomorphic maps between complete Reinhardt domains in A_n-variety must be of the special form: permutation of coordinates modulo scalar multiplication. This result is much stronger than the corresponding result obtained by Sunada [17]. In principle, the method introduced by Du and Yau [4] could be used to study the biholomorphic equivalence problem or moduli problem for more general pseudoconvex domains in A_n-variety.

The purpose of this paper is to show that Yau's Bergman function theory can also solve the biholomorphic equivalence problem or moduli problem for complete Reinhardt pseudoconvex domains in $\mathbb{C}^n$ for all $n \geqslant 2$. In order to describe the complete biholomorphic invariants of bounded complete Reinhardt domains in $\mathbb{C}^n$, we introduce the following notations. Let S_n be the symmetric group of degree n. Recall that the group ring $\mathbb{R}[S_n]$ is a ring of the form $\mathbb{R}[\tau_1, \tau_2, \ldots, \tau_{n!}]$ with $\tau_i \in S_n$ for $1 \leqslant i \leqslant n!$. Let $\sum_i x_i\tau_i$ and $\sum_j y_j\tau_j$, where x_i, y_j are in $\mathbb{R}$, be two elements in $\mathbb{R}[S_n]$. Then $(\sum_i x_i\tau_i)$ $(\sum_j y_j\tau_j) := \sum_{i,j} x_iy_j(\tau_i \cdot \tau_j)$, where $\tau_i \cdot \tau_j$ is the product in the group S_n. We shall consider $\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n]$ the product of the group ring with itself. Such a product has a natural S_n-module structure in the following manner. Let $\sigma \in S_n$ and $(\sum_i x_i\tau_i, \ldots, \sum y_i\tau_i) \in \mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n]$. Then

$$\sigma\left(\sum_i x_i\tau_i, \ldots, \sum y_i\tau_i\right) := \left(\sum_i x_i(\tau_i\sigma), \ldots, \sum y_i(\tau_i\sigma)\right). \tag{1.1}$$

Definition 1.1. Two elements f, g in $\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n]$ are said to be equivalent and denoted by $f \sim g$ if there exists a $\sigma \in S_n$ such that $\sigma(f) = g$.

Before we can describe our main results, we need the following notations. Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$ be an n-tuple of non-negative integers. Denote $\phi_{\vec{\alpha}} = (\prod_{i=1}^n z_i^{\alpha_i})\, dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$. For a domain D in $\mathbb{C}^n$, we shall use notation $\|\phi_{\vec{\alpha}}\|_D^2 = \int_D \phi_{\vec{\alpha}} \wedge \phi_{\vec{\alpha}}$. In this paper, we show that all the biholomorphic invariants of a bounded complete Reinhardt domain are contained in $(\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n])/\sim$ where there are $n!$ copies of $\mathbb{R}[S_n]$ and $\sim$ is the equivalent relation defined in Definition 1.1.

Theorem A. *Let D be a bounded complete Reinhardt domain in $\mathbb{C}^n$. Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$ be a n-tuple of non-negative integers. For any $\tau \in S_n$, denote*

$$g_D^\tau(\vec{\alpha}) = \frac{\|\phi_{\vec{0}}\|_D^{\Sigma\alpha_i - 1}\|\phi_{\tau(\vec{\alpha})}\|_D}{\prod_{i=1}^n \|\phi_{\vec{e}_i}\|_D^{\alpha_{\tau(i)}}} \tag{1.2}$$

where $\tau(\vec{\alpha}) = (\alpha_{\tau(1)}, \ldots, \alpha_{\tau(n)})$ and $\vec{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the ith component. Then for all n-tuples of non-negative integers $\vec{\beta}_1, \ldots, \vec{\beta}_{n!}$, $\xi_D^{\vec{\beta}_1,\ldots,\vec{\beta}_{n!}} = (\sum_{\tau\in S_n} g_D^\tau(\vec{\beta}_1)\tau, \ldots, \sum_{\tau\in S_n} g_D^\tau(\vec{\beta}_{n!})\tau)$ as an element in $(\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n])/\sim$ is a biholomorphic invariant. In fact, if D_1 and D_2 are two such domains which are biholomorphically equivalent, then there exists a $\sigma \in S_n$ such that $g_{D_1}^\tau(\vec{\alpha}) = g_{D_2}^{\tau\cdot\sigma}(\vec{\alpha})$ $\forall\tau \in S_n$ and $\forall\vec{\alpha}$ n-tuple of non-negative integers.

The invariants in Theorem A are complete invariants for bounded complete Reinhardt pseudoconvex domains with C^1 boundaries.

Theorem B. *Let D_i, $i = 1, 2$, be two bounded complete Reinhardt pseudoconvex domains in $\mathbb{C}^n$ with C^1 boundaries. If for all $\vec{\alpha}, \ldots, \vec{\alpha}_{n!}$ n-tuples of non-negative integers, $\xi_{D_1}^{\vec{\alpha},\ldots,\vec{\alpha}_{n!}} = \xi_{D_2}^{\vec{\alpha}_1,\ldots,\vec{\alpha}_{n!}}$ in $(\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n])/\sim$, where $\xi_D^{\vec{\alpha}_1,\ldots,\vec{\alpha}_{n!}} = (\sum_{\tau\in S_n} g_D^\tau(\vec{\alpha}_1)\tau, \ldots, \sum_{\tau\in S_n} g_{D_i}^\tau(\vec{\alpha}_{n!})\tau)$, then there exists*

$\sigma \in S_n$ and a biholomorphic map

$$\Psi_\sigma(z_1, \dots, z_n) = \left(a_1 z_{\sigma(1)}, \dots, a_n z_{\sigma(n)}\right),$$

where $a_i = \frac{\|\phi_{\vec{0}}\|_{D_1}\|\phi_{\vec{e}_i}\|_{D_2}}{\|\phi_{\vec{e}_{\sigma(i)}}\|_{D_1}\|\phi_{\vec{0}}\|_{D_2}}$, such that Ψ_σ sends D_1 onto D_2.

Theorem A and Theorem B above give a complete characterization of two bounded complete Reinhardt domains in $\mathbb{C}^n$ to be biholomorphically equivalent in terms of the quotient of group ring $(\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n])/\sim$. In case $n = 2$, we can actually write down the complete numerical invariants for two bounded complete Reinhardt in $\mathbb{C}^2$ to be biholomorphically equivalent.

Theorem C. *Let D_1, D_2 be two bounded complete Reinhardt pseudoconvex domains in $\mathbb{C}^2$ with C^1 boundaries. Then D_1 is biholomorphic to D_2 if and only if*

$$g_{D_1}(\alpha_1, \alpha_2) + g_{D_1}(\alpha_2, \alpha_1) = g_{D_2}(\alpha_1, \alpha_2) + g_{D_2}(\alpha_2, \alpha_1), \tag{1.3}$$

$$g_{D_1}(\alpha_1, \alpha_2) g_{D_1}(\alpha_2, \alpha_1) = g_{D_2}(\alpha_1, \alpha_2) g_{D_2}(\alpha_2, \alpha_1), \tag{1.4}$$

$$\begin{aligned} &(g_{D_1}(\alpha_1, \alpha_2) - g_{D_1}(\alpha_2, \alpha_1))\,(g_{D_1}(\beta_1, \beta_2) - g_{D_1}(\beta_2, \beta_1)) \\ &\quad = (g_{D_2}(\alpha_1, \alpha_2) - g_{D_2}(\alpha_2, \alpha_1))\,(g_{D_2}(\beta_1, \beta_2) - g_{D_2}(\beta_2, \beta_1)) \end{aligned} \tag{1.5}$$

for all non-negative integers α_i, β_i, where

$$g_{D_i}(\alpha_1, \alpha_2) = \frac{\|\phi_{\vec{0}}\|_{D_i}^{\alpha_1+\alpha_2-1}\|\phi_{(\alpha_1,\alpha_2)}\|_{D_i}}{\prod_{j=1}^{2}\|\phi_{\vec{e}_j}\|_{D_i}^{\alpha_j}}.$$

Corollary D. *The moduli space of bounded complete Reinhardt pseudoconvex domains with C^1 boundaries in $\mathbb{C}^2$ can be constructed explicitly as the image of the complete family of numerical invariants:*

$$\begin{gathered} g_D(\alpha_1, \alpha_2) + g_D(\alpha_2, \alpha_1), \\ g_D(\alpha_1, \alpha_2) g_D(\alpha_2, \alpha_1), \end{gathered}$$

and

$$(g_D(\alpha_1, \alpha_2) - g_D(\alpha_2, \alpha_1)) \cdot (g_D(\beta_1, \beta_2) - g_D(\beta_2, \beta_1))$$

$\forall\ \alpha_i, \beta_i$ non-negative integers.

In order to find the complete numerical biholomorphic invariants of bounded complete Reinhardt domain in $\mathbb{C}^n$ for $n \geqslant 3$, we need to consider the finite symmetric group $S_n = \{\sigma_1, \sigma_2, \ldots, \sigma_{n!}\}$ of degree n acting on the affine space $\mathbb{C}^{n!n!} = \mathbb{C}^{n!} \times \cdots \times \mathbb{C}^{n!}$, which is the product of $n!$ copies of $\mathbb{C}^{n!}$, in the following manner. Let $\tau \in S_n$ and $(x_{\sigma_1}, \ldots, x_{\sigma_{n!}}; \ldots; y_{\sigma_1}, \ldots, y_{\sigma_{n!}}) \in \mathbb{C}^{n!} \times \cdots \times \mathbb{C}^{n!} = \mathbb{C}^{n!n!}$. Then $\tau(x_{\sigma_1}, \ldots, x_{\sigma_{n!}}; \ldots; y_{\sigma_1}, \ldots, y_{\sigma_{n!}}) = (x_{\sigma_1\tau}, \ldots, x_{\sigma_{n!}\tau}; \ldots; y_{\sigma_1\tau}, \ldots, y_{\sigma_{n!}\tau})$. Since S_n is linearly reductive, by Hilbert theorem, the ring of invariants $\mathbb{C}[x_{\sigma_1}, \ldots, x_{\sigma_{n!}}; \ldots; y_{\sigma_1}, \ldots, y_{\sigma_{n!}}]^{S_n}$ is finitely generated. Moreover, the generators can be listed explicitly by Göbel's theorem [6]. Before we give the statement of Göbel's theorem, we will introduce some definitions first.

Definition 1.2. Suppose that a finite group G acts as permutations on a finite set X. We then refer to X together with the G-action as a finite G-set. A subset $B \subset X$ is called an orbit if G permutes the elements of B among themselves and the induced permutation action of G on B is transitive.

Definition 1.3. If $K = (k_1, \ldots, k_n)$ is an n-tuple of non-negative integers, then K is called an exponent sequence. The associated partition of K is the ordered set consisting of the n numbers $k_1, \ldots, k_n$ rearranged in weakly decreasing order. We denote by $\lambda(K)$ the partition associated to K, so

$$\lambda(K) = (\lambda_1(K) \geq \lambda_2(K) \geq \cdots \geq \lambda_n(K))$$

and the n-tuple $(\lambda_1(K), \lambda_2(K) \cdots \lambda_n(K))$ is a permutation of $k_1, \ldots, k_n$. The monomial x^K is called special if the associated partition $\lambda(K)$ of the exponent sequence K satisfies

(1) $\lambda_i(K) - \lambda_{i+1}(K) \leq 1$ for all $i = 1, \ldots, n-1$, and

(2) $\lambda_n(K) = 0$.

Notice that if two exponent sequences A and B are permutations of each other, then $\lambda(A) = \lambda(B)$.

Theorem 1.1 (M. Göbel)**.** *Let G be a finite group, X a finite G-set, and R a commutative ring. Then the ring of invariants $R[X]^G$ is generated as an algebra by $e_{|X|} = \prod_{x \in X} x$, the top degree elementary symmetric polynomial in the elements of X, and the orbit sums of special monomials.*

Theorem E. *Let $f_1, \ldots, f_N \in \mathbb{C}[x_{\sigma_1}, \ldots x_{\sigma_{n!}}; \ldots; y_{\sigma_1}, \ldots y_{\sigma_{n!}}]^{S_n}$ be the generators of the ring of invariant polynomials computed by Theorem 1.1. Let*

D be a bounded complete Reinhardt domain in $\mathbb{C}^n$. Then, for $\vec{\alpha}_1, \vec{\alpha}_2, \ldots, \vec{\alpha}_{n!}$ n-tuples of non-negative integers,

$$f_1(g_D^\sigma(\vec{\alpha}_1), \ldots, g_D^\sigma(\vec{\alpha}_{n!}))_{\sigma\in S_n}, \ldots, f_N(g_D^\sigma(\vec{\alpha}_1), \ldots, g_D^\sigma(\vec{\alpha}_{n!}))_{\sigma\in S_n}$$

are biholomorphic invariants, where

$$g_D^\sigma(\vec{\beta}) = \frac{\|\phi_{\vec{0}}\|_D^{\Sigma\beta_i - 1}\|\phi_{\sigma(\vec{\beta})}\|_D}{\prod_{i=1}^n \|\phi_{\vec{e}_i}\|_D^{\beta_{\sigma(i)}}}, \quad \vec{\beta} = (\beta_1, \beta_2, \ldots, \beta_n).$$

The following theorem says that the above invariants are actually complete in case the domain D is pseudoconvex.

Theorem F. *Let D_i, $i = 1, 2$, be two bounded complete Reinhardt pseudoconvex domains in $\mathbb{C}^n$ with C^1 boundaries. Let $f_1, \ldots, f_N \in \mathbb{C}[x_{\sigma_1}, \ldots, x_{\sigma_{n!}}; \ldots; y_{\sigma_i}, \ldots, y_{\sigma_{n!}}]^{S_n}$ be the generators of the ring of invariant polynomials computed by Theorem 1.1. If for all $\vec{\alpha}_1, \ldots, \vec{\alpha}_{n!}$ n-tuples of non-negative integers*

$$f_i(g_{D_1}^\sigma(\vec{\alpha}_1), \ldots, g_{D_1}^\sigma(\vec{\alpha}_{n!}))_{\sigma\in S_n} = f_i\left(g_{D_2}^\sigma(\vec{\alpha}_1), \ldots, g_{D_2}^\sigma(\vec{\alpha}_{n!})\right)_{\sigma\in S_n}, \quad i = 1, 2, \ldots, N,$$

then there exists $\tau \in S_n$ and a biholomorphic map $\Psi_\tau \colon \mathbb{C}^n \to \mathbb{C}^n$, $\Psi_\tau(z_1, \ldots, z_n) = (a_1 z_{\tau(1)}, \ldots, a_n z_{\tau(n)})$, where

$$a_i = \frac{\|\phi_{\vec{0}}\|_{D_1}\|\phi_{\vec{e}_i}\|_{D_2}}{\|\phi_{\vec{e}_{\tau(i)}}\|_{D_1}\|\phi_{\vec{0}}\|_{D_2}},$$

such that Ψ_τ sends D_1 onto D_2.

Corollary G. *The moduli space of bounded complete Reinhardt pseudoconvex domains with C^1 boundaries in $\mathbb{C}^n$ can be constructed explicitly as the image of the complete family of numerical invariants:*

$$f_i(g_D^\sigma(\vec{\alpha}_1), \ldots, g_D^\sigma(\vec{\alpha}_{n!}))_{\sigma\in S_n}, \quad 1 \leqslant i \leqslant N,$$

where $\vec{\alpha}_1, \ldots, \vec{\alpha}_{n!}$ are all possible n-tuples of non-negative integers.

Remark. One can compute explicitly the relation of the generators $f_1, \ldots, f_N \in \mathbb{C}[x_{\sigma_1}, \ldots x_{\sigma_{n!}}; \ldots; y_{\sigma_1}, \ldots y_{\sigma_{n!}}]^{S_n}$. These relations define an algebraic variety in $\mathbb{R}^\infty$ where the moduli space lies.

For complete Reinhardt pseudoconvex domains with real analytic boundaries, we can use fewer numerical invariants to characterize these domains. More precisely, we have the following theorems.

Theorem B′. *Let D_i, $i = 1, 2$, be two bounded complete Reinhardt pseudoconvex domains in $\mathbb{C}^n$ with real analytic boundaries. Then D_1 is biholomorphically equivalent to D_2 if and only if for all $\vec{\alpha}$ n-tuple of non-negative integers, $\xi^{\vec{\alpha}}_{D_1} = \xi^{\vec{\alpha}}_{D_2}$ in $\mathbb{R}[S_n]/\sim$, where $\xi^{\vec{\alpha}}_{D_i} = \sum_{\tau \in S_n} g^{\tau}_{D_i}(\vec{\alpha})\tau$. In this case, there exists $\sigma \in S_n$ and a biholomorphic map*

$$\Psi_\sigma(z_1, \ldots, z_n) = (a_1 z_{\sigma(1)}, \ldots, a_n z_{\sigma(n)}),$$

where $a_i = \dfrac{\|\phi_{\vec{0}}\|_{D_1}\|\phi_{\vec{e}}\|_{D_2}}{\|\phi_{\vec{e}_{\sigma(i)}}\|_{D_1}\|\phi_{\vec{0}}\|_{D_2}}$, *such that Ψ_σ sends D_1 onto D_2.*

Theorem C′. *Let D_1, D_2 be two bounded complete Reinhardt pseudoconvex domains in $\mathbb{C}^2$ with real analytic boundaries. Then D_1 is biholomorphic to D_2 if and only if*

$$g_{D_1}(\alpha_1, \alpha_2) + g_{D_1}(\alpha_2, \alpha_1) = g_{D_2}(\alpha_1, \alpha_2) + g_{D_2}(\alpha_2, \alpha_1),$$
$$g_{D_1}(\alpha_1, \alpha_2) g_{D_1}(\alpha_2, \alpha_1) = g_{D_2}(\alpha_1, \alpha_2) g_{D_2}(\alpha_2, \alpha_1)$$

for all non-negative integers α_1, α_2, where

$$g_{D_i}(\alpha_1, \alpha_2) = \frac{\|\phi_{\vec{0}}\|^{\alpha_1+\alpha_2-1}_{D_i}\|\phi_{(\alpha_1,\alpha_2)}\|_{D_i}}{\prod_{j=1}^{2}\|\phi_{\vec{e}_j}\|^{\alpha_j}_{D_i}}.$$

Theorem F′. *Let D_i, $i = 1, 2$, be two bounded complete Reinhardt pseudoconvex domains in $\mathbb{C}^n$ with real analytic boundaries. Let*

$$f_1, \ldots, f_N \in \mathbb{C}[x_{\sigma_1}, \ldots, x_{\sigma_{n!}}]^{S_n}$$

be the generators of the ring of invariant polynomials computed by Theorem 1.1. Then D_1 is biholomorphically equivalent to D_2 if and only if for all $\vec{\alpha}$ n-tuples of non-negative integers

$$f_i(g^{\sigma}_{D_1}(\vec{\alpha}))_{\sigma\in S_n} = f_i(g^{\sigma}_{D_2}(\vec{\alpha}))_{\sigma\in S_n}, \quad i = 1, 2, \ldots, N.$$

In this case, there exists $\tau \in S_n$ and a biholomorphic map $\Psi_\tau \colon \mathbb{C}^n \to \mathbb{C}^n$ $\Psi_\tau(z_1, \ldots, z_n) = (a_1 z_{\tau(1)}, \ldots, a_n z_{\tau(n)})$, where $a_i = \dfrac{\|\phi_{\vec{0}}\|_{D_1}\|\phi_{\vec{e}_i}\|_{D_2}}{\|\phi_{\vec{e}_{\sigma(i)}}\|_{D_1}\|\phi_{\vec{0}}\|_{D_2}}$, *such that Ψ_τ sends D_1 onto D_2.*

Our paper is organized as follows. In Section 2 we recall the basic notion of the Bergman function and some well-known results which are needed for later discussion. In Section 3, we construct continuous invariants of bounded complete Reinhardt domains in $\mathbb{C}^n$ and prove Theorem A and Theorem B (B′). In Section 4, the Hilbert 14th problem and Göbel's theorem are discussed and complete continuous numerical invariants of bounded complete Reinhardt pseudoconvex domains are constructed. Theorem C (C′), Theorem E and Theorem F (F′) are proved in this section. In Section 5, we give applications to some concrete examples.

2. Preliminaries

In this section, we shall recall some basic definitions and results in our previous papers [4, 20] which will facilitate our subsequent discussion. Let M be a pseudoconvex complex manifold and A be a compact complex analytic variety in the interior of M.

Definition 2.1. Let F_M (respectively, $F_{M,A}$) be the space of all L^2-integrable holomorphic n-form on M (respectively, vanishing at the compact analytic subset A in M). Let $\{\omega_j\}$ (respectively, $\{\omega_j^A\}$) be a complete orthonormal basis of F_M (respectively, $F_{M,A}$). The Bergman kernel (respectively, Bergman kernel vanishing at A) is defined to be $K_M(z) = \sum_j \omega_j(z) \wedge \overline{\omega_j(z)}$ (respectively, $K_{M,A}(z) = \sum_j \omega_j^A(z) \wedge \overline{\omega_j^A(z)}$).

Lemma 2.1. *(a) Bergman kernel $K_{M,A}(z)$ vanishing at the compact analytic subset A is independent of the choice of the complete orthonormal basis of $F_{M,A}$.*

(b) Let $\Phi\colon (M_1, A_1) \to (M_2, A_2)$ be a biholomorphic map such that $\Phi(A_1) = A_2$. Then $K_{M_1,A_1}(z) = \Phi^* K_{M_2,A_2}(z)$.

Definition 2.2. The Bergman function $B_{M,A}$ on M is defined to be $K_{M,A}(z)/K_M(z)$.

The following Theorem 2.1 can be found in [20].

Theorem 2.1. *Let A_1 (respectively A_2) be compact analytic variety in complex manifold M_1 (respectively M_2). If $\Phi\colon (M_1, A_1) \to (M_2, A_2)$ is a biholomorphic map, then $B_{M_1,A_1}(z) = B_{M_2,A_2}(\Phi(z))$.*

In what follows, we shall recall the following beautiful theorem of Sunada.

Theorem 2.2 (Sunada [17]). *Two n-dimensional bounded Reinhardt domains D_1 and D_2 are biholomorphically equivalent if and only if there exists a linear transformation $\Phi\colon \mathbb{C}^n \to \mathbb{C}^n$ given by $z_i \mapsto r_i z_{\sigma(i)}$ ($r_i > 0$, $i = 1, \ldots, n$ and σ being a permutation of the indices i) such that $\Phi(D_1) = D_2$.*

The following proposition, which is a corollary to Satz 1 in [13], is stated as Proposition 1 in [12].

Proposition 2.1 (Pflug [13]). *The Bergman kernel blows up at every boundary point in a pseudoconvex domain with C^1-bounding in $\mathbb{C}^n$.*

3. Continuous invariants of bounded complete Reinhardt domains in $\mathbb{C}^n$

In what follows, we shall use the following notations $\vec{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the ith position.

$$\mathbb{Z}_+ = \text{set of non-negative integers}$$
$$\vec{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n.$$

Proposition 3.1. *Let D be a bounded complete Reinhardt domain in $\mathbb{C}^n$. Let $\phi_{\vec{\alpha}} = \prod_{i=1}^n z_i^{\alpha_i}\, dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$, $\alpha_i \in \mathbb{Z}_+$. Then $\left\{ \frac{\phi_{\vec{\alpha}}}{\|\phi_{\vec{\alpha}}\|_D} \right\}$ is a complete orthonormal base of F_D, and $\left\{ \frac{\phi_{\vec{\alpha}}}{\|\phi_{\vec{\alpha}}\|_D} : \vec{\alpha} \neq 0 \right\}$ is a complete orthonormal basis of $F_{D,0}$. The Bergman kernel $K_{D,0}$ vanishes at the origin and the Bergman kernel K_D are given by*

$$K_{D,0} = \Theta_D\, dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \tag{3.1}$$

and

$$K_D = \left(\frac{1}{\|\phi_{\vec{0}}\|_D^2} + \Theta_D \right) dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \tag{3.2}$$

where

$$\Theta_D = \sum_{\vec{\alpha} \neq 0} \frac{\prod_{i=1}^n |z_i|^{2\alpha_i}}{\|\phi_{\vec{\alpha}}\|_D^2}. \tag{3.3}$$

Proof. This is a consequence of the proof of Proposition 3.2 of [20]. □

Proposition 3.2. *Let D be a bounded complete Reinhardt domain in $\mathbb{C}^n$. With the notations in the above proposition, $\|\phi_{\vec{0}}\|_D^2 \Theta_D$ is invariant under biholomorphic maps which send the origin to the origin.*

Proof. Let $\Psi\colon D_1 \to D_2$ be a biholomorphic map between two bounded complete Reinhardt domains such that $\Psi(0) = 0$. By Theorem 2.1 and Proposition 3.1, we have

$$\begin{aligned} B_{D_1}(z) &= B_{D_2}(\Psi(z)) \\ &\Rightarrow \frac{\Theta_{D_1}(z)}{\frac{1}{\|\phi_{\vec{0}}\|_{D_1}^2} + \Theta_{D_1}(z)} = \frac{\Theta_{D_2}(\Psi(z))}{\frac{1}{\|\phi_{\vec{0}}\|_{D_2}^2} + \Theta_{D_2}(\Psi(z))} \\ &\Rightarrow \|\phi_{\vec{0}}\|_{D_1}^2 \Theta_{D_1}(z) = \|\phi_{\vec{0}}\|_{D_2}^2 \Theta_{D_2}(\Psi(z)). \end{aligned}$$

□

Theorem 3.1. *Let D_i, $i = 1, 2$, be two bounded complete Reinhardt domains in $\mathbb{C}^n$. If D_1 is biholomorphically equivalent to D_2, then there exists a biholomorphic map Ψ_σ of the following form:*

$$\Psi_\sigma(z) = \left(a_1 z_{\sigma(1)}, \ldots, a_n z_{\sigma(n)}\right), \tag{3.4}$$

where σ is in S_n, a symmetric group of order n, and

$$a_i = \frac{\|\phi_{\vec{0}}\|_{D_1} \|\phi_{\vec{e}_i}\|_{D_2}}{\|\phi_{\vec{e}_{\sigma(i)}}\|_{D_1} \|\phi_{\vec{0}}\|_{D_2}}. \tag{3.5}$$

Proof. In view of Theorem 2.2, there exists a biholomorphism $\Psi_\sigma\colon D_1 \to D_2$ of the form (3.4). By Proposition 3.2

$$\|\phi_{\vec{0}}\|_{D_1}^2 \Theta_{D_1}(z) = \|\phi_{\vec{0}}\|_{D_2}^2 \Theta_{D_2}(\Psi_\sigma(z)),$$

$$\Rightarrow \|\phi_{\vec{0}}\|_{D_1}^2 \sum_{\vec{\alpha} \neq 0} \frac{\prod_{i=1}^n |z_n|^{2\alpha_i}}{\|\phi_{\vec{\alpha}}\|_{D_1}^2} = \|\phi_{\vec{0}}\|_{D_2}^2 \sum_{\vec{\alpha} \neq 0} \frac{\prod_{i=1}^n |a_i z_{\sigma(i)}|^{2\alpha_i}}{\|\phi_{\vec{\alpha}}\|_{D_2}^2}. \tag{3.6}$$

Comparing the coefficient of $|z_i|^2$ in (3.6), we can get

$$\begin{aligned} \|\phi_{\vec{0}}\|_{D_1}^2 \frac{1}{\|\phi_{\vec{e}_i}\|_{D_1}^2} &= \frac{\|\phi_{\vec{0}}\|_{D_2}^2 |a_{\sigma^{-1}(i)}|^2}{\|\phi_{\vec{e}_{\sigma^{-1}(i)}}\|_{D_2}^2} \\ \Rightarrow a_{\sigma^{-1}(i)} &= \frac{\|\phi_{\vec{0}}\|_{D_1} \|\phi_{\vec{e}_{\sigma^{-1}(i)}}\|_{D_2}}{\|\phi_{\vec{e}_i}\|_{D_1} \|\phi_{\vec{0}}\|_{D_2}} \\ \text{i.e., } a_i &= \frac{\|\phi_{\vec{0}}\|_{D_1} \|\phi_{\vec{e}_i}\|_{D_2}}{\|\phi_{\vec{e}_{\sigma(i)}}\|_{D_1} \|\phi_{\vec{0}}\|_{D_2}}. \end{aligned}$$

□

Now we are ready to prove Theorem A.

Proof of Theorem A. Let D_1 and D_2 be biholomorphically equivalent bounded complete Reinhardt domains. By Theorem 3.1, there exists a biholomorphic map

$$\begin{aligned}\Psi_\sigma\colon D_1 &\longrightarrow D_2\\ (z_1,\dots,z_n) &\longrightarrow (a_1 z_{\sigma(1)},\dots,a_n z_{\sigma(n)}),\end{aligned}$$

where

$$a_i = \frac{\|\phi_{\vec{0}}\|_{D_1}\|\phi_{\vec{e}_i}\|_{D_2}}{\|\phi_{\vec{e}_{\sigma(i)}}\|_{D_1}\|\phi_{\vec{0}}\|_{D_2}}.$$

By Proposition 3.2

$$\begin{aligned}\|\phi_{\vec{0}}\|^2_{D_1}\sum_{\vec{\alpha}\neq 0}\frac{\prod_{i=1}^n |z_i|^{2\alpha_i}}{\|\phi_{\vec{\alpha}}\|^2_{D_1}} &= \|\phi_{\vec{0}}\|^2_{D_1}\Theta_{D_1}(z)\\ &= \|\phi_{\vec{0}}\|^2_{D_2}\Theta_{D_2}\left(\Psi_\sigma(z)\right)\\ &= \|\phi_{\vec{0}}\|^2_{D_2}\sum_{\vec{\alpha}\neq 0}\frac{\prod_{i=1}^n |a_i z_{\sigma(i)}|^{2\alpha_i}}{\|\phi_{\vec{\alpha}}\|^2_{D_2}}.\end{aligned}$$

Comparing the coefficient of $|z_1|^{2\alpha_1}|z_2|^{2\alpha_2}\dots|z_n|^{2\alpha_n}$ in the identity above, we get

$$\begin{aligned}\frac{\|\phi_{\vec{0}}\|^2_{D_1}}{\|\phi_{\vec{\alpha}}\|^2_{D_1}} &= \frac{\|\phi_{\vec{0}}\|^2_{D_2}\cdot\prod_{i=1}^n |a_{\sigma^{-1}(i)}|^{2\alpha_i}}{\|\phi_{\sigma(\vec{\alpha})}\|^2_{D_2}}\\ &\Rightarrow \frac{\|\phi_{\vec{0}}\|^2_{D_1}}{\|\phi_{\vec{\alpha}}\|^2_{D_1}}\\ &= \frac{\|\phi_{\vec{0}}\|^2_{D_2}}{\|\phi_{\sigma(\vec{\alpha})}\|^2_{D_2}}\prod_{i=1}^n\left(\frac{\|\phi_{\vec{0}}\|_{D_1}\|\phi_{\vec{e}_{\sigma-1(i)}}\|_{D_2}}{\|\phi_{\vec{e}_i}\|_{D_1}\|\phi_{\vec{0}}\|_{D_2}}\right)^{2\alpha_i}\\ &\Rightarrow \frac{\|\phi_{\vec{0}}\|^{\Sigma\alpha_i-1}_{D_1}\cdot\|\phi_{\vec{\alpha}}\|_{D_1}}{\prod_{i=1}^n\|\phi_{\vec{e}_i}\|^{\alpha_i}_{D_1}}\\ &= \frac{\|\phi_{\vec{0}}\|^{\Sigma\alpha_i-1}_{D_2}\cdot\|\phi_{\sigma(\vec{\alpha})}\|_{D_2}}{\prod_{i=1}^n\|\phi_{\vec{e}_{\sigma^{-1}(i)}}\|^{\alpha_i}_{D_2}}\\ &= \frac{\|\phi_{\vec{0}}\|^{\sum\alpha_i-1}_{D_2}\|\phi_{\sigma(\vec{\alpha})}\|_{D_2}}{\prod_{i=1}^n\|\phi_{\vec{e}_i}\|^{\alpha_{\sigma(i)}}_{D_2}}\end{aligned}\tag{3.7}$$

i.e., $g^{\mathrm{Id}}_{D_1}(\vec{\alpha}) = g^{\sigma}_{D_2}(\vec{\alpha})$.

Similarly, by comparing the coefficient of $|z_1|^{2\alpha_{\tau(1)}}|z_2|^{2\alpha_{\tau(2)}}\ldots|z_n|^{2\alpha_{\tau(n)}}$, we know

$$g_{D_1}^{\tau}(\vec{\alpha}) = g_{D_2}^{\tau\cdot\sigma}(\vec{\alpha}),$$

which implies

$$\begin{aligned}\sum_{\tau\in S_n} g_{D_1}^{\tau}(\vec{\alpha})\tau &= \sum_{\tau\in S_n} g_{D_2}^{\tau\cdot\sigma}(\vec{\alpha})\tau \\ &= \sum_{\tau\in S_n} g_{D_2}^{\tau\cdot\sigma}(\vec{\alpha})\tau\cdot\sigma\cdot\sigma^{-1} \\ &= \sigma^{-1}\left(\sum_{\tau\in S_n} g_{D_2}^{\tau}(\vec{\alpha})\tau\right). \qquad (3.8)\end{aligned}$$

For the same reason, we have

$$\sum_{\tau\in S_n} g_{D_1}^{\tau}(\vec{\alpha}_i)\tau = \sigma^{-1}\left(\sum_{\tau\in S_n} g_{D_2}^{\tau}(\vec{\alpha}_i)\tau\right), \quad i=1,\ldots,n!. \qquad (3.9)$$

From (3.9), we have $\xi_{D_1}^{\vec{\alpha},\ldots,\vec{\alpha}_{n!}} \sim \xi_{D_2}^{\vec{\alpha},\ldots,\vec{\alpha}_{n!}}$.

Next we claim that $\exists\sigma\in S_n$ such that $g_{D_1}^{\tau}(\vec{\alpha}) = g_{D_2}^{\tau\cdot\sigma}(\vec{\alpha})$ $\forall\tau\in S_n$ and $\forall\vec{\alpha}$ n-tuple of non-negative integers. For any $\vec{\alpha}$ n-tuple of non-negative integers, let

$$I_{\vec{\alpha}} = \{\sigma\in S_n\colon g_{D_1}^{\tau}(\vec{\alpha}) = g_{D_2}^{\tau\cdot\sigma}(\vec{\alpha}), \quad \forall\tau\in S_n\}.$$

If our claim is not true, then $\forall\sigma_i\in S_n=\{\sigma_1,\ldots,\sigma_{n!}\}$, $\exists\vec{\alpha}_i$, n-tuple of non-negative integers such that $\sigma_i\notin I_{\vec{\alpha}_i}$. It follows that

$$\bigcup_{i=1}^{n!}(S_n\setminus I_{\vec{\alpha}_i}) = S_n,$$

which implies $\bigcap_{i=1}^{n!} I_{\vec{\alpha}_i} = \emptyset$. On the other hand for these n-tuples of integers $\vec{\alpha}_1,\ldots,\vec{\alpha}_{n!}$, we have $\xi_{D_1}^{\vec{\alpha}_1,\ldots,\vec{\alpha}_{n!}} = \xi_{D_2}^{\vec{\alpha}_1,\ldots,\vec{\alpha}_{n!}}$, i.e., $\exists\sigma^{\vec{\alpha}_1,\ldots,\vec{\alpha}_{n!}}\in S_n$ such that (3.10) hold. Let $\tau_{(\vec{\alpha}_1,\ldots,\vec{\alpha}_{n!})} = (\sigma^{\vec{\alpha}_1,\ldots,\vec{\alpha}_{n!}})^{-1}$. In view of (3.10), one easily sees that $\tau_{(\vec{\alpha}_1,\ldots,\vec{\alpha}_{n!})}\in\bigcap_{i=1}^{n!} I_{\alpha_i}$. This leads to a contradiction and our claim is proven. $\square$

In order to prove Theorem B, we need to establish the following theorem.

Theorem 3.2. *Let D_i, $i = 1, 2$, be two bounded complete Reinhardt domains in $\mathbb{C}^n$. Suppose for all $\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}$ n-tuples of non-negative integers, $\xi_{D_1}^{\vec{\alpha}_1,\dots,\vec{\alpha}_{n!}} = \xi_{D_2}^{\vec{\alpha}_1,\dots,\vec{\alpha}_{n!}}$ in $(\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n])/\sim$, i.e., $\exists \sigma^{\vec{\alpha}_1,\dots,\vec{\alpha}_{n!}} \in S_n$ such that*

$$\xi_{D_1}^{\vec{\alpha}_1,\dots,\vec{\alpha}_{n!}} = \sigma^{\vec{\alpha}_1,\dots,\vec{\alpha}_{n!}}\left(\xi_{D_2}^{\vec{\alpha}_1,\dots,\vec{\alpha}_{n!}}\right). \tag{3.10}$$

Then, there exists $\sigma \in S_n$ and a biholomorphic map

$$\begin{aligned}\Psi_\sigma \colon \mathbb{C}^n &\longrightarrow \mathbb{C}^n \\ (z_1, \dots, z_n) &\longrightarrow \left(a_1 z_{\sigma(1)}, \dots, a_n z_{\sigma(n)}\right),\end{aligned}$$

where $a_i = \dfrac{\|\phi_{\vec{0}}\|_{D_1} \|\phi_{\vec{e}_i}\|_{D_2}}{\|\phi_{\vec{e}_{\sigma(i)}}\|_{D_1} \|\phi_{\vec{0}}\|_{D_2}}$, such that $B_{D_1}(z) = B_{D_2}(\Psi_\sigma(z))$.

Proof. By Theorem A, we now take σ in the claim of Theorem A and let

$$\begin{aligned}\Psi_\sigma \colon \mathbb{C}^n &\longrightarrow \mathbb{C}^n \\ (z_1, \dots, z_n) &\longrightarrow \left(a_1 z_{\sigma(1)}, \dots, a_n z_{\sigma(n)}\right),\end{aligned}$$

where $a_i = \dfrac{\|\phi_{\vec{0}}\|_{D_1} \|\phi_{\vec{e}_i}\|_{D_2}}{\|\phi_{\vec{e}_{\sigma(i)}}\|_{D_1} \|\phi_{\vec{0}}\|_{D_2}}$. After computation as in the proof of Theorem A, we get

$$\|\phi_{\vec{0}}\|_{D_1}^2 \sum \frac{|z_1|^{2\alpha_1} \dots |z_n|^{2\alpha_n}}{\|\phi_{\vec{\alpha}}\|_{D_1}} = \|\phi_{\vec{0}}\|_{D_2}^2 \sum \frac{\prod_{i=1}^n |a_i z_{\sigma(i)}|^{2\alpha_i}}{\|\phi_{\sigma(\vec{\alpha})}\|_{D_2}^2}.$$

It follows that $B_{D_1}(z) = B_{D_2}(\Psi_\sigma(z))$. □

Now we are ready to prove Theorem B.

Proof of Theorem B. It is easy to see that the Bergman function B_{D_i} is zero at the origin and $0 < B_{D_i} < 1$ on $D_i \setminus \{(0, 0, \dots, 0)\}$. In view of Proposition 2.1, B_{D_i} is identically equal to 1 on ∂D_i. By Theorem 3.2, there exists a biholomorphic map Ψ_σ from $\mathbb{C}^n$ to $\mathbb{C}^n$ such that $B_{D_1}(z) = B_{D_2}(\Psi_\sigma(z))$. In particular Ψ_σ preserves the level sets of the Bergman functions. It follows that Ψ_σ sends ∂D_1 to ∂D_2. □

4. Complete continuous numerical invariants

We shall now construct the complete continuous numerical invariants for complete Reinhardt domains with C^1 boundaries. For $n = 2$, we have Theorem C stated in Section 1.

Proof of Theorem C. Since (1.3), (1.4) and (1.5) hold for all non-negative integers α_i, β_i, we have either

$$(1)\quad \begin{aligned} &g_{D_1}(\alpha_1,\alpha_2) = g_{D_2}(\alpha_1,\alpha_2), \quad g_{D_1}(\alpha_2,\alpha_1) = g_{D_2}(\alpha_2,\alpha_1),\\ &g_{D_1}(\beta_1,\beta_2) = g_{D_2}(\beta_1,\beta_2), \quad\ \ g_{D_1}(\beta_2,\beta_1) = g_{D_2}(\beta_2,\beta_1),\end{aligned}$$

or

$$(2)\quad \begin{aligned} &g_{D_1}(\alpha_1,\alpha_2) = g_{D_2}(\alpha_2,\alpha_1), \quad g_{D_1}(\alpha_2,\alpha_1) = g_{D_2}(\alpha_1,\alpha_2),\\ &g_{D_1}(\beta_1,\beta_2) = g_{D_2}(\beta_2,\beta_1), \quad\ \ g_{D_1}(\beta_2,\beta_1) = g_{D_2}(\beta_1,\beta_2),\end{aligned}$$

for all non-negative integers α_i, β_i. It is easy to see that in both cases, we get

$$\begin{aligned} \xi_{D_1}^{\vec{\alpha},\vec{\beta}} &:= \Big(g_{D_1}(\alpha_1,\alpha_2)\cdot \mathrm{Id} + g_{D_1}(\alpha_2,\alpha_1)\cdot\sigma,\\ &\qquad\qquad g_{D_1}(\beta_1,\beta_2)\cdot \mathrm{Id} + g_{D_2}(\beta_2,\beta_1)\cdot\sigma\Big)\\ \sim \xi_{D_2}^{\vec{\alpha},\vec{\beta}} &:= \Big(g_{D_2}(\alpha_1,\alpha_2)\cdot \mathrm{Id} + g_{D_2}(\alpha_2,\alpha_1)\cdot\sigma,\\ &\qquad\qquad g_{D_2}(\beta_1,\beta_2)\cdot \mathrm{Id} + g_{D_2}(\beta_2,\beta_1)\cdot\sigma\Big),\end{aligned} \tag{4.1}$$

where $S_2 = \{\mathrm{Id}, \sigma\}$. In view of Theorem B, we know that D_1 is biholomorphic to D_2.

Conversely if D_1 is biholomorphic to D_2, then by Theorem A, (4.1) holds. Then it is easy to check that (1.3), (1.4) and (1.5) hold. □

Theorem C says that

$$\begin{gathered} g_D(\alpha_1,\alpha_2) + g_D(\alpha_2,\alpha_1),\\ g_D(\alpha_1,\alpha_2)\cdot g_D(\alpha_2,\alpha_1)\end{gathered}$$

and

$$\left(g_D(\alpha_1,\alpha_2) - g_D(\alpha_2,\alpha_1)\right)\left(g_D(\beta_1,\beta_2) - g_D(\beta_2,\beta_1)\right),$$

$\forall \alpha_i, \beta_i \geqslant 0$, $\alpha_i, \beta_i \in \mathbb{Z}_+$, are complete numerical biholomorphic invariants of bounded complete Reinhardt pseudoconvex domains in $\mathbb{C}^n$ with C^1 boundaries. If we want to find the complete numerical biholomorphic invariants for bounded complete Reinhardt pseudoconvex domains in $\mathbb{C}^n$, $n \geqslant 3$, we need to consider the following problem.

Let S_n be the symmetric group of order n. S_n acts on $\mathbb{C}[x_\sigma, \dots, y_\sigma]$ in the following manner:

$$\tau \in S_n, \quad \tau(x_\sigma) = x_{\sigma\cdot\tau}, \dots, \tau(y_\sigma) = y_{\sigma\cdot\tau}$$

Let R be subring of invariants in $\mathbb{C}[x_\sigma, \dots, y_\sigma]$. In other words,

$$R := \mathbb{C}[x_\sigma, \dots, y_\sigma]^{S_n} = \{f \in \mathbb{C}[x_\sigma, \dots, y_\sigma] \colon \forall \tau \in S_n,\ \tau(f) = f\}.$$

We want to know whether R is finitely generated and what are the generators of R. Actually the first problem is a special case of Hilbert's 14^{th} problem. At the International Congress of Mathematicians at Paris in 1900, David Hilbert asked the following question.

Hilbert 14th Problem. If an algebraic group acts linearly on a polynomial ring in finitely many variables, is the ring of invariants always finitely generated?

Recall that an algebraic group G is said to be linearly reductive if, for every epimorphism $\phi \colon V \to W$ of G representations, the induced map on G-invariants $\phi^G \colon V^G \to W^G$ is surjective. The answer to the Hilbert 14th problem is positive if the group G is linearly reductive [7]. In fact, the answer is also positive if G is additive (non-reductive) group $\mathbb{C}$ (or, more generally, the field k. See [16, 19]). In view of the following lemma, we know that finding the complete numerical biholomorphic invariants is equivalent to finding the generators of R.

Lemma 4.1. *Let the symmetric group $S_n = \{\sigma_1, \dots, \sigma_{n!}\}$ act on $\mathbb{C}^{n!n!}$ via the following formula:*

$$\tau \in S_n, \quad p = (x_{\sigma_1}, \dots, x_{\sigma_{n!}}, \dots, y_{\sigma_1}, \dots, y_{\sigma_{n!}}) \in \mathbb{C}^{n!n!},$$
$$\tau(p) = (x_{\sigma_1\tau}, \dots, x_{\sigma_{n!}\tau}, \dots, y_{\sigma_1\tau}, \dots, y_{\sigma_{n!}\tau}) \,.$$

Let $\pi \colon \mathbb{C}^{n!n!} \to \mathbb{C}^{n!n!}/S_n$ be the quotient map. For any p and p' in $\mathbb{C}^{n!n!}$, the following two statements are equivalent:

(1) $\pi(p) = \pi(p')$.

(2) For any $f \in R = \mathbb{C}\left[x_{\sigma_1}, \dots, x_{\sigma_{n!}}, y_{\sigma_1}, \dots, y_{\sigma_{n!}}\right]^{S_n}$, $f(p) = f(p')$.

Proof. See [11, p. 167, Theorem 5.16]. □

In view of Lemma 4.1, we want to show that $R = \mathbb{C}[x_1, \dots, x_{n!}, \dots, y_1, \dots, y_{n!}]^{S_n}$ is finitely generated.

Proposition 4.1. *Every finite group is linearly reductive.*

Theorem 4.1 (Hilbert [7])**.** *For linearly reductive group, the ring of invariant polynomials is finitely generated.*

From Proposition 4.1 and Theorem 4.1, we know that

$$R = \mathbb{C}[x_{\sigma_1}, \ldots, x_{\sigma_{n!}}, \ldots, y_{\sigma_1}, \ldots, y_{\sigma_{n!}}]^{S_n}$$

is finitely generated. Now we assume that the generators of R are $f_1(x_\sigma, \ldots, y_\sigma), \ldots, f_N(x_\sigma, \ldots, y_\sigma)$ where σ runs over all the elements in S_n. We want to show that

$$f_1\left(g_D^\sigma(\vec{\alpha}_1), \ldots, g_D^\sigma(\vec{\alpha}_{n!})\right)_{\sigma \in S_n}, \ldots, f_N\left(g_D^\sigma(\vec{\alpha}_1), \ldots, g_D^\sigma\left(\vec{\alpha}_{n!}\right)\right)_{\sigma \in S_n},$$

for all n-tuples of non-negative integers $\vec{\alpha}_1, \ldots, \vec{\alpha}_{n!}$, are complete numerical invariants of bounded complete Reinhardt pseudoconvex domains in $\mathbb{C}^n$.

Proof of Theorem E. Let D_1, D_1 be two bounded complete Reinhardt domains in $\mathbb{C}^n$. If D_1 is biholomorphic to D_2, then there exists $\tau \in S_n$ such that $g_{D_1}^\sigma(\vec{\alpha}) = g_{D_2}^{\sigma\tau}(\vec{\alpha})$, $\forall \sigma \in S_n$, $\forall \vec{\alpha}$ n-tuple of non-negative integers. We take

$$P_i = \left(g_{D_i}^{\sigma_1}(\vec{\alpha}_1), \ldots, g_{D_i}^{\sigma_{n!}}\left(\vec{\alpha}_1\right), \ldots, g_{D_i}^{\sigma_1}\left(\vec{\alpha}_{n!}\right), \ldots, g_{D_i}^{\sigma_{n!}}(\vec{\alpha}_{n!})\right) \in \mathbb{C}^{n!n!},$$

where $i = 1, 2$ and $S_n = \{\sigma_1, \sigma_2, \ldots, \sigma_{n!}\}$. Then $\pi(P_1) = \pi(P_2)$ where $\pi\colon \mathbb{C}^{n!n!} \to \mathbb{C}^{n!n!}/S_n$ is the quotient map in Lemma 4.1. It follows that

$$f_i\left(g_{D_1}^\sigma(\vec{\alpha}_1), \ldots, g_D^\sigma(\vec{\alpha}_{n!})\right)_{\sigma \in S_n} = f_i\left(g_{D_2}^\sigma(\vec{\alpha}_1), \ldots, g_{D_2}^\sigma(\vec{\alpha}_{n!})\right)_{\sigma \in S_n}, \quad 1 \leqslant i \leqslant N.$$

□

Proof of Theorem F. Since

$$f_i\left(g_{D_1}^\sigma(\vec{\alpha}_1), \ldots, g_{D_1}^\sigma(\vec{\alpha}_{n!})\right)_{\sigma \in S_n} = f_i\left(g_{D_2}^\sigma(\vec{\alpha}_1), \ldots, g_{D_2}^\sigma(\vec{\alpha}_{n!})\right)_{\sigma \in S_n}, \quad i = 1, 2, \ldots, N$$

and $\{f_1, \ldots, f_N\}$ generates $\mathbb{C}[x_{\sigma_1}, \ldots x_{\sigma_{n!}}, \ldots, y_{\sigma_1}, \ldots, y_{\sigma_{n!}}]^{S_n}$, we know by Lemma 4.1 that

$$P_1 = \left(g_{D_1}^{\sigma_1}(\vec{\alpha}_1), \ldots, g_{D_1}^{\sigma_{n!}}(\vec{\alpha}_1), \ldots, g_{D_1}^{\sigma_1}(\vec{\alpha}_{n!}), \ldots, g_{D_1}^{\sigma_{n!}}(\vec{\alpha})\right)$$

and $P_2 = \left(g_{D_2}^{\sigma_1}(\vec{\alpha}_1), \ldots, g_{D_2}^{\sigma_{n!}}(\vec{\alpha}_{n!}), \ldots, g_{D_2}^{\sigma_1}(\vec{\alpha}_{n!}), \ldots, g_{D_2}^{\sigma_{n!}}(\vec{\alpha}_{n!})\right)$ are in the same S_n orbit. Hence there exists $\sigma^{\vec{\alpha}_1,\ldots,\vec{\alpha}_{n!}} \in S_n$ such that

$$g_{D_1}^{\sigma}(\vec{\alpha}_1) = g_{D_2}^{\sigma\cdot\sigma^{\vec{\alpha}_1,\ldots,\vec{\alpha}_{n!}}}(\vec{\alpha}_1), \ldots, g_{D_1}^{\sigma}(\vec{\alpha}_{n!}) = g_{D_2}^{\sigma\cdot\sigma^{\vec{\alpha}_1,\ldots,\vec{\alpha}_{n!}}}(\vec{\alpha}_{n!}). \tag{4.2}$$

Equation (4.2) implies that $\xi_{D_1}^{\vec{\alpha}_1,\ldots,\vec{\alpha}_{n!}} = \xi_{D_2}^{\vec{\alpha}_1,\ldots,\vec{\alpha}_{n!}}$ in $(\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n])/\sim$. By Theorem 3.2, there exists $\sigma \in S_n$ and a biholomorphic map $\Psi_\sigma\colon \mathbb{C}^n \to \mathbb{C}^n$ such that $B_{D_1}(z) = B_{D_2}(\Psi_\sigma(z))$. The proof of Theorem B shows that Ψ_σ sends D_1 onto D_2. □

For complete Reinhardt pseudoconvex domains with real analytic boundaries, we can use fewer numerical invariants to characterize these domains.

Proof of Theorem B′.

"⇒" This follows from the proof of Theorem A.

"⇐" Suppose that for all $\vec{\alpha}$ n-tuple non-negative integers,

$$\xi_{D_1}^{\vec{\alpha}} = \xi_{D_2}^{\vec{\alpha}} \text{ in } R[S_n]/\sim .$$

Then $\exists \sigma_{\vec{\alpha}} \in S_n$ such that

$$\begin{aligned}
\sum_{\tau\in S_n} g_{D_1}^{\tau}(\vec{\alpha})\tau &= \sigma_{\vec{\alpha}}\left(\sum_{\tau\in S_n} g_{D_2}^{\tau}(\vec{\alpha})\tau\right)\\
&= \sum_{\tau\in S_n} g_{D_2}^{\tau}(\vec{\alpha})\tau\cdot\sigma_{\vec{\alpha}}\\
&= \sum_{\tau\in S_n} g_{D_2}^{\tau\cdot\sigma_{\vec{\alpha}}^{-1}}(\vec{\alpha})\tau\\
&\Rightarrow g_{D_1}^{\tau}(\vec{\alpha}) = g_{D_2}^{\tau\cdot\sigma_{\vec{\alpha}}^{-1}}.
\end{aligned} \tag{4.3}$$

Let $S_n = \{\tau_1, \ldots, \tau_{n!}\}$. Denote $I_j = \{\vec{\alpha} \in (\mathbb{Z}_+)^n \colon \sigma_{\vec{\alpha}}^{-1} = \tau_j\}$, $j = 1, 2, \ldots, n!$. We shall introduce the concept of partial Bergman function in the following manner. Let

$$\Theta_D^{(j)}(z) = \sum_{\vec{\alpha}\in I_j} \frac{\prod_{h=1}^n |z_k|^{2\alpha_k}}{\|\phi_{\vec{\alpha}}\|_D^2}, \quad B_D^{(j)}(z) = \frac{\Theta_D^{(j)}(z)}{\frac{1}{\|\phi_{\vec{\alpha}}\|_D^2} + \Theta_D^{(j)}(z)}.$$

Clearly, we have

$$\Theta_D(z) = \sum_{\vec{\alpha}\neq 0} \frac{\prod_{k=1}^{n} |z_k|^{2\alpha_k}}{\|\phi_{\vec{\alpha}}\|_D^2} = \sum_{j=1}^{n!} \Theta_D^{(j)}(z). \tag{4.4}$$

In view of (4.3), by routine computation we get

$$\|\Phi_{\vec{0}}\|_{D_1}^2 \Theta_{D_1}^{(j)}(z) = \|\phi_{\vec{0}}\|_{D_2}^2 \Theta_{D_2}^{(j)}(\Psi_{\tau_j}(z)) \tag{4.5}$$

and

$$B_{D_1}^{(j)}(z) = B_{D_2}^{(j)}(\Psi_{\tau_j}(z)), \tag{4.6}$$

where

$$\Psi_{\tau_j} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$$
$$(z_1, \ldots, z_n) \longrightarrow \left(a_1 z_{\tau_j(1)}, \ldots, a_n z_{\tau_j(n)}\right)$$
$$a_i = \frac{\|\phi_{\vec{0}}\|_{D_1} \|\phi_{\vec{e}_i}\|_{D_2}}{\|\phi_{\vec{e}_{\tau_j(i)}}\|_{D_1} \|\phi_{\vec{0}}\|_{D_2}}$$

By Proposition 2.1, the Bergman kernel and hence Θ_{D_1} blows up at the boundary points. It follows from (4.4) that there exists some m such that $\Theta_{D_1}^{(m)}$ is infinite on a non-empty open set $\mathcal{U}_1$ of ∂D_1. It follows that $B_{D_1}^{(m)}(z)$ is equal to 1 in the non-empty open set $\mathcal{U}_1$ of ∂D_1. Recall that by the definition, $B_{D_i}^{(m)}$ is zero at the origin and is $0 < B_{D_i}^{(m)} < 1$ on $D_i \setminus \{(0, \ldots, 0)\}$. Since $B_{D_1}^{(m)}(z) = B_{D_2}^{(m)}(\Psi_{\tau_m}(z))$, we see immediately that Ψ_{τ_m} sends $\mathcal{U}_1 \subseteq \partial D_1$ to an open subset of ∂D_2. In view of the fact that ∂D_i, $i = 1, 2$, are real analytic and compact, it follows easily that Ψ_{τ_m} sends ∂D_1 to ∂D_2. □

Proof of Theorem C′. This follows immediately from Theorem B′. □

Proof of Theorem F′.

"⇒" This follows from the proof of Theorem E.

"⇐" Suppose that for all $\vec{\alpha}$ n-tuple non-negative integers, we have

$$f_i\left(g_{D_1}^{\sigma}(\vec{\alpha})\right)_{\sigma\in S_n} = f_i\left(g_{D_2}^{\sigma}(\vec{\alpha})\right)_{\sigma\in S_n}, \quad i = 1, \ldots, N.$$

Then by the analogy of Lemma 4.1, we have

$$\pi\left(g_{D_1}^{\tau_1}(\vec{\alpha}), \ldots, g_{D_1}^{\tau_{n!}}(\vec{\alpha})\right) = \pi\left(g_{D_2}^{\tau_1}(\vec{\alpha}), \ldots, g_{D_2}^{\tau_{n!}}(\vec{\alpha})\right)$$

when $\pi\colon \mathbb{C}^{n!} \longrightarrow \mathbb{C}^{n!}/S_n$. Therefore, there exist $\sigma_{\vec{\alpha}}$ such that $g^{\sigma}_{D_1}(\vec{\alpha}) = g^{\sigma\cdot\sigma_{\vec{\alpha}}}_{D_2}(\vec{\alpha})$ for all $\sigma \in S_n$. By the proof similar to the proof of Theorem B′, we get a biholomorphic map Ψ_τ which sends D_1 onto D_2. □

Actually, the generators of $R = \mathbb{C}\left[x_{\sigma_1}, \ldots, x_{\sigma_{n!}}; \ldots; y_{\sigma_1}, \ldots, y_{\sigma_{n!}}\right]^{S_n}$ can be listed explicitly by a theorem in Section 1 discovered by Göbel [6].

5. Application in concrete examples

In [9], Huang communicated to us the following state of art of determining when two complete Reinhardt domains is biholomorphically equivalent. Assume that D_1, D_2 are two bounded Reinhardt domains. D_i is defined by real analytic function $f_i(|z_1|, \ldots, |z_n|) < 0$. Let $T(D_i) = \{(|z_1|, \ldots, |z_n|)\colon (z_1, \ldots, z_n) \in D_i\}$. By Sunada's theorem, D_1 is biholomorphic to D_2 if and only if there is a permutation $\sigma \in S_n$ such that $\sigma(T(D_1)) = T(D_2)$. Thus the biholomorphic problem for Reinhardt domains is converted to the identification problem up to permutation for domains in $(\mathbb{R}^+)^n$ with points $P_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ (in the jth component) being the boundary points. This method is effective when f_i, $i = 1, 2$ are irreducible polynomial. For example, let

$$\begin{aligned} D_1 = \{(z_1, z_2) \in \mathbb{C}^2\colon & \\ & f(|z_1|, |z_2|) = |z_1|^N + |z_2|^N + \sum_{j=1}^{N-1} c_j |z_1|^j |z_2|^{N-j} - 1 < 0\}, \\ D_2 = \{(z_1, z_2) \in \mathbb{C}^2\colon & \\ & f(|z_1|, |z_2|) = |z_1|^N + |z_2|^N + \sum_{j=1}^{N-1} e_j |z_1|^j |z_2|^{N-j} - 1 < 0\}, \end{aligned}$$

under the assumption that f_1, f_2 are irreducible, we get that D_1 is biholomorphic to D_2 if and only if either $(e_1, \ldots, e_{N-1}) = (c_1, \ldots, c_{N-1})$ or $(e_1, \ldots, e_{N-1}) = (c_{N-1}, c_{N-2}, \ldots, c_1)$. But this method cannot handle the case when f_i is not irreducible.

Example 5.1. Let

$$\begin{aligned} D_1 = \{z \in \mathbb{C}^2\colon f_1(|z_1|, |z_2|) = & \\ & (|z_1||z_2| + |z_1| + |z_2| - 1)\left(\frac{1}{2}|z_1||z_2| + |z_1| + |z_2| - 1\right) < 0\}, \\ D_2 = \{z \in \mathbb{C}^2\colon f_2(|z_1|, |z_2|) = & \\ & (|z_1||z_2| + |z_1| + |z_2| - 1)\left(\frac{1}{3}|z_1||z_2| + |z_1| + |z_2| - 1\right) < 0\}. \end{aligned}$$

Notice that $P_1 = (1,0)$ and $P_2 = (0,1)$ are both the outermost boundary points for D_1 and D_2. Also, $T(D_j)$ are invariant under the action of S_2. Hence, D_1 and D_2 are holomorphically equivalent if and only if $T(D_1) = T(D_2)$ in $(\mathbb{R}^+)^n$. But this is impossible, for $(1/3, 4/7)$ is a boundary point of $T(D_1)$ while it is an interior point of $T(D_2)$. From this example , we can see that there is no easy way for this method to handle the biholomorphic equivalent problem for general Reinhardt domains.

However, we can easily use our theory to show that D_1 is not biholomorphic to D_2 as follows. Let

$$\begin{aligned}\|\phi_{\vec{\alpha}}\|_D^2 &= \int_D \phi_{\vec{\alpha}} \wedge \overline{\phi}_{\vec{\alpha}} = \int_D |z_1|^{2\alpha_1}|z_2|^{2\alpha_2}\, dz_1 \wedge dz_2 \wedge d\overline{z}_1 \wedge d\overline{z}_2 \\ &= (4\pi)^2 \iint_{\{(r_1,r_2):\ f(r_1,r_2)<0\}} r_1^{2\alpha_1+1} r_2^{2\alpha_2+1}\, dr_1\, dr_2.\end{aligned}$$

By computation, we get

$$\begin{aligned}
\|\phi_{\vec{0}}\|_{D_1}^2 &= \left(-\frac{57}{4} + 42\ln 3 - 46\ln 2\right)\cdot(4\pi)^2,\\
\|\phi_{(1,0)}\|_{D_1}^2 = \|\phi_{(0,1)}\|_{D_1}^2 &= \left(-\frac{2903}{24} + 312\ln 3 - 320\ln 2\right)\cdot(4\pi)^2,\\
\|\phi_{(2,0)}\|_{D_1}^2 = \|\phi_{(0,2)}\|_{D_1}^2 &= \left(1824\ln 3 + 836\ln 2 - \frac{2925}{4}\right)\cdot(4\pi)^2,\\
\|\phi_{\vec{0}}\|_{D_2}^2 &= (-49 + 356\ln 2 - 180\ln 3)\cdot(4\pi)^2,\\
\|\phi_{(1,0)}\|_{D_2}^2 = \|\phi_{(0,1)}\|_{D_2}^2 &= \left(5824\ln 2 - \frac{2500}{3} - 2916\ln 3\right)\cdot(4\pi)^2,\\
\|\phi_{(2,0)}\|_{D_2}^2 = \|\phi_{(0,2)}\|_{D_2}^2 &= \left(-\frac{163457}{45} + 75804\ln 2 - 37908\ln 3\right)\cdot(4\pi)^2.
\end{aligned}$$

Let $S_2 = \{\tau_1 = \mathrm{Id},\ \tau_2 = (1,2)\}$ be the symmetric group of degree 2. We shall take $\vec{\beta} = (2,0)$. Then

$$\begin{aligned}
g_{D_i}(2,0) &= \frac{\|\phi_{\vec{0}}\|_{D_i}\|\phi_{(2,0)}\|_{D_i}}{\|\phi_{(1,0)}\|_{D_i}^2}, \quad i = 1,2,\\
g_{D_i}(0,2) &= \frac{\|\phi_{\vec{0}}\|_{D_i}\|\phi_{(0,2)}\|_{D_i}}{\|\phi_{(0,1)}\|_{D_i}^2} = g_{D_i}(2,0), \quad i = 1,2.
\end{aligned}$$

Since $g_{D_1}(2,0) + g_{D_1}(0,2)$, which is approximately equal to 25.277 is not equal to $g_{D_2}(2,0) + g_{D_2}(0,2)$, which is approximately equal to 2.637, we conclude that D_1 is not biholomorphic to D_2 by Theorem C.

In 1907 Poincaré discovered the following theorem [15, p. 24].

Theorem 5.1 (Poincaré)**.** *Let* $B_n = \{z \in \mathbb{C}^n \mid \sum_{i=1}^n |z_i|^2 < 1\}$ *be the unit ball and* $\Delta_n = \{z \in \mathbb{C}^n \mid |z_i| < 1, i = 1, \ldots, n\}$ *be the unit polydisc in* $\mathbb{C}^n$*, then there exists no biholomorphic map between* Δ_n *and* B_n*.*

Using our method, we can easily get this result.

Notation: $\vec{0} = (0, \ldots, 0)$, $\vec{e}_i = (0, \ldots, 1, \ldots, 0)$, $\vec{\alpha} = (2, 0, \ldots, 0)$.

Proof of Theorem 5.1. For any $\vec{\beta} \in \mathbb{Z}^n$, $D = \{z \in \mathbb{C}^n \mid f(z_1, \ldots, z_n) < 0\}$ is the bounded Reinhardt domain.

$$\begin{aligned}\|\phi_{\vec{\beta}}\|_D &= \int_D \phi_{\vec{\beta}} \wedge \overline{\phi}_{\vec{\beta}} \\ &= \int_D |z_1|^{2\beta_1} \ldots |z_n|^{2\beta_n} dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n \\ &= (4\pi)^n \underbrace{\int \cdots \int}_{\substack{n \\ \{\vec{r}:\, f(r_1,\ldots,r_n)<0\}}} \prod_{j=1}^n r_j^{2\beta_j+1}\, dr_1 \wedge \cdots \wedge dr_n.\end{aligned}$$

After computation, we get

$$\begin{aligned}\|\phi_{\vec{0}}\|^2_{\Delta_n} &= (4\pi^n)\int\cdots\int_{\Delta_n} r_1 r_2 \ldots r_n\, dr_1 \ldots dr_n = (2\pi)^n, \\ \|\phi_{\vec{e}_1}\|^2_{\Delta_n} &= (4\pi^n)\int\cdots\int_{\Delta_n} r_1^3 r_2 \ldots r_n\, dr_1 \ldots dr_n = \frac{1}{2}\cdot(2\pi)^n, \\ \|\phi_{\vec{\alpha}}\|^2_{\Delta_n} &= (4\pi^n)\int\cdots\int_{\Delta_n} r_1^5 r_2 \ldots r_n\, dr_1 \ldots dr_n = \frac{1}{3}\cdot(2\pi)^n, \\ \|\phi_{\vec{0}}\|^2_{B_n} &= (4\pi^n)\int\cdots\int_{B_n} r_1 r_2 \ldots r_n\, dr_1 \ldots dr_n = \frac{(4\pi)^n}{\prod_{i=1}^n 2i}, \\ \|\phi_{\vec{e}_1}\|^2_{B_n} &= (4\pi^n)\int\cdots\int_{B_n} r_1^3 r_2 \ldots r_n\, dr_1 \ldots dr_n = \frac{(4\pi)^n}{\prod_{i=1}^{n+1} 2i}, \\ \|\phi_{\vec{\alpha}}\|^2_{B_n} &= (4\pi^n)\int\cdots\int_{B_n} r_1^5 r_2 \ldots r_n\, dr_1 \ldots dr_n = \frac{(4\pi)^n}{\prod_{i=3}^{n+2} 2i}.\end{aligned}$$

For Δ_n and B_n are symmetric domains

$$\|\phi_{\vec{e}_i}\|_{\Delta_n} = \|\phi_{\vec{e}_j}\|_{\Delta_n}, \quad \|\phi_{\vec{e}_i}\|_{B_n} = \|\phi_{\vec{e}_j}\|_{B_n}, \quad \forall i, j \in \{1, \ldots, n\},$$

$$\|\phi_{\vec{\alpha}}\|_{\Delta_n} = \|\phi_{\tau(\vec{\alpha})}\|_{\Delta_n}, \quad \|\phi_{\vec{\alpha}}\|_{B_n} = \|\phi_{\tau(\vec{\alpha})}\|_{B_n}, \quad \forall \tau \in S_n,$$

$$g^{\tau}_{\Delta_n}(\vec{\alpha}) = g^{\mathrm{Id}}_{\Delta_n}(\vec{\alpha}) = \frac{\|\phi_{\vec{0}}\|_{\Delta_n}^{\sum \alpha_i - 1}\|\phi_{\vec{\alpha}}\|_{\Delta_n}}{\prod_{i=1}^{n}\|\phi_{\vec{e}_i}\|_{\Delta_n}^{\alpha_i}} = \frac{2\sqrt{3}}{3},$$

$$g^{\tau}_{B_n}(\vec{\alpha}) = g^{\mathrm{Id}}_{B_n}(\vec{\alpha}) = \frac{\|\phi_{\vec{0}}\|_{B_n}^{\sum \alpha_i - 1}\|\phi_{\vec{\alpha}}\|_{B_n}}{\prod_{i=1}^{n}\|\phi_{\vec{e}_i}\|_{B_n}^{\alpha_i}} = \frac{\sqrt{2(n+1)(n+2)}}{n+2}. \tag{5.1}$$

So $\sum_{\tau\in S_n} g^{\tau}_{\Delta_n}(\vec{\alpha})\cdot\tau \neq \sum_{\tau\in S_n} g^{\tau}_{B_n}(\vec{\alpha})\cdot\tau$ in $\mathbb{R}[S_n]/\sim$.

Then Δ_n is not biholomorphic to B_n by Theorem A. □

Example 5.2. In addition, we can show $A_n = \{z \in \mathbb{C}^n \mid \sum_{i=1}^{n}|z_n| < 1\}$ is not biholomorphic to B_n

$$\|\phi_{\vec{0}}\|_{A_n} = (4\pi^n)\int\cdots\int_{A_n} r_1 r_2 \dots r_n \, dr_1 \dots dr_n = \frac{(4\pi)^n}{\prod_{i=1}^{2n} i},$$

$$\|\phi_{\vec{e}_1}\|_{A_n} = (4\pi^n)\int\cdots\int_{A_n} r_1^3 r_2 \dots r_n \, dr_1 \dots dr_n = \frac{(4\pi)^n}{\prod_{i=4}^{2(n+1)} i},$$

$$\|\phi_{\vec{\alpha}}\|_{A_n} = (4\pi^n)\int\cdots\int_{A_n} r_1^5 r_2 \dots r_n \, dr_1 \dots dr_n = \frac{(4\pi)^n}{\prod_{i=6}^{2(n+2)} i}.$$

Using the same method, we get

$$g^{\tau}_{A_n}(\vec{\alpha}) = g^{\mathrm{Id}}_{A_n}(\vec{\alpha}) = \frac{\|\phi_{\vec{0}}\|_{A_n}^{\sum \alpha_i - 1}\|\phi_{\vec{\alpha}}\|_{A_n}}{\prod_{i=1}^{n}\|\phi_{\vec{e}_i}\|_{A_n}^{\alpha_i}} = \frac{\sqrt{5(2n+1)(2n+2)}}{\sqrt{3(2n+3)(2n+4)}}$$

So $\sum_{\tau\in S_n} g^{\tau}_{B_n}(\vec{\alpha})\cdot\tau \neq \sum_{\tau\in S_n} g^{\tau}_{A_n}(\vec{\alpha})\cdot\tau$ in $\mathbb{R}[S_n]/\sim$.

Then B_n is not biholomorphic to A_n by Theorem A.

Example 5.3. Δ_n is not biholomorphic to A_n.

According to the computation above,

$$\sum_{\tau\in S_n} g^{\tau}_{\Delta_n}(\vec{\alpha})\cdot\tau \neq \sum_{\tau\in S_n} g^{\tau}_{A_n}(\vec{\alpha})\cdot\tau \text{ in } \mathbb{R}[S_n]/\sim.$$

So Δ_n is not biholomorphic to A_n.

From the examples above, it is reasonable to suspect the problem lies with the fact that B_n has a C^∞ smooth boundary, whereas Δ_n and A_n have "corners" in its boundaries. However, Δ_n is not biholomorphic to A_n

though their boundaries both have "corners". So the biholomorphic equivalence problem of domains in $\mathbb{C}^n$ is very complicated even in the Reinhardt domain case.

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Department of Mathematics
East China Normal University
No. 500, Dongchuan Road
Shanghai 200241
P. R. of China
E-mail address: rdu@math.ecnu.edu.cn

Department of Mathematics
Shanghai Jiao Tong University
Shanghai 200240
P. R. of China
E-mail address: gaoyunmath@sjtu.edu.cn

Institute of Mathematics
East China Normal University
Shanghai
P. R. of China

Current address: Department of Mathematics
Statistics and Computer Science
SEO Rm 322, M/C 249
University of Illinois at Chicago
851 S. Morgan Street
Chicago, IL 60607-7045
USA
E-mail address: yau@uic.edu

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Dedicate to Professor Richard Melrose on the occasion of his 60th birthday.