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**SEVERAL SPLITTING CRITERIA FOR
VECTOR BUNDLES AND REFLEXIVE SHEAVES**

STEPHEN S.-T. YAU AND FEI YE

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In this paper, we show two splitting criteria for vector bundles on complex projective spaces by analytic method. We also prove a splitting criterion for reflexive sheaves on Horrocks schemes by algebraic method.

1. Introduction

Vector bundles are favored objects studied in algebraic geometry and commutative algebra. We say that a vector bundle splits if it is isomorphic to a direct sum of line bundles. A very interesting problem is whether there are nonsplitting vector bundles of small ranks on complex projective spaces. Although such vector bundles exist in lower dimensions, they seem to be extremely rare as the dimension increases. In fact, Hartshorne conjectured:

Conjecture 1.1 [Hartshorne 1974]. *If $n \geq 7$, all rank-2 vector bundles on the projective space \mathbb{P}^n split.*

Under some additional conditions, the conjecture was proved. However, the conjecture is still open. The most well-known condition is the vanishing of certain intermediate (local) cohomology groups. The first splitting criterion of this type is attributed to Horrocks.

Theorem 1.2 [Horrocks 1964]. *Let E be a vector bundle on the projective space \mathbb{P}^n with $n \geq 2$. Then E splits if and only if $H^i(\mathbb{P}^n, E(k)) = 0$ for all $k \in \mathbb{Z}$ and $1 \leq i \leq n - 1$.*

A standard proof is to apply induction to n and use the following so-called “restriction criterion”.

Theorem 1.3. *Let E be a rank- r vector bundle over \mathbb{P}^n , with $n \geq 3$. E splits if and only if its restriction $E|_H$ to some hyperplane $H \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$ splits.*

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Evans and Griffith improved Horrocks' criterion in the 1980's.

Theorem 1.4 [Evans and Griffith 1981]. *Let E be a vector bundle on the projective space \mathbb{P}^n of rank $r < n$. Then E splits if and only if $H^i(\mathbb{P}^n, E(k)) = 0$ for all $k \in \mathbb{Z}$ and $1 \leq i \leq r - 1$.*

Kumar, Peterson and Rao obtained another improvement of Horrocks' theorem.

Theorem 1.5 [Kumar et al. 2003]. *Let E be a vector bundle on \mathbb{P}^n . If $\text{rank } E < 2[n/2]$, then E splits if and only if $H^i(\mathbb{P}^n, E(k)) = 0$ for all $k \in \mathbb{Z}$ and $1 < i < n - 1$.*

Another type of splitting criteria involves extensibility of vector bundles. Let X be an algebraic variety and Y be a subvariety of X . A vector bundle E on Y is said to extend to X if there exists a vector bundle F on X such that $F|_Y = E$. Barth and van de Ven [1974] showed that a rank-2 vector bundle E on \mathbb{P}^n splits if and only if E extends to \mathbb{P}^N for all $N > n$. Their result was generalized to vector bundles of arbitrary rank by Sato [1977].

For any coherent sheaf \mathcal{F} , we denote the dual by $\mathcal{F}^* = \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$. The next theorem combines extensibility and vanishing of cohomology groups.

Theorem 1.6 [Kempf 1990]. *Let E be a vector bundle on the projective space \mathbb{P}^n with $n \geq 2$ and E^* be its dual. Then E splits if and only if the following two conditions are satisfied:*

- (1) E extends to \mathbb{P}^{n+1} .
- (2) $H^1(\mathbb{P}^n, E \otimes E^*(-k)) = 0$ for all positive integer k .

Proofs of the above mentioned theorems are all algebraic. However, the following remarkable criterion uses an analytic method.

Theorem 1.7 [Luk and Yau 1993]. *Let E be a holomorphic vector bundle on $\mathbb{C}\mathbb{P}^n$ with $n \geq 2$ and E^* be its dual. Then E splits if and only if $H^1(\mathbb{C}\mathbb{P}^n, E \otimes E^*(k))$ vanish for all $k \in \mathbb{Z}$.*

In this paper, we employ Luk and Yau's idea to provide analytic proofs of some splitting criteria of vector bundles on complex projective spaces.

Let $p : \mathbb{C}\mathbb{P}^{n+1} \setminus \{\xi\} \rightarrow \mathbb{C}\mathbb{P}^n$ be the projection from a point $\xi \in \mathbb{C}\mathbb{P}^{n+1} \setminus \mathbb{C}\mathbb{P}^n$. We prove the following theorems in this paper.

Theorem A. *A holomorphic vector bundle E on $\mathbb{C}\mathbb{P}^n$ splits if and only if p^*E extends to a vector bundle on $\mathbb{C}\mathbb{P}^{n+1}$.*

Theorem B. *Let E be a holomorphic vector bundle on $\mathbb{C}\mathbb{P}^n$. If $\text{rank } E < 2[n/2]$, then E splits if and only if the local cohomology groups $H_{\{\xi\}}^i(\mathbb{P}^{n+1}, \tilde{E}(k)) = 0$ for all $k \in \mathbb{Z}$ and $1 < i < n$, where \tilde{E} is the extension of p^*E on $\mathbb{C}\mathbb{P}^{n+1}$.*

Kumar observed that condition (2) in Theorem 1.6 implies that p^*E extends to a vector bundle on $\mathbb{C}\mathbb{P}^{n+1}$. Thus Theorem A implies the following theorem.

Theorem 1.8 [Kumar 2003]. *Let E be a holomorphic vector bundle on $\mathbb{C}\mathbb{P}^n$ with $n \geq 2$ and E^* be its dual. Then E splits if and only if $H^1(\mathbb{C}\mathbb{P}^n, E \otimes E^*(-k)) = 0$ for all positive integer k .*

On the other hand, generalizations of splitting criteria to reflexive sheaves and more general varieties have been obtained. Abe and Yoshinaga [2008] generalized the restriction criterion for reflexive sheaves on projective spaces. On the other hand, Bakhtary [2011] generalized the restriction criterion to Horrocks varieties.

Definition 1.9. A coherent sheaf \mathcal{F} on X is reflexive if $\mathcal{F} \cong \mathcal{F}^{**}$. It is normal if for every open set $U \subset X$ and every closed subset $Y \subset U$ of codimension ≥ 2 , the restriction map $\mathcal{F}(U) \rightarrow \mathcal{F}(U \setminus Y)$ is bijective. We define the singular locus of \mathcal{F} as $\text{Sing}(\mathcal{F}) := \{x \in X \mid \mathcal{F}_x \text{ is not locally free}\}$.

Definition 1.10. An algebraic variety X is called a splitting variety if $H^1(X, L) = 0$ for any line bundle L on X . A Horrocks variety is a splitting variety X with $H^2(X, L) = 0$ for any line bundle L on X .

In this paper, we prove a generalization of both the theorem of Bakhtary and the theorem of Abe and Yoshinaga.

Theorem C. *Let H be an effective ample divisor on a smooth projective variety X of dimension $\dim X \geq 4$. Assume that X is a Horrocks variety. Then a reflexive sheaf \mathcal{F} on X is splitting if and only if the restriction $\mathcal{F}|_H$ is splitting.*

Theorems A and B will be proved in Section 2 and Theorem C will be proved in the last section.

2. Splitting criteria via connections

Let E be a rank- r complex vector bundle over a complex n -dimensional manifold M . Denote by \mathcal{A}^q and $\mathcal{A}^q(E)$ the sheaves of smooth q -forms on M and smooth q -forms on M with coefficients in E respectively.

Definition 2.1. A connection on E is a \mathbb{C} -linear morphism

$$D : \mathcal{A}^0(E) \rightarrow \mathcal{A}^1(E)$$

such that for any open subset U of M

$$D(f\gamma) = df \otimes \gamma + fD(\gamma),$$

for any $f \in \Gamma(U, \mathcal{A}^0)$ and $\gamma \in \Gamma(U, \mathcal{A}^0(E))$.

A connection D of a vector bundle E localized over any open subset U is determined by a matrix $\omega = (\omega_j^k)$ of smooth 1-forms, called a connection matrix of D over U . It is well known that if the complex vector bundle E is holomorphic, then we have a connection D which can be decomposed into $D' + \bar{\partial}$ such that

$D' : \mathcal{A}^{(0,0)}(E) \rightarrow \mathcal{A}^{(1,0)}(E)$ and $\bar{\partial} : \mathcal{A}^{(0,0)}(E) \rightarrow \mathcal{A}^{(0,1)}(E)$, where $\mathcal{A}^{(p,q)}(E)$ are the sheaves of smooth (p, q) -forms with coefficients in E .

Definition 2.2. Let E be a holomorphic vector bundle over $\mathbb{C}^{n+1} \setminus \{0\}$ with a connection $D = D' + \bar{\partial}$. Denote by $(\sum_{j=0}^n \theta_{ij}^k dz_j)$ the connection matrix of D' . We say D is holomorphic in the direction z_l for some $0 \leq l \leq n$ if the θ_{il}^k are holomorphic for all $1 \leq i, k \leq r$.

Theorem 2.8 of [Luk and Yau 1993] says that the existence of such a connection will force the cohomology groups $H^i(\mathbb{C}^{n+1} \setminus \{0\}, E)$ to vanish for $0 < i < n$. To prove our theorems, we will look for such a connection.

Let L be the line in \mathbb{C}^{n+2} defined by $z_0 = z_1 = \dots = z_n = 0$ and $\xi = [0, \dots, 0, 1]$ be a point in $\mathbb{C}\mathbb{P}^{n+1} \setminus \mathbb{C}\mathbb{P}^n$. Set $X = \mathbb{C}^{n+2} \setminus L$ and $U = \mathbb{C}\mathbb{P}^{n+1} \setminus \{\xi\}$. Consider the projections $\pi : X \rightarrow U$, $\pi((z_0, \dots, z_{n+1})) = [z_0, \dots, z_{n+1}]$ and $p : U \rightarrow \mathbb{C}\mathbb{P}^n$, $p([z_0, \dots, z_{n+1}]) = [z_0, \dots, z_n]$. It is well known that p defines a line bundle structure on U . In fact, U is the total space of the line bundle $\mathbb{C}_{\mathbb{C}\mathbb{P}^n}(1)$.

Assume that E is a holomorphic vector bundle on $\mathbb{C}\mathbb{P}^n$. We claim that $\pi^* p^* E$ has a connection which is holomorphic in the direction z_{n+1} . Let $D = D' + \bar{\partial}$ be a connection on E . Assume that D' has a connection matrix $\omega^i = (\omega_\beta^\alpha)$ over $U_i = \{z_i \neq 0\} \subset \mathbb{C}\mathbb{P}^n$, and $\omega_\beta^\alpha = \sum_{j=0}^n \theta_{\beta j}^\alpha d(z_j/z_i)$. Pulling back E and the connection D to X , we see that $\pi^* p^* E$ admits a connection whose D' -part is defined by the connection matrix

$$\left(\sum_{j=0}^n (\theta_{\beta j}^\alpha \circ p \circ \pi) \frac{1}{z_i} dz_j - \sum_{j=0}^n (\theta_{\beta j}^\alpha \circ p \circ \pi) \frac{z_j}{(z_i)^2} dz_i \right),$$

which is clearly holomorphic in the direction z_{n+1} .

Now we are ready to prove Theorems A and B. The next result is a special case of Lemma 2.2 in [Luk and Yau 1993].

Lemma 2.3.
$$H^i(X, \pi^* p^* E) = \bigoplus_{k=-\infty}^{+\infty} H^i(U, p^* E(k)).$$

Proof. Notice that the projection $\pi : X \rightarrow U$ admits a bundle structure whose fibers are the punctured complex line \mathbb{C}^* . In fact, one can check that X is the total space of $\mathbb{C}_U(1)$ with the zero section U removed. Applying Lemma 2.2 of [Luk and Yau 1993], we get the equality. \square

Denote by $\iota : \mathbb{C}\mathbb{P}^{n+1} \setminus \{\xi\} \hookrightarrow \mathbb{C}\mathbb{P}^{n+1}$ the inclusion and $\tilde{E} = \iota_* p^* E$ the extension of E . The following proposition is the key to prove Theorems A and B.

Proposition 2.4. *Assume that the local cohomology groups*

$$H_{\{\xi\}}^i(\mathbb{C}\mathbb{P}^{n+1}, \tilde{E}(k)) = H_{\{\xi\}}^{i+1}(\mathbb{C}\mathbb{P}^{n+1}, \tilde{E}(k)) = 0,$$

for $0 < i < n$ and all $k \in \mathbb{Z}$. Then $H^i(X, \pi^* p^* E) = 0$ for $0 < i < n$.

Proof. By the assumption and the exact sequence of local cohomology

$$\begin{aligned}
 H^i_{\{\xi\}}(\mathbb{C}\mathbb{P}^{n+1}, \tilde{E}(k)) &\rightarrow H^i(\mathbb{C}\mathbb{P}^{n+1}, \tilde{E}(k)) \\
 &\rightarrow H^i(U, p^*E(k)) \rightarrow H^{i+1}_{\{\xi\}}(\mathbb{C}\mathbb{P}^{n+1}, \tilde{E}(k)),
 \end{aligned}$$

we see that

$$H^i(U, p^*E(k)) = H^i(\mathbb{C}\mathbb{P}^{n+1}, \tilde{E}(k)).$$

Hence $H^i(U, p^*E(k))$ is of finite dimension. By Serre vanishing theorem, there exists an integer N such that $H^i(\mathbb{C}\mathbb{P}^{n+1}, \tilde{E}(-k)) = 0$ for all integers i, k with $|k| \geq N$ and $1 \leq i \leq n$. Therefore there are only finitely many $H^i(U, p^*E(-k)) \neq 0$. Thus $H^i(X, \pi^*p^*E) = \bigoplus_{k=-N}^N H^i(U, p^*E(-k))$ is of finite dimension by Lemma 2.3.

We know that π^*p^*E admits a connection which is holomorphic in the direction z_{n+1} . Applying the same argument as in the proof of Theorem 2.8 in [Luk and Yau 1993], we conclude that $H^i(X, \pi^*p^*E) = 0$. □

Theorem A follows very easily from Proposition 2.4.

Theorem A. *Let E be a holomorphic vector bundle on $\mathbb{C}\mathbb{P}^n$. Then E splits if and only if p^*E extends to a vector bundle on $\mathbb{C}\mathbb{P}^{n+1}$.*

Proof. Assume that $E = \bigoplus_a \mathcal{O}_{\mathbb{C}\mathbb{P}^n}(a)$. Then $p^*\mathcal{O}_{\mathbb{C}\mathbb{P}^n}(a)$ extends uniquely to $\mathcal{O}_{\mathbb{C}\mathbb{P}^{n+1}}(a)$. Hence \tilde{E} is a vector bundle, moreover \tilde{E} splits. Conversely, assume that \tilde{E} is a vector bundle, then $\text{depth}_x \tilde{E} = \text{depth } \tilde{E}_x = \dim \mathcal{O}_x = n+1$. By [Hartshorne 1967, Proposition 1.4 and Theorem 3.8], we know that $H^i_{\{\xi\}}(\mathbb{C}\mathbb{P}^{n+1}, \tilde{E}(-k)) = 0$ for all $k \in \mathbb{Z}$ and $i \leq n$. By Proposition 2.4 and Lemma 2.3, we see that $H^i(U, p^*E(-k)) = 0$ for all $k \in \mathbb{Z}$ and $1 \leq i \leq n$. Hence $H^i(\mathbb{C}\mathbb{P}^{n+1}, \tilde{E}(-k)) = 0$ for all $k \in \mathbb{Z}$ and $1 \leq i \leq n$. It follows from Theorem 1.2 that \tilde{E} splits, and so does E . □

Another consequence of Proposition 2.4 is the following local cohomology version of Theorem 1.5 of Kumar, Peterson and Rao.

Theorem B. *Let E be a vector bundle on $\mathbb{C}\mathbb{P}^n$. If $\text{rank } E < 2[n/2]$, then E splits if and only if for all $k \in \mathbb{Z}$ and $1 < i < n$ the local cohomology groups $H^i_{\{\xi\}}(\mathbb{C}\mathbb{P}^{n+1}, \tilde{E}(k)) = 0$.*

Proof. It is clear that if E splits then $H^i_{\{\xi\}}(\mathbb{C}\mathbb{P}^{n+1}, \tilde{E}(k)) = 0$ for all $k \in \mathbb{Z}$ and $1 < i < n$. Assume that $H^i_{\{\xi\}}(\mathbb{C}\mathbb{P}^{n+1}, \tilde{E}(k)) = 0$ for all $k \in \mathbb{Z}$ and $1 < i < n$. By Proposition 2.4,

$$H^i(X, \pi^*p^*E) = \bigoplus_{k=-\infty}^{+\infty} H^i(U, p^*E(-k)) = 0$$

for $2 \leq i \leq n-2$. In particular, $H^i(U, p^*E) = \bigoplus_{k=-\infty}^{+\infty} H^i(\mathbb{C}\mathbb{P}^n, E(-k)) = 0$ for $2 \leq i \leq n-2$. By Theorem 1.5, we see that E splits. □

Another local version of [Theorem 1.5](#) was obtained as one of the main theorems in [\[Majidi-Zolbanin 2005\]](#).

3. Splitting of reflexive sheaves

The proof of [Theorem C](#) relies on the following two propositions.

Proposition 3.1. *Let \mathcal{F} be a reflexive sheaf on a smooth projective variety X and H be an effective ample divisor X . Assume that H is a splitting variety. If $\mathcal{F}|_H$ splits into a direct sum of line bundles, then*

$$H^1(X, \mathcal{F}(kH)) = 0 \quad \text{for all } k \in \mathbb{Z}.$$

Proof. By assumption, we have a surjective morphism

$$(*) \quad H^1(X, \mathcal{F}((k-1)H)) \rightarrow H^1(X, \mathcal{F}(kH)) \rightarrow 0.$$

By Serre duality, $H^1(X, \mathcal{F}(kH)) \cong \text{Hom}(\text{Ext}^{n-1}(\mathcal{F}(kH), \omega_X), k)$. It suffices to show that $\text{Ext}^{n-1}(\mathcal{F}(kH), \omega_X) = 0$ for $k \ll 0$. Consider the spectral sequence of local and global Ext functors

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(\mathcal{F}, \omega_X)) \Rightarrow E^{p+q} = \text{Ext}^{p+q}(\mathcal{F}, \omega_X).$$

Since $\mathcal{F}|_H$ is free, then $\text{Sing}(\mathcal{F}|_H) = \emptyset$, which implies that $\text{Sing}(\mathcal{F}) \cap H = \emptyset$. Note that the singular locus $\text{Sing}(\mathcal{F})$ is a closed subset of X and H is ample. If $\dim \text{Sing}(\mathcal{F}) = d > 0$, then $\text{Sing}(\mathcal{F}) \cdot H^d > 0$. In particular, $\text{Sing}(\mathcal{F}) \cap H \neq \emptyset$. Therefore, $\dim \text{Sing}(\mathcal{F}) = 0$. Since a coherent sheaf \mathcal{F} is free at a point $p \in X$ if and only if the stalk $(\mathcal{E}xt^q(\mathcal{F}, \mathcal{G}))_p = \text{Ext}^q(\mathcal{F}_p, \mathcal{G}_p) = 0$ for all $q > 0$ and any coherent sheaf \mathcal{G} , we see that $\dim \text{Supp}(\mathcal{E}xt^q(\mathcal{F}, \omega_X)) = 0$ for $q > 0$. Hence, $H^p(X, \mathcal{E}xt^q(\mathcal{F}, \omega_X)) = 0$ for $p > 0$ and $q > 0$. Now there are only two E^2 -terms $H^0(X, \mathcal{E}xt^{n-1}(\mathcal{F}, \omega_X))$ and $H^{n-1}(X, \mathcal{H}om(\mathcal{F}, \omega_X))$ which may contribute to $\text{Ext}^{n-1}(\mathcal{F}, \omega_X)$. Since \mathcal{F} is reflexive, then $\text{depth}_x \mathcal{F} \geq 2$ for all $x \in X$ by [\[Hartshorne 1980, Proposition 1.3\]](#). Thus $H^1_{\{x\}}(\mathcal{F}_x) = 0$. By local duality, we get $\mathcal{E}xt^{n-1}(\mathcal{F}_x, \omega_{X,x}) = H^1_{\{x\}}(\mathcal{F}_x) = 0$. Hence $\mathcal{E}xt^{n-1}(\mathcal{F}, \omega_X) = 0$. The spectral sequence then tells us that there is a surjective morphism

$$(**) \quad H^{n-1}(X, \mathcal{H}om(\mathcal{F}, \omega_X)) \twoheadrightarrow \text{Ext}^{n-1}(\mathcal{F}, \omega_X).$$

Since $H^{n-1}(X, \mathcal{F}^*(-kH) \otimes \omega_X) = 0$ for $-k \gg 0$ by Serre vanishing theorem, replacing \mathcal{F} by $\mathcal{F}(kH)$ in (**), we conclude that

$$\text{Ext}^{n-1}(\mathcal{F}(kH), \omega_X) = 0 \quad \text{for all } k \ll 0. \quad \square$$

Theorem 3.2. *Let \mathcal{E} be a vector bundle and \mathcal{F} be a reflexive sheaf over a smooth projective variety X of dimension $\dim X \geq 4$. Let H be an effective ample divisor on X . If $\mathcal{F}|_H \cong \mathcal{E}|_H$ and $H^1(X, \mathcal{E}^* \otimes \mathcal{F}(-H)) = 0$, then $\mathcal{F} \cong \mathcal{E}$.*

Proof. By the assumption, we see that $\mathcal{H}om(\mathcal{E}, \mathcal{F}) = \mathcal{E}^* \otimes \mathcal{F}$ is also reflexive. Since a reflexive sheaf is torsion free, then the following sequence is exact:

$$0 \rightarrow \mathcal{E}^* \otimes \mathcal{F}(-H) \rightarrow \mathcal{E}^* \otimes \mathcal{F} \rightarrow (\mathcal{E}^* \otimes \mathcal{F})|_H \rightarrow 0.$$

Since $H^1(X, \mathcal{E}^* \otimes \mathcal{F}(-H)) = 0$, then $H^0(X, \mathcal{E}^* \otimes \mathcal{F}) \rightarrow H^0(H, (\mathcal{E}^* \otimes \mathcal{F})|_H)$ is surjective. Note that $\text{Hom}(\mathcal{E}, \mathcal{F}) = H^0(X, \mathcal{H}om(\mathcal{E}, \mathcal{F}))$. Therefore, there is a morphism $\varphi: \mathcal{E} \rightarrow \mathcal{F}$ extending the isomorphism $\phi: \mathcal{E}|_H \rightarrow \mathcal{F}|_H$. We need to show that φ is an isomorphism. Consider $\det \varphi: \det \mathcal{E} \rightarrow \det \mathcal{F}$. Since $\mathcal{E}|_H \cong \mathcal{F}|_H$ and $\text{Pic}(X) = \text{Pic}(H)$ by Grothendieck–Lefschetz theorem, we conclude that $\det \mathcal{E} = \mathcal{O}_X(c_1(\mathcal{E})) = \mathcal{O}_X(c_1(\mathcal{F})) = \det \mathcal{F}$. Therefore $\det \varphi \in H^0(X, \det \mathcal{E}^* \otimes \det \mathcal{F}) = \mathbb{C}$ because a reflexive rank-1 sheaf is a line bundle (see [Okonek et al. 1980, Lemma 1.1.15]). Clearly, $\det \varphi$ is a nonzero constant, since $\det \phi$ is a nonzero constant. Thus at each $x \in X \setminus (\text{Sing}(\mathcal{E}^* \otimes \mathcal{F}))$, the morphism φ_x is an isomorphism. Since $\mathcal{E}^* \otimes \mathcal{F}$ is reflexive, then $\text{codim}(\text{Sing}(\mathcal{E}^* \otimes \mathcal{F})) \geq 3$ and hence φ is an isomorphism by [Hartshorne 1980, Proposition 1.6]. \square

Theorem C follows easily from Proposition 3.1 and Theorem 3.2.

Theorem C. *Let H be an effective ample divisor on a smooth projective variety X of dimension $\dim X \geq 4$. Assume that X is a Horrocks variety. Then a reflexive sheaf \mathcal{F} on X is splitting if and only if the restriction $\mathcal{F}|_H$ is splitting.*

Proof. By [Bakhtary 2011, Proposition 4.13], X is Horrocks if and only if X and all effective ample divisors are splitting. Clearly, if \mathcal{F} is splitting, then $\mathcal{F}|_H$ is splitting. Conversely, assume that $\mathcal{F}|_H$ is splitting. By Grothendieck–Lefschetz theorem, there is a splitting vector bundle \mathcal{E} on X such that $\mathcal{E}|_H = \mathcal{F}|_H$. By Proposition 3.1, we know that $H^1(X, \mathcal{E}^* \otimes \mathcal{F}(-H)) = 0$. Therefore, $\mathcal{E} \cong \mathcal{F}$ by Theorem 3.2. \square

In [Abe and Yoshinaga 2008], the authors also generalized Horrocks cohomology criterion for reflexive sheaves on projective spaces. However, Horrocks cohomology criterion may not hold on Horrocks varieties in general. There exist smooth hypersurfaces in \mathbb{P}^5 with nonsplit vector bundles satisfying the Horrocks cohomology condition (see Remarks in the introduction of [Kumar et al. 2007]). It will be very interesting to know under what cohomology conditions a vector bundle on a Horrocks variety splits.

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