SCIENCE CHINA
Mathematics

• ARTICLES •

January 2014 Vol. 57 No. 1: 31–41
doi: 10.1007/s11425-013-4657-2

On variance of exponents for isolated surface singularities with modality $\leqslant 2$

In Memory of Professor Philip Wagreich

YAU Stephen S. T.[1,*] & ZUO HuaiQing[2]

[1]*Department of Mathematical Science, Tsinghua University, Beijing* 100084*, China;*
[2]*Mathematical Sciences Center, Tsinghua University, Beijing* 100084*, China*

Email: yau@uic.edu, hqzuo@math.tsinghua.edu.cn

Received December 11, 2012; accepted December 19, 2012; published online September 10, 2013

Abstract Using the theory of the mixed Hodge structure one can define a notion of exponents of a singularity. In 2000, Hertling proposed a conjecture about the variance of the exponents of a singularity. Here, we prove that the Hertling conjecture is true for isolated surface singularities with modality $\leqslant 2$.

Keywords singularities, modality, exponents

MSC(2010) 32S25, 32S35

Citation: Yau S S T, Zuo H Q. On variance of exponents for isolated surface singularities with modality $\leqslant 2$. Sci China Math, 2014, 57: 31–41, doi: 10.1007/s11425-013-4657-2

1 Introduction

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function with an isolated singularity. Using Steenbrink's theory [17] of mixed Hodge structure on the cohomology of the Milnor fiber, we can define the *exponents* (or spectrum, up to the shift by one, in the terminology of Varchenko [21]) to be μ rational numbers $\{\alpha_1, \dots, \alpha_\mu\}$ such that $\exp(2\pi\mathrm{i}\sqrt{-1}\alpha_i)$ are the eigenvalues of the Milnor monodromy and their integral part is determined by the Hodge filtration of the mixed Hodge structure. This notion was first introduced by Steenbrink [17] and it is important discrete invariants of an isolated hypersurface singularity. Its main properties have been established by Steenbrink and Varchenko. They express the vanishing order (up to the shift by one) of the period integrals of holomorphic forms on vanishing cycles. See [20, 21]. It is known that the exponents are constant under μ-constant deformation of f. See [22]. In particular, they depend only on $f^{-1}(0)$.

In fact, the exponents are μ-rational numbers between 0 and $n + 1$. The variance measures the distribution of these numbers with respect to the middle point (see Proposition 2.2) and is defined by

$$\frac{1}{\mu}\sum_{i=1}^{\mu}\left(\alpha_i - \frac{n+1}{2}\right)^2,$$

where $\alpha_i, 1 \leqslant i \leqslant \mu$ are the exponents with $\alpha_1 \leqslant \cdots \leqslant \alpha_\mu$.

*Corresponding author

It came as a great surprise when Hertling, at the Summer Institute on Singularities, Newton Institute, Cambridge 2000, proposed the following conjecture.

Conjecture. Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function with an isolated singularity at 0, then

$$\frac{1}{\mu}\sum_{i=1}^{\mu}\left(\alpha_i - \frac{n+1}{2}\right)^2 \leqslant \frac{\alpha_\mu - \alpha_1}{12}.$$

This conjecture was supported at the time by the case of weighted homogeneous singularities where one has in fact equality (see [7] for a conceptual proof involving Frobenius manifolds and [6] for elementary proof based on some formulas in [17]). Soon after this, Saito [13] showed that the conjecture holds for all irreducible plane curve singularities. The conjecture was proved by Brelivet for curve singularities with nondegenerated Newton boundary (see [3]) and for all curve singularities (see [4]).

In this paper, we prove the Hertling conjecture is true for isolated surface singularities with modality $\leqslant 2$ (for the definition of modality see Definition 3.1).

Main Theorem. *Let $(V, 0)$ be an isolated hypersurface singularity in $\mathbb{C}^3$ defined by the zero set of a holomorphic function $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ with modality $\leqslant 2$, μ be the milnor number and $0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_\mu \leqslant 3$ be the exponents of $(V, 0)$. Then*

$$\frac{1}{\mu}\sum_{i=1}^{\mu}\left(\alpha_i - \frac{3}{2}\right)^2 \leqslant \frac{\alpha_\mu - \alpha_1}{12}.$$

2 Mixed Hodge structures and exponents

In this section, we review briefly the theory of the mixed Hodge structure and exponets.

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function with an isolated singularity at the origin. Then we can assume that f is a polynomial and $Y_0 := \overline{\{x \in \mathbb{C}^{n+1} : f(x) = 0\}} \subset \mathbb{P}^{n+1}$ is nonsingular away from 0. Define

$$Y := \overline{\{(x, t) \in \mathbb{C}^{n+1} \times S : f(x) = t\}} \subset \mathbb{P}^{n+1} \times S,$$

and

$$Y_t := Y \cap (\mathbb{P}^{n+1} \times \{t\}) \quad \text{for} \quad t \in S,$$

where $S := \{t \in \mathbb{C} : |t| < \eta\}$ and the number η is sufficiently small such that the projection $Y \to S$ is smooth away from 0.

Let $B := \{x \in \mathbb{C}^{n+1} : \|x\| < \epsilon\}$ be an open ball with radius ϵ. If ϵ and η are sufficently small, the projection

$$X := Y \cap (B \times S) \to S$$

is smooth away from 0 and its restriction over $S^* := S - \{0\}$ is a topological fibration and the topological type of a general fiber X_t does not depend on ϵ. If $i \neq 0, n$, $H^i(X_t, \mathbb{C}) = 0$ and $\dim_{\mathbb{C}} H^n(X_t, \mathbb{C})$ is called the Milnor number and is denoted by μ. Since $\bigcup_{t\in S^*} H^n(X_t, \mathbb{C})$ is a flat vector bundle, the monodromy transformation T acts on $H^n(X_t, \mathbb{C})$, and we call it the local monodromy.

In the same way T acts on $H^i(Y_t, \mathbb{C})$ and the restriction morphism $H^i(Y_t, \mathbb{C}) \to H^i(Y_t, \mathbb{C})$ is equivariant with respect to T.

Let S_∞ be a universal covering space of S^* and set $X_\infty := X \times_S S_\infty$ and $Y_\infty := Y \times_S S_\infty$.

According to Deligne and Steenbrink [17], we can put mixed Hodge structures on $\tilde{H}^i(Y_0, \mathbb{C})$, $H^i(Y_\infty, \mathbb{C})$ and $H^i(X_\infty, \mathbb{C})$ such that

$$\cdots \to \tilde{H}^i(Y_0, \mathbb{C}) \to H^i(Y_\infty, \mathbb{C}) \to H^i(X_\infty, \mathbb{C}) \to \tilde{H}^{i+1}(Y_0, \mathbb{C}) \to \cdots$$

is an exact sequence of mixed Hodge structures, where $\tilde{H}^i$ is the reduced cohomology.

Let $(H^i(X_\infty, \mathbb{C}), F, W)$ be the mixed Hodge structure on $H^i(X_\infty, \mathbb{C})$. For $\lambda \in \mathbb{C}$, we put

$$H^i(X_\infty, \mathbb{C})_\lambda := \{u \in H^i(X_\infty, \mathbb{C}) : (T - \lambda)^k u = 0, \exists\, k \in \mathbb{Z}_+\},$$

$$H_\lambda^{p,q} := \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H^n(X_\infty, \mathbb{C})_\lambda, \quad h_\lambda^{p,q} := \dim_{\mathbb{C}} H_\lambda^{p,q}.$$

Since the direct decomposition $H^n(X_\infty) = \bigoplus_\lambda H^n(X_\infty)_\lambda$ is compatible with both filtrations, we have $\sum_{\lambda,p,q} h_\lambda^{p,q} = \mu$ $(= \dim_{\mathbb{C}} H^n(X_\infty))$. Due to the monodromy theorem, if $h_\lambda^{p,q} \neq 0$, λ is a root of unity.

Definition 2.1 (See [17])**.** Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function with an isolated singularity at the origin. The exponents of f are μ-rational numbers $\{\alpha_1, \ldots, \alpha_\mu\}$ such that

$$0 < \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_\mu < n+1$$

and are defined by the following condition:

$$\forall\, \lambda \in \mathbb{C}, \forall\, p \in \mathbb{Z}, \lambda \neq 1 \Rightarrow \sharp\{j : e(-\alpha_j) = \lambda, [\alpha_j] = n - p\} = \sum_q h_\lambda^{p,q},$$
$$\lambda = 1 \Rightarrow \sharp\{j : \alpha_j = n - p + 1\} = \sum_q h_1^{p,q},$$

where $[\alpha] = \max\{i \in \mathbb{Z} : i \leqslant \alpha\}$, $e(\alpha) = \exp(2\pi \mathrm{i} \alpha)$. This is well defined because of $\sum_{\lambda,p,q} h_\lambda^{p,q} = \mu$.

Let $\chi_f(t) = \sum_{1 \leqslant i \leqslant \mu} t^{\alpha_i}$ which is also called the characteristic function of spectrum of f by a recent terminology.

Proposition 2.2 (See [17])**.** *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function with an isolated singularity, and $0 < \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_\mu < n+1$ be the exponents of f. Then the exponents are symmetric with center $(n+1)/2$, i.e., $\alpha_i + \alpha_{\mu+1-i} = n+1$.*

Remark 2.3. By the symmetry of the exponents, we have

$$\chi_f(t) = t^{n+1} \chi_f(t^{-1}).$$

Example 2.4. Let f be a quasihomogeneous polynomial with weight $(w_0, \ldots, w_n)$, i.e., f is a linear combination of monomials $x_0^{m_0} \cdots x_n^{m_n}$ such that $\sum w_i m_i = 1$. Then

$$\Omega_f := \Omega_{X,0}^{n+1} / df \wedge \Omega_{X,0}^n$$

is a graded μ-dimensional vector space whose grading is induced by the weight of the coordinates and is indexed by $\mathbb{Q}$. Let $P(\Omega_f, t)$ be the Poincaré polynomial of Ω_f. By [18], we have

$$P(\Omega_f, t) = \chi_f(t).$$

Furthermore $P(\Omega_f, t)$ can be written explicitly in terms of the weights, and we get the well-known formula (see [18])

$$\chi_f(t) = \prod_{i=0}^n \frac{t^{w_i} - t}{1 - t^{w_i}}.$$

This formula is generalized to the nondegenerate Newton polyhedron case (see [15, 17]) by using the Newton filtration on f.

3 Classification of critical points

The critical points of a holomorphic function are the points where the differential vanishes. A critical point is nondegenerate if the second differential is a nondegenerate quadratic form. In some neighbourhood of a nondegenerate critical point the function can be represented in the Morse normal form

$$f = \pm x_1^2 \pm \cdots \pm x_n^2 + \mathrm{const}$$

using suitable local coordinates.

Every degenerate critical point bifurcates into some nondegenerate points after an arbitrarily small deformation ("morsification"). So generically, functions have no degenerate critical points.

Degenerate critical points appear naturally when the function depends upon parameters. For example, the function $f(x) = x^3 - tx$ has a degenerate critical point for the value $t = 0$ of the parameter. Every family of functions close enough to this one-parameter family has a similar degenerate critical point for some small value of the parameter,

When the parameters are few, only the simplest degeneracies appear generically, and one can explicitly list them, giving normal forms for functions and families. When the number of parameters increases, more complicated degeneracies appear, and their classification seems hopeless.

Let f be a germ of a holomorphic function at a critical point 0. The Milnor number μ of the critical point can also be defined as the number of nondegenerate critical points to which 0 bifurcates after a morsification (compare the definition in Subsection 2.1).

Two germs of analytic functions are *equivalent* if one of them can be transformed into the other by a local biholomorphism of the domain space. In fact, every germ of an analytic function at a critical point of finite Milnor number (i.e., isolated critical point) is equivalent to a germ of a polynomial.

So, the classification problem for isolated critical points is reduced to an algebraic problem dealing with linear action of a Lie group on finite dimensional space of jets. The first step in solving this algebraic problem was taken by Thom [19], Mather [9–11], and Siersma [16].

Let G be a Lie group, X a smooth manifold, and $G \times X \to X$ an action of the group on the manifold. We consider any point x of X. The orbits of the action of G on X may form a finite stratification or a continuous family in the neighbourhood of x. The modality of a point x is the number of parameters necessary for numbering the orbits in the neighbourhood of x.

Definition 3.1. The modality of a point $x \in X$ under the action of G on X is the smallest number m such that a sufficiently small neighbourhood of x in X intersects at most finitely many at most m-parameter families of orbits of G.

The point x is called *simple* if its modality is 0, i.e., if some neighbourhood of x intersects only a finite number of orbits. The modality of the germ of a function at a critical point is the modality of its sufficient jet in the space of jets of functions with critical point 0 and critical value 0.

Two germs of holomorphic functions (of possibly different numbers of variables) are said to be *stably equivalent* if they become equivalent after taking their direct sum with nondegenerate quadratic forms; the stable equivalence of f and g means the ordinary equivalence

$$f(x_1, \dots, x_k) + x_{k+1}^2 + \cdots + x_n^2 \sim g(y_1, \dots, y_l) + y_{l+1}^2 + \cdots + y_n^2.$$

Theorem 3.2 (See [1])**.** *Up to stable equivalence, simple germs of holomorphic functions are exactly the following germs*:

$$A_k : x^{n+1}, \quad D_k : x^2y + y^{n-1}, \quad E_6 : x^3 + y^4, \quad E_7 : x^3 + xy^3, \quad E_8 : x^3 + y^5.$$

Theorem 3.3 (See [2])**.** *Unimodal germs* (*that is, germs of modality $m = 1$*) *of holomorphic functions belong* (*up to stable equivalence*) *either to the following series of one-parameter families of functions*:

$$T_{p,q,t} : f(x, y, z) = axyz + x^p + y^q + z^t, \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{t} < 1, \quad a \neq 0,$$

or to one of the following three families:

$$\begin{aligned} P_8 &:= T_{3,3,3} : f(x, y, z) = x^3 + y^3 + z^3 + axyz, \quad a^3 + 27 \neq 0, \\ X_9 &:= T_{2,4,4} : f(x, y, z) = x^4 + y^4 + z^2 + ax^2y^2, \quad a^2 \neq 4, \\ J_{10} &:= T_{2,3,6} : f(x, y, z) = x^3 + y^6 + z^2 + ax^2y^2, \quad 4a^3 + 27 \neq 0, \end{aligned}$$

or to one of the fourteen "exceptional" one-parameter families, given in one of the following 14 *exceptional one-parameter families* (*throughout, the parameter a runs over* $\mathbb{C}$):

$$Q_{10} : x^2z + y^3 + ayz^3 + z^4, \quad Q_{11} : x^2z + y^3 + yz^3 + az^5, \quad Q_{12} : x^2z + y^3 + ayz^4 + z^5,$$

$$S_{11}: x^2z + yz^2 + y^4 + ay^3z, \quad S_{12}: x^2z + yz^2 + xy^3 + ay^5, \quad U_{12}: x^3 + y^3 + z^4 + 3axyz^2,$$
$$Z_{11}: x^3y + 3axy^4 + y^5, \quad Z_{12}: x^3y + 3xy^4 + ay^6, \quad Z_{13}: x^3y + 3axy^5 + y^6,$$
$$W_{12}: x^3 + y^5 + 2ax^2y^3, \quad W_{13}: x^4 + 4xy^4 + ay^6, \quad E_{12}: x^3 + axy^5 + y^7,$$
$$E_{13}: x^3 + axy^5 + y^8, \quad E_{14}: x^3 + xy^6 + ay^8.$$

Theorem 3.4 (See [2]). *Bimodal germs (that is, germs of modality $m = 2$) of holomorphic functions belong (up to stable equivalence) either to the following 8 infinite series and 14 exceptional families. Let $a = a_0 + a_1y$.*

I) *Infinite series of bimodal singularities of corank 2:*

$$J_{3,0}: x^3 + bx^2y^3 + y^9 + cxy^7, \quad 4b^3 + 27 \neq 0, \quad \mu = 16,$$
$$J_{3,P}: x^3 + x^2y^3 + ay^{9+p}, \quad p > 0, \quad a_0 \neq 0, \quad \mu = 16 + p,$$
$$Z_{1,0}: x^3y + dx^2y^3 + cxy^6 + y^7, \quad 4d^3 + 27 \neq 0, \quad \mu = 15,$$
$$Z_{1,p}: x^3y + x^2y^3 + ay^{7+p}, \quad p > 0, \quad a_0 \neq 0, \quad \mu = 15 + p,$$
$$W_{1,0}: x^4 + ax^2y^3 + ay^6, \quad a_0^2 \neq 4, \quad \mu = 15,$$
$$W_{1,p}: x^4 + x^2y^3 + ay^{6+p}, \quad p > 0, \quad a_0 \neq 0, \quad \mu = 15 + p,$$
$$W^\sharp_{1,2q-1}: (x^2 + y^3)^2 + axy^{4+q}, \quad q > 0, \quad a_0 \neq 0, \quad \mu = 15 + 2q - 1,$$
$$W^\sharp_{1,2q}: (x^2 + y^3)^2 + ax^2y^{3+q}, \quad q > 0, \quad a_0 \neq 0, \quad \mu = 15 + 2q.$$

II) *Infinite series of bimodal singularities of corank 3:*

$$Q_{2,0}: x^3 + yz^2 + ax^2y^2 + xy^4, \quad a_0^2 \neq 4, \quad \mu = 14,$$
$$Q_{2,P}: x^3 + yz^2 + x^2y^2 + ay^{6+p}, \quad p > 0, \quad a_0 \neq 0, \quad \mu = 14 + p,$$
$$S_{1,0}: x^2z + yz^2 + y^5 + azy^3, \quad a_0^2 \neq 4, \quad \mu = 14,$$
$$S_{1,p}: x^2z + yz^2 + x^2y^2 + ay^{5+p}, \quad p > 0, \quad a_0 \neq 0, \quad \mu = 14 + p,$$
$$S^\sharp_{1,2q-1}: x^2z + yz^2 + zy^3 + axy^{3+p}, \quad q \neq 0, \quad a_0 \neq 0, \quad \mu = 14 + 2q - 1,$$
$$S^\sharp_{1,2q}: x^2z + yz^2 + zy^3 + ax^2y^{2+q}, \quad q > 0, \quad a_0 \neq 0, \quad \mu = 14 + 2q,$$
$$U_{1,0}: x^3 + xz^2 + xy^3 + ay^3z, \quad a_0(a_0^2 + 1) \neq 0, \quad \mu = 14,$$
$$U_{1,2q-1}: x^3 + xz^2 + xy^3 + ay^{1+q}z^2, \quad q > 0, \quad a_0 \neq 0, \quad \mu = 14 + 2q - 1,$$
$$U_{1,2q}: x^3 + xz^2 + xy^3 + ay^{3+q}z, \quad q > 0, \quad a_0 \neq 0, \quad \mu = 14 + 2q.$$

III) 14 *exceptional families*:

$$E_{18}: x^3 + y^{10} + axy^7, \quad E_{19}: x^3 + xy^7ay^{11},$$
$$E_{20}: x^3 + y^{11} + axy^8, \quad Z_{17}: x^3y + y^8 + axy^6,$$
$$Z_{18}: x^3y + xy^6 + ay^9, \quad Z_{19}: x^3y + y^9 + axy^7,$$
$$W_{17}: x^4 + xy^5 + ay^7, \quad W_{18}: x^4 + y^7 + ax^2y^4,$$
$$Q_{16}: x^3 + yz^2 + y^7 + axy^5, \quad Q_{17}: x^3 + yz^2 + xy^5 + ay^8,$$
$$Q_{18}: x^3 + yz^2 + y^8 + axy^6, \quad S_{16}: x^2z + yz^2 + xy^4 + ay^6,$$
$$S_{17}: x^2z + yz^2 + y^6 + azy^4, \quad U_{16}: x^3 + xz^2 + y^5 + ax^2y^2.$$

4 Proof of the main theorem

Let $(V, 0)$ be an isolated hypersurface singularity in $\mathbb{C}^3$ defined by the zero set of a holomorphic function $f: (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$. Then we give the following Tables 1–3, compiled by Goryunov [8], of the exponents of

Table 1 Exponents of simple singularities (i.e., singularities with modality 0)

Class	N	(L_r)	Class	N	(L_r)
A_μ	$\mu+1$	$\mu+1+k,\ 1\leqslant k\leqslant\mu$	E_6	12	13 16 17
D_μ	$2\mu-2$	$3\mu-3\ \ 2\mu-1+2k$	E_7	18	19 23 25
		$0\leqslant k\leqslant\mu-2$	E_8	30	31 37 41 43

Table 2 Exponents of unimodal singularities

Class	N	(L_r)	Class	N	(L_r)
P_8	3	3 4 4 4	Z_{12}	22	21 25 27 29 31 33
X_9	4	4 5 5 6	Z_{18}	18	17 20 22 23 25 26
J_{10}	6	6 7 8 8 9	W_{12}	20	19 23 24 27 28 29
Q_{10}	24	23 29 31 32 35	W_{13}	16	15 18 19 21 22 23
Q_{11}	18	17 21 23 24 25	E_{12}	42	41 47 53 55 59 61
Q_{12}	15	14 17 19 20 20 22	E_{13}	30	29 33 37 39 41 43
S_{11}	16	15 19 20 21 23	E_{14}	24	23 26 29 31 32 34 35
S_{12}	13	12 15 16 17 18 19	$T_{p,q,t}$	pqt	$pqt\ 2pqt\ (p+k_1)qt$
U_{12}	12	11 14 15 15 17 18			$(q+k_2)pt\ \ (t+k_3)pq$
Z_{11}	30	29 35 37 41 45			$1\leqslant k_1<p,\ \ 1\leqslant k_2<q,\ \ 1\leqslant k_3<t$

Table 3 Exponents of bimodal singularities

Class	N	(L_r)
$J_{3,p}$	$18(p+9)$	$17(p+9)\ \ 23(p+9)\ \ 25(p+9)\ \ 9(2p+17+2k),\ 1\leqslant k\leqslant(p+10)/2$
$Z_{1,p}$	$14(p+7)$	$13(p+7)\ \ 17(p+7)\ \ 19(p+7)\ \ 21(p+7)\ \ 7(2p+13+2k),\ 1\leqslant k\leqslant(p+7)/2$
$W_{1,p}$	$12(p+6)$	$11(p+6)\ \ 14(p+6)\ \ 16(p+6)\ \ 17(p+6)\ \ 6(2p+11+2k),\ 1\leqslant k\leqslant(p+7)/2$
$W^\sharp_{1,p}$	$12(p+12)$	$11(p+12)\ \ 17(p+12)\ \ 12(p+12+k),\ 1\leqslant k\leqslant(p+11)/2$
$Q_{2,p}$	$12(p+6)$	$15(p+6)\ \ 16(p+6)\ \ 17(p+6)\ \ 6(2p+11+2k),\ 1\leqslant k\leqslant(p+6)/2$
$S_{1,p}$	$10(p+5)$	$9(p+5)\ \ 12(p+5)\ \ 13(p+5)\,14(p+5)\ \ 5(2p+9+2k),\ 1\leqslant k\leqslant(p+6)/2$
$S^\sharp_{1,p}$	$10(p+10)$	$9(p+10)\ \ 13(p+10)\ \ 10(p+10+k),\ 1\leqslant k\leqslant(p+10)/2$
$U_{1,p}$	$9(p+9)$	$8(p+9)\ \ 11(p+9)\ \ 13(p+9)\ \ 9(p+9+k),\ 1\leqslant k\leqslant(p+8)/2$

Class	N	(L_r)	Class	N	(L_r)
Q_{16}	21	19 22 25 26 28 28 29 31	W_{18}	28	25 29 32 33 36 37 39 40 41
Q_{17}	30	27 31 35 37 39 40 41 43	Z_{17}	24	22 25 28 29 31 32 34 35
Q_{18}	48	43 49 55 59 61 64 65 67 71	Z_{18}	34	31 35 39 41 43 45 47 49 51
S_{16}	17	15 18 20 21 22 23 24 25	Z_{19}	54	49 55 61 65 67 71 73 77 79
S_{17}	24	21 25 28 29 31 32 33 35	E_{18}	30	28 31 34 37 38 40 41 43 44
U_{16}	15	13 16 18 18 19 21 21 22	E_{19}	42	39 43 47 51 53 55 57 59 61
W_{17}	20	18 21 23 24 26 27 28 29	E_{20}	66	61 67 73 79 83 85 89 91 95 97

simple, unimodal and bimodal critial points for $n=3$. In these tables, there is indicated for each point the numbers N, L_r; the exponents $\{\alpha_r\}$ are given by the formula

$$\alpha_r = L_r/N.$$

In view of the symmetry relative to the number $\frac{3}{2}$ all the exponents, except the exponents of the points $A_\mu, D_\mu, T_{p,q,t}$, are written up to the half-way mark, that is for $r\leqslant\mu/2$.

Let $w=(w_1,\dots,w_n)$ be an n-tuple of positive rational numbers. A polynomial $f(z_1,\dots,z_n)$ is said to be a weighted homogeneous polynomial with weights w if each monomial $\alpha z_1^{a_1}z_2^{a_2}\cdots z_n^{a_n}$ of f satisfies $a_1/w_1+\cdots+a_n/w_n=1$. Recall that a polynomial $f(z_1,\dots,z_n)$ is called quasi-homogeneous if f is in the Jacobian ideal of f i.e., $f\in(\partial f/\partial z_1,\dots,\partial f/\partial z_n)$. By a theorem of Saito [12], if f is quasihomogeneous with an isolated critical point at 0, then after a biholomorphic change of coordinates, f becomes a weighted homogeneous polynomial.

Recall that Hertling proved the following beautiful theorem.

Theorem 4.1 (See [7]). *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of a quasihomogeneous polynomial with an isolated singularity at 0. Then the vairance is*

$$\frac{1}{\mu}\sum_{i=1}^{\mu}\left(\alpha_i - \frac{n+1}{2}\right)^2 = \frac{\alpha_\mu - \alpha_1}{12}.$$

The following important theorem is due to Varchenko [21] and can be used to simplify the calculation in the proof of our main theorem.

Theorem 4.2 (See [22]). *The exponents are constant under a deformation of isolated hypersurface singularities with constant Milnor number (under a μ-const deformation).*

Proof of Main Theorem. Since simple singularities are quasihomogeneous singularities, by Theorem 4.1, the main theorem is true for all simple singularities (i.e., $A_\mu, D_\mu, E_6, E_7, E_8$). Except for $T_{p,q,t}$, $J_{3,p}$, $Z_{1,p}, W_{1,p}, W^\sharp_{1,p}, Q_{2,p}, S_{1,p}, S^\sharp_{1,p}, U_{1,p}$ the other cases can be deformed to quasihomogeneous singularities without change the constant Milnor. By Theorems 4.1 and 4.2, the main theorem is true in these cases (in fact, one can also check directly and see that the inequality in the main theorem becomes equality for all these cases). Furthermore, $Z_{1,p}, W_{1,p}, W^\sharp_{1,p}, Q_{2,p}$ are curve singularities, and the main theorem is true for curve singularities [4,13]. So we only need to check the remaining $T_{p,q,t}, J_{3,p}, S_{1,p}, S^\sharp_{1,p}, U_{1,p}$ case by case.

Case 1. $T_{p,q,t}$.

It is well known that the Milnor number $\mu = p+q+t-1$ for $T_{p,q,t}$. By Proposition 2.2, i.e., symmetry of spectrum, and Table 2 of exponents of $T_{p,q,t}$ above, we have

$$\begin{aligned}
D_{T_{p,q,t}} &:= \frac{1}{\mu}\sum_{i=1}^{\mu}\left(\alpha_i - \frac{3}{2}\right)^2 - \frac{\alpha_\mu - \alpha_1}{12} = \frac{1}{p+q+t-1}\sum_{i=1}^{p+q+t-1}\left(\alpha_i - \frac{3}{2}\right)^2 - \frac{2-1}{12} \\
&= \frac{1}{p+q+t-1}\left[\left(1-\frac{3}{2}\right)^2 + \sum_{k_1=1}^{p-1}\left(\frac{k_1}{p} - \frac{1}{2}\right)^2 + \sum_{k_2=1}^{q-1}\left(\frac{k_2}{q} - \frac{1}{2}\right)^2 + \sum_{k_3=1}^{t-1}\left(\frac{k_1}{t} - \frac{1}{2}\right)^2 \right. \\
&\quad \left. + \left(2 - \frac{3}{2}\right)^3\right] - \frac{1}{12} \\
&= \frac{1}{p+q+t-1}\left[\frac{1}{2} + \sum_{k_1=1}^{p-1}\left(\frac{k_1^2}{p^2} - \frac{k_1}{p} + \frac{1}{4}\right) + \sum_{k_2=1}^{p-1}\left(\frac{k_2^2}{p^2} - \frac{k_2}{p} + \frac{1}{4}\right) + \sum_{k_3=1}^{p-1}\left(\frac{k_3^2}{p^2} - \frac{k_3}{p} + \frac{1}{4}\right)\right] - \frac{1}{12} \\
&= \frac{1}{p+q+t-1}\left[\frac{p+q+t-1}{4} - \frac{p+q+t-3}{2} + \sum_{k_1=1}^{p-1}\frac{k_1^2}{p^2} + \sum_{k_2=1}^{q-1}\frac{k_2^2}{q^2} + \sum_{k_3=1}^{t-1}\frac{k_1^2}{t^2}\right] - \frac{1}{12} \\
&= \frac{1}{p+q+t-1}\left[\frac{5-(p+q+t)}{4} + \frac{(p-1)(2p-1)}{6p} + \frac{(q-1)(2q-1)}{6q} + \frac{(r-1)(2r-1)}{6r}\right] - \frac{1}{12} \\
&= \frac{1}{12(p+q+t-1)}\left[15 - 3(p+q+t) + \frac{2(2p^2-3p+1)}{p} + \frac{2(2q^2-3q+1)}{q} + \frac{2(2t^2-3t+1)}{t} \right. \\
&\quad \left. - (p+q+t) + 1\right] \\
&= \frac{1}{6(p+q+t-1)}\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{t} - 1\right) < 0,
\end{aligned}$$

where the last "<" follows from the defintion of $T_{p,q,t}$ which needs $\frac{1}{p} + \frac{1}{q} + \frac{1}{t} < 1$ and $p+q+t > 1$. Therefore the main theorem is true in this case.

Case 2. $J_{3,p}$.

The Milnor number $\mu = 16 + p$ for $J_{3,p}$ (see Theorem 3.4). By symmetry of spectrum (see Proposition 2.2)) and Table 3 of exponents of $J_{3,p}$, we have

$$D_{J_{3,p}} := \frac{1}{\mu}\sum_{i=1}^{\mu}\left(\alpha_i - \frac{3}{2}\right)^2 - \frac{\alpha_\mu - \alpha_1}{12} = \frac{1}{16+p}\sum_{i=1}^{10+p}\left(\alpha_i - \frac{3}{2}\right)^2 - \frac{2(\frac{3}{2} - \frac{17(p+9)}{18(p+9)})}{12}$$

$$= \frac{2}{16+p}\Bigg[\left(\frac{17(p+9)}{18(p+9)} - \frac{3}{2}\right)^2 + \left(\frac{23(p+9)}{18(p+9)} - \frac{3}{2}\right)^2 + \left(\frac{25(p+9)}{18(p+9)} - \frac{3}{2}\right)^2$$

$$+ \sum_{k=1}^{[\frac{10+p}{2}]}\left(\frac{9(2p+17+2k)}{18(p+9)} - \frac{3}{2}\right)^2\Bigg] - \frac{5}{54}$$

(where $[\frac{10+p}{2}]$ is Gauss symbol, i.e., the integral part of $\frac{10+p}{2}$)

$$= \frac{2}{16+p}\Bigg[\frac{10}{27} + \sum_{k=1}^{[\frac{10+p}{2}]}\left(\frac{(2k-1)}{2(p+9)} - \frac{1}{2}\right)^2\Bigg] - \frac{5}{54}$$

$$= \frac{2}{16+p}\Bigg[\frac{10}{27} + \sum_{k=1}^{[\frac{10+p}{2}]}\left(\frac{(2k-1)^2}{4(p+9)^2} - \frac{2k-1}{2(p+9)} + \frac{1}{4}\right)\Bigg] - \frac{5}{54}$$

$$= \frac{2}{16+p}\left(\frac{10}{27} + \frac{1}{4}\left[\frac{10+p}{2}\right] + \sum_{k=1}^{[\frac{10+p}{2}]}\frac{(2k-1)^2}{4(p+9)^2} - \sum_{k=1}^{[\frac{10+p}{2}]}\frac{2k-1}{2(p+9)}\right) - \frac{5}{54}$$

$$= \frac{2}{16+p}\left(\frac{10}{27} + \frac{1}{4}\left[\frac{10+p}{2}\right] + \frac{[\frac{10+p}{2}](2[\frac{10+p}{2}]-1)(2[\frac{10+p}{2}]+1)}{12(p+9)^2} - \frac{[\frac{10+p}{2}]^2}{2(p+9)}\right) - \frac{5}{54}.$$

If p is even, then the right-hand side of the last formula for $D_{J_{3,p}}$

$$= \frac{2}{16+p}\left(\frac{10}{27} + \frac{10+p}{8} + \frac{(p+10)(p+11)}{24(p+9)} - \frac{(p+10)^2}{8(p+9)}\right) - \frac{5}{54}$$

$$= \frac{2}{16+p}\left(\frac{10}{27} + \frac{10+p}{8} + \frac{(p+10)(p+11)}{24(p+9)} - \frac{(p+10)^2}{8(p+9)} - \frac{5}{54}\frac{16+p}{2}\right)$$

$$= \frac{-p(p+8)}{108(p+16)(p+9)} \leqslant 0.$$

Therefore the main theorem is true in this subcase and the equality holds if $p = 0$. If p is odd, then the right-hand side of the last formula for $D_{J_{3,p}}$

$$= \frac{2}{16+p}\left(\frac{10}{27} + \frac{9+p}{8} + \frac{(p+8)(p+10)}{24(p+9)} - \frac{(p+9)^2}{8(p+9)}\right) - \frac{5}{54}$$

$$= \frac{2}{16+p}\left(\frac{10}{27} + \frac{9+p}{8} + \frac{(p+8)(p+10)}{24(p+9)} - \frac{(p+9)^2}{8(p+9)} - \frac{5}{54}\frac{16+p}{2}\right)$$

$$= \frac{-p(p+8)}{108(p+16)(p+9)} < 0.$$

Therefore the main theorem is true for $J_{3,p}$.

Case 3. $S_{1,p}$.

The Milnor number $\mu = 14 + p$ for $S_{1,p}$ (see Theorem 3.4). By symmetry of spectrum and Table 3 of exponents of $S_{1,p}$, we have

$$D_{S_{1,p}} := \frac{1}{\mu}\sum_{i=1}^{\mu}\left(\alpha_i - \frac{3}{2}\right)^2 - \frac{\alpha_\mu - \alpha_1}{12} = \frac{1}{14+p}\sum_{i=1}^{14+p}\left(\alpha_i - \frac{3}{2}\right)^2 - \frac{2(\frac{3}{2} - \frac{9(p+5)}{10(p+5)})}{12}$$

$$= \frac{2}{14+p}\Bigg[\left(\frac{9(p+5)}{10(p+5)} - \frac{3}{2}\right)^2 + \left(\frac{12(p+5)}{10(p+5)} - \frac{3}{2}\right)^2 + \left(\frac{13(p+5)}{10(p+5)} - \frac{3}{2}\right)^2 + \left(\frac{14(p+5)}{10(p+5)} - \frac{3}{2}\right)^2$$

$$+\sum_{k=1}^{[\frac{p+6}{2}]}\left(\frac{5(2p+9+2k)}{10(p+5)}-\frac{3}{2}\right)^2\Bigg]-\frac{1}{10}$$

$$=\frac{2}{14+p}\left[\frac{1}{2}+\sum_{k=1}^{[\frac{p+6}{2}]}\left(\frac{(2k-1)}{2(p+5)}-\frac{1}{2}\right)^2\right]-\frac{1}{10}$$

$$=\frac{2}{14+p}\left[\frac{1}{2}+\sum_{k=1}^{[\frac{p+6}{2}]}\left(\frac{(2k-1)^2}{4(p+5)^2}-\frac{2k-1}{2(p+5)}+\frac{1}{4}\right)\right]-\frac{1}{10}$$

$$=\frac{2}{14+p}\left(\frac{1}{2}+\frac{1}{4}\left[\frac{p+6}{2}\right]+\sum_{k=1}^{[\frac{p+6}{2}]}\frac{(2k-1)^2}{4(p+5)^2}-\sum_{k=1}^{[\frac{p+6}{2}]}\frac{2k-1}{2(p+5)}\right)-\frac{1}{10}$$

$$=\frac{2}{14+p}\left(\frac{1}{2}+\frac{1}{4}\left[\frac{p+6}{2}\right]+\frac{[\frac{p+6}{2}](2[\frac{p+6}{2}]-1)(2[\frac{p+6}{2}]+1)}{12(p+5)^2}-\frac{[\frac{p+6}{2}]^2}{2(p+5)}\right)-\frac{1}{10}.$$

If p is even, then the right-hand side of the last formula for $D_{S_{1,p}}$

$$\begin{aligned}&=\frac{2}{14+p}\left(\frac{1}{2}+\frac{p+6}{8}+\frac{(p+6)(p+7)}{24(p+5)}-\frac{(p+6)^2}{8(p+5)}\right)-\frac{1}{10}\\&=\frac{2}{14+p}\left(\frac{1}{2}+\frac{p+6}{8}+\frac{(p+6)(p+7)}{24(p+5)}-\frac{(p+6)^2}{8(p+5)}-\frac{1}{10}\frac{14+p}{2}\right)\\&=\frac{-p(p+4)}{60(p+5)(p+14)}\leqslant 0.\end{aligned}$$

Therefore the main theorem is true in this subcase and the equality holds if $p=0$. If p is odd, then the right-hand side of the last formula for $D_{S_{1,p}}$

$$\begin{aligned}&=\frac{2}{14+p}\left(\frac{1}{2}+\frac{p+5}{8}+\frac{(p+4)(p+6)}{24(p+5)}-\frac{(p+5)^2}{8(p+5)}\right)-\frac{1}{10}\\&=\frac{2}{14+p}\left(\frac{1}{2}+\frac{p+5}{8}+\frac{(p+4)(p+6)}{24(p+5)}-\frac{(p+5)^2}{8(p+5)}-\frac{1}{10}\frac{14+p}{2}\right)\\&=\frac{-p(p+4)}{60(p+5)(p+14)}<0.\end{aligned}$$

Therefore the main theorem is true for $S_{1,p}$.

Case 4. $S^\sharp_{1,p}$.

The Milnor number $\mu=14+p$ for $S^\sharp_{1,p}$ (see Theorem 3.4). By symmetry of spectrum and Table 3 of exponents of $S^\sharp_{1,p}$, we have

$$D_{S^\sharp_{1,p}}:=\frac{1}{\mu}\sum_{i=1}^{\mu}\left(\alpha_i-\frac{3}{2}\right)^2-\frac{\alpha_\mu-\alpha_1}{12}=\frac{1}{14+p}\sum_{i=1}^{14+p}\left(\alpha_i-\frac{3}{2}\right)^2-\frac{2(\frac{3}{2}-\frac{9(p+10)}{10(p+10)})}{12}$$

$$=\frac{2}{14+p}\left[\left(\frac{9(p+12)}{10(p+12)}-\frac{3}{2}\right)^2+\left(\frac{13(p+10)}{10(p+13)}-\frac{3}{2}\right)^2+\sum_{k=1}^{[\frac{p+10}{2}]}\left(\frac{10(p+10+k)}{10(p+10)}-\frac{3}{2}\right)^2\right]-\frac{1}{10}$$

$$=\frac{2}{14+p}\left[\frac{2}{5}+\sum_{k=1}^{[\frac{p+10}{2}]}\left(\frac{(k)}{p+10}-\frac{1}{2}\right)^2\right]-\frac{1}{10}$$

$$=\frac{2}{14+p}\left[\frac{2}{5}+\sum_{k=1}^{[\frac{p+10}{2}]}\left(\frac{k^2}{(p+10)^2}-\frac{k}{p+10}+\frac{1}{4}\right)\right]-\frac{1}{10}$$

$$=\frac{2}{14+p}\left(\frac{2}{5}+\frac{1}{4}\left[\frac{p+10}{2}\right]+\sum_{k=1}^{[\frac{p+10}{2}]}\frac{k^2}{(p+10)^2}-\sum_{k=1}^{[\frac{p+10}{2}]}\frac{k}{p+10}\right)-\frac{1}{10}$$

$$= \frac{2}{14+p}\left(\frac{2}{5} + \frac{1}{4}\left[\frac{p+10}{2}\right] + \frac{[\frac{p+10}{2}]([\frac{p+10}{2}]+1)(2[\frac{p+10}{2}]+1)}{6(p+10)^2} - \frac{[\frac{p+10}{2}]([\frac{p+10}{2}]+1)}{2(p+10)}\right) - \frac{1}{10}.$$

If p is even, then the right-hand side of the last formula for $D_{S_1^\sharp,p}$

$$\begin{aligned}
&= \frac{2}{14+p}\left(\frac{2}{5} + \frac{p+10}{8} + \frac{(p+11)(p+12)}{24(p+10)} - \frac{(p+10)(p+12)}{8(p+10)}\right) - \frac{1}{10}\\
&= \frac{2}{14+p}\left(\frac{2}{5} + \frac{p+10}{8} + \frac{(p+11)(p+12)}{24(p+10)} - \frac{(p+10)(p+12)}{8(p+10)} - \frac{1}{10}\frac{14+p}{2}\right)\\
&= \frac{-p(p+11)}{60(p+10)(p+14)} \leqslant 0.
\end{aligned}$$

Therefore the main theorem is true in this subcase and the equality holds if $p = 0$. If p is odd, then the right-hand side of the last formula for $D_{S_1^\sharp,p}$

$$\begin{aligned}
&= \frac{2}{14+p}\left(\frac{2}{5} + \frac{p+9}{8} + \frac{(p+9)(p+11)}{24(p+10)} - \frac{(p+9)(p+11)}{8(p+10)}\right) - \frac{1}{10}\\
&= \frac{2}{14+p}\left(\frac{2}{5} + \frac{p+9}{8} + \frac{(p+9)(p+11)}{24(p+10)} - \frac{(p+9)(p+11)}{8(p+10)} - \frac{1}{10}\frac{14+p}{2}\right)\\
&= \frac{-p(p+11)}{60(p+10)(p+14)} < 0.
\end{aligned}$$

Therefore the main theorem is true for $S_{1,p}^\sharp$.

Case 5. $U_{1,p}$.

The Milnor number $\mu = 14+p$ for $U_{1,p}$ (see Theorem 3.4). By symmetry of spectrum and Table 3 of exponents of $U_{1,p}$, we have

$$\begin{aligned}
D_{U_{1,p}} &:= \frac{1}{\mu}\sum_{i=1}^{\mu}\left(\alpha_i - \frac{3}{2}\right)^2 - \frac{\alpha_\mu - \alpha_1}{12} = \frac{1}{14+p}\sum_{i=1}^{14+p}\left(\alpha_i - \frac{3}{2}\right)^2 - \frac{2(\frac{3}{2} - \frac{8(p+9)}{9(p+9)})}{12}\\
&= \frac{2}{14+p}\left[\left(\frac{8(p+9)}{9(p+9)} - \frac{3}{2}\right)^2 + \left(\frac{11(p+9)}{9(p+9)} - \frac{3}{2}\right)^2 + \left(\frac{13(p+9)}{9(p+9)} - \frac{3}{2}\right)^2\right.\\
&\quad \left. + \sum_{k=1}^{[\frac{p+8}{2}]}\left(\frac{9(p+9+k)}{9(p+9)} - \frac{3}{2}\right)^2\right] - \frac{11}{108}\\
&= \frac{2}{14+p}\left[\frac{49}{108} + \sum_{k=1}^{[\frac{p+8}{2}]}\left(\frac{(k)}{p+9} - \frac{1}{2}\right)^2\right] - \frac{11}{108}\\
&= \frac{2}{14+p}\left[\frac{49}{108} + \sum_{k=1}^{[\frac{p+8}{2}]}\left(\frac{k^2}{(p+9)^2} - \frac{k}{p+9} + \frac{1}{4}\right)\right] - \frac{11}{108}\\
&= \frac{2}{14+p}\left(\frac{49}{108} + \frac{1}{4}\left[\frac{p+8}{2}\right] + \sum_{k=1}^{[\frac{p+8}{2}]}\frac{k^2}{(p+9)^2} - \sum_{k=1}^{[\frac{p+8}{2}]}\frac{k}{p+9}\right) - \frac{11}{108}\\
&= \frac{2}{14+p}\left(\frac{49}{108} + \frac{1}{4}\left[\frac{p+8}{2}\right] + \frac{[\frac{p+8}{2}]([\frac{p+8}{2}]+1)(2[\frac{p+8}{2}]+1)}{6(p+9)^2} - \frac{[\frac{p+8}{2}]([\frac{p+8}{2}]+1)}{2(p+9)}\right) - \frac{11}{108}.
\end{aligned}$$

If p is even, then the right-hand side of the last formula for $D_{U_{1,p}}$

$$\begin{aligned}
&= \frac{2}{14+p}\left(\frac{49}{108} + \frac{p+8}{8} + \frac{(p+8)(p+10)}{24(p+9)} - \frac{(p+8)(p+10)}{8(p+9)}\right) - \frac{11}{108}\\
&= \frac{2}{14+p}\left(\frac{49}{108} + \frac{p+8}{8} + \frac{(p+8)(p+10)}{24(p+9)} - \frac{(p+8)(p+10)}{8(p+9)} - \frac{11}{108}\frac{14+p}{2}\right)
\end{aligned}$$

$$= \frac{-p(p+10)}{54(p+9)(p+14)} \leqslant 0.$$

Therefore the main theorem is true in this subcase and the equality holds if $p = 0$. If p is odd, then the right-hand side of the last formula for $D_{U_{1,p}}$

$$= \frac{2}{14+p}\left(\frac{49}{108} + \frac{p+7}{8} + \frac{(p+7)(p+8)}{24(p+9)} - \frac{(p+7)(p+9)}{8(p+9)}\right) - \frac{11}{108}$$
$$= \frac{2}{14+p}\left(\frac{49}{108} + \frac{p+7}{8} + \frac{(p+7)(p+8)}{24(p+9)} - \frac{(p+7)(p+9)}{8(p+9)} - \frac{11}{108}\frac{14+p}{2}\right)$$
$$= \frac{-p(p+10)}{54(p+9)(p+14)} < 0.$$

Therefore the main theorem is true for $U_{1,p}$.

Acknowledgements This work was supported by Start-up Fund of Tsinghua University. The first author would like to thank National Center for Theoretical Sciences (Mathematics Division, Taipei Office) for providing an excellent working environment during his visit while part of the work was done.

References

1 Arnold V I. Normal forms of functions near degenerate critical points, the Weyl group A_k, D_k, E_k and Lagrange singularities. Funct Anal Appl, 1972, 6: 254–274

2 Arnold V I. Some remarks on the stationary phase method and Coxeter numbers. Uspehi Mat Nauk, 1973, 28: 17–44; Russian Math Surveys, 1973, 28: 19–48

3 Brelivet T. Variance of spectral numbers and Newton polygons. Bull Sci Math, 2002, 126: 333–342

4 Brelivet T. The Hertling conjecture in dimension 2. ArXiv:math.AG/0405489

5 Deligne P, Thèorie de Hodge I. Actes Congrés. Intern Math, 1970: 425–430; II. Publ Math IHES, 1971, 40: 5–58; III. ibid, 1974, 44: 5–77

6 Dimca A. Monodromy and Hodge theory of regular functions. In: New Developments in Singularity Theory. Berlin: Springer, 2001, 257–278

7 Hertling C. Frobenius manifolds and variance of the spectral numbers. In: New Developments in Singularity Theory. Berlin: Springer, 2001, 235–255

8 Goryunov V V. Adjacencies of spectra of certain singularities. Vestnik MGU Ser Math, 1981, 4: 19–22

9 Mather J N. Finitely determined map germs. Inst Hautes Tudes Sci Pub Math, 1968, 35: 279–308

10 Mather J N. Stability of C^∞ mappings, I: The division theorem. Ann Math, 1968, 87: 89–104

11 Mather J N. Infinitesimal stability implies stability. Ann Math, 1969, 89: 254–292

12 Saito K. Quasihomogene isolierte Singularitäten von Hyperflächen. Invent Math, 1971, 14: 123–142

13 Saito M. Exponents of an irreducible plane curve singularity. ArXiv:math/0009133

14 Saito M. On the exponents and the geometric genus of an isolated hypersurface singularity. Proc Symp Pure Math, 1983, 40: 465–472

15 Saito M. Exponents and Newton polyhedra of isolated hypersurface singularities. Math Ann, 1988, 281: 411–417

16 Siersma D. The singularities of C^∞-functions of right-codimension smaller than or equal to eight. Indag Math, 1973, 35: 31–37

17 Steenbrink J. Mixed Hodge structure on the vanishing cohomology. In: Real and Complex Singularities. Proceedings of the Nordic Summer School. Germantown: Sijthoff and Noordhoff International Publishers, 1976, 525–563

18 Steenbrink J. Intersection form for quasi-homogeneous singularities. Compos Math, 1977, 34: 211–223

19 Thom R. Stabilité Structurelle et Morphogénèse: Essai D'une Thèorie Générale des Modèles. New York: Benjamin, 1971

20 Varchenko A. The asymptotics of holomorphic forms determine a mixed Hodge structure. Soviet Math Dokl, 1980, 22: 772–775

21 Varchenko A. Asymptotic Hodge structure in the vanishing cohomology. Math USSR Izv, 1982, 18: 465–512

22 Varchenko A. The complex exponent of a singularity does not change along strata $\mu =$const. Func Anal Appl, 1982, 16: 1–9