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Ke-Pao Lin · Xue Luo · Stephen S.-T. Yau · Huaiqing Zuo

On a number-theoretic conjecture on positive integral points in a 5-dimensional tetrahedron and a sharp estimate of the Dickman–de Bruijn function

Dedicated to Professor Ronald Graham on the occasion of his 80th birthday

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Abstract. It is well known that getting an estimate of the number of integral points in right-angled simplices is equivalent to getting an estimate of the Dickman–de Bruijn function $\psi(x, y)$ which is the number of positive integers $\leq x$ and free of prime factors $> y$. Motivated by the Yau Geometric Conjecture, the third author formulated a number-theoretic conjecture which gives a sharp polynomial upper estimate on the number of positive integral points in n-dimensional ($n \geq 3$) real right-angled simplices. In this paper, we prove this conjecture for $n = 5$. As an application, we give a sharp estimate of the Dickman–de Bruijn function $\psi(x, y)$ for $5 \leq y < 13$.

Keywords. Tetrahedron, Yau number-theoretic conjecture, upper estimate

1. Introduction

Let $\Delta(a_1, \dots, a_n)$ be an n-dimensional simplex described by

$$\frac{x_1}{a_1} + \cdots + \frac{x_n}{a_n} \leq 1, \qquad x_1, \dots, x_n \geq 0, \tag{1.1}$$

where $a_1 \geq \cdots \geq a_n \geq 1$ are real numbers. Let $P_n = P(a_1, \dots, a_n)$ and $Q_n = Q(a_1, \dots, a_n)$ be the numbers of positive and of nonnegative integral solutions of (1.1) respectively. They are related by the formula

$$Q(a_1, \dots, a_n) = P(a_1(1 + a), \dots, a_n(1 + a)), \tag{1.2}$$

where $a = 1/a_1 + \cdots + 1/a_n$. Estimates of the number of integral points have many applications in number theory, complex geometry, toric varieties and tropical geometry.

K.-P. Lin: Department of Art and Science, Chang Gung University of Science and Technology, 261 Wen Hwa 1 Road, Kwei-Shan, Tao-Yuan, Taiwan; e-mail: kplin@gw.cgust.edu.tw

X. Luo: School of Mathematics and Systems Science, Beihang University, Beijing, 100191, P. R. China; e-mail: xluo@buaa.edu.cn

S. S.-T. Yau (corresponding author): Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, P. R. China; e-mail: yau@uic.edu

H. Q. Zuo: Mathematical Sciences Center, Tsinghua University, Beijing, 100084, P. R. China; e-mail: hqzuo@math.tsinghua.edu.cn

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One of the central topics in computational number theory is to estimate $\psi(x, y)$, the Dickman–de Bruijn function (see [4], [5], [6], [10]). Let $S(x, y)$ be the set of positive integers $\leq x$ composed only of prime factors $\leq y$. The *Dickman–de Bruijn function* $\psi(x, y)$ is the cardinality of this set. It turns out that the computation of $\psi(x, y)$ is equivalent to computing the number of integral points in an n-dimensional tetrahedron $\Delta(a_1, \dots, a_n)$ with real vertices $(a_1, 0, \dots, 0), \dots, (0, \dots, 0, a_n)$. Let $p_1 < \cdots < p_n$ denote all the primes up to y. It is clear that $p_1^{l_1} \cdots p_n^{l_n} \leq x$ if and only if $l_1 \log p_1 + \cdots + l_n \log p_n \leq \log x$. Therefore, $\psi(x, y)$ is precisely the number Q_n of (integer) lattice points inside the n-dimensional tetrahedron (1.1) with $a_i = \log x / \log p_i$, $1 \leq i \leq n$.

Counting the number Q_n has been a challenging problem for many years. Much effort has been put into developing an exact formula when $a_1, \dots, a_n$ are positive integers (see [2], [1], [7], [14]). Mordell [21] gave a formula for Q_3, expressed in terms of three Dedekind sums, in the case that a_1, a_2, a_3 are pairwise relatively prime; Pommersheim [22] extended this formula to arbitrary a_1, a_2, a_3 using toric varieties, and so forth. Moreover, the problem of counting the number of integral points in an n-dimensional tetrahedron with real vertices is a classical subject which has attracted a lot of famous mathematicians. Hardy and Littlewood wrote several papers that applied Diophantine approximation [11]–[13]. A more general approximation of Q_n was obtained by D. C. Spencer [23], [24] via complex function-theoretic methods. Also in the context of estimating the Dickman–de Bruijn function, a_i, $1 \leq i \leq n$, are not always integers.

According to Granville [9], an upper polynomial estimate of P_n is a key topic in number theory. Such an estimate could be applied to finding large gaps between primes, to Waring's problem, to primality testing and factoring algorithms, and to bounds for the least prime k-th power residues and non-residues modulo n. Granville [9] obtained the following estimate:

$$P_n \leq \frac{1}{n!} a_1 \cdots a_n. \tag{1.3}$$

This estimate of $P(a_1, \dots, a_n)$ is interesting, but not strong enough to be useful, particularly when many of the a_i's are small [9].

In geometry and singularity theory, estimating P_n for real right-angled simplices is related to the Durfee Conjecture [27]. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of a complex analytic function with an isolated critical point at the origin. Let $V = \{(z_1, \dots, z_n) \in \mathbb{C}^n : f(z_1, \dots, z_n) = 0\}$. The *Milnor number* of the singularity $(V, 0)$ is defined as

$$\mu = \dim \mathbb{C}\{z_1, \dots, z_n\}/(f_{z_1}, \dots, f_{z_n}),$$

and the *geometric genus* p_g of $(V, 0)$ is defined as

$$p_g = \dim H^{n-2}(M, \Omega^{n-1})$$

where M is a resolution of V and Ω^{n-1} is the sheaf of germs of holomorphic $n-1$-forms on M. In 1978, Durfee [8] made the following conjecture:

Durfee Conjecture. *$n! p_g \leq \mu$, with equality only when $\mu = 0$.*

If $f(z_1,\dots,z_n)$ is a weighted homogeneous polynomial of type $(a_1,\dots,a_n)$ with an isolated singularity at the origin, Milnor and Orlik [20] proved that $\mu=(a_1-1)\cdots(a_n-1)$. On the other hand, Merle and Teissier [19] showed that $p_g=P_n$. Finding a sharp estimate of P_n will lead to a resolution of the Durfee Conjecture.

Starting from the early 1990's, the authors of [16], [26] and [28] tried to get sharp upper estimates of P_n where a_i are positive real numbers. They were successful for $n=3,\ 4$, and 5:

$$\begin{aligned}
3!P_3 &\le f_3 = a_1a_2a_3-(a_1a_2+a_1a_3+a_2a_3)+a_1+a_2,\\
4!P_4 &\le f_4 = a_1a_2a_3a_4-\tfrac{3}{2}(a_1a_2a_3+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4)\\
&\qquad +\tfrac{11}{3}(a_1a_2+a_1a_3+a_2a_3)-2(a_1+a_2+a_3),\\
5!P_5 &\le f_5 = a_1a_2a_3a_4a_5\\
&\qquad -2(a_1a_2a_3a_4+a_1a_2a_3a_5+a_1a_2a_4a_5+a_1a_3a_4a_5+a_2a_3a_4a_5)\\
&\qquad +\tfrac{35}{4}(a_1a_2a_3+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4)\\
&\qquad -\tfrac{50}{6}(a_1a_2+a_1a_3+a_1a_4+a_2a_3+a_2a_4+a_2a_5)+6(a_1+a_2+a_3+a_4).
\end{aligned}$$

They then proposed a general conjecture:

Conjecture 1.1 (Granville–Lin–Yau (GLY) Conjecture). *Let P_n be the number of elements of $\{(x_1,\dots,x_n)\in\mathbb{Z}_+^n : x_1/a_1+\cdots+x_n/a_n\le 1\}$, where $\mathbb{Z}_+=\{1,2,\dots\}$. Let $n\ge 3$.*

(1) *Sharp Estimate: If $a_1\ge\cdots\ge a_n\ge n-1$, then*

$$n!P_n \le f_n := A_0^n+\frac{s(n,n-1)}{n}A_1^n+\sum_{l=1}^{n-2}\frac{s(n,n-1-l)}{\binom{n-1}{l}}A_l^{n-1}, \tag{1.4}$$

where $s(n,k)$ is the Stirling number of the first kind defined by the generating function

$$x(x-1)\cdots(x-n+1)=\sum_{k=0}^{n}s(n,k)x^k,$$

and A_k^n is defined as

$$A_k^n=\Big(\prod_{i=1}^n a_i\Big)\Big(\sum_{1\le i_1<\cdots<i_k\le n}\frac{1}{a_{i_1}\cdots a_{i_k}}\Big)$$

for $k=1,\dots,n-1$. Equality holds if and only if $a_1=\cdots=a_n=$ integer.

(2) *Weak Estimate: If $a_1\ge\cdots\ge a_n>1$ then*

$$n!P_n < q_n := \prod_{i=1}^n (a_i-1). \tag{1.5}$$

These estimates are all polynomial in a_i. They are sharp because equality holds true if and only if all the a_i take the same integer value. The weak estimate in (1.5) has recently been proven [29]. Before, [15], [16], [26], [28] showed that (1.5) holds for $3\le n\le 5$. The sharp estimate conjecture was first formulated in [17]. In a private communication to the third author, Granville formulated that conjecture independently after reading [15]. Again, the Sharp GLY Conjecture has been proven individually for $n=3,4,5$ in [27],

[28], [16] respectively. It has also been proven generally for $n \leq 6$ in [25]. However, for $n = 7$, a counter-example has been given.

Counter-example to the Sharp GLY Conjecture. Take $n = 7$. Let $a_1 = \cdots = a_6 = 2000$ and $a_7 = 6.09$. Consider the following 7-dimensional tetrahedron:

$$x_i > 0, \quad 1 \leq i \leq 7, \quad \frac{x_1}{2000} + \cdots + \frac{x_6}{2000} + \frac{x_7}{6.09} \leq 1.$$

P_7 has been computed to be $3.9656226290532420 \cdot 10^{16}$. Meanwhile, $f_7 = 1.99840413 \cdot 10^{20}$ when $a_1 = \cdots = a_6 = 2000$, $a_7 = 6.09$. Thus,

$$f_7 - 7!P_7 = -2.69675 \cdot 10^{16}.$$

This implies that the sharp estimate of the GLY Conjecture fails in the case $n = 7$.

The breakthrough in the subject is the following theorem by Yau and Zhang [29] which states that the Weak GLY Conjecture holds for all $n \geq 3$.

Theorem 1.1 (Yau–Zhang [29]). *For $n \geq 3$, let $a_1 \geq \cdots \geq a_n > 1$ be real numbers. Then*

$$n!P_n \leq (a_1 - 1) \cdots (a_n - 1),$$

and equality holds if and only if $a_n = 1$.

Theorem 1.1 implies the Durfee Conjecture for weighted homogeneous singularities. However, the Yau–Zhang estimate is not sharp. It is not good enough to characterize the homogeneous polynomial with an isolated singularity. For that, the third author made the following conjecture in 1995.

Conjecture 1.2 (Yau Geometric Conjecture). *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of a weighted homogeneous polynomial with an isolated critical point at the origin. Let μ, P_g and ν be respectively the Milnor number, geometric genus and multiplicity of the singularity $V = \{z : f(z) = 0\}$. Then*

$$\mu - h(\nu) \geq (n+1)!P_g, \tag{1.6}$$

where $h(\nu) = (\nu - 1)^{n+1} - \nu(\nu - 1) \cdots (\nu - n)$, and equality holds if and only if f is a homogeneous polynomial.

The Yau Geometric Conjecture was confirmed for $n = 3, 4, 5$ in [27], [16] and [3] respectively.

In order to overcome the difficulty that the Sharp GLY Conjecture is only true if a_n is larger than $y(n)$, a positive integer depending on n, the third author proposes to prove a new sharp polynomial estimate conjecture which is motivated by the Yau Geometric Conjecture. The importance of this conjecture is that we only need $a_n > 1$ and hence the conjecture will give a sharp upper estimate of the Dickman–de Bruijn function $\psi(x, y)$.

Conjecture 1.3. *Assume that $a_1 \geq \cdots \geq a_n > 1$, $n \geq 3$. If $P_n > 0$, then*

$$n!P_n \leq (a_1 - 1) \cdots (a_n - 1) - (a_n - 1)^n + a_n(a_n - 1) \cdots (a_n - (n - 1)), \tag{1.7}$$

and equality holds if and only if $a_1 = \cdots = a_n =$ integer.

Obviously, there is an intimate relation between the Yau Geometric Conjecture (1.6) and the number-theoretic conjecture (1.7). Recall that if $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a weighted homogeneous polynomial with an isolated singularity at the origin, then the multiplicity ν of f at the origin is given by $\inf\{n \in \mathbb{Z}_+ : n \geq \inf\{w_1, \dots, w_n\}\}$, where w_i is the weight of x_i. Notice that in general, w_i is only a rational number. In case the minimal weight is an integer, the Yau Geometric Conjecture (1.6) and the number-theoretic conjecture (1.7) are the same. In general, these two conjectures do not imply each other, although they are intimately related.

The number-theoretic conjecture (1.7) is much sharper than the Weak GLY Conjecture (1.5). The estimate in (1.7) is optimal in the sense that equality occurs precisely when $a_1 = \cdots = a_n =$ integer. Moreover, the Sharp GLY Conjecture (1.4) does not hold for $n = 7$ as the counter-example shows. However, the number-theoretic conjecture (1.7) does hold for this example.

By the previous works of Xu and Yau [26], [28], it was shown that the number-theoretic conjecture is true for $n = 3$. The case $n = 4$ has been shown in our previous work [18]. The purpose of this paper is to prove that the number-theoretic conjecture is true for $n = 5$. The basic strategy for $n = 4$ and $n = 5$ is the same. But the feasibility of the strategy has been a challenge, even if the dimension has only increased by 1. As we will see in our proof, the number of subcases has increased from 4 (when $n = 4$) to 11 (when $n = 5$). Showing all subcases one by one would require tremendously involved computations. And it would be tedious to the reader. In this paper, we simplify the 11 subcases to five major classes ($k = 1, 2, 3, 4$ and $a_5 \geq 5$), and modify the former four classes with a delicate analysis of A_i's domain, where $A_i = a_i(1 - k/a_5)$, $i = 1, 2, 3, 4$, to deal with the subcases one by one. Furthermore, we give an explicit formula for the the Dickman–de Bruijn function $\psi(x, y)$ when $5 \leq y < 13$. Mathematica 4.0 is adopted for some involved computations. The following are our main theorems.

Theorem 1.2 (Number-theoretic conjecture for $n = 5$). *Let $a_1 \geq \cdots \geq a_5 > 1$ be real numbers. Let P_5 be the number of positive integral solutions of $x_1/a_1 + \cdots + x_5/a_5 \leq 1$. If $P_5 > 0$, then*

$$\begin{aligned} 120P_5 \leq\ & (a_1 - 1)\cdots(a_5 - 1) - (a_5 - 1)^5 \\ & + a_5(a_5 - 1)\cdots(a_5 - 4), \end{aligned}$$

and equality holds if and only if $a_1 = \cdots = a_5 =$ integer. This can also be expressed as

$$\begin{aligned} 120P_5 \leq\ & a_1a_2a_3a_4a_5 - (a_1a_2a_3a_4 + a_1a_2a_4a_5 + a_2a_3a_4a_5 + a_1a_3a_4a_5) - 5a_5^4 \\ & + (a_1a_2a_3 + a_1a_2a_4 + a_1a_2a_5 + a_1a_3a_4 + a_1a_3a_5 + a_1a_4a_5 \\ & \quad + a_2a_3a_4 + a_2a_3a_5 + a_2a_4a_5 + a_3a_4a_5) + 25a_5^3 \\ & + (a_1a_2 + a_1a_3 + a_1a_4 + a_1a_5 + a_2a_3 + a_2a_4 + a_2a_5 + a_3a_4 + a_3a_5 + a_4a_5) \\ & - 40a_5^2 - (a_1 + a_2 + a_3 + a_4) + 20a_5. \end{aligned} \tag{1.8}$$

Theorem 1.3 (Estimate of the Dickman–de Bruijn function). *Let $\psi(x, y)$ be the Dickman–de Bruijn function. We have the following upper estimate for $5 \leq y < 13$:*

(i) *when $5 \leq y < 7$ and $x > 5$, we have*

$$\begin{aligned}\psi(x, y) \leq \frac{1}{6}\Bigg\{ &\frac{1}{\log 2 \log 3 \log 5}(\log x + \log 15)(\log x + \log 10)(\log x + \log 6)\\ &- \frac{1}{\log^3 5}[(\log x + \log 6)^3\\ &- (\log x + \log 6 + \log 5)(\log x + \log 6)(\log x + \log 6 - \log 5)]\Bigg\};\end{aligned}$$

(ii) *when $7 \leq y < 11$ and $x > 11$, we have*

$$\begin{aligned}\psi(x, y) \leq \frac{1}{24}\Bigg\{ &\frac{1}{\log 2 \log 3 \log 5 \log 7}(\log x + \log 105)(\log x + \log 70)\\ &\cdot (\log x + \log 42)(\log x + \log 30)\\ &- \frac{1}{\log^4 7}[(\log x + \log 30)^4 - (\log x + \log 7 + \log 30)(\log x + \log 30)\\ &\cdot (\log x + \log 30 - \log 7)(\log x + \log 30 - 2\log 7)]\Bigg\};\end{aligned}$$

(iii) *when $11 \leq y < 13$ and $x > 13$, we have*

$$\begin{aligned}\psi(x, y) \leq \frac{1}{120}\Bigg\{ &\frac{1}{\log 2 \log 3 \log 5 \log 7 \log 11}(\log x + \log 1155)(\log x + \log 770)\\ &\cdot (\log x + \log 462)(\log x + \log 330)(\log x + \log 210)\\ &- \frac{1}{\log^5 11}[(\log x + \log 210)^5 - (\log x + \log 11 + \log 210)\\ &\cdot (\log x + \log 210)(\log x + \log 210 - \log 11)\\ &\cdot (\log x + \log 210 - 2\log 11)(\log x + \log 210 - 3\log 11)]\Bigg\}.\end{aligned}$$

2. Proofs of theorems

2.1. Proof of Theorem 1.2

Our strategy is to divide the proof into five cases:

(1) $a_5 \geq 5$;
(2) $5 > a_5 > 4$;
(3) $4 \geq a_5 > 3$;
(4) $3 \geq a_5 > 2$;
(5) $2 \geq a_5 > 1$.

To prove case (1), we only need to recall the main theorem in [16].

Theorem 2.1 ([16]). *Let $a_1 \geq \cdots \geq a_5 \geq 4$ be real numbers. Then*

$$\begin{aligned}120P_5 \leq\ & a_1a_2a_3a_4a_5 - 2(a_1a_2a_3a_4 + a_1a_2a_4a_5 + a_2a_3a_4a_5 + a_1a_3a_4a_5 + a_1a_2a_3a_5)\\ &+ \tfrac{35}{4}(a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4)\\ &- \tfrac{50}{6}(a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4) + 6(a_1 + a_2 + a_3 + a_4), \qquad (2.9)\end{aligned}$$

and equality is attained if and only if $a_1 = \cdots = a_5 =$ integer.

Case (1) is solved by showing that our sharp upper bound is larger than or equal to theirs, and equality holds if and only if $a_1 = \cdots = a_5$.

Lemma 2.1. *When $a_5 \geq 5$, R.H.S. of (1.8) $\geq$ R.H.S. of (2.9).*

Proof. Let $A_i = a_i/a_5$, $i = 1, 2, 3, 4$. From the condition $a_1 \geq \cdots \geq a_5 > 1$, we have $A_i \geq 1$, $i = 1, 2, 3, 4$. Now, subtracting R.H.S. of (2.9) from R.H.S. of (1.8), and expressing the result in terms of A_i, $i = 1, 2, 3, 4$, and a_5, we obtain

$$\begin{aligned}\triangle_1 :=\ & \text{R.H.S. of (1.8)} - \text{R.H.S. of (2.9)}\\ =\ & A_1A_2A_3A_4a_5^4 + (A_1A_2A_3 + A_1A_2A_4 + A_1A_3A_4 + A_2A_3A_4)\big(a_5^4 - \tfrac{31}{4}a_5^3\big)\\ &+ (A_1A_2 + A_1A_3 + A_1A_4 + A_2A_3 + A_2A_4 + A_3A_4)\big(a_5^3 + \tfrac{22}{3}a_5^2\big)\\ &+ (A_1 + A_2 + A_3 + A_4)(-a_5^2 - 5a_5) + (-5a_5^4 + 25a_5^3 - 40a_5^2 + 20a_5). \qquad (2.10)\end{aligned}$$

The idea is to show that for all $a_5 \geq 5$, the minimum of $\triangle_1$ in $A_1 \geq A_2 \geq A_3 \geq A_4 \geq 1$ occurs at $A_1 = A_2 = A_3 = A_4 = 1$, and $\triangle_1|_{A_1=A_2=A_3=A_4=1} = 0$ for all $a_5 \geq 5$. Note that $\triangle_1$ is symmetric with respect to A_1, A_2, A_3, A_4. We have

$$\frac{\partial^4 \triangle_1}{\partial A_1 \partial A_2 \partial A_3 \partial A_4} = a_5^4 > 0$$

for $a_5 > 1$. It follows that $\frac{\partial^3 \triangle_1}{\partial A_1 \partial A_2 \partial A_3}$ is an increasing function of A_4 for $a_5 > 1$ and $A_4 \geq 1$. Hence its minimum occurs at $A_4 = 1$, and

$$\frac{\partial^3 \triangle_1}{\partial A_1 \partial A_2 \partial A_3}\bigg|_{A_4=1} = \big[A_4a_5^4 + \big(a_5^4 - \tfrac{31}{4}a_5^3\big)\big]\big|_{A_4=1} = a_5^3\big(2a_5 - \tfrac{31}{4}\big) > 0$$

for $a_5 > \frac{31}{8}$. It follows that $\frac{\partial^3 \triangle_1}{\partial A_1 \partial A_2 \partial A_3} > 0$ for $A_4 \geq 1$ and $a_5 > \frac{31}{8}$. Note that $\frac{\partial^2 \triangle_1}{\partial A_1 \partial A_2}$ is symmetric with respect to A_3 and A_4. Thus, $\frac{\partial^3 \triangle_1}{\partial A_1 \partial A_2 \partial A_4} > 0$ for $A_3 \geq 1$ and $a_5 > \frac{31}{8}$. Moreover, $\frac{\partial^2 \triangle_1}{\partial A_1 \partial A_2}$ is increasing with respect to A_3 and A_4 for $A_3 \geq A_4 \geq 1$ and $a_5 > \frac{31}{8}$. Hence its minimum occurs at $A_3 = A_4 = 1$, and

$$\begin{aligned}\frac{\partial^2 \triangle_1}{\partial A_1 \partial A_2}\bigg|_{A_3=A_4=1} &= \big[A_3A_4a_5^4 + (A_3 + A_4)\big(a_5^4 - \tfrac{31}{4}a_5^3\big) + \big(a_5^3 + \tfrac{22}{3}a_5^2\big)\big]\big|_{A_3=A_4=1}\\ &= 3a_5^4 - \tfrac{29}{2}a_5^3 + \tfrac{22}{3}a_5^2 = a_5^2\big(3a_5^2 - \tfrac{29}{2}a_5 + \tfrac{22}{3}\big) > 0\end{aligned}$$

for $a_5 \geq 5$, since the largest solution to $3a_5^2 - \frac{29}{2}a_5 + \frac{22}{3} = 0$ is around 4.26. It follows that $\frac{\partial^2 \triangle_1}{\partial A_1 \partial A_2} > 0$ for $A_3, A_4 \geq 1$ and $a_5 \geq 5$. As $\frac{\partial \triangle_1}{\partial A_1}$ is symmetric with respect to A_2, A_3, A_4, we also get $\frac{\partial \triangle_1}{\partial A_1 \partial A_3} > 0$ for $A_2, A_4 \geq 1$ and $a_5 \geq 5$, and $\frac{\partial \triangle_1}{\partial A_1 \partial A_4} > 0$ for $A_2, A_3 \geq 1$ and $a_5 \geq 5$. Therefore, $\frac{\partial \triangle_1}{\partial A_1}$ is an increasing function of A_2, A_3, A_4 for $A_2, A_3, A_4 \geq 1$ and $a_5 \geq 5$. Hence its minimum occurs at $A_2 = A_3 = A_4 = 1$, and

$$\begin{aligned}\left.\frac{\partial \triangle_1}{\partial A_1}\right|_{A_2=A_3=A_4=1} &= \big[A_2A_3A_4a_5^4 + (A_2A_3 + A_2A_4 + A_3A_4)(a_5^4 - \tfrac{31}{4}a_5^3) \\ &\quad + (A_2 + A_3 + A_4)\big(a_5^3 + \tfrac{22}{3}a_5^2\big) + (-a_5^2 - 5a_5)\big]\big|_{A_2=A_3=A_4=1} \\ &= 4a_5^4 - \tfrac{81}{4}a_5^3 + 21a_5^2 - 5a_5 = a_5\big(4a_5^3 - \tfrac{79}{4}a_5^2 + 21a_5 - 5\big) > 0\end{aligned}$$

for $a_5 \geq 5$, since $4a_5^3 - \frac{81}{4}a_5^2 + 21a_5 - 5 \geq a_5\big(4a_5^2 - \frac{81}{4}a_5 + 20\big) = 4a_5\big[\big(a_5 - \frac{81}{32}\big)^2 - \frac{1441}{1024}\big]$, and $f(a_5) := \big(a_5 - \frac{81}{32}\big)^2 - \frac{1441}{1024} > f(5) = \frac{75}{16} > 0$ for $a_5 \geq 5$. It follows that $\frac{\partial \triangle_1}{\partial A_1} > 0$ for $A_2, A_3, A_4 \geq 1$ and $a_5 \geq 5$. As $\triangle_1$ is symmetric with respect to A_1, A_2, A_3, A_4, its minimum occurs at $A_1 = A_2 = A_3 = A_4 = 1$, and

$$\triangle_1|_{A_1=A_2=A_3=A_4=1} = 0$$

for $a_5 \geq 5$. Therefore, $\triangle_1 \geq 0$ when $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq 5$, and $\triangle_1 = 0$ if and only if $a_1 = a_2 = a_3 = a_4 = a_5$. Equality holds in (2.9) if and only if $a_1 = \cdots = a_5 =$ integer, as also does equality in (1.8). □

For cases (2) to (5), we adopt a similar strategy: the 5-dimensional tetrahedron will be partitioned into 4-dimensional ones [25]. We have

$$\begin{aligned}&\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{k}{a_5} \leq 1, \\ &\frac{x_1}{a_1(1 - k/a_5)} + \frac{x_2}{a_2(1 - k/a_5)} + \frac{x_3}{a_3(1 - k/a_5)} + \frac{x_4}{a_4(1 - k/a_5)} \leq 1,\end{aligned} \tag{2.11}$$

for $k = 1, \ldots, \lfloor a_5 \rfloor$, where $\lfloor \circ \rfloor$ is the largest integer less than or equal to $\circ$. Let $P_4(k)$ be the number of positive integral solutions of (2.11). Then

$$P_5 = \sum_{k=1}^{\lfloor a_5 \rfloor} P_4(k). \tag{2.12}$$

According to Theorem 1.1 in [18], if $P_4(k) > 0$, then

$$\begin{aligned}5!P_4(k) \leq 5[&(a_1(1-k/a_5)-1)(a_2(1-k/a_5)-1)(a_3(1-k/a_5)-1)(a_4(1-k/a_5)-1) \\ &- (a_4(1-k/a_5)-1)^4 \\ &+ a_4(1-k/a_5)(a_4(1-k/a_5)-1)(a_4(1-k/a_5)-2)(a_4(1-k/a_5)-3)].\end{aligned}$$

Suppose there exists some k_0, $1 \leq k_0 \leq \lfloor a_5 \rfloor$, which is the largest integer such that $P_4(k_0) > 0$ and $P_4(k) = 0$ for all $k_0 < k \leq \lfloor a_5 \rfloor$. In fact, the integer k_0 does exist due to the condition $P_5 > 0$. By (2.12), we have

$$
\begin{aligned}
5!P_5 &= 5!\sum_{k=1}^{k_0} P_4(k)\\
&\le 5\sum_{k=1}^{k_0}[(a_1(1-k/a_5)-1)(a_2(1-k/a_5)-1)(a_3(1-k/a_5)-1)(a_4(1-k/a_5)-1)\\
&\quad - (a_4(1-k/a_5)-1)^4\\
&\quad + a_4(1-k/a_5)(a_4(1-k/a_5)-1)(a_4(1-k/a_5)-2)(a_4(1-k/a_5)-3)]. \qquad (2.13)
\end{aligned}
$$

In order to prove (1.8), it is sufficient to show that R.H.S. of (1.8) $\geq$ R.H.S. of (2.13). For cases (2) to (5), equality in (1.8) is never attained: On the one hand, $P_5 > 0$ will not be satisfied if $a_1 = a_2 = a_3 = a_4 = a_5 < 5$. On the other hand, we could show that R.H.S. of (1.8) is strictly larger than R.H.S. of (2.13) in these cases. Therefore, no such $a_1 \geq \cdots \geq a_5$ and $a_5 \in (1, 5)$ could make the equality in (1.8) happen.

Now, for case (5), there are two levels $k = 1$ and $k = 2$. It is easy to see that $P_4(2) = 0$. From the condition $P_5 > 0$, we know that the level $k = 1$ can have no positive integral solution, i.e. $P_4(1) = P_5 > 0$. It is also implied that the smallest positive integral solution $(1, 1, 1, 1, 1)$ must be its solution, which gives $1/a_1 + 1/a_2 + 1/a_3 + 1/a_4 \leq 1 - 1/a_5 =: \alpha \in \left(0, \frac{1}{2}\right]$, since $a_5 \in (1, 2]$. Let $A_i = a_i\alpha$, $i = 1, 2, 3, 4$, and notice that

$$
A_1 \geq 4, \qquad A_2 \geq 3, \qquad A_3 \geq 2, \qquad A_4 \geq 1, \qquad (2.14)
$$

since $1/A_4 \leq 1$, $2/A_3 \leq 1/A_3 + 1/A_4 \leq 1$, $3/A_2 \leq 1/A_2 + 1/A_3 + 1/A_4 \leq 1$ and $4/A_1 \leq 1/A_1 + 1/A_2 + 1/A_3 + 1/A_4 \leq 1$. (2.13) can be rewritten as

$$
\begin{aligned}
5!P_5 = 5!P_4(1) \leq 5[&(A_1-1)(A_2-1)(A_3-1)(A_4-1) - (A_4-1)^4\\
&+ A_4(A_4-1)(A_4-2)(A_4-3)]. \qquad (2.15)
\end{aligned}
$$

To prove (1.8) in this case, it is sufficient to show:

Lemma 2.2. *When $1 < a_5 \leq 2$, R.H.S. of (1.8) $>$ R.H.S. of (2.15).*

Proof. Subtracting R.H.S. of (2.15) from R.H.S. of (1.8), writing the expression in terms of A_i, $i = 1, 2, 3, 4$, and α, and multiplying it by $(1-\alpha)^4$, we get

$$
\begin{aligned}
\triangle_2 := {} & A_1A_2A_3A_4\left(\frac{1}{\alpha^3} - \frac{3}{\alpha^2} + \frac{3}{\alpha} - 6 + 20\alpha - 30\alpha^2 + 20\alpha^3 - 5\alpha^4\right)\\
&+ (A_1A_2A_3 + A_1A_2A_4 + A_1A_3A_4 + A_2A_3A_4)\\
&\qquad \cdot\left(-\frac{1}{\alpha^2} + \frac{3}{\alpha} + 2 - 19\alpha + 30\alpha^2 - 20\alpha^3 + 5\alpha^4\right)\\
&+ (A_1A_2 + A_1A_3 + A_1A_4 + A_2A_3 + A_2A_4 + A_3A_4)\\
&\qquad \cdot\left(\frac{1}{\alpha} - 8 + 23\alpha - 31\alpha^2 + 20\alpha^3 - 5\alpha^4\right)\\
&+ (A_1 + A_2 + A_3)(4 - 17\alpha + 27\alpha^2 - 19\alpha^3 + 5\alpha^4)\\
&+ A_4^3(10 - 40\alpha + 60\alpha^2 - 40\alpha^3 + 10\alpha^4)\\
&+ A_4^2(-25 + 100\alpha - 150\alpha^2 + 100\alpha^3 - 25\alpha^4)\\
&+ A_4(14 - 57\alpha + 87\alpha^2 - 59\alpha^3 + 15\alpha^4) + (-5\alpha + 20\alpha^2 - 20\alpha^3).
\end{aligned}
$$

The idea is to show that for all $\alpha \in (0, 1/2]$, the minimum of $\triangle_2$ in $A_1 \geq 4$, $A_2 \geq 3$, $A_3 \geq 2$, $A_4 \geq 1$ occurs at $A_1 = 4$, $A_2 = 3$, $A_3 = 2$, $A_4 = 1$, and $\triangle_2|_{A_1=4,A_2=3,A_3=2,A_4=1} > 0$ for all $\alpha \in (0, 1/2]$. We have

$$\begin{aligned}\frac{\partial^4 \triangle_2}{\partial A_1 \partial A_2 \partial A_3 \partial A_4} &= \frac{1}{\alpha^3} - \frac{3}{\alpha^2} + \frac{3}{\alpha} - 6 + 20\alpha - 30\alpha^2 + 20\alpha^3 - 5\alpha^4 \\ &= \frac{1}{\alpha^3}(1-\alpha)^3(1 - 5\alpha^3 + 5\alpha^4) > 0 \end{aligned} \tag{2.16}$$

for $\alpha \in (0, 1)$. In fact, let $f(\alpha) := 1 - 5\alpha^3 + 5\alpha^4$. Then $f'(\alpha) = 20\alpha^3 - 15\alpha^2 = 5\alpha^2(4\alpha - 3)$, which implies that $f'(\alpha) \leq 0$ for $\alpha \in \left(0, \frac{3}{4}\right]$, while $f'(\alpha) > 0$ for $\alpha \in \left(\frac{3}{4}, 1\right)$. Thus, $\min_{\alpha \in (0,1)} f(\alpha) = f\left(\frac{3}{4}\right) = \frac{121}{256} > 0$. Therefore, $f(\alpha) > 0$ for $\alpha \in (0, 1)$. It follows that $\frac{\partial^3 \triangle_2}{\partial A_1 \partial A_2 \partial A_3}$ is an increasing function of A_4 for $\alpha \in (0, 1)$ and $A_4 \geq 1$. Hence its minimum occurs at $A_4 = 1$, and

$$\begin{aligned}\left.\frac{\partial^3 \triangle_2}{\partial A_1 \partial A_2 \partial A_3}\right|_{A_4=1} &= \left[A_4\left(\frac{1}{\alpha^3} - \frac{3}{\alpha^2} + \frac{3}{\alpha} - 6 + 20\alpha - 30\alpha^2 + 20\alpha^3 - 5\alpha^4\right)\right. \\ &\quad \left.\left. + \left(-\frac{1}{\alpha^2} + \frac{3}{\alpha} + 2 - 19\alpha + 30\alpha^2 - 20\alpha^3 + 5\alpha^4\right)\right]\right|_{A_4=1} \\ &= \frac{1}{\alpha^3}(\alpha - 1)^4 > 0 \end{aligned} \tag{2.17}$$

for $\alpha \in (0, 1)$. It follows that $\frac{\partial^3 \triangle_2}{\partial A_1 \partial A_2 \partial A_3} > 0$ for $A_4 \geq 1$ and $\alpha \in (0, 1)$. Note that $\frac{\partial^2 \triangle_2}{\partial A_1 \partial A_2}$ is symmetric with respect to A_3 and A_4. Thus, $\frac{\partial^3 \triangle_2}{\partial A_1 \partial A_2 \partial A_4} > 0$ for $A_3 \geq 1$ and $\alpha \in (0, 1)$. Moreover, $\frac{\partial^2 \triangle_2}{\partial A_1 \partial A_2}$ is increasing with respect to A_3 and A_4 for $A_3 \geq A_4 \geq 1$ and $\alpha \in (0, 1)$. Hence its minimum occurs at $A_3 = A_4 = 1$, and

$$\begin{aligned}\left.\frac{\partial^2 \triangle_2}{\partial A_1 \partial A_2}\right|_{A_3=A_4=1} &= \left[A_3 A_4\left(\frac{1}{\alpha^3} - \frac{3}{\alpha^2} + \frac{3}{\alpha} - 6 + 20\alpha - 30\alpha^2 + 20\alpha^3 - 5\alpha^4\right)\right. \\ &\quad + (A_3 + A_4)\left(-\frac{1}{\alpha^2} + \frac{3}{\alpha} + 2 - 19\alpha + 30\alpha^2 - 20\alpha^3 + 5\alpha^4\right) \\ &\quad \left.\left. + \left(\frac{1}{\alpha} - 8 + 23\alpha - 31\alpha^2 + 20\alpha^3 - 5\alpha^4\right)\right]\right|_{A_3=A_4=1} \\ &= -\frac{1}{\alpha^3}(-1 + \alpha)^5 > 0 \end{aligned} \tag{2.18}$$

for $\alpha \in (0, 1)$. It follows that $\frac{\partial^2 \triangle_2}{\partial A_1 \partial A_2} > 0$ for $A_3 \geq A_4 \geq 1$ and $\alpha \in (0, 1)$. As $\frac{\partial \triangle_2}{\partial A_1}$ is symmetric with respect to A_2, A_3, A_4, we also get $\frac{\partial^2 \triangle_2}{\partial A_1 \partial A_3} > 0$ for $A_2 \geq A_4 \geq 1$ and $\alpha \in (0, 1)$, and $\frac{\partial^2 \triangle_2}{\partial A_1 \partial A_4} > 0$ for $A_2 \geq A_3 \geq 1$ and $\alpha \in (0, 1)$. Therefore, $\frac{\partial \triangle_2}{\partial A_1}$ is an increasing function of A_2, A_3, A_4 for $A_2 \geq A_3 \geq A_4 \geq 1$ and $\alpha \in (0, 1)$. Hence its minimum occurs at $A_2 = A_3 = A_4 = 1$, and

$$\begin{aligned}\frac{\partial \triangle_2}{\partial A_1}\bigg|_{A_2=A_3=A_4=1} &= \bigg[A_2A_3A_4\Big(\frac{1}{\alpha^3} - \frac{3}{\alpha^2} + \frac{3}{\alpha} - 6 + 20\alpha - 30\alpha^2 + 20\alpha^3 - 5\alpha^4\Big)\\ &\quad + (A_2A_3 + A_2A_4 + A_3A_4)\Big(-\frac{1}{\alpha^2} + \frac{3}{\alpha} + 2 - 19\alpha + 30\alpha^2 - 20\alpha^3 + 5\alpha^4\Big)\\ &\quad + (A_2 + A_3 + A_4)\Big(\frac{1}{\alpha} - 8 + 23\alpha - 31\alpha^2 + 20\alpha^3 - 5\alpha^4\Big)\\ &\quad + (4 - 17\alpha + 27\alpha^2 - 19\alpha^3 + 5\alpha^4)\bigg]\bigg|_{A_2=A_3=A_4=1}\\ &= \frac{1}{\alpha^3}(-1 + \alpha)^6 > 0 \end{aligned} \tag{2.19}$$

for $\alpha \in (0, 1)$. It follows that $\frac{\partial \triangle_2}{\partial A_1} > 0$ for $A_2 \geq A_3 \geq A_4 \geq 1$ and $\alpha \in (0, 1)$. As $\triangle_2$ is symmetric with respect to A_1, A_2, A_3, we also have $\frac{\partial \triangle_2}{\partial A_2} > 0$ for $A_1 \geq A_3 \geq A_4 \geq 1$, and $\frac{\partial \triangle_2}{\partial A_3} > 0$ for $A_1 \geq A_2 \geq A_4 \geq 1$. Moreover,

$$\frac{\partial^3 \triangle_2}{\partial A_4^3} = 10(-1 + \alpha)^4 > 0 \tag{2.20}$$

for $\alpha \in (0, 1)$. It follows that $\frac{\partial^2 \triangle_2}{\partial A_4^2}$ is an increasing function of A_4 for $A_4 \geq 1$ and $\alpha \in (0, 1)$. Thus, its minimum occurs at $A_4 = 1$, and

$$\begin{aligned}\frac{\partial^2 \triangle_2}{\partial A_4^2}\bigg|_{A_4=1} &= \big[6A_4(10 - 40\alpha + 60\alpha^2 - 40\alpha^3 + 10\alpha^4)\\ &\quad + 2(-25 + 100\alpha - 150\alpha^2 + 100\alpha^3 - 25\alpha^4)\big]\big|_{A_4=1}\\ &= 10(-1 + \alpha)^4 > 0 \end{aligned} \tag{2.21}$$

for $\alpha \in (0, 1)$. It follows that $\frac{\partial^2 \triangle_2}{\partial A_4^2} > 0$ for $A_4 \geq 1$ and $\alpha \in (0, 1)$. Thus, $\frac{\partial \triangle_2}{\partial A_4}$ is an increasing function of A_4 for $A_4 \geq 1$ and $\alpha \in (0, 1)$. Moreover, it is an increasing function with respect to A_1, A_2, A_3, A_4 for $A_1 \geq A_2 \geq A_3 \geq A_4 \geq 1$, $\alpha \in (0, 1)$, since it is symmetric with respect to A_1, A_2, A_3. Taking condition (2.14) into consideration, the minimum of $\frac{\partial \triangle_2}{\partial A_4}$ occurs at $A_1 = 4$, $A_2 = 3$, $A_3 = 2$, $A_4 = 1$, and

$$\begin{aligned}\left.\frac{\partial\triangle_2}{\partial A_4}\right|_{A_1=4,A_2=3,A_3=2,A_4=1} &= \Bigg[A_1A_2A_3\left(\frac{1}{\alpha^3}-\frac{3}{\alpha^2}+\frac{3}{\alpha}-6+20\alpha-30\alpha^2+20\alpha^3-5\alpha^4\right)\\ &\quad+(A_1A_2+A_1A_3+A_2A_3)\left(-\frac{1}{\alpha^2}+\frac{3}{\alpha}+2-19\alpha+30\alpha^2-20\alpha^3+5\alpha^4\right)\\ &\quad+(A_1+A_2+A_3)\left(\frac{1}{\alpha}-8+23\alpha-31\alpha^2+20\alpha^3-5\alpha^4\right)\\ &\quad+(14-57\alpha+87\alpha^2-59\alpha^3+15\alpha^4)\\ &\quad+3A_4^2(10-40\alpha+60\alpha^2-40\alpha^3+10\alpha^4)\\ &\quad+2A_4(-25+100\alpha-150\alpha^2+100\alpha^3-25\alpha^4)\Bigg]\Bigg|_{A_1=4,A_2=3,A_3=2,A_4=1}\\ &= -\frac{1}{\alpha^3}(-1+\alpha)^3(24-26\alpha+9\alpha^2-41\alpha^3+40\alpha^4)>0 \qquad (2.22)\end{aligned}$$

for $\alpha\in\left(0,\frac{4}{5}\right)$. In fact, let $g(\alpha):=24-26\alpha+9\alpha^2-41\alpha^3+40\alpha^4$. Then $g'(\alpha)=-26+18\alpha-123\alpha^2+160\alpha^3<-8\alpha-123\alpha^2+160\alpha=\alpha(-8-123\alpha+160\alpha^2)$. Moreover

$$h(\alpha)=-8-123\alpha+160\alpha^2=160\left(\alpha-\tfrac{123}{320}\right)^2-\tfrac{20249}{640}<0 \quad \text{for } \alpha\in\left(0,\tfrac{4}{5}\right),$$

since $h(0)=-8$ and $h\left(\frac{4}{5}\right)=-4$, $\max_{\alpha\in\left(0,\frac{4}{5}\right)}h(\alpha)=-4<0$. Thus, $g'(\alpha)<0$ for $\alpha\in\left(0,\frac{4}{5}\right)$. It follows that $g(\alpha)$ is a decreasing function in $\alpha\in\left(0,\frac{4}{5}\right)$. Moreover, $g(\alpha)\ge g\left(\frac{4}{5}\right)=\frac{544}{125}>0$ for $\alpha\in\left(0,\frac{4}{5}\right)$. It follows that $\frac{\partial\triangle_2}{\partial A_4}>0$ for $A_1\ge 4$, $A_2\ge 3$, $A_3\ge 2$, $A_4\ge 1$ and $\alpha\in\left(0,\frac{4}{5}\right)$. Therefore, $\triangle_2$ is an increasing function of A_1, A_2, A_3, A_4 for $A_1\ge 4$, $A_2\ge 3$, $A_3\ge 2$, $A_4\ge 1$ and $\alpha\in\left(0,\frac{4}{5}\right)$. Thus, its minimum occurs at $A_1=4$, $A_2=3$, $A_3=2$, $A_4=1$, and

$$\begin{aligned}&\triangle_2|_{A_1=4,A_2=3,A_3=2,A_4=1}\\ &\qquad=-\frac{1}{\alpha^3}(-24+122\alpha-257\alpha^2+289\alpha^3-180\alpha^4+45\alpha^5+10\alpha^6)>0\end{aligned}$$

for $\alpha\in\left(0,\frac{1}{2}\right]$. Indeed, let $f(\alpha):=-24+122\alpha-257\alpha^2+289\alpha^3-180\alpha^4+45\alpha^5+10\alpha^6$. Then $f^{(3)}(\alpha)=1734-4320\alpha+2700\alpha^2+1200\alpha^3>1734-4320\alpha+2700\alpha^2=2700\left(\alpha-\frac{4}{5}\right)^2+6>0$ for $\alpha\in\left(0,\frac{1}{2}\right]$. Thus, $f''(\alpha)$ is increasing in $\alpha\in\left(0,\frac{1}{2}\right]$ and $f''(\alpha)<f''\left(\frac{1}{2}\right)=-\frac{223}{4}<0$. So $f'(\alpha)$ is decreasing in $\alpha\in\left(0,\frac{1}{2}\right]$ and $f'(\alpha)>f'\left(\frac{1}{2}\right)=\frac{123}{16}>0$. This implies that $f(\alpha)$ is increasing in $\alpha\in\left(0,\frac{1}{2}\right]$ and $f(\alpha)<f\left(\frac{1}{2}\right)=-\frac{13}{16}<0$. Therefore, $f(\alpha)<0$ for $\alpha\in\left(0,\frac{1}{2}\right]$. It follows that $\triangle_2>0$ for $A_1\ge 4$, $A_2\ge 3$, $A_3\ge 2$, $A_4\ge 1$ and $\alpha\in\left(0,\frac{1}{2}\right]$. □

For case (4), there are three levels: $k=1$, $k=2$ and $k=3$. Also it is easy to see that $P_4(3)=0$. The condition $P_5>0$ guarantees that $P_4(1)>0$, but the positivity of $P_4(2)$ is unknown. Therefore, we split this case into the following two subcases:

(4a) $P_4(2) = 0$ (i.e. $k_0 = 1$ in (2.13));
(4b) $P_4(2) > 0$ (i.e. $k_0 = 2$ in (2.13)).

For subcase (4a), the proof is quite similar to that in case (5). In the present case $P_5 = P_4(1) > 0$, thus $(1, 1, 1, 1, 1)$ is the smallest positive integral solution, i.e. $1/a_1 + 1/a_2 + 1/a_3 + 1/a_4 \le 1 - 1/a_5 =: \alpha \in \left(\frac{1}{2}, \frac{2}{3}\right]$, since $a_5 \in (2, 3]$. This yields $A_1 \ge 4$, $A_2 \ge 3$, $A_3 \ge A_4 \ge \frac{\alpha}{1-\alpha}$, since $A_i = a_i\alpha \ge a_5\alpha = \frac{\alpha}{1-\alpha}$. With $\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right]$, it is easy to check that $1 < \frac{\alpha}{1-\alpha} \le 2$. Therefore, it is sufficient to show that $\triangle_2 > 0$ in the range

$$A_1 \ge 4, \qquad A_2 \ge 3, \qquad A_3 \ge 2, \qquad A_4 \ge \frac{\alpha}{1-\alpha} \tag{2.23}$$

for $\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right]$. In the proof of Lemma 2.2, all the computations of the partial derivatives in (2.16)–(2.22) are valid in the even larger range (2.14) for $\alpha \in \left(0, \frac{4}{5}\right)$, so they hold in the new range (2.23) for $\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right]$. We only need to show the positivity of $\triangle_2|_{A_1=4, A_2=3, A_3=2, A_4=\frac{\alpha}{1-\alpha}}$ for $\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right]$. Taking condition (2.23) instead for (2.14) for A_i, $i = 1, 2, 3, 4$, yields

$$\triangle_2|_{A_1=4, A_2=3, A_3=2, A_4=\frac{\alpha}{1-\alpha}} = \frac{1}{\alpha}(24 - 39\alpha - 82\alpha^2 + 223\alpha^3 - 152\alpha^4 + 20\alpha^5) > 0$$

for $\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right]$. In fact, let $f(\alpha) := 24 - 39\alpha - 82\alpha^2 + 223\alpha^3 - 152\alpha^4 + 20\alpha^5$. Then $f''(\alpha) = -162 + 1338\alpha - 1824\alpha^2 + 400\alpha^3 > -162 + 1338\alpha - 1624\alpha^2 = -1624\left(\alpha - \frac{669}{1624}\right)^2 + \frac{184473}{1624} =: g(\alpha)$, and $g(\alpha) > g(\frac{2}{3}) = \frac{74}{9} > 0$. So $f''(\alpha) > 0$ for $\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right]$. Thus, $f'(\alpha)$ is increasing in $\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right]$, i.e. $f'(\alpha) < f'\left(\frac{2}{3}\right) = -\frac{815}{81} < 0$. So $f'(\alpha) < 0$ for $\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right]$. It follows that $f(\alpha)$ is decreasing in $\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right]$, and $f(\alpha) > f\left(\frac{2}{3}\right) = \frac{166}{243} > 0$. Therefore, $f(\alpha) > 0$, for $\alpha \in \left(\frac{1}{2}, \frac{2}{3}\right]$.

For subcase (4b), $P_4(2) > 0$ which implies that $(1, 1, 1, 1, 2)$ is the smallest positive integral solution for the level $k = 2$. So we have $1/a_1 + 1/a_2 + 1/a_3 + 1/a_4 \le 1 - 2/a_5 =: \beta \in \left(0, \frac{1}{3}\right]$, since $a_5 \in (2, 3]$. Let $A_i = a_i\beta$, $i = 1, 2, 3, 4$, and notice that condition (2.14) still holds here. (2.13) can be written as

$$\begin{aligned}
&5!P_5 = 5!(P_4(1) + P_4(2)) \\
&\le 5\Bigg[\left(A_1\frac{1+\beta}{2\beta} - 1\right)\left(A_2\frac{1+\beta}{2\beta} - 1\right)\left(A_3\frac{1+\beta}{2\beta} - 1\right)\left(A_4\frac{1+\beta}{2\beta} - 1\right) \\
&\quad - \left(A_4\frac{1+\beta}{2\beta} - 1\right)^4 + A_4\frac{1+\beta}{2\beta}\left(A_4\frac{1+\beta}{2\beta} - 1\right)\left(A_4\frac{1+\beta}{2\beta} - 2\right)\left(A_4\frac{1+\beta}{2\beta} - 3\right) \\
&\quad + (A_1-1)(A_2-1)(A_3-1)(A_4-1) - (A_4-1)^4 + A_4(A_4-1)(A_4-2)(A_4-3)\Bigg].
\end{aligned} \tag{2.24}$$

It is sufficient to show:

Lemma 2.3. *When $2 < a_5 \le 3$, R.H.S. of (1.8) > R.H.S. of (2.24).*

Proof. Subtracting R.H.S. of (2.24) from R.H.S. of (1.8), writing the expression in terms of A_i, $i = 1, 2, 3, 4$, and β, and multiplying by $(1-\beta)^5\beta^4$, yields

$$\begin{aligned}
\triangle_3 := {} & A_1A_2A_3A_4 \\
& \cdot \big(\tfrac{11}{16} - \tfrac{43}{16}\beta + \tfrac{13}{4}\beta^2 + \tfrac{3}{4}\beta^3 - \tfrac{79}{8}\beta^4 + \tfrac{223}{8}\beta^5 - \tfrac{195}{4}\beta^6 + \tfrac{195}{4}\beta^7 - \tfrac{405}{16}\beta^8 + \tfrac{85}{16}\beta^9\big) \\
[.5pt] & + (A_1A_2A_3 + A_1A_2A_4 + A_1A_3A_4 + A_2A_3A_4) \\
& \cdot \big(-\tfrac{3}{8}\beta + \tfrac{7}{4}\beta^2 - \tfrac{13}{4}\beta^3 + \tfrac{27}{4}\beta^4 - 22\beta^5 + \tfrac{181}{4}\beta^6 - \tfrac{195}{4}\beta^7 + \tfrac{105}{4}\beta^8 - \tfrac{45}{8}\beta^9\big) \\
[.5pt] & + (A_1A_2 + A_1A_3 + A_1A_4 + A_2A_3 + A_2A_4 + A_3A_4) \\
& \cdot \big(-\tfrac{1}{4}\beta^2 + \tfrac{3}{4}\beta^3 - \tfrac{17}{4}\beta^4 + \tfrac{83}{4}\beta^5 - \tfrac{187}{4}\beta^6 + \tfrac{209}{4}\beta^7 - \tfrac{115}{4}\beta^8 + \tfrac{25}{4}\beta^9\big) \\
[.5pt] & + (A_1 + A_2 + A_3)\big(\tfrac{3}{2}\beta^3 - 2\beta^4 - \tfrac{29}{2}\beta^5 + 48\beta^6 - \tfrac{119}{2}\beta^7 + 34\beta^8 - \tfrac{15}{2}\beta^9\big) \\
& + A_4^3\big(\tfrac{5}{4}\beta - \tfrac{5}{2}\beta^2 - \tfrac{5}{2}\beta^3 + \tfrac{35}{2}\beta^4 - 50\beta^5 + \tfrac{185}{2}\beta^6 - \tfrac{195}{2}\beta^7 + \tfrac{105}{2}\beta^8 - \tfrac{45}{4}\beta^9\big) \\
& + A_4^2\big(-\tfrac{25}{4}\beta^2 + \tfrac{75}{4}\beta^3 - \tfrac{125}{4}\beta^4 + \tfrac{375}{4}\beta^5 - \tfrac{875}{4}\beta^6 + \tfrac{1025}{4}\beta^7 - \tfrac{575}{4}\beta^8 + \tfrac{125}{4}\beta^9\big) \\
& + A_4\big(\tfrac{13}{2}\beta^3 - 12\beta^4 - \tfrac{79}{2}\beta^5 + 148\beta^6 - \tfrac{369}{2}\beta^7 + 104\beta^8 - \tfrac{45}{2}\beta^9\big) \\
& + (-40\beta^6 + 40\beta^8).
\end{aligned}$$

The idea is to show that for all $\beta \in \big(0, \frac{1}{3}\big]$, the minimum of $\triangle_3$ in $A_1 \geq 4$, $A_2 \geq 3$, $A_3 \geq 2$ and $A_4 \geq 1$ occurs at $A_1 = 4$, $A_2 = 3$, $A_3 = 2$ and $A_4 = 1$, and $\triangle_3|_{A_1=4,A_2=3,A_3=2,A_4=1} > 0$ for all $\beta \in \big(0, \frac{1}{3}\big]$. We have

$$\begin{aligned}
& \frac{\partial^4 \triangle_3}{\partial A_1 \partial A_2 \partial A_3 \partial A_4} \\
& \quad = \tfrac{11}{16} - \tfrac{43}{16}\beta + \tfrac{13}{4}\beta^2 + \tfrac{3}{4}\beta^3 - \tfrac{79}{8}\beta^4 + \tfrac{223}{8}\beta^5 - \tfrac{195}{4}\beta^6 + \tfrac{195}{4}\beta^7 - \tfrac{405}{16}\beta^8 + \tfrac{85}{16}\beta^9 \\
& \quad = \tfrac{1}{16}(-1+\beta)^4(11 + \beta - 10\beta^2 + 10\beta^3 - 65\beta^4 + 85\beta^5) > 0
\end{aligned} \tag{2.25}$$

for $\beta \in \big(0, \frac{3}{5}\big)$. In fact, $11+\beta-10\beta^2+10\beta^3-65\beta^4+85\beta^5 > 11-9\beta-55\beta^3+85\beta^5 =: f(\beta)$ for $\beta \in \big(0, \frac{3}{5}\big)$. Then $f'(\beta) = -9-165\beta^2+425\beta^4 = 425\big(\beta^2 - \frac{33}{170}\big)^2 - \frac{1701}{68}$. Thus, $f'(\beta) < f'\big(\frac{3}{5}\big) = -\frac{333}{25} < 0$ for $\beta \in \big(0, \frac{3}{5}\big)$, which implies $f(\beta)$ is a decreasing function, i.e. $f(\beta) > f\big(\frac{3}{5}\big) = \frac{206}{625} > 0$ for $\beta \in \big(0, \frac{3}{5}\big)$. It follows that $\frac{\partial^3 \triangle_3}{\partial A_1 \partial A_2 \partial A_3}$ is an increasing function of A_4 for $\beta \in \big(0, \frac{3}{5}\big)$ and $A_4 \geq 1$. Hence its minimum occurs at $A_4 = 1$, and

$$\begin{aligned}
& \frac{\partial^3 \triangle_3}{\partial A_1 \partial A_2 \partial A_3}\bigg|_{A_4=1} \\
& = \Big[A_4\big(\tfrac{11}{16} - \tfrac{43}{16}\beta + \tfrac{13}{4}\beta^2 + \tfrac{3}{4}\beta^3 - \tfrac{79}{8}\beta^4 + \tfrac{223}{8}\beta^5 - \tfrac{195}{4}\beta^6 + \tfrac{195}{4}\beta^7 - \tfrac{405}{16}\beta^8 + \tfrac{85}{16}\beta^9\big) \\
& \qquad + \big(-\tfrac{3}{8}\beta + \tfrac{7}{4}\beta^2 - \tfrac{13}{4}\beta^3 + \tfrac{27}{4}\beta^4 - 22\beta^5 + \tfrac{181}{4}\beta^6 - \tfrac{195}{4}\beta^7 + \tfrac{105}{4}\beta^8 - \tfrac{45}{8}\beta^9\big)\Big]\Big|_{A_4=1} \\
& = -\tfrac{1}{16}(-1+\beta)^5(1+\beta)(11 - 5\beta + 5\beta^2 + 5\beta^3) > 0
\end{aligned} \tag{2.26}$$

for $\beta \in \left(0, \frac{3}{5}\right)$, since $11 - 5\beta + 5\beta^2 + 5\beta^3 > 8 + 5\beta^2 + 5\beta^3 > 0$ for $\beta \in \left(0, \frac{3}{5}\right)$. It follows that $\frac{\partial^3 \triangle_3}{\partial A_1 \partial A_2 \partial A_3} > 0$ for $A_4 \geq 1$ and $\beta \in \left(0, \frac{3}{5}\right)$. Note that $\frac{\partial^2 \triangle_3}{\partial A_1 \partial A_2}$ is symmetric with respect to A_3 and A_4. Thus, $\frac{\partial^3 \triangle_3}{\partial A_1 \partial A_2 \partial A_4} > 0$ for $A_3 \geq 1$ and $\beta \in \left(0, \frac{3}{5}\right)$. Moreover, $\frac{\partial^2 \triangle_3}{\partial A_1 \partial A_2}$ is increasing with respect to A_3 and A_4 for $A_3 \geq A_4 \geq 1$ and $\beta \in \left(0, \frac{3}{5}\right)$. Hence its minimum occurs at $A_3 = A_4 = 1$, and

$$\begin{aligned}
&\left.\frac{\partial^2 \triangle_3}{\partial A_1 \partial A_2}\right|_{A_3=A_4=1} \\
&= \Big[A_3A_4\left(\tfrac{11}{16} - \tfrac{43}{16}\beta + \tfrac{13}{4}\beta^2 + \tfrac{3}{4}\beta^3 - \tfrac{79}{8}\beta^4 + \tfrac{223}{8}\beta^5 - \tfrac{195}{4}\beta^6 + \tfrac{195}{4}\beta^7 - \tfrac{405}{16}\beta^8 + \tfrac{85}{16}\beta^9\right) \\
&\quad + (A_3 + A_4)\left(-\tfrac{3}{8}\beta + \tfrac{7}{4}\beta^2 - \tfrac{13}{4}\beta^3 + \tfrac{27}{4}\beta^4 - 22\beta^5 + \tfrac{181}{4}\beta^6 - \tfrac{195}{4}\beta^7 + \tfrac{105}{4}\beta^8 - \tfrac{45}{8}\beta^9\right) \\
&\quad + \left(-\tfrac{1}{4}\beta^2 + \tfrac{3}{4}\beta^3 - \tfrac{17}{4}\beta^4 + \tfrac{83}{4}\beta^5 - \tfrac{187}{4}\beta^6 + \tfrac{209}{4}\beta^7 - \tfrac{115}{4}\beta^8 + \tfrac{25}{4}\beta^9\right)\Big]\Big|_{A_3=A_4=1} \\
&= \tfrac{1}{16}(-1+\beta)^6(1+\beta)(11+5\beta^2) > 0
\end{aligned} \tag{2.27}$$

for $\beta \in \left(0, \frac{3}{5}\right)$. It follows that $\frac{\partial^2 \triangle_3}{\partial A_1 \partial A_2} > 0$ for $A_3 \geq A_4 \geq 1$ and $\beta \in \left(0, \frac{3}{5}\right)$. As $\frac{\partial \triangle_3}{\partial A_1}$ is symmetric with respect to A_2, A_3, A_4, we also get $\frac{\partial^2 \triangle_3}{\partial A_1 \partial A_3} > 0$ for $A_2 \geq A_4 \geq 1$ and $\beta \in \left(0, \frac{3}{5}\right)$, and $\frac{\partial^2 \triangle_3}{\partial A_1 \partial A_4} > 0$ for $A_2 \geq A_3 \geq 1$ and $\beta \in \left(0, \frac{3}{5}\right)$. Therefore, $\frac{\partial \triangle_3}{\partial A_1}$ is an increasing function of A_2, A_3, A_4 for $A_2 \geq A_3 \geq A_4 \geq 1$ and $\beta \in \left(0, \frac{3}{5}\right)$. Hence its minimum occurs at $A_2 = A_3 = A_4 = 1$, and

$$\begin{aligned}
&\left.\frac{\partial \triangle_3}{\partial A_1}\right|_{A_2=A_3=A_4=1} \\
&= \Big[A_2A_3A_4\left(\tfrac{11}{16} - \tfrac{43}{16}\beta + \tfrac{13}{4}\beta^2 + \tfrac{3}{4}\beta^3 - \tfrac{79}{8}\beta^4 + \tfrac{223}{8}\beta^5 - \tfrac{195}{4}\beta^6 + \tfrac{195}{4}\beta^7 - \tfrac{405}{16}\beta^8 + \tfrac{85}{16}\beta^9\right) \\
&\quad + (A_2A_3 + A_2A_4 + A_3A_4) \\
&\qquad \cdot \left(-\tfrac{3}{8}\beta + \tfrac{7}{4}\beta^2 - \tfrac{13}{4}\beta^3 + \tfrac{27}{4}\beta^4 - 22\beta^5 + \tfrac{181}{4}\beta^6 - \tfrac{195}{4}\beta^7 + \tfrac{105}{4}\beta^8 - \tfrac{45}{8}\beta^9\right) \\
&\quad + (A_2 + A_3 + A_4)\left(-\tfrac{1}{4}\beta^2 + \tfrac{3}{4}\beta^3 - \tfrac{17}{4}\beta^4 + \tfrac{83}{4}\beta^5 - \tfrac{187}{4}\beta^6 + \tfrac{209}{4}\beta^7 - \tfrac{115}{4}\beta^8 + \tfrac{25}{4}\beta^9\right) \\
&\quad + \left(\tfrac{3}{2}\beta^3 - 2\beta^4 - \tfrac{29}{2}\beta^5 + 48\beta^6 - \tfrac{119}{2}\beta^7 + 34\beta^8 - \tfrac{15}{2}\beta^9\right)\Big]\Bigg|_{A_2=A_3=A_4=1} \\
&= \tfrac{1}{16}(-1+\beta)^6(1+\beta)(11+5\beta^2) > 0
\end{aligned} \tag{2.28}$$

for $\beta \in \left(0, \frac{3}{5}\right)$. It follows that $\frac{\partial \triangle_3}{\partial A_1} > 0$ for $A_2 \geq A_3 \geq A_4 \geq 1$ and $\beta \in \left(0, \frac{3}{5}\right)$. As $\triangle_3$ is symmetric with respect to A_1, A_2, A_3, we also have $\frac{\partial \triangle_3}{\partial A_2} > 0$ for $A_1 \geq A_3 \geq A_4 \geq 1$, and $\frac{\partial \triangle_3}{\partial A_3} > 0$ for $A_1 \geq A_2 \geq A_4 \geq 1$. Moreover,

$$\frac{\partial^3\triangle_3}{\partial A_4^3} = 6\big(\tfrac{5}{4}\beta - \tfrac{5}{2}\beta^2 - \tfrac{5}{2}\beta^3 + \tfrac{35}{2}\beta^4 - 50\beta^5 + \tfrac{185}{2}\beta^6 - \tfrac{195}{2}\beta^7 + \tfrac{105}{2}\beta^8 - \tfrac{45}{4}\beta^9\big)$$
$$= -\tfrac{15}{2}(-1+\beta)^5(1+3\beta)(1+3\beta^2)\beta > 0 \tag{2.29}$$

for $\beta \in \big(0, \frac{3}{5}\big)$. It follows that $\frac{\partial^2\triangle_3}{\partial A_4^2}$ is an increasing function of A_4 for $A_4 \geq 1$ and $\beta \in \big(0, \frac{3}{5}\big)$. Thus, its minimum occurs at $A_4 = 1$, and

$$\frac{\partial^2\triangle_3}{\partial A_4^2}\bigg|_{A_4=1}$$
$$= \big[6A_4\big(\tfrac{5}{4}\beta - \tfrac{5}{2}\beta^2 - \tfrac{5}{2}\beta^3 + \tfrac{35}{2}\beta^4 - 50\beta^5 + \tfrac{185}{2}\beta^6 - \tfrac{195}{2}\beta^7 + \tfrac{105}{2}\beta^8 - \tfrac{45}{4}\beta^9\big)$$
$$+2\big(-\tfrac{25}{4}\beta^2 + \tfrac{75}{4}\beta^3 - \tfrac{125}{4}\beta^4 + \tfrac{375}{4}\beta^5 - \tfrac{875}{4}\beta^6 + \tfrac{1025}{4}\beta^7 - \tfrac{575}{4}\beta^8 + \tfrac{125}{4}\beta^9\big)\big]\big|_{A_4=1}$$
$$= -\tfrac{5}{2}(-1+\beta)^5\beta(3+4\beta-\beta^2+2\beta^3) > 0 \tag{2.30}$$

for $\beta \in \big(0, \frac{3}{5}\big)$, since $3 + 4\beta - \beta^2 + 2\beta^3 > 3 + 4\beta - \frac{3}{5}\beta = 3 + \frac{17}{5}\beta > 0$ for $\beta \in \big(0, \frac{3}{5}\big)$. It follows that $\frac{\partial^2\triangle_3}{\partial A_4^2} > 0$ for $A_4 \geq 1$ and $\beta \in \big(0, \frac{3}{5}\big)$. Thus, $\frac{\partial\triangle_3}{\partial A_4}$ is an increasing function of A_4 for $A_4 \geq 1$ and $\beta \in \big(0, \frac{3}{5}\big)$. Moreover, it is an increasing function with respect to A_1, A_2, A_3, A_4 for $A_1 \geq A_2 \geq A_3 \geq A_4 \geq 1$, $\beta \in \big(0, \frac{3}{5}\big)$, since it is symmetric with respect to A_1, A_2, A_3. Hence its minimum occurs at $A_1 = A_2 = A_3 = A_4 = 1$, and

$$\frac{\partial\triangle_3}{\partial A_4}\bigg|_{A_1=A_2=A_3=A_4=1}$$
$$= \big[A_1A_2A_3$$
$$\cdot\big(\tfrac{11}{16} - \tfrac{43}{16}\beta + \tfrac{13}{4}\beta^2 + \tfrac{3}{4}\beta^3 - \tfrac{79}{8}\beta^4 + \tfrac{223}{8}\beta^5 - \tfrac{195}{4}\beta^6 + \tfrac{195}{4}\beta^7 - \tfrac{405}{16}\beta^8 + \tfrac{85}{16}\beta^9\big)$$
$$+ (A_1A_2 + A_1A_3 + A_2A_3)$$
$$\cdot\big(-\tfrac{3}{8}\beta + \tfrac{7}{4}\beta^2 - \tfrac{13}{4}\beta^3 + \tfrac{27}{4}\beta^4 - 22\beta^5 + \tfrac{181}{4}\beta^6 - \tfrac{195}{4}\beta^7 + \tfrac{105}{4}\beta^8 - \tfrac{45}{8}\beta^9\big)$$
$$+ (A_1 + A_2 + A_3)\big(-\tfrac{1}{4}\beta^2 + \tfrac{3}{4}\beta^3 - \tfrac{17}{4}\beta^4 + \tfrac{83}{4}\beta^5 - \tfrac{187}{4}\beta^6 + \tfrac{209}{4}\beta^7 - \tfrac{115}{4}\beta^8 + \tfrac{25}{4}\beta^9\big)$$
$$+ \big(\tfrac{13}{2}\beta^3 - 12\beta^4 - \tfrac{79}{2}\beta^5 + 148\beta^6 - \tfrac{369}{2}\beta^7 + 104\beta^8 - \tfrac{45}{2}\beta^9\big)$$
$$+ 3A_4^2\big(\tfrac{5}{4}\beta - \tfrac{5}{2}\beta^2 - \tfrac{5}{2}\beta^3 + \tfrac{35}{2}\beta^4 - 50\beta^5 + \tfrac{185}{2}\beta^6 - \tfrac{195}{2}\beta^7 + \tfrac{105}{2}\beta^8 - \tfrac{45}{4}\beta^9\big)$$
$$+ 2A_4\big(-\tfrac{25}{4}\beta^2 + \tfrac{75}{4}\beta^3 - \tfrac{125}{4}\beta^4 + \tfrac{375}{4}\beta^5$$
$$- \tfrac{875}{4}\beta^6 + \tfrac{1025}{4}\beta^7 - \tfrac{575}{4}\beta^8 + \tfrac{125}{4}\beta^9\big)\big]\big|_{A_1=A_2=A_3=A_4=1}$$
$$= -\tfrac{1}{16}(-1+\beta)^5(11 + 54\beta + 64\beta^2 + 66\beta^3 + 285\beta^4) > 0 \tag{2.31}$$

for $\beta \in \big(0, \frac{3}{5}\big)$. It follows that $\frac{\partial\triangle_3}{\partial A_4} > 0$ for $A_1 \geq A_2 \geq A_3 \geq A_4 \geq 1$ and $\beta \in \big(0, \frac{3}{5}\big)$. Therefore, $\triangle_3$ is an increasing function of A_1, A_2, A_3, A_4, for $A_1 \geq A_2 \geq A_3 \geq A_4 \geq 1$ and $\beta \in \big(0, \frac{3}{5}\big)$. Thus, its minimum occurs at $A_1 = A_2 = A_3 = A_4 = 1$, and taking condition (2.14) into consideration, we have

$$\triangle_3|_{A_1=4,A_2=3,A_3=2,A_4=1} > \triangle_3|_{A_1=A_2=A_3=A_4=1}$$
$$= -\tfrac{1}{16}(-1+\beta)(1+\beta)(-1+3\beta)(-11+14\beta+31\beta^2-172\beta^3+115\beta^4-322\beta^5+25\beta^6)$$
$$> 0$$

for $\beta \in \left(0, \frac{1}{3}\right]$, since $-11 + 14\beta + 31\beta^2 - 172\beta^3 + 115\beta^4 - 322\beta^5 + 25\beta^6 < -11 + \frac{14}{3} + \frac{31}{9} - 172\beta^3 + \frac{115}{3}\beta^3 - 322\beta^5 + \frac{25}{3}\beta^5 = -\frac{26}{9} - \frac{401}{3}\beta^3 - \frac{941}{3}\beta^5 < 0$ for $\beta \in \left(0, \frac{1}{3}\right]$. It follows that $\triangle_3 > 0$ for $A_1 \geq 4$, $A_2 \geq 3$, $A_3 \geq 2$, $A_4 \geq 1$ and $\beta \in \left(0, \frac{1}{3}\right]$. □

For case (3), there are four levels: $k = 1$, $k = 2$, $k = 3$ and $k = 4$. It is easy to see that $P_4(4) = 0$. From the condition $P_5 > 0$, we know that $P_4(1) > 0$, but the positivity of $P_4(2)$ and $P_4(3)$ is unknown. Therefore, we split this case into three subcases:

(3a) $P_4(2) = P_4(3) = 0$ (i.e. $k_0 = 1$ in (2.13));
(3b) $P_4(2) > 0$, $P_4(3) = 0$ (i.e. $k_0 = 2$ in (2.13));
(3c) $P_4(2) > 0$, $P_4(3) > 0$ (i.e. $k_0 = 3$ in (2.13)).

For subcase (3a), the argument is nearly the same as that in subcase (4a) and case (5). In the present case $P_5 = P_4(1) > 0$, thus $(1, 1, 1, 1, 1)$ is the smallest positive integral solution, i.e. $1/a_1 + 1/a_2 + 1/a_3 + 1/a_4 \leq 1 - 1/a_5 =: \alpha \in \left(\frac{2}{3}, \frac{3}{4}\right]$, since $a_5 \in (3, 4]$. The new range of α helps us to improve condition (2.14) to

$$A_1 \geq 4, \quad A_2 \geq 3, \quad A_3 \geq A_4 \geq \frac{\alpha}{1-\alpha}, \tag{2.32}$$

since $A_i = a_i\alpha \geq a_5\alpha = \frac{\alpha}{1-\alpha}$, and $2 < \frac{\alpha}{1-\alpha} \leq 3$ for $\alpha \in \left(\frac{2}{3}, \frac{3}{4}\right]$. Therefore, it is sufficient to show that $\triangle_2 > 0$ in (2.32) for $\alpha \in \left(\frac{2}{3}, \frac{3}{4}\right]$. As argued in subcase (4a), all the computations (2.16)–(2.22) hold in the range (2.14) for $\alpha \in \left(0, \frac{4}{5}\right)$, so they hold in (2.32) for $\alpha \in \left(\frac{2}{3}, \frac{3}{4}\right]$. It only remains to show the positivity of $\triangle_2|_{A_1=4,A_2=3,A_3=A_4=\frac{\alpha}{1-\alpha}}$. Taking condition (2.32) instead of (2.14) yields

$$\triangle_2|_{A_1=4,A_2=3,A_3=A_4=\frac{\alpha}{1-\alpha}} = -25 + 176\alpha - 411\alpha^2 + 415\alpha^3 - 160\alpha^4 > 0$$

for $\alpha \in \left(\frac{2}{3}, \frac{3}{4}\right]$. In fact, let $f(\alpha) := -25 + 176\alpha - 411\alpha^2 + 415\alpha^3 - 160\alpha^4$. Then $f''(\alpha) = -822 + 2490\alpha - 1920\alpha^2 = -1920\left(\alpha - \frac{83}{128}\right)^2 - \frac{1881}{128} < 0$, for $\alpha \in \left(\frac{2}{3}, \frac{3}{4}\right]$. Thus, $f'(\alpha)$ is decreasing in α, and $f'(\alpha) \leq f'\left(\frac{2}{3}\right) = -\frac{224}{27} < 0$. Therefore, $f(\alpha)$ is decreasing in $\alpha \in \left(\frac{2}{3}, \frac{3}{4}\right]$ and $f(\alpha) \geq f\left(\frac{3}{4}\right) = \frac{17}{64} > 0$ for $\alpha \in \left(\frac{2}{3}, \frac{3}{4}\right]$.

For subcase (3b), the proof is nearly the same as in subcase (4b). In the present case $P_5 = P_4(1) + P_4(2) > 0$, thus $(1, 1, 1, 1, 2)$ is the smallest positive integral solution for the level $k = 2$, i.e. $1/a_1 + 1/a_2 + 1/a_3 + 1/a_4 \leq 1 - 2/a_5 =: \beta \in \left(\frac{1}{3}, \frac{1}{2}\right]$, since $a_5 \in (3, 4]$. Let $A_i = a_i\beta$, $i = 1, 2, 3, 4$. The new range of β helps us to improve condition (2.14) to

$$A_1 \geq 4, \quad A_2 \geq 3, \quad A_3 \geq 2, \quad A_4 \geq \frac{2\beta}{1-\beta}, \tag{2.33}$$

since $A_i = a_i\beta \geq a_5\beta = \frac{2\beta}{1-\beta}$. With $\beta \in \left(\frac{1}{3}, \frac{1}{2}\right]$, it is easy to check that $1 < \frac{2\beta}{1-\beta} \leq 2$. Therefore, it is sufficient to show that $\triangle_3 > 0$ in (2.33) for $\beta \in \left(\frac{1}{3}, \frac{1}{2}\right]$. In the proof of Lemma 2.3, all the computations of the partial derivatives in (2.25)–(2.31) are valid in the even larger range (2.14) for $\beta \in \left(0, \frac{3}{5}\right)$, so they hold in the new range (2.33) for $\beta \in \left(\frac{1}{3}, \frac{1}{2}\right]$. We only need to show the positivity of $\triangle_2|_{A_1=4, A_2=3, A_3=2, A_4=\frac{2\beta}{1-\beta}}$ for $\beta \in \left(\frac{1}{3}, \frac{1}{2}\right]$. Taking condition (2.33) instead of (2.14) yields

$$\begin{aligned}&\triangle_3|_{A_1=4, A_2=3, A_3=2, A_4=\frac{2\beta}{1-\beta}}\\ &\quad = \beta(-1+\beta)(-24+56\beta-26\beta^2-65\beta^3+222\beta^4-256\beta^5+28\beta^6+145\beta^7) > 0\end{aligned}$$

for $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$. Indeed, $-24+56\beta-26\beta^2-65\beta^3+222\beta^4-256\beta^5+28\beta^6+145\beta^7 \le -24+56\beta-26\beta^2-65\beta^3+222\beta^4-\frac{823}{4}\beta^5 < -24+56\beta-26\beta^2-65\beta^3+\frac{1841}{12}\beta^4 \le -24+56\beta-26\beta^2+\frac{281}{24}\beta^3 \le -24+56\beta-\frac{967}{48}\beta^2 =: f(\beta)$, and $f(\beta) = -\frac{967}{48}\left(\beta-\frac{1344}{967}\right)^2+\frac{14424}{967}$. Thus, $f(\beta) < f\left(\frac{1}{2}\right) = -\frac{199}{192} < 0$ for $\beta \in \left(\frac{1}{3}, \frac{1}{2}\right]$.

For subcase (3c), $P_4(3) > 0$, which implies that $(1, 1, 1, 1, 3)$ is the smallest positive integral solution for the level $k = 3$. So we have $1/a_1+1/a_2+1/a_3+1/a_4 \le 1-3/a_5 =: \gamma \in \left(0, \frac{1}{4}\right]$, since $a_5 \in (3, 4]$. Let $A_i = a_i\gamma$, $i = 1, 2, 3, 4$, and notice that condition (2.14) still holds here. (2.13) can be written as

$$\begin{aligned}5!P_5 &= 5!(P_4(1)+P_4(2)+P_4(3))\\
&\le 5\Bigg[\left(A_1\frac{2+\gamma}{3\gamma}-1\right)\left(A_2\frac{2+\gamma}{3\gamma}-1\right)\left(A_3\frac{2+\gamma}{3\gamma}-1\right)\left(A_4\frac{2+\gamma}{3\gamma}-1\right)\\
&\quad -\left(A_4\frac{2+\gamma}{3\gamma}-1\right)^4+A_4\frac{2+\gamma}{3\gamma}\left(A_4\frac{2+\gamma}{3\gamma}-1\right)\left(A_4\frac{2+\gamma}{3\gamma}-2\right)\left(A_4\frac{2+\gamma}{3\gamma}-3\right)\\
&\quad +\left(A_1\frac{1+2\gamma}{3\gamma}-1\right)\left(A_2\frac{1+2\gamma}{3\gamma}-1\right)\left(A_3\frac{1+2\gamma}{3\gamma}-1\right)\left(A_4\frac{1+2\gamma}{3\gamma}-1\right)\\
&\quad -\left(A_4\frac{1+2\gamma}{3\gamma}-1\right)^4+A_4\frac{1+2\gamma}{3\gamma}\left(A_4\frac{1+2\gamma}{3\gamma}-1\right)\left(A_4\frac{1+2\gamma}{3\gamma}-2\right)\left(A_4\frac{1+2\gamma}{3\gamma}-3\right)\\
&\quad +(A_1-1)(A_2-1)(A_3-1)(A_4-1)-(A_4-1)^4+A_4(A_4-1)(A_4-2)(A_4-3)\Bigg].\end{aligned} \tag{2.34}$$

It is sufficient to prove

Lemma 2.4. *When* $3 < a_5 \le 4$, *R.H.S. of* (1.8) $>$ *R.H.S. of* (2.34).

Proof. Subtracting R.H.S. of (2.34) from R.H.S. of (1.8), writing the expression in terms of A_i, $i = 1, 2, 3, 4$, and γ, and multiplying by $(1-\gamma)^5\gamma^4$, yields

$$\begin{aligned}\triangle_4 :=& A_1A_2A_3A_4\\
&\cdot\left(\tfrac{77}{81}-\tfrac{38}{9}\gamma+\tfrac{62}{9}\gamma^2-\tfrac{104}{27}\gamma^3-\tfrac{53}{9}\gamma^4+\tfrac{224}{9}\gamma^5-\tfrac{1300}{27}\gamma^6+\tfrac{460}{9}\gamma^7-\tfrac{250}{9}\gamma^8+\tfrac{490}{81}\gamma^9\right)\\
&+(A_1A_2A_3+A_1A_2A_4+A_1A_3A_4+A_2A_3A_4)\\
&\quad\cdot\left(-\tfrac{1}{3}\gamma+2\gamma^2-\tfrac{14}{3}\gamma^3+\tfrac{26}{3}\gamma^4-23\gamma^5+\tfrac{142}{3}\gamma^6-\tfrac{160}{3}\gamma^7+30\gamma^8-\tfrac{20}{3}\gamma^9\right)\end{aligned}$$

$$\begin{aligned}
&+(A_1A_2+A_1A_3+A_1A_4+A_2A_3+A_2A_4+A_3A_4)\\
&\qquad\cdot\big(-\tfrac{7}{9}\gamma^2+\tfrac{22}{9}\gamma^3-\tfrac{16}{3}\gamma^4+\tfrac{182}{9}\gamma^5-\tfrac{443}{9}\gamma^6+\tfrac{178}{3}\gamma^7-\tfrac{310}{9}\gamma^8+\tfrac{70}{9}\gamma^9\big)\\
&+(A_1+A_2+A_3)(3\gamma^3-8\gamma^4-8\gamma^5+52\gamma^6-73\gamma^7+44\gamma^8-10\gamma^9)\\
&+A_4^3\big(\tfrac{10}{3}\gamma-10\gamma^2+\tfrac{20}{3}\gamma^3+\tfrac{40}{3}\gamma^4-50\gamma^5+\tfrac{290}{3}\gamma^6-\tfrac{320}{3}\gamma^7+60\gamma^8-\tfrac{40}{3}\gamma^9\big)\\
&+A_4^2\big(-\tfrac{125}{9}\gamma^2+\tfrac{425}{9}\gamma^3-\tfrac{200}{3}\gamma^4+\tfrac{1000}{9}\gamma^5-\tfrac{2125}{9}\gamma^6+\tfrac{875}{3}\gamma^7-\tfrac{1550}{9}\gamma^8+\tfrac{350}{9}\gamma^9\big)\\
&+A_4(13\gamma^3-38\gamma^4-8\gamma^5+152\gamma^6-223\gamma^7+134\gamma^8-30\gamma^9)\\
&+(-30\gamma^4-105\gamma^5-45\gamma^6+120\gamma^7+60\gamma^8).
\end{aligned}$$

The idea is to show that for all $\gamma \in \big(0, \frac{1}{4}\big]$, the minimum of $\triangle_4$ in $A_1 \geq 4$, $A_2 \geq 3$, $A_3 \geq 2$ and $A_4 \geq 1$ occurs at $A_1 = 4$, $A_2 = 3$, $A_3 = 2$, $A_4 = 1$, and $\triangle_4|_{A_1=4,A_2=3,A_3=2,A_4=1} > 0$ for all $\gamma \in \big(0, \frac{1}{4}\big]$. We have

$$\frac{\partial^4\triangle_4}{\partial A_1\partial A_2\partial A_3\partial A_4} = \tfrac{1}{81}(-1+\gamma)^4(77-34\gamma-40\gamma^2+40\gamma^3-290\gamma^4+490\gamma^5) > 0 \tag{2.35}$$

for $\gamma \in \big(0, \frac{2}{5}\big)$, since $77-34\gamma-40\gamma^2+40\gamma^3-290\gamma^4+490\gamma^5 > 77-34\cdot\frac{2}{5}-40\cdot\big(\frac{2}{5}\big)^2-290\cdot\big(\frac{2}{5}\big)^4+40\gamma^3+490\gamma^5 = \frac{6197}{125}+40\gamma^3+490\gamma^5 > 0$. It follows that $\frac{\partial^3\triangle_4}{\partial A_1\partial A_2\partial A_3}$ is an increasing function of A_4 for $\gamma \in \big(0, \frac{2}{5}\big)$ and $A_4 \geq 1$. Hence its minimum occurs at $A_4 = 1$, and

$$\begin{aligned}
&\frac{\partial^3\triangle_4}{\partial A_1\partial A_2\partial A_3}\bigg|_{A_4=1}\\
&=\big[A_4\big(\tfrac{77}{81}-\tfrac{38}{9}\gamma+\tfrac{62}{9}\gamma^2-\tfrac{104}{27}\gamma^3-\tfrac{53}{9}\gamma^4+\tfrac{224}{9}\gamma^5-\tfrac{1300}{27}\gamma^6+\tfrac{460}{9}\gamma^7-\tfrac{250}{9}\gamma^8+\tfrac{490}{81}\gamma^9\big)\\
&\qquad+(-\tfrac{1}{3}\gamma+2\gamma^2-\tfrac{14}{3}\gamma^3+\tfrac{26}{3}\gamma^4-23\gamma^5+\tfrac{142}{3}\gamma^6-\tfrac{160}{3}\gamma^7+30\gamma^8-\tfrac{20}{3}\gamma^9)\big]\big|_{A_4=1}\\
&=-\tfrac{1}{81}(-1+\gamma)^5(77+16\gamma+30\gamma^2+70\gamma^3+50\gamma^4) > 0
\end{aligned}\tag{2.36}$$

for $\gamma \in \big(0, \frac{2}{5}\big)$. It follows that $\frac{\partial^3\triangle_4}{\partial A_1\partial A_2\partial A_3} > 0$ for $A_4 \geq 1$ and $\gamma \in \big(0, \frac{2}{5}\big)$. Note that $\frac{\partial^2\triangle_4}{\partial A_1\partial A_2}$ is symmetric with respect to A_3 and A_4. Thus, $\frac{\partial^3\triangle_4}{\partial A_1\partial A_2\partial A_4} > 0$ for $A_3 \geq 1$ and $\beta \in \big(0, \frac{3}{5}\big)$. Moreover, $\frac{\partial^2\triangle_4}{\partial A_1\partial A_2}$ is increasing with respect to A_3 and A_4 for $A_3 \geq A_4 \geq 1$ and $\gamma \in \big(0, \frac{2}{5}\big)$. Hence its minimum occurs at $A_3 = A_4 = 1$, and

$$\begin{aligned}
&\frac{\partial^2\triangle_4}{\partial A_1\partial A_2}\bigg|_{A_3=A_4=1}\\
&=\big[A_3A_4\big(\tfrac{77}{81}-\tfrac{38}{9}\gamma+\tfrac{62}{9}\gamma^2-\tfrac{104}{27}\gamma^3-\tfrac{53}{9}\gamma^4+\tfrac{224}{9}\gamma^5-\tfrac{1300}{27}\gamma^6+\tfrac{460}{9}\gamma^7-\tfrac{250}{9}\gamma^8+\tfrac{490}{81}\gamma^9\big)\\
&\qquad+(A_3+A_4)\big(-\tfrac{1}{3}\gamma+2\gamma^2-\tfrac{14}{3}\gamma^3+\tfrac{26}{3}\gamma^4-23\gamma^5+\tfrac{142}{3}\gamma^6-\tfrac{160}{3}\gamma^7+30\gamma^8-\tfrac{20}{3}\gamma^9\big)\\
&\qquad+\big(-\tfrac{7}{9}\gamma^2+\tfrac{22}{9}\gamma^3-\tfrac{16}{3}\gamma^4+\tfrac{182}{9}\gamma^5-\tfrac{443}{9}\gamma^6+\tfrac{178}{3}\gamma^7-\tfrac{310}{9}\gamma^8+\tfrac{70}{9}\gamma^9\big)\big]\big|_{A_3=A_4=1}\\
&=\tfrac{1}{81}(-1+\gamma)^6(77+66\gamma+60\gamma^2+40\gamma^3) > 0
\end{aligned}\tag{2.37}$$

for $\gamma \in \left(0, \frac{2}{5}\right)$. It follows that $\frac{\partial^2 \triangle_4}{\partial A_1 \partial A_2} > 0$ for $A_3 \geq A_4 \geq 1$ and $\gamma \in \left(0, \frac{2}{5}\right)$. As $\frac{\partial \triangle_4}{\partial A_1}$ is symmetric with respect to A_2, A_3, A_4, we also get $\frac{\partial^2 \triangle_4}{\partial A_1 \partial A_3} > 0$ for $A_2 \geq A_4 \geq 1$ and $\gamma \in \left(0, \frac{2}{5}\right)$, and $\frac{\partial^2 \triangle_4}{\partial A_1 \partial A_4} > 0$ for $A_2 \geq A_3 \geq 1$ and $\gamma \in \left(0, \frac{2}{5}\right)$. Therefore, $\frac{\partial \triangle_4}{\partial A_1}$ is an increasing function of A_2, A_3, A_4 for $A_2 \geq A_3 \geq A_4 \geq 1$ and $\gamma \in \left(0, \frac{2}{5}\right)$. Hence its minimum occurs at $A_2 = A_3 = A_4 = 1$, and

$$\begin{aligned}
&\left.\frac{\partial \triangle_4}{\partial A_1}\right|_{A_2=A_3=A_4=1} \\
&= \Big[A_2 A_3 A_4 \\
&\qquad \cdot \left(\tfrac{77}{81} - \tfrac{38}{9}\gamma + \tfrac{62}{9}\gamma^2 - \tfrac{104}{27}\gamma^3 - \tfrac{53}{9}\gamma^4 + \tfrac{224}{9}\gamma^5 - \tfrac{1300}{27}\gamma^6 + \tfrac{460}{9}\gamma^7 - \tfrac{250}{9}\gamma^8 + \tfrac{490}{81}\gamma^9\right) \\
&\quad + (A_2 A_3 + A_2 A_4 + A_3 A_4) \\
&\qquad \cdot \left(-\tfrac{1}{3}\gamma + 2\gamma^2 - \tfrac{14}{3}\gamma^3 + \tfrac{26}{3}\gamma^4 - 23\gamma^5 + \tfrac{142}{3}\gamma^6 - \tfrac{160}{3}\gamma^7 + 30\gamma^8 - \tfrac{20}{3}\gamma^9\right) \\
&\quad + (A_2 + A_3 + A_4)\left(-\tfrac{7}{9}\gamma^2 + \tfrac{22}{9}\gamma^3 - \tfrac{16}{3}\gamma^4 + \tfrac{182}{9}\gamma^5 - \tfrac{443}{9}\gamma^6 + \tfrac{178}{3}\gamma^7 - \tfrac{310}{9}\gamma^8 + \tfrac{70}{9}\gamma^9\right) \\
&\quad + \left(3\gamma^3 - 8\gamma^4 - 8\gamma^5 + 52\gamma^6 - 73\gamma^7 + 44\gamma^8 - 10\gamma^9\right)\Big]\Big|_{A_2=A_3=A_4=1} \\
&= -\tfrac{1}{81}(-1+\gamma)^7(77 + 116\gamma + 50\gamma^2) > 0
\end{aligned} \tag{2.38}$$

for $\gamma \in \left(0, \frac{2}{5}\right)$. It follows that $\frac{\partial \triangle_4}{\partial A_1} > 0$ for $A_2 \geq A_3 \geq A_4 \geq 1$ and $\gamma \in \left(0, \frac{2}{5}\right)$. Since $\triangle_4$ is symmetric with respect to A_1, A_2, A_3, we also have $\frac{\partial \triangle_4}{\partial A_2} > 0$ for $A_1 \geq A_3 \geq A_4 \geq 1$, and $\frac{\partial \triangle_4}{\partial A_3} > 0$ for $A_1 \geq A_2 \geq A_4 \geq 1$. Moreover,

$$\frac{\partial^3 \triangle_4}{\partial A_4^3} = -20(-1+\gamma)^5 \gamma (1 + 2\gamma)(1 + 2\gamma^2) > 0 \tag{2.39}$$

for $\gamma \in \left(0, \frac{2}{5}\right)$. It follows that $\frac{\partial^2 \triangle_4}{\partial A_4^2}$ is an increasing function of A_4 for $A_4 \geq 1$ and $\gamma \in \left(0, \frac{2}{5}\right)$. Thus, its minimum occurs at $A_4 = 1$, and

$$\begin{aligned}
&\left.\frac{\partial^2 \triangle_4}{\partial A_4^2}\right|_{A_4=1} \\
&= \Big[6A_4\left(\tfrac{10}{3}\gamma - 10\gamma^2 + \tfrac{20}{3}\gamma^3 + \tfrac{40}{3}\gamma^4 - 50\gamma^5 + \tfrac{290}{3}\gamma^6 - \tfrac{320}{3}\gamma^7 + 60\gamma^8 - \tfrac{40}{3}\gamma^9\right) \\
&\quad + 2\left(-\tfrac{125}{9}\gamma^2 + \tfrac{425}{9}\gamma^3 - \tfrac{200}{3}\gamma^4 + \tfrac{1000}{9}\gamma^5 - \tfrac{2125}{9}\gamma^6 + \tfrac{875}{3}\gamma^7 - \tfrac{1550}{9}\gamma^8 + \tfrac{350}{9}\gamma^9\right)\Big]\Big|_{A_4=1} \\
&= -\tfrac{10}{9}(-1+\gamma)^5 \gamma (18 + 11\gamma - 4\gamma^2 + 2\gamma^3) > 0
\end{aligned} \tag{2.40}$$

for $\gamma \in \left(0, \frac{2}{5}\right)$, since $18 + 11\gamma - 4\gamma^2 + 2\gamma^3 > 18 + 11\gamma - \frac{8}{5}\gamma + 2\gamma^3 = 18 + \frac{47}{5}\gamma + 2\gamma^3 > 0$ for $\gamma \in \left(0, \frac{2}{5}\right)$. It follows that $\frac{\partial^2 \triangle_4}{\partial A_4^2} > 0$ for $A_4 \geq 1$ and $\gamma \in \left(0, \frac{2}{5}\right)$. Thus, $\frac{\partial \triangle_4}{\partial A_4}$ is an increasing function of A_4 for $A_4 \geq 1$ and $\gamma \in \left(0, \frac{2}{5}\right)$. Moreover, it is an increasing function of A_1, A_2, A_3, A_4 for $A_1 \geq A_2 \geq A_3 \geq A_4 \geq 1$ and $\gamma \in \left(0, \frac{2}{5}\right)$, since it is symmetric with respect to A_1, A_2, A_3. Therefore its minimum occurs at $A_1 = A_2 =$

$A_3 = A_4 = 1$, and

$$\begin{aligned}
&\frac{\partial \triangle_4}{\partial A_4}\bigg|_{A_1=A_2=A_3=A_4=1} \\
&= \big[A_1A_2A_3 \\
&\qquad \cdot \big(\tfrac{77}{81} - \tfrac{38}{9}\gamma + \tfrac{62}{9}\gamma^2 - \tfrac{104}{27}\gamma^3 - \tfrac{53}{9}\gamma^4 + \tfrac{224}{9}\gamma^5 - \tfrac{1300}{27}\gamma^6 + \tfrac{460}{9}\gamma^7 - \tfrac{250}{9}\gamma^8 + \tfrac{490}{81}\gamma^9\big) \\
&\quad + (A_1A_2 + A_1A_3 + A_2A_3) \\
&\qquad \cdot \big(-\tfrac{1}{3}\gamma + 2\gamma^2 - \tfrac{14}{3}\gamma^3 + \tfrac{26}{3}\gamma^4 - 23\gamma^5 + \tfrac{142}{3}\gamma^6 - \tfrac{160}{3}\gamma^7 + 30\gamma^8 - \tfrac{20}{3}\gamma^9\big) \\
&\quad + (A_1 + A_2 + A_3)\big(-\tfrac{7}{9}\gamma^2 + \tfrac{22}{9}\gamma^3 - \tfrac{16}{3}\gamma^4 + \tfrac{182}{9}\gamma^5 - \tfrac{443}{9}\gamma^6 + \tfrac{178}{3}\gamma^7 - \tfrac{310}{9}\gamma^8 + \tfrac{70}{9}\gamma^9\big) \\
&\quad + (13\gamma^3 - 38\gamma^4 - 8\gamma^5 + 152\gamma^6 - 223\gamma^7 + 134\gamma^8 - 30\gamma^9) \\
&\quad + 3A_4^2\big(\tfrac{10}{3}\gamma - 10\gamma^2 + \tfrac{20}{3}\gamma^3 + \tfrac{40}{3}\gamma^4 - 50\gamma^5 + \tfrac{290}{3}\gamma^6 - \tfrac{320}{3}\gamma^7 + 60\gamma^8 - \tfrac{40}{3}\gamma^9\big) \\
&\quad + 2A_4\big(-\tfrac{125}{9}\gamma^2 + \tfrac{425}{9}\gamma^3 - \tfrac{200}{3}\gamma^4 + \tfrac{1000}{9}\gamma^5 \\
&\qquad\qquad - \tfrac{2125}{9}\gamma^6 + \tfrac{875}{3}\gamma^7 - \tfrac{1550}{9}\gamma^8 + \tfrac{350}{9}\gamma^9\big)\big]\big|_{A_1=A_2=A_3=A_4=1} \\
&= \tfrac{1}{81}(-1+\gamma)^5(-77 - 772\gamma + 735\gamma^2 + 1154\gamma^3 + 1390\gamma^4) > 0 \qquad (2.41)
\end{aligned}$$

for $\gamma \in \big(0, \frac{2}{5}\big)$, since $-77-772\gamma+735\gamma^2+1154\gamma^3+1390\gamma^4 < -77-773\gamma+735\cdot\frac{2}{5}\gamma + 1154\cdot\big(\frac{2}{5}\big)^2\gamma + 1390\cdot\big(\frac{2}{5}\big)^4 = -\frac{5177}{125} - \frac{7334}{25}\gamma < 0$ for $\gamma \in \big(0, \frac{2}{5}\big)$. It follows that $\frac{\partial\triangle_4}{\partial A_4} > 0$ for $A_1 \geq A_2 \geq A_3 \geq A_4 \geq 1$ and $\gamma \in \big(0, \frac{2}{5}\big)$. Therefore, $\triangle_4$ is an increasing function of A_1, A_2, A_3, A_4, for $A_1 \geq A_2 \geq A_3 \geq A_4 \geq 1$ and $\gamma \in \big(0, \frac{2}{5}\big)$. Thus, its minimum occurs at $A_1 = A_2 = A_3 = A_4 = 1$, and taking condition (2.14) into consideration, we have

$$\begin{aligned}
&\triangle_4|_{A_1=4, A_2=3, A_3=2, A_4=1} \\
&\qquad = -\tfrac{1}{27}(-1+\gamma)(616 - 2480\gamma + 3304\gamma^2 - 647\gamma^3 - 3023\gamma^4 - 2180\gamma^5 \\
&\qquad\qquad - 4235\gamma^6 - 2570\gamma^7 + 280\gamma^8) > 0
\end{aligned}$$

for $\gamma \in \big(0, \frac{1}{4}\big]$. In fact, $616 - 2480\gamma + 3304\gamma^2 - 647\gamma^3 - 3023\gamma^4 - 2180\gamma^5 - 4235\gamma^6 - 2570\gamma^7 + 280\gamma^8 > 616 - 2480\gamma + (3304 - 647\cdot\frac{1}{4} - 3023\cdot\big(\frac{1}{4}\big)^2 - 2180\cdot\big(\frac{1}{4}\big)^3 - 4235\cdot\big(\frac{1}{4}\big)^4 - 2570\cdot\big(\frac{1}{4}\big)^5)\gamma^2 = 616 - 2480\gamma + \frac{1484901}{8192}\gamma^2 =: f(\gamma)$. Since $f(\gamma) = \frac{1484901}{8192}\big(\gamma - \frac{10158080}{1484901}\big)^2 - \frac{11681320184}{1484901}$, we have $f(\gamma) > f\big(\frac{1}{4}\big) = \frac{960613}{131072} > 0$ for $\gamma \in \big(0, \frac{1}{4}\big]$. It follows that $\triangle_4 > 0$ for $A_1 \geq 4$, $A_2 \geq 3$, $A_3 \geq 2$, $A_4 \geq 1$ and $\gamma \in \big(0, \frac{1}{4}\big]$. □

For case (2), there are four levels: $k = 1$, $k = 2$, $k = 3$ and $k = 4$. From the condition $P_5 > 0$, we know that $P_4(1) > 0$, but the positivity of $P_4(2)$, $P_4(3)$ and $P_4(4)$ is unknown. Therefore, we split this case into four subcases:

(2a) $P_4(2) = P_4(3) = P_4(4) = 0$ (i.e. $k_0 = 1$ in (2.13));
(2b) $P_4(2) > 0$, $P_4(3) = P_4(4) = 0$ (i.e. $k_0 = 2$ in (2.13));
(2c) $P_4(2) > 0$, $P_4(3) > 0$, $P_4(4) = 0$ (i.e. $k_0 = 3$ in (2.13));
(2d) $P_4(2) > 0$, $P_4(3) > 0$, $P_4(4) > 0$ (i.e. $k_0 = 4$ in (2.13)).

For subcase (2a), the proof is nearly the same as in subcase (3a), subcase (4a) and case (5). In the present case $P_5 = P_4(1) > 0$, thus $(1, 1, 1, 1, 1)$ is the smallest positive integral solution, i.e. $1/a_1 + 1/a_2 + 1/a_3 + 1/a_4 \le 1 - 1/a_5 =: \alpha \in \left(\frac{3}{4}, \frac{4}{5}\right)$, since $a_5 \in (4, 5)$. The new range of α helps us to improve condition (2.14) to

$$A_1 \ge 4, \qquad A_2 \ge A_3 \ge A_4 \ge \frac{\alpha}{1-\alpha}, \tag{2.42}$$

since $A_i = a_i\alpha \ge a_5\alpha = \frac{\alpha}{1-\alpha}$. With $\alpha \in \left(\frac{3}{4}, \frac{4}{5}\right)$, it is easy to check that $3 < \frac{\alpha}{1-\alpha} < 4$. Therefore, it is sufficient to show that $\triangle_2 > 0$ in (2.42) for $\alpha \in \left(\frac{3}{4}, \frac{4}{5}\right)$. In the proof of Lemma 2.2, all the computations of the partial derivatives in (2.16)–(2.22) are valid in the even larger range (2.14) for $\alpha \in \left(0, \frac{4}{5}\right)$, so they hold in the new range (2.42) for $\alpha \in \left(\frac{3}{4}, \frac{4}{5}\right)$. We only need to show the positivity of $\triangle_2|_{A_1=4, A_2=A_3=A_4=\frac{\alpha}{1-\alpha}}$ for $\alpha \in \left(\frac{3}{4}, \frac{4}{5}\right)$. Taking condition (2.42) instead of (2.14) yields

$$\triangle_2|_{A_1=4, A_2=A_3=A_4=\frac{\alpha}{1-\alpha}} = (-4 + 5\alpha)(-5 + 24\alpha - 37\alpha^2 + 16\alpha^3) > 0$$

for $\alpha \in \left(\frac{3}{4}, \frac{4}{5}\right)$. In fact, $-5 + 24\alpha - 37\alpha^2 + 16\alpha^3 < -5 + 24\alpha - 37\alpha^2 + 16 \cdot \frac{4}{5}\alpha^2 = -5 + 24\alpha - \frac{121}{5}\alpha^2 =: f(\alpha)$ for $\alpha \in \left(\frac{3}{4}, \frac{4}{5}\right)$, and $f(\alpha) = -\frac{121}{5}\left(\alpha - \frac{60}{121}\right)^2 + \frac{115}{121}$. So $f(\alpha)$ is decreasing in $\alpha \in \left(\frac{3}{4}, \frac{4}{5}\right)$. Thus, $f(\alpha) < f\left(\frac{3}{4}\right) = -\frac{49}{80}$ for $\alpha \in \left(\frac{3}{4}, \frac{4}{5}\right)$.

For subcase (2b), the proof is nearly the same as in subcase (4b) and subcase (3b). In the present case $P_5 = P_4(1) + P_4(2) > 0$, thus $(1, 1, 1, 1, 2)$ is the smallest positive integral solution for the level $k = 2$, i.e. $1/a_1 + 1/a_2 + 1/a_3 + 1/a_4 \le 1 - 2/a_5 =: \beta \in \left(\frac{1}{2}, \frac{3}{5}\right)$, since $a_5 \in (4, 5)$. Let $A_i = a_i\beta$, $i = 1, 2, 3, 4$. The new range of β helps us to improve condition (2.14) to

$$A_1 \ge 4, \qquad A_2 \ge 3, \qquad A_3 \ge A_4 \ge \frac{2\beta}{1-\beta}, \tag{2.43}$$

since $A_i = a_i\beta \ge a_5\beta = \frac{2\beta}{1-\beta}$. With $\beta \in \left(\frac{1}{2}, \frac{3}{5}\right)$, it is easy to check that $2 < \frac{2\beta}{1-\beta} < 3$. Therefore, it is sufficient to show that $\triangle_3 > 0$ in (2.43) for $\beta \in \left(\frac{1}{2}, \frac{3}{5}\right)$. In the proof of Lemma 2.3, all the computations of the partial derivatives in (2.25)–(2.31) are valid in the even larger range (2.14) for $\beta \in \left(0, \frac{3}{5}\right)$, so they hold in the new range (2.43) for $\beta \in \left(\frac{1}{2}, \frac{3}{5}\right)$. We only need to show the positivity of $\triangle_2|_{A_1=4, A_2=3, A_3=A_4=\frac{2\beta}{1-\beta}}$ for $\beta \in \left(\frac{1}{2}, \frac{3}{5}\right)$. Taking condition (2.43) instead of (2.14) yields

$$\begin{aligned}\triangle_3&|_{A_1=4, A_2=3, A_3=A_4=\frac{2\beta}{1-\beta}} \\ &= -\beta^2(-12 - 17\beta + 99\beta^2 - 270\beta^3 + 718\beta^4 - 953\beta^5 + 515\beta^6) > 0\end{aligned}$$

for $\beta \in \left(\frac{1}{2}, \frac{3}{5}\right)$. Indeed, let $f(\beta) := -12 - 17\beta + 99\beta^2 - 270\beta^3 + 718\beta^4 - 953\beta^5 + 515\beta^6$. Then $f^{(3)}(\beta) = 12(-135 + 1436\beta - 4765\beta^2 + 5150\beta^3) > 12(-270\beta + 1436\beta - 4765\beta^2 + 5150\beta^3) = 12\beta(1166 - 4765\beta + 5150\beta^2) = 61800\beta\left(\beta - \frac{953}{2060}\right)^2 + \frac{52575}{824} > 0$ for $\beta \in \left(\frac{1}{2}, \frac{3}{5}\right)$. This implies that $f''(\beta)$ is increasing in $\beta \in \left(\frac{1}{2}, \frac{3}{5}\right)$. So $f''(\beta) > f''\left(\frac{1}{2}\right) =$

$\frac{1001}{8} > 0$, which tells us that $f'(\beta)$ is also increasing in $\beta \in \left(\frac{1}{2}, \frac{3}{5}\right)$. Thus, $f'(\beta) > f'\left(\frac{1}{2}\right) = \frac{149}{4} > 0$ for $\beta \in \left(\frac{1}{2}, \frac{3}{5}\right)$. It follows that $f(\beta)$ is increasing in $\beta \in \left(\frac{1}{2}, \frac{3}{5}\right)$. So $f(\beta) < f\left(\frac{3}{5}\right) = -\frac{5952}{3125} < 0$. Therefore, $\triangle_3 > 0$ for $A_1 \geq 4$, $A_2 \geq 3$, $A_3 \geq A_4 \geq \frac{2\beta}{1-\beta}$ and $\beta \in \left(\frac{1}{2}, \frac{3}{5}\right)$.

For subcase (2c), the proof is nearly the same as in subcase (3c). In the present case $P_5 = P_4(1) + P_4(2) + P_4(3) > 0$, thus $(1, 1, 1, 1, 3)$ is the smallest positive integral solution for the level $k = 2$, i.e. $1/a_1 + 1/a_2 + 1/a_3 + 1/a_4 \leq 1 - 3/a_5 =: \gamma \in \left(\frac{1}{4}, \frac{2}{5}\right)$, since $a_5 \in (4, 5)$. Let $A_i = a_i\gamma$, $i = 1, 2, 3, 4$. The new range of γ helps us to improve condition (2.14) to

$$A_1 \geq 4, \quad A_2 \geq 3, \quad A_3 \geq 2, \quad A_4 \geq \frac{3\gamma}{1-\gamma}, \tag{2.44}$$

since $A_i = a_i\gamma \geq a_5\gamma = \frac{3\gamma}{1-\gamma}$. With $\gamma \in \left(\frac{1}{4}, \frac{2}{5}\right)$, it is easy to check that $1 < \frac{3\gamma}{1-\gamma} < 2$. Therefore, it is sufficient to show that $\triangle_4 > 0$ in (2.44) for $\gamma \in \left(\frac{1}{4}, \frac{2}{5}\right)$. In the proof of Lemma 2.4, all the computations of the partial derivatives in (2.35)–(2.41) are valid in the even larger range (2.14) for $\alpha \in \left(0, \frac{2}{5}\right)$, so they hold in the new range (2.44) for $\alpha \in \left(\frac{1}{4}, \frac{2}{5}\right)$. We only need to show the positivity of $\triangle_4|_{A_1=4, A_2=3, A_3=2, A_4=\frac{3\gamma}{1-\gamma}}$ for $\alpha \in \left(\frac{1}{4}, \frac{2}{5}\right)$. Taking condition (2.44) instead of (2.14) yields

$$\begin{aligned}
&\triangle_4|_{A_1=4, A_2=3, A_3=2, A_4=\frac{3\gamma}{1-\gamma}} \\
&= \tfrac{1}{9}\gamma(-1+\gamma)(-544+1560\gamma-1572\gamma^2+539\gamma^3+2349\gamma^4-1353\gamma^5-2974\gamma^6+5640\gamma^7) \\
&> 0
\end{aligned}$$

for $\gamma \in \left(\frac{1}{4}, \frac{2}{5}\right)$. Indeed, $-544+1560\gamma-1572\gamma^2+539\gamma^3+2349\gamma^4-1353\gamma^5-2974\gamma^6+5640\gamma^7 < -544 + 1560\gamma + \left(-1572 + 539 \cdot \frac{2}{5} + 2349 \cdot \left(\frac{2}{5}\right)^2 - 1353 \cdot \left(\frac{1}{4}\right)^3 - 2974 \cdot \left(\frac{1}{4}\right)^4 + 5640 \cdot \left(\frac{2}{5}\right)^5\right)\gamma^2 = -544 + 1560\gamma - \frac{76445137}{80000}\gamma^2 =: f(\gamma)$. We have $f(\gamma) = -\frac{76445137}{80000}\left(\gamma - \frac{62400000}{76445137}\right)^2 + \frac{7085845472}{76445137}$. So $f(\gamma) < f\left(\frac{2}{5}\right) = -\frac{36445137}{500000} < 0$ for $\gamma \in \left(\frac{1}{4}, \frac{2}{5}\right)$. Therefore, $\triangle_4 > 0$ for $A_1 \geq 4$, $A_2 \geq 3$, $A_3 \geq 2$, $A_4 \geq \frac{3\gamma}{1-\gamma}$ and $\gamma \in \left(\frac{1}{4}, \frac{2}{5}\right)$.

For subcase (2d), $P_4(4) > 0$, which implies that $(1, 1, 1, 1, 4)$ is the smallest positive integral solution to the level $k = 4$. So we have $1/a_1 + 1/a_2 + 1/a_3 + 1/a_4 \leq 1 - 4/a_5 =: \delta \in \left(0, \frac{1}{5}\right)$, since $a_5 \in (4, 5)$. Let $A_i = a_i\delta$, $i = 1, 2, 3, 4$, and notice that condition (2.14) still holds here. (2.13) can be written as

$$\begin{aligned}
5!P_5 &= 5!(P_4(1)+P_4(2)+P_4(3)+P_4(4)) \\
&\leq 5\Bigg[\left(A_1\frac{3+\delta}{4\delta}-1\right)\left(A_2\frac{3+\delta}{4\delta}-1\right)\left(A_3\frac{3+\delta}{4\delta}-1\right)\left(A_4\frac{3+\delta}{4\delta}-1\right) \\
&\quad -\left(A_4\frac{3+\delta}{4\delta}-1\right)^4 + A_4\frac{3+\delta}{4\delta}\left(A_4\frac{3+\delta}{4\delta}-1\right)\left(A_4\frac{3+\delta}{4\delta}-2\right)\left(A_4\frac{3+\delta}{4\delta}-3\right) \\
&\quad +\left(A_1\frac{1+\delta}{2\delta}-1\right)\left(A_2\frac{1+\delta}{2\delta}-1\right)\left(A_3\frac{1+\delta}{2\delta}-1\right)\left(A_4\frac{1+\delta}{2\delta}-1\right)
\end{aligned}$$

$$
\begin{aligned}
&-\left(A_4\frac{1+\delta}{2\delta}-1\right)^4+A_4\frac{1+\delta}{2\delta}\left(A_4\frac{1+\delta}{2\delta}-1\right)\left(A_4\frac{1+\delta}{2\delta}-2\right)\left(A_4\frac{1+\delta}{2\delta}-3\right)\\
&+\left(A_1\frac{1+3\delta}{4\delta}-1\right)\left(A_2\frac{1+3\delta}{4\delta}-1\right)\left(A_3\frac{1+3\delta}{4\delta}-1\right)\left(A_4\frac{1+3\delta}{4\delta}-1\right)\\
&-\left(A_4\frac{1+3\delta}{4\delta}-1\right)^4+A_4\frac{1+3\delta}{4\delta}\left(A_4\frac{1+3\delta}{4\delta}-1\right)\left(A_4\frac{1+3\delta}{4\delta}-2\right)\left(A_4\frac{1+3\delta}{4\delta}-3\right)\\
&+(A_1-1)(A_2-1)(A_3-1)(A_4-1)-(A_4-1)^4+A_4(A_4-1)(A_4-2)(A_4-3)\Big].
\end{aligned}
\tag{2.45}
$$

It is sufficient to show

Lemma 2.5. *When* $4 < a_5 < 5$*, R.H.S. of* (1.8) $>$ *R.H.S. of* (2.45).

Proof. Subtracting R.H.S. of (2.45) from R.H.S. of (1.8), writing the expression in terms of A_i, $i = 1, 2, 3, 4$, and δ, and multiplying by $(1-\delta)^5\delta^4$, yields

$$
\begin{aligned}
\triangle_5 :=\,& A_1A_2A_3A_4\big(\tfrac{139}{128} - \tfrac{643}{128}\delta + \tfrac{283}{32}\delta^2 - \tfrac{207}{32}\delta^3 - \tfrac{219}{64}\delta^4 + \tfrac{1499}{64}\delta^5\\
&\qquad - \tfrac{1585}{32}\delta^6 + \tfrac{1765}{32}\delta^7 - \tfrac{3965}{128}\delta^8 + \tfrac{885}{128}\delta^9\big)\\
&+ (A_1A_2A_3 + A_1A_2A_4 + A_1A_3A_4 + A_2A_3A_4)\\
&\qquad \cdot \big(-\tfrac{3}{16}\delta + \tfrac{13}{8}\delta^2 - \tfrac{37}{8}\delta^3 + \tfrac{73}{8}\delta^4 - 24\delta^5 + \tfrac{407}{8}\delta^6 - \tfrac{475}{8}\delta^7 + \tfrac{275}{8}\delta^8 - \tfrac{125}{16}\beta^9\big)\\
&+ (A_1A_2 + A_1A_3 + A_1A_4 + A_2A_3 + A_2A_4 + A_3A_4)\\
&\qquad \cdot \big(-\tfrac{11}{8}\delta^2 + \tfrac{37}{8}\delta^3 - \tfrac{63}{8}\delta^4 + \tfrac{177}{8}\delta^5 - \tfrac{433}{8}\delta^6 + \tfrac{543}{8}\delta^7 - \tfrac{325}{8}\delta^8 + \tfrac{75}{8}\delta^9\big)\\
&+ (A_1 + A_2 + A_3)\big(\tfrac{9}{2}\delta^3 - 14\delta^4 - \tfrac{3}{2}\delta^5 + 56\delta^6 - \tfrac{173}{2}\delta^7 + 54\delta^8 - \tfrac{25}{2}\delta^9\big)\\
&+ A_4^3\big(\tfrac{45}{8}\delta - \tfrac{75}{4}\delta^2 + \tfrac{75}{4}\delta^3 + \tfrac{25}{4}\delta^4 - 50\delta^5 + \tfrac{415}{4}\delta^6 - \tfrac{475}{4}\delta^7 + \tfrac{275}{4}\delta^8 - \tfrac{125}{8}\delta^9\big)\\
&+ A_4^2\big(-\tfrac{175}{8}\delta^2 + \tfrac{625}{8}\delta^3 - \tfrac{875}{8}\delta^4 + \tfrac{1125}{8}\delta^5 - \tfrac{2125}{8}\delta^6 + \tfrac{2675}{8}\delta^7 - \tfrac{1625}{8}\delta^8 + \tfrac{375}{8}\delta^9\big)\\
&+ A_4\big(\tfrac{39}{2}\delta^3 - 64\delta^4 + \tfrac{47}{2}\delta^5 + 156\delta^6 - \tfrac{523}{2}\delta^7 + 164\delta^8 - \tfrac{75}{2}\delta^9\big)\\
&+ (-240\delta^4 - 320\delta^5 + 160\delta^6 + 320\delta^7 + 80\delta^8).
\end{aligned}
$$

The idea is to show that for all $\delta \in \left(0, \frac{1}{5}\right)$, the minimum of $\triangle_5$ in $A_1 \geq 4$, $A_2 \geq 3$, $A_3 \geq 2$ and $A_4 \geq 1$ occurs at $A_1 = 4$, $A_2 = 3$, $A_3 = 2$, $A_4 = 1$, and $\triangle_5|_{A_1=4,A_2=3,A_3=2,A_4=1} > 0$ for all $\delta \in \left(0, \frac{1}{5}\right)$. We have

$$
\frac{\partial^4\triangle_5}{\partial A_1\partial A_2\partial A_3\partial A_4} = \tfrac{1}{128}(-1+\delta)^4(139 - 87\delta - 50\delta^2 + 50\delta^3 - 425\delta^4 + 885\delta^5) > 0
$$

for $\delta \in \left(0, \frac{1}{5}\right)$, since $139-87\delta-50\delta^2+50\delta^3-425\delta^4+885\delta^5 > 139-87\cdot\frac{1}{5}-50\cdot\left(\frac{1}{5}\right)^2 + \left(50 - 425\cdot\frac{1}{5}\right)\delta^3 + 885\delta^5 = \frac{598}{5} - 35\delta^3 + 885\delta^5 > \frac{2983}{25} + 885\delta^5 > 0$ for $\delta \in \left(0, \frac{1}{5}\right)$. It follows that $\frac{\partial^3\triangle_5}{\partial A_1\partial A_2\partial A_3}$ is an increasing function of A_4 for $\delta \in \left(0, \frac{1}{5}\right)$ and $A_4 \geq 1$. Hence its minimum occurs at $A_4 = 1$, and

$$\left.\frac{\partial^3\triangle_5}{\partial A_1\partial A_2\partial A_3}\right|_{A_4=1}$$
$$= \big[A_4\big(\tfrac{139}{128} - \tfrac{643}{128}\delta + \tfrac{283}{32}\delta^2 - \tfrac{207}{32}\delta^3 - \tfrac{219}{64}\delta^4 + \tfrac{1499}{64}\delta^5 - \tfrac{1585}{32}\delta^6 + \tfrac{1765}{32}\delta^7 - \tfrac{3965}{128}\delta^8 + \tfrac{885}{128}\delta^9\big) + \big(-\tfrac{3}{16}\delta + \tfrac{13}{8}\delta^2 - \tfrac{37}{8}\delta^3 + \tfrac{73}{8}\delta^4 - 24\delta^5 + \tfrac{407}{8}\delta^6 - \tfrac{475}{8}\delta^7 + \tfrac{275}{8}\delta^8 - \tfrac{125}{16}\beta^9\big)\big]\big|_{A_4=1}$$
$$= -\tfrac{1}{128}(-1+\delta^5)(139 + 28\delta + 90\delta^2 + 140\delta^3 + 115\delta^4) > 0$$

for $\delta \in \big(0, \frac{1}{5}\big)$. It follows that $\frac{\partial^3\triangle_5}{\partial A_1\partial A_2\partial A_3} > 0$ for $A_4 \geq 1$ and $\delta \in \big(0, \frac{1}{5}\big)$. Note that $\frac{\partial^2\triangle_5}{\partial A_1\partial A_2}$ is symmetric with respect to A_3 and A_4. Thus, $\frac{\partial^3\triangle_5}{\partial A_1\partial A_2\partial A_4} > 0$ for $A_3 \geq 1$ and $\delta \in \big(0, \frac{1}{5}\big)$. Moreover, $\frac{\partial^2\triangle_5}{\partial A_1\partial A_2}$ is increasing with respect to A_3 and A_4 for $A_3 \geq A_4 \geq 1$ and $\delta \in \big(0, \frac{1}{5}\big)$. Hence its minimum occurs at $A_3 = A_4 = 1$, and

$$\left.\frac{\partial^2\triangle_5}{\partial A_1\partial A_2}\right|_{A_3=A_4=1}$$
$$= \big[A_3A_4\big(\tfrac{139}{128} - \tfrac{643}{128}\delta + \tfrac{283}{32}\delta^2 - \tfrac{207}{32}\delta^3 - \tfrac{219}{64}\delta^4 + \tfrac{1499}{64}\delta^5 - \tfrac{1585}{32}\delta^6 + \tfrac{1765}{32}\delta^7 - \tfrac{3965}{128}\delta^8 + \tfrac{885}{128}\delta^9\big) + (A_3+A_4)\big(-\tfrac{3}{16}\delta + \tfrac{13}{8}\delta^2 - \tfrac{37}{8}\delta^3 + \tfrac{73}{8}\delta^4 - 24\delta^5 + \tfrac{407}{8}\delta^6 - \tfrac{475}{8}\delta^7 + \tfrac{275}{8}\delta^8 - \tfrac{125}{16}\beta^9\big) + \big(-\tfrac{11}{8}\delta^2 + \tfrac{37}{8}\delta^3 - \tfrac{63}{8}\delta^4 + \tfrac{177}{8}\delta^5 - \tfrac{433}{8}\delta^6 + \tfrac{543}{8}\delta^7 - \tfrac{325}{8}\delta^8 + \tfrac{75}{8}\delta^9\big)\big]\big|_{A_3=A_4=1}$$
$$= \tfrac{1}{128}(-1+\delta)^6(139 + 143\delta + 145\delta^2 + 85\delta^3) > 0$$

for $\delta \in \big(0, \frac{1}{5}\big)$. It follows that $\frac{\partial^2\triangle_5}{\partial A_1\partial A_2} > 0$ for $A_3 \geq A_4 \geq 1$ and $\delta \in \big(0, \frac{1}{5}\big)$. As $\frac{\partial\triangle_5}{\partial A_1}$ is symmetric with respect to A_2, A_3, A_4, we also get $\frac{\partial^2\triangle_5}{\partial A_1\partial A_3} > 0$ for $A_2 \geq A_4 \geq 1$ and $\delta \in \big(0, \frac{1}{5}\big)$, and $\frac{\partial^2\triangle_5}{\partial A_1\partial A_4} > 0$ for $A_2 \geq A_3 \geq 1$ and $\delta \in \big(0, \frac{1}{5}\big)$. Therefore, $\frac{\partial\triangle_5}{\partial A_1}$ is an increasing function of A_2, A_3, A_4 for $A_2 \geq A_3 \geq A_4 \geq 1$ and $\delta \in \big(0, \frac{1}{5}\big)$. Hence its minimum occurs at $A_2 = A_3 = A_4 = 1$, and

$$\left.\frac{\partial\triangle_5}{\partial A_1}\right|_{A_2=A_3=A_4=1}$$
$$= \big[A_2A_3A_4\big(\tfrac{139}{128} - \tfrac{643}{128}\delta + \tfrac{283}{32}\delta^2 - \tfrac{207}{32}\delta^3 - \tfrac{219}{64}\delta^4 + \tfrac{1499}{64}\delta^5 - \tfrac{1585}{32}\delta^6 + \tfrac{1765}{32}\delta^7 - \tfrac{3965}{128}\delta^8 + \tfrac{885}{128}\delta^9\big) + (A_2A_3 + A_2A_4 + A_3A_4) \cdot \big(-\tfrac{3}{16}\delta + \tfrac{13}{8}\delta^2 - \tfrac{37}{8}\delta^3 + \tfrac{73}{8}\delta^4 - 24\delta^5 + \tfrac{407}{8}\delta^6 - \tfrac{475}{8}\delta^7 + \tfrac{275}{8}\delta^8 - \tfrac{125}{16}\beta^9\big) + (A_2+A_3+A_4)\big(-\tfrac{11}{8}\delta^2 + \tfrac{37}{8}\delta^3 - \tfrac{63}{8}\delta^4 + \tfrac{177}{8}\delta^5 - \tfrac{433}{8}\delta^6 + \tfrac{543}{8}\delta^7 - \tfrac{325}{8}\delta^8 + \tfrac{75}{8}\delta^9\big) + \big(\tfrac{9}{2}\delta^3 - 14\delta^4 - \tfrac{3}{2}\delta^5 + 56\delta^6 - \tfrac{173}{2}\delta^7 + 54\delta^8 - \tfrac{25}{2}\delta^9\big)\big]\big|_{A_2=A_3=A_4=1}$$
$$= -\tfrac{1}{128}(-1+\delta)^7(139 + 258\delta + 115\delta^2) > 0$$

for $\delta \in \left(0, \frac{1}{5}\right)$. It follows that $\frac{\partial \triangle_5}{\partial A_1} > 0$ for $A_2 \geq A_3 \geq A_4 \geq 1$ and $\delta \in \left(0, \frac{1}{5}\right)$. Since $\triangle_5$ is symmetric with respect to A_1, A_2, A_3, we also have $\frac{\partial \triangle_5}{\partial A_2} > 0$ for $A_1 \geq A_3 \geq A_4 \geq 1$, and $\frac{\partial \triangle_5}{\partial A_3} > 0$ for $A_1 \geq A_2 \geq A_4 \geq 1$. Moreover,

$$\frac{\partial^3 \triangle_5}{\partial A_4^3} = -\frac{15}{4}(-1+\delta)^5\delta(3+5\delta)(3+5\delta^2) > 0$$

for $\delta \in \left(0, \frac{1}{5}\right)$. It follows that $\frac{\partial^2 \triangle_5}{\partial A_4^2}$ is an increasing function of A_4 for $A_4 \geq 1$ and $\delta \in \left(0, \frac{1}{5}\right)$. Thus, its minimum occurs at $A_4 = 1$, and

$$\begin{aligned}
&\left.\frac{\partial^2 \triangle_5}{\partial A_4^2}\right|_{A_4=1} \\
&= \Big[6A_4\big(\tfrac{45}{8}\delta - \tfrac{75}{4}\delta^2 + \tfrac{75}{4}\delta^3 + \tfrac{25}{4}\delta^4 - 50\delta^5 + \tfrac{415}{4}\delta^6 - \tfrac{475}{4}\delta^7 + \tfrac{275}{4}\delta^8 - \tfrac{125}{8}\delta^9\big) \\
&\quad + 2\big(-\tfrac{175}{8}\delta^2 + \tfrac{625}{8}\delta^3 - \tfrac{875}{8}\delta^4 + \tfrac{1125}{8}\delta^5 - \tfrac{2125}{8}\delta^6 + \tfrac{2675}{8}\delta^7 - \tfrac{1625}{8}\delta^8 + \tfrac{375}{8}\delta^9\big)\Big]\Big|_{A_4=1} \\
&= \tfrac{5}{4}(-1+\delta)^5\delta(-27-10\delta+5\delta^2) > 0
\end{aligned}$$

for $\delta \in \left(0, \frac{1}{5}\right)$. In fact, let $f(\delta) := -27 - 10\delta + 5\delta^2 = 5(\delta-1)^2 - 32$. Then $f(\delta) < f(0) = -27 < 0$ for $\gamma \in \left(0, \frac{2}{5}\right)$. It follows that $\frac{\partial^2 \triangle_5}{\partial A_4^2} > 0$ for $A_4 \geq 1$ and $\delta \in \left(0, \frac{1}{5}\right)$. Thus, $\frac{\partial \triangle_5}{\partial A_4}$ is an increasing function of A_4 for $A_4 \geq 1$ and $\delta \in \left(0, \frac{1}{5}\right)$. Moreover, it is an increasing function with respect to A_1, A_2, A_3, A_4 for $A_1 \geq A_2 \geq A_3 \geq A_4 \geq 1$ and $\delta \in \left(0, \frac{1}{5}\right)$, since it is symmetric with respect to A_1, A_2, A_3. Hence its minimum occurs at $A_1 = A_2 = A_3 = A_4 = 1$, and

$$\begin{aligned}
&\left.\frac{\partial \triangle_5}{\partial A_4}\right|_{A_1=A_2=A_3=A_4=1} \\
&= \Big[A_1A_2A_3\big(\tfrac{139}{128} - \tfrac{643}{128}\delta + \tfrac{283}{32}\delta^2 - \tfrac{207}{32}\delta^3 - \tfrac{219}{64}\delta^4 + \tfrac{1499}{64}\delta^5 \\
&\qquad - \tfrac{1585}{32}\delta^6 + \tfrac{1765}{32}\delta^7 - \tfrac{3965}{128}\delta^8 + \tfrac{885}{128}\delta^9\big) \\
&\quad + (A_1A_2 + A_1A_3 + A_2A_3) \\
&\qquad \cdot \big(-\tfrac{3}{16}\delta + \tfrac{13}{8}\delta^2 - \tfrac{37}{8}\delta^3 + \tfrac{73}{8}\delta^4 - 24\delta^5 + \tfrac{407}{8}\delta^6 - \tfrac{475}{8}\delta^7 + \tfrac{275}{8}\delta^8 - \tfrac{125}{16}\beta^9\big) \\
&\quad + (A_1+A_2+A_3)\big(-\tfrac{11}{8}\delta^2 + \tfrac{37}{8}\delta^3 - \tfrac{63}{8}\delta^4 + \tfrac{177}{8}\delta^5 - \tfrac{433}{8}\delta^6 + \tfrac{543}{8}\delta^7 - \tfrac{325}{8}\delta^8 + \tfrac{75}{8}\delta^9\big) \\
&\quad + 3A_4^2\big(\tfrac{45}{8}\delta - \tfrac{75}{4}\delta^2 + \tfrac{75}{4}\delta^3 + \tfrac{25}{4}\delta^4 - 50\delta^5 + \tfrac{415}{4}\delta^6 - \tfrac{475}{4}\delta^7 + \tfrac{275}{4}\delta^8 - \tfrac{125}{8}\delta^9\big) \\
&\quad + 2A_4\big(-\tfrac{175}{8}\delta^2 + \tfrac{625}{8}\delta^3 - \tfrac{875}{8}\delta^4 + \tfrac{1125}{8}\delta^5 \\
&\qquad - \tfrac{2125}{8}\delta^6 + \tfrac{2675}{8}\delta^7 - \tfrac{1625}{8}\delta^8 + \tfrac{375}{8}\delta^9\big)\Big]\Big|_{A_1=A_2=A_3=A_4=1} \\
&= \tfrac{1}{128}(-1+\delta)^5(-139 - 2140\delta + 2262\delta^2 + 2452\delta^3 + 2685\delta^4) > 0
\end{aligned}$$

for $\delta \in \left(0, \frac{1}{5}\right)$, since $-139 - 2140\delta + 2262\delta^2 + 2452\delta^3 + 2685\delta^4 < -139 + (-2140 + 2262 \cdot \frac{1}{5} + 2452 \cdot \left(\frac{1}{5}\right)^2 + 2685 \cdot \left(\frac{1}{5}\right)^3)\delta = -139 - \frac{39201}{25}\delta < 0$ for $\delta \in \left(0, \frac{1}{5}\right)$. It follows that $\frac{\partial \triangle_5}{\partial A_4} > 0$ for $A_1 \geq A_2 \geq A_3 \geq A_4 \geq 1$ and $\delta \in \left(0, \frac{1}{5}\right)$. Therefore, $\triangle_5$ is an increasing function of A_1, A_2, A_3, A_4 for $A_1 \geq A_2 \geq A_3 \geq A_4 \geq 1$ and $\delta \in \left(0, \frac{1}{5}\right)$. Thus, its minimum occurs at $A_1 = A_2 = A_3 = A_4 = 1$ and taking condition (2.14) into consideration, we have

$$\begin{aligned}
&\triangle_5|_{A_1=4, A_2=3, A_3=2, A_4=1} \\
&\quad = \frac{1}{16}\delta(417 - 1572\delta + 1704\delta^2 + 620\delta^3 - 6334\delta^4 - 7660\delta^5 - 5760\delta^6 - 2140\delta^7 + 245\delta^8) \\
&\quad > 0
\end{aligned}$$

for $\delta \in \left(0, \frac{1}{5}\right)$, since $417 - 1572\delta + 1704\delta^2 + 620\delta^3 - 6334\delta^4 - 7660\delta^5 - 5760\delta^6 - 2140\delta^7 + 245\delta^8 > 417 - 1572 \cdot \frac{1}{5} + 1704\delta^2 + (620 - 6334 \cdot \frac{1}{5} - 7660 \cdot \left(\frac{2}{5}\right)^2 - 5760 \cdot \left(\frac{2}{5}\right)^3 - 2140 \cdot \left(\frac{2}{5}\right)^4)\delta^3 + 245\delta^8 = \frac{513}{5} + 1704\delta^2 - \frac{125338}{125}\delta^3 + 245\delta^8 > \frac{513}{5} + (1704 - \frac{125338}{125} \cdot \frac{1}{5})\delta^2 + 245\delta^8 = \frac{513}{5} + \frac{939662}{625}\delta^2 + 245\delta^8 > 0$ for $\delta \in \left(0, \frac{1}{5}\right)$. It follows that $\triangle_5 > 0$ for $A_1 \geq 4$, $A_2 \geq 3$, $A_3 \geq 2$, $A_4 \geq 1$ and $\delta \in \left(0, \frac{1}{5}\right)$. □

2.2. *Proof of Theorem 1.3*

As we stated in the introduction, an estimate of the Dickman–de Bruijn function $\psi(x, y)$ is equivalent to a sharp estimate of Q_n (or P_n by (1.2)). We have already got an estimate of P_n, $n \leq 5$, so we can apply it to estimating $\psi(x, y)$. In detail, let $p_1 < \cdots < p_5$ be 5 primes $\leq y$. Clearly $p_1^{l_1} \cdots p_5^{l_5} \leq x$ if and only if $\frac{l_1}{\frac{\log x}{\log p_1}} + \cdots + \frac{l_n}{\frac{\log x}{\log p_n}} \leq 1$. Therefore, $\psi(x, y)$ is precisely the number Q_n of (1.1) with $a_i = \frac{\log x}{\log p_i}$, $1 \leq i \leq n$. Moreover, by (1.2), $\psi(x, y)$ is also precisely the number $P(a_1(1 + a), \ldots, a_n(1 + a))$, where $a = 1/a_1 + \cdots + 1/a_n$.

According to the number of prime numbers $\leq y$, we split the proof of Theorem 1.3 into three cases:

(i) $5 \leq y < 7$;
(ii) $7 \leq y < 11$;
(iii) $11 \leq y < 13$.

Cases (i) and (ii) have been proven in [18]. For (iii), we have five prime numbers $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$ and $p_5 = 11$, thus

$$a = \frac{\log 2 + \log 3 + \log 5 + \log 7 + \log 11}{\log x}.$$

Therefore,

$$\begin{aligned}
\psi(x, y) &= Q_5 \\
&= P\left(\frac{\log x}{\log 2}\left(1 + \frac{\log(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11)}{\log x}\right), \frac{\log x}{\log 3}\left(1 + \frac{\log(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11)}{\log x}\right),\right. \\
&\qquad \frac{\log x}{\log 5}\left(1 + \frac{\log(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11)}{\log x}\right), \frac{\log x}{\log 7}\left(1 + \frac{\log(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11)}{\log x}\right), \\
&\qquad \left.\frac{\log x}{\log 11}\left(1 + \frac{\log(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11)}{\log x}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{5!}\left\{\left(\frac{\log x}{\log 2}+\frac{\log(3\cdot 5\cdot 7\cdot 11)}{\log 2}\right)\left(\frac{\log x}{\log 3}+\frac{\log(2\cdot 5\cdot 7\cdot 11)}{\log 3}\right)\right.\\
&\qquad\cdot\left(\frac{\log x}{\log 5}+\frac{\log(2\cdot 3\cdot 7\cdot 11)}{\log 5}\right)\left(\frac{\log x}{\log 7}+\frac{\log(2\cdot 3\cdot 5\cdot 11)}{\log 7}\right)\\
&\qquad\cdot\left(\frac{\log x}{\log 11}+\frac{\log(2\cdot 3\cdot 5\cdot 7)}{\log 11}\right)\\
&\qquad-\left[\left(\frac{\log x}{\log 11}+\frac{\log(2\cdot 3\cdot 5\cdot 7)}{\log 11}\right)^5\right.\\
&\qquad\quad-\left(\frac{\log x}{\log 11}+1+\frac{\log(2\cdot 3\cdot 5\cdot 7)}{\log 11}\right)\left(\frac{\log x}{\log 11}+\frac{\log(2\cdot 3\cdot 5\cdot 7)}{\log 11}\right)\\
&\qquad\quad\cdot\left(\frac{\log x}{\log 11}+\frac{\log(2\cdot 3\cdot 5\cdot 7)}{\log 11}-1\right)\left(\frac{\log x}{\log 11}+\frac{\log(2\cdot 3\cdot 5\cdot 7)}{\log 11}-2\right)\\
&\qquad\quad\left.\left.\cdot\left(\frac{\log x}{\log 11}+\frac{\log(2\cdot 3\cdot 5\cdot 7)}{\log 11}-3\right)\right]\right\}\\
&=\frac{1}{120}\left\{\frac{1}{\log 2\log 3\log 5\log 7\log 11}(\log x+\log 1155)(\log x+\log 770)(\log x+\log 462)\right.\\
&\qquad\qquad\cdot(\log x+\log 330)(\log x+\log 210)\\
&\qquad-\frac{1}{\log^5 11}\Big[(\log x+\log 210)^5\\
&\qquad\qquad-(\log x+\log 11+\log 210)(\log x+\log 210)(\log x+\log 210-\log 11)\\
&\qquad\qquad\left.\cdot(\log x+\log 210-2\log 11)(\log x+\log 210-3\log 11)\Big]\right\}. \qquad\square
\end{aligned}
$$

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