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On higher dimensional complex Plateau problem

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Abstract Let X be a compact connected strongly pseudoconvex *CR* manifold of real dimension $2n-1$ in $\mathbb{C}^N$. It has been an interesting question to find an intrinsic smoothness criteria for the complex Plateau problem. For $n \geq 3$ and $N = n+1$, Yau found a necessary and sufficient condition for the interior regularity of the Harvey–Lawson solution to the complex Plateau problem by means of Kohn–Rossi cohomology groups on X in 1981. For $n = 2$ and $N \geq n+1$, the first and third authors introduced a new CR invariant $g^{(1,1)}(X)$ of X. The vanishing of this invariant will give the interior regularity of the Harvey–Lawson solution up to normalization. For $n \geq 3$ and $N > n+1$, the problem still remains open. In this paper, we generalize the invariant $g^{(1,1)}(X)$ to higher dimension as $g^{(\Lambda^n 1)}(X)$ and show that if $g^{(\Lambda^n 1)}(X) = 0$, then the interior has at most finite number of rational singularities. In particular, if X is Calabi–Yau of real dimension 5, then the vanishing of this invariant is equivalent to give the interior regularity up to normalization.

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1 Introduction

One of the natural fundamental questions of complex geometry is to study the boundaries of complex varieties. For example, the famous classical complex Plateau problem asks which odd dimensional real sub-manifolds of $\mathbb{C}^N$ are boundaries of complex sub-manifolds in $\mathbb{C}^N$. In their beautiful seminal paper, Harvey and Lawson [14] proved that for any compact connected *CR* manifold X of real dimension $2n-1$, $n \geq 2$, in $\mathbb{C}^N$, there is a unique complex variety V in $\mathbb{C}^N$ such that the boundary of V is X. In fact, Harvey and Lawson proved the following theorem.

Theorem (Harvey–Lawson [14,15]) *Let X be an embeddable strongly pseudoconvex CR manifold. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain in $\mathbb{C}^N$. Then X can be CR embedded in some $\mathbb{C}^N$ and X bounds a Stein variety $V \subseteq \mathbb{C}^N$ with at most isolated singularities.*

The above theorem is one of the important theorems in complex geometry. It relates theory of strongly pseudoconvex *CR* manifolds on the one hand and the theory of isolated normal singularities on the other hand.

The next fundamental question is to determine when X is the boundary of a complex sub-manifold in $\mathbb{C}^N$, i.e., when V is smooth. In 1981, Yau [26] solved this problem for the case $n \geq 3$ by calculation of Kohn–Rossi cohomology groups $H^{p,q}_{KR}(X)$. More precisely, suppose X is a compact connected strongly pseudoconvex *CR* manifold of real dimension $2n-1$, $n \geq 3$, in the boundary of a bounded strongly pseudoconvex domain D in $\mathbb{C}^{n+1}$. Then X is the boundary of a complex sub-manifold $V \subset D - X$ if and only if Kohn–Rossi cohomology groups $H^{p,q}_{KR}(X)$ are zeros for $1 \leq q \leq n-2$ (see Theorem 5.1).

For $n = 2$, i.e. X is a 3-dimensional *CR* manifold, the intrinsic smoothness criteria for the complex Plateau problem remains unsolved for over a quarter of a century even for the hypersurface case. The main difficulty is that the Kohn–Rossi cohomology groups are infinite dimensional in this case. Let V be a complex variety with X as its boundary. Then the singularities of V are surface singularities. In [9], the first and the third authors introduced a new *CR* invariant $g^{(1,1)}(X)$ to solve the regularity problem of the Harvey–Lawson solution to the complex Plateau problem. More precisely, they showed that if X is a strongly pseudoconvex compact Calabi–Yau *CR* manifold of dimension 3 and X is contained in the boundary of a strongly pseudoconvex bounded domain D in $\mathbb{C}^N$ with holomorphic De Rham cohomology $H^2_h(X) = 0$, then X is the boundary of a complex sub-manifold up to normalization $V \subset D - X$ with boundary regularity if and only if $g^{(1,1)}(X) = 0$. In particular, if $N = 3$, then X is the boundary of a complex sub-manifold $V \subset D - X$ if and only if $g^{(1,1)}(X) = 0$.

For $n \geq 3$ and $N > n+1$, i.e., non-hypersurface type, the complex Plateau problem still remains open. In this paper, we generalize the invariant $g^{(1,1)}(X)$ to higher dimension as $g^{(\Lambda^n 1)}(X)$ and show that if $g^{(\Lambda^n 1)}(X) = 0$, then the interior which X bounds has at most finite number of rational singularities. In particular, if X is Calabi–Yau of real dimension 5, i.e., $n = 3$, then the vanishing of this invariant is equivalent to give the interior regularity.

Theorem A *Let X be a strongly pseudoconvex compact CR manifold of dimension $2n-1$, where $n > 2$. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in $\mathbb{C}^N$. Then X is the boundary of a variety $V \subset D - X$ with boundary regularity and the number of non-rational singularities (up to normalization) is not great than $g^{(\Lambda^n 1)}(X)$. In particular, if $g^{(\Lambda^n 1)}(X) = 0$, then V has at most finite number of rational singularities.*

Theorem B *Let X be a strongly pseudoconvex compact Calabi–Yau CR manifold of dimension* 5*. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in $\mathbb{C}^N$. Then X is the boundary of a complex sub-manifold (up to normalization) $V \subset D - X$ with boundary regularity if and only if $g^{(\Lambda^3 1)}(X) = 0$.*

In Sect. 2, we shall recall the definition Kohn–Rossi cohomology and holomorphic De Rham cohomology for a *CR* manifold. In Sect. 3 , we survey some known results about the conjecture of minimal discrepancy and properties of terminal and rational Gorenstein threefolds singularities. In Sect. 4, we generalize the invariant of singularities $g^{(1,1)}$ to higher dimension as $g^{(\Lambda^n 1)}$ and and study some properties of $g^{(\Lambda^n 1)}$. In Sect. 5, we use the results in Sect. 4 to solve our main theorems in this paper.

2 Strongly pseudoconvex CR manifolds

Kohn–Rossi cohomology was first introduced by Kohn–Rossi. Following Tanaka [25], we reformulate the definition in a way independent of the interior manifold.

Definition 2.1 Let X be a connected orientable manifold of real dimension $2n - 1$. A *CR* structure on X is an $(n-1)$-dimensional subbundle S of $\mathbb{C}T(X)$ (complexified tangent bundle) such that

(1) $S \bigcap \bar{S} = \{0\}$,
(2) If L, L' are local sections of S, then so is $[L, L']$.

Such a manifold with a *CR* structure is called a *CR* manifold. There is a unique subbundle $\mathcal{H}$ of $T(X)$ such that $\mathbb{C}\mathcal{H} = S \bigoplus \bar{S}$. Furthermore, there is a unique homomorphism $J : \mathcal{H} \longrightarrow \mathcal{H}$ such that $J^2 = -1$ and $S = \{v - iJv : v \in \mathcal{H}\}$. The pair $(\mathcal{H}, J)$ is called the real expression of the *CR* structure.

Let X be a *CR* manifold with structure S. For a complex valued C^∞ function u defined on X, the section $\bar{\partial}_b u \in \Gamma(\bar{S}^*)$ is defined by

$$\bar{\partial}_b u(\bar{L}) = \bar{L}(u), L \in S.$$

The differential operator $\bar{\partial}_b$ is called the (tangential) Cauchy–Riemann operator, and a solution u of the equation $\bar{\partial}_b u = 0$ is called a holomorphic function.

Definition 2.2 A complex vector bundle E over X is said to be holomorphic if there is a differential operator

$$\bar{\partial}_E : \Gamma(E) \longrightarrow \Gamma\left(E \otimes \bar{S}^*\right)$$

satisfying the following conditions:

1. $\bar{\partial}_E(fu)(\bar{L}_1) = (\bar{\partial}_b f)(\bar{L}_1)u + f(\bar{\partial}_E u)(\bar{L}_1) = (\bar{L}_1 f)u + f(\bar{\partial}_E u)(\bar{L}_1)$,
2. $(\bar{\partial}_E u)[\bar{L}_1, \bar{L}_2] = \bar{\partial}_E(\bar{\partial}_{Eu}(\bar{L}_2))(\bar{L}_1) - \bar{\partial}_E(\bar{\partial}_E u(\bar{L}_1))(\bar{L}_2)$,

where $u \in \Gamma(E)$, $f \in C^\infty(X)$ and $L_1, L_2 \in \Gamma(S)$.

The operator $\bar{\partial}_E$ is called the Cauchy–Riemann operator and a solution u of the equation $\bar{\partial}_E u = 0$ is called a holomorphic cross section.

A basic holomorphic vector bundle over a CR manifold X is the vector bundle $\widehat{T}(X) = \mathbb{C}T(X)/\bar{S}$. The corresponding operator $\bar{\partial} = \bar{\partial}_{\widehat{T}(X)}$ is defined as follows. Let p be the projection from $\mathbb{C}T(X)$ to $\widehat{T}(X)$. Take any $u \in \Gamma(\widehat{T}(X))$ and express it as $u = p(Z)$, $Z \in \Gamma(\mathbb{C}T(X))$. For any $L \in \Gamma(S)$, define a cross section $(\bar{\partial}u)(\bar{L})$ of $\widehat{T}(X)$ by $(\bar{\partial}u)(\bar{L}) = p([\bar{L}, Z])$. One can show that $(\bar{\partial}u)(\bar{L})$ does not depend on the choice of Z and that $\bar{\partial}u$ gives a cross section of $\widehat{T}(X) \otimes \bar{S}^*$. Furthermore one can show that the operator $u \longmapsto \bar{\partial}u$ satisfies (1) and (2) of Definition 2.2, using the Jacobi identity in the Lie algebra $\Gamma(\mathbb{C}T(X))$. The resulting holomorphic vector bundle $\widehat{T}(X)$ is called the holomorphic tangent bundle of X.

If X is a real hypersurface in a complex manifold M, we may identify $\widehat{T}(X)$ with the holomorphic vector bundle of all $(1, 0)$ tangent vectors to M and $\widehat{T}(X)$ with the restriction of $\widehat{T}(M)$ to X. In fact, since the structure S of X is the bundle of all $(1, 0)$ tangent vectors to X, the inclusion map $\mathbb{C}T(X) \longrightarrow \mathbb{C}T(M)$ induces a natural map $\widehat{T}(X) \xrightarrow{\phi} \widehat{T}(M)|_X$ which is a bundle isomorphism satisfying $\bar{\partial}(\phi(u))(\bar{L}) = \phi(\bar{\partial}u(\bar{L}))$, $u \in \Gamma(\widehat{T}(X))$, $L \in S$.

For a holomorphic vector bundle E over X, set

$$C^q(X, E) = E \otimes \wedge^q \bar{S}^*, \mathscr{C}^q(X, E) = \Gamma(C^q(X, E))$$

and define a differential operator

$$\bar{\partial}^q_E : \mathscr{C}^q(X, E) \longrightarrow \mathscr{C}^{q+1}(X, E)$$

by

$$\begin{aligned}\left(\bar{\partial}^q_E\phi\right)\left(\bar{L}_1, \ldots, \bar{L}_{q+1}\right) &= \sum_i (-1)^{i+1}\bar{\partial}_E\left(\phi(\bar{L}_1, \ldots, \widehat{\bar{L}_i}, \ldots, \bar{L}_{q+1})\right)(\bar{L}_i)\\ &\quad + \sum_{i<j}(-1)^{i+j}\phi\left([\bar{L}_i, \bar{L}_j], \bar{L}_1, \ldots, \widehat{\bar{L}_i}, \ldots, \bar{L}_{q+1}\right)\end{aligned}$$

for all $\phi \in \mathscr{C}^q(X, E)$ and $L_1, \ldots, L_{q+1} \in \Gamma(S)$. One shows by standard arguments that $\bar{\partial}^q_E\phi$ gives an element of $\mathscr{C}^{q+1}(X, E)$ and that $\bar{\partial}^{q+1}_E\bar{\partial}^q_E = 0$. The cohomology groups of the resulting complex $\{\mathscr{C}^q(X, E), \bar{\partial}^q_E\}$ is denoted by $H^q(X, E)$.

Let $\{\mathscr{A}^k(X), d\}$ be the De Rham complex of X with complex coefficients, and let $H^k(X)$ be the De Rham cohomology groups. There is a natural filtration of the De Rham complex, as follows. For any integer p and k, put $A^k(X) = \wedge^k(\mathbb{C}T(X)^*)$ and denote by $F^p(A^k(X))$ the subbundle of $A^k(X)$ consisting of all $\phi \in A^k(X)$ which satisfy the equality

$$\phi\left(Y_1, \ldots, Y_{p-1}, \bar{Z}_1, \ldots, \bar{Z}_{k-p+1}\right) = 0$$

for all $Y_1, \ldots, Y_{p-1} \in \mathbb{C}T(X)_x$ and $Z_1, \ldots, Z_{k-p+1} \in S_x$, $x \in X$. Then

$$\begin{aligned}A^k(X) = F^0\left(A^k(X)\right) &\supset F^1\left(A^k(X)\right) \supset \cdots\\ &\supset F^k\left(A^k(X)\right) \supset F^{k+1}\left(A^k(X)\right) = 0.\end{aligned}$$

Setting $F^p(\mathscr{A}^k(X)) = \Gamma(F^p(A^k(X)))$, we have

$$\begin{aligned}\mathscr{A}^k(X) = F^0\left(\mathscr{A}^k(X)\right) &\supset F^1\left(\mathscr{A}^k(X)\right) \supset \cdots\\ &\supset F^k\left(\mathscr{A}^k(X)\right) \supset F^{k+1}\left(\mathscr{A}^k(X)\right) = 0.\end{aligned}$$

Since clearly $dF^p(\mathscr{A}^k(X)) \subseteq F^p(\mathscr{A}^{k+1}(X))$, the collection $\{F^p(\mathscr{A}^k(X))\}$ gives a filtration of the De Rham complex.

We denote by $H_{KR}^{p,q}(X)$ the groups $E_1^{p,q}(X)$ of the spectral sequence $\{E_r^{p,q}(X)\}$ associated with the filtration $\{F^p(\mathscr{A}^k(X))\}$. We call $H_{KR}^{p,q}(X)$ the Kohn–Rossi cohomology group of type (p,q). More explicitly, let

$$A^{p,q}(X) = F^p(A^{p+q}(X)), \mathscr{A}^{p,q}(X) = \Gamma(A^{p,q}(X)),$$
$$C^{p,q}(X) = A^{p,q}(X)/A^{p+1,q-1}(X), \mathscr{C}^{p,q}(X) = \Gamma(C^{p,q}(X)).$$

Since $d : \mathscr{A}^{p,q}(X) \longrightarrow \mathscr{A}^{p,q+1}(X)$ maps $\mathscr{A}^{p+1,q-1}(X)$ into $\mathscr{A}^{p+1,q}(X)$, it induces an operator $d'' : \mathscr{C}^{p,q}(X) \longrightarrow \mathscr{C}^{p,q+1}(X)$. $H_{KR}^{p,q}(X)$ are then the cohomology groups of the complex $\{\mathscr{C}^{p,q}(X), d''\}$.

Alternatively $H_{KR}^{p,q}(X)$ may be described in terms of the vector bundle $E^p = \wedge^p(\widehat{T}(X)^*)$. If for $\phi \in \Gamma(E^p)$, $u_1, \ldots, u_p \in \Gamma(\widehat{T}(X))$, $Y \in S$, we define $(\bar{\partial}_{E^p}\phi)(\bar{Y}) = \bar{Y}\phi$ by

$$\bar{Y}\phi(u_1, \ldots, u_p) = \bar{Y}(\phi(u_1, \ldots, u_p)) + \sum_i (-1)^i \phi\left(\bar{Y}u_i, u_1, \ldots, \widehat{u_i}, \ldots, u_p\right)$$

where $\bar{Y}u_i = (\bar{\partial}_{\widehat{T}(X)}u_i)(\bar{Y})$, then we easily verify that E^p with $\bar{\partial}_{E^p}$ is a holomorphic vector bundle. Tanaka [25] proves that $C^{p,q}(X)$ may be identified with $C^q(X, E^p)$ in a natural manner such that

$$d''\phi = (-1)^p \bar{\partial}_{E^p}\phi, \phi \in \mathscr{C}^{p,q}(X).$$

Thus, $H_{KR}^{p,q}(X)$ may be identified with $H^q(X, E^p)$.

We denote by $H_h^k(X)$ the groups $E_2^{k,0}(X)$ of the spectral sequence $\{E_r^{p,q}(X)\}$ associated with the filtration $\{F^p(\mathscr{A}^k(X))\}$. We call $H_h^k(X)$ the holomorphic De Rham cohomology groups. The groups $H_h^k(X)$ are the cohomology groups of the complex $\{\mathscr{S}^k(X), d\}$, where we put $\mathscr{S}^k(X) = E_1^{k,0}(X)$ and $d = d_1 : E_1^{k,0} \longrightarrow E_1^{k+1,0}$. Recall that $\mathscr{S}^k(X)$ is the kernel of the following mapping:

$$d_0 : E_0^{k,0} = F^k\mathscr{A}^k = \mathscr{A}^{k,0}(X)$$
$$\to E_0^{k,1} = F^k\mathscr{A}^{k+1}/F^{k+1}\mathscr{A}^{k+1} = \mathscr{A}^{k,1}(X)/\mathscr{A}^{k+1,0}.$$

Note that $\mathscr{S}$ may be characterized as the space of holomorphic k-forms, namely holomorphic cross sections of E^k. Thus the complex $\{\mathscr{S}^k(X), d\}$ (respectively, the groups $H_h^k(X)$ will be called the holomorphic De Rham complex (respectively, the holomorphic De Rham cohomology groups).

Definition 2.3 Let $L_1, \ldots, L_{n-1}$ be a local frame of the *CR* structure S on X so that $\bar{L}_1, \ldots, \bar{L}_{n-1}$ is a local frame of $\bar{S}$. Since $S \oplus \bar{S}$ has complex codimension one in $\mathbb{C}T(X)$, we may choose a local section N of $\mathbb{C}T(X)$ such that $L_1, \ldots, L_{n-1}, \bar{L}_1, \ldots, \bar{L}_{n-1}, N$ span $\mathbb{C}T(X)$. We may assume that N is purely imaginary. Then the matrix (c_{ij}) defined by

$$[L_i, \bar{L}_j] = \sum_k a_{i,j}^k L_k + \sum_k b_{i,j}^k \bar{L}_k + c_{i,j}N$$

is Hermitian, and is called the Levi form of X.

Proposition 2.4 *The number of non-zero eigenvalues and the absolute value of the signature of (c_{ij}) at each point are independent of the choice of $L_1, \ldots, L_{n-1}, N$.*

Definition 2.5 X is said to be strongly pseudoconvex if the Levi form is positive definite at each point of X.

Definition 2.6 Let X be a CR manifold of real dimension $2n-1$. X is said to be Calabi–Yau if there exists a nowhere vanishing holomorphic section in $\Gamma(\wedge^n \widehat{T}(X)^*)$, where $\widehat{T}(X)$ is the holomorphic tangent bundle of X.

Remark

1. Let X be a *CR* manifold of real dimension $2n-1$ in $\mathbb{C}^n$. Then X is a Calabi–Yau *CR* manifold.
2. Let X be a strongly pseudoconvex *CR* manifold of real dimension $2n-1$ contained in the boundary of bounded strongly pseudoconvex domain in $\mathbb{C}^{n+1}$. Then X is a Calabi–Yau *CR* manifold.

The proof of these two statements is essentially the fact that any hypersurface singularities are Gorenstein and with the same arguments as Lemma 4.6 to get a nowhere vanishing holomorphic section of the holomorphic tangent bundle of X.

3 Minimal discrepancy and 3-dimensional canonical Gorenstein singularities

Canonical singularities appear as singularities of the canonical model of a projective variety, and terminal singularities are special cases that appear as singularities of minimal models. They were introduced by Reid in 1980 ([20]). Terminal singularities are important in the minimal model program because smooth minimal models do not always exist, and thus one must allow certain singularities, namely the terminal singularities.

Suppose that X is a normal variety such that its canonical class K_X is $\mathbb{Q}$-Cartier, and let $f : Y \to X$ be a resolution of the singularities of X. Then

$$K_Y = f^* K_X + \sum_i a_i E_i,$$

where the sum is over the irreducible exceptional divisors, and the a_i are rational numbers, called the discrepancies.

Then the singularities of X are called:

$$\begin{cases} \text{terminal} & a_i > 0 \quad \text{for all i} \\ \text{canonical} & a_i \geq 0 \quad \text{for all i} \\ \text{log-terminal} & a_i > -1 \quad \text{for all i} \\ \text{log-canonical} & a_i \geq -1 \quad \text{for all i.} \end{cases} \tag{3.1}$$

Definition 3.1 The minimal discrepancy of a variety X at 0, denoted by $Md_0(X)$ (or $Md(X)$ for short), is the minimum of all discrepancies of discrete valuations of $\mathbb{C}(X)$, whose center on X is 0.

Remark 3.2 The minimal discrepancy only exists when X has log-canonical singularities (see, e.g. [4]). Whenever $Md(X)$ exists it is at least -1.

Shokurov conjecture that the minimal discrepancy is bounded above in term of the dimension of a variety.

Conjecture 3.3 (*Shokurov* [22]) The minimal discrepancy $Md_0(X)$ of a variety X at 0 of dimension n is at most $n-1$. Moreover, if $Md_0(X) = n-1$, then $(X, 0)$ is nonsingular.

The conjecture was confirmed for surfaces ([1]) and 3-dimensional singularities after the explicit classification ([21]) of Gorenstein terminal threefold singularities with [11] or [18]. If X is a local complete intersection, then the conjecture also holds (see [10] and [11]).

In this paper, we are going to consider the 3-dimensional singularities. Mori ([19]), Cutkosky ([5]) and Brenton ([3]) had some classification theorems about special 3-folds singularities. The following two theorems will be used to prove our main theorems.

Theorem 3.4 ([20] Corollary 2.14, Corollary 2.12) *Let* $0 \in V$ *be a 3-dimensional rational Gorenstein singularity, then there exists a partial resolution* $\pi : \tilde{V} \to V$ *such that* $\pi^{-1}\{0\}$ *is a union of nonsingular rational or elliptic ruled surfaces,* $\tilde{V}$ *only has terminal singularities and* $K_{\tilde{V}} = \pi^* K_V$.

Remark 3.5 The information of the irreducible and reduced components of the exceptional set can also be found in [3].

Definition 3.6 A singular point 0 is called compound Du Val (*cDV*) if for a general section H through 0, $0 \in H$ is a Du Val singularities.

Remark 3.7 A cDV singularity is formally equivalent to the germ of a hypersurface singularity $(\{f = 0\}, 0)$ in $\mathbb{C}^4$, where

$$f(x_0, x_1, x_2, x_3) = f_{X_n}(x_0, x_1, x_2) + x_3 g(x_0, x_1, x_2, x_3), \tag{3.2}$$

where X_n stands for A_n, D_n or E_n, $g(0, 0, 0, 0) = 0$ and f_{X_n} is one of the following polynomials:

$$f_{A_n} = x_0^2 + x_1^2 + x_2^{n+1} \quad (n \geq 1) \tag{3.3}$$

$$f_{D_n} = x_0^2 + x_1^2 x_2 + x_2^{n-1} \quad (n \geq 4) \tag{3.4}$$

$$f_{E_6} = x_0^2 + x_1^3 + x_2^4 \tag{3.5}$$

$$f_{E_7} = x_0^2 + x_1^3 + x_1 x_2^3 \tag{3.6}$$

$$f_{E_8} = x_0^2 + x_1^3 + x_2^5. \tag{3.7}$$

Theorem 3.8 ([21] Theorem 1.1) *Let* $0 \in V$ *be a* 3*-dimensional singularity. Then* 0 *is an isolated cDV singularity if and only if* 0 *is Gorenstein terminal.*

4 New invariants of singularities and new *CR*-invariants

Let V be an n-dimensional complex analytic subvariety in $\mathbb{C}^N$ with only isolated singularities. In [27], Yau considered four kinds of sheaves of germs of holomorphic p-forms

1. $\bar{\Omega}_V^p := \pi_* \Omega_M^p$, where $\pi : M \longrightarrow V$ is a resolution of singularities of V.
2. $\bar{\bar{\Omega}}_V^p := \theta_* \Omega_{V \backslash V_{sing}}^p$ where $\theta : V \backslash V_{sing} \longrightarrow V$ is the inclusion map and V_{sing} is the singular set of V.
3. $\Omega_V^p := \Omega_{\mathbb{C}^N}^p / \mathscr{K}^p$, where $\mathscr{K}^p = \{f\alpha + dg \wedge \beta : \alpha \in \Omega_{\mathbb{C}^N}^p; \beta \in \Omega_{\mathbb{C}^N}^{p-1}; f, g \in \mathscr{I}\}$ and $\mathscr{I}$ is the ideal sheaf of V in $\mathbb{C}^N$.
4. $\widetilde{\Omega}_V^p := \Omega_{\mathbb{C}^N}^p / \mathscr{H}^p$, where $\mathscr{H}^p = \{\omega \in \Omega_{\mathbb{C}^N}^p : \omega|_{V \backslash V_{sing}} = 0\}$.

Clearly Ω_V^p, $\widetilde{\Omega}_V^p$ are coherent. $\bar{\Omega}_V^p$ is a coherent sheaf because π is a proper map. $\bar{\bar{\Omega}}_V^p$ is also a coherent sheaf by a theorem of Siu (cf Theorem A of [23]). In case V is a normal variety, the dualizing sheaf ω_V of Grothendieck is actually the sheaf $\bar{\bar{\Omega}}_V^n$.

Definition 4.1 Let V be an n-dimensional Stein space with 0 as its only singular point. Let $\pi : (M, A) \to (V, 0)$ be a resolution of the singularity with A as exceptional set. The geometric genus p_g and the irregularity q of the singularity are defined as follows (cf. [27], [24]):

$$p_g := \dim \Gamma(M\backslash A, \Omega^n)/\,\Gamma(M, \Omega^n), \tag{4.1}$$

$$q := \dim \Gamma\left(M\backslash A, \Omega^{n-1}\right)/\,\Gamma\left(M, \Omega^{n-1}\right), \tag{4.2}$$

$$g^{(p)} := \dim \Gamma\left(M, \Omega^p_M\right)/\pi^*\Gamma\left(V, \Omega^p_V\right). \tag{4.3}$$

Let X be a compact connected strongly pseudoconvex *CR* manifold of real dimension $2n-1$, in the boundary of a bounded strongly pseudoconvex domain D in $\mathbb{C}^N$. By a result of Harvey and Lawson, there is a unique complex variety V in $\mathbb{C}^N$ such that the boundary of V is X. Let $\pi : (M, A_1, \ldots, A_k) \to (V, 0_1, \ldots, 0_k)$ be a resolution of the singularities with $A_i = \pi^{-1}(0_i)$, $1 \le i \le k$, as exceptional sets.

In order to solve the classical complex Plateau problem, we need to find some *CR*-invariant which can be calculated directly from the boundary X and the vanishing of this invariant will give regularity of Harvey-Lawson solution to complex Plateau problem.

For this purpose, we define a new sheaf $\bar{\bar{\Omega}}^{1,1}_V$, new invariant of surface singularities $g^{(1,1)}$ and new *CR* invariant $g^{(1,1)}(X)$ in [9]. Now, we are going to generalize them to higher dimension for dealing with general complex Plateau problem.

Definition 4.2 Let $(V, 0)$ be a Stein germ of an n-dimensional analytic space with an isolated singularity at 0. Define a sheaf of germs $\bar{\bar{\Omega}}^{\Lambda^p 1}_V$ by the sheaf associated with the presheaf

$$U \mapsto \left\langle \Lambda^p \Gamma\left(U, \bar{\bar{\Omega}}^1_V\right)\right\rangle,$$

where U is an open set of V and $2 \le p \le n$.

Lemma 4.3 *Let V be an n-dimensional Stein space with* 0 *as its only singular point in* $\mathbb{C}^N$. *Let $\pi : (M, A) \to (V, 0)$ be a resolution of the singularity with A as exceptional set. Then* $\bar{\bar{\Omega}}^{\Lambda^p 1}_V$ *is coherent and there is a short exact sequence*

$$0 \longrightarrow \bar{\bar{\Omega}}^{\Lambda^p 1}_V \longrightarrow \bar{\bar{\Omega}}^p_V \longrightarrow \mathscr{G}^{(\Lambda^p 1)} \longrightarrow 0 \tag{4.4}$$

where $\mathscr{G}^{(\Lambda^p 1)}$ *is a sheaf supported on the singular point of V. Let*

$$G^{(\Lambda^p 1)}(M\backslash A) := \Gamma\left(M\backslash A, \Omega^p_M\right)/\left\langle \Lambda^p \Gamma\left(M\backslash A, \Omega^1_M\right)\right\rangle, \tag{4.5}$$

then $\dim \mathscr{G}^{(\Lambda^p 1)}_0 = \dim G^{(\Lambda^p 1)}(M\backslash A)$.

Proof Since the sheaf of germ $\bar{\bar{\Omega}}^p_V$ is coherent by a theorem of Siu (cf Theorem A of [23]), for any point $w \in V$ there exists an open neighborhood U of w in V such that $\Gamma(U, \bar{\bar{\Omega}}^1_V)$ is finitely generated over $\Gamma(U, \mathscr{O}_V)$. So $\Gamma(U, \Lambda^p \bar{\bar{\Omega}}^1_V))$ is finitely generated over $\Gamma(U, \mathscr{O}_V)$, which means $\Gamma(U, \bar{\bar{\Omega}}^{\Lambda^p 1}_V)$ is finitely generated over $\Gamma(U, \mathscr{O}_V)$, i.e. $\bar{\bar{\Omega}}^{\Lambda^p 1}_V$ is a sheaf of finite type. It is obvious that $\bar{\bar{\Omega}}^{\Lambda^p 1}_V$ is a subsheaf of $\bar{\bar{\Omega}}^p_V$ which is also coherent. So $\bar{\bar{\Omega}}^{\Lambda^p 1}_V$ is coherent.

Notice that the stalk of $\bar{\bar{\Omega}}^{\Lambda^p 1}_V$ and $\bar{\bar{\Omega}}^p_V$ coincide at each point different from the singular point 0, $\mathscr{G}^{(\Lambda^p 1)}$ is supported at 0. And from Cartan Theorem B

$$\dim \mathscr{G}^{(\Lambda^p 1)}_0 = \dim \Gamma\left(V, \bar{\bar{\Omega}}^p_V\right)/\,\Gamma\left(V, \bar{\bar{\Omega}}^{\Lambda^p 1}_V\right) = \dim G^{(\Lambda^p 1)}(M\backslash A).$$

□

Thus, from Lemma 4.3, we can define a local invariant of a singularity which is independent of resolution.

Definition 4.4 Let V be an n-dimensional Stein space with 0 as its only singular point. Let $\pi : (M, A) \to (V, 0)$ be a resolution of the singularity with A as exceptional set. Let

$$g^{(\Lambda^p 1)}(0) := \dim \mathscr{G}_0^{(\Lambda^p 1)} = \dim G^{(\Lambda^p 1)}(M\backslash A). \tag{4.6}$$

We will omit 0 in $g^{(\Lambda^p 1)}(0)$ if there is no confusion from the context.

Let $\pi : (M, A_1, \ldots, A_k) \to (V, 0_1, \ldots, 0_k)$ be a resolution of the singularities with $A_i = \pi^{-1}(0_i)$, $1 \le i \le k$, as exceptional sets. In this case we still let

$$G^{(\Lambda^p 1)}(M\backslash A) := \Gamma\left(M\backslash A, \Omega_M^p\right) / \left\langle \Lambda^p \Gamma\left(M\backslash A, \Omega_M^1\right)\right\rangle,$$

where $A = \cup_i A_i$.

Definition 4.5 If X is a compact connected strongly pseudoconvex CR manifold of real dimension $2n-1$, in the boundary of a bounded strongly pseudoconvex domain D in $\mathbb{C}^N$. Suppose V in $\mathbb{C}^N$ such that the boundary of V is X. Let $\pi : (M, A = \cup_i A_i) \to (V, 0_1, \ldots, 0_k)$ be a resolution of the singularities with $A_i = \pi^{-1}(0_i)$, $1 \le i \le k$, as exceptional sets. Let

$$G^{(\Lambda^p 1)}(M\backslash A) := \Gamma\left(M\backslash A, \Omega_M^p\right) / \left\langle \Lambda^p \Gamma\left(M\backslash A, \Omega_M^1\right)\right\rangle \tag{4.7}$$

and

$$G^{(\Lambda^p 1)}(X) := \mathscr{S}^p(X)/\langle \Lambda^p \mathscr{S}^1(X)\rangle, \tag{4.8}$$

where $\mathscr{S}^q$ are holomorphic cross sections of $\wedge^q(\widehat{T}(X)^*)$. Then we set

$$g^{(\Lambda^p 1)}(M\backslash A) := \dim G^{(\Lambda^p 1)}(M\backslash A), \tag{4.9}$$

$$g^{(\Lambda^p 1)}(X) := \dim G^{(\Lambda^p 1)}(X). \tag{4.10}$$

Lemma 4.6 *Let X be a compact connected strongly pseudoconvex CR manifold of real dimension $2n-1$ which bounds a bounded strongly pseudoconvex variety V with only isolated singularities $\{0_1, \ldots, 0_k\}$ in $\mathbb{C}^N$. Let $\pi : (M, A_1, \ldots, A_k) \to (V, 0_1, \ldots, 0_k)$ be a resolution of the singularities with $A_i = \pi^{-1}(0_i)$, $1 \le i \le k$, as exceptional sets. Then $g^{(\Lambda^p 1)}(X) = g^{(\Lambda^p 1)}(M\backslash A)$, where $A = \cup A_i$, $1 \le i \le k$.*

Proof Take a one-convex exhausting function ϕ on M such that $\phi \ge 0$ on M and $\phi(y) = 0$ if and only if $y \in A$. Set $M_r = \{y \in M, \phi(y) \ge r\}$. Since $X = \partial M$ is strictly pseudoconvex, any holomorphic q-form $\theta \in \mathscr{S}^q(X)$ can be extended to a one side neighborhood of X in M. Hence θ can be thought of as holomorphic q-form on M_r, i.e. an element in $\Gamma(M_r, \Omega_{M_r}^q)$. By Andreotti and Grauert ([2]), $\Gamma(M_r, \Omega_{M_r}^q)$ is isomorphic to $\Gamma(M\backslash A, \Omega_M^q)$. So $g^{(\Lambda^p 1)}(X) = g^{(\Lambda^p 1)}(M\backslash A)$. □

By Lemma 4.6 and the proof of Lemma 4.3, we can get the following lemma easily.

Lemma 4.7 *Let X be a compact connected strongly pseudoconvex CR manifold of real dimension $2n-1$, which bounds a bounded strongly pseudoconvex variety V with only isolated singularities $\{0_1, \ldots, 0_k\}$ in $\mathbb{C}^N$. Then $g^{(\Lambda^p 1)}(X) = \sum_i g^{(\Lambda^p 1)}(0_i) = \sum_i \dim \mathscr{G}_{0_i}^{(\Lambda^p 1)}$.*

The following proposition is to show that $g^{(\Lambda^p 1)}$ is bounded above.

Proposition 4.8 *Let V be an n-dimensional Stein space with 0 as its only singular point. Then*

$$g^{(\Lambda^p 1)} \leq \begin{cases} g^{(p)}, & p \leq n-2; \\ g^{(n-1)} + q, & p = n-1; \\ g^{(n)} + p_g, & p = n. \end{cases}$$

Proof Since

$$\begin{aligned} g^{(\Lambda^p 1)} &= \dim \Gamma\left(M \backslash A, \Omega^p_M\right) / \left\langle \Lambda^p \Gamma\left(M \backslash A, \Omega^1_M\right)\right\rangle, \\ g^{(p)} &= \dim \Gamma(M, \Omega^p)/\pi^* \Gamma(V, \Omega^p_V), \\ \dim \Gamma\left(M \backslash A, \Omega^p_M\right) / \Gamma\left(M, \Omega^p_M\right) &= \begin{cases} 0, & p \leq n-2; \\ q, & p = n-1; \\ p_g, & p = n. \end{cases} \end{aligned}$$

and

$$\begin{aligned} \pi^* \Gamma\left(V, \Omega^p_V\right) &= \left\langle \pi^* \left(\Lambda^p \Gamma(V, \Omega^1_V)\right)\right\rangle \\ &\subseteq \Lambda^p \Gamma(M, \Omega^1_M) \\ &\subseteq \Lambda^p \Gamma(M \backslash A, \Omega^1_M), \end{aligned} \tag{4.11}$$

the result follows easily. □

The following theorem is the crucial part for solving the classical complex Plateau problem of real dimension 3.

Theorem 4.9 ([9]) *Let V be a 2-dimensional Stein space with 0 as its only normal singular point with $\mathbb{C}^*$-action. Let $\pi : (M, A) \to (V, 0)$ be a minimal good resolution of the singularity with A as exceptional set, then $g^{(\Lambda^2 1)} \geq 1$.*

Remark 4.10 We also show that $g^{(\Lambda^2 1)}$ is strictly positive for rational singularities ([6]) and minimal elliptic singularities ([7]) and exact 1 for rational double points, triple points and quotient singularities ([8]).

Similarly, the following theorem is the crucial part for solving the classical complex Plateau problem of real dimension 5.

Theorem 4.11 *Let V be an n-dimensional Stein space with 0 as its only non-rational singular point, where $n > 2$, then $g^{(\Lambda^n 1)} \geq 1$.*

Proof Suppose $\pi : M \to V$ be any resolution of the singularity 0 with E as its exceptional set. By a result of Greuel ([13], Proposition 2.3), for every holomorphic 1 form η on $V - 0$, $\pi^*(\eta)$ can extends holomorphically to M. Since 0 is not rational, there exists a holomorphic n form ω on $V - 0$ such that $\pi^*(\omega)$ can not extends holomorphically to M. So

$$g^{(\Lambda^n 1)} \geq \dim \Gamma\left(M - E, \Omega^n_M\right) / \Lambda^n \Gamma\left(M, \Omega^1_M\right) > 0.$$

□

Theorem 4.12 *Let V be a 3-dimensional Stein space with 0 as its only normal Gorenstein singular point, then $g^{(\Lambda^3 1)} \geq 1$.*

Proof If 0 is non-rational, then $g^{(\Lambda^3 1)} \geq 1$ by Theorem 4.11. So we only need to show that the result is true for rational Gorenstein singularities. It is well known that rational Gorenstein singularities are canonical (see [16] Corollary 5.24).

We are going to separate our argument into two cases to finish the proof.

Case i If 0 is terminal, by Theorem 3.8, 0 is a cDV singularity defined by $f(x_0, x_1, x_2, x_3) = 0$. Take a typical blowing-up $\sigma : V' \to V$ at 0, the exceptional set, i.e., the projectivised tangent cone is a subscheme of degree 2 in $\mathbb{P}^3$ whose irreducible and reduced component denoted by F is a nonsingular rational surface after desingularization. Consider

$$s = \frac{dx_1 \wedge dx_2 \wedge dx_3}{\partial f / \partial x_0}.$$

A typical piece of the blowing-up of $\mathbb{C}^4$ at 0 has coordinates y_0, y_1, y_2 and y_3 with $x_0 = y_0, x_1 = y_0 y_1, x_2 = y_0 y_2, x_3 = y_0 y_3$, and in this piece the nonsingular proper transform is given by $f(y_0, y_0 y_1, y_0 y_2, y_0 y_3)/y_0^2$. Then

$$dx_i = y_i dy_0 + y_0 dy_i, \quad i = 1, 2, 3$$

and the vanishing order of $\partial f(y_0, y_0 y_1, y_0 y_2, y_0 y_3)/\partial y_0$ along y_0 is 1. Then

$$\begin{aligned} \sigma^* s &= \frac{\bigwedge_{i=1}^3 (y_i dy_0 + y_0 dy_i)}{\partial f(y_0, y_0 y_1, y_0 y_2, y_0 y_3)/\partial y_0} \\ &= \frac{y_0^2 \Theta(y_0, y_1, y_2, y_3)}{\partial f(y_0, y_0 y_1, y_0 y_2, y_0 y_3)/\partial y_0}, \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} &\Theta(y_0, y_1, y_2, y_3) \\ &\quad = y_1 dy_0 \wedge dy_2 \wedge dy_3 + y_2 dy_1 \wedge dy_0 \wedge dy_3 \\ &\qquad + y_3 dy_1 \wedge dy_2 \wedge dy_0 + y_0 dy_1 \wedge dy_2 \wedge dy_3. \end{aligned} \tag{4.13}$$

So the vanishing order of $\sigma^* s$ along F is 1.

Let $\pi : M \to V'$ be a resolution consists of a series of blowing-ups with $E = \cup E_i$ as the exceptional set of $\pi \circ \sigma$, where each E_i is the non-singular irreducible component. We can assume without loss of generality that the exceptional set E is a divisor with normal crossings. So $(\pi \circ \sigma)^* s \in \Gamma(M, \Omega_M^3)$ vanishes along some $E_j = \pi^* F \subseteq E$ of order 1, which is a nonsingular rational surface, i.e. $\mathrm{Ord}_{E_j} (\pi \circ \sigma)^* s = 1$. Take a tubular neighborhood M_j of E_j such that $M_j \subset M$. Consider the exact sequence ([12])

$$0 \to \Omega_{M_j}^1 (\log E_j)(-E_j) \to \Omega_{M_j}^1 \to \Omega_{E_j}^1 \to 0. \tag{4.14}$$

By taking global sections we have

$$0 \to \Gamma\left(M_j, \Omega_{M_j}^1 (\log E_j)(-E_j)\right) \to \Gamma\left(M_j, \Omega_{M_j}^1\right) \to \Gamma\left(E_j, \Omega_{E_j}^1\right). \tag{4.15}$$

Since E_j is a nonsingular rational surface, $h^1(E_j, \mathcal{O}_{E_j})$, the irregularity of E_j, is 0. Then $\Gamma(E_j, \Omega_{E_j}^1) = 0$ by Hodge symmetry. Therefore

$$\Gamma\left(M_j, \Omega_{M_j}^1 (\log E_j)(-E_j)\right) = \Gamma\left(M_j, \Omega_{M_j}^1\right) \tag{4.16}$$

from (4.15).

Suppose $\eta \in \Gamma(M_j, \Omega^1_{M_j})$, then $\eta \in \Gamma(M_j, \Omega^1_{M_j}(\log E_j)(-E_j))$ by (4.16). Chose a point P in E_j which is a smooth point in E. Let (x_1, x_2, x_3) be a coordinate system center at P such that E_j is given locally by $x_1 = 0$. Write η locally around P : $\eta \mathrel{\mathring{=}} f_1 dx_1 + f_2 x_1 dx_2 + f_3 x_1 dx_3$, where f_1, f_2 and f_3 are holomorphic functions and "$\mathring{=}$" means local equality around P. So the vanishing order of any elements in $\Lambda^3\Gamma(M, \Omega^1_M)$ along the irreducible exceptional set E_j is at least 2 by noticing $\Gamma(M, \Omega^1_M) \subseteq \Gamma(M_j, \Omega^1_{M_j})$ under natural restriction. So

$$g^{(\Lambda^3 1)} = \dim \Gamma\left(M, \Omega^3_M\right) / \Lambda^3 \Gamma\left(M, \Omega^1_M\right) \geq 1.$$

Case ii If 0 is canonical but not terminal, then by Theorem 3.4, there exists a partial resolution $\rho : \tilde{V} \to V$ such that $\rho^{-1}\{0\} = \cup_i F_i$ is a union of nonsingular rational or elliptic ruled surfaces, $\tilde{V}$ only has terminal singularities and $K_{\tilde{V}} = \rho^* K_V$. So if we let $\pi : M \to \tilde{V}$ be a resolution consists a series of blowing-ups, then the discrepancy of some $\pi^* F_j$ for V is 0. Therefore there is a section $s \in \Gamma(M, \Omega^3_M)$ such that s does not vanish along some irreducible exceptional set $E_j := \pi^* F_j$, i.e. $\mathrm{Ord}_{E_j} s = 0$. Take a tubular neighborhood M_j of E_j such that $M_j \subset M$. Consider the same exact sequence as in Case i:

$$0 \to \Omega^1_{M_j}(\log E_j)(-E_j) \to \Omega^1_{M_j} \to \Omega^1_{E_j} \to 0. \tag{4.17}$$

By taking global sections we have

$$0 \to \Gamma\left(M_j, \Omega^1_{M_j}(\log E_j)(-E_j)\right) \to \Gamma\left(M_j, \Omega^1_{M_j}\right) \to \Gamma\left(E_j, \Omega^1_{E_j}\right). \tag{4.18}$$

We know that E_j must be a rational surface or elliptic ruled surface. If E_j is rational, we have $h^1(E_j, \mathcal{O}_{E_j})$, the irregularity of E_j, is 0. Then $\Gamma(E_j, \Omega^1_{E_j}) = 0$ by Hodge symmetry. Therefore

$$\Gamma\left(M_j, \Omega^1_{M_j}(\log E_j)(-E_j)\right) = \Gamma\left(M_j, \Omega^1_{M_j}\right) \tag{4.19}$$

from (4.18). Then by the same local argument as in Case i and $\Gamma(M, \Omega^1_M) \subseteq \Gamma(M_j, \Omega^1_{M_j})$, we have

$$g^{(\Lambda^3 1)} = \dim \Gamma\left(M, \Omega^3_M\right) / \Lambda^3 \Gamma\left(M, \Omega^1_M\right) \geq 1.$$

If E_j is elliptic ruled surface, the only difference is that $h^1(E_j, \mathcal{O}_{E_j})$, the irregularity of E_j, is 1. Then $\Gamma(E_j, \Omega^1_{E_j}) = 1$ by Hodge symmetry. Therefore

$$\dim \Gamma(M_j, \Omega^1_{M_j}) / \Gamma(M_j, \Omega^1_{M_j}(\log E_j)(-E_j)) \leq 1 \tag{4.20}$$

from (4.15).

If we take three $\mathbb{C}$-linear independent holomorphic 1-forms in $\Gamma(M, \Omega^1_M)$, then there must exist two elements $\eta_1, \eta_2 \in \Gamma(M, \Omega^1_M)$ such that $\eta_1|_{M_j}, \eta_2|_{M_j} \in \Gamma(M_j, \Omega^1_{M_j}(\log E_j)(-E_j))$ from (4.20). Similarly, chose a point P in E_j which is a smooth point in E. Let (x_1, x_2, x_3) be a coordinate system center at P such that E_j is given locally by $x_1 = 0$. Write η_1 and η_2 locally around P :

$$\eta_1 \mathrel{\mathring{=}} f_1 dx_1 + f_2 x_1 dx_2 + f_3 x_1 dx_3$$

and

$$\eta_2 \mathrel{\mathring{=}} g_1 dx_1 + g_2 x_1 dx_2 + g_3 x_1 dx_3,$$

where f_i and g_i $(1 \le i \le 3)$ are holomorphic functions and "$\stackrel{\circ}{=}$" means local equality around P. So the vanishing order of any elements in $\Lambda^3\Gamma(M_j, \Omega^1_{M_j})$ along the irreducible exceptional set E_j is at least 1. So by noticing $\Gamma(M, \Omega^1_M) \subseteq \Gamma(M_j, \Omega^1_{M_j})$, we have

$$g^{(\Lambda^3 1)} = \dim \Gamma\left(M, \Omega^3_M\right) / \Lambda^3 \Gamma\left(M, \Omega^1_M\right) \ge 1.$$

□

5 The classical complex Plateau problem

In 1981, Yau [26] solved the classical complex Plateau problem for the case $n \ge 3$.

Theorem 5.1 ([26]) *Let X be a compact connected strongly pseudoconvex CR manifold of real dimension $2n-1$, $n \ge 3$, in the boundary of a bounded strongly pseudoconvex domain D in $\mathbb{C}^{n+1}$. Then X is the boundary of a complex sub-manifold $V \subset D - X$ if and only if Kohn–Rossi cohomology groups $H^{p,q}_{KR}(X)$ are zeros for $1 \le q \le n-2$*

When $n = 2$, the Plateau problem remains unsolved for many years even there are no any criterion to judge whether X is the boundary of a complex manifold. In [9], the first and the third authors used *CR* invariant $g^{(1,1)}(X)$ to give the sufficient and necessary condition for the variety bounded by a Calabi–Yau CR manifold X being smooth if $H^2_h(X) = 0$.

Theorem 5.2 ([9]) *Let X be a strongly pseudoconvex compact Calabi–Yau CR manifold of dimension 3. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in $\mathbb{C}^N$ with $H^2_h(X) = 0$. Then X is the boundary of a complex sub-manifold (up to normalization) $V \subset D - X$ with boundary regularity if and only if $g^{(1,1)}(X) = 0$.*

Theorem 5.3 ([9]) *Let X be a strongly pseudoconvex compact CR manifold of dimension 3. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in $\mathbb{C}^3$ with $H^2_h(X) = 0$. Then X is the boundary of a complex sub-manifold $V \subset D - X$ if and only if $g^{(1,1)}(X) = 0$.*

We will use the new *CR* invariant $g^{(\Lambda^n 1)}(X)$ to deal with complex Plateau problem of X in general type.

Theorem 5.4 *Let X be a strongly pseudoconvex compact CR manifold of dimension $2n-1$, where $n > 2$. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in $\mathbb{C}^N$. Then X is the boundary of a variety $V \subset D - X$ with boundary regularity and the number of non-rational singularities is not great than $g^{(\Lambda^n 1)}(X)$. In particular, if $g^{(\Lambda^n 1)}(X) = 0$, then V has at most finite number of rational singularities.*

Proof It is well known that X is the boundary of a variety V in D with boundary regularity ([15,17]). Suppose $\{0_1, \ldots 0_k\}$ be k non-rational singularities in V. The the result follows easily from Theorem 4.11 and Lemma 4.7. □

When X is a Calabi–Yau *CR* manifold of dimension 5, we give the following necessary and sufficient condition for the variety bounded by X being smooth.

Theorem 5.5 *Let X be a strongly pseudoconvex compact Calabi–Yau CR manifold of dimension 5. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in $\mathbb{C}^N$. Then X is the boundary of a complex sub-manifold (up to normalization) $V \subset D - X$ with boundary regularity if and only if $g^{(\Lambda^3 1)}(X) = 0$.*

Proof (⇒) : Since V is smooth, $g^{(\Lambda^3 1)}(X) = 0$ follows from Lemma 4.7.

(⇐) : It is well known that X is the boundary of a variety V in D with boundary regularity ([15,17]). The the result follows easily from Theorem 4.12 and Lemma 4.7. □

Corollary 5.6 *Let X be a strongly pseudoconvex compact Calabi–Yau CR manifold of dimension 5. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in $\mathbb{C}^4$. Then X is the boundary of a complex sub-manifold $V \subset D - X$ with boundary regularity if and only if $g^{(\Lambda^3 1)}(X) = 0$.*

Proof The result follows easily from the fact that isolated hypersurface singularities are normal and Gorenstein. □

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