Journal of Number Theory 160 (2016) 254–286

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On the polynomial sharp upper estimate conjecture in 7-dimensional simplex ☆

Stephen S.-T. Yau [a,*], Beihui Yuan [b], Huaiqing Zuo [c]

[a] *Department of Mathematical Sciences, Tsinghua University, Beijing 100084, PR China*
[b] *Department of Mathematics, Cornell University, Ithaca, NY, 14853, USA*
[c] *Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, PR China*

A R T I C L E I N F O

Article history:
Received 4 May 2015
Received in revised form 14 May 2015
Accepted 31 August 2015
Available online 13 October 2015
Communicated by David Goss

Dedicated to Professor John Henry Coates on the occasion of his 70th birthday

MSC:
primary 11P21
secondary 11Y99

Keywords:
Sharp estimate
Integral points
Simplex

A B S T R A C T

Because of its importance in number theory and singularity theory, the problem of finding a polynomial sharp upper estimate of the number of positive integral points in an n-dimensional ($n \geq 3$) polyhedron has received attention by a lot of mathematicians. The first named author proposed the Number Theoretic Conjecture for the upper estimate. The previous results on the Number Theoretic Conjecture in low dimension cases ($n < 7$) are proved by using the sharp GLY conjecture which is true only for low dimensional case. Thus the proof cannot be generalized to high dimension. In this paper, we offer a uniform approach to prove the Number Theoretic Conjecture for all dimensions by simply using the induction method and the Yau–Zhang [19] estimates (see Lemmas 2.3–2.5). As a result, the Number Theoretic Conjecture is proven for $n = 7$. An important estimate for all dimensions is also obtained (Propositions 3.1 and 3.2) which will be useful to prove the general case of the Number Theoretic Conjecture. As an application, we give a sharper

☆ This work was partially supported by NSFC (grant nos. 11401335, 11531007), Tsinghua University Initiative Scientific Research Program and Tsinghua University start-up fund.

* Corresponding author.
E-mail addresses: yau@uic.edu (S.S.-T. Yau), by238@cornell.edu (B. Yuan), hqzuo@math.tsinghua.edu.cn (H. Zuo).

http://dx.doi.org/10.1016/j.jnt.2015.08.012

estimate of the Dickman–De Bruijn function $\psi(x,y)$ for $5 \leq y < 19$, compared with the result obtained by Ennola.

1. Introduction

Let $T(a_1, a_2, \ldots, a_n)$ be an n-dimensional simplex described by

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leq 1, x_1, x_2, \ldots, x_n \geq 0 \tag{1}$$

where $a_1 \geq a_2 \geq \cdots \geq a_n \geq 1$ are real numbers. Let $P_n = P(a_1, a_2, \ldots, a_n)$ and $Q_n = Q(a_1, a_2, \ldots, a_n)$ be the number of positive integer solutions and nonnegative integer solutions of (1), respectively. We can see there is a relation

$$Q(a_1, \ldots, a_n) = P(a_1(1+a), \ldots, a_n(1+a))$$

where $a = \frac{1}{a_1} + \cdots + \frac{1}{a_n}$.

The estimate of P_n and Q_n can be applied in number theory. A *smooth number* is a number with only small prime factors. Smooth numbers play important roles in factoring and primality testing [12]. Given an integer y, the number $m = p_1^{l_1} p_2^{l_2} \ldots p_n^{l_n}$ is called y-smooth if all its prime factors $p_i \leq y$ for $i = 1, \ldots, n$. Number theorists want to know the number of y-smooth integers less than or equal to x, which is denoted by $\psi(x,y)$, called the Dickman–De Bruijn function. One of the central topics in computational number theory is the estimate of $\psi(x,y)$ (see [5]). It turns out that the computation of $\psi(x,y)$ is equivalent to compute the number of integral points in an k-dimensional tetrahedron $\triangle(a_1, a_2, \cdots, a_k)$ with real vertices $(a_1, 0, \cdots, 0), \cdots, (0, \cdots, 0, a_k)$. Let $p_1 < p_2 < \cdots < p_k$ denote the primes up to y. It is clear that $p_1^{l_1} p_2^{l_2} \cdots p_k^{l_k} \leq x$ which is also equivalent to counting the number of $(l_1, l_2, \cdots, l_k) \in \mathbb{Z}^n_{\geq 0}$ such that

$$\frac{l_1}{a_1} + \frac{l_2}{a_2} + \cdots + \frac{l_k}{a_k} \leq 1, \text{ where } a_i = \frac{\log x}{\log p_i}.$$

Therefore, $\psi(x,y)$ is precisely the number Q_k of (integer) lattice points inside the n-dimensional tetrahedron (1) with $a_i = \frac{\log x}{\log p_i}$, $n = k$, and $1 \leq i \leq k$. In [4], Ennola gave both lower and upper bounds for the $\psi(x,y)$:

$$\frac{(\log x)^k}{k! \prod_{i=1}^{k} \log p_i} < \psi(x,y) \leq \frac{(\log x + \sum_{i=1}^{k} \log p_i)^k}{k! \prod_{i=1}^{k} \log p_i} \tag{2}$$

which yields the following result.

Theorem 1.1. *(See Ennola [4].) Uniformly for* $2 \leq y \leq \sqrt{\log x \log_2 x}$*, we have that*

$$\psi(x,y) = \frac{1}{k!}\prod_{p\leq y}(\frac{\log x}{\log p})[1+O(\frac{y^2}{\log x \log y})].$$

Numbers P_n and Q_n also have applications in geometry and singularity theory. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function with isolated critical point at the origin and $V = \{z \in \mathbb{C}^n : f(z) = 0\}$. The geometric genus p_g is defined to be $\dim \Gamma(V - \{0\}, \Omega^{n-1})/L^2(V - \{0\}, \Omega^{n-1})$, where Ω^{n-1} is the sheaf of germs of holomorphic $(n-1)$-forms on $V - \{0\}$. If $f(z_1, \dots, z_n)$ is weighted homogeneous of type $(w_1, \dots, w_n)$, where $w_1, \dots, w_n$ are fixed positive rational numbers, i.e., f can be expressed as a linear combination of monomials $z_1^{i_1} \dots z_n^{i_n}$ for which $i_1/w_1 + \dots + i_n/w_n = 1$, then Merle and Teissier [9] showed that p_g is exactly the number $P(w_1, \dots, w_n)$.

There are a lot of papers on finding the exact formula for P_n or Q_n, in case $a_1, \dots, a_n$ are integers. For example, Mordell [11] gave an exact formula for Q_3 with a_1, a_2 and a_3 relatively prime. Pommersheim [13] extended this result to arbitrary integers a_1, a_2 and a_3 using toric variety techniques and a result of Ehrhart [3]. The exact formula is complicated, it involves generalized Dedekind sum. It is hard to figure out how large the sum is from the exact formula. Therefore, sometimes we want to get a sharp upper estimate of P_n in terms of a polynomial in $a_1, \dots, a_n$. Such a polynomial upper estimate have many important applications. For example, it can be used in the following Durfee Conjecture:

Conjecture. *(See Durfee [2].) Let* $(V, 0)$ *be an isolated hypersurface singularity defined by a holomorphic function* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$*. Let*

$$\mu = \dim \mathbb{C}\{z_1, \dots, z_n\}/(f_{z_1}, \dots, f_{z_n})$$

be the Milnor number of the singularity. Then $n!p_g \leq \mu$ *where* p_g *is the geometric genus of* $(V, 0)$*.*

If f is weighted homogeneous of type $(w_1, \dots, w_n)$, Milnor and Orlik [10] proved that $\mu = (w_1 - 1)\dots(w_n - 1)$. Therefore Durfee conjecture is a special case of the following theorem, which was proved by Yau and Zhang [19]:

Theorem 1.2 *(GLY rough estimate). Let* $a_1, \dots, a_n$ *be positive real numbers greater than or equal to* 1 *and* $n \geq 3$*. Then*

$$n!P(a_1, \dots, a_n) < (a_1 - 1)(a_2 - 1)\dots(a_n - 1). \tag{3}$$

The estimate in the above theorem is nice. However, it is not sharp enough to provide a solution of the following problem:

Problem. (See [21,20].) Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function with an isolated critical point at the origin. Find an intrinsic characterization for f to be a homogeneous polynomial.

In 1971, Saito [14] gave an intrinsic characterization for f to be a weighted homogeneous polynomial

Theorem 1.3 *(Saito). Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function with isolated critical point at the origin. Then f is a weighted homogeneous polynomial after a biholomorphic change of coordinates if and only if $\mu = \tau$, where*

$$\mu = \dim \mathbb{C}\{z_1, \dots, z_n\}/(f_{z_1}, \dots, f_{z_n})$$

and

$$\tau = \dim \mathbb{C}\{z_1, \dots, z_n\}/(f, f_{z_1}, \dots, f_{z_n}).$$

To find a necessary and sufficient condition for f to be a homogeneous polynomial, Yau made the following conjecture in 1995:

Conjecture *(Yau Geometric Conjecture). Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a weighted homogeneous polynomial with an isolated singularity at the origin. Let μ, p_g and ν be the Milnor number, geometric genus and multiplicity of singularity $V = \{z : f(z) = 0\}$, respectively. Then*

$$\mu - h(\nu) \geq n! p_g \tag{4}$$

where $h(\nu) = (\nu - 1)^n - \nu(\nu - 1) \dots (\nu - n + 1)$. The equality holds if and only if f is a homogeneous polynomial after a biholomorphic change of coordinates.

The Yau Geometric Conjecture together with Theorem 1.3 will give an intrinsic characterization for a holomorphic function f to be a homogeneous polynomial after a biholomorphic change of coordinates. In order to prove Yau Geometric Conjecture, Lin, Yau [6] and Granville have formulated GLY Rough Estimate and the following GLY Sharp Conjecture:

Conjecture *(GLY Sharp Estimate). Let $n \geq 3$. If $a_1 \geq a_2 \geq \dots \geq a_n \geq n - 1$. Then*

$$n! P_n \leq f_n := A_0^n + \frac{s(n, n-1)}{n} A_1^n + \sum_{l=1}^{n-2} \frac{s(n, n-1-l)}{\binom{n-1}{l}} A_l^{n-1} \tag{5}$$

where $s(n, k)$ is the Stirling number of the first kind defined by the following generating function:

$$x(x-1)\dots(x-n+1)=\sum_{k=0}^{n}s(n,k)x^k$$

and A_k^n is defined as

$$A_k^n=(\prod_{i=1}^{n}a_i)(\sum_{1\le i_1\le i_2\le\cdots\le i_k\le n}\frac{1}{a_{i_1}a_{i_2}\dots a_{i_k}})$$

for $k=1,2,\dots,n-1$. Equality in (5) holds if and only if $a_1=\cdots=a_n$ are integers.

The above GLY Sharp Estimate is true for $n=4,5,6$ (cf. [22,1]) and there is a counter-example for $n=7$ [16]. In [16], Wang and Yau also modify GLY Conjecture as follows:

Conjecture *(Modified GLY Conjecture). There exists an integer $y(n)$ which depends only on n such that the sharp estimate (5) holds when $a_1\ge a_2\ge\cdots\ge a_n\ge y(n)$.*

In order to overcome the difficulty that GLY Sharp Estimate is only true when a_n is larger than $y(n)$, Yau proposed a new sharp upper estimate which is motived for the Yau Geometric Conjecture:

Conjecture *(Yau Number Theoretic Conjecture). Let*

$$P_n=P_n(a_1,a_2,\dots,a_n)=\#\{(x_1,\dots,x_n)\in\mathbb{Z}_+^n:\frac{x_1}{a_1}+\frac{x_2}{a_2}+\cdots+\frac{x_n}{a_n}\le 1\},$$

where $n\ge 3$, $a_1\ge a_2\ge\cdots\ge a_n>1$ are real numbers. If $P_n>0$, then

$$n!P_n\le(a_1-1)(a_2-1)\dots(a_n-1)-(a_n-1)^n+a_n(a_n-1)\dots(a_n-(n-1)) \quad (6)$$

and equality holds if and only if $a_1=a_2=\cdots=a_n$ are integers.

There is an intimate relation between the Yau Geometric Conjecture and the Yau Number Theoretic Conjecture. Let $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be a weighted homogeneous polynomial with an isolated singularity at the origin, then the multiplicity ν of f at the origin is given by $\inf\{n\in\mathbb{Z}_+: n\ge\inf\{w_1,\dots,w_n\}\}$, where w_i is the weight of x_i. In general, the weight w_i is a rational number. In case the minimal weight is an integer, the Yau Geometric Conjecture and Yau Number Theoretic Conjecture are the same. However, in general, these two conjectures do not imply each other.

The Yau Number Theoretic Conjecture has already been verified for $n=3$ by Xu and Yau [17,18] and for $n=4,5$ by Lin, Luo, Yau and Zuo [7,8]. Liang, Yau and Zuo [5] gave the following result for $n=6$. In this paper, we will prove the conjecture for $n=7$:

Theorem 1.4 *(Main Theorem). Let* $P_7 = P_7(a_1, a_2, \dots, a_7) = \#\{(x_1, \dots, x_7) \in \mathbb{Z}_+^7 : \frac{x_1}{a_1} + \frac{x_2}{a_2} + \dots + \frac{x_7}{a_7} \le 1\}$*, where* $a_1 \ge a_2 \ge \dots \ge a_7 > 1$ *are real numbers. If* $P_7 > 0$*, then*

$$7!P_7 \le (a_1 - 1)(a_2 - 1)\dots(a_7 - 1) - (a_7 - 1)^7 + a_7(a_7 - 1)\dots(a_7 - 6) \tag{7}$$

and equality holds if and only if $a_1 = a_2 = \dots = a_7$ *are integers.*

Let

$$g_n(a_1, \dots, a_n) := (a_1 - 1)\dots(a_n - 1) - (a_n - 1)^n + a_n(a_n - 1)\dots(a_n - (n - 1))$$

be the right hand of (6). In case $n = 7$,

$$g_7(a_1, \dots, a_n) := (a_1 - 1)\dots(a_7 - 1) - (a_7 - 1)^7 + a_7(a_7 - 1)\dots(a_7 - 6).$$

In [7], we can see that, when $n = 5$, the number of subcases increases from 4 (when $n = 4$) to 11. The authors of [7] ([5] resp.) simplify those 11 (21 resp.) subcases into 5 (6 resp.) major classes. They divide the whole range into five intervals and classify those subcases by which interval the last variable a_n is in. The benefit of this classification is that the number of classes will increase only by 1 as the dimension increase by 1. However, the proofs of $n \le 6$ relied on the GLY Sharp Estimate, which is only true for $n \le 6$. Therefore the proof cannot be generalized to higher dimension. In this paper, we avoid entirely the GLY Sharp Estimate, and we will prove our main theorem purely by induction. This is a significant improvement since it suggests a way to prove the general case.

As an application, we will also prove that

Theorem 1.5 *(Estimate of* $\psi(x, y)$*). Let* $\psi(x, y)$ *be the function as before. We have the following upper estimate for* $5 \le y < 17$*:*

(I) when $5 \le y < 7$ *and* $x > 5$*, we have*

$$\begin{aligned}\psi(x, y) \le \frac{1}{6}\{&\frac{1}{\log 2 \log 3 \log 5}(\log x + \log 15)(\log x + \log 10)(\log x + \log 6)\\ &- \frac{1}{\log^3 5}[(\log x + \log 6)^3\\ &- (\log x + \log 6 + \log 5)(\log x + \log 6)(\log x + \log 6 - \log 5)]\};\end{aligned}$$

(II) when $7 \le y < 11$ *and* $x > 7$*, we have*

$$\begin{aligned}\psi(x, y) \le \frac{1}{24}\{&\frac{1}{\log 2 \log 3 \log 5 \log 7}(\log x + \log 105)(\log x + \log 70)\\ &\cdot (\log x + \log 42)(\log x + \log 30)\\ &- \frac{1}{\log^4 7}[(\log x + \log 30)^4\end{aligned}$$

$$- (\log x + \log 7 + \log 30)(\log x + \log 30) \cdot (\log x + \log 30 - \log 7)(\log x + \log 30 - 2\log 7)]\};$$

(III) when $11 \le y < 13$ *and* $x > 11$, *we have*

$$\psi(x,y) \le \frac{1}{120}\{\frac{1}{\log 2 \log 3 \log 5 \log 7 \log 11}(\log x + \log 1155)(\log x + \log 770) \cdot (\log x + \log 462)(\log x + \log 330)(\log x + \log 210) - \frac{1}{\log^5 11}[(\log x + \log 210)^5 - (\log x + \log 11 + \log 210)(\log x + \log 210) \cdot (\log x + \log 210 - \log 11)(\log x + \log 210 - 2\log 11) \cdot (\log x + \log 210 - 3\log 11)]\}.$$

(IV) when $13 \le y < 17$ *and* $x > 13$, *we have*

$$\psi(x,y) \le \frac{1}{720}\{\frac{1}{\log 2 \log 3 \log 5 \log 7 \log 11 \log 13}(\log x + \log 15015)(\log x + \log 10010) \cdot (\log x + \log 6006)(\log x + \log 4290)(\log x + \log 2730) \cdot (\log x + \log 2310) - \frac{1}{\log^6 13}[(\log x + \log 2310)^6 - (\log x + \log 13 + \log 2310)(\log x + \log 2310)(\log x + \log 2310 - \log 13) \cdot (\log x + \log 2310 - 2\log 13)(\log x + \log 2310 - 3\log 13) (\log x + \log 2310 - 4\log 13)]\}.$$

(V) when $17 \le y < 19$ *and* $x > 17$, *we have*

$$\psi(x,y) \le \frac{1}{5040}\{\frac{1}{\log 2 \log 3 \log 5 \log 7 \log 11 \log 13 \log 17}(\log x + \log 255255) (\log x + \log 170170)(\log x + \log 102102)(\log x + \log 72930)(\log x + \log 46410) \cdot (\log x + \log 39270)(\log x + \log 30030) - \frac{1}{\log^7 17}[(\log x + \log 30030)^7 - (\log x + \log 17 + \log 30030)(\log x + \log 30030)(\log x + \log 30030 - \log 17) \cdot (\log x + \log 30030 - 2\log 17)(\log x + \log 30030 - 3\log 17) \cdot (\log x + \log 30030 - 4\log 17)(\log x + \log 30030 - 5\log 17)]\}.$$

Remark. For comparison, we list the Ennola's upper bounds (see (2)) for $5 \le y < 19$ as follows:

(1): $5 \leq y < 7$ and $x > 5$,

$$\psi(x, y) \leq \frac{(\log x + \log 30)^3}{6\log 2\log 3\log 5}$$

(2): $7 \leq y < 11$ and $x > 7$,

$$\psi(x, y) \leq \frac{(\log x + \log 210)^4}{24\log 2\log 3\log 5\log 7}$$

(3): $11 \leq y < 13$ and $x > 11$,

$$\psi(x, y) \leq \frac{(\log x + \log 2310)^5}{120\log 2\log 3\log 5\log 7\log 11}$$

(4): $13 \leq y < 17$ and $x > 13$,

$$\psi(x, y) \leq \frac{(\log x + \log 30030)^6}{720\log 2\log 3\log 5\log 7\log 11\log 13}$$

(5): $17 \leq y < 19$ and $x > 17$,

$$\psi(x, y) \leq \frac{(\log x + \log 255255)^7}{5040\log 2\log 3\log 5\log 7\log 11\log 13\log 17}$$

It is easy to see that our upper bound of $\psi(x, y)$ is substantially better than the one obtained by Ennola. For example, in $17 \leq y < 19$ and $x > 17$ case, though the coefficient of $(\log x)^7$ in our estimate is same as Ennola's, but our coefficient of $(\log x)^6$ is

$$\frac{1}{5040}\Big(\frac{\log 255255 + \log 170170 + \log 102102 + \log 72930 + \log 46410 + \log 39270 + \log 30030}{\log 2\log 3\log 5\log 7\log 11\log 13\log 17} - \frac{14}{\log^6 17}\Big) \approx 0.00599482679$$

which is smaller than Ennola's

$$\frac{1}{5040}\frac{7\log 255255}{\log 2\log 3\log 5\log 7\log 11\log 13\log 17} \approx 0.0620120969.$$

We use the symbolic computation software, Sage, to deal with tremendous involved computation. Besides, we have found a quick way to judge the positivity of a polynomial in a restricted domain. We also simplify the process of computation by making use of some characteristics of those polynomials.

2. Some lemmas

We will frequently use the following two lemmas to decide the positivity of polynomials in some restricted domains.

Lemma 2.1. *(See [16] Lemma 3.1.) Let $f(\beta)$ be a polynomial defined by*

$$f(\beta) = \sum_{i=0}^{n} c_i \beta^i \tag{8}$$

where $\beta \in (0, 1)$. If for any $k = 0, 1, \ldots, n$

$$\sum_{i=0}^{k} c_i \geq 0 \tag{9}$$

then $f(\beta) \geq 0$ for $\beta \in (0, 1)$.

Lemma 2.1 is easy to use. However, the condition of Lemma 2.1 may not be satisfied in some situation. In that case, we shall make use of the following lemma.

Lemma 2.2 *(Sturm's Theorem). Starting from a given polynomial $X = f(x)$, let the polynomials $X_1, X_2, \ldots, X_r$ be determined by Euclidean algorithm as follows:*

$$\begin{aligned} X_1 &= f'(x) \qquad , \\ X &= Q_1 X_1 - X_2, \\ X_1 &= Q_2 X_2 - X_3, \\ \cdots \quad &\cdots \quad \cdots\cdots \\ X_{r-1} &= Q_r X_r \end{aligned} \tag{10}$$

where $\deg X_k > \deg X_{k+1}$ for $k = 1, \ldots, r - 1$. For every real number a which is not a root of $f(x)$ let $w(a)$ be the number of variations in sign in the number sequence

$$X(a), X_1(a), \ldots, X_r(a)$$

in which all zeros are omitted. If b and c are any numbers ($b < c$) for which $f(x)$ does not vanish, then the number of the various roots in the interval $b \leq x \leq c$ (multiple roots to be counted only once) is equal to

$$w(b) - w(c).$$

Proof. See [15]. □

The condition of Lemma 2.2 is necessary and sufficient, so it can be applied to judge the positivity of any such polynomials in some intervals. The computation in Lemma 2.2 is more complicated than that in Lemma 2.1. Therefore, we prefer Lemma 2.1 when it works.

Lemma 2.3. *(See [19] Proposition 3.1.) Given any positive real number β where $0 < \beta < 1$, let $a > 1$ be any number such that $\beta = a - \lfloor a \rfloor$, where $\lfloor a \rfloor$ denotes the greatest positive integer less than or equal to a. If $n \geq 3$, then*

$$a - 1 > (n+1) \sum_{k=0}^{\lfloor a \rfloor - 1} \frac{(k+\beta)^n}{a^n}. \tag{11}$$

Lemma 2.4. *(See [19] Lemma 3.3.) Let $a_{j-1}, a_j, \dots, a_{n+1}$ be real numbers and $\beta = a_{n+1} - \lfloor a_{n+1} \rfloor$. Assume that $a_{j-1} > 1$ and $a_j \geq a_{j+1} \geq \dots \geq a_n \geq a_{n+1} > 1$. If $\frac{a_n}{a_{n+1}}\beta \geq 1$, and*

$$\prod_{i=j}^{n+1}(a_i - 1) > (n+1) \sum_{k=0}^{\lfloor a_{n+1} \rfloor - 1} [\frac{(k+\beta)^{j-1}}{a_{n+1}^{j-1}} \prod_{i=j}^{n}(\frac{a_i}{a_{n+1}}(k+\beta) - 1)] \tag{12}$$

then

$$\prod_{i=j-1}^{n+1}(a_i - 1) > (n+1) \sum_{k=0}^{\lfloor a_{n+1} \rfloor - 1} [\frac{(k+\beta)^{j-2}}{a_{n+1}^{j-2}} \prod_{i=j-1}^{n}(\frac{a_i}{a_{n+1}}(k+\beta) - 1)]. \tag{13}$$

Lemma 2.5. *(See [19] Lemma 3.4.) Let $a_{j-1}, a_j, \dots, a_{n+1}$ be real numbers and $\beta = a_{n+1} - \lfloor a_{n+1} \rfloor$. Assume that $a_{j-1} > 1$ and $a_j \geq a_{j+1} \geq \dots \geq a_n \geq a_{n+1} > 1$. If $\frac{a_n}{a_{n+1}}\beta < 1$, and*

$$\prod_{i=j}^{n+1}(a_i - 1) > (n+1) \sum_{k=1}^{\lfloor a_{n+1} \rfloor - 1} [\frac{(k+\beta)^{j-1}}{a_{n+1}^{j-1}} \prod_{i=j}^{n}(\frac{a_i}{a_{n+1}}(k+\beta) - 1)] \tag{14}$$

then

$$\prod_{i=j-1}^{n+1}(a_i - 1) > (n+1) \sum_{k=1}^{\lfloor a_{n+1} \rfloor - 1} [\frac{(k+\beta)^{j-2}}{a_{n+1}^{j-2}} \prod_{i=j-1}^{n}(\frac{a_i}{a_{n+1}}(k+\beta) - 1)]. \tag{15}$$

3. Proof of the main theorem

We will prove the main theorem by induction. Notice that P_n can be obtained by recursion: let k be the possible integer such that $1 \leq k \leq \lfloor a_n \rfloor$, where $\lfloor a_n \rfloor$ is the biggest integer less than or equal to a_n. For each k, we have an $(n-1)$-dimensional simplex

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{k}{a_n} \leq 1, x_1, x_2, \ldots, x_{n-1} \geq 0. \tag{16}$$

Let $P_{n-1}^{(k)}$ be the number of positive integer solution of (16). Clearly,

$$P_n = \sum_{k=1}^{\lfloor a_n \rfloor} P_{n-1}^{(k)}. \tag{17}$$

Therefore, since we already know that the Yau Number Theoretic Conjecture is true for $n = 6$ [5], we want to prove that $g_7(a_1, \ldots, a_7)$ is greater than or equal to the sum of g_6's, the upper estimate of 6-dimensional layers in $T(a_1, a_2, \ldots, a_7)$.

Let m be number of 6-dimensional layers in 7-dimensional simplex, i.e. $P_6^{(m)} > 0$ and $P_6^{(m+1)} = 0$, where $P_6^{(k)} = \#\{(x_1, \ldots, x_6) \in \mathbb{Z}_+^6 : \frac{x_1}{a_1} + \cdots + \frac{x_6}{a_6} + \frac{k}{a_7} \leq 1\}$, where $1 \leq k \leq m$, $a_1 \geq a_2 \geq \cdots \geq a_6 \geq 1$ are real numbers. Let

$$\begin{aligned}\Delta_m &:= g_7(a_1, \ldots, a_7) - 7\sum_{k=1}^{m} g_6(a_1(1 - \frac{k}{a_7}), \ldots, a_6(1 - \frac{k}{a_7}))\\ &= (a_1 - 1)\ldots(a_7 - 1) - (a_7 - 1)^7 + a_7(a_7 - 1)\ldots(a_7 - 6)\\ &\quad - 7[\sum_{k=1}^{m}(a_1(1 - \frac{k}{a_7}) - 1)\ldots(a_6(1 - \frac{k}{a_7}) - 1) - (a_6(1 - \frac{k}{a_7}) - 1)^6\\ &\quad + a_6(1 - \frac{k}{a_7})(a_6(1 - \frac{k}{a_7}) - 1)\ldots(a_6(1 - \frac{k}{a_7}) - 5)]\end{aligned}$$

be the difference between $g_7(a_1, \ldots, a_7)$ and the sum of g_6's. We should prove that $\Delta_m \geq 0$ under the condition of main theorem.

Since $P_6^{(m)} = \#\{(x_1, \ldots, x_6) \in \mathbb{Z}_+^6 : \frac{x_1}{a_1} + \cdots + \frac{x_6}{a_6} + \frac{m}{a_7} \leq 1\}$, let $\alpha = 1 - \frac{m}{a_7} \in (0, 1)$, $A_i = a_i\alpha$, for $i = 1, \ldots, 6$, then we have

$$\frac{x_1}{A_1} + \frac{x_2}{A_2} + \cdots + \frac{x_6}{A_6} \leq 1 \tag{18}$$

and

$$g_6(m) := \sum_{k=1}^{m} g_6(\frac{m - k + k\alpha}{m\alpha} A_1, \ldots, \frac{m - k + k\alpha}{m\alpha} A_6)$$

$$\Delta_m(A_1, \ldots, A_6, \alpha) = g_7(\frac{A_1}{\alpha}, \ldots, \frac{A_6}{\alpha}, \frac{m}{1 - \alpha}) - 7g_6(m).$$

Let $B_{6,k}$ be $e_k(A_1, \ldots, A_6)$, the elementary symmetric polynomial, for $k = 0, \ldots, 6$, that is, $B_{6,k} = A_1 \cdots A_6 \sum_{1 \leq i_1 < \cdots \leq i_k \leq 6} \frac{1}{A_{i_1} \ldots A_{i_k}}$. For example, $B_{6,0} = A_1 \cdots A_6$, $B_{6,5} = A_1 + \cdots + A_6$ and $B_{6,6} = 1$. Then

$$\begin{aligned}
g_6(m) &= \sum_{k=1}^{m}(\frac{m-k+k\alpha}{m\alpha}A_1-1)\cdots(\frac{m-k+k\alpha}{m\alpha}A_6-1)-\sum_{k=1}^{m}(\frac{m-k+k\alpha}{m\alpha}A_6-1)^6 \\
&\quad+\sum_{k=1}^{m}\frac{m-k+k\alpha}{m\alpha}A_6(\frac{m-k+k\alpha}{m\alpha}A_6-1)\cdots(\frac{m-k+k\alpha}{m\alpha}A_6-5) \\
&= \sum_{k=1}^{m}[(\frac{m-k+k\alpha}{m\alpha})^6B_{6,0}-(\frac{m-k+k\alpha}{m\alpha})^5B_{6,1}+(\frac{m-k+k\alpha}{m\alpha})^4B_{6,2} \\
&\quad-(\frac{m-k+k\alpha}{m\alpha})^3B_{6,3}+(\frac{m-k+k\alpha}{m\alpha})^2B_{6,4}-(\frac{m-k+k\alpha}{m\alpha})B_{6,5}+B_{6,6}] \\
&\quad+\sum_{k=1}^{m}[-9(\frac{m-k+k\alpha}{m\alpha})^5A_6^5+70(\frac{m-k+k\alpha}{m\alpha})^4A_6^4-205(\frac{m-k+k\alpha}{m\alpha})^3A_6^3 \\
&\quad+259(\frac{m-k+k\alpha}{m\alpha})^2A_6^2-114(\frac{m-k+k\alpha}{m\alpha})A_6-1].
\end{aligned}$$

To make $g_6(m)$ a polynomial of m, we must transform the function to avoid the appearance of m in the sum symbol. Let

$$S_q := \sum_{k=1}^{m}(\frac{m-k+k\alpha}{m\alpha})^q, \qquad \text{for } q=1,\ldots,6.$$

We will use the first six S_q in the later computation:

$$\begin{aligned}
S_1 &= \frac{1}{m\alpha}[\frac{1}{2}m(m+1)\alpha+\frac{1}{2}m(m-1)] \\
S_2 &= (\frac{1}{m\alpha})^2[\frac{1}{6}m(m+1)(2m+1)(\alpha-1)^2+m^2(m+1)(\alpha-1)+m^3] \\
S_3 &= (\frac{1}{m\alpha})^3[\frac{1}{4}m^2(m+1)^2(\alpha-1)^3+\frac{1}{2}m^2(m+1)(2m+1)(\alpha-1)^2 \\
&\quad+\frac{3}{2}m^3(m+1)(\alpha-1)+m^4] \\
S_4 &= (\frac{1}{m\alpha})^4[\frac{1}{30}m(m+1)(2m+1)(3m^2+3m-1)(\alpha-1)^4+m^3(m+1)^2(\alpha-1)^3 \\
&\quad+m^3(m+1)(2m+1)(\alpha-1)^2+2m^4(m+1)(\alpha-1)+m^5] \\
S_5 &= (\frac{1}{m\alpha})^5[\frac{1}{12}m^2(m+1)^2(2m^2+2m-1)(\alpha-1)^5 \\
&\quad+\frac{1}{6}m^2(m+1)(2m+1)(3m^2+3m-1)(\alpha-1)^4+\frac{5}{2}m^4(m+1)^2(\alpha-1)^3 \\
&\quad+\frac{5}{3}m^4(m+1)(2m+1)(\alpha-1)^2+\frac{5}{2}m^5(m+1)(\alpha-1)+m^6] \\
S_6 &= (\frac{1}{m\alpha})^6[\frac{1}{42}m(m+1)(2m+1)(3m^4+6m^3-3m+1)\alpha^6 \\
&\quad+\frac{1}{2}m^3(m+1)^2(2m^2+2m-1)(\alpha-1)^5
\end{aligned}$$

$$+\frac{1}{2}m^3(m+1)(2m+1)(3m^2+3m-1)(\alpha-1)^4+5m^5(m+1)^2(\alpha-1)^3$$
$$+\frac{5}{2}m^5(m+1)(2m+1)(\alpha-1)^2+3m^6(m+1)(\alpha-1)+m^7].$$

In large part of this paper, we determine the positivity of the polynomial in some restricted domain by using the initial value of all partial derivatives. To make this point clear, we introduce the following lemmas:

Lemma 3.1. *Let $f(m)$ be a polynomial of m, whose degree is s. If*

(1) $\frac{\partial^s f}{\partial m^s}>0$,
(2) $\frac{\partial^k f}{\partial m^k}|_{m=m_0}>0$ *for* $k=0,\dots,s-1$.

Then $f(m)>0$ *for* $m\geq m_0$.

Proof. It is trivial. □

Lemma 3.2. *Consider α and m as parameters and let $\Delta_m(A_1,A_2,\dots,A_6,\alpha)$ be a polynomial of $A_1,\dots,A_6$. If*

(1) $\Delta_m(A_1^{(0)},\dots,A_6^{(0)},\alpha)\geq 0$,
(2) $\frac{\partial\Delta_m}{\partial A_i}\geq 0$, $\frac{\partial^2\Delta_m}{\partial A_i\partial A_6}\geq 0$ *and* $\frac{\partial^5\Delta_m}{\partial A_6^5}\geq 0$ *for all* $1\leq i\leq 5$, $A_1\geq A_1^{(0)},\dots,A_6\geq A_6^{(0)}$,
(3) $\frac{\partial^k\Delta_m}{\partial A_6^k}|_{A_1=A_1^{(0)},\dots,A_6=A_6^{(0)}}\geq 0$ *for all* $1\leq k\leq 4$.

Then $\Delta_m(A_1,A_2,\dots,A_6,\alpha)\geq 0$ *for* $A_1\geq A_1^{(0)},\dots,A_6\geq A_6^{(0)}$.

Proof. Suppose $f(A_1,\dots,A_6)$ is a polynomial of $A_1,\dots,A_6$. To prove $f\geq 0$ for $A_1\geq A_1^{(0)},\dots,A_6\geq A_6^{(0)}$, we only need to show

(1) $f(A_1^{(0)},\dots,A_6^{(0)})\geq 0$ and
(2) $\frac{\partial f}{\partial A_i}\geq 0$, for all $1\leq i\leq 6$, $A_1\geq A_1^{(0)},\dots,A_6\geq A_6^{(0)}$.

In particular, we can apply this method to show $\Delta_m\geq 0$ and $\frac{\partial^k\Delta_m}{\partial A_{i_1}\dots\partial A_{i_k}}\geq 0$, where $1\leq i_1\leq\cdots\leq i_k\leq 6$. In order to show $\frac{\partial^k\Delta_m}{\partial A_{i_1}\dots\partial A_{i_k}}\geq 0$, we only need to show

(1) $\frac{\partial^k\Delta_m}{\partial A_{i_1}\dots\partial A_{i_k}}|_{A_1=A_1^{(0)},\dots,A_6=A_6^{(0)}}\geq 0$ and
(2) $\frac{\partial}{\partial A_j}(\frac{\partial^k\Delta_m}{\partial A_{i_1}\dots\partial A_{i_k}})\geq 0$, for all $1\leq j\leq 6$, $A_1\geq A_1^{(0)},\dots,A_6\geq A_6^{(0)}$.

Notice that for $k \geq 2$, $\frac{\partial^k \Delta_m}{\partial A_6^k}$ only contains one variable A_6, i.e., $\frac{\partial^{k+1} \Delta_m}{\partial A_i \partial^k A_6} = 0$ for $1 \leq i \leq 5$. Therefore, given the three conditions in the proposition statement, by induction we can prove that $\Delta_m(A_1, A_2, \dots, A_6, \alpha) \geq 0$ for $A_1 \geq A_1^{(0)}, \dots, A_6 \geq A_6^{(0)}$. □

So we can use the initial value of all partial derivatives to determine the sign of Δ_m by applying Lemma 3.2. The following proposition gives results about the sign of some partial derivatives of Δ_m in general n-dimensional case. This proposition can save us some labor of computing.

Proposition 3.1. *Let*

$$g_n(a_1, \dots, a_n) := (a_1 - 1) \dots (a_n - 1) - (a_n - 1)^n + a_n(a_n - 1) \dots (a_n - (n - 1))$$

be the polynomial upper estimate of $P_n(a_1, \dots, a_n)$ in the Yau Number Theoretic Conjecture. And let m be the number of $(n-1)$-dimensional layers in the n-dimensional simplex, i.e., $P_{n-1}(m) > 0$ and $P_{n-1}(m+1) = 0$. Let $\alpha = 1 - \frac{m}{a_n} \in (0, 1)$, $A_i = a_i \alpha$, for $i = 1, \dots, n-1$ and

$$g_{n-1}(m) := \sum_{k=1}^{m} g_{n-1}\left(\frac{m - k + k\alpha}{m\alpha} A_1, \dots, \frac{m - k + k\alpha}{m\alpha} A_{n-1}\right)$$

$$\Delta_m(A_1, \dots, A_{n-1}, \alpha) = g_n\left(\frac{A_1}{\alpha}, \dots, \frac{A_{n-1}}{\alpha}, \frac{m}{1-\alpha}\right) - n g_{n-1}(m)$$

then

$$\frac{\partial \Delta_m}{\partial A_i} > 0$$

and

$$\frac{\partial^2 \Delta_m}{\partial A_i \partial A_{n-1}} > 0$$

for all $i = 1, \dots, n-2$, $A_1 \geq \dots \geq A_{n-1} \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (0, 1)$.

Proof. Notice that $A_1, \dots, A_{n-2}$ are symmetric in the polynomial. Therefore we only need to prove $\frac{\partial \Delta_m}{\partial A_1} > 0$ and $\frac{\partial^2 \Delta_m}{\partial A_1 \partial A_{n-1}} > 0$ for $A_1 \geq \dots \geq A_{n-1} \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (0, 1)$. Let $k' = \lfloor a_n \rfloor - k$, $\beta = a_n - \lfloor a_n \rfloor$,

$$\frac{\partial \Delta_m}{\partial A_1} = \frac{1}{\alpha}\{\prod_{i=2}^{n}(a_i - 1) - n \sum_{k'=\lfloor a_n \rfloor - m}^{\lfloor a_n \rfloor - 1} [(\frac{k' + \beta}{a_n}) \prod_{i=2}^{n-1}(\frac{a_i}{a_n}(k' + \beta) - 1)]\}$$

and

$$\frac{\partial^2 \Delta_m}{\partial A_1 \partial A_{n-1}} = \frac{1}{\alpha^2}\{\prod_{i=2}^{n-2}(a_i-1)(a_n-1) - n\sum_{k'=\lfloor a_n\rfloor-m}^{\lfloor a_n\rfloor-1}[(\frac{k'+\beta}{a_n})^2\prod_{i=2}^{n-2}(\frac{a_i}{a_n}(k'+\beta)-1)]\}.$$

Our goal is to show that

$$\prod_{i=2}^{n}(a_i-1) - n\sum_{k'=\lfloor a_n\rfloor-m}^{\lfloor a_n\rfloor-1}[(\frac{k'+\beta}{a_n})\prod_{i=2}^{n-1}(\frac{a_i}{a_n}(k'+\beta)-1)] > 0 \tag{19}$$

and

$$\prod_{i=2}^{n-2}(a_i-1)(a_n-1) - n\sum_{k'=\lfloor a_n\rfloor-m}^{\lfloor a_n\rfloor-1}[(\frac{k'+\beta}{a_n})^2\prod_{i=2}^{n-2}(\frac{a_i}{a_n}(k'+\beta)-1)] > 0. \tag{20}$$

We are going to consider two cases.
Case 1: $a_n - 1 < m < a_n$
In this case, $m = \lfloor a_n\rfloor$. Since $P_n(m) > 0$, $\frac{1}{a_1} + \cdots + \frac{1}{a_{n-1}} + \frac{m}{a_n} \le 1$ must holds. Thus $\frac{1}{a_{n-1}} + \frac{m}{a_n} \le 1$ and it is equivalent to $\frac{1}{a_{n-1}} + \frac{a_n-\beta}{a_n} \le 1$, so $\frac{a_{n-1}}{a_n}\beta \ge 1$. By Lemma 2.3,

$$a_n - 1 > n\sum_{k'=0}^{\lfloor a_n\rfloor-1}\frac{(k'+\beta)^{n-1}}{a_n^{n-1}}. \tag{21}$$

Since we have $a_1 \ge a_2 \ge \cdots \ge a_{n-2} \ge a_{n-1} \ge a_n > 1$, if we repeatedly apply Lemma 2.4 to (21), then after $n-2$ times we will have

$$\prod_{i=2}^{n}(a_i-1) - n\sum_{k'=0}^{\lfloor a_n\rfloor-1}[(\frac{k'+\beta}{a_n})\prod_{i=2}^{n-1}(\frac{a_i}{a_n}(k'+\beta)-1)] > 0.$$

Notice that we also have $a_1 \ge a_2 \ge \cdots \ge a_{n-2} \ge a_n > 1$, so if we repeatedly apply Lemma 2.4 to (21) only $n-3$ times we will have

$$\prod_{i=2}^{n-2}(a_i-1)(a_n-1) - n\sum_{k'=0}^{\lfloor a_n\rfloor-1}[(\frac{k'+\beta}{a_n})^2\prod_{i=2}^{n-2}(\frac{a_i}{a_n}(k'+\beta)-1)] > 0.$$

Notice that for $k' \ge 0$,

$$(\frac{k'+\beta}{a_n})\prod_{i=2}^{n-1}(\frac{a_i}{a_n}(k'+\beta)-1) \ge 0 \tag{22}$$

$$(\frac{k'+\beta}{a_n})^2\prod_{i=2}^{n-2}(\frac{a_i}{a_n}(k'+\beta)-1) \ge 0. \tag{23}$$

Thus we have (19) and (20).

Case 2: $m \le a_n - 1$ In this case, $\lfloor a_n \rfloor - m \ge 1$. By Lemma 2.3,

$$a_n - 1 > n \sum_{k'=0}^{\lfloor a_n \rfloor - 1} \frac{(k'+\beta)^{n-1}}{a_n^{n-1}}$$
$$> n \sum_{k'=1}^{\lfloor a_n \rfloor - 1} \frac{(k'+\beta)^{n-1}}{a_n^{n-1}}. \tag{24}$$

Since we have $a_1 \ge a_2 \ge \cdots \ge a_{n-2} \ge a_{n-1} \ge a_n > 1$, if we repeatedly apply Lemma 2.5 to (24), then after $n-2$ times we will have

$$\prod_{i=2}^{n}(a_i - 1) - n \sum_{k'=1}^{\lfloor a_n \rfloor - 1} [(\frac{k'+\beta}{a_n}) \prod_{i=2}^{n-1}(\frac{a_i}{a_n}(k'+\beta) - 1)] > 0.$$

Notice that we also have $a_1 \ge a_2 \ge \cdots \ge a_{n-2} \ge a_n > 1$, so if we repeatedly apply Lemma 2.5 to (24) only $n-3$ times we will have

$$\prod_{i=2}^{n-2}(a_i - 1)(a_n - 1) - n \sum_{k'=1}^{\lfloor a_n \rfloor - 1} [(\frac{k'+\beta}{a_n})^2 \prod_{i=2}^{n-2}((k'+\beta)\frac{a_i}{a_n} - 1)] > 0.$$

Notice that $a_i > a_n$ for $i = 1, \ldots, n-1$, thus for $k' \ge 1$, (22) and (23) holds. Thus we get (19) and (20) in this case. Therefore, $\frac{\partial \Delta_m}{\partial A_1} > 0$ and $\frac{\partial^2 \Delta_m}{\partial A_1 \partial A_{n-1}} > 0$ for $A_1 \ge \cdots \ge A_{n-1} \ge \frac{m\alpha}{1-\alpha}$, $\alpha \in (0,1)$. □

Proposition 3.2. *Let*

$$g_n(a_1, \ldots, a_n) := (a_1 - 1)\ldots(a_n - 1) - (a_n - 1)^n + a_n(a_n - 1)\ldots(a_n - (n-1))$$

be the polynomial upper estimate of $P_n(a_1, \ldots, a_n)$ in the Yau Number Theoretic Conjecture. And let m be the number of $(n-1)$-dimensional layers in the n-dimensional simplex, i.e., $P_{n-1}(m) > 0$ and $P_{n-1}(m+1) = 0$. Let $\alpha = 1 - \frac{m}{a_n} \in (0,1)$, $A_i = a_i\alpha$, for $i = 1, \ldots, n-1$ and

$$g_{n-1}(m) := \sum_{k=1}^{m} g_{n-1}(\frac{m-k+k\alpha}{m\alpha}A_1, \ldots, \frac{m-k+k\alpha}{m\alpha}A_{n-1})$$
$$\Delta_m(A_1, \ldots, A_{n-1}, \alpha) = g_n(\frac{A_1}{\alpha}, \ldots, \frac{A_{n-1}}{\alpha}, \frac{m}{1-\alpha}) - ng_{n-1}(m)$$

then

$$\frac{\partial^{n-2}\Delta_m}{\partial A_{n-1}^{n-2}} > 0$$

for all $n \ge 5$, $\alpha \in (0,1)$, $m \in \mathbb{Z}^+$.

Proof. In fact, for $n \geq 5$,

$$\frac{\partial^{n-2} g_n(\frac{A_1}{\alpha}, \ldots, \frac{A_{n-1}}{\alpha}, \frac{m}{1-\alpha})}{\partial A_{n-1}^{n-2}} = 0$$

hence,

$$\frac{\partial^{n-2}\Delta_m}{\partial A_{n-1}^{n-2}} = -n\frac{\partial^{n-2} g_{n-1}(m)}{\partial A_{n-1}^{n-2}}$$

$$= n\frac{\partial^{n-2}}{\partial A_{n-1}^{n-2}}[\sum_{k=1}^{m}(\frac{m-k+k\alpha}{m\alpha}A_{n-1}-1)^{n-1}$$

$$-\sum_{k=1}^{m}\frac{m-k+k\alpha}{m\alpha}A_{n-1}(\frac{m-k+k\alpha}{m\alpha}A_{n-1}-1)\ldots(\frac{m-k+k\alpha}{m\alpha}A_{n-1}-(n-2))]$$

$$= n\frac{\partial^{n-2}}{\partial A_{n-1}^{n-2}}\sum_{k=1}^{m}[-(n-1)(\frac{m-k+k\alpha}{m\alpha}A_{n-1})^{n-2} + \frac{(n-1)(n-2)}{2}(\frac{m-k+k\alpha}{m\alpha}A_{n-1})^{n-2}$$

$$+ \text{lower degree terms of } A_{n-1}]$$

$$= \sum_{k=1}^{m}\frac{(n-4)\cdot n!}{2}(\frac{m-k+k\alpha}{m\alpha})^{n-2} > 0$$

for $m \in \mathbb{Z}^+$, $\alpha \in (0,1)$. □

The proof of the main theorem is divided into 7 cases:

Case 1: $a_7 \in (m, m+1]$;
Case 2: $a_7 \in (m+1, m+2]$;
Case 3: $a_7 \in (m+2, m+3]$;
Case 4: $a_7 \in (m+3, m+4]$;
Case 5: $a_7 \in (m+4, m+5]$;
Case 6: $a_7 \in (m+5, m+6)$;
Case 7: $a_7 \geq m+6$.

Remark. For cases 1 to 6, the equality in (7) cannot be attained by any chance, because in these cases Δ_m is positive. On the other hand, $a_1 = \cdots = a_7$ cannot hold in these cases, it can only hold in case 7.

The proof will begin with case (1) and solve the rest of cases in numerical order. For conciseness, we shall give the detailed proof of cases (1), (6) and (7). Since the proofs of cases (2)–(5) are akin to that of cases (1) and (6), we shall only list the subcases in each cases.

3.1. Case 1: $a_7 \in (m, m+1]$

For $a_7 \in (m, m+1]$, $\alpha \in (0, \frac{1}{m+1}]$. Since $x_1 = \cdots = x_6 = 1$, $x_7 = m$ is a solution of the inequality, we know that

$$\frac{1}{A_1} + \cdots + \frac{1}{A_6} \leq 1 \tag{25}$$

and $A_1 \geq A_2 \geq \cdots \geq A_6$. So we just need to show that $\Delta_m \geq 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq 2, A_6 \geq 1$. Notice that in this case, $\frac{m\alpha}{1-\alpha} \in (0, 1]$, so by Proposition 3.1, $\frac{\partial \Delta_m}{\partial A_i} > 0$ and $\frac{\partial^2 \Delta_m}{\partial A_i \partial A_6} > 0$ for all $i = 1, \ldots, 5$, $A_1 \geq \cdots \geq A_{n-1} \geq 1$, $\alpha \in (0, \frac{1}{m+1}]$.

By Proposition 3.2, $\frac{\partial^5 \Delta_m}{\partial A_6^5} > 0$, for $\alpha \in (0, 1)$, $m \geq 2$, m integer.

(i) $\frac{\partial^4 \Delta_m}{\partial A_6^4}|_{A_6=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer.

$$\begin{aligned}
&\frac{\partial^4 \Delta_m}{\partial A_6^4}|_{A_6=1} \\
&= \frac{1}{\alpha^5 m^3}[-14(m+1)(78m^3 + 72m^2 - 17m + 17)\alpha^5 \\
&\quad - 14(m-1)(m+1)(78m^2 + 23)\alpha^4 \\
&\quad - 1092(m-1)(m+1)(m^2+1)\alpha^3 - 28(m-1)(m+1)(39m^2 - 101)\alpha^2 \\
&\quad - 14(m-1)(78m^3 - 342m^2 + 163m + 163)\alpha + 630(m-1)^2(2m^2 - 2m - 1)].
\end{aligned}$$

For $m \geq 4$, the coefficients of $\alpha, \ldots, \alpha^5$ are less than 0, and

$$\begin{aligned}
&\frac{\partial^4 \Delta_m}{\partial A_6^4}|_{A_6=1, \alpha=\frac{1}{m+1}} \\
&= 14\frac{(1+m)^5}{m^3}(90m^4 + 12m^3 - 207m^2 - 73m + 28)\frac{m^4}{(1+m)^4} \\
&= 14(1+m)m(90m^4 + 12m^3 - 207m^2 - 73m + 28) \\
&> 0.
\end{aligned}$$

Thus $\frac{\partial^4 \Delta_m}{\partial A_6^4}|_{A_6=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 4$.

For $m = 2$,

$$\begin{aligned}
&\frac{\partial^4 \Delta_2}{\partial A_6^4}|_{A_6=1} \\
&= -\frac{105}{4\alpha^5}(179\alpha^5 + 67\alpha^4 + 78\alpha^3 + 22\alpha^2 - 17\alpha - 9) \\
&> 0 \qquad\qquad \text{for } \alpha \in (0, \frac{1}{3}].
\end{aligned}$$

For $m = 3$,

$$\begin{aligned}&\frac{\partial^4 \Delta_2}{\partial A_6^4}|_{A_6=1}\\&= -\frac{280}{27\alpha^5}(544\alpha^5 + 290\alpha^4 + 312\alpha^3 + 200\alpha^2 - 32\alpha - 99)\\&> 0 \qquad\qquad \text{for } \alpha \in (0, \frac{1}{4}].\end{aligned}$$

These two ">"s can be proved by Lemma 2.1, you may need to replace α with, for example, $\beta = (m+1)\alpha$, $\beta \in (0, 1]$, for both $m = 2$ and $m = 3$ cases. Thus $\frac{\partial^4 \Delta_m}{\partial A_6^4}|_{A_6=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer.

(ii) $\frac{\partial^3 \Delta_m}{\partial A_6^3}|_{A_6=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer. Let $\beta = (1 + m)\alpha \in (0, 1]$.

$$\begin{aligned}&\frac{\partial^3 \Delta_m}{\partial A_6^3}|_{A_6=1}\\&= \frac{1+m}{\beta^5 m^3}[\frac{7}{2}(123m^3 - 33m^2 - 22m + 22)\beta^5 + \frac{7}{2}(m-1)(m+1)(123m^2 + 178)\beta^4\\&+ \frac{861}{2}(m-1)(m+1)^2(m-2)(m+2)\beta^3\\&+ \frac{7}{2}(m-1)(m+1)^2(123m^3 - 1107m^2 + 628m + 628)\beta^2\\&- 7(m-1)(m+1)^3(246m^3 - 594m^2 + 191m + 191)\beta\\&+ 315(m-1)^2(m+1)^4(2m^2 - 2m - 1)].\end{aligned}$$

The function Δ_m can be extended to a function of m for $m \in \mathbb{R}^+$. We still denote this extended function by Δ_m.

$$\begin{aligned}&\frac{\partial^8}{\partial m^8}(\frac{\beta^5 m^3}{1+m}\frac{\partial^3 \Delta_m}{\partial A_6^3}|_{A_6=1})|_{m=2}\\&= 25401600 > 0 \qquad\qquad \text{for } \beta \in (0, 1]\\&\frac{\partial^7}{\partial m^7}(\frac{\beta^5 m^3}{1+m}\frac{\partial^3 \Delta_m}{\partial A_6^3}|_{A_6=1})|_{m=2}\\&= -211680(41\beta - 255) > 0 \qquad\qquad \text{for } \beta \in (0, 1]\\&\frac{\partial^6}{\partial m^6}(\frac{\beta^5 m^3}{1+m}\frac{\partial^3 \Delta_m}{\partial A_6^3}|_{A_6=1})|_{m=2}\\&= 7560(41\beta^2 - 2228\beta + 7350) > 0 \qquad\qquad \text{for } \beta \in (0, 1]\\&\frac{\partial^5}{\partial m^5}(\frac{\beta^5 m^3}{1+m}\frac{\partial^3 \Delta_m}{\partial A_6^3}|_{A_6=1})|_{m=2}\\&= 420(123\beta^3 + 492\beta^2 - 36886\beta + 87480) > 0 \qquad\qquad \text{for } \beta \in (0, 1]\end{aligned}$$

$$\frac{\partial^4}{\partial m^4}(\frac{\beta^5 m^3}{1+m}\frac{\partial^3\Delta_m}{\partial A_6^3}|_{A_6=1})|_{m=2}$$
$$= 84(123\beta^4 + 1353\beta^3 - 3062\beta^2 - 105742\beta + 207630) > 0 \qquad \text{for } \beta \in (0,1]$$
$$\frac{\partial^3}{\partial m^3}(\frac{\beta^5 m^3}{1+m}\frac{\partial^3\Delta_m}{\partial A_6^3}|_{A_6=1})|_{m=2}$$
$$= 21(123\beta^5 + 984\beta^4 + 5289\beta^3 - 22256\beta^2 - 167064\beta + 299700) > 0 \qquad \text{for } \beta \in (0,1]$$
$$\frac{\partial^2}{\partial m^2}(\frac{\beta^5 m^3}{1+m}\frac{\partial^3\Delta_m}{\partial A_6^3}|_{A_6=1})|_{m=2}$$
$$= 7(705\beta^5 + 3007\beta^4 + 8487\beta^3 - 49101\beta^2 - 142344\beta + 255150) > 0 \qquad \text{for } \beta \in (0,1]$$
$$\frac{\partial}{\partial m}(\frac{\beta^5 m^3}{1+m}\frac{\partial^3\Delta_m}{\partial A_6^3}|_{A_6=1})|_{m=2}$$
$$= 7(661\beta^5 + 2078\beta^4 + 2214\beta^3 - 22158\beta^2 - 29619\beta + 58320) > 0 \qquad \text{for } \beta \in (0,1]$$
$$(\frac{\beta^5 m^3}{1+m}\frac{\partial^3\Delta_m}{\partial A_6^3}|_{A_6=1})|_{m=2}$$
$$= 35(83\beta^5 + 201\beta^4 - 1404\beta^2 - 891\beta + 2187) > 0 \qquad \text{for } \beta \in (0,1].$$

The ">"s can be proved by Lemma 2.1 or Lemma 2.2. Thus by Lemma 3.1, $\frac{\partial^3\Delta_m}{\partial A_6^3}|_{A_6=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer.

(iii) $\frac{\partial^2\Delta_m}{\partial A_6^2}|_{A_6=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer. Let $\beta = (1+m)\alpha \in (0,1]$.

$$\frac{\partial^2\Delta_m}{\partial A_6^2}|_{A_6=1}$$
$$= \frac{1+m}{\beta^5 m^3}[\frac{7}{6}(19m^3 - 175m^2 + 78m - 78)\beta^5 - \frac{7}{6}(m-1)(m+1)(19m^2 - 402)\beta^4$$
$$- \frac{7}{6}(m-1)(m+1)(19m^3 - 1535m^2 + 828m + 828)\beta^3$$
$$+ \frac{7}{2}(m-1)(m-2)(m+1)^2(339m^2 - 213m - 142)\beta^2$$
$$- 7(m-1)(m+1)^3(138m^3 - 282m^2 + 73m + 73)\beta$$
$$+ 105(m-1)^2(m+1)^4(2m^2 - 2m - 1)]$$

Similarly as case 1(ii), we can show that

$$\frac{\partial^i}{\partial m^i}(\frac{\beta^5 m^3}{1+m}\frac{\partial^2\Delta_m}{\partial A_6^2}|_{A_6=1})|_{m=2} > 0 \qquad \text{for } 1 \leq i \leq 8,$$
$$(\frac{\beta^5 m^3}{1+m}\frac{\partial^2\Delta_m}{\partial A_6^2}|_{A_6=1})|_{m=2}$$
$$= \frac{7}{3}(235\beta^5 + 489\beta^4 + 5256\beta^2 - 15795\beta + 10935) > 0 \qquad \text{for } \beta \in (0,1].$$

Thus by Lemma 3.1, $\frac{\partial^2\Delta_m}{\partial A_6^2}|_{A_6=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer.

(iv) $\frac{\partial\Delta_m}{\partial A_6}|_{A_1=6,A_2=5,A_3=4,A_4=3,A_5=2,A_6=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer. Let $\beta = (1+m)\alpha \in (0,1]$.

$$\begin{aligned}
&\frac{\partial\Delta_m}{\partial A_6}|_{A_1=6,A_2=5,A_3=4,A_4=3,A_5=2,A_6=1} \\
&= \frac{1+m}{\beta^6 m^5(1+m-\beta)}[\frac{1}{12}(1906m^5 + 4142m^4 + 3425m^3 - 3425m^2 - 1440m + 1440)\beta^7 \\
&\quad - \frac{1}{6}(615m^5 + 11333m^4 - 10934m^2 + 5040)\beta^6 \\
&\quad + \frac{1}{12}(m+1)(4818m^6 - 27174m^5 + 388081m^4 - 46445m^2 + 30240)\beta^5 \\
&\quad - \frac{1}{3}(m+1)^2(3706m^6 - 13209m^5 + 12588m^4 - 18620m^2 + 12600)\beta^4 \\
&\quad + \frac{5}{12}(m+1)^3(2862m^6 - 11928m^5 + 20489m^4 - 20111m^2 + 10080)\beta^3 \\
&\quad - \frac{1}{6}(m+1)^4(3744m^6 - 31185m^5 + 65275m^5 - 46690m^2 + 15120)\beta^2 \\
&\quad + \frac{3}{4}(m+1)^5(302m^6 - 6090m^5 + 10955m^4 - 5327m^2 + 1120)\beta \\
&\quad + 120(m-1)(m+1)^6(15m^4 - 6m^3 - 6m^2 + m + 1)]
\end{aligned}$$

Similarly as case 1(ii), we can show that

$$\begin{aligned}
&\frac{\partial^i}{\partial m^i}(\frac{\beta^6 m^5(1+m-\beta)}{1+m}\frac{\partial\Delta_m}{\partial A_6}|_{A_1=6,A_2=5,A_3=4,A_4=3,A_5=2,A_6=1}) > 0 \text{ for } 1 \leq i \leq 11, \\
&(\frac{\beta^6 m^5(1+m-\beta)}{1+m}\frac{\partial\Delta_m}{\partial A_6}|_{A_1=6,A_2=5,A_3=4,A_4=3,A_5=2,A_6=1})|_{m=2} \\
&= 11627\beta^7 - 27052\beta^6 - 23985\beta^5 - 25272\beta^4 + 662985\beta^3 - 1545156\beta^2 \\
&\quad - 3728835\beta + 14959080 > 0 \qquad \text{for } \beta \in (0,1]
\end{aligned}$$

Thus by Lemma 3.1, $\frac{\partial\Delta_m}{\partial A_6}|_{A_1=6,A_2=5,A_3=4,A_4=3,A_5=2,A_6=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer. By Lemma 3.2, we conclude that

Proposition 3.3. $\frac{\partial\Delta_m}{\partial A_6} > 0$, *for* $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq 2, A_6 \geq 1$, $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, *m integer.*

(v) $\Delta_m > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq 2, A_6 \geq 1$, $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer. Let $\beta = (1+m)\alpha \in (0,1]$.

$$
\begin{aligned}
&\Delta_m|_{A_1=6,A_2=5,A_3=4,A_4=3,A_5=2,A_6=1}\\
=&\frac{1+m}{\beta^6 m^5(1+m-\beta)^6}[\frac{1}{60}(m-1)(4904m^4-4861m^2+7200)\beta^{12}\\
&-\frac{1}{12}(13472m^6+5418m^5-19747m^4+26453m^2-17280)\beta^{11}\\
&-\frac{1}{12}(m+1)(20916m^7-47230m^6-48636m^5+103285m^4-141939m^2+95040)\beta^{10}\\
&-\frac{1}{12}(m+1)^2(19068m^8-83664m^7+58346m^6+208698m^5-381563m^4\\
&+495159m^2-316800)\beta^9\\
&-\frac{1}{6}(m+1)^3(4200m^9-28602m^8+62748m^7+12388m^6-292950m^5+544817m^4\\
&-619941m^2+356400)\beta^8\\
&-\frac{1}{30}(m+1)^4(4620m^10-42000m^9+143010m^8-209160m^7-436126m^6\\
&+3067470m^5-6013175m^4+5767461m^2-2851200)\beta^7\\
&-\frac{1}{6}(m+1)^5(84m^11-924m^10+4200m^9-9534m^8+10458m^7+138376m^6\\
&-1001958m^5+2003561m^4-1605387m^2+665280)\beta^6\\
&+\frac{1}{6}(m+1)^6(136430m^6-1262178m^5+2463055m^4-1659567m^2+570240)\beta^5\\
&-\frac{1}{12}(m+1)^7(190904m^6-2364948m^5+4370653m^4-2505069m^2+712800)\beta^4\\
&+\frac{1}{12}(m+1)^8(92316m^6-1572090m^5+2713319m^4-1338617m^2+316800)\beta^3\\
&-\frac{1}{60}(m+1)^9(136206m^6-3477240m^5+5584285m^4-2396611m^2+475200)\beta^2\\
&+\frac{3}{4}(m+1)^{10}(406m^6-20322m^5+30415m^4-11459m^2+1920)\beta\\
&+120(m-1)(m+1)^{11}(15m^4-6m^3-6m^2+m+1)]
\end{aligned}
$$

Similarly as case 1(ii), we can show that

$$
\frac{\partial^i}{\partial m^i}(\frac{\beta^6 m^5(1+m-\beta)^6}{1+m}\Delta_m|_{A_1=6,A_2=5,A_3=4,A_4=3,A_5=2,A_6=1})>0 \qquad \text{for } 1\le i\le 16,
$$

$$
\begin{aligned}
&(\frac{\beta^6 m^5(1+m-\beta)^6}{1+m}\Delta_m|_{A_1=6,A_2=5,A_3=4,A_4=3,A_5=2,A_6=1})|_{m=2}\\
=&\frac{1}{3}(\beta+3)(3311\beta^{11}-211974\beta^{10}+1177407\beta^9-3855600\beta^8-207522\beta^7+103332348\beta^6\\
&-719244522\beta^5+2771068968\beta^4-6761799405\beta^3+10405811610\beta^2-9253982133\beta\\
&+3635056440)>0 \qquad \text{for } \beta\in(0,1]
\end{aligned}
$$

Thus $\Delta_m|_{A_1=6,A_2=5,A_3=4,A_4=3,A_5=2,A_6=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer. By Lemma 3.2, we know that $\Delta_m > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq 2, A_6 \geq 1$, $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer.

(vi) $\Delta_m > 0$, for $m = 1$, $1 < a_7 \leq 2$, $\alpha = 1 - \frac{1}{a_7} \in (0, \frac{1}{2}]$. By Proposition 3.1, $\frac{\partial \Delta_1}{\partial A_i} > 0$, for $1 \leq i \leq 5$, $A_1 \geq 1, \ldots, A_6 \geq 1$, $\alpha \in (0, 1)$.

And

$$\frac{\partial^2 \Delta_1}{\partial A_6^2} = 1260A_6^3 - 5880A_6^2 + 8610A_6 - 3626 > 0 \qquad \text{for } A_6 \geq 1.$$

$$\begin{aligned}&\frac{\partial \Delta_1}{\partial A_6}|_{A_1=6,A_2=5,A_3=4,A_4=3,A_5=2,A_6=1}\\ &= \frac{1}{(1-\alpha)\alpha^6}(1008\alpha^7 - 1009\alpha^6 + 19\alpha^5 - 135\alpha^4 + 425\alpha^3 - 464\alpha^2 - 324\alpha + 720)\\ &> 0 \qquad \text{for } \alpha \in (0, 1)\end{aligned}$$

Thus, we conclude that

Proposition 3.4. $\frac{\partial \Delta_1}{\partial A_6} > 0$, *for* $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq 2, A_6 \geq 1$, $\alpha \in (0, 1)$.

And since

$$\begin{aligned}&\Delta_1|_{A_1=6,A_2=5,A_3=4,A_4=3,A_5=2,A_6=1}\\ &= -\frac{1}{(1-\alpha)^6\alpha^5}(693\alpha^{10} - 1547\alpha^9 - 63\alpha^8 + 6510\alpha^7 - 18970\alpha^6 + 34026\alpha^5\\ &\quad - 41330\alpha^4 + 33695\alpha^3 - 17644\alpha^2 + 5364\alpha - 720)\\ &> 0 \qquad \text{for } \alpha \in (0, \frac{1}{2}],\end{aligned}$$

thus by Lemma 3.2, $\Delta_1 > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq 2, A_6 \geq 1$, $\alpha \in (0, \frac{1}{2}]$.

Therefore $\Delta_m > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq 2, A_6 \geq 1$, $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 1$, m integer.

3.2. Case 2: $a_7 \in (m+1, m+2]$

In this case, $\frac{m\alpha}{1-\alpha} \in (1, 2]$, so $A_1 \geq A_2 \geq \cdots \geq A_6 \geq \frac{m\alpha}{1-\alpha}$. By Proposition 3.1, $\frac{\partial \Delta_m}{\partial A_i} > 0$, $\frac{\partial^2 \Delta_m}{\partial A_j \partial A_6} > 0$, for $1 \leq i, j \leq 5$, $A_1 \geq A_2 \geq \cdots \geq A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (0, 1)$, $m \geq 1$, m integer.

By Proposition 3.2, $\frac{\partial^5 \Delta_m}{\partial A_6^5} > 0$, for $\alpha \in (0, 1)$, $m \geq 2$, m integer.

The same argument as case 1, we can show that

(i) $\frac{\partial^4 \Delta_m}{\partial A_6^4}|_{A_6=\frac{m\alpha}{1-\alpha}} > 0$, for $\alpha \in (\frac{1}{m+1}, 1]$, $m \geq 2$, m integer,
(ii) $\frac{\partial^3 \Delta_m}{\partial A_6^3}|_{A_6=\frac{m\alpha}{1-\alpha}} > 0$, for $\alpha \in (\frac{1}{m+1}, 1]$, $m \geq 2$, m integer,
(iii) $\frac{\partial^2 \Delta_m}{\partial A_6^2}|_{A_6=\frac{m\alpha}{1-\alpha}} > 0$, for $\alpha \in (\frac{1}{m+1}, 1]$, $m \geq 2$, m integer,
(iv) $\frac{\partial \Delta_m}{\partial A_6}|_{A_1=6,A_2=5,A_3=4,A_4=3,A_5=2,A_6=\frac{m\alpha}{1-\alpha}} > 0$, for $\alpha \in (\frac{1}{m+1}, 1]$, $m \geq 2$, m integer.

By the above observation, we conclude that

Proposition 3.5. *In case $n = 7$, $\frac{\partial \Delta_m}{\partial A_6} > 0$ for $A_1 \geq 6, A_2 \geq 5, \ldots, A_5 \geq 2, A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{1}{m+1}, 1]$, $m \geq 2$, m integer.*

The same argument as case 6, we can show that

(v) $\Delta_m > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq 2, A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{1}{m+1}, \frac{2}{m+2}]$, $m \geq 2$, m integer.
Thus by Lemma 3.1, $\Delta_m|_{A_1=6,A_2=5,A_3=4,A_4=3,A_5=2,A_6=\frac{m\alpha}{1-\alpha}} > 0$, for $\alpha \in (\frac{1}{m+1}, \frac{2}{m+2}]$, $m \geq 2$, m integer. By Lemma 3.2, we know that $\Delta_m > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq 2, A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{1}{m+1}, \frac{2}{m+2}]$, $m \geq 2$, m integer.
(vi) $\Delta_m > 0$ for $m = 1$, $a_7 \in (2, 3]$, $\alpha = 1 - \frac{1}{a_7} \in (\frac{1}{2}, \frac{2}{3}]$. By Proposition 3.1, $\frac{\partial \Delta_1}{\partial A_i} > 0$, for $1 \leq i \leq 5$, $A_1 \geq 1, \ldots, A_6 \geq 1$, $\alpha \in (0, 1)$.
By Proposition 3.4, $\frac{\partial \Delta_1}{\partial A_6} > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq 2, A_6 \geq 1$, $\alpha \in (0, 1)$. Since $\frac{\alpha}{1-\alpha} \in (1, 2]$ here, $\frac{\partial \Delta_1}{\partial A_6} > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq 2, A_6 \geq \frac{\alpha}{1-\alpha}$, $\alpha \in (0, 1)$. And we have

$$\begin{aligned}
&\Delta_1|_{A_1=6,A_2=5,A_3=4,A_4=3,A_5=2,A_6=\frac{\alpha}{1-\alpha}} \\
&= -\frac{1}{(1-\alpha)^6\alpha^3}(2912\alpha^9 - 6613\alpha^8 + 952\alpha^7 + 9116\alpha^6 - 7481\alpha^5 - 3772\alpha^4 \\
&\quad + 10772\alpha^3 - 9076\alpha^2 + 3924\alpha - 720) > 0 \qquad \text{for } \alpha \in (\frac{1}{2}, \frac{2}{3}]
\end{aligned}$$

Thus, $\Delta_1 > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq 2, A_6 \geq \frac{\alpha}{1-\alpha}$, $\alpha \in (\frac{1}{2}, \frac{2}{3}]$.

Therefore, $\Delta_m > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq 2, A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{1}{m+1}, \frac{2}{m+2}]$, $m \geq 1$, m integer.

3.3. Case 3: $a_7 \in (m + 2, m + 3]$

For $a_7 \in (m+2, m+3]$, $\alpha \in (\frac{2}{m+2}, \frac{3}{m+3}]$. $A_1 \geq 6, A_2 \geq 5, \ldots, A_4 \geq 3, A_5 \geq A_6 \geq \frac{m\alpha}{1-\alpha}$.
In this case, $\frac{m\alpha}{1-\alpha} \in (2, 3]$, so $A_1 \geq A_2 \geq \cdots \geq A_6 \geq \frac{m\alpha}{1-\alpha}$. By Proposition 3.1, $\frac{\partial \Delta_m}{\partial A_i} > 0$, $\frac{\partial^2 \Delta_m}{\partial A_j \partial A_6} > 0$, for $1 \leq i, j \leq 5$, $A_1 \geq A_2 \geq \cdots \geq A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (0, 1)$, $m \geq 1$, m integer.

By Proposition 3.5, we know that $\frac{\partial \Delta_m}{\partial A_6} > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3$, $A_5 \geq 2, A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{1}{m+1}, 1]$, $m \geq 2$, m integer. Since $\frac{m\alpha}{1-\alpha} \in (2, 3]$ here, $\frac{\partial \Delta_m}{\partial A_6} > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{2}{m+2}, \frac{3}{m+3}]$, $m \geq 2$, m integer. The same argument as case 6, we can show that

(i) $\Delta_m > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{2}{m+2}, \frac{3}{m+3}]$, $m \geq 2$, m integer.
Thus $\Delta_m|_{A_1=6,A_2=5,A_3=4,A_4=3,A_5=\frac{m\alpha}{1-\alpha},A_6=\frac{m\alpha}{1-\alpha}} > 0$, for $\alpha \in (\frac{2}{m+2}, \frac{3}{m+3}]$, $m \geq 2$, m integer. By Lemma 3.2, we know that $\Delta_m > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{2}{m+2}, \frac{3}{m+3}]$, $m \geq 2$, m integer.

(ii) $\Delta_m > 0$, for $m = 1$, $a_7 \in (3, 4]$, $\alpha = 1 - \frac{1}{a_7} \in (\frac{2}{3}, \frac{3}{4}]$. By Proposition 3.1, $\frac{\partial \Delta_1}{\partial A_i} > 0$, for $1 \leq i \leq 5$, $A_1 \geq 1, \ldots, A_6 \geq 1$, $\alpha \in (0, 1)$.
By Proposition 3.4, $\frac{\partial \Delta_1}{\partial A_6} > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq 2, A_6 \geq 1$, $\alpha \in (0, 1)$. Since $\frac{\alpha}{1-\alpha} \in (2, 3]$ here, $\frac{\partial \Delta_1}{\partial A_6} > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq \frac{\alpha}{1-\alpha}, A_6 \geq \frac{\alpha}{1-\alpha}$, $\alpha \in (0, 1)$. And we have

$$\begin{aligned}
&\Delta_1|_{A_1=6,A_2=5,A_3=4,A_4=3,A_5=A_6=\frac{\alpha}{1-\alpha}} \\
&= -\frac{1}{(1-\alpha)^6\alpha}(7952\alpha^7 - 32650\alpha^6 + 56327\alpha^5 - 52474\alpha^4 + 28545\alpha^3 - 9581\alpha^2 \\
&\quad + 2255\alpha - 360) \\
&> 0 \qquad \text{for } \alpha \in (\frac{2}{3}, \frac{3}{4}]
\end{aligned}$$

Thus, $\Delta_1 > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq \frac{\alpha}{1-\alpha}, A_6 \geq \frac{\alpha}{1-\alpha}$, $\alpha \in (\frac{2}{3}, \frac{3}{4}]$.

Therefore, $\Delta_m > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{2}{m+2}, \frac{3}{m+3}]$, $m \geq 1$, m integer.

3.4. Case 4: $a_7 \in (m+3, m+4]$

For $a_7 \in (m+3, m+4]$, $\alpha \in (\frac{3}{m+3}, \frac{4}{m+4}]$. $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq A_5 \geq A_6 \geq \frac{m\alpha}{1-\alpha}$.

In this case, $\frac{m\alpha}{1-\alpha} \in (3, 4]$, so $A_1 \geq A_2 \geq \cdots \geq A_6 \geq \frac{m\alpha}{1-\alpha}$. By Proposition 3.1, $\frac{\partial \Delta_m}{\partial A_i} > 0$, $\frac{\partial^2 \Delta_m}{\partial A_j \partial A_6} > 0$, for $1 \leq i, j \leq 5$, $A_1 \geq A_2 \geq \cdots \geq A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (0, 1)$, $m \geq 1$, m integer.

By Proposition 3.5, we know that $\frac{\partial \Delta_m}{\partial A_6} > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3$, $A_5 \geq 2, A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{1}{m+1}, 1]$, $m \geq 2$, m integer. Since $\frac{m\alpha}{1-\alpha} \in (3, 4]$ here, $\frac{\partial \Delta_m}{\partial A_6} > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq A_5 \geq A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{3}{m+3}, \frac{4}{m+4}]$, $m \geq 2$, m integer. The same argument as case 6, we can show that

(i) $\Delta_m > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq \frac{m\alpha}{1-\alpha}, A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{3}{m+3}, \frac{4}{m+4}]$, $m \geq 2$, m integer.

Thus by Lemma 3.1, $\Delta_m|_{A_1=6,A_2=5,A_3=4,A_4=A_5=A_6=\frac{m\alpha}{1-\alpha}} > 0$, for $\alpha \in (\frac{3}{m+3}, \frac{4}{m+4}]$, $m \geq 2$, m integer. By Lemma 3.2, we know that $\Delta_m > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq \frac{m\alpha}{1-\alpha}, A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{3}{m+3}, \frac{4}{m+4}]$, $m \geq 2$, m integer.

(ii) $\Delta_m > 0$, for $m = 1$ $a_7 \in (4, 5]$, $\alpha = 1 - \frac{1}{a_7} \in (\frac{3}{4}, \frac{4}{5}]$. By Proposition 3.1, $\frac{\partial \Delta_1}{\partial A_i} > 0$, for $1 \leq i \leq 5$, $A_1 \geq 1, \ldots, A_6 \geq 1$, $\alpha \in (0, 1)$.
By Proposition 3.4, $\frac{\partial \Delta_1}{\partial A_6} > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq 2, A_6 \geq 1$, $\alpha \in (0, 1)$. Since $\frac{\alpha}{1-\alpha} \in (3, 4]$ here, $\frac{\partial \Delta_1}{\partial A_6} > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq \frac{\alpha}{1-\alpha}, A_5 \geq \frac{\alpha}{1-\alpha}, A_6 \geq \frac{\alpha}{1-\alpha}$, $\alpha \in (0, 1)$. And we have

$$\begin{aligned}
&\Delta_1|_{A_1=6,A_2=5,A_3=4,A_4=A_5=A_6=\frac{\alpha}{1-\alpha}} \\
&= -\frac{1}{(1-\alpha)^6}(1232\alpha^6 - 726\alpha^5 - 5904\alpha^4 + 11417\alpha^3 - 8602\alpha^2 + 3024\alpha - 427) \\
&> 0 \qquad \text{for } \alpha \in (\frac{3}{4}, \frac{4}{5}]
\end{aligned}$$

Thus, $\Delta_1 > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq \frac{\alpha}{1-\alpha}, A_5 \geq \frac{\alpha}{1-\alpha}, A_6 \geq \frac{\alpha}{1-\alpha}$, $\alpha \in (\frac{3}{4}, \frac{4}{5}]$.

Therefore, $\Delta_m > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq \frac{m\alpha}{1-\alpha}, A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{3}{m+3}, \frac{4}{m+4}]$, $m \geq 1$, m integer.

3.5. Case 5: $a_7 \in (m + 4, m + 5]$

For $a_7 \in (m+4, m+5]$, $\alpha \in (\frac{4}{m+4}, \frac{5}{m+5}]$. $A_1 \geq 6, A_2 \geq 5, A_3 \geq A_4 \geq A_5 \geq A_6 \geq \frac{m\alpha}{1-\alpha}$.

In this case, $\frac{m\alpha}{1-\alpha} \in (4, 5]$, so $A_1 \geq A_2 \geq \cdots \geq A_6 \geq \frac{m\alpha}{1-\alpha}$. By Proposition 3.1, $\frac{\partial \Delta_m}{\partial A_i} > 0$, $\frac{\partial^2 \Delta_m}{\partial A_j \partial A_6} > 0$, for $1 \leq i, j \leq 5$, $A_1 \geq A_2 \geq \cdots \geq A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (0, 1)$, $m \geq 1$, m integer.

By Proposition 3.5, we know that $\frac{\partial \Delta_m}{\partial A_6} > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3$, $A_5 \geq 2, A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{1}{m+1}, 1]$, $m \geq 2$, m integer. Since $\frac{m\alpha}{1-\alpha} \in (4, 5]$ here, $\frac{\partial \Delta_m}{\partial A_6} > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq A_4 \geq A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{4}{m+4}, \frac{5}{m+5}]$, $m \geq 2$, m integer.

The same argument as case 6, we can show that

(i) $\Delta_m > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq \frac{m\alpha}{1-\alpha}, A_4 \geq \frac{m\alpha}{1-\alpha}, A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{4}{m+4}, \frac{5}{m+5}]$, $m \geq 2$, m integer.
Thus by Lemma 3.1, $\Delta_m|_{A_1=6,A_2=5,A_3=\frac{m\alpha}{1-\alpha},A_4=\frac{m\alpha}{1-\alpha},A_5=\frac{m\alpha}{1-\alpha},A_6=\frac{m\alpha}{1-\alpha}} > 0$, for $\alpha \in (\frac{4}{m+4}, \frac{5}{m+5}]$, $m \geq 2$, m integer. By Lemma 3.2, we know that $\Delta_m > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq \frac{m\alpha}{1-\alpha}, A_4 \geq \frac{m\alpha}{1-\alpha}, A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{4}{m+4}, \frac{5}{m+5}]$, $m \geq 2$, m integer.

(ii) $\Delta_m > 0$, for $m = 1$, $a_7 \in (5, 6]$, $\alpha = 1 - \frac{1}{a_7} \in (\frac{4}{5}, \frac{5}{6}]$. By Proposition 3.1, $\frac{\partial \Delta_1}{\partial A_i} > 0$, for $1 \leq i \leq 5$, $A_1 \geq 1, \ldots, A_6 \geq 1$, $\alpha \in (0, 1)$.

By Proposition 3.4, $\frac{\partial \Delta_1}{\partial A_6} > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq 2, A_6 \geq 1$, $\alpha \in (0,1)$. Since $\frac{\alpha}{1-\alpha} \in (4,5]$ here, $\frac{\partial \Delta_1}{\partial A_6} > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq \frac{\alpha}{1-\alpha}, A_4 \geq \frac{\alpha}{1-\alpha}, A_5 \geq \frac{\alpha}{1-\alpha}, A_6 \geq \frac{\alpha}{1-\alpha}$, $\alpha \in (0,1)$. And we have

$$\begin{aligned} &\Delta_1|_{A_1=6,A_2=5,A_3=\cdots=A_6=\frac{\alpha}{1-\alpha}} \\ &= -\frac{1}{(1-\alpha)^6}(6832\alpha^6 - 24801\alpha^5 + 36312\alpha^4 - 27110\alpha^3 + 10684\alpha^2 - 2036\alpha + 133) \\ &> 0 \qquad \text{for } \alpha \in (\frac{4}{5}, \frac{5}{6}]. \end{aligned}$$

Thus, $\Delta_1 > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq \frac{\alpha}{1-\alpha}, A_4 \geq \frac{\alpha}{1-\alpha}, A_5 \geq \frac{\alpha}{1-\alpha}, A_6 \geq \frac{\alpha}{1-\alpha}$, $\alpha \in (\frac{4}{5}, \frac{5}{6}]$.

Therefore, $\Delta_m > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq \frac{m\alpha}{1-\alpha}, A_4 \geq \frac{m\alpha}{1-\alpha}, A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{4}{m+4}, \frac{5}{m+5}]$, $m \geq 1$, m integer.

3.6. Case 6: $a_7 \in (m+5, m+6)$

We shall give the detailed proof in this case. For $a_7 \in (m+5, m+6)$, $\alpha \in (\frac{5}{m+5}, \frac{6}{m+6})$, $A_1 \geq 6, A_2 \geq \frac{m\alpha}{1-\alpha}, A_3 \geq \frac{m\alpha}{1-\alpha}, A_4 \geq \frac{m\alpha}{1-\alpha}, A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}$.

In this case, $\frac{m\alpha}{1-\alpha} \in (5,6)$, so $A_1 \geq A_2 \geq \cdots \geq A_6 \geq \frac{m\alpha}{1-\alpha}$. By Proposition 3.1, $\frac{\partial \Delta_m}{\partial A_i} > 0$, $\frac{\partial^2 \Delta_m}{\partial A_j \partial A_6} > 0$, for $1 \leq i,j \leq 5$, $A_1 \geq A_2 \geq \cdots \geq A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (0,1)$, $m \geq 1$, m integer.

By Proposition 3.5, we know that $\frac{\partial \Delta_m}{\partial A_6} > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3$, $A_5 \geq 2, A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{1}{m+1}, 1]$, $m \geq 2$, m integer. Since $\frac{m\alpha}{1-\alpha} \in (5,6)$ here, $\frac{\partial \Delta_m}{\partial A_6} > 0$, for $A_1 \geq 6, A_2 \geq A_3 \geq A_4 \geq A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{5}{m+5}, \frac{6}{m+6}]$, $m \geq 2$, m integer.

(i) $\Delta_m > 0$, for $A_1 \geq 6, A_2 \geq \frac{m\alpha}{1-\alpha}, A_3 \geq \frac{m\alpha}{1-\alpha}, A_4 \geq \frac{m\alpha}{1-\alpha}, A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{5}{m+5}, \frac{6}{m+6})$, $m \geq 2$, m integer. Let $\eta = \frac{\alpha - \frac{5}{m+5}}{\frac{6}{m+6} - \frac{5}{m+5}} \in (0,1)$.

$$\begin{aligned} &\Delta_m|_{A_1=6,A_2=\cdots=A_6=\frac{m\alpha}{1-\alpha}} \\ &= \frac{(1-\eta)(m+6)}{(m-\eta+6)^6(m\eta+5m+30)}[\frac{1}{12}m(m+1)(m+2)(68m^3+396m^2+571m-43)\eta^6 \\ &\quad + \frac{1}{6}(m+6)(720m^5+5605m^4+15257m^3+16439m^2+5317m-30)\eta^5 \\ &\quad + \frac{1}{12}(m+6)^2(12395m^4+81586m^3+179893m^2+146450m+34500)\eta^4 \\ &\quad + \frac{1}{3}(m+6)^3(14021m^3+74127m^2+122545m+62775)\eta^3 \\ &\quad + \frac{1}{12}(m+6)^4(565m^4+9362m^3+193643m^2+650710m+577800)\eta^2 \\ &\quad + \frac{1}{6}(m+6)^5(120m^5+2435m^4+19669m^3+79075m^2+244793m+289200)\eta \end{aligned}$$

$$+\frac{1}{12}(m+6)^6(40m^6+960m^5+9565m^4+50654m^3+150415m^2+237526m+245760)]$$

$$\frac{\partial^{12}}{\partial m^{12}}\left(\frac{(m-\eta+6)^6(m\eta+5m+30)}{(1-\eta)(m+6)}\Delta_m|_{A_1=6,A_2=\frac{m\alpha}{1-\alpha},A_3=\frac{m\alpha}{1-\alpha},A_4=\frac{m\alpha}{1-\alpha},A_5=\frac{m\alpha}{1-\alpha},A_6=\frac{m\alpha}{1-\alpha}}\right)$$
$$=1596672000>0 \qquad \text{for } \eta\in(0,1)$$

$$\frac{\partial^{11}}{\partial m^{11}}\left(\frac{(m-\eta+6)^6(m\eta+5m+30)}{(1-\eta)(m+6)}\Delta_m|_{A_1=6,A_2=\frac{m\alpha}{1-\alpha},A_3=\frac{m\alpha}{1-\alpha},A_4=\frac{m\alpha}{1-\alpha},A_5=\frac{m\alpha}{1-\alpha},A_6=\frac{m\alpha}{1-\alpha}}\right)|_{m=2}$$
$$=11176704000>0 \qquad \text{for } \eta\in(0,1)$$

$$\frac{\partial^{10}}{\partial m^{10}}\left(\frac{(m-\eta+6)^6(m\eta+5m+30)}{(1-\eta)(m+6)}\Delta_m|_{A_1=6,A_2=\frac{m\alpha}{1-\alpha},A_3=\frac{m\alpha}{1-\alpha},A_4=\frac{m\alpha}{1-\alpha},A_5=\frac{m\alpha}{1-\alpha},A_6=\frac{m\alpha}{1-\alpha}}\right)|_{m=2}$$
$$=1512000(48\eta+25817)>0 \qquad \text{for } \eta\in(0,1)$$

$$\frac{\partial^{9}}{\partial m^{9}}\left(\frac{(m-\eta+6)^6(m\eta+5m+30)}{(1-\eta)(m+6)}\Delta_m|_{A_1=6,A_2=\frac{m\alpha}{1-\alpha},A_3=\frac{m\alpha}{1-\alpha},A_4=\frac{m\alpha}{1-\alpha},A_5=\frac{m\alpha}{1-\alpha},A_6=\frac{m\alpha}{1-\alpha}}\right)|_{m=2}$$
$$=-423360(1205\eta+214221)>0 \qquad \text{for } \eta\in(0,1)$$

$$\frac{\partial^{8}}{\partial m^{8}}\left(\frac{(m-\eta+6)^6(m\eta+5m+30)}{(1-\eta)(m+6)}\Delta_m|_{A_1=6,A_2=\frac{m\alpha}{1-\alpha},A_3=\frac{m\alpha}{1-\alpha},A_4=\frac{m\alpha}{1-\alpha},A_5=\frac{m\alpha}{1-\alpha},A_6=\frac{m\alpha}{1-\alpha}}\right)|_{m=2}$$
$$=3360(565\eta^2+532298\eta+46930651)>0 \qquad \text{for } \eta\in(0,1)$$

$$\frac{\partial^{7}}{\partial m^{7}}\left(\frac{(m-\eta+6)^6(m\eta+5m+30)}{(1-\eta)(m+6)}\Delta_m|_{A_1=6,A_2=\frac{m\alpha}{1-\alpha},A_3=\frac{m\alpha}{1-\alpha},A_4=\frac{m\alpha}{1-\alpha},A_5=\frac{m\alpha}{1-\alpha},A_6=\frac{m\alpha}{1-\alpha}}\right)|_{m=2}$$
$$=840(15981\eta^2+4963889\eta+260535253)>0 \qquad \text{for } \eta\in(0,1)$$

$$\frac{\partial^{6}}{\partial m^{6}}\left(\frac{(m-\eta+6)^6(m\eta+5m+30)}{(1-\eta)(m+6)}\Delta_m|_{A_1=6,A_2=\frac{m\alpha}{1-\alpha},A_3=\frac{m\alpha}{1-\alpha},A_4=\frac{m\alpha}{1-\alpha},A_5=\frac{m\alpha}{1-\alpha},A_6=\frac{m\alpha}{1-\alpha}}\right)|_{m=2}$$
$$=60(68\eta^6+1440\eta^5+12395\eta^4+56084\eta^3+924559\eta^2+121371922\eta+4209174216)$$
$$>0 \qquad \text{for } \eta\in(0,1)$$

$$\frac{\partial^{5}}{\partial m^{5}}\left(\frac{(m-\eta+6)^6(m\eta+5m+30)}{(1-\eta)(m+6)}\Delta_m|_{A_1=6,A_2=\frac{m\alpha}{1-\alpha},A_3=\frac{m\alpha}{1-\alpha},A_4=\frac{m\alpha}{1-\alpha},A_5=\frac{m\alpha}{1-\alpha},A_6=\frac{m\alpha}{1-\alpha}}\right)|_{m=2}$$
$$=20(708\eta^6+18565\eta^5+189533\eta^4+989514\eta^3+8235757\eta^2+509759278\eta+12463623296)$$
$$>0 \qquad \text{for } \eta\in(0,1)$$

$$\frac{\partial^{4}}{\partial m^{4}}\left(\frac{(m-\eta+6)^6(m\eta+5m+30)}{(1-\eta)(m+6)}\Delta_m|_{A_1=6,A_2=\frac{m\alpha}{1-\alpha},A_3=\frac{m\alpha}{1-\alpha},A_4=\frac{m\alpha}{1-\alpha},A_5=\frac{m\alpha}{1-\alpha},A_6=\frac{m\alpha}{1-\alpha}}\right)|_{m=2}$$
$$=2(11975\eta^6+382674\eta^5+4652105\eta^4+28309636\eta^3+184400896\eta^2+5989254880\eta$$
$$+107446195712)>0 \qquad \text{for } \eta\in(0,1)$$

$$\frac{\partial^{3}}{\partial m^{3}}\left(\frac{(m-\eta+6)^6(m\eta+5m+30)}{(1-\eta)(m+6)}\Delta_m|_{A_1=6,A_2=\frac{m\alpha}{1-\alpha},A_3=\frac{m\alpha}{1-\alpha},A_4=\frac{m\alpha}{1-\alpha},A_5=\frac{m\alpha}{1-\alpha},A_6=\frac{m\alpha}{1-\alpha}}\right)|_{m=2}$$
$$=26251\eta^6+1011277\eta^5+14639831\eta^4+104750378\eta^3+640625536\eta^2+12228276992\eta$$
$$+164515139584>0 \qquad \text{for } \eta\in(0,1)$$

$$\frac{\partial^{2}}{\partial m^{2}}\left(\frac{(m-\eta+6)^6(m\eta+5m+30)}{(1-\eta)(m+6)}\Delta_m|_{A_1=6,A_2=\frac{m\alpha}{1-\alpha},A_3=\frac{m\alpha}{1-\alpha},A_4=\frac{m\alpha}{1-\alpha},A_5=\frac{m\alpha}{1-\alpha},A_6=\frac{m\alpha}{1-\alpha}}\right)|_{m=2}$$

$$= \frac{1}{6}(125585\eta^6 + 5783850\eta^5 + 99645980\eta^4 + 843940320\eta^3 + 5316157440\eta^2 + 66873626624\eta + 680812576768) > 0 \qquad \text{for } \eta \in (0,1)$$

$$\frac{\partial}{\partial m}\left(\frac{(m-\eta+6)^6(m\eta+5m+30)}{(1-\eta)(m+6)}\Delta_m|_{A_1=6,A_2=\frac{m\alpha}{1-\alpha},A_3=\frac{m\alpha}{1-\alpha},A_4=\frac{m\alpha}{1-\alpha},A_5=\frac{m\alpha}{1-\alpha},A_6=\frac{m\alpha}{1-\alpha}}\right)|_{m=2}$$

$$= \frac{1}{6}(77603\eta^6 + 4240072\eta^5 + 86918048\eta^4 + 876552064\eta^3 + 5989519360\eta^2 + 55514390528\eta + 429013336064) > 0 \qquad \text{for } \eta \in (0,1)$$

$$\left(\frac{(m-\eta+6)^6(m\eta+5m+30)}{(1-\eta)(m+6)}\Delta_m|_{A_1=6,A_2=\frac{m\alpha}{1-\alpha},A_3=\frac{m\alpha}{1-\alpha},A_4=\frac{m\alpha}{1-\alpha},A_5=\frac{m\alpha}{1-\alpha},A_6=\frac{m\alpha}{1-\alpha}}\right)|_{m=2}$$

$$= 14(461\eta^6 + 29632\eta^5 + 723040\eta^4 + 8734976\eta^3 + 66748416\eta^2 + 505266176\eta + 2986606592) > 0 \qquad \text{for } \eta \in (0,1).$$

Thus by Lemma 3.1, $\Delta_m|_{A_1=6,A_2=\frac{m\alpha}{1-\alpha},A_3=\frac{m\alpha}{1-\alpha},A_4=\frac{m\alpha}{1-\alpha},A_5=\frac{m\alpha}{1-\alpha},A_6=\frac{m\alpha}{1-\alpha}} > 0$, for $\alpha \in (\frac{5}{m+5}, \frac{6}{m+6})$, $m \geq 2$, m integer. By Lemma 3.2, we know that $\Delta_m > 0$, for $A_1 \geq 6, A_2 \geq \frac{m\alpha}{1-\alpha}, A_3 \geq \frac{m\alpha}{1-\alpha}, A_4 \geq \frac{m\alpha}{1-\alpha}, A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{5}{m+5}, \frac{6}{m+6})$, $m \geq 2$, m integer.

(ii) $\Delta_m > 0$, for $m = 1$, $a_7 \in (6,7)$, $\alpha = 1 - \frac{1}{a_7} \in (\frac{5}{6}, \frac{6}{7})$. By Proposition 3.1, $\frac{\partial \Delta_1}{\partial A_i} > 0$, for $1 \leq i \leq 5$, $A_1 \geq 1, \dots, A_6 \geq 1$, $\alpha \in (0,1)$.

By Proposition 3.4, $\frac{\partial \Delta_1}{\partial A_6} > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq 2, A_6 \geq 1$, $\alpha \in (0,1)$. Since $\frac{\alpha}{1-\alpha} \in (5,6)$ here, $\frac{\partial \Delta_1}{\partial A_6} > 0$, for $A_1 \geq 6, A_2 \geq \frac{\alpha}{1-\alpha}, A_3 \geq \frac{\alpha}{1-\alpha}, A_4 \geq \frac{\alpha}{1-\alpha}, A_5 \geq \frac{\alpha}{1-\alpha}, A_6 \geq \frac{\alpha}{1-\alpha}$, $\alpha \in (0,1)$.

And

$$\Delta_1|_{A_1=6,A_2=\cdots=A_6=\frac{\alpha}{1-\alpha}} = -\frac{1}{(1-\alpha)^6}(7\alpha-6)(496\alpha^5 - 1277\alpha^4 + 1207\alpha^3 - 474\alpha^2 + 50\alpha + 7) > 0 \qquad \text{for } \alpha \in (\frac{5}{6}, \frac{6}{7}).$$

Thus, $\Delta_1 > 0$, for $A_1 \geq 6, A_2 \geq \frac{\alpha}{1-\alpha}, A_3 \geq \frac{\alpha}{1-\alpha}, A_4 \geq \frac{\alpha}{1-\alpha}, A_5 \geq \frac{\alpha}{1-\alpha}, A_6 \geq \frac{\alpha}{1-\alpha}$, $\alpha \in (\frac{5}{6}, \frac{6}{7})$.

Therefore, $\Delta_m > 0$, for $A_1 \geq 6, A_2 \geq \frac{m\alpha}{1-\alpha}, A_3 \geq \frac{m\alpha}{1-\alpha}, A_4 \geq \frac{m\alpha}{1-\alpha}, A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{5}{m+5}, \frac{6}{m+6})$, $m \geq 1$, m integer.

3.7. Case 7: $a_7 \geq m+6$

For $a_7 \geq m+6$, $\alpha \in [\frac{6}{m+6}, 1)$.

In this case, $\frac{m\alpha}{1-\alpha} \geq 6$, so $A_1 \geq A_2 \geq \cdots \geq A_6 \geq \frac{m\alpha}{1-\alpha}$. By Proposition 3.1, $\frac{\partial \Delta_m}{\partial A_i} > 0$, $\frac{\partial^2 \Delta_m}{\partial A_j \partial A_6} > 0$, for $1 \leq i,j \leq 5$, $A_1 \geq A_2 \geq \cdots \geq A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (0,1)$, $m \geq 1$, m integer.

By Proposition 3.5, we know that $\frac{\partial \Delta_m}{\partial A_6} > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq 2, A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{1}{m+1}, 1]$, $m \geq 2$, m integer. Since $\frac{m\alpha}{1-\alpha} \geq 6$ here, $\frac{\partial \Delta_m}{\partial A_6} > 0$, for $A_1 \geq A_2 \geq \cdots \geq A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in [\frac{6}{m+6}1)$, $m \geq 2$, m integer.

(i) $\Delta_m \geq 0$, for $A_1 \geq \frac{m\alpha}{1-\alpha}, A_2 \geq \frac{m\alpha}{1-\alpha}, A_3 \geq \frac{m\alpha}{1-\alpha}, A_4 \geq \frac{m\alpha}{1-\alpha}, A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in [\frac{6}{m+6}, 1)$, $m \geq 2$, m integer.

$$\begin{aligned}&\Delta_m|_{A_1=A_2=\cdots=A_6=\frac{m\alpha}{1-\alpha}}\\&=\frac{m\alpha}{1-\alpha}(\frac{m\alpha}{1-\alpha}-1)(\frac{m\alpha}{1-\alpha}-2)(\frac{m\alpha}{1-\alpha}-3)(\frac{m\alpha}{1-\alpha}-4)(\frac{m\alpha}{1-\alpha}+-5)(\frac{m\alpha}{1-\alpha}-6)\end{aligned}$$

Thus $\Delta_m|_{A_1=A_2=\cdots=A_6=\frac{m\alpha}{1-\alpha}} \geq 0$, for $\alpha \in [\frac{6}{m+6}, 1)$, $m \geq 2$, m integer. By Lemma 3.2, $\Delta_m \geq 0$, for $A_1 \geq \frac{m\alpha}{1-\alpha}, A_2 \geq \frac{m\alpha}{1-\alpha}, A_3 \geq \frac{m\alpha}{1-\alpha}, A_4 \geq \frac{m\alpha}{1-\alpha}, A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in [\frac{6}{m+6}, 1)$, $m \geq 2$, m integer. Equality holds if and only if $A_1 = A_2 = \cdots = A_6 = \frac{m\alpha}{1-\alpha}$ and $\alpha = \frac{6}{m+6}$, or equivalently, $a_1 = a_2 = \cdots = a_7 = m + 6$.

(ii) $\Delta_m \geq 0$, for $m = 1$, $a_7 \geq 7$, $\alpha = 1 - \frac{1}{a_7} \in [\frac{6}{7}, 1)$. By Proposition 3.1, $\frac{\partial \Delta_1}{\partial A_i} > 0$, for $1 \leq i \leq 5$, $A_1 \geq 1, \ldots, A_6 \geq 1$, $\alpha \in (0, 1)$.
By Proposition 3.4, $\frac{\partial \Delta_1}{\partial A_6} > 0$, for $A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq 2, A_6 \geq 1$, $\alpha \in (0, 1)$. Since $\frac{\alpha}{1-\alpha} > 6$ here, $\frac{\partial \Delta_1}{\partial A_6} > 0$, for $A_1 \geq \frac{\alpha}{1-\alpha}, A_2 \geq \frac{\alpha}{1-\alpha}, A_3 \geq \frac{\alpha}{1-\alpha}, A_4 \geq \frac{\alpha}{1-\alpha}, A_5 \geq \frac{\alpha}{1-\alpha}, A_6 \geq \frac{\alpha}{1-\alpha}$, $\alpha \in (0, 1)$.
And

$$\begin{aligned}&\Delta_1|_{A_1=\cdots=A_6=\frac{\alpha}{1-\alpha}}\\&=\frac{\alpha}{(1-\alpha)^7}(2\alpha-1)(3\alpha-2)(4\alpha-3)(5\alpha-4)(6\alpha-5)(7\alpha-6)\\&>0 \qquad \text{for } \alpha \in [\frac{6}{7}, 1).\end{aligned}$$

Thus, $\Delta_1 \geq 0$, for $A_1 \geq \frac{\alpha}{1-\alpha}, A_2 \geq \frac{\alpha}{1-\alpha}, A_3 \geq \frac{\alpha}{1-\alpha}, A_4 \geq \frac{\alpha}{1-\alpha}, A_5 \geq \frac{\alpha}{1-\alpha}, A_6 \geq \frac{\alpha}{1-\alpha}$, $\alpha \in [\frac{6}{7}, 1)$. Equality holds if and only if $A_1 = A_2 = \cdots = A_6 = \frac{\alpha}{1-\alpha}$ and $\alpha = \frac{6}{7}$, or equivalently, $a_1 = a_2 = \cdots = a_7 = 7$.

Therefore, $\Delta_m \geq 0$, for $A_1 \geq \frac{m\alpha}{1-\alpha}, A_2 \geq \frac{m\alpha}{1-\alpha}, A_3 \geq \frac{m\alpha}{1-\alpha}, A_4 \geq \frac{m\alpha}{1-\alpha}, A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in [\frac{6}{m+6}, 1)$, $m \geq 1$, m integer. Equality holds if and only if $A_1 = \cdots = A_6 = \frac{m\alpha}{1-\alpha}$ and $\alpha = \frac{6}{m+6}$, or equivalently, $a_1 = a_2 = \cdots = a_7 = m + 6$.

3.8. Completion of the proof

We still need to show the necessary and sufficient condition for the equality in (7) to hold. We have proved that $\Delta_m = g_7 - 7\sum_{k=1}^m g_6(k) \geq 0$, with the equality holds only if $a_1 = a_2 = \cdots = a_7 = m + 6$. Then we have,

$$7!P_7 = 7!\sum_{k=1}^m P_6(k) = 7\sum_{k=1}^m 6!P_6(k) \leq 7\sum_{k=1}^m g_6(k) \leq g_7. \tag{26}$$

For the last "$\leq$", the equality holds only if $a_1 = \cdots = a_7 = m + 6$.

To complete the last statement of Theorem 1.4, we only need the following result which was given by Wang and Yau [16]:

Theorem 3.1. *Let P_n and $a_1, \dots, a_n$ be as the same as in the Yau Number Theoretic Conjecture. If $a_1 = \cdots = a_n =$ integer, then the equality in (6) holds.*

Thus we have our main theorem proved.

3.9. Proof of Theorem 1.5

Due to the fact that $\psi(x, y) = Q_n$, we can apply our sharp estimate of P_7 to the function in order to obtain an estimate. Let $p_1 < p_2 < \cdots < p_7$ be the first seven prime numbers up to y. If $p_1^{l_1} p_2^{l_2} \cdots p_7^{l_7} \le x$, then $\frac{l_1}{\frac{log x}{log p_1}} + \frac{l_2}{\frac{log x}{log p_2}} + \cdots + \frac{l_7}{\frac{log x}{log p_7}} \le 1$. It follows that $a_i = \frac{\log x}{\log p_i}$ and $x_i = l_i$, $1 \le i \le 7$. Note that $Q_7 = P(a_1(1+a), a_2(1+a), \cdots, a_7(1+a))$, where $a = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_7}$. We split the estimate into five cases:

(I) $5 \le y < 7$
(II) $7 \le y < 11$
(III) $11 \le y < 13$
(IV) $13 \le y < 17$
(V) $17 \le y < 19$.

Cases (I) through (IV) have been proven through the estimates of P_3, P_4, P_5, and P_6, respectively [5]. Case (V) involves the first seven prime numbers: $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11$, $p_6 = 13$ and $p_7 = 17$. Set $\lambda = \log 2 + \log 3 + \log 5 + \log 7 + \log 11 + \log 13 + \log 17$, consequently, $a = \frac{\log 2 + \log 3 + \log 5 + \log 7 + \log 11 + \log 13 + \log 17}{\log x} = \frac{\lambda}{\log x}$ and

$$
\begin{aligned}
\psi(x, y) &= Q_7 \\
&= P\Big(\frac{\log x}{\log 2}(1 + \frac{\lambda}{\log x}), \frac{\log x}{\log 3}(1 + \frac{\lambda}{\log x}), \frac{\log x}{\log 5}(1 + \frac{\lambda}{\log x}), \frac{\log x}{\log 7}(1 + \frac{\lambda}{\log x}), \\
&\qquad \frac{\log x}{\log 11}(1 + \frac{\lambda}{\log x}), \frac{\log x}{\log 13}(1 + \frac{\lambda}{\log x}), \frac{\log x}{\log 17}(1 + \frac{\lambda}{\log x})\Big) \\
&\le \frac{1}{7!}\Big[\Big(\frac{\log x}{\log 2} + \frac{\lambda - \log 2}{\log 2}\Big)\Big(\frac{\log x}{\log 3} + \frac{\lambda - \log 3}{\log 3}\Big)\Big(\frac{\log x}{\log 5} + \frac{\lambda - \log 5}{\log 5}\Big) \\
&\quad \Big(\frac{\log x}{\log 7} + \frac{\lambda - \log 7}{\log 7}\Big)\Big(\frac{\log x}{\log 11} + \frac{\lambda - \log 11}{\log 11}\Big) \\
&\quad \Big(\frac{\log x}{\log 13} + \frac{\lambda - \log 13}{\log 13}\Big)\Big(\frac{\log x}{\log 17} + \frac{\lambda - \log 17}{\log 17}\Big) \\
&\quad - \Big\{\Big(\frac{\log x}{\log 17} + \frac{\lambda - \log 17}{\log 17}\Big)^7 - \Big(\frac{\log x}{\log 17} + \frac{\lambda - \log 17}{\log 17} + 1\Big)\Big(\frac{\log x}{\log 17} + \frac{\lambda - \log 17}{\log 17}\Big)
\end{aligned}
$$

$$(\frac{\log x}{\log 17} + \frac{\lambda - \log 17}{\log 17} - 1)(\frac{\log x}{\log 17} + \frac{\lambda - \log 17}{\log 17} - 2)(\frac{\log x}{\log 17} + \frac{\lambda - \log 17}{\log 17} - 3)$$
$$(\frac{\log x}{\log 17} + \frac{\lambda - \log 17}{\log 17} - 4)(\frac{\log x}{\log 17} + \frac{\lambda - \log 17}{\log 17} - 5)\}]$$
$$= \frac{1}{5040}\{\frac{1}{\log 2 \log 3 \log 5 \log 7 \log 11 \log 13 \log 17}(\log x + \log 255255)$$
$$(\log x + \log 170170)(\log x + \log 102102)(\log x + \log 72930)(\log x + \log 46410)$$
$$\cdot (\log x + \log 39270)(\log x + \log 30030) - \frac{1}{\log^7 17}[(\log x + \log 30030)^7$$
$$- (\log x + \log 17 + \log 30030)(\log x + \log 30030)(\log x + \log 30030 - \log 17)$$
$$\cdot (\log x + \log 30030 - 2\log 17)(\log x + \log 30030 - 3\log 17)$$
$$\cdot (\log x + \log 30030 - 4\log 17)(\log x + \log 30030 - 5\log 17)]\}.$$

References

[1] I. Chen, K.-P. Lin, S.S.-T. Yau, H.Q. Zuo, Coordinate-free characterization of homogeneous polynomials with isolated singularities, Comm. Anal. Geom. 19 (4) (2011) 661–704.
[2] A.H. Durfee, The signature of smoothings of complex surface singularities, Math. Ann. 232 (1) (1978) 85–98.
[3] E. Ehrhart, Sur un problème de gèomètrie diophantienne linèaire II, J. Reine Angew. Math. 227 (1967) 25–49.
[4] V. Ennola, On Numbers with Small Prime Divisors, Ann. Acad. Sci. Fenn. Ser. AI, vol. 440, 1969, 16 pp.
[5] A. Liang, S.S.-T. Yau, H.Q. Zuo, A sharp estimate of positive integral points in 6-dimensional polyhedra and a sharp estimate of smooth numbers, Sci. China Math. (2015), http://dx.doi.org/10.1007/s11425-015-5061-x, in press.
[6] K.-P. Lin, S.S.-T. Yau, Counting the number of integral points in general n-dimensional tetrahedra and Bernoulli polynomials, Canad. Math. Bull. 46 (2) (2003) 229–241.
[7] K.-P. Lin, X. Luo, S.S.-T. Yau, H.Q. Zuo, On number theoretic conjecture of positive integral points in 5-dimensional tetrahedron and a sharp estimation of Dickman–De Bruijn function, J. Eur. Math. Soc. 16 (2014) 90–102.
[8] X. Luo, S.S.-T. Yau, H.Q. Zuo, A sharp estimate of Dickman–De Bruijn function and a sharp polynomial estimate of positive integral points in 4-dimension tetrahedron, Math. Nachr. 288 (1) (2015) 61–75.
[9] M. Merle, B. Teissier, Conditions d'adjonction, d'après du val, in: Séminaire sur les Singularités des Surfaces, Springer, 1980, pp. 229–245.
[10] J. Milnor, P. Orlik, Isolated singularities defined by weighted homogeneous polynomials, Topology 9 (4) (1970) 385–393.
[11] L.J. Mordell, Lattice points in a tetrahedron and generalize Dedekind sums, J. Indian Math. 15 (1951) 41–46.
[12] C. Pomerance, The role of smooth numbers in number theoretic algorithms, in: ICM, Zürich, 1994, pp. 411–422.
[13] J.E. Pommersheim, Toric varieties, lattice points and Dedekind sums, Math. Ann. 295 (1) (1993) 1–24.
[14] K. Saito, Quasihomogene isolierte singularitäten von hyperflächen, Invent. Math. 14 (2) (1971) 123–142.
[15] B.L. van der Waerden, Modern Algebra, vol. I, Frederick Unger Publishing Co., 1950.
[16] X.-J. Wang, S.S.-T. Yau, On the GLY conjecture of upper estimate of positive integral points in real right-angled simplices, J. Number Theory 122 (1) (2007) 184–210.
[17] Y.-J. Xu, S.S.-T. Yau, A sharp estimate of the number of integral points in a tetrahedron, J. Reine Angew. Math. 423 (1992) 199–219.

[18] Y.-J. Xu, S.S.-T. Yau, A sharp estimate of the number of integral points in a 4-dimensional tetrahedra, J. Reine Angew. Math. 473 (1996) 69–86.
[19] S.S.-T. Yau, L.-T. Zhang, An upper estimate of integral points in real simplices with an application to singularity theory, Math. Res. Lett. 13 (2006) 911–921.
[20] S.S.-T. Yau, H.Q. Zuo, Complete characterization of isolated homogeneous hypersurface singularities, Pacific J. Math. 260 (1) (2012) 245–255.
[21] S.S.-T. Yau, H.Q. Zuo, Lower estimate of Milnor number and characterization of isolated homogeneous hypersurface singularities, Pacific J. Math. 273 (1) (2015) 213–224.
[22] S.S.-T. Yau, L. Zhao, H.Q. Zuo, Biggest sharp polynomial estimate of integral points in right-angled simplices, in: Topology of Algebraic Varieties and Singularities, in: Contemp. Math., vol. 538, Amer. Math. Soc., Providence, RI, 2011, pp. 433–467.