Journal of Algebra 457 (2016) 18–25

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Derivations of the moduli algebras of weighted homogeneous hypersurface singularities

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ARTICLE INFO

Article history:
Received 29 December 2014
Available online 21 March 2016
Communicated by Steven Dale Cutkosky

Keywords:
Derivations
Moduli algebra
Isolated singularity

ABSTRACT

Let $R = \mathbb{C}[x_1, x_2, \cdots, z_n]/(f)$ where f is a weighted homogeneous polynomial defining an isolated singularity at the origin. Then R, and $Der(R)$, the Lie algebra of derivations on R, are graded. It is well-known that $Der(R)$ has no negatively graded component [10]. J. Wahl conjectured that the above fact is still true in higher codimensional case provided that $R = \mathbb{C}[x_1, x_2, \cdots, x_n]/(f_1, f_2, \cdots, f_m)$ is an isolated, normal and complete intersection singularity and $f_1, f_2, \cdots, f_m$ are weighted homogeneous polynomials with the same weight type $(w_1, w_2, \cdots, w_n)$. On the other hand the first author Yau conjectured that the moduli algebra $A(V) = \mathbb{C}[x_1, x_2, \cdots, x_n]/(\partial f/\partial x_1, \cdots, \partial f/\partial x_n)$ has no negatively weighted derivations where f is a weighted homogeneous polynomial defining an isolated singularity at the origin. Assuming this conjecture has a positive answer, he gave a characterization of weighted homogeneous hypersurface singularities only using the Lie algebra $Der(A(V))$ of derivations on $A(V)$. The conjecture of Yau can be thought as an Artinian analogue of J. Wahl's conjecture. For the low embedding dimension, the Yau conjecture has a positive answer. In this paper we prove this conjecture for any high-dimensional sin-

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[1] This work is supported by NSFC (grant no. 11531007) and the start-up fund from Tsinghua University.
[2] This work is supported by the NSFC (grant nos. 11401335, 11531007), Tsinghua University Initiative Scientific Research Program and the start-up fund from Tsinghua University.

http://dx.doi.org/10.1016/j.jalgebra.2016.03.003

gularities under the condition that the lowest weight is bigger than or equal to half of the highest weight.

1. Introduction

Let A be a weighted zero dimensional complete intersection, i.e., a commutative algebra of the form

$$A = \mathbb{C}[x_1, x_2, \cdots, x_n]/I$$

where the ideal I is generated by a regular sequence of length n, $(f_1, f_2, \cdots, f_n)$. Here the variables have strictly positive integral weights, denoted by $wt(x_i) = w_i$, $1 \leq i \leq n$, and the equations are weighted homogeneous with respect to these weights. They are arranged for future convenience in the decreasing order of the degrees: $p_i := deg f_i$, $i = 1, 2, \cdots, n$ and $p_1 \geq p_2 \geq \cdots \geq p_n$. Consequently the algebra A is graded and one may speak about its homogeneous degree k derivations (k is an integer). A linear map $D : A \to A$ is a derivation if $D(ab) = D(a)b + aD(b)$, for any $a, b \in A$. D belongs to $Der^k(A)$ if $D : A^* \to A^{*+k}$. One of the most important open problems in rational homotopy theory is related to the vanishing of the above derivations in strictly negative degrees:

Halperin Conjecture. *(See [5].) If A is as above, then $Der^{<0}(A) = 0$.*

Assuming that all the weights w_i are even, this has the following topological interpretation. If a space X has $H^*(X, \mathbb{C}) = A$ as graded algebras, then it is known that the vanishing of $Der^{<0}(A) = 0$ implies the collapsing at the E_2-term of the Serre spectral sequence with $\mathbb{C}$-coefficients of any orientable fibration having X as fiber. Actually the above collapsing properties also implies vanishing properties when $\mathbb{C}$ is replaced by $\mathbb{Q}$ and X a rational space, see e.g. [5]. The Halperin Conjecture has been verified in several particular cases [10]:

1) equal weights ($w_1 = w_2 = \cdots = w_n$), see [14];
2) $n = 2, 3$, see [9,3];
3) "fibered" algebras see [4];
4) assuming $\mathbb{C}[x_1, x_2, \cdots, x_n]/(f_1, f_2, \cdots, f_{n-1})$ is reduced, see [6].
5) homogeneous spaces of equal rank compact connected Lie groups ($A = H^*(G/K)$), see [8].

On the other hand S.S.-T. Yau discovered independently the following conjecture on the nonexistence of the negative weight derivation from his work on *Lie* algebras of derivations of the moduli algebras of isolated hypersurface singularities, and especially his work on micro-local characterization of quasi-homogeneous hypersurface singularities ([1,11–13]).

Yau Conjecture. *Let $(V,0) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n\colon f(x_1, x_2, \ldots, x_n) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial $f(x_1, x_2, \ldots, x_n)$. Then there is no non-zero negative weight derivation on the moduli algebra (= Milnor algebra) $A(V) = \mathbb{C}[x_1, x_2, \ldots, x_n]/(\partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n)$.*

In case f is a homogeneous polynomial, then it was shown in [11] that $L(V)$ is a graded Lie algebra without negative weight. In fact they proved the following theorem.

Theorem 1.1. *(See [11].) Let $A = \oplus_{i=0}^{t} A_i$ be a commutative Artinian local algebra with $A_0 = \mathbb{C}$. Suppose that maximal ideal of A is generated by A_j for some $j > 0$. Then $Der(A)$ is a nonnegatively graded Lie Algebra $\oplus_{k=0}^{t} Der^k(A)$.*

This conjecture was proved in the low-dimensional case $n \leq 4$ ([1,2]) by explicit calculations.

Theorem 1.2. *(See [1].) Let $f(x_1, x_2, x_3)$ be a weighted homogeneous polynomial of type $(w_1, w_2, w_3; d)$ with isolated singularity at the origin. Assume that $d \geq 2w_1 \geq 2w_2 \geq 2w_3$. Let D be a derivation of the moduli algebra*

$$\mathbb{C}[x_1, x_2, x_3]/(\partial f/\partial x_1, \partial f/\partial x_2, \partial f/\partial x_3).$$

Then $D \equiv 0$ if D is negatively weighted.

Theorem 1.3. *(See [2].) Let $f(x_1, x_2, x_3, x_4)$ be a weighted homogeneous polynomial of type $(w_1, w_2, w_3, w_4; d)$ with isolated singularity at the origin. Assume that $d \geq 2w_1 \geq 2w_2 \geq 2w_3 \geq 2w_4$. Let D be a derivation of the moduli algebra*

$$\mathbb{C}[x_1, x_2, x_3, x_4]/(\partial f/\partial x_1, \partial f/\partial x_2, \partial f/\partial x_3, \partial f/\partial x_4).$$

Then $D \equiv 0$ if D is negatively weighted.

Let $(V,0) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n\colon f(x_1, x_2, \ldots, x_n) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial $f(x_1, x_2, \ldots, x_n)$ of type $(d : w_1, w_2, \cdots, w_n)$. Then by a result of Saito (see Theorem 2.1), we can always assume without loss of generality that $2w_i \leq d$ for all $1 \leq i \leq n$. If we give the variable x_i weight w_i for $1 \leq i \leq n$, the moduli algebra $A(V) = \mathbb{C}[x_1, \cdots, x_n]/(\partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n)$ is a graded algebra $\bigoplus_{i=0}^{\infty} A_i(V)$ and the Lie algebra of derivations $Der(A(V))$ is also graded.

In this paper, we deal with Yau conjecture for high-dimensional singularities. We prove the following result which answers Yau conjecture positively for some high-dimensional singularities.

Main Theorem. *Let* $(V,0) = \{(x_1, x_2, \dots, x_n) \in \mathbb{C}^n : f(x_1, x_2, \dots, x_n) = 0\}$ *be an isolated singularity defined by the weighted homogeneous polynomial* $f(x_1, x_2, \dots, x_n)$ *of type* $(d : w_1, w_2, \cdots, w_n)$*. Assume that* $d \geq 2w_1 \geq 2w_2 \geq \cdots \geq 2w_n$ *without loss of generality. Let* $Der(A(V))$ *be the Lie algebra of derivations of the moduli algebra*

$$A(V) = \mathbb{C}[x_1, \cdots, x_n]/(\partial f/\partial x_1, \partial f/\partial x_2, \dots, \partial f/\partial x_n).$$

If $w_n \geq w_1/2$*, then* $Der^{<0}(A(V)) = 0$*.*

In §2, we recall some definitions and results which are necessary to prove the main theorems. In §3, we shall give the proof of our main theorem.

2. Preliminary

In this section, we recall some known results which are needed to prove the Main Theorem.

Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a finitely generated integral domain over $\mathbb{C}$, with $A_0 = \mathbb{C}$. Let A have homogeneous generators $x_i \in A_{n_i}$, $i = 1, \cdots, s$; then A is graded quotient

$$\mathbb{C}[x_1, x_2, \cdots, x_n]/I, wt(x_i) = n_i.$$

A derivation D of A (i.e. $D \in Der(A)$) can be viewed as a derivation of $\mathbb{C}[x_1, x_2, \cdots, x_n]$ which preserve I. So, by abuse of notation, we shall write

$$D = \sum a_i \partial/\partial x_i \quad (a_i \in A).$$

The module $Der(A)$ of derivations is graded by saying D as above has weight k if a_i are weighted homogeneous with $wt(a_i) = n_i + k$, or equivalently, $D(A_i) \subset A_{i+k}$ for all i. In particular, the Euler derivation

$$\Delta = \sum n_i x_i \partial/\partial x_i$$

(which is a derivation of A since I is graded) has weight 0. We write

$$Der(A) = \bigoplus_{k \in \mathbb{Z}} Der^k(A).$$

(Clearly $Der^{-k}(A) = 0$ for $k > max\{n_i, i = 1, \cdots, s\}$.)

Definition 2.1. A polynomial $f(x_1, x_2, \cdots, x_n)$ is weighted homogeneous of type $(d : w_1, w_2, \cdots, w_n)$ where d and $w_1, w_2, \cdots, w_n$ are fixed positive integers, if it can be expressed as a linear combination of monomials $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ for which $w_1 i_1 + w_2 i_2 + \cdots + w_n i_n = d$. Here, d is called the degree of f.

Let f be a weighted homogeneous polynomial of type $(d : w_1, w_2, \cdots, w_n)$ with an isolated singularity at the origin. Then the moduli algebra

$$A(V) = \mathbb{C}[x_1, \cdots, x_n]/(\partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n)$$

is a graded algebra $\bigoplus_{i=0}^{\infty} A_i(V)$ and the Lie algebra of derivations $Der(A(V))$ is also graded.

Lemma 2.1. *(See [13].) Let (A, m) be a commutative Artinian local algebra and $D \in Der(A)$ be any derivation of A. Then D preserve the m-adic filtration of A, i.e., $D(m) \subset m$.*

Definition 2.2. Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be germs of holomorphic functions defining an isolated hypersurface singularities at the origin respectively. Let $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a germ of biholomorphic map. f is called right equivalent to g, if $g = f \circ \phi$.

Theorem 2.1. *(See [7].) Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be the germ of a complex analytic function with an isolated critical point at the origin.*

(a) f is right equivalent to a weighted homogeneous polynomial if and only if $\mu = \tau$ or

$$f \in J_f := (\partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n)$$

(b) If f is weighted homogeneous with normalized weight system $(d : w_1, \cdots, w_n)$ with $0 < w_n \leq \cdots \leq w_1 < d$ and if $f \in \boldsymbol{m}_{\mathbb{C}^n,0}^3$, then the weight system is unique and $0 < w_n \leq \cdots \leq w_1 < \frac{d}{2}$

(c) If $f \in J_f$ then f is right equivalent to a weighted homogeneous polynomial $z_1^2 + \cdots + z_k^2 + g(z_{k+1}, \cdots, z_n)$ with $g \in \boldsymbol{m}_{\mathbb{C}^n,0}^3$. Especially, its normalized weight system satisfies $0 < w_n \leq \cdots \leq w_{k+1} < w_k = \cdots = w_1 = \frac{d}{2}$

(d) If f and $\bar{f} \in \mathcal{O}_{\mathbb{C}^n,0}$ are right equivalent and weighted homogeneous with normalized weight systems $(d : w_1, \cdots, w_n)$ and $(d : \bar{w_1}, \cdots, \bar{w_n})$ with $0 < w_n \leq \cdots \leq w_1 \leq \frac{d}{2}$ and $0 < \bar{w_n} \leq \cdots \leq \bar{w_1} \leq \frac{d}{2}$ then $w_i = \bar{w_i}$.

3. Proof of the Main Theorem

Proof. Since the Halperin conjecture is true for $n = 2$, by Theorem 1.2 and Theorem 1.3, we only need to prove the main Theorem for $n \geq 5$.

Since f is a weighted homogeneous polynomial,

$$A(V) = \mathbb{C}[x_1, \cdots, x_n]/(\partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n)$$

is graded, so $Der(A(V)) = \bigoplus_k Der^k(A(V))$ is also graded. Any $D \in Der^k(A(V))$ is of the form $\sum_{i=1}^n p_i(x_1, x_2, \cdots, x_n)\partial/\partial x_i$ therefore $k \geq -w_1$. If $Der(A(V))$ has negative graded part we can take $Der^q(A(V))$ to be the most negatively graded part, that is $Der(A(V)) = \bigoplus_{k \geq q} Der^k(A(V))$ and $Der^q(A(V)) \neq 0$. What we want to prove is that any $D \in Der^q(A(V))$ has to be 0.

Lemma 3.1. *Let $f \in \mathbb{C}[x_1, \cdots, x_n]$ be a weighted polynomial of type $(d : w_1, w_2, \cdots, w_n)$ with isolated singularity at the origin. Assume that $d \geq 2w_1 \geq 2w_2 \geq \cdots \geq 2w_n$ without loss of generality. Let $D \in Der^q(A(V))$ be a negative weight derivation as above. Then for $i \geq 2$,*

$$\partial(Df)/\partial x_i \in (\partial f/\partial x_1, \partial f/\partial x_2, \cdots, \partial f/\partial x_{i-1}).$$

Proof. By comparison of the weighted degrees it is clear that a negative weight derivation D has to be the following form

$$D = p_1(x_2, x_3, \cdots, x_n)\partial/\partial x_1 + p_2(x_3, x_4, \cdots, x_n)\partial/\partial x_2 + \cdots + c_1 x_n^k \partial/\partial x_{n-1} + c_2 \partial/\partial x_n$$

where $deg p_i = w_i + q < w_i$ and c_1, c_2 are constants. By Lemma 2.1 $c_2 = 0$ and $k \geq 1$. Therefore the commutator $[\partial/\partial x_1, D] = 0$ and $[\partial/\partial x_i, D]$, $i = 2, 3 \cdots, n$ is of the following form by a direct computation.

$$[\partial/\partial x_i, D] = \partial p_1/\partial x_i \cdot \partial/\partial x_1 + \partial p_2/\partial x_i \cdot \partial/\partial x_2 + \cdots + \partial p_{i-1}/\partial x_i \cdot \partial/\partial x_{i-1}. \quad (3.1)$$

Let $J_f = (\partial f/\partial x_1, \partial f/\partial x_2, \cdots, \partial f/\partial x_n)$ be the Jacobian ideal. Since D is a derivation on $A(V)$ it has to preserve J_f, that is, $D(J_f) \subset J_f$. By the relation $d \geq 2w_1 \geq 2w_2 \geq \cdots \geq 2w_n$ and a comparison of weighted degrees we have

$$D(\partial f/\partial x_1) = 0 \text{ and for } i \geq 2,\ D(\partial f/\partial x_i) \in (\partial f/\partial x_1, \partial f/\partial x_2, \cdots, \partial f/\partial x_{i-1}). \quad (3.2)$$

By (3.1) we have

$$\begin{aligned}\partial(Df)/\partial x_i = D(\partial f/\partial x_i) + \partial p_1/\partial x_i \cdot \partial f/\partial x_1 + \partial p_2/\partial x_i \cdot \partial f/\partial x_2 + \cdots \\ + \partial p_{i-1}/\partial x_i \cdot \partial f/\partial x_{i-1}.\end{aligned} \quad (3.3)$$

The right-hand side of (3.3) is in $(\partial f/\partial x_1, \partial f/\partial x_2, \cdots, \partial f/\partial x_{i-1})$ by (3.2). □

The following Lemma is an immediate consequence from the proof of the Lemma 3.1.

Lemma 3.2. *Under the same conditions with Lemma 3.1, then we have $\partial(Df)/\partial x_1 = 0$.*

Proof. It is easy to see that $\partial(Df)/\partial x_1 = D(\partial f/\partial x_1) = 0$. $\square$

Now we continue to prove our main Theorem. By Lemma 3.2, $\partial(Df)/\partial x_1 = 0$, it is clear that Df does not depend on x_1. Then by Lemma 3.1, $\partial(Df)/\partial x_2 = q(x_2, x_3, \cdots, x_n)\partial f/\partial x_1$. If $q(x_2, x_3, \cdots, x_n)$ is not 0, then $degq(x_2, x_3, \cdots, x_n) \geq wt(x_n) = w_n$. It follows that $wt(q(x_2, x_3, \cdots, x_n)\partial f/\partial x_1) \geq w_n + degf - w_1$. Since $wt(\partial(Df)/\partial x_2) = wt(D) + degf - w_2$. By assumption $w_n \geq w_1/2$, we have $wt(\partial(Df)/\partial x_2) = wt(D) + degf - w_2 \leq wt(D) + degf - w_n < degf - w_n \leq w_n + degf - w_1 \leq wt(q(x_2, x_3, \cdots, x_n)\partial f/\partial x_1)$, a contradiction. Therefore $q(x_2, x_3, \cdots, x_n) \equiv 0$ and $\partial(Df)/\partial x_2 = 0$ which implies Df does not depend on x_2. For $3 \leq i \leq n$. By Lemma 3.1, $\partial(Df)/\partial x_i = q_1\partial f/\partial x_1 + q_2\partial f/\partial x_2 + \cdots + q_{i-1}\partial f/\partial x_{i-1}$. Observe that $q_j, 1 \leq j \leq i-1$ depends only on $x_{j+1}, x_{j+2}, \cdots, x_n$ variables by weight consideration since D has negative weight. If there exists a q_j, $j = 1, \cdots, i-1$ which is not 0, then

$$wt(q_1(x_2, x_3, \cdots, x_n)\partial f/\partial x_1 + q_2(x_3, x_4, \cdots, x_n)\partial f/\partial x_2 + \cdots \\ + q_{i-1}(x_i, x_{i+1}, \cdots, x_n)\partial f/\partial x_{i-1}) \geq w_n + degf - w_1.$$

On the other hand, $wt(\partial(Df)/\partial x_i) = wt(D) + degf - w_i \leq wt(D) + degf - w_n < degf - w_n \leq w_n + degf - w_1 \leq wt(q_1(x_2, x_3, \cdots, x_n)\partial f/\partial x_1 + q_2(x_3, x_4, \cdots, x_n)\partial f/\partial x_2 + \cdots + q_{i-1}(x_i, x_{i+1}, \cdots, x_n)\partial f/\partial x_{i-1})$, a contradiction. Therefore, $q_j \equiv 0$ for all $j = 1, \cdots, i-1$. Thus we have $\partial(Df)/\partial x_i = 0$ for $i = 1, \cdots, n$. It follows that Df is a constant and $wt(D) + d = 0$ where d is the degree of f. Since $wt(D) \geq -w_1$, we have that $d \leq w_1$ which contradicts with our assumption $d \geq 2w_1$. The main Theorem is proved. $\square$

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