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A Sharp Upper Estimate Conjecture for the Yau Number of a Weighted Homogeneous Isolated Hypersurface Singularity

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Dedicated to Professor Eduard Looijenga on the occasion of his 69th birthday

Abstract: Let V be a hypersurface with an isolated singularity at the origin defined by the function of $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. Let $L(V)$ be the Lie algebra of derivations of the moduli algebra $A(V) := \mathbb{C}\{x_1, \cdots, x_n\}/(f, \partial f/\partial x_1, \cdots, \partial f/\partial x_n)$. It is known that $L(V)$ is a finite dimensional solvable Lie algebra ([Ya1], [Ya2]). $L(V)$ is called the Yau algebra of V in [Yu] and [Khi] in order to distinguish from Lie algebras of other types appearing in singularity theory ([AVZ], [AM], [BY]). $\dim L(V)$ is called Yau number in [EK],[Khi]. This number is an analytic invariant. In case V is a weighted homogeneous singularity, it is a natural question whether Yau number can be bounded by a number which depends only on the weight. In this paper we formulate a sharp upper estimate conjecture for the Yau number of weighted homogeneous isolated hypersurface singularities. We prove this conjecture for a large class of singularities.
Keywords: Yau number, isolated hypersurface singularity, derivation Lie algebra.

1. Introduction

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function with an isolated critical point at the origin. Then $V = \{f = 0\}$ is an isolated hypersurface singularity germ. It was Yau ([Ya1], [Ya2], [Ya3], [XY], [CCYZ], [CYZ], [YZ]) who

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first systematically studied $L(V)$ which is the Lie algebras of derivations of the moduli algebra

$$A(V) := \mathbb{C}\{x_1, \cdots, x_n\}/(f, \partial f/\partial x_1, \cdots, \partial f/\partial x_n),$$

i.e., $L(V) = \mathrm{Der}_{\mathbb{C}}(A(V), A(V))$. According to [Ya3], $L(V)$ is a finite-dimensional solvable Lie algebra. Yu [Yu] and Khimshiashvili [Khi] call $L(V)$ the Yau algebra of V in order to distinguish from Lie algebras of other types appearing in singularity theory [AVZ], [AM], [BY]. Its dimension is called the Yau number by Elashvili and Khimshiashvili [EK] and will be denoted $\lambda(V)$ (see Definition 2.1).

The multiplicity *mult*(f) of the singularity $(V, 0)$ is defined to be the order of the lowest nonvanishing term in the power series expansion of f at 0. The Milnor number μ and the Tjurina number of the singularity $(V, 0)$ are defined respectively by

$$\begin{aligned} \mu &= \dim \mathbb{C}\{x_1, x_2, \cdots, x_n\}/(\partial f/\partial x_1, \partial f/\partial x_2, \cdots, \partial f/\partial x_n), \\ \tau &= \dim \mathbb{C}\{x_1, x_2, \cdots, x_n\}/(f, \partial f/\partial x_1, \partial f/\partial x_2, \cdots, \partial f/\partial x_n). \end{aligned}$$

Recall that a polynomial $f \in \mathbb{C}[x_1, \cdots, x_n]$ is called weighted homogeneous if there exist positive rational numbers $w_1, \cdots, w_n$ (weights of $x_1, \cdots, x_n$) and d such that, for each monomial $\prod x_i^{a_i}$ appearing in f with nonzero coefficient, one has $\sum a_i w_i = d$. The number d is called the weighted homogeneous degree (w-degree) of f with respect to weights w_j. Obviously, without loss of generality one can asssume that w-deg$f = 1$ and we willl often do so in the sequel. The collection $(w_1, w_2, \cdots, w_n; d)$ is called the weight type of f. It is well known that the Milnor number of a weighted homogeneous isolated hypersurface singularities can be calculated by only using the weight type.

Theorem 1.1. (Milnor-Orlik [MO]). *Let $f(x_1, x_2, \cdots, x_n)$ be a weighted homogeneous polynomial of type $(w_1, w_2, \cdots, w_n; 1)$ with an isolated singularity at the origin. Then the Milnor number $\mu = (\frac{1}{w_1} - 1)(\frac{1}{w_2} - 1) \cdots (\frac{1}{w_n} - 1)$.*

In 1971, Saito proved the following theorem which gives a necessary and sufficient condition for V to be defined by a weighted homogeneous polynomial.

Theorem 1.2. (Saito [Sa]) *f is a weighted homogeneous polynomial after a biholomorphic change of coordinates if and only if $\mu = \tau$.*

It is well known that the weight types are toplogical invariants for 1 or 2-dimensional weighted homogeneous hypersurface singularities ([YS], [Sae]). A natural question is: can we bound the analytic invariant Yau numbers by only using the topological invariant weight types of the weighted homogeneous isolated hypersurface singularities? We propose the following conjecture:

Conjecture. *Let* $(V,0)=\{(x_1,x_2,\ldots,x_n)\in\mathbb{C}^n:\ f(x_1,x_2,\ldots,x_n)=0\}$ $(n\geq 2)$ *be an isolated singularity defined by the weighted homogeneous polynomial* $f(x_1,x_2,\ldots,x_n)$ *of weight type* $(w_1,w_2,\cdots,w_n;1)$. *Then the Yau number*

$$\lambda(V)\leq n\mu-\sum_i^n(\frac{1}{w_1}-1)(\frac{1}{w_2}-1)\cdots(\widehat{\frac{1}{w_i}-1})\cdots(\frac{1}{w_n}-1),$$

where $(\widehat{\frac{1}{w_i}-1})$ *means that* $\frac{1}{w_i}-1$ *is omitted and* μ *is the Milnor number.*

Remark: Combining with Theorem 1.1, we have,

$$\lambda(V)\leq n\prod_{i=1}^n(\frac{1}{w_i}-1)-\sum_i^n(\frac{1}{w_1}-1)(\frac{1}{w_2}-1)\cdots(\widehat{\frac{1}{w_i}-1})\cdots(\frac{1}{w_n}-1).$$

In this paper, we obtain the following results which give positive answer to the conjecture for a large class of singularities.

Main Theorem A. *Let* $(V,0)$ *be a binomial singularity (see definition 3.1) defined by the weighted homogeneous polynomial* $f(x_1,x_2)$ *with weight type* $(w_1,w_2;1)$. *Then* $\lambda(V)\leq 2(\frac{1}{w_1}-1)(\frac{1}{w_2}-1)-(\frac{1}{w_1})-(\frac{1}{w_2})+2$.

Let $f(x_1,x_2,\cdots,x_n)=0$ of weight type $(w_1,w_2\cdots,w_n;1)$ and $g(y_1,y_2,\cdots,y_m)=0$ of weight type $(w_{n+1},w_{n+2},\cdots,w_{n+m};1)$ be two weighted homogeneous polynomials which define two isolated hypersurface singularities $(V_f,0)\subset(\mathbb{C}^n,0)$ and $(V_g,0)\subset(\mathbb{C}^m,0)$. It is clear that $f(x_1,\cdots,x_n)+g(y_1,\cdots,y_m)=0$ (which is called addition of Thom-Sebastiani) has weight type $(w_1,w_2,\cdots,w_{n+m};1)$ and define an weighted homogeneous isolated singularity $(V_{f+g},0)\subset(\mathbb{C}^{m+n},0)$.

Main Theorem B. *Let* $(V_f,0)\subset(\mathbb{C}^n,0)$ *and* $(V_g,0)\subset(\mathbb{C}^m,0)$ *be defined by weighted homogeneous polynomials* $f(x_1,x_2,\cdots,x_n)=0$ *of weight type* $(w_1,w_2\cdots,w_n;1)$ *and* $g(y_1,y_2,\cdots,y_m)=0$ *of weight type* $(w_{n+1},w_{n+2},\cdots,w_{n+m};1)$ *respectively. Let* $\mu(V_f)$, $\mu(V_g)$, $A(V_f)$ *and* $A(V_g)$

be the Milnor numbers and moduli algebras of $(V_f, 0)$ *and* $(V_g, 0)$ *respectively. Then*

$$\lambda(V_{f+g}) = \mu(V_f)\lambda(V_g) + \mu(V_g)\lambda(V_f). \tag{1.1}$$

and furthermore if both f *and* g *satisfy the conjecture, then* $f+g$ *also satisfies the conjecture.*

Main Theorem C. *Let* $f \in \mathbb{C}\{x_1, \cdots, x_n\}$ *be a weighted homogeneous isolated singularity which is a Thom-Sebastiani summation of the following three types with* $mult(f) \geq 3$*:*

1) $x_1^{a_1} + x_2^{a_2} + \cdots + x_{m-1}^{a_{m-1}} + x_m^{a_m}$, $m \geq 1$,

2) $x_1^{a_1} x_2 + x_2^{a_2}$,

3) $x_1^{a_1} x_2 + x_2^{a_2} x_1$.

Then f *satisfies the conjecture.*

2. Derivations

In this section, we recall the definition of derivation and some known results which are needed to prove the Main Theorems.

Recall that a derivation of commutative associative algebra A is defined as a linear endomorphism D of A satisfying the Leibniz rule: $D(ab) = D(a)b + aD(b)$. The set of its derivations on A is denoted by $\mathrm{Der}(A, A)$ (sometimes use $\mathrm{Der}A$). It has a natural Lie algebra structure with Lie bracket defined by the commutator of linear endo-morphisms. In particular, for a singularity $(V, 0)$ defined by a polynomial f, one can consider the Lie algebras $\mathrm{Der}(A(V), A(V))$ where $A(V)$ is the moduli algebra of $(V, 0)$ and $A(V) = \mathbb{C}\{x_1, \cdots, x_n\}/(f, \partial f/\partial x_1, \cdots, \partial f/\partial x_n)$.

Definition 2.1. ([EK], [Khi]) *Let* $V = \{f = 0\}$ *be a germ of an isolated hypersurface singularity at the origin in* $\mathbb{C}^n$ *defined by a complex polynomial* $f(x_1, \cdots, x_n)$*. The Lie algebra* $Der_{\mathbb{C}}(A(V), A(V))$ *of derivations of the moduli algebra* $A(V)$ *is called the Yau algebra of* $(V, 0)$ *and denoted by* $L(V)$*. Its dimension is called the Yau number of* $(V, 0)$ *and denoted* $\lambda(V)$*.*

The following concepts and results enable one to compute the Yau algebras of many concrete singularities. Let A, B be associative algebras over $\mathbb{C}$. Recall that the multiplication algebra $M(A)$ of A is defined as the subalgebra of endomorphisms of A generated by the identity element and left and right multiplications by elements of A. The centroid $C(A)$ is the set of endomorphisms of A which commute with all elements of $M(A)$. Clearly, $C(A)$ is a unital subalgebra of End(A). The following statement is a particular case of a general result from Proposition 1.2 of [Bl]. Let $S = A \otimes B$ be a tensor product of finite dimensional associative algebras with units. Then

$$\mathrm{Der}S \cong (\mathrm{Der}A) \otimes C(B) + C(A) \otimes (\mathrm{Der}B).$$

We will only use this result for commutative associative algebras with unit, in which case the centroid coincides with the algebra itself. Thus for commutative associative algebras A, B one has:

Theorem 2.1. ([Bl]) *For commutative associative algebras A, B,*

$$DerS \cong (DerA) \otimes B + A \otimes (DerB). \tag{2.1}$$

This formula will be used in the sequel.

Definition 2.2. *For an ideal J in an analytic algebra S, denote by $Der_J S \subseteq Der_{\mathbb{C}} S$ the Lie subalgebra of all $\sigma \in Der_{\mathbb{C}} S$ for which $\sigma(J) \subset J$.*

We shall use the following well-known result to compute the derivations.

Theorem 2.2. *Let J be an ideal in $R = \mathbb{C}\{x_1, \cdots, x_n\}$. Then there is a natural isomorphism of Lie algebras*

$$(Der_J R)/(J \cdot Der_{\mathbb{C}} R) \cong Der_{\mathbb{C}}(R/J).$$

Proof. By definition, there is a map $\varphi : \mathrm{Der}_J R \to \mathrm{Der}_{\mathbb{C}}(R/J)$ whose kernel contains $J \cdot \mathrm{Der}_{\mathbb{C}} R$. Note that $\mathrm{Der}_{\mathbb{C}} R$ is a free R-module with basis $\partial/\partial x_1, \cdots, \partial/\partial x_n$ and that the coefficient of $\partial/\partial x_i$ in $\sigma \in \mathrm{Der}_{\mathbb{C}} R$ is $\sigma(x_i)$. So if $\sigma \in \mathrm{Ker}\varphi$, then $\sigma(x_i) \in J$ and hence $\sigma \in J \cdot \mathrm{Der}_{\mathbb{C}} R$. This proves injectivity. By a result of Scheja and Wiebe [SW], any $\bar{\sigma} \in \mathrm{Der}_{\mathbb{C}}(R/J)$ lifts to a $\sigma \in \mathrm{Der}_{\mathbb{C}} R$ which is then necessarily in $\mathrm{Der}_J R$. This proves surjectivity and the claim follows. Q.E.D.

3. Weighted Homogenous Fewnomial Isolated Singularities

We first recall the setting of the so-called fewnomials introduced in [Kh]. Let us first establish precise terminology which will be different from the setting of [Kh] where the term fewnomial was introduced. Let P be a polynomial in n variables. We shall say that P is a fewnomial if the number of monomials in P does not exceed n. Obviously, the number of monomials in

P may depend on the system of coordinates. In order to obtain a rigorous concept we shall only admit linear changes of coordinates and say that P (or rather its germ at the origin) is a k-nomial if k is the smallest integer such that P becomes a polynomial consisting of k monomials after (possibly) a linear change of coordinates. For linguistic flexibility it is convenient to say in such case that the nomiality $\mathrm{nom}(P)$ of P is equal to k. Nomiality may be considered as a sort of elementary complexity measure of polynomials which appears relevant in some problems of enumerative algebraic geometry [Kh]. An isolated hypersurface singularity $(V, 0)$ is called k-nomial if there exists an isolated hypersurface singularity Y analytically isomorphic to V which can be defined by a k-nomial and k is the smallest such number. It turns out then, except for some non-interesting cases, that a singularity defined by a fewnomial P can be isolated only if $\mathrm{nom}(P) = n$, i.e., if P is a n-nomial in n variables. We shall formulate this result separately for further reference.

Lemma 3.1. ([Sa]) *Fix an $i \in \{1, \cdots, n\}$. For an isolated singularity f, at least one of the monomials of the form $x_i^a x_j, a \geq 1, j = 1, \cdots, n$ appears in the series f with a nonzero coefficient.*

Lemma 3.2. ([Khi]) *A k-nomial P in n variables which has multiplicity at least three, cannot have an isolated critical point at the origin if $k < n$.*

Proof. If P has an isolated critical point at the origin, then by lemma 3.1 for each $1 \leq i \leq n$ there exists a monomial $x_i^a x_j$ entering in P. For each i, fix a monomial of such form with the minimal $j = j(i)$. Since P has multiplicity at least three, so there are no monomials of degree two in P. Two such monomials for two different numbers $i_1 \neq i_2$ cannot coincide, i.e., $x_{i_1}^{a_{i_1}} x_{j(i_1)} \neq x_{i_2}^{a_{i_2}} x_{j(i_2)}$. This obviously implies that the number of monomials in P cannot be less than the number of coordinates n, which gives the result. Q.E.D.

Remark: Using terminology of [AVZ], the requirement that there are no quadratic terms can be expressed by saying that P is of (maximal) corank n at the origin (here the corank meaning that corank of the second differential of P, cf. p187,[AGV]). The reason why we have to exclude quadratic terms, is that otherwise the formulation given above would not be correct. Indeed, a stabilization of A_1 singularity can be defined by a polynomial in $2k$ variables of the form $x_1x_2 + x_3x_4 + \cdots + x_{2k-1}x_{2k}$ which contains only k monomials. Notice that polynomials of the form $x_1^{a_1} + x_2^{a_2} + \cdots + x_n^{a_n}$ give evident examples of n-nomials with isolated singularity at the origin of $\mathbb{C}^n$.

We introduce some terminology.

Definition 3.1. *We say that an isolated hypersurface singularity in* $\mathbb{C}^n$ *is fewnomial if it can be defined by a* n*-nomial in* n *variables and we say that it is a weighted homogenous fewnomial isolated singularity if it can be defined by a weighted homogenous fewnomial.* 2*-nomial isolated hypersurface singularity is also called binomial singularity.*

Notice that a direct sum of weighted homogenous fewnomial isolated singularity is also a weighted homogenous fewnomial isolated singularity. Moreover, according to (2.1) derivation algebras of direct sums can be easily computed. For this reason our strategy will be to prove Main Theorem A for certain series of weighted homogeneous fewnomial isolated singularities and then extend it to direct sums of singularities from those series.

Proposition 3.1. *Let* f *be a weighted homogeneous fewnomial isolated singularity with mult*$(f) \geq 3$. *Then* f *analytically equivalent to a linear combination of the following three series:*

Type A. $x_1^{a_1} + x_2^{a_2} + \cdots + x_{n-1}^{a_{n-1}} + x_n^{a_n}$, $n \geq 1$,
Type B. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}$, $n \geq 2$,
Type C. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1$, $n \geq 2$.

Proof. let's first introduce a Lemma which is crucial part of the proof of the proposition.

From now on only singularities of multiplicity at least 3 will be considered.

Definition 3.2. *The support of a polynomial*

$$f = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha \cdot x^\alpha \in \mathbb{C}[x_1, \cdots, x_n] \text{ where } x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

and $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$, *is supp*$(f) = \{x^\alpha \mid \alpha \in \mathbb{N}_0^n, a_\alpha \neq 0\}$.

Lemma 3.3. *let* $f(x_1, \cdots, x_n)$ *be a weighted homogeneous fewnomial which define an isolated singularity at origin with multiplicity at least 3. Then* $x_{i_1}^{a_{i_1}} x_{j(i_1)}$, $x_{i_2}^{a_{i_2}} x_{j(i_2)}$, *where* $j(i_1) \neq i_1$, $j(i_2) \neq i_2$, $i_1 \neq i_2$ *and* $j(i_1) = j(i_2)$, *cannot both appear in supp*(f).

Proof. It is enough to prove that $x_1^{a_1}x_3$ and $x_2^{a_2}x_3$ cannot both appear in supp(f). Since f is weighted homogeneous fewnomial with isolated critical point at the origin, in view of Lemma 3.1, f can be written as

$$f = c_1 x_1^{a_1} x_3 + c_2 x_2^{a_2} x_3 + \sum_{k=3}^{n} c_k x_k^{a_k} x_{j(k)},\ c_i \neq 0, \text{ for } 1 \leq i \leq n.$$

Then

$$\begin{aligned}
\partial f/\partial x_1 =& c_1 a_1 x_1^{a_1-1} x_3 + g_1(x_3, \cdots, x_n),\\
\partial f/\partial x_2 =& c_2 a_2 x_2^{a_2-1} x_3 + g_2(x_3, \cdots, x_n),\\
\partial f/\partial x_3 =& c_1 x_1^{a_1} + c_2 x_2^{a_2} + c_3 a_3 x_3^{a_3-1} x_{j(3)} + g_3(x_3, \cdots, x_n),\\
\partial f/\partial x_k =& c_k a_k x_k^{a_k-1} x_{j(k)} + g_k(x_3, \cdots, x_n), \text{ for } k \geq 4,
\end{aligned}$$

where multiplicity of g_i is at least one if g_i is a nonzero function for $1 \leq i \leq n$. Clearly the singular set of f is given by $\{(x_1, x_2, 0 \cdots, 0) : c_1 x_1^{a_1} + c_2 x_2^{a_2} = 0\}$ which is not an isolated singularity. Hence $x_1^{a_1} x_3$ and $x_2^{a_2} x_3$ cannot both appear in the support of f. Q.E.D.

Then the Proposition 3.1 is an immediate corollary of Lemma 3.1, Lemma 3.2 and Lemma 3.3 up to nonzero coefficients. After a rescaling, we can make all the coefficients of the monomials in f to be 1. Q.E.D.

Proposition 3.1 has an immediate corollary.

Corollary 3.1. *Each binomial isolated singularity is analytically equivalent to one from the three series: A) $x_1^{a_1} + x_2^{a_2}$, B) $x_1^{a_1} x_2 + x_2^{a_2}$, C) $x_1^{a_1} x_2 + x_2^{a_2} x_1$.*

4. Proof of Main Theorems

In order to prove Main Theorem A, we first prove some propositions. Indeed the following Proposition 4.1 and part of Proposition 4.2 and 4.3 can also be found in [EK] and [Khi], however, the following proof of Propositions 4.1 is more simple and direct than in [EK]. For convenience of the reader we give the detailed proofs of these propositions.

Proposition 4.1. *Let $(V, 0)$ be a weighted homogeneous fewnomial isolated singularity of type A which is defined by $f = x_1^{a_1} + x_2^{a_2} + \cdots + x_n^{a_n}$ $(a_i \geq$*

3, $1 \leq i \leq n$) *with weight type* $(\frac{1}{a_1}, \frac{1}{a_2}, \cdots, \frac{1}{a_n}; 1)$. *Then the Yau number*

$$\lambda(V) = n\prod_{i=1}^{n}(a_i - 1) - \sum_{i}^{n}(a_1 - 1)(a_2 - 1)\cdots\widehat{(a_i - 1)}\cdots(a_n - 1),$$

where $\widehat{(a_i - 1)}$ *means that* $a_i - 1$ *is omitted.*

Proof. Since the moduli algebra

$$\begin{aligned} A(V) :=& \mathbb{C}\{x_1, \cdots, x_n\}/(f_{x_1}, f_{x_2} \cdots, f_{x_n}) \\ =& \mathbb{C}\{x_1, \cdots, x_n\}/(a_1 x_1^{a_1 - 1}, a_2 x_2^{a_2 - 1} \cdots, a_n x_n^{a_n - 1}) \\ \cong& \mathbb{C}\{x_1\}/(x_1^{a_1-1}) \otimes \mathbb{C}\{x_2\}/(x_2^{a_2-1}) \cdots \otimes \mathbb{C}\{x_n\}/(x_1^{a_n-1}), \end{aligned}$$

by (2.1), we have

$$\begin{aligned} \lambda(V) =& \dim(Der A(V)) \\ =& \sum_{k=1}^{n} \dim(\mathbb{C}\{x_1\}/(x_1^{a_1-1})) \cdots \dim(\mathbb{C}\{x_{k-1}\}/(x_{k-1}^{a_{k-1}-1})) \\ & \cdot \dim(Der(\mathbb{C}\{x_k\}/(x_k^{a_k-1}))) \cdot \dim(\mathbb{C}\{x_{k+1}\}/(x_{k+1}^{a_{k+1}-1})) \\ & \cdots \dim(\mathbb{C}\{x_n\}/(x_n^{a_n-1})). \end{aligned}$$

By Theorem 2.2, $\mathrm{Der}(\mathbb{C}\{x_k\}/(x_k^{a_k-1}))$ is spanned by $x_k^i \partial x_k$, $1 \leq i \leq a_k - 2$. So, $\dim(Der(\mathbb{C}\{x_k\}/(x_k^{a_k-1}))) = a_k - 2$, for $1 \leq k \leq n$. Notice that $\mathbb{C}\{x_k\}/(x_k^{a_k-1})$ has a basis $x_k^i, 0 \leq i \leq a_k - 2$. So, $\dim(\{x_k\}/(x_k^{a_k-1})) = a_k - 1$, $1 \leq k \leq n$. The result follows immediately. Q.E.D.

Proposition 4.2. *Let* $(V, 0)$ *be a binomial isolated singularity of type B which is defined by* $f = x_1^{a_1}x_2 + x_2^{a_2}$ *with weight type* $(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}; 1)$. *Then the Yau number*

$$\lambda(V) = 2a_1a_2 - 2a_1 - 3a_2 + 5.$$

And

$$\lambda(V) \leq 2(\frac{a_1a_2}{a_2 - 1} - 1)(a_2 - 1) - \frac{a_1a_2}{a_2 - 1} - a_2 + 2.$$

Proof. It is easy to see that the moduli algebra $A(V) = \mathbb{C}\{x_1, x_2\}/(f_{x_1}, f_{x_2})$ has dimension $a_2(a_1 - 1) + 1$ and has a monomial basis of the form ([AGV])

$$(4.1) \qquad \{x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq a_1 - 2; 0 \leq i_2 \leq a_2 - 1; x_1^{a_1-1}\},$$

with the following relations:

$$x_1^{a_1-1}x_2=0, \tag{4.2}$$
$$x_1^{a_1}+a_2x_2^{a_2-1}=0. \tag{4.3}$$

From the above (4.2) and (4.3) we get

$$x_1^{a_1+i}=-a_2x_1^ix_2^{a_2-1},\ 0\le i\le a_1-2, \tag{4.4}$$
$$x_1^i=0,\ i\ge 2a_1-1, \tag{4.5}$$
$$x_2^i=0,\ i\ge a_2. \tag{4.6}$$

In order to compute a derivation D of $A(V)$ it suffices to indicate its values on the generators x_1,x_2 which can be written in terms of the basis (4.1). Thus we can write

$$Dx_j=\sum_{i_1=0}^{a_1-2}\sum_{i_2=0}^{a_2-1}c^j_{i_1,i_2}x_1^{i_1}x_2^{i_2}+c^j_{a_1-1,0}x_1^{a_1-1},\ j=1,2.$$

Using the relations (4.2)-(4.6) one easily finds the necessary and sufficient conditions defining a derivation of $A(V)$ as follows:

$$c^1_{0,0}=c^1_{0,1}=\cdots=c^1_{0,a_2-3}=0; \tag{4.7}$$
$$c^2_{0,0}=c^2_{1,0}=\cdots=c^1_{a_1-2,0}=0; \tag{4.8}$$

(4.9)
$$a_1c^1_{1,0}=(a_2-1)c^2_{0,1},a_1c^1_{2,0}=(a_2-1)c^2_{1,1},\cdots,a_1c^1_{a_1-1,0}=(a_2-1)c^2_{a_1-2,1},$$
$$(a_1-1)c^1_{0,a_2-2}=a_2c^2_{a_1-1,0}. \tag{4.10}$$

Using (4.7)-(4.10) we obtain the following description of the Yau algebra in question.

The derivations represented by the following vector fields form a basis in Der$A(V)$:

$$\begin{gathered}(a_2-1)x_1\partial_1+a_1x_2\partial_2,(a_2-1)x_1^2\partial_1+a_1x_1x_2\partial_2,\cdots\\ ,(a_2-1)x_1^{a_1-1}\partial_1+a_1x_1^{a_1-2}x_2\partial_2,\\ a_2x_2^{a_2-2}\partial_1+(a_1-1)x_1^{a_1-1}\partial_2;\\ x_1^{i_1}x_2^{i_2}\partial_1,1\le i_1\le a_1-2,1\le i_2\le a_2-1;x_2^{a_2-1}\partial_1;\\ x_1^{i_1}x_2^{i_2}\partial_2,0\le i_1\le a_1-2,2\le i_2\le a_2-1.\end{gathered}$$

Therefore we have $\lambda(V) = 2a_1a_2 - 2a_1 - 3a_2 + 5$.

For the second part in question, we only need to prove that

$$2a_1a_2 - 2a_1 - 3a_2 + 5 \leq 2(\frac{a_1a_2}{a_2 - 1} - 1)(a_2 - 1) - \frac{a_1a_2}{a_2 - 1} - a_2 + 2,$$

which can be simplified to

$$(a_1 - 1)(a_2 - 2) \geq 1.$$

Since f defines a binomial singularity of type B, so $mult(f) \geq 3$, we have $a_1 \geq 2$ and $a_2 \geq 3$. Thus $(a_1 - 1)(a_2 - 2) \geq 1$, which gives the result. The equality happens if and only if $a_1 = 2$ and $a_2 = 3$.

Q.E.D.

Proposition 4.3. *Let* $(V, 0)$ *be a binomial isolated singularity of type C which is defined by* $f = x_1^{a_1}x_2 + x_2^{a_2}x_1$ *with weight type* $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}; 1)$. *If* $mult(f) \geq 4$, *i.e.,* $a_1, a_2 \geq 3$, *then the Yau number*

$$\lambda(V) = 2a_1a_2 - 2a_1 - 2a_2 + 6.$$

If $mult(f) = 3$, *i.e.,* $f = x_1^2x_2 + x_2^{a_2}x_1$, *then the Yau number is* $\lambda(V) = 2a_2$. *Furthermore,*

$$\lambda(V) \leq 2(\frac{a_1a_2 - 1}{a_1 - 1} - 1)(\frac{a_1a_2 - 1}{a_2 - 1} - 1) - \frac{a_1a_2 - 1}{a_1 - 1} - \frac{a_1a_2 - 1}{a_2 - 1} + 2.$$

Proof. It is easy to see that the moduli algebra $A(V) = \mathbb{C}\{x_1, x_2\}/(f_{x_1}, f_{x_2})$ has dimension a_1a_2 and has monomial basis of the form ([AGV])

$$\{x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq a_1 - 1; 0 \leq i_2 \leq a_2 - 1\}, \tag{4.11}$$

with the following relations:

$$a_1x_1^{a_1-1}x_2 + x_2^{a_2} = 0, \tag{4.12}$$

$$x_1^{a_1} + a_2x_1x_2^{a_2-1} = 0. \tag{4.13}$$

From the above (4.12) and (4.13) we get

$$x_1^{a_1}x_2 = x_1x_2^{a_2} = 0, \tag{4.14}$$

$$x_1^i = 0, \ i \geq 2a_1 - 1, \tag{4.15}$$

$$x_2^i = 0, \ i \geq 2a_2 - 1. \tag{4.16}$$

In order to compute a derivation D of $A(V)$ it suffices to indicate its values on the generators x_1, x_2 which can be written in the basis (4.11). Thus we can write

$$Dx_j = \sum_{i_1=0}^{a_1-1} \sum_{i_2=0}^{a_2-1} c^j_{i_1,i_2} x_1^{i_1} x_2^{i_2},\ j = 1, 2.$$

We assume that $mult(f) \geq 4$, then $a_1, a_2 \geq 3$. Using the relations (4.12)-(4.16) one finds the conditions defining a derivation of $A(V)$ as follows:

$$c^1_{0,0} = c^1_{0,1} = \cdots = c^1_{0,a_2-3} = 0; \tag{4.17}$$

$$c^2_{0,0} = c^2_{1,0} = \cdots = c^2_{a_1-3,0} = 0; \tag{4.18}$$

$$\begin{gathered}(a_1-1)c^1_{1,0} = (a_2-1)c^2_{0,1}, (a_1-1)c^1_{1,1} = (a_2-1)c^2_{0,2},\\ \cdots, (a_1-1)c^1_{1,a_2-2} = (a_2-1)c^2_{0,a_2-1};\end{gathered} \tag{4.19}$$

$$\begin{gathered}(a_1-1)c^1_{2,0} = (a_2-1)c^2_{1,1}, (a_1-1)c^1_{3,0} = (a_2-1)c^2_{2,1},\\ \cdots, (a_1-1)c^1_{a_1-1,0} = (a_2-1)c^2_{a_1-2,1};\end{gathered} \tag{4.20}$$

$$a_1c^1_{1,a_2-2} = a_2c^2_{a_1-1,0}, a_1c^1_{0,a_2-2} = a_2c^2_{a_1-2,0}, a_1c^1_{0,a_2-1} = a_2c^2_{a_1-2,0}. \tag{4.21}$$

Using (4.17)-(4.21) we obtain the following description of the Yau algebra in question.

The derivations represented by the following vector fields form a basis in $\mathrm{Der}A(V)$:

$$\begin{gathered}
x_1^{i_1}x_2^{i_2}\partial_1,\ 2 \leq i_1 \leq a_1-1, 1 \leq i_2 \leq a_2-2; x_1^{i_1}x_2^{a_2-1}\partial_1,\ 1 \leq i_1 \leq a_1-1;\\
x_1^{j_1}x_2^{j_2}\partial_2,\ 1 \leq j_1 \leq a_1-2, 2 \leq j_2 \leq a_2-1; x_1^{a_1-1}x_2^{j_2}\partial_2,\ 1 \leq j_2 \leq a_2-1;\\
(a_2-1)x_1x_2^{i}\partial_1 + (a_1-1)x_2^{i+1}\partial_2,\ 0 \leq i \leq a_2-2;\\
(a_2-1)x_1^{j+1}\partial_1 + (a_1-1)x_1^{j}x_2\partial_2,\ 1 \leq j \leq a_1-2;\\
a_2x_1x_2^{a_2-2}\partial_1 + a_1x_1^{a_1-1}\partial_2;\\
a_2x_2^{a_2-2}\partial_1 + a_1x_1^{a_1-2}\partial_2;\\
a_2x_2^{a_2-1}\partial_1 + a_1x_1^{a_1-2}x_2\partial_2.
\end{gathered}$$

Therefore we have $\lambda(V) = 2a_1a_2 - 2(a_1+a_2) + 6$.

We assume that $mult(f) = 3$, then $a_1 = 2, a_2 \geq 2$. Similarly, one obtains the derivations represented by the following vector fields form a basis in $\mathrm{Der}A(V)$:

$$x_1x_2^{a_2-1}\partial_1;$$
$$x_1x_2^j\partial_2,\ 1\le j\le a_2-1;$$
$$(a_2-1)x_1x_2^i\partial_1+x_2^{i+1}\partial_2,\ 0\le i\le a_2-2;$$
$$a_2x_1x_2^{a_2-2}\partial_1+2x_1\partial_2.$$

Therefore we have $\lambda(V)=2a_2$.

For the second part in question, we divide it into two cases to discuss.

Case 1. For $mult(f)\ge 4$, we only need to prove that

$$2a_1a_2-2(a_1+a_2)+6\le 2(\frac{a_1a_2-1}{a_1-1}-1)(\frac{a_1a_2-1}{a_2-1}-1) -\frac{a_1a_2-1}{a_1-1}-\frac{a_1a_2-1}{a_2-1}+2,$$

which can be simplified to

$$a_1+a_2-4\ge\frac{a_2-1}{a_1-1}+\frac{a_1-1}{a_2-1}.$$

Since $mult(f)\ge 4$, so we have $a_1\ge 3, a_2\ge 3$. It follows that

$$\frac{a_2-1}{a_1-1}+\frac{a_1-1}{a_2-1}\le\frac{a_2-1}{2}+\frac{a_1-1}{2}\le\frac{a_1+a_2-2}{2}\le a_1+a_2-4,$$

which gives the result. The equality happens if and only if $a_1=3$ and $a_2=3$.

Case 2, For $mult(f)=3$, we only need to prove that

$$2a_2\le 4a_2-(2a_2-1)-\frac{2a_2-1}{a_2-1}+2,$$

which can be simplified to $1-\frac{1}{a_2-1}\ge 0$ which is obvious true since we have $a_2\ge 2$. The equality happens if and only if $a_2=2$. Q.E.D.

Proof of Main Theorem A.

Proof. Since $f\in\mathbb{C}\{x_1,x_2\}$ is a binomial isolated singularity, by Corollary 3.1, f can be classified into the following three types:

Type A. $x_1^{a_1}+x_2^{a_2}$,

Type B. $x_1^{a_1}x_2+x_2^{a_2}$,

Type C. $x_1^{a_1}x_2+x_2^{a_2}x_1$.

The Main Theorem A is an immediate corollary of Propositions 4.1, 4.2, and 4.3. Q.E.D.

Proof of Main Theorem B.

Proof.

$$\begin{aligned}
&\lambda(V_{f+g})\\
&= \dim \mathrm{Der}\mathbb{C}\{x_1,\cdots,x_n,y_1,\cdots,y_m\}\Big/\Big(f+g,\frac{\partial(f+g)}{\partial x_1},\cdots,\frac{\partial(f+g)}{\partial y_m}\Big)\\
&= \dim \mathrm{Der}\mathbb{C}\{x_1,\cdots,x_n,y_1,\cdots,y_m\}\Big/\Big(f+g,\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n},\\
&\qquad\qquad \frac{\partial g}{\partial x_1}\cdots,\frac{\partial g}{\partial y_m}\Big)\\
&= \dim \mathrm{Der}\mathbb{C}\{x_1,\cdots,x_n,y_1,\cdots,y_m\}\Big/\Big(\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n},\frac{\partial g}{\partial x_1}\cdots,\frac{\partial g}{\partial y_m}\Big)\\
&= \dim \mathrm{Der}(\mathbb{C}\{x_1,\cdots,x_n\}\Big/\Big(\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n}\Big)\\
&\qquad\qquad \otimes\mathbb{C}\{y_1,\cdots,y_m\}\Big/\Big(\frac{\partial g}{\partial y_1},\cdots,\frac{\partial g}{\partial y_m}\Big))\\
&= \dim\Big\{\mathrm{Der}(\mathbb{C}\{x_1,\cdots,x_n\}\Big/\Big(\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n}\Big))\\
&\qquad\qquad \otimes\mathbb{C}\{y_1,\cdots,y_m\}\Big/\Big(\frac{\partial g}{\partial y_1},\cdots,\frac{\partial g}{\partial y_m}\Big)\\
&+\mathbb{C}\{x_1,\cdots,x_n\}\Big/\Big(\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n}\Big)\\
&\qquad\qquad \otimes\mathrm{Der}(\mathbb{C}\{y_1,\cdots,y_m\}\Big/\Big(\frac{\partial g}{\partial y_1},\cdots,\frac{\partial g}{\partial y_m}\Big))\Big\}\\
&= \mu(V_f)\lambda(V_g)+\mu(V_g)\lambda(V_f).
\end{aligned}$$

The third equality above comes from the fact that f, g are weighted homogeneous while the fifth equality follows from the Theorem 2.1.

Since both f and g saitisfy the conjecture, we have

$$\lambda(V_f)\le n\mu(f)-\sum_i^n(\frac{1}{w_1}-1)(\frac{1}{w_2}-1)\cdots\widehat{(\frac{1}{w_i}-1)}\cdots(\frac{1}{w_n}-1), \tag{4.22}$$

where $\widehat{(\frac{1}{w_i}-1)}$ denotes the omission of $\frac{1}{w_i}-1$.

(4.23)

$$\lambda(V_g) \leq m\mu(g) - \sum_{i}^{n}(\frac{1}{w_{n+1}} - 1)(\frac{1}{w_{n+2}} - 1)\cdots(\widehat{\frac{1}{w_{n+i}} - 1})\cdots(\frac{1}{w_{n+m}} - 1).$$

By Theorem 1.1, we have

$$\mu(V_f) = \prod_{i=1}^{n}(\frac{1}{w_i} - 1), \tag{4.24}$$

$$\mu(V_g) = \prod_{i=1}^{n+m}(\frac{1}{w_{n+i}} - 1). \tag{4.25}$$

From above we can see that $A(V_{f+g}) = A(V_f) \otimes A(V_g)$, thus $\mu(V_{f+g}) = \mu(V_f)\mu(V_g)$. Combining this fact with the (4.22), (4.23), (4.24), and (4.25), we have

$$\begin{aligned}\lambda(V_{f+g}) &= \mu(V_f)\lambda(V_g) + \mu(V_g)\lambda(V_f) \\ &\leq (n+m)\mu(f+g) - \sum_{i}^{n+m}(\frac{1}{w_1} - 1)(\frac{1}{w_2} - 1)\cdots(\widehat{\frac{1}{w_i} - 1}) \\ &\qquad \cdots(\frac{1}{w_{n+m}} - 1),\end{aligned}$$

which shows that $f+g$ satisfy the conjecture. Q.E.D.

Proof of Main Theorem C.

Proof. It follows from Proposition 4.1, Main Theorems A and B immediatelty. Q.E.D.

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