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On the Sharp Polynomial Upper Estimate Conjecture in Eight-Dimensional Simplex

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Dedicated to Professor Eduard Looijenga on the occasion of his 70th birthday

Abstract: The estimate of integral points in an n-dimensional polyhedron has many applications in singularity theory, number theory and toric geometry. The third author formulated Yau Number Theoretic Conjecture which gives a sharp polynomial upper estimate on the number of positive integral points in n-dimensional ($n \geq 3$) real right-angled simplices. The previous results on the conjecture in low dimension cases ($n \leq 6$) have been proved by using the sharp GLY conjecture. However, it is only valid up to 6 dimension. The Yau Number Theoretic Conjecture for $n = 7$ has been shown with a completely new method in [22]. In this paper, on the one hand, we use similar method to prove the conjecture for $n = 8$ case, but with more meticulous analyses. The main method of proof is summing existing sharp upper bounds for the number of points in seven-dimensional simplex over the cross sections of eight-dimensional simplex. This is a significant progress since it sheds light on proving the Yau Number Theoretic Conjecture in general case. On the other hand, we give a new sharper estimate of the Dickman-De Bruijn function $\psi(x, y)$ for $5 \leq y < 23$, compared with the result obtained by Ennola [5].
Keywords: Sharp upper estimate, integral points, prime decomposition, simplex.

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1. Introduction

Let $T(a_1, a_2, \ldots, a_n)$ be an n-dimensional simplex defined by the inequality

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leq 1, \tag{1}$$

where $x_1, x_2, \ldots, x_n \geq 0$ and $a_1 \geq a_2 \geq \cdots \geq a_n \geq 1$ are real numbers. Let $P_n = P(a_1, a_2, \ldots, a_n)$ and $Q_n = Q(a_1, a_2, \ldots, a_n)$ be defined as the number of positive and nonnegative integer solutions of (1), respectively. If we let $a = \frac{1}{a_1} + \cdots + \frac{1}{a_n}$, then

$$Q(a_1, \ldots, a_n) = P(a_1(1+a), \ldots, a_n(1+a)).$$

Therefore, the study of P_n and Q_n are equivalent.

The general problems of counting the numbers P_n and Q_n have been a challenging problem for many years. Tremendous researches have being done to develop the exact formula for P_n or Q_n, in case $a_1, \ldots, a_n$ are integers. Mordell [12] gave an exact formula for Q_3 with a_1, a_2 and a_3 relatively prime. Pommersheim [14] extended this result to arbitrary integers a_1, a_2 and a_3 using toric variety techniques and a result of Ehrhart [4]. The exact formula is complicated, it involves large generalized Dedekind sum. Therefore, we need a sharp upper estimate of P_n in terms of a polynomial in $a_1, \ldots, a_n$. The polynomial upper estimate of P_n is a very active research topic and has many applications ([6], [8], [9], [10]). On another seemly unrelated research field, such estimates have important applications in singularity theory ([1], [7], [8]).

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function with an isolated critical point at the origin and $V = \{z \in \mathbb{C}^n : f(z) = 0\}$. The geometric genus p_g is defined to be $\dim \Gamma(V - \{0\}, \Omega^{n-1})/L^2(V - \{0\}, \Omega^{n-1})$, where Ω^{n-1} is the sheaf of germs of holomorphic $(n-1)$-forms on $V - \{0\}$. And the Milnor number $\mu := \dim \mathbb{C}\{z_1, \ldots, z_n\}/(f_{z_1}, \ldots, f_{z_n})$ where $f_{z_i} := \partial f / \partial z_i$.

In 1978, Durfee [3] made the following conjecture:

Durfee Conjecture. *Let $(V, 0)$ be an isolated hypersurface singularity defined by a holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. Let μ be the Milnor number and p_g be the geometric genus of $(V, 0)$. Then $n! p_g \leq \mu$.*

If $f(z_1, \ldots, z_n)$ is a weighted homogeneous polynomial of type $(w_1, \ldots, w_n)$, where $w_1, \ldots, w_n$ are fixed positive rational numbers, i.e., f can be expressed as a linear combination of monomials $z_1^{i_1} \ldots z_n^{i_n}$ for which $i_1/w_1 + \cdots + i_n/w_n = 1$, then Merle and Teissier [13] showed that p_g is exactly the number $P(w_1, \ldots, w_n)$. Milnor and Orlik [11] proved that $\mu = (w_1 - 1) \ldots (w_n - 1)$. Therefore Durfee Conjecture is a special case of the following theorem, which was proved by Yau and Zhang [21]:

Theorem 1.1 (GLY rough estimate). *Let $a_1, \ldots, a_n$ be positive real numbers greater than or equal to 1 and $n \geq 3$. Then*

$$n!P(a_1, \ldots, a_n) < (a_1 - 1)(a_2 - 1) \ldots (a_n - 1). \tag{2}$$

However, the estimate in the above theorem is not sharp enough to provide a solution of the following problem:

Problem. *(cf. [24], [25]) Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function with an isolated critical point at the origin. Find an intrinsic characterization for f to be a homogeneous polynomial after a biholomorphic change of coordinates.*

In 1971, Saito [16] gave an intrinsic characterization for f to be a weighted homogeneous polynomial.

Theorem 1.2 (Saito). *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function with an isolated critical point at the origin. Then f is a weighted homogeneous polynomial after a biholomorphic change of coordinates if and only if $\mu = \tau$, where μ is the Milnor number and*

$$\tau = \dim \mathbb{C}\{z_1, \ldots, z_n\}/(f, f_{z_1}, \ldots, f_{z_n})$$

is the Tjurina number of the singularity.

The above Saito's theorem is generalized to isolated smoothable Gorenstein surface singularities in [2].

To find a necessary and sufficient condition for f to be a homogeneous polynomial, the third author made the following conjecture [7, 21]:

Yau Geometric Conjecture. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a weighted homogeneous polynomial with an isolated singularity at the origin. Let μ,*

p_g and ν be the Milnor number, geometric genus and multiplicity of singularity $V = \{z : f(z) = 0\}$, respectively. Then

$$\mu - h(\nu) \geq n!p_g \tag{3}$$

where $h(\nu) = (\nu-1)^n - \nu(\nu-1)\dots(\nu-n+1)$. The equality holds if and only if f is a homogeneous polynomial after a biholomorphic change of coordinates.

This conjecture together with Theorem 1.2 give an intrinsic characterization for a holomorphic function f to be a homogeneous polynomial after a biholomorphic change of coordinates. In order to prove this conjecture, Lin, Yau [8] and Granville formulated

Conjecture (GLY Sharp Estimate). *Let $n \geq 3$. If $a_1 \geq a_2 \geq \cdots \geq a_n \geq n-1$. Then*

$$n!P_n \leq f_n := A_0^n + \frac{s(n,n-1)}{n}A_1^n + \sum_{l=1}^{n-2} \frac{s(n,n-1-l)}{\binom{n-1}{l}} A_l^{n-1} \tag{4}$$

where $s(n,k)$ is the Stirling number of the first kind defined by the following generating function:

$$x(x-1)\dots(x-n+1) = \sum_{k=0}^{n} s(n,k)x^k$$

and A_k^n is defined as

$$A_k^n = (\prod_{i=1}^{n} a_i)(\sum_{1\leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n} \frac{1}{a_{i_1}a_{i_2}\dots a_{i_k}})$$

for $k = 1, 2, \dots, n-1$. Equality in (4) holds if and only if $a_1 = \cdots = a_n$ are integers.

The GLY Sharp Estimate conjecuture is verified for $n = 4, 5, 6$ (cf. [23], [1]) and there is a counter-example for $n = 7$ [18]. Therefore, Wang and Yau [18] modified it as follows:

Conjecture (Modified GLY Conjecture). *There exists an integer $y(n)$ which depends only on n such that the sharp estimate (4) holds when $a_1 \geq a_2 \geq \cdots \geq a_n \geq y(n)$.*

In order to overcome the difficulty that GLY sharp estimate is only true when a_n is larger than $y(n)$, a positive integer depending on n, the third named author proposed a new sharp upper estimate which is motivated by the Yau Geometric Conjecture, The importance of this conjecture is that we only need $a_n > 1$ and hence it is more useful.

Yau Number Theoretic Conjecture. *Let*

$$P_n = P_n(a_1, a_2, \ldots, a_n) = \#\{(x_1, \ldots, x_n) \in \mathbb{Z}_+^n : \frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leq 1\},$$

where $n \geq 3$, $a_1 \geq a_2 \geq \cdots \geq a_n > 1$ are real numbers. If $P_n > 0$, then

$$n!P_n \leq \prod_{i=1}^{n}(a_i - 1) - (a_n - 1)^n + \prod_{i=0}^{n-1}(a_n - i) \tag{5}$$

and equality holds if and only if $a_1 = a_2 = \cdots = a_n$ are integers.

Obviously, there is an intimate relation between the Yau Geometric Conjecture and the Yau Number Theoretic Conjecture. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a weighted homogeneous polynomial with an isolated singularity at the origin. The multiplicity ν of f at the origin is given by $\inf\{n \in \mathbb{Z}_+ : n \geq \inf\{w_1, \ldots, w_n\}\}$, where w_i is the weight of x_i and a rational number. In case the minimal weight is an integer, the two conjectures are the same. However, in general, these two conjectures do not imply each other, although they are intimately related.

The Yau Number Theoretic Conjecture has already been verified for $n \leq 7$ (cf. [19], [20], [6], [9], [10], [22]).

Theorem 1.3 ([22]). *Let $a_1 \geq a_2 \geq \cdots \geq a_7 > 1$ be real numbers. Let P_7 be the number of positive integral solutions of $\frac{x_1}{a_1} + \cdots + \frac{x_7}{a_7} \leq 1$. If $P_7 > 0$, then*

$$5040P_7 \leq g_7 := \prod_{i=1}^{7}(a_i - 1) - (a_7 - 1)^7 + \prod_{j=0}^{6}(a_7 - j) \tag{6}$$

and equality holds if and only if $a_1 = a_2 = \cdots = a_7 \in \mathbb{Z}$.

In this paper, we will prove the conjecture for $n = 8$:

Theorem 1.4 (Main Theorem A). *Let* $P_8 = P_8(a_1, a_2, \ldots, a_8) = \#\{(x_1, \ldots, x_8) \in \mathbb{Z}_+^8 : \frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_8}{a_8} \leq 1\}$, *where* $a_1 \geq a_2 \geq \cdots \geq a_8 > 1$ *are real numbers. If* $P_8 > 0$, *then*

$$\begin{aligned} 8!P_8 \leq &(a_1-1)(a_2-1)\ldots(a_8-1)-(a_8-1)^8 \\ &+a_8(a_8-1)\ldots(a_8-7) \end{aligned} \tag{7}$$

and equality holds if and only if $a_1 = a_2 = \cdots = a_8$ *are integers.*

The proofs of the Yau Number Theoretic Conjecture for $n \leq 6$ relied on the GLY sharp estimate, which is only true for $n \leq 6$. Therefore the proofs cannot be generalized to higher dimension. The third and fourth author [22] verified the conjecture for $n = 7$ using a completely new method. In this paper, the similar method has been applied to prove the conjecture for $n = 8$, but with more meticulous analyses. This significant progress takes us one step closer to prove the Yau Number Theoretic Conjecture in full generality.

On the other hand, such estimates could also be applied in number theory. Recall that a *smooth number* is a number with only small prime factors. Given an integer y, the number $m = p_1^{l_1} p_2^{l_2} \ldots p_n^{l_n}$ is called y-smooth if all its prime factor $p_i \leq y$ for $i = 1, \ldots, n$. Smooth numbers play important roles in factoring and primality testing [15]. It is of great importance to know the number of y-smooth integers less than or equal to x, which is denoted by $\psi(x, y)$, called the Dickman-De Bruijn function. One of the central topics in computational number theory is the estimate of $\psi(x, y)$ ([10]). In fact, the computation of $\psi(x, y)$ is equivalent to compute the number of integral points in an k-dimensional tetrahedron $\triangle(a_1, a_2, \cdots, a_k)$ with real vertices $(a_1, 0, \cdots, 0), \cdots, (0, \cdots, 0, a_k)$. Let $p_1 < p_2 < \cdots < p_k$ denote the primes up to y. Suppose that $p_1^{l_1} p_2^{l_2} \cdots p_k^{l_k} \leq x$, computation of $\psi(x, y)$ is equivalent to counting the number of $(l_1, l_2, \cdots, l_k) \in \mathbb{Z}_{\geq 0}^n$ such that

$$\frac{l_1}{a_1} + \frac{l_2}{a_2} + \cdots + \frac{l_k}{a_k} \leq 1, \text{ where } a_i = \frac{\log x}{\log p_i}.$$

Therefore, $\psi(x, y)$ is precisely the number Q_k of (integer) lattice points inside the n-dimensional tetrahedron (1) with $a_i = \frac{\log x}{\log p_i}$, $n = k$, and $1 \leq i \leq k$.

Ennola [5] gave both lower and upper bounds for the $\psi(x,y)$:

$$\frac{(\log x)^k}{k!\prod_{i=1}^{k}\log p_i} < \psi(x,y) \le \frac{(\log x+\sum_{i=1}^{k}\log p_i)^k}{k!\prod_{i=1}^{k}\log p_i} \tag{8}$$

which yields

Theorem 1.5. *(Ennola, [5]) Uniformly for $2 \le y \le \sqrt{\log x \log_2 x}$, we have that*

$$\psi(x,y) = \frac{1}{k!}\prod_{p\le y}\left(\frac{\log x}{\log p}\right)\left[1+O\left(\frac{y^2}{\log x \log y}\right)\right].$$

As an application, we give a new sharper estimate of the Dickman-De Bruijn function $\psi(x,y)$ for $5 \le y < 23$, compared with the result obtained by Ennola. We prove that

Theorem 1.6. *(Main Theorem B, Estimate of $\psi(x,y)$) Let $\psi(x,y)$ be the function as before. We have the following upper estimate for $5 \le y < 23$:*

(I) when $5 \le y < 7$ and $x > 5$, we have

$$\begin{aligned}\psi(x,y) \le &\frac{1}{6}\{\frac{1}{\log 2\log 3\log 5}(\log x+\log 15)(\log x+\log 10)(\log x+\log 6)\\ &-\frac{1}{\log^3 5}[(\log x+\log 6)^3-(\log x+\log 6+\log 5)(\log x+\log 6)\\ &(\log x+\log 6-\log 5)]\};\end{aligned}$$

(II) when $7 \le y < 11$ and $x > 7$, we have

$$\begin{aligned}\psi(x,y) \le &\frac{1}{24}\{\frac{1}{\log 2\log 3\log 5\log 7}(\log x+\log 105)(\log x+\log 70)\\ &(\log x+\log 42)(\log x+\log 30)\\ &-\frac{1}{\log^4 7}[(\log x+\log 30)^4-(\log x+\log 7+\log 30)\\ &(\log x+\log 30)(\log x+\log 30-\log 7)\\ &(\log x+\log 30-2\log 7)]\};\end{aligned}$$

(III) when $11 \le y < 13$ and $x > 11$, we have

$$\psi(x,y) \le \frac{1}{120}\{\frac{1}{\log 2\log 3\log 5\log 7\log 11}(\log x+\log 1155)$$

$$(\log x+\log 770)(\log x+\log 462)(\log x+\log 330)(\log x+\log 210)$$
$$-\frac{1}{\log^5 11}[(\log x+\log 210)^5-(\log x+\log 11+\log 210)$$
$$(\log x+\log 210)(\log x+\log 210-\log 11)(\log x+\log 210-2\log 11)$$
$$(\log x+\log 210-3\log 11)]\}.$$

(IV) when $13 \le y < 17$ and $x > 13$, we have

$$\psi(x,y) \le \frac{1}{720}\{\frac{1}{\log 2\log 3\log 5\log 7\log 11\log 13}(\log x+\log 15015)$$
$$(\log x+\log 10010)(\log x+\log 6006)(\log x+\log 4290)$$
$$(\log x+\log 2730)(\log x+\log 2310)$$
$$-\frac{1}{\log^6 13}[(\log x+\log 2310)^6-(\log x+\log 13+\log 2310)$$
$$(\log x+\log 2310)(\log x+\log 2310-\log 13)$$
$$(\log x+\log 2310-2\log 13)(\log x+\log 2310-3\log 13)$$
$$(\log x+\log 2310-4\log 13)]\}.$$

(V) when $17 \le y < 19$ and $x > 17$, we have

$$\psi(x,y) \le \frac{1}{5040}\{\frac{1}{\log 2\log 3\log 5\log 7\log 11\log 13\log 17}(\log x+\log 255255)$$
$$(\log x+\log 170170)(\log x+\log 102102)(\log x+\log 72930)$$
$$(\log x+\log 46410)(\log x+\log 39270)(\log x+\log 30030)$$
$$(\log x+\log 30030)^7-(\log x+\log 17+\log 30030)(\log x+\log 30030)$$
$$(\log x+\log 30030-\log 17)(\log x+\log 30030-2\log 17)$$
$$(\log x+\log 30030-3\log 17)(\log x+\log 30030-4\log 17)$$
$$(\log x+\log 30030-5\log 17)]\}.$$

(VI) when $19 \le y < 23$ and $x > 19$, we have

$$\psi(x,y) \le \frac{1}{40320}\{\frac{1}{\log 2\log 3\log 5\log 7\log 11\log 13\log 17\log 19}$$
$$(\log x+\log 4849845)(\log x+\log 3233230)(\log x+\log 1939938)$$
$$(\log x+\log 1385670)(\log x+\log 881790)$$
$$(\log x+\log 746130)(\log x+\log 570570)$$
$$(\log x+\log 510510)-\frac{1}{\log^8 19}[(\log x+\log 570570)^8-$$

$(\log x+\log 19+\log 570570)(\log x+\log 570570)$
$(\log x+\log 570570-\log 19)(\log x+\log 570570-2\log 19)$
$(\log x+\log 570570-3\log 19)(\log x+\log 570570-4\log 19)$
$(\log x+\log 570570-5\log 19)(\log x+\log 570570-6\log 19)]\}.$

Remark. *For comparison, the Ennola's upper bounds (see (8)) for* $5\le y<23$ *are listed:*

(1): $5\le y<7$ *and* $x>5$,

$$\psi(x,y)\le\frac{(\log x+\log 30)^3}{6\log 2\log 3\log 5}$$

(2): $7\le y<11$ *and* $x>7$,

$$\psi(x,y)\le\frac{(\log x+\log 210)^4}{24\log 2\log 3\log 5\log 7}$$

(3): $11\le y<13$ *and* $x>11$,

$$\psi(x,y)\le\frac{(\log x+\log 2310)^5}{120\log 2\log 3\log 5\log 7\log 11}$$

(4): $13\le y<17$ *and* $x>13$,

$$\psi(x,y)\le\frac{(\log x+\log 30030)^6}{720\log 2\log 3\log 5\log 7\log 11\log 13}$$

(5): $17\le y<19$ *and* $x>17$,

$$\psi(x,y)\le\frac{(\log x+\log 510510)^7}{5040\log 2\log 3\log 5\log 7\log 11\log 13\log 17}$$

(6): $19\le y<23$ *and* $x>19$,

$$\psi(x,y)\le\frac{(\log x+\log 9699690)^8}{40320\log 2\log 3\log 5\log 7\log 11\log 13\log 17\log 19}.$$

It is easy to see that our upper bound of $\psi(x,y)$ *is substantially better than the one obtained by Ennola. For example, in* $19\le y<23$ *and* $x>19$ *case, though the coefficient of* $(\log x)^8$ *in our estimate is same as Ennola's, but our coefficient of* $(\log x)^7$ *is*

$$\frac{1}{40320}[\frac{1}{\log 2\log 3\ldots\log 19}(\log 4849845+\log 3233230+\log 1939938+\log 1385670$$

$$+\log 881790+\log 746130+\log 570570+\log 510510)-\frac{20}{\log^7 19}]\approx 0.007744154691$$

which is smaller than Ennola's

$$\frac{1}{40320}\frac{8\log 9699690}{\log 2\log 3\log 5\log 7\log 11\log 13\log 17\log 19}\approx 0.008950128404.$$

We use Maple 18, a symbolic computation software, to deal with tremendous involved computation.

2. Two Lemmas

The following two lemmas can be used to identify the positivity of polynomials in some restricted domains. We shall use it frequently in the proof of the Main Theorem A.

Lemma 2.1 ([18] Lemma 3.1). *Let $f(\beta)$ be a polynomial defined by*

$$f(\beta)=\sum_{i=0}^{n}c_i\beta^i \tag{9}$$

where $\beta\in(0,1)$. If for any $k=0,1,\dots,n$

$$\sum_{i=0}^{k}c_i\geq 0 \tag{10}$$

then $f(\beta)\geq 0$ for $\beta\in(0,1)$.

Lemma 2.1 is easy to use. However, the condition of Lemma 2.1 is too strong. In general case, we can use the following lemma.

Lemma 2.2 (Sturm's Theorem). *Starting from a given polynomial $X=f(x)$, let the polynomials X_1, X_2, $\dots$, X_r be determined by Euclidean algorithm as follows:*

$$\begin{array}{rcl} X_1 & = & f'(x) \qquad , \\ X & = & Q_1X_1-X_2, \\ X_1 & = & Q_2X_2-X_3, \\ \cdots & \cdots & \cdots\cdots \end{array} \tag{11}$$

$$X_{r-1} \quad = \quad Q_r X_r$$

where $\deg X_k > \deg X_{k+1}$ *for* $k = 1, \ldots, r-1$. *For every real number* a *which is not a root of* $f(x)$ *let* $w(a)$ *be the number of variations in sign in the number sequence*

$$X(a), X_1(a), \ldots, X_r(a)$$

in which all zeros are omitted. If b *and* c *are any numbers* $(b < c)$ *for which* $f(x)$ *does not vanish, then the number of the various roots in the interval* $b \le x \le c$*(multiple roots to be counted only once) is equal to*

$$w(b) - w(c).$$

Proof. See [17]. □

It can be seen that the computation in Lemma 2.2 is more complicated than that in Lemma 2.1. We prefer Lemma 2.1 when it works.

3. Proof of the Main Theorem A

We will prove the Main Theorem A (i.e. Theorem 1.4) by induction. Let

$$\begin{aligned} g_n(a_1, \ldots, a_n) := (a_1 - 1) \ldots (a_n - 1) - (a_n - 1)^n \\ + a_n(a_n - 1) \ldots (a_n - (n-1)) \end{aligned}$$

be the right hand of (5). Let k be the possible integer such that $1 \le k \le \lfloor a_n \rfloor$, where $\lfloor a_n \rfloor$ is the biggest integer less than or equal to a_n. For each k, we have an $(n-1)$-dimensional simplex

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{k}{a_n} \le 1, x_1, x_2, \ldots, x_{n-1} \ge 0. \tag{12}$$

Let $P_{n-1}^{(k)}$ be the number of positive integer solution of (12). Clearly,

$$P_n = \sum_{k=1}^{\lfloor a_n \rfloor} P_{n-1}^{(k)}. \tag{13}$$

Since we already know that the Yau Number Theoretic Conjecture is true for $n = 7$ by Theorem 1.3, we want to prove that $g_8(a_1, \ldots, a_7)$ is greater than or equal to the sum of g_7's, the upper estimate of seven-dimensional layers in $T(a_1, a_2, \ldots, a_8)$.

Let m be number of seven-dimensional layers in eight-dimensional simplex, i.e. $P_7^{(m)} > 0$ and $P_7^{(m+1)} = 0$, where $P_7^{(k)} = \#\{(x_1, \ldots, x_7) \in \mathbb{Z}_+^7 : \frac{x_1}{a_1} + \cdots + \frac{x_7}{a_7} + \frac{k}{a_8} \leq 1\}$, where $1 \leq k \leq m$, $a_1 \geq a_2 \geq \cdots \geq a_8 > 1$ are real numbers. Let

$$\begin{aligned}
\Delta_m \quad := \quad & g_8(a_1, \ldots, a_8) - 8\sum_{k=1}^{m} g_7(a_1(1-\frac{k}{a_8}), \ldots, a_7(1-\frac{k}{a_8})) \\
= \quad & (a_1-1)\ldots(a_8-1) - (a_8-1)^8 + a_8(a_8-1)\ldots(a_8-7) \\
& -8[\sum_{k=1}^{m}(a_1(1-\frac{k}{a_8})-1)\ldots(a_7(1-\frac{k}{a_8})-1) - (a_7(1-\frac{k}{a_8})-1)^7 \\
& +a_7(1-\frac{k}{a_8})(a_7(1-\frac{k}{a_8})-1)\ldots(a_7(1-\frac{k}{a_8})-6)]
\end{aligned}$$

be the difference between $g_8(a_1, \ldots, a_8)$ and the sum of g_7's. We should prove that $\Delta_m \geq 0$ under the condition of the Main Theorem A.

Since $P_7^{(m)} = \#\{(x_1, \ldots, x_7) \in \mathbb{Z}_+^7 : \frac{x_1}{a_1} + \cdots + \frac{x_7}{a_7} + \frac{m}{a_8} \leq 1\}$, let $\alpha = 1 - \frac{m}{a_8} \in (0,1)$, $A_i = a_i\alpha$, for $i = 1, \ldots, 7$, then we have

$$\frac{x_1}{A_1} + \frac{x_2}{A_2} + \cdots + \frac{x_7}{A_7} \leq 1 \tag{14}$$

and

$$\begin{aligned}
g_7(m) := & \sum_{k=1}^{m} g_7(\frac{m-k+k\alpha}{m\alpha}A_1, \ldots, \frac{m-k+k\alpha}{m\alpha}A_7) \\
\Delta_m(A_1, \ldots, A_7, \alpha) = & \; g_8(\frac{A_1}{\alpha}, \ldots, \frac{A_7}{\alpha}, \frac{m}{1-\alpha}) - 8g_7(m).
\end{aligned}$$

Let $B_{7,k}$ be $e_k(A_1, \ldots, A_7)$, the elementary symmetric polynomial, for $k = 0, \ldots, 7$, that is, $B_{7,k} = A_1 \ldots A_7 \sum_{1 \leq i_1 < \cdots \leq i_k \leq 7} \frac{1}{A_{i_1} \ldots A_{i_k}}$. For example, $B_{7,0} = A_1 \ldots A_7$, $B_{7,6} = A_1 + \cdots + A_7$ and $B_{7,7} = 1$. Then

$$\begin{aligned}
g_7(m) = & \sum_{k=1}^{m}(\frac{m-k+k\alpha}{m\alpha}A_1 - 1)\ldots(\frac{m-k+k\alpha}{m\alpha}A_7 - 1) \\
& - \sum_{k=1}^{m}(\frac{m-k+k\alpha}{m\alpha}A_7 - 1)^7
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{m}\frac{m-k+k\alpha}{m\alpha}A_7(\frac{m-k+k\alpha}{m\alpha}A_7-1)\dots(\frac{m-k+k\alpha}{m\alpha}A_7-6)\\
=&\sum_{k=1}^{m}[(\frac{m-k+k\alpha}{m\alpha})^7B_{7,0}-(\frac{m-k+k\alpha}{m\alpha})^6B_{7,1}+(\frac{m-k+k\alpha}{m\alpha})^5B_{7,2}\\
&-(\frac{m-k+k\alpha}{m\alpha})^4B_{7,3}+(\frac{m-k+k\alpha}{m\alpha})^3B_{7,4}-(\frac{m-k+k\alpha}{m\alpha})^2B_{7,5}\\
&+(\frac{m-k+k\alpha}{m\alpha})B_{7,6}+B_{7,7}]\\
&+\sum_{k=1}^{m}[-14(\frac{m-k+k\alpha}{m\alpha})^6A_7^6+154(\frac{m-k+k\alpha}{m\alpha})^5A_7^5\\
&-700(\frac{m-k+k\alpha}{m\alpha})^4A_7^4+1589(\frac{m-k+k\alpha}{m\alpha})^3A_7^3\\
&-1743(\frac{m-k+k\alpha}{m\alpha})^2A_7^2+713(\frac{m-k+k\alpha}{m\alpha})A_7+1].
\end{aligned}
$$

To make $g_7(m)$ a polynomial of m, we must transform the function to avoid the appearance of m in the sum symbol. Let

$$S_q:=\sum_{k=1}^{m}(\frac{m-k+k\alpha}{m\alpha})^q,$$

for $q=1,\dots,7$. We will use the first seven S_q in the later computation:

$$
\begin{aligned}
S_1=&\frac{1}{m\alpha}[\frac{1}{2}m(m+1)\alpha+\frac{1}{2}m(m-1)]\\
S_2=&(\frac{1}{m\alpha})^2[\frac{1}{6}m(m+1)(2m+1)(\alpha-1)^2+m^2(m+1)(\alpha-1)+m^3]\\
S_3=&(\frac{1}{m\alpha})^3[\frac{1}{4}m^2(m+1)^2(\alpha-1)^3+\frac{1}{2}m^2(m+1)(2m+1)(\alpha-1)^2\\
&+\frac{3}{2}m^3(m+1)(\alpha-1)+m^4]\\
S_4=&(\frac{1}{m\alpha})^4[\frac{1}{30}m(m+1)(2m+1)(3m^2+3m-1)(\alpha-1)^4\\
&+m^3(m+1)^2(\alpha-1)^3+m^3(m+1)(2m+1)(\alpha-1)^2\\
&+2m^4(m+1)(\alpha-1)+m^5]\\
S_5=&(\frac{1}{m\alpha})^5[\frac{1}{12}m^2(m+1)^2(2m^2+2m-1)(\alpha-1)^5\\
&+\frac{1}{6}m^2(m+1)(2m+1)(3m^2+3m-1)(\alpha-1)^4
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{5}{2}m^4(m+1)^2(\alpha-1)^3+\frac{5}{3}m^4(m+1)(2m+1)(\alpha-1)^2\\
&+\frac{5}{2}m^5(m+1)(\alpha-1)+m^6]\\
S_6=&(\frac{1}{m\alpha})^6[\frac{1}{42}m(m+1)(2m+1)(3m^4+6m^3-3m+1)\alpha^6\\
&+\frac{1}{2}m^3(m+1)^2(2m^2+2m-1)(\alpha-1)^5\\
&+\frac{1}{2}m^3(m+1)(2m+1)(3m^2+3m-1)(\alpha-1)^4\\
&+5m^5(m+1)^2(\alpha-1)^3+\frac{5}{2}m^5(m+1)(2m+1)(\alpha-1)^2\\
&+3m^6(m+1)(\alpha-1)+m^7]\\
S_7=&(\frac{1}{m\alpha})^7[\frac{1}{24}m^2(3m^4+6m^3-m^2-4m+2)(m+1)^2(\alpha-1)^7\\
&+\frac{1}{6}m^2(2m+1)(m+1)(3m^4+6m^3-3m+1)(\alpha-1)^6\\
&+\frac{7}{4}m^4(2m^2+2m-1)(m+1)^2(\alpha-1)^5\\
&+\frac{7}{6}m^4(2m+1)(m+1)(3m^2+3m-1)(\alpha-1)^4\\
&+\frac{35}{4}m^6(m+1)^2(\alpha-1)^3+\frac{7}{2}m^6(2m+1)(m+1)(\alpha-1)^2\\
&+\frac{7}{2}m^7(m+1)(\alpha-1)+m^8].
\end{aligned}
$$

We need the following technical lemmas.

Lemma 3.1. *Let $f(m)$ be a polynomial of m, whose degree is s. If*

(1) $\frac{\partial^s f}{\partial m^s}>0$,

(2) $\frac{\partial^k f}{\partial m^k}|_{m=m_0}>0$ *for* $k=0,\dots,s-1$.

Then $f(m)>0$ for $m\geq m_0$.

Proof. It is obvious. □

Lemma 3.2. *Consider α and m as parameters and let $\Delta_m(A_1,A_2,\dots,A_7,\alpha)$ be a polynomial of $A_1,\dots,A_7$. If*

(1) $\Delta_m(A_1^{(0)},\dots,A_7^{(0)},\alpha)\geq 0$,

(2) $\frac{\partial \Delta_m}{\partial A_i} \geq 0$, $\frac{\partial^2 \Delta_m}{\partial A_i \partial A_7} \geq 0$ *and* $\frac{\partial^6 \Delta_m}{\partial A_7^6} \geq 0$ *for all* $1 \leq i \leq 5$, $A_1 \geq A_1^{(0)}, \ldots, A_7 \geq A_7^{(0)}$,

(3) $\frac{\partial^k \Delta_m}{\partial A_7^k}|_{A_1=A_1^{(0)},\ldots,A_7=A_7^{(0)}} \geq 0$ *for all* $1 \leq k \leq 4$.

Then $\Delta_m(A_1, A_2, \ldots, A_7, \alpha) \geq 0$ *for* $A_1 \geq A_1^{(0)}, \ldots, A_7 \geq A_7^{(0)}$.

Proof. Suppose $f(A_1, \ldots, A_7)$ is a polynomial of $A_1, \ldots, A_7$. To prove $f \geq 0$ for $A_1 \geq A_1^{(0)}, \ldots, A_7 \geq A_7^{(0)}$, we only need to show

(1) $f(A_1^{(0)}, \ldots, A_7^{(0)}) \geq 0$ and

(2) $\frac{\partial f}{\partial A_i} \geq 0$, for all $1 \leq i \leq 7$, $A_1 \geq A_1^{(0)}, \ldots, A_7 \geq A_7^{(0)}$.

In particular, we can apply this method to show $\Delta_m \geq 0$ and $\frac{\partial^k \Delta_m}{\partial A_{i_1} \ldots \partial A_{i_k}} \geq 0$, where $1 \leq i_1 \leq \cdots \leq i_k \leq 7$. In order to show $\frac{\partial^k \Delta_m}{\partial A_{i_1} \ldots \partial A_{i_k}} \geq 0$, we only need to show

(1) $\frac{\partial^k \Delta_m}{\partial A_{i_1} \ldots \partial A_{i_k}}|_{A_1=A_1^{(0)},\ldots,A_7=A_7^{(0)}} \geq 0$ and

(2) $\frac{\partial}{\partial A_j}(\frac{\partial^k \Delta_m}{\partial A_{i_1} \ldots \partial A_{i_k}}) \geq 0$, for all $1 \leq j \leq 7$, $A_1 \geq A_1^{(0)}, \ldots, A_7 \geq A_7^{(0)}$.

Notice that for $k \geq 2$, $\frac{\partial^k \Delta_m}{\partial A_7^k}$ only contains one variable A_7, i.e., $\frac{\partial^{k+1} \Delta_m}{\partial A_i \partial^k A_7} = 0$ for $1 \leq i \leq 6$. Therefore, given the three conditions in the proposition statement, by induction we can prove that $\Delta_m(A_1, A_2, \ldots, A_7, \alpha) \geq 0$ for $A_1 \geq A_1^{(0)}, \ldots, A_7 \geq A_7^{(0)}$. □

So we can use the initial value of all partial derivatives to determine the sign of Δ_m by applying Lemma 3.2. We shall use the following propositions which had been proven in [22].

Proposition 3.1. *[22] Let* $g_n(a_1, \ldots, a_n)$ *be the polynomial upper estimate of* $P_n(a_1, \ldots, a_n)$ *in the Yau Number Theoretic Conjecture. And let* m *be the number of* $(n-1)$*-dimensional layers in the* n*-dimensional simplex, i.e.,* $P_{n-1}(m) > 0$ *and* $P_{n-1}(m+1) = 0$*. Let* $\alpha = 1 - \frac{m}{a_n} \in (0, 1)$, $A_i = a_i \alpha$, *for* $i = 1, \ldots, n-1$ *and*

$$g_{n-1}(m) := \sum_{k=1}^{m} g_{n-1}(\frac{m-k+k\alpha}{m\alpha} A_1, \ldots, \frac{m-k+k\alpha}{m\alpha} A_{n-1})$$

$$\Delta_m(A_1, \ldots, A_{n-1}, \alpha) = g_n(\frac{A_1}{\alpha}, \ldots, \frac{A_{n-1}}{\alpha}, \frac{m}{1-\alpha}) - n g_{n-1}(m)$$

then

$$\frac{\partial \Delta_m}{\partial A_i} > 0 \qquad \text{and} \qquad \frac{\partial^2 \Delta_m}{\partial A_i \partial A_{n-1}} > 0$$

for all $i = 1, \ldots, n-2$, $A_1 \geq \cdots \geq A_{n-1} \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (0,1)$.

Proposition 3.2. *[22] Let* $g_n(a_1, \ldots, a_n)$ *be the polynomial upper estimate of* $P_n(a_1, \ldots, a_n)$ *in the Yau Number Theoretic Conjecture. And let* m *be the number of* $(n-1)$*-dimensional layers in the* n*-dimensional simplex, i.e.,* $P_{n-1}(m) > 0$ *and* $P_{n-1}(m+1) = 0$*. Let* $\alpha = 1 - \frac{m}{a_n} \in (0,1)$, $A_i = a_i\alpha$, *for* $i = 1, \ldots, n-1$ *and*

$$g_{n-1}(m) := \sum_{k=1}^{m} g_{n-1}\left(\frac{m-k+k\alpha}{m\alpha}A_1, \ldots, \frac{m-k+k\alpha}{m\alpha}A_{n-1}\right)$$

$$\Delta_m(A_1, \ldots, A_{n-1}, \alpha) = g_n\left(\frac{A_1}{\alpha}, \ldots, \frac{A_{n-1}}{\alpha}, \frac{m}{1-\alpha}\right) - n g_{n-1}(m)$$

then

$$\frac{\partial^{n-2} \Delta_m}{\partial A_{n-1}^{n-2}} > 0$$

for all $n \geq 5$, $\alpha \in (0,1)$, $m \in \mathbb{Z}^+$.

We divide the proof of the Main Theorem A into 8 cases:

Case 1: $a_8 \in (m, m+1]$;

Case 2: $a_8 \in (m+1, m+2]$;

Case 3: $a_8 \in (m+2, m+3]$;

Case 4: $a_8 \in (m+3, m+4]$;

Case 5: $a_8 \in (m+4, m+5]$;

Case 6: $a_8 \in (m+5, m+6]$;

Case 7: $a_8 \in (m+6, m+7)$;

Case 8: $a_8 \geq m+7$.

Note that the division into different cases depends on the vaule of a_8 because it is treated as a parameter in reducing the 8 dimensional simplex into 7 dimension simplices. And different case has different vaule of boundary data on the domain to be considered. We shall give the detailed proof of (1), (2), (7) and (8). Since the proof of cases (3)-(6) are akin to that of cases (7), so we shall only list the key informaion in each cases (3)-(6).

For case 1 to 7, the equality in (7) in Theorem 1.4 cannot be attained by any chance, because in these cases Δ_m is positive. Moreover, $a_1 = \cdots = a_8$ cannot hold in these cases, it can only hold in case 8.

3.1. Case 1: $a_8 \in (m, m+1]$

Since $a_8 \in (m, m+1]$, $\alpha \in (0, \frac{1}{m+1}]$, so $x_1 = \cdots = x_6 = x_7 = 1$, $x_8 = m$ is a solution of the inequality, thus we have

$$\frac{1}{A_1} + \cdots + \frac{1}{A_7} \leq 1 \tag{15}$$

and $A_1 \geq A_2 \geq \cdots \geq A_7$. We just need to show that $\Delta_m \geq 0$, for $A_1 \geq 7, A_2 \geq 6, A_3 \geq 5, A_4 \geq 4, A_5 \geq 3, A_6 \geq 2, A_7 \geq 1$. Notice that in this case, $\frac{m\alpha}{1-\alpha} \in (0,1]$, so by Proposition 3.1, $\frac{\partial \Delta_m}{\partial A_i} > 0$ and $\frac{\partial^2 \Delta_m}{\partial A_i \partial A_7} > 0$ for all $i = 1, \ldots, 6$, $A_1 \geq \cdots \geq A_7 \geq 1$, $\alpha \in (0, \frac{1}{m+1}]$.

By Proposition 3.2, $\frac{\partial^6 \Delta_m}{\partial A_7^6} > 0$, for $\alpha \in (0,1)$, $m \geq 2$, m integer.

(i) $\frac{\partial^5 \Delta_m}{\partial A_7^5}|_{A_7=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer.

$$\begin{aligned} & \frac{\partial^5 \Delta_m}{\partial A_7^5}|_{A_7=1} \\ = \; & \frac{1}{\alpha^6 m^5}[-160(m+1)(85m^5 + 128m^4 + 5m^3 - 5m^2 + 12m - 12)\alpha^6 \\ & -160(m-1)(m+1)(82m^4 - 51m^2 - 72)\alpha^5 \\ & -320(m-1)(m+1)(41m^4 + 41m^2 + 90)\alpha^4 \\ & -320(m-1)(m+1)(41m^4 + 41m^2 - 120)\alpha^3 \\ & -160(m-1)(m+1)(82m^4 - 303m^2 + 180)\alpha^2 \\ & -160(m-1)(82m^5 - 380m^4 + 257m^3 + 257m^2 - 72m - 72)\alpha \\ & +1920(m-1)(2m-1)(3m^4 - 6m^3 + 3m + 1)]. \end{aligned}$$

For $m \geq 4$, the coefficients of $\alpha, \ldots, \alpha^6$ are less than 0, and

$$\frac{\partial^5 \Delta_m}{\partial A_7^5}|_{A_7=1, \alpha=\frac{1}{m+1}}$$

$$
\begin{aligned}
&= 160\frac{(1+m)^7}{m^8}(72m^5+26m^4-236m^3-149m^2+65m+12)\frac{m^9}{(1+m)^6}\\
&= 160(1+m)m(72m^5+26m^4-236m^3-149m^2+65m+12)>0.
\end{aligned}
$$

Thus $\frac{\partial^5\Delta_m}{\partial A_7^5}|_{A_7=1}>0$, for $\alpha\in(0,\frac{1}{m+1}]$, $m\geq 4$.
For $m=2$,

$$
\frac{\partial^5\Delta_2}{\partial A_7^5}|_{A_7=1}=-\frac{420}{\alpha^6}(168\alpha^6+37\alpha^5+65\alpha^4+50\alpha^3+10\alpha^2-7\alpha-3)>0,
$$

for $\alpha\in(0,\frac{1}{3}]$. For $m=3$,

$$
\begin{aligned}
&\frac{\partial^5\Delta_3}{\partial A_7^5}|_{A_7=1}\\
= &-\frac{4480}{81\alpha^6}(1448\alpha^6+582\alpha^5+720\alpha^4+680\alpha^3+390\alpha^2-45\alpha-130)>0\\
&\text{for } \alpha\in(0,\frac{1}{4}].
\end{aligned}
$$

These two ">"s can be proved by Lemma 2.1, you may need to replace α with, for example, $\beta=\alpha/3$, $\beta\in(0,1]$, for $m=3$. Thus $\frac{\partial^5\Delta_m}{\partial A_7^5}|_{A_7=1}>0$, for $\alpha\in(0,\frac{1}{m+1}]$, $m\geq 2$, m integer.

(ii) $\frac{\partial^4\Delta_m}{\partial A_7^4}|_{A_7=1}>0$, for $\alpha\in(0,\frac{1}{m+1}]$, $m\geq 2$, m integer. Let $\beta=(1+m)\alpha\in(0,1]$.

$$
\begin{aligned}
\frac{\partial^4\Delta_m}{\partial A_7^4}|_{A_7=1}=&\frac{160(1+m)}{\beta^6m^5}[(50m^5+34m^4-13m^3+13m^2-6m+6)\beta^6\\
&+(50m^6-21m^4+7m^2-36)\beta^5\\
&+(50m^7+50m^6-140m^3-140m^2+90m+90)\beta^4\\
&+(50m^8+100m^7-230m^6-560m^5\\
&+70m^4+700m^3+230m^2-240m-120)\beta^3\\
&+(50m^9-270m^8-445m^7+785m^6\\
&+1190m^5-490m^4-1065m^3-115m^2+270m+90)\beta^2\\
&+(-118m^{10}-10m^9+629m^8+256m^7-1133m^6\\
&-770m^5+671m^4+668m^3-13m^2-144m-36)\beta\\
&+36m^{11}+54m^{10}-144m^9-270m^8+138m^7\\
&+456m^6+90m^5-264m^4-150m^3+18m^2+30m+6].
\end{aligned}
$$

The function Δ_m can be extended to a function of m for $m \in \mathbb{R}^+$. We still denote this extended function by Δ_m.

$$\frac{\partial^{11}}{\partial m^{11}}\left(\frac{\beta^6 m^5}{160(1+m)}\frac{\partial^4\Delta_m}{\partial A_7^4}|_{A_7=1}\right)$$
$$=1437004800 > 0,$$
$$\frac{\partial^{10}}{\partial m^{10}}\left(\frac{\beta^6 m^5}{160(1+m)}\frac{\partial^4\Delta_m}{\partial A_7^4}|_{A_7=1}\right)|_{m=2}$$
$$=-428198400\beta + 3069964800 > 0,$$
$$\frac{\partial^{9}}{\partial m^{9}}\left(\frac{\beta^6 m^5}{160(1+m)}\frac{\partial^4\Delta_m}{\partial A_7^4}|_{A_7=1}\right)|_{m=2}$$
$$=18144000\beta^2 - 860025600\beta + 3213665280 > 0,$$
$$\frac{\partial^{8}}{\partial m^{8}}\left(\frac{\beta^6 m^5}{160(1+m)}\frac{\partial^4\Delta_m}{\partial A_7^4}|_{A_7=1}\right)|_{m=2}$$
$$=2016000\beta^3 + 25401600\beta^2 - 838293120\beta + 2192520960 > 0,$$

$$\frac{\partial^{7}}{\partial m^{7}}\left(\frac{\beta^6 m^5}{160(1+m)}\frac{\partial^4\Delta_m}{\partial A_7^4}|_{A_7=1}\right)|_{m=2}$$
$$=252000\beta^4 + 4536000\beta^3 + 12272400\beta^2 - 526176000\beta + 1093690080 > 0,$$
$$\frac{\partial^{6}}{\partial m^{6}}\left(\frac{\beta^6 m^5}{160(1+m)}\frac{\partial^4\Delta_m}{\partial A_7^4}|_{A_7=1}\right)|_{m=2}$$
$$=36000\beta^5 + 540000\beta^4 + 4874400\beta^3 - 1501200\beta^2 - 237816720\beta + 424111680 > 0,$$
$$\frac{\partial^{5}}{\partial m^{5}}\left(\frac{\beta^6 m^5}{160(1+m)}\frac{\partial^4\Delta_m}{\partial A_7^4}|_{A_7=1}\right)|_{m=2}$$
$$=6000\beta^6 + 72000\beta^5 + 576000\beta^4 + 3297600\beta^3 - 5631600\beta^2 - 81933840\beta + 132695280 > 0$$
$$\frac{\partial^{4}}{\partial m^{4}}\left(\frac{\beta^6 m^5}{160(1+m)}\frac{\partial^4\Delta_m}{\partial A_7^4}|_{A_7=1}\right)|_{m=2}$$
$$=12816\beta^6 + 71496\beta^5 + 408000\beta^4 + 1552080\beta^3 - 4005360\beta^2 - 22202136\beta + 34320672 > 0$$
$$\frac{\partial^{3}}{\partial m^{3}}\left(\frac{\beta^6 m^5}{160(1+m)}\frac{\partial^4\Delta_m}{\partial A_7^4}|_{A_7=1}\right)|_{m=2}$$
$$=13554\beta^6 + 46992\beta^5 + 215160\beta^4 + 525960\beta^3$$

$$
\begin{aligned}
&\qquad - 1776150\beta^2 - 4810536\beta + 7462044 > 0,\\
&\frac{\partial^2}{\partial m^2}\left(\frac{\beta^6 m^5}{160(1+m)}\frac{\partial^4\Delta_m}{\partial A_7^4}|_{A_7=1}\right)|_{m=2}\\
&\qquad =9502\beta^6 + 23006\beta^5 + 89240\beta^4 + 125820\beta^3\\
&\qquad - 574290\beta^2 - 839322\beta + 1381212 > 0,\\
&\frac{\partial}{\partial m}\left(\frac{\beta^6 m^5}{160(1+m)}\frac{\partial^4\Delta_m}{\partial A_7^4}|_{A_7=1}\right)|_{m=2}\\
&\qquad =9502\beta^6 + 23006\beta^5 + 89240\beta^4 + 125820\beta^3\\
&\qquad - 574290\beta^2 - 839322\beta + 1381212 > 0.
\end{aligned}
$$

for $\beta \in (0, 1]$. The ">"s can be proved by Lemma 2.1 or Lemma 2.2. Thus by Lemma 3.1, $\frac{\partial^4\Delta_m}{\partial A_7^4}|_{A_7=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer.

(iii) $\frac{\partial^3\Delta_m}{\partial A_7^3}|_{A_7=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer. Let $\beta = (1+m)\alpha \in (0, 1]$.

$$
\begin{aligned}
&\frac{\partial^3\Delta_m}{\partial A_7^3}|_{A_7=1}\\
&=\frac{1+m}{\beta^6 m^5}[(-2588m^5 + 1412m^4 + 240m^3 - 240m^2 - 320m + 320)\beta^6\\
&\quad - (4(m-1))(m+1)(647m^4 + 1060m^2 - 480)\beta^5\\
&\quad - (4(m-1))(647m^4 - 4120m^2 + 1200)(m+1)^2\beta^4\\
&\quad - (4(m-1))(647m^5 - 8887m^4 + 7080m^3\\
&\qquad\qquad + 7080m^2 - 1600m - 1600)(m+1)^2\beta^3\\
&\quad + (80(m-1))(m-2)(206m^4 - 222m^3 - 133m^2\\
&\qquad\qquad + 45m + 30)(m+1)^3\beta^2\\
&\quad - (80(m-1))(130m^5 - 332m^4 + 137m^3\\
&\qquad\qquad + 137m^2 - 24m - 24)(m+1)^4\beta\\
&\quad + (320(m-1))(2m-1)(3m^4 - 6m^3 + 3m + 1)(m+1)^5].
\end{aligned}
$$

Similarly as case 1(ii), we can show that

$$
\frac{\partial^i}{\partial m^i}\left(\frac{\beta^6 m^5}{1+m}\frac{\partial^3\Delta_m}{\partial A_7^3}|_{A_7=1}\right)|_{m=2} > 0,
$$

for $1 \le i \le 11$.

$$\left(\frac{\beta^6 m^5}{1+m}\frac{\partial^3 \Delta_m}{\partial A_7^3}|_{A_7=1}\right)|_{m=2} = -59584\beta^6 - 169344\beta^5 + 177408\beta^4 + 1487808\beta^3 - 2721600\beta + 1632960 > 0.$$

for $\beta \in (0,1]$. Thus by Lemma 3.1, $\frac{\partial^3 \Delta_m}{\partial A_7^3}|_{A_7=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \ge 2$, m integer.

(iv) $\frac{\partial^2 \Delta_m}{\partial A_7^2}|_{A_7=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \ge 2$, m integer. Let $\beta = (1+m)\alpha \in (0,1]$.

$$\begin{aligned}
&\frac{\partial^2 \Delta_m}{\partial A_7^2}|_{A_7=1}\\
=&\frac{1+m}{3\beta^6 m^5}[(124m^5 - 3820m^4 + 2000m^3 - 2000m^2 - 240m + 240)\beta^6\\
&+ (4(m-1))(m+1)(31m^4 - 3000m^2 + 360)\beta^5\\
&+ (4(m-1))(m+1)(31m^5 - 10427m^4 + 7360m^3 + 7360m^2 - 900m - 900)\beta^4\\
&- (4(m-1))(6941m^5 - 21661m^4 + 9440m^3\\
&\qquad + 9440m^2 - 1200m - 1200)(m+1)^2\beta^3\\
&+ (80(m-1))(368m^5 - 892m^4 + 333m^3 + 333m^2 - 45m - 45)(m+1)^3\beta^2\\
&- (160(m-1))(68m^5 - 163m^4 + 61m^3 + 61m^2 - 9m - 9)(m+1)^4\beta\\
&+ (240(m-1))(2m-1)(3m^4 - 6m^3 + 3m + 1)(m+1)^5].
\end{aligned}$$

Similarly as case 1(ii), we can show that

$$\frac{\partial^i}{\partial m^i}\left(\frac{3\beta^6 m^5}{1+m}\frac{\partial^2 \Delta_m}{\partial A_7^2}|_{A_7=1}\right)|_{m=2} > 0,$$

for $1 \le i \le 11$,

$$\begin{aligned}
&\left(\frac{3\beta^6 m^5}{1+m}\frac{\partial^2 \Delta_m}{\partial A_7^2}|_{A_7=1}\right)|_{m=2}\\
&\qquad = -49392\beta^6 - 133728\beta^5 - 962640\beta^4 + 532224\beta^3 + 2948400\beta^2\\
&\qquad\quad - 3538080\beta + 1224720 > 0,
\end{aligned}$$

for $\beta \in (0,1]$. Thus by Lemma 3.1, $\frac{\partial^2 \Delta_m}{\partial A_7^2}|_{A_7=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \ge 2$, m integer.

(v) $\frac{\partial\Delta_m}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer. Let $\beta = (1+m)\alpha \in (0,1]$.

$$
\begin{aligned}
&\frac{\partial\Delta_m}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1}\\
&=\frac{1+m}{21\beta^7 m^5(1+m-\beta)}[(1452m^5+48948m^4+38944m^3\\
&\qquad -38944m^2-38112m+38112)\beta^8\\
&+(-110355m^5-77420m^4+283220m^2-196224)\beta^7\\
&-(7(m+1))(8565m^6-50640m^5+45920m^4+40012m^2-43776)\beta^6\\
&+(7(28023m^6-101934m^5+110516m^4-56840m^2+21120))(m+1)^2\beta^5\\
&-(7(29487m^6-108444m^5+16055m^4-238840m^2+162240))(m+1)^3\beta^4\\
&+(7(16437m^6-103761m^5+250544m^4-386372m^2+238464))(m+1)^4\beta^3\\
&-(84(682m^6-10703m^5+26999m^4-29435m^2+14464))(m+1)^5\beta^2\\
&+(12(2175m^6-69195m^5+139104m^4-101248m^2+37984))(m+1)^6\beta\\
&+(35280(m-1))(9m^4-5m^3-5m^2+2m+2)(m+1)^7].
\end{aligned}
$$

Similarly as case 1(ii), we can show that

$$
\frac{\partial^i}{\partial m^i}\Big(\frac{21\beta^7 m^5(1+m-\beta)}{1+m}\frac{\partial\Delta_m}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1}\Big)|_{m=2} > 0,
$$

for $1 \leq i \leq 12$,

$$
\begin{aligned}
&\Big(\frac{21\beta^7 m^5(1+m-\beta)}{1+m}\frac{\partial\Delta_m}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1}\Big)|_{m=2}\\
&=947296\beta^8-3833424\beta^7+4647888\beta^6+5896800\beta^5\\
&\quad -36415008\beta^4+245678832\beta^3-609502320\beta^2\\
&\quad -1892927232\beta+6944162400>0,
\end{aligned}
$$

for $\beta \in (0,1]$. Thus by Lemma 3.1, $\frac{\partial\Delta_m}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer. By Lemma 3.2, we conclude that

Proposition 3.3. $\frac{\partial\Delta_m}{\partial A_7} > 0$, *for* $A_1 \geq 7, A_2 \geq 6, A_3 \geq 5, A_4 \geq 4, A_5 \geq 3, A_6 \geq 2, A_7 \geq 1$, $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m *integer.*

(vi) $\Delta_m > 0$, for $A_1 \geq 7, A_2 \geq 6, A_3 \geq 5, A_4 \geq 4, A_5 \geq 3, A_6 \geq 2, A_7 \geq 1$, $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer. Let $\beta = (1+m)\alpha \in (0,1]$.

$$
\begin{aligned}
&\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A-7=1}\\
&=\frac{1+m}{105\beta^7 m^5(1+m-\beta)^7}[(2(m-1))(21459m^4+16524m^2-45580)\beta^{14}\\
&\quad+(-785973m^6-308280m^5+368060m^4+1032948m^2-832280)\beta^{13}\\
&\quad-(10(m+1))(136920m^7-350628m^6\\
&\quad-299145m^5+333445m^4+256802m^2-252408)\beta^{12}\\
&\quad-(5(274596m^8-1369200m^7+1278253m^6\\
&\quad+2718618m^5-3340232m^4+1127532m^2-289328))(m+1)^2\beta^{11}\\
&\quad-(5(140679m^9-1098384m^8+2738400m^7-556608m^6\\
&\quad-7942557m^5+11825856m^4-12082532m^2+7144280))(m+1)^3\beta^{10}\\
&\quad-(3(66640m^{10}-703395m^9+2745960m^8-4564000m^7-3309318m^6\\
&\quad+29375745m^5-54592720m^4+72994768m^2-44976360))(m+1)^4\beta^9\\
&\quad-(3(10290m^{11}-133280m^{10}+703395m^9-1830640m^8+2282000m^7\\
&\qquad+8045368m^6-55031795m^5+120792700m^4-165755128m^2\\
&\qquad+99207680))(m+1)^5\beta^8\\
&\quad-(15(140m^{12}-2058m^{11}+13328m^{10}-46893m^9+91532m^8-91280m^7\\
&\qquad-2101490m^6+17881129m^5-42020608m^4\\
&\qquad+53077640m^2-29931616))(m+1)^6\beta^7\\
&\quad-(15(2063454m^6-24184461m^5+56176610m^4\\
&\qquad-62376776m^2+32538792))(m+1)^7\beta^6\\
&\quad+(5(4906227m^6-77088417m^5+169624476m^4\\
&\qquad-163110332m^2+77774840))(m+1)^8\beta^5\\
&\quad-(14975472m^6-306341385m^5+628239290m^4\\
&\qquad-522903612m^2+226180240)(m+1)^9\beta^4\\
&\quad+(6396627m^6-173969355m^5+331434040m^4-\\
&\qquad240101932m^2+93790320)(m+1)^{10}\beta^3\\
&\quad-(20(83369m^6-3319848m^5+5887721m^4\\
&\qquad-3740541m^2+1316666))(m+1)^{11}\beta^2\\
&\quad+(20(9885m^6-761775m^5+1262562m^4-708974m^2+224762))(m+1)^{12}\beta\\
&\quad+(176400(m-1))(9m^4-5m^3-5m^2+2m+2)(m+1)^{13}].
\end{aligned}
$$

Similarly as case 1(ii), we can show that

$$\frac{\partial^i}{\partial m^i}\left(\frac{105\beta^7 m^5(1+m-\beta)^7}{1+m}\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1}\right)|_{m=2} > 0,$$

for $1 \leq i \leq 18$,

$$\begin{aligned}&\left(\frac{105\beta^7 m^5(1+m-\beta)^7}{1+m}\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1}\right)|_{m=2}\\ &\quad 727720\beta^{14} - 50978760\beta^{13} + 251314560\beta^{12}\\ &\quad - 657901440\beta^{11} + 54046440\beta^{10} + 17344960920\beta^9\\ &\quad - 139196776320\beta^8 + 596005089120\beta^7 - 1312716744360\beta^6\\ &\quad - 443054094840\beta^5 + 12953376001440\beta^4 - 42592516092000\beta^3\\ &\quad + 72068650237080\beta^2 - 65412968183640\beta + 25311471948000 > 0,\end{aligned}$$

for $\beta \in (0,1]$. Thus $\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1} > 0$, for $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer. By Lemma 3.2, we know that $\Delta_m > 0$, for $A_1 \geq 7, A_2 \geq 6, A_3 \geq 5, A_4 \geq 4, A_5 \geq 3, A_6 \geq 2, A_7 \geq 1$, $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 2$, m integer.

(vii) $\Delta_m > 0$, for $m = 1$, $1 < a_8 \leq 2$, $\alpha = 1 - \frac{1}{a_8} \in (0, \frac{1}{2}]$. By Proposition 3.1, $\frac{\partial \Delta_1}{\partial A_i} > 0$, for $1 \leq i \leq 6$, $A_1 \geq \cdots \geq A_7 \geq 1$, $\alpha \in (0,1)$.
And

$$\frac{\partial^2 \Delta_1}{\partial A_7^2} = 3360A_7^4 - 24640A_7^3 + 67200A_7^2 - 76272A_7 + 27888.$$

Since $A_7 \geq 1$, so set $\frac{\partial^2 \Delta_1}{\partial A_7^2} = 0$, we have $A_7 = 1.822613576$.

$$\begin{aligned}&\frac{\partial \Delta_1}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1.822613576}\\ &= \frac{1}{\alpha^6(1-\alpha)}(\alpha^6 - 27\alpha^5 + 295\alpha^4 - 1665\alpha^3 + 5014\alpha^2 - 8028\alpha + 5040)\\ &\quad - 6157.912873 > 0,\end{aligned}$$

for $\alpha \in (0, \frac{1}{2}]$. Thus, we conclude that

Proposition 3.4. $\frac{\partial \Delta_1}{\partial A_7} > 0$, *for* $A_1 \geq 7, A_2 \geq 6, A_3 \geq 5, A_4 \geq 4, A_5 \geq 3, A_6 \geq 2, A_7 \geq 1$, $\alpha \in (0,1)$.

And since

$$\Delta_1|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=1}$$
$$=\frac{1}{(\alpha-1)^7\alpha^6}(\alpha^{13}-34\alpha^{12}+505\alpha^{11}-4332\alpha^{10}+23934\alpha^9-90012\alpha^8+237593\alpha^7$$
$$-442129\alpha^6+585344\alpha^5-553123\alpha^4+366501\alpha^3-161776\alpha^2+42588\alpha-5040)$$
$$>0,$$

for $\alpha \in (0, \frac{1}{2}]$. Thus by Lemma 3.2, $\Delta_1 > 0$, for $A_1 \geq 7, A_2 \geq 6, A_3 \geq 5, A_4 \geq 4, A_5 \geq 3, A_6 \geq 2, A_7 \geq 1$, $\alpha \in (0, \frac{1}{2}]$.

Therefore $\Delta_m > 0$, for $A_1 \geq 7, A_2 \geq 6, A_3 \geq 5, A_4 \geq 4, A_5 \geq 3, A_6 \geq 2, A_7 \geq 1$, $\alpha \in (0, \frac{1}{m+1}]$, $m \geq 1$, m integer.

3.2. Case 2: $a_8 \in (m+1, m+2]$

In this case, $\frac{m\alpha}{1-\alpha} \in (1, 2]$, so $A_1 \geq A_2 \geq \cdots \geq A_7 \geq \frac{m\alpha}{1-\alpha}$. By Proposition 3.1, $\frac{\partial \Delta_m}{\partial A_i} > 0, \frac{\partial^2 \Delta_m}{\partial A_j \partial A_7} > 0$, for $1 \leq i, j \leq 6$, $A_1 \geq \cdots \geq A_7 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (0, 1)$, $m \geq 1$, m integer.

By Proposition 3.2, $\frac{\partial^6 \Delta_m}{\partial A_7^6} > 0$, for $\alpha \in (0, 1)$, $m \geq 2$, m integer.

(i) $\frac{\partial^5 \Delta_m}{\partial A_7^5}|_{A_7=\frac{m\alpha}{1-\alpha}} > 0$, for $\alpha \in (\frac{1}{m+1}, 1]$, $m \geq 2$, m integer.

$$\frac{\partial^5 \Delta_m}{\partial A_7^5}|_{A_7=\frac{m\alpha}{1-\alpha}}$$
$$=\frac{1}{\alpha^5 m^4(1-\alpha)}[(160(m+1))(72m^5+334m^4+380m^3+5m^2-89m+12)\alpha^6$$
$$+(320(m+1))(36m^5-36m^4-321m^3-64m^2+190m-36)\alpha^5$$
$$+(160(m-1))(m+1)(72m^4+72m^2+385m-180)\alpha^4$$
$$+(3840(m-1))(m+1)(3m^4+3m^2+10)\alpha^3$$
$$+(160(m-1))(m+1)(72m^4+72m^2-385m-180)\alpha^2$$
$$+(320(m-1))(36m^5+36m^4-321m^3+64m^2+190m+36)\alpha$$
$$+(160(m-1))(72m^5-334m^4+380m^3-5m^2-89m-12)].$$

Similarly as case 1(ii), we can show that

$$\frac{\partial^i}{\partial m^i}(m^4\alpha^5(1-\alpha)\frac{\partial^5 \Delta_m}{\partial A_7^5}|_{A_7=1})|_{m=3} > 0,$$

for $1 \le i \le 6$,

$$
\begin{aligned}
&(m^4\alpha^5(1-\alpha)\frac{\partial^5\Delta_m}{\partial A_7^5}|_{A_7=1})|_{m=3}\\
&\quad 34944000\alpha^6 =- 3682560\alpha^5 + 9542400\alpha^4 + 8601600\alpha^3 + 6585600\alpha^2\\
&\quad + 2674560\alpha + 120960 > 0,
\end{aligned}
$$

for $\alpha \in (\frac{1}{m+1}, 1]$. Thus by Lemma 3.1, $\frac{\partial^5\Delta_m}{\partial A_7^5}|_{A_7=\frac{m\alpha}{1-\alpha}} > 0$, for $\alpha \in (\frac{1}{m+1}, 1]$, $m \ge 3$. And we can check that

$$
\begin{aligned}
(\frac{\partial^5\Delta_m}{\partial A_7^5}&|_{A_7=\frac{2\alpha}{1-\alpha}})|_{m=2}\\
&= -\frac{420}{\alpha^5(\alpha-1)}(753\alpha^6 - 272\alpha^5 + 145\alpha^4 + 120\alpha^3\\
&\qquad + 35\alpha^2 - 8\alpha - 5) > 0,
\end{aligned}
$$

for $\alpha \in (\frac{1}{m+1}, 1]$. Thus, $\frac{\partial^5\Delta_m}{\partial A_7^5}|_{A_7=\frac{m\alpha}{1-\alpha}} > 0$, for $\alpha \in (\frac{1}{m+1}, 1]$, $m \ge 2$, m integer.

(ii) $\frac{\partial^4\Delta_m}{\partial A_7^4}|_{A_7=\frac{m\alpha}{1-\alpha}} > 0$, for $\alpha \in (\frac{1}{m+1}, 1]$, $m \ge 2$, m integer.

$$
\begin{aligned}
&\frac{\partial^4\Delta_m}{\partial A_7^4}|_{A_7=\frac{m\alpha}{1-\alpha}}\\
&\quad=\frac{160}{m^3\alpha^4(1-\alpha)^2}[(m+1)(36m^5 + 244m^4 + 512m^3\\
&\qquad + 293m^2 - 55m - 22)\alpha^6\\
&\quad + 2(m+1)(18m^5 - 18m^4 - 360m^3 - 445m^2 + 88m + 66)\alpha^5\\
&\quad + (m+1)(36m^5 - 36m^4 + 36m^3 + 769m^2 - 55m - 330)\alpha^4\\
&\quad + 4(m-1)(m+1)(9m^4 + 9m^2 - 110)\alpha^3\\
&\quad + (m-1)(36m^5 + 36m^4 + 36m^3 - 769m^2 - 55m + 330)\alpha^2\\
&\quad + 2(m-1)(18m^5 + 18m^4 - 360m^3 + 445m^2 + 88m - 66)\alpha\\
&\quad + (m-1)(36m^5 - 244m^4 + 512m^3 - 293m^2 - 55m + 22)].
\end{aligned}
$$

Similarly as case 1(ii), we can show that

$$\frac{\partial^i}{\partial m^i}(\frac{m^3\alpha^4(1-\alpha)^2}{160}\frac{\partial^4\Delta_m}{\partial A_7^4}|_{A_7=1})|_{m=3} > 0,$$

for $1 \le i \le 6$,

$$\begin{aligned}(\frac{m^3\alpha^4(1-\alpha)^2}{160}\frac{\partial^4\Delta_m}{\partial A_7^4}|_{A_7=1})|_{m=3}\\ =&179144\alpha^6 - 83832\alpha^5 + 52920\alpha^4 + 22400\alpha^3\\ &+ 11760\alpha^2 + 1260\alpha + 56 > 0,\end{aligned}$$

for $\alpha \in (\frac{1}{m+1}, 1]$. Thus by Lemma 3.1, $\frac{\partial^4\Delta_m}{\partial A_7^4}|_{A_7=\frac{m\alpha}{1-\alpha}} > 0$, for $\alpha \in (\frac{1}{m+1}, 1]$, $m \ge 3$. And we can check that for $m = 2$

$$\begin{aligned}&\frac{\partial^4\Delta_2}{\partial A_7^4}|_{A_7=\frac{2\alpha}{1-\alpha}}\\ &=\frac{1680}{(1-\alpha)^2\alpha^4}(364\alpha^6 - 295\alpha^5 + 125\alpha^4 + 10\alpha^3 - 10\alpha^2 - 3\alpha + 1) > 0,\end{aligned}$$

for $\alpha \in (\frac{1}{3}, 1]$. Thus $\frac{\partial^4\Delta_m}{\partial A_7^4}|_{A_7=\frac{m\alpha}{1-\alpha}} > 0$, for $\alpha \in (\frac{1}{m+1}, 1]$, $m \ge 2$, m integer.

(iii) $\frac{\partial^3\Delta_m}{\partial A_7^3}|_{A_7=\frac{m\alpha}{1-\alpha}} > 0$, for $\alpha \in (\frac{1}{m+1}, 1]$, $m \ge 2$, m integer.

$$\begin{aligned}&\frac{\partial^3\Delta_m}{\partial A_7^3}|_{A_7=\frac{m\alpha}{1-\alpha}}\\ &=\frac{4}{m^2\alpha^3(1-\alpha)^3}[(m+1)(480m^5 + 4280m^4 + 13360m^3\\ &\qquad + 15907m^2 + 4267m - 1040)\alpha^6\\ &\quad + 2(m+1)(240m^5 - 240m^4 - 8580m^3 - 20687m^2 - 9574m + 3120)\alpha^5\\ &\quad + (m+1)(480m^5 - 480m^4 + 480m^3 + 28787m^2 + 31735m - 15600)\alpha^4\\ &\quad + (480m^6 - 40348m^2 + 20800)\alpha^3\\ &\quad + (m-1)(480m^5 + 480m^4 + 480m^3 - 28787m^2 + 31735m + 15600)\alpha^2\\ &\quad + 2(m-1)(240m^5 + 240m^4 - 8580m^3 + 20687m^2 - 9574m - 3120)\alpha\\ &\quad + (m-1)(480m^5 - 4280m^4 + 13360m^3 - 15907m^2 + 4267m + 1040)].\end{aligned}$$

Similarly as case 1(ii), we can show that

$$\frac{\partial^i}{\partial m^i}(\frac{m^2\alpha^3(1-\alpha)^3}{4}\frac{\partial^3\Delta_m}{\partial A_7^3}|_{A_7=\frac{m\alpha}{1-\alpha}})|_{m=3} > 0,$$

for $1 \leq i \leq 6$,

$$\begin{aligned}\frac{m^2\alpha^3(1-\alpha)^3}{4}&\Delta_m|_{A_7=\frac{m\alpha}{1-\alpha}})|_{m=3}\\ &=3915856\alpha^6 - 3236520\alpha^5 + 1717632\alpha^4\\ &\quad + 7588\alpha^3 + 40404\alpha^2 + 1764\alpha + 2716 > 0,\end{aligned}$$

for $\alpha \in (\frac{1}{m+1}, 1]$. Thus by Lemma 3.1, $\frac{\partial^3\Delta_m}{\partial A_7^3}|_{A_7=\frac{m\alpha}{1-\alpha}} > 0$, for $\alpha \in (\frac{1}{m+1}, 1]$, $m \geq 3$. And we can check that for $m = 2$

$$\begin{aligned}(\frac{\partial^3\Delta_2}{\partial A_7^3}&|_{A_7=\frac{2\alpha}{1-\alpha}})|_{m=2}\\ &= -\frac{42}{\alpha^3(\alpha-1)^3}(18703\alpha^6 - 23368\alpha^5 + 12467\alpha^4\\ &\quad - 2616\alpha^3 - 219\alpha^2 + 160\alpha - 7) > 0,\end{aligned}$$

for $\alpha \in (\frac{1}{3}, 1]$. Thus, $\frac{\partial^3\Delta_m}{\partial A_7^3}|_{A_7=\frac{m\alpha}{1-\alpha}} > 0$, for $\alpha \in (\frac{1}{m+1}, 1]$, $m \geq 2$, m integer.

(iv) $\frac{\partial^2\Delta_m}{\partial A_7^2}|_{A_7=\frac{m\alpha}{1-\alpha}} > 0$, for $\alpha \in (\frac{1}{m+1}, 1]$, $m \geq 2$, m integer.

$$\begin{aligned}\frac{\partial^2\Delta_m}{\partial A_7^2}&|_{A_7=\frac{m\alpha}{1-\alpha}}\\ =&\frac{4}{3m\alpha^2(\alpha-1)^4}[(m+1)(360m^5 + 3980m^4 + 16600m^3 + 30601m^2\\ &\quad + 21353m + 1866)\alpha^6\\ &+ 2(m+1)(180m^5 - 180m^4 - 10110m^3 - 37091m^2 - 40840m - 5598)\alpha^5\\ &+ (360m^6 + 47201m^3 + 155862m^2 + 116095m + 27990)\alpha^4\\ &+ (360m^6 - 103908m^2 - 37320)\alpha^3\\ &+ (360m^6 - 47201m^3 + 155862m^2 - 116095m + 27990)\alpha^2\\ &+ 2(m-1)(180m^5 + 180m^4 - 10110m^3 + 37091m^2 - 40840m + 5598)\alpha\end{aligned}$$

$$+ (m-1)(360m^5 - 3980m^4 + 16600m^3 - 30601m^2 + 21353m - 1866)].$$

Similarly as case 1(ii), we can show that

$$\frac{\partial^i}{\partial m^i}\left(\frac{m\alpha^2(1-\alpha)^4}{4}\frac{\partial^2\Delta_m}{\partial A_7^2}|_{A_7=\frac{m\alpha}{1-\alpha}}\right)|_{m=3} > 0,$$

for $1 \le i \le 6$,

$$\begin{aligned}\frac{m\alpha^2(1-\alpha)^4}{4}\frac{\partial^2\Delta_m}{\partial A_7^2}|_{A_7=\frac{m\alpha}{1-\alpha}})|_{m=3}\\ =&4797576\alpha^6 - 5645976\alpha^5 + 3315900\alpha^4 - 710052\alpha^3\\ &+ 70476\alpha^2 + 8988\alpha + 168 > 0,\end{aligned}$$

for $\alpha \in (\frac{1}{m+1}, 1]$. Thus by Lemma 3.1, $\frac{\partial^2\Delta_m}{\partial A_7^2}|_{A_7=\frac{m\alpha}{1-\alpha}} > 0$, for $\alpha \in (\frac{1}{m+1}, 1]$, $m \ge 3$. And we can check that for $m = 2$

$$\begin{aligned}&\left(\frac{\partial^2\Delta_m}{\partial A_7^2}|_{A_7=\frac{2\alpha}{1-\alpha}}\right)|_{m=2}\\ &= -\frac{56}{\alpha^2(1-\alpha)^4}(13392\alpha^6 - 22403\alpha^5 + 15289\alpha^4\\ &\quad - 5118\alpha^3 + 770\alpha^2 + \alpha - 11) > 0,\end{aligned}$$

for $\alpha \in (\frac{1}{3}, 1]$. Thus, $\frac{\partial^2\Delta_m}{\partial A_7^2}|_{A_7=\frac{m\alpha}{1-\alpha}} > 0$, for $\alpha \in (\frac{1}{m+1}, 1]$, $m \ge 2$, m integer.

(v) $\frac{\partial\Delta_m}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=\frac{m\alpha}{1-\alpha}} > 0$, for $\alpha \in (\frac{1}{m+1}, 1]$, $m \ge 2$, m integer.

$$\begin{aligned}&\frac{\partial\Delta_m}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=\frac{m\alpha}{1-\alpha}}\\ &=\frac{1}{21\alpha^7m^5(1-\alpha)^5}[2(m+1)(1008m^{10} + 13300m^9 + 69608m^8 + 175049m^7\\ &\qquad + 199997m^6 + 76627m^5 + 24691m^4\\ &\qquad + 22930m^3 - 22930m^2 - 19224m + 19224)\alpha^{12}\\ &+ (2016m^{11} - 165816m^9 - 978628m^8 - 2250276m^7\\ &\quad - 2212992m^6 - 1155399m^5 - 570696m^4 + 665140m^2 - 352368)\alpha^{11}\\ &+ (2016m^{11} + 489314m^8 + 2250276m^7 + 2777275m^6\end{aligned}$$

$$
\begin{aligned}
&\quad + 2739156m^5 + 1411704m^4 - 2214576m^2 + 1338480)\alpha^{10} \\
&+ (2016m^{11} - 1500184m^7 - 43195m^6 \\
&\quad - 3475640m^5 - 1725920m^4 + 3663744m^2 - 2463120)\alpha^9 \\
&+ (2016m^{11} - 489314m^8 + 2250276m^7 - 2710005m^6 \\
&\quad + 2650200m^5 + 310940m^4 - 1333416m^2 + 1045440)\alpha^8 \\
&+ (2016m^{11} - 165816m^9 + 978628m^8 - 2250276m^7 \\
&\quad + 2220281m^6 - 1962618m^5 + 3904320m^4 - 8202600m^2 + 5521824)\alpha^7 \\
&+ (2016m^{11} - 28616m^{10} + 165816m^9 - 489314m^8 + 750092m^7 \\
&\quad - 702807m^6 + 3499048m^5 - 11859176m^4 + 22666224m^2 - 14835744)\alpha^6 \\
&+ (318595m^6 - 7205436m^5 + 21698096m^4 - 33152784m^2 + 19910880)\alpha^5 \\
&+ (-400815m^6 + 10403820m^5 - 26826870m^4 + 31866156m^2 - 16940880)\alpha^4 \\
&+ (308395m^6 - 9945495m^5 + 22230600m^4 - 20704684m^2 + 9527760)\alpha^3 \\
&+ (-132064m^6 + 6025572m^5 - 11812192m^4 + 8784608m^2 - 3457872)\alpha^2 \\
&+ (24084m^6 - 2093364m^5 + 3637872m^4 - 2200464m^2 + 737712)\alpha \\
&+ 35280(m-1)(9m^4 - 5m^3 - 5m^2 + 2m + 2)].
\end{aligned}
$$

Similarly as case 1(ii), we can show that

$$\frac{\partial^i}{\partial m^i}(21\alpha^7 m^5 (1-\alpha)^5 \frac{\partial \Delta_m}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=\frac{m\alpha}{1-\alpha}})|_{m=3} > 0,$$

for $1 \le i \le 11$,

$$
\begin{aligned}
21\alpha^7 m^5 (1-\alpha)^5 &\frac{\partial \Delta_m}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=\frac{m\alpha}{1-\alpha}}|_{m=3} \\
&= 10621023840\alpha^{12} - 16183385505\alpha^{11} + 11274874821\alpha^{10} \\
&\quad - 3909132675\alpha^9 + 750728601\alpha^8 - 17586261\alpha^7 - 72310077\alpha^6 \\
&\quad - 39583593\alpha^5 + 332812179\alpha^4 - 568071126\alpha^3 + 486755388\alpha^2 \\
&\quad - 215529048\alpha + 39301920 > 0,
\end{aligned}
$$

for $\alpha \in (\frac{1}{m+1}, 1]$. Thus by Lemma 3.1, $\frac{\partial \Delta_m}{\partial A_7}|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=\frac{m\alpha}{1-\alpha}} > 0$, for $\alpha \in (\frac{1}{m+1}, 1]$, $m \ge 3$, m integer. And we can check that for $m = 2$

$$
\begin{aligned}
&(\frac{\partial \Delta_m}{\partial A_7}|_{A_7=\frac{2\alpha}{1-\alpha}})|_{m=2} \\
&= -\frac{1}{2\alpha^7(\alpha-1)^5}(1139056\alpha^{12} - 2395120\alpha^{11} + 2077067\alpha^{10} - 944353\alpha^9
\end{aligned}
$$

$$+ 234978\alpha^8 - 11312\alpha^7 - 48978\alpha^6 + 72278\alpha^5$$
$$- 34036\alpha^4 - 47976\alpha^3 + 80511\alpha^2 - 45549\alpha + 9450) > 0,$$

for $\alpha \in (\frac{1}{3}, 1]$. Thus, $\frac{\partial \Delta_m}{\partial A_7}|_{A_7=\frac{m\alpha}{1-\alpha}} > 0$, for $\alpha \in (\frac{1}{m+1}, 1]$, $m \geq 2$, m integer.

By the above observation, we conclude that

Proposition 3.5. *In case* $n = 8$, $\frac{\partial \Delta_m}{\partial A_7} > 0$ *for* $A_1 \geq 7, A_2 \geq 6, \ldots, A_6 \geq 2, A_7 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{1}{m+1}, 1]$,$m \geq 2$, *$m$ integer.*

(vi) $\Delta_m > 0$, for $A_1 \geq 7, A_2 \geq 6, A_3 \geq 5, A_4 \geq 4, A_5 \geq 3, A_6 \geq 2, A_7 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{1}{m+1}, \frac{2}{m+2}]$, $m \geq 2$, m integer.

$$\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=\frac{m\alpha}{1-\alpha}}$$
$$=\frac{m+1}{105m^5(\beta+1)^6(\beta-m)^7}[2(m+2)(840m^{10} + 11200m^9 + 58800m^8$$
$$+ 141995m^7 + 131145m^6 + 27535m^5 + 98366m^4 + 38927m^3$$
$$- 99628m^2 + 10260m + 25200)\beta^{13}$$
$$- m(5600m^{10} + 41440m^9 - 66430m^8 - 1351630m^7 - 3743910m^6$$
$$- 2220308m^5 + 1445709m^4 - 2663674m^3 - 3181116m^2$$
$$+ 1905464m + 765360)\beta^{12}$$
$$- m^2(10080m^9 + 273630m^8 + 1728510m^7 + 2066845m^6 - 7570415m^5$$
$$- 10823016m^4 + 12525794m^3 + 6539686m^2 - 12285636m - 186128)\beta^{11}$$
$$+ m^3(57610m^8 + 354970m^7 - 2548570m^6 - 15079555m^5 - 1271290m^4$$
$$+ 41229978m^3 - 8789493m^2 - 36354242m + 17402172)\beta^{10}$$
$$+ 5m^4(4361m^8 + 3619m^7 + 407051m^6 + 1665465m^5 + 5992508m^4$$
$$- 10168756m^3 + 17930223m^2 + 12204343m - 16995480)\beta^9$$
$$+ m^5(11760m^9 - 18375m^8 + 146265m^7 - 187055m^6 - 7425209m^5 +$$
$$57968849m^4 + 27675486m^3 - 231423611m^2 - 71933485m + 206306690)\beta^8$$
$$+ m^5(3430m^{11} - 6370m^{10} + 76195m^9 - 114905m^8 - 637000m^7$$
$$+ 24317544m^6 - 94440842m^5 - 62275438m^4 + 346870028m^3$$
$$+ 113526328m^2 - 300594685m + 75495)\beta^7$$
$$+ m^6(420m^{12} - 910m^{11} + 21490m^{10} - 38885m^9 + 210875m^8 + 2091970m^7$$
$$- 56074920m^6 + 130273338m^5 + 200704218m^4 - 342938338m^3$$
$$- 217017492m^2 + 274870659m + 151095)\beta^6$$

$$
\begin{aligned}
&+ m^7(2520m^{11} - 5460m^{10} + 56910m^9 - 99540m^8 - 3273335m^7 + 90842725m^6 \\
&\quad - 110981095m^5 - 332409531m^4 + 197211601m^3 + 281683739m^2 \\
&\quad - 155260909m - 1047165)\beta^5 \\
&+ 5m^8(1260m^{10} - 2730m^9 + 16450m^8 + 681735m^7 - 20748361m^6 + 6177014m^5 \\
&+ 63574717m^4 - 3523509m^3 - 43494287m^2 + 9715229m + 340011)\beta^4 \\
&+ m^9(8400m^9 - 18200m^8 - 2132645m^7 + 81940867m^6 + 39204318m^5 \\
&\quad - 175323448m^4 - 58865962m^3 + 97304740m^2 - 4318220m - 1289400)\beta^3 \\
&+ m^{10}(6300m^8 + 767090m^7 - 42501367m^6 - 47503744m^5 + 46818164m^4 \\
&\quad + 38259731m^3 - 23866114m^2 - 2199900m + 487620)\beta^2 \\
&- m^{11}(117900m^7 - 13020400m^6 - 21299501m^5 + 927289m^4 + 8435386m^3 \\
&\quad - 3021236m^2 - 805068m + 75600)\beta \\
&- m^{13}(1783200m^5 + 3656990m^4 + 1576777m^3 - 276422m^2 + 248002m + 105988)],
\end{aligned}
$$

where $\beta = (m+1)\alpha - 1 \in (0,1)$. Similarly as case 1(ii), we can show that

$$
\frac{\partial^i}{\partial m^i}\Big(\frac{105m^5(\beta-m)^7(\beta+1)^6}{m+1}\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=\frac{m\alpha}{1-\alpha}}\Big)|_{m=3} > 0,
$$

for $1 \le i \le 19$,

$$
\begin{aligned}
&\frac{105m^5(\beta-m)^7(\beta+1)^6}{m+1}\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=\frac{m\alpha}{1-\alpha}}|_{m=3} \\
=& - 1215110400\beta^{13} - 2880296160\beta^{12} + 1379085120\beta^{11} \\
&+ 8794440480\beta^{10} + 4120868640\beta^9 - 5386711680\beta^8 \\
&- 9054954720\beta^7 - 202222964160\beta^6 + 414303603840\beta^5 \\
&+ 783606432000\beta^4 - 5112740290560\beta^3 + 9783137710080\beta^2 \\
&- 8792667340800\beta + 3137980661760 > 0,
\end{aligned}
$$

for $\beta \in (0,1)$. Thus by Lemma 3.1, $\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=\frac{m\alpha}{1-\alpha}} > 0$, for $\alpha \in (\frac{1}{m+1}, \frac{2}{m+2}]$, $m \ge 2$, m integer. By Lemma 3.2, we know that $\frac{\partial \Delta_m}{\partial A_7} > 0$, for $A_1 \ge 7, A_2 \ge 6, A_3 \ge 5, A_4 \ge 4, A_5 \ge 3, A_6 \ge 2, A_7 \ge \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{1}{m+1}, \frac{2}{m+2}]$, $m \ge 2$, m integer.

(vii) $\Delta_m > 0$ for $m = 1$, $a_8 \in (2,3]$, $\alpha = 1 - \frac{1}{a_8} \in (\frac{1}{2}, \frac{2}{3}]$. By Proposition 3.1, $\frac{\partial \Delta_1}{\partial A_i} > 0$, for $1 \le i \le 6$, $A_1 \ge \cdots \ge A_7 \ge 1$, $\alpha \in (0,1)$.
By Proposition 3.4, $\frac{\partial \Delta_1}{\partial A_7} > 0$, for $A_1 \ge 7, A_2 \ge 6, A_3 \ge 5, A_4 \ge 4, A_5 \ge$

$3, A_6 \geq 2, A_7 \geq 1$, $\alpha \in (0,1)$. Since $\frac{\alpha}{1-\alpha} \in (1,2]$ here, $\frac{\partial \Delta_1}{\partial A_7} > 0$, for $A_1 \geq 7, A_2 \geq 6, A_3 \geq 5, A_4 \geq 4, A_5 \geq 3, A_6 \geq 2, A_7 \geq \frac{\alpha}{1-\alpha}$, $\alpha \in (0,1)$. And

$$\begin{aligned}
&\Delta_1|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=2,A_7=\frac{\alpha}{1-\alpha}}\\
&= -\frac{1}{(1-\alpha)^7\alpha^4}(50816\alpha^{11} - 243985\alpha^{10} + 509763\alpha^9 - 606517\alpha^8\\
&\quad + 456486\alpha^7 - 251440\alpha^6 + 165380\alpha^5 - 170892\alpha^4\\
&\quad + 157865\alpha^3 - 95644\alpha^2 + 33228\alpha - 5040) > 0
\end{aligned}$$

for $\alpha \in (\frac{1}{2}, \frac{2}{3}]$. Thus, $\Delta_1 > 0$, for $A_1 \geq 7, A_2 \geq 6, A_3 \geq 5, A_4 \geq 4, A_5 \geq 3, A_6 \geq 2, A_7 \geq \frac{\alpha}{1-\alpha}$, $\alpha \in (\frac{1}{2}, \frac{2}{3}]$.

Therefore, $\Delta_m > 0$, for $A_1 \geq 7, A_2 \geq 6, A_3 \geq 5, A_4 \geq 4, A_5 \geq 3, A_6 \geq 2, A_7 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{1}{m+1}, \frac{2}{m+2}]$, $m \geq 1$, m integer.

3.3. Case 3: $a_8 \in (m+2, m+3]$

Since $a_8 \in (m+2, m+3]$, so $\alpha \in (\frac{2}{m+2}, \frac{3}{m+3}]$. In this case $A_1 \geq 7, A_2 \geq 6, \ldots, A_5 \geq 3, A_6 \geq A_7 \geq \frac{m\alpha}{1-\alpha}$.

Since $\frac{m\alpha}{1-\alpha} = a_8 - m \in (2,3]$, so $A_1 \geq A_2 \geq \cdots \geq A_7 \geq \frac{m\alpha}{1-\alpha}$. By Proposition 3.1, $\frac{\partial \Delta_m}{\partial A_i} > 0, \frac{\partial^2 \Delta_m}{\partial A_j \partial A_7} > 0$, for $1 \leq i, j \leq 6$, $A_1 \geq \cdots \geq A_7 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (0,1)$, $m \geq 1$, m integer.

By Proposition 3.5, we know that $\frac{\partial \Delta_m}{\partial A_7} > 0$, for $A_1 \geq 7, A_2 \geq 6, A_3 \geq 5, A_4 \geq 4, A_5 \geq 3, A_6 \geq 2, a_7 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{2}{m+2}, \frac{3}{m+3}]$, $m \geq 2$, m integer. Since $\frac{m\alpha}{1-\alpha} \in (2,3]$ here, $\frac{\partial \Delta_m}{\partial A_7} > 0$, for $A_1 \geq 7, A_2 \geq 6, A_3 \geq 5, A_4 \geq 4, A_5 \geq 3, A_6 \geq \frac{m\alpha}{1-\alpha}, A_7 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{2}{m+2}, \frac{3}{m+3}]$, $m \geq 2$, m integer.

(i) $\Delta_m > 0$, for $A_1 \geq 7, A_2 \geq 6, A_3 \geq 5, A_4 \geq 4, A_5 \geq 3, A_6 \geq \frac{m\alpha}{1-\alpha}, A_7 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{2}{m+2}, \frac{3}{m+3}]$, $m \geq 2$, m integer.

The same argument as Case 7(i), we can show that

$$\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=\frac{m\alpha}{1-\alpha},A_7=\frac{m\alpha}{1-\alpha}} > 0,$$

for $\alpha \in (\frac{2}{m+2}, \frac{3}{m+3}]$, $m \geq 2$, m integer. By Lemma 3.2, we know that $\Delta_m > 0$, for $A_1 \geq 7, A_2 \geq 6, A_3 \geq 5, A_4 \geq 4, A_5 \geq 3, A_6 \geq \frac{m\alpha}{1-\alpha}, A_7 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{2}{m+2}, \frac{3}{m+3}]$, $m \geq 2$, m integer.

(ii) $\Delta_m > 0$, for $m = 1$, $a_8 \in (3, 4]$, $\alpha = 1 - \frac{1}{a_8} \in (\frac{2}{3}, \frac{3}{4}]$. By Proposition 3.1, $\frac{\partial \Delta_1}{\partial A_i} > 0$, for $1 \le i \le 6$, $A_1 \ge \cdots \ge A_7 \ge 1$, $\alpha \in (0, 1)$.
By Proposition 3.4, $\frac{\partial \Delta_1}{\partial A_7} > 0$, for $A_1 \ge 7, A_2 \ge 6, A_3 \ge 5, A_4 \ge 4, A_5 \ge 3, A_6 \ge 2, A_7 \ge 1$, $\alpha \in (0, 1)$. Since $\frac{\alpha}{1-\alpha} \in (2, 3]$ here, $\frac{\partial \Delta_1}{\partial A_7} > 0$, for $A_1 \ge 7, A_2 \ge 6, A_3 \ge 5, A_4 \ge 4, A_5 \ge 3, A_6 \ge \frac{\alpha}{1-\alpha}, A_7 \ge \frac{\alpha}{1-\alpha}$, $\alpha \in (0, 1)$.
And

$$\begin{aligned}
&\Delta_1|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=3,A_6=A_7=\frac{\alpha}{1-\alpha}} \\
&= \frac{1}{\alpha^2(\alpha-1)^7}(16256\alpha^9 - 30862\alpha^8 - 49046\alpha^7 + 200994\alpha^6 - 242343\alpha^5 \\
&\qquad + 131613\alpha^4 - 15363\alpha^3 - 21543\alpha^2 + 12834\alpha - 2520) > 0,
\end{aligned}$$

for $\alpha \in (\frac{2}{3}, \frac{3}{4}]$. Thus, $\Delta_1 > 0$, for $A_1 \ge 7, A_2 \ge 6, A_3 \ge 5, A_4 \ge 4, A_5 \ge 3, A_6 \ge \frac{\alpha}{1-\alpha}, A_7 \ge \frac{\alpha}{1-\alpha}$, $\alpha \in (\frac{2}{3}, \frac{3}{4}]$.

Therefore, $\Delta_m > 0$, for $A_1 \ge 7, A_2 \ge 6, A_3 \ge 5, A_4 \ge 4, A_5 \ge 3, A_6 \ge \frac{m\alpha}{1-\alpha}, A_7 \ge \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{2}{m+2}, \frac{3}{m+3}]$, $m \ge 1$, m integer.

3.4. Case 4: $a_8 \in (m+3, m+4]$

For $a_8 \in (m+3, m+4]$, $\alpha \in (\frac{3}{m+3}, \frac{4}{m+4}]$, $A_1 \ge 7, A_2 \ge 6, A_3 \ge 5, A_4 \ge 4, A_5 \ge A_6 \ge A_7 \ge \frac{m\alpha}{1-\alpha}$.

In this case, $\frac{m\alpha}{1-\alpha} \in (3, 4]$, so $A_1 \ge A_2 \ge \cdots \ge A_7 \ge \frac{m\alpha}{1-\alpha}$. By Proposition 3.1, $\frac{\partial \Delta_m}{\partial A_i} > 0, \frac{\partial^2 \Delta_m}{\partial A_j \partial A_7} > 0$, for $1 \le i, j \le 6$, $A_1 \ge \cdots \ge A_7 \ge \frac{m\alpha}{1-\alpha}$, $\alpha \in (0, 1)$, $m \ge 1$, m integer.

By Proposition 3.5, we know that $\frac{\partial \Delta_m}{\partial A_7} > 0$, for $A_1 \ge 7, A_2 \ge 6, A_3 \ge 5, A_4 \ge 4, A_5 \ge 3, A_6 \ge 2, A_7 \ge \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{1}{m+1}, 1]$, $m \ge 2$, m integer. Since $\frac{m\alpha}{1-\alpha} \in (3, 4]$ here, $\frac{\partial \Delta_m}{\partial A_7} > 0$, for $A_1 \ge 7, A_2 \ge 6, A_3 \ge 5, A_4 \ge 4, A_5 \ge A_6 \ge A_7 \ge \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{3}{m+3}, \frac{4}{m+4}]$, $m \ge 2$, m integer.

(i) $\Delta_m > 0$, for $A_1 \ge 7, A_2 \ge 6, A_3 \ge 5, A_4 \ge 4, A_5 \ge \frac{m\alpha}{1-\alpha}, A_6 \ge \frac{m\alpha}{1-\alpha}, A_7 \ge \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{3}{m+3}, \frac{4}{m+4}]$, $m \ge 2$, m integer.
The same argument as Case 7(i), we can show that

$$\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=\frac{m\alpha}{1-\alpha},A_6=\frac{m\alpha}{1-\alpha},A_7=\frac{m\alpha}{1-\alpha}} > 0,$$

for $\alpha \in (\frac{3}{m+3}, \frac{4}{m+4}]$, $m \ge 2$, m integer. By Lemma 3.2, we know that $\Delta_m > 0$, for $A_1 \ge 7, A_2 \ge 6, A_3 \ge 5, A_4 \ge 4, A_5 \ge \frac{m\alpha}{1-\alpha}, A_6 \ge \frac{m\alpha}{1-\alpha}, A_7 \ge \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{3}{m+3}, \frac{4}{m+4}]$, $m \ge 2$, m integer.

(ii) $\Delta_m > 0$, for $m = 1$ $a_8 \in (4,5]$, $\alpha = 1 - \frac{1}{a_8} \in (\frac{3}{4}, \frac{4}{5}]$. By Proposition 3.1, $\frac{\partial \Delta_1}{\partial A_i} > 0$, for $1 \le i \le 6$, $A_1 \ge \cdots \ge A_7 \ge \frac{m\alpha}{1-\alpha}$, $\alpha \in (0,1)$.
By Proposition 3.4, $\frac{\partial \Delta_1}{\partial A_7} > 0$, for $A_1 \ge 7, A_2 \ge 6, A_3 \ge 5, A_4 \ge 4, A_5 \ge 3, A_6 \ge 2, A_7 \ge 1$, $\alpha \in (0,1)$. Since $\frac{\alpha}{1-\alpha} \in (3,4]$ here, $\frac{\partial \Delta_1}{\partial A_7} > 0$, for $A_1 \ge 7, A_2 \ge 6, A_3 \ge 5, A_4 \ge 4, A_5 \ge \frac{\alpha}{1-\alpha}, A_6 \ge \frac{\alpha}{1-\alpha}, A_7 \ge \frac{\alpha}{1-\alpha}$, $\alpha \in (0,1)$.
And

$$\begin{aligned}
&\Delta_1|_{A_1=7,A_2=6,A_3=5,A_4=4,A_5=A_6=A_7=\frac{\alpha}{1-\alpha}} \\
&= -\frac{1}{(1-\alpha)^7}(62336\alpha^7 - 295818\alpha^6 + 595971\alpha^5 - 659059\alpha^4 \\
&\qquad + 431625\alpha^3 - 168209\alpha^2 + 36886\alpha - 3712) > 0,
\end{aligned}$$

for $\alpha \in (\frac{3}{4}, \frac{4}{5}]$. Thus, $\Delta_1 > 0$, for $A_1 \ge 7, A_2 \ge 6, A_3 \ge 5, A_4 \ge 4, A_5 \ge \frac{\alpha}{1-\alpha}, A_6 \ge \frac{\alpha}{1-\alpha}, A_7 \ge \frac{\alpha}{1-\alpha}$, $\alpha \in (\frac{3}{4}, \frac{4}{5}]$.

Therefore, $\Delta_m > 0$, for $A_1 \ge 7, A_2 \ge 6, A_3 \ge 5, A_4 \ge 4, A_5 \ge \frac{m\alpha}{1-\alpha}, A_6 \ge \frac{m\alpha}{1-\alpha}, A_7 \ge \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{3}{m+3}, \frac{4}{m+4}]$, $m \ge 1$, m integer.

3.5. Case 5: $a_8 \in (m+4, m+5]$

For $a_8 \in (m+4, m+5]$, $\alpha \in (\frac{4}{m+4}, \frac{5}{m+5}]$, $A_1 \ge 7, A_2 \ge 6, A_3 \ge 5, A_4 \ge \frac{m\alpha}{1-\alpha}, A_5 \ge \frac{m\alpha}{1-\alpha}, A_6 \ge \frac{m\alpha}{1-\alpha}, A_7 \ge \frac{m\alpha}{1-\alpha}$.

In this case, $\frac{m\alpha}{1-\alpha} \in (4,5]$, so $A_1 \ge A_2 \ge \cdots \ge A_7 \ge \frac{m\alpha}{1-\alpha}$. By Proposition 3.1, $\frac{\partial \Delta_m}{\partial A_i} > 0, \frac{\partial^2 \Delta_m}{\partial A_j \partial A_7} > 0$, for $1 \le i,j \le 6$, $A_1 \ge \cdots \ge A_7 \ge \frac{m\alpha}{1-\alpha}$, $\alpha \in (0,1)$, $m \ge 1$, m integer.

By Proposition 3.5, we know that $\frac{\partial \Delta_m}{\partial A_7} > 0$, for $A_1 \ge 7, A_2 \ge 6, A_3 \ge 5, A_4 \ge 4, A_5 \ge 3, A_6 \ge 2, A_7 \ge \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{1}{m+1}, 1]$, $m \ge 2$, m integer. Since $\frac{m\alpha}{1-\alpha} \in (4,5]$ here, $\frac{\partial \Delta_m}{\partial A_7} > 0$, for $A_1 \ge 7, A_2 \ge 6, A_3 \ge 5, A_4 \ge \frac{m\alpha}{1-\alpha}, A_5 \ge \frac{m\alpha}{1-\alpha}, A_6 \ge \frac{m\alpha}{1-\alpha}, A_7 \ge \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{4}{m+4}, \frac{5}{m+5}]$, $m \ge 2$, m integer.

(i) $\Delta_m > 0$, for $A_1 \ge 7, A_2 \ge 6, A_3 \ge 5, A_4 \ge \frac{m\alpha}{1-\alpha}, A_5 \ge \frac{m\alpha}{1-\alpha}, A_6 \ge \frac{m\alpha}{1-\alpha}, A_7 \ge \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{4}{m+4}, \frac{5}{m+5}]$, $m \ge 2$, m integer.
The same argument as Case 7(i), we can show that

$$\Delta_m|_{A_1=7,A_2=6,A_3=5,A_4=\frac{m\alpha}{1-\alpha},A_5=\frac{m\alpha}{1-\alpha},A_6=\frac{m\alpha}{1-\alpha},A_7=\frac{m\alpha}{1-\alpha}} > 0,$$

for $\alpha \in (\frac{4}{m+4}, \frac{5}{m+5}]$, $m \ge 2$, m integer. By Lemma 3.2, we know that $\Delta_m > 0$, for $A_1 \ge 7, A_2 \ge 6, A_3 \ge 5, A_4 \ge \frac{m\alpha}{1-\alpha}, A_5 \ge \frac{m\alpha}{1-\alpha}, A_6 \ge \frac{m\alpha}{1-\alpha}, A_7 \ge \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{4}{m+4}, \frac{5}{m+5}]$, $m \ge 2$, m integer.

(ii) $\Delta_m > 0$, for$m = 1$, $a_8 \in (5, 6]$, $\alpha = 1 - \frac{1}{a_8} \in (\frac{4}{5}, \frac{5}{6}]$. By Proposition 3.1, $\frac{\partial \Delta_1}{\partial A_i} > 0$, for $1 \le i \le 6$, $A_1 \ge \cdots \ge A_7 \ge 1$, $\alpha \in (0, 1)$.
By Proposition 3.4, $\frac{\partial \Delta_1}{\partial A_7} > 0$, for $A_1 \ge 7, A_2 \ge 6, A_3 \ge 5, A_4 \ge 4, A_5 \ge 3, A_6 \ge 2, A_7 \ge 1$, $\alpha \in (0, 1)$. Since $\frac{\alpha}{1-\alpha} \in (4, 5]$ here, $\frac{\partial \Delta_1}{\partial A_7} > 0$, for $A_1 \ge 7, A_2 \ge 6, A_3 \ge 5, A_4 \ge \frac{\alpha}{1-\alpha}, A_5 \ge \frac{\alpha}{1-\alpha}, A_6 \ge \frac{\alpha}{1-\alpha}, A_7 \ge \frac{\alpha}{1-\alpha}$, $\alpha \in (0, 1)$.
And

$$\begin{aligned} &\Delta_1|_{A_1=7,A_2=6,A_3=5,A_4=A_5=A_6=A_7=\frac{\alpha}{1-\alpha}} \\ &= -\frac{1}{(1-\alpha)^7}(23936\alpha^7 - 92293\alpha^6 + 140827\alpha^5 - 102419\alpha^4 \\ &\qquad + 28519\alpha^3 + 6114\alpha^2 - 5632\alpha + 968) > 0, \end{aligned}$$

for $\alpha \in (\frac{4}{5}, \frac{5}{6}]$. Thus, $\Delta_1 > 0$, for $A_1 \ge 7, A_2 \ge 6, A_3 \ge 5, A_4 \ge \frac{\alpha}{1-\alpha}, A_5 \ge \frac{\alpha}{1-\alpha}, A_6 \ge \frac{\alpha}{1-\alpha}, A_7 \ge \frac{\alpha}{1-\alpha}$, $\alpha \in (\frac{4}{5}, \frac{5}{6}]$.

Therefore, $\Delta_m > 0$, for $A_1 \ge 7, A_2 \ge 6, A_3 \ge 5, A_4 \ge \frac{m\alpha}{1-\alpha}, A_5 \ge \frac{m\alpha}{1-\alpha}, A_6 \ge \frac{m\alpha}{1-\alpha}, A_7 \ge \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{4}{m+4}, \frac{5}{m+5}]$, $m \ge 1$, m integer.

3.6. Case 6: $a_8 \in (m+5, m+6)$

For $a_8 \in (m+5, m+6)$, $\alpha \in (\frac{5}{m+5}, \frac{6}{m+6})$, $A_1 \ge 7, A_2 \ge 6, A_3 \ge \frac{m\alpha}{1-\alpha}, A_4 \ge \frac{m\alpha}{1-\alpha}, A_5 \ge \frac{m\alpha}{1-\alpha}, A_6 \ge \frac{m\alpha}{1-\alpha}, A_7 \ge \frac{m\alpha}{1-\alpha}$.

In this case, $\frac{m\alpha}{1-\alpha} \in (5, 6)$, so $A_1 \ge A_2 \ge \cdots \ge A_7 \ge \frac{m\alpha}{1-\alpha}$. By Proposition 3.1, $\frac{\partial \Delta_m}{\partial A_i} > 0, \frac{\partial^2 \Delta_m}{\partial A_j \partial A_7} > 0$, for $1 \le i, j \le 6$, $A_1 \ge \cdots \ge A_7 \ge \frac{m\alpha}{1-\alpha}$, $\alpha \in (0, 1)$, $m \ge 1$, m integer.

By Proposition 3.5, we know that $\frac{\partial \Delta_m}{\partial A_7} > 0$, for $A_1 \ge 7, A_2 \ge 6, A_3 \ge \frac{m\alpha}{1-\alpha}, A_4 \ge \frac{m\alpha}{1-\alpha}, A_5 \ge \frac{m\alpha}{1-\alpha}, A_6 \ge \frac{m\alpha}{1-\alpha}, A_7 \ge \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{1}{m+1}, 1]$, $m \ge 2$, m integer. Since $\frac{m\alpha}{1-\alpha} \in (5, 6)$ here, $\frac{\partial \Delta_m}{\partial A_7} > 0$, for $A_1 \ge 7, A_2 \ge 6, A_3 \ge \frac{m\alpha}{1-\alpha}, A_4 \ge \frac{m\alpha}{1-\alpha}, A_5 \ge \frac{m\alpha}{1-\alpha}, A_6 \ge \frac{m\alpha}{1-\alpha}, A_7 \ge \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{5}{m+5}, \frac{6}{m+6}]$, $m \ge 2$, m integer.

(i) $\Delta_m > 0$, for $A_1 \ge 7, A_2 \ge 6, A_3 \ge \frac{m\alpha}{1-\alpha}, A_4 \ge \frac{m\alpha}{1-\alpha}, A_5 \ge \frac{m\alpha}{1-\alpha}, A_6 \ge \frac{m\alpha}{1-\alpha}, A_7 \ge \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{5}{m+5}, \frac{6}{m+6})$, $m \ge 2$, m integer.
The same argument as Case 7(i), we can show that

$$\Delta_m|_{A_1=7,A_2=6,A_3=\frac{m\alpha}{1-\alpha},A_4=\frac{m\alpha}{1-\alpha},A_5=\frac{m\alpha}{1-\alpha},A_6=\frac{m\alpha}{1-\alpha},A_7=\frac{m\alpha}{1-\alpha}} > 0,$$

for $\alpha \in (\frac{5}{m+5}, \frac{6}{m+6})$, $m \geq 2$, m integer. By Lemma 3.2, we know that $\Delta_m > 0$, for $A_1 \geq 7, A_2 \geq 6, A_3 \geq \frac{m\alpha}{1-\alpha}, A_4 \geq \frac{m\alpha}{1-\alpha}, A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}, A_7 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{5}{m+5}, \frac{6}{m+6})$, $m \geq 2$, m integer.

(ii) $\Delta_m > 0$, for $m = 1$, $a_8 \in (6, 7)$, $\alpha = 1 - \frac{1}{a_8} \in (\frac{5}{6}, \frac{6}{7})$. By Proposition 3.1, $\frac{\partial \Delta_1}{\partial A_i} > 0$, for $1 \leq i \leq 6$, $A_1 \geq \cdots \geq A_7 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (0, 1)$.
By Proposition 3.4, $\frac{\partial \Delta_1}{\partial A_7} > 0$, for $A_1 \geq 7, A_2 \geq 6, A_3 \geq 5, A_4 \geq 4, A_5 \geq 3, A_6 \geq 2, A_7 \geq 1$, $\alpha \in (0, 1)$. Since $\frac{\alpha}{1-\alpha} \in (5, 6)$ here, $\frac{\partial \Delta_1}{\partial A_7} > 0$, for $A_1 \geq 7, A_2 \geq 6, A_3 \geq \frac{\alpha}{1-\alpha}, A_4 \geq \frac{\alpha}{1-\alpha}, A_5 \geq \frac{\alpha}{1-\alpha}, A_6 \geq \frac{\alpha}{1-\alpha}, A_7 \geq \frac{\alpha}{1-\alpha}$, $\alpha \in (0, 1)$.
And

$$\begin{aligned}
&\Delta_1|_{A_1=7, A_2=6, A_3=A_4=A_5=A_6=A_7=\frac{\alpha}{1-\alpha}} \\
&= -\frac{1}{(1-\alpha)^7}(46976\alpha^7 - 203647\alpha^6 + 367298\alpha^5 - 353539\alpha^4 \\
&\qquad + 192632\alpha^3 - 57276\alpha^2 + 7808\alpha - 232) > 0,
\end{aligned}$$

for $\alpha \in (\frac{5}{6}, \frac{6}{7})$. Thus, $\Delta_1 > 0$, for $A_1 \geq 7, A_2 \geq 6, A_3 \geq \frac{\alpha}{1-\alpha}, A_4 \geq \frac{\alpha}{1-\alpha}, A_5 \geq \frac{\alpha}{1-\alpha}, A_6 \geq \frac{\alpha}{1-\alpha}, A_7 \geq \frac{\alpha}{1-\alpha}$, $\alpha \in (\frac{5}{6}, \frac{6}{7})$.

Therefore, $\Delta_m > 0$, for $A_1 \geq 7, A_2 \geq 6, A_3 \geq \frac{m\alpha}{1-\alpha}, A_4 \geq \frac{m\alpha}{1-\alpha}, A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}, A_7 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{5}{m+5}, \frac{6}{m+6})$, $m \geq 1$, m integer.

3.7. Case 7: $a_8 \in (m+6, m+7)$

For $a_8 \in (m+6, m+7)$, $\alpha \in (\frac{6}{m+6}, \frac{7}{m+7})$, $A_1 \geq 7, A_2 \geq \frac{m\alpha}{1-\alpha}, A_3 \geq \frac{m\alpha}{1-\alpha}, A_4 \geq \frac{m\alpha}{1-\alpha}, A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}, A_7 \geq \frac{m\alpha}{1-\alpha}$.

In this case, $\frac{m\alpha}{1-\alpha} \in (6, 7)$, so $A_1 \geq A_2 \geq \cdots \geq A_7 \geq \frac{m\alpha}{1-\alpha}$. By Proposition 3.1, $\frac{\partial \Delta_m}{\partial A_i} > 0, \frac{\partial^2 \Delta_m}{\partial A_j \partial A_7} > 0$, for $1 \leq i, j \leq 6$, $A_1 \geq A_2 \geq \cdots \geq A_7 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (0, 1)$, $m \geq 1$, m integer.

By Proposition 3.5, we know that $\frac{\partial \Delta_m}{\partial A_7} > 0$, for $A_1 \geq 7, A_2 \geq 6, A_3 \geq 5, A_4 \geq 4, A_5 \geq 3, A_6 \geq 2, A_7 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{1}{m+1}, 1]$, $m \geq 2$, m integer. Since $\frac{m\alpha}{1-\alpha} \in (6, 7)$ here, $\frac{\partial \Delta_m}{\partial A_7} > 0$, for $A_1 \geq 7, A_2 \geq \frac{m\alpha}{1-\alpha}, A_3 \geq \frac{m\alpha}{1-\alpha}, A_4 \geq \frac{m\alpha}{1-\alpha}, A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}, A_7 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{6}{m+6}, \frac{7}{m+7}]$, $m \geq 2$, m integer.

(i) $\Delta_m > 0$, for $A_1 \geq 7, A_2 \geq \frac{m\alpha}{1-\alpha}, A_3 \geq \frac{m\alpha}{1-\alpha}, A_4 \geq \frac{m\alpha}{1-\alpha}, A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}, A_7 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{6}{m+6}, \frac{7}{m+7})$, $m \geq 2$, m integer. Let

$$\beta = \frac{\alpha - \frac{6}{m+6}}{\frac{7}{m+7} - \frac{6}{m+6}} \in (0,1).$$

$$\begin{aligned}
&\Delta_m|_{A_1=7,A_2=A_3=A_4=A_5=A_6=A_7=\frac{m\alpha}{1-\alpha}} \\
&= \frac{(m+7)(\beta-1)}{21(\beta m+6m+42)(\beta-m-7)^7}[m(m+2)(m+1) \\
&\quad (213m^4+2014m^3+6531m^2+7172m-82)\beta^7 \\
&\quad +7(m+7)(899m^6+9930m^5+42513m^4+85864m^3+78634m^2 \\
&\qquad +24329m-18)\beta^6 \\
&\quad +21(3739m^5+35479m^4+127120m^3+205211m^2 \\
&\qquad +137470m+25704)(m+7)^2\beta^5 \\
&\quad +7(76691m^4+597557m^3+1677445m^2+1939303m+744174)(m+7)^3\beta^4 \\
&\quad +28(92m^4+79615m^3+477598m^2+946244m+607644)(m+7)^4\beta^3 \\
&\quad +21(81m^5+2062m^4+20930m^3+355468m^2+1270445m \\
&\qquad +1267326)(m+7)^5\beta^2 \\
&\quad +7(83m^6+2502m^5+31344m^4+208892m^3+781177m^2 \\
&\qquad +2550130m+3297240)(m+7)^6\beta \\
&\quad +(81m^7+2821m^6+42021m^5+347074m^4+1716883m^3 \\
&\qquad +5087131m^2+8360831m+9843750)(m+7)^7]
\end{aligned}$$

$$\frac{\partial^{14}}{\partial m^{14}}\left(\frac{21(\beta m+6m+42)(\beta-m-7)^7}{(m+7)(\beta-1)}\Delta_m|_{A_1=7,A_2=A_3=A_4=A_5=A_6=A_7=\frac{m\alpha}{1-\alpha}}\right)$$
$$=7061441587200>0,$$

$$\frac{\partial^{13}}{\partial m^{13}}\left(\frac{21(\beta m+6m+42)(\beta-m-7)^7}{(m+7)(\beta-1)}\Delta_m|_{A_1=7,A_2=A_3=A_4=A_5=A_6=A_7=\frac{m\alpha}{1-\alpha}}\right)|_{m=2}$$
$$=56404354406400>0,$$

$$\frac{\partial^{12}}{\partial m^{12}}\left(\frac{21(\beta m+6m+42)(\beta-m-7)^7}{(m+7)(\beta-1)}\Delta_m|_{A_1=7,A_2=A_3=A_4=A_5=A_6=A_7=\frac{m\alpha}{1-\alpha}}\right)|_{m=2}$$
$$=278299929600\beta+224950168396800>0,$$

$$\frac{\partial^{11}}{\partial m^{11}}\left(\frac{21(\beta m+6m+42)(\beta-m-7)^7}{(m+7)(\beta-1)}\Delta_m|_{A_1=7,A_2=A_3=A_4=A_5=A_6=A_7=\frac{m\alpha}{1-\alpha}}\right)|_{m=2}$$
$$=2229752448000\beta+597236957856000>0,$$

$$\frac{\partial^{10}}{\partial m^{10}}\left(\frac{21(\beta m+6m+42)(\beta-m-7)^7}{(m+7)(\beta-1)}\Delta_m|_{A_1=7,A_2=A_3=A_4=A_5=A_6=A_7=\frac{m\alpha}{1-\alpha}}\right)|_{m=2}$$

$$=6172588800\beta^2+8918019129600\beta+1187517484339200>0,$$

$$\frac{\partial^9}{\partial m^9}(\frac{21(\beta m+6m+42)(\beta-m-7)^7}{(m+7)(\beta-1)}\Delta_m|_{A_1=7,A_2=A_3=A_4=A_5=A_6=A_7=\frac{m\alpha}{1-\alpha}})|_{m=2}$$
$$=49662668160\beta^2+23739801926400\beta+1886206336963200>0,$$

$$\frac{\partial^8}{\partial m^8}(\frac{21(\beta m+6m+42)(\beta-m-7)^7}{(m+7)(\beta-1)}\Delta_m|_{A_1=7,A_2=A_3=A_4=A_5=A_6=A_7=\frac{m\alpha}{1-\alpha}})|_{m=2}$$
$$=103864320\beta^3+199416107520\beta^2+47317910250240\beta+2492972654941440>0,$$

$$\frac{\partial^7}{\partial m^7}(\frac{21(\beta m+6m+42)(\beta-m-7)^7}{(m+7)(\beta-1)}\Delta_m|_{A_1=7,A_2=A_3=A_4=A_5=A_6=A_7=\frac{m\alpha}{1-\alpha}})|_{m=2}$$
$$=1073520\beta^7+31716720\beta^6+395735760\beta^5$$
$$+2705658480\beta^4+11806522560\beta^3+559236968640\beta^2$$
$$+75359096346240\beta+2820053922305760>0,$$

$$\frac{\partial^6}{\partial m^6}(\frac{21(\beta m+6m+42)(\beta-m-7)^7}{(m+7)(\beta-1)}\Delta_m|_{A_1=7,A_2=A_3=A_4=A_5=A_6=A_7=\frac{m\alpha}{1-\alpha}})|_{m=2}$$
$$=4057200\beta^7+145197360\beta^6+2119385520\beta^5$$
$$+16539979680\beta^4+78520055040\beta^3+1265773914720\beta^2$$
$$+100042761057120\beta+2787262594900560>0,$$

$$\frac{\partial^5}{\partial m^5}(\frac{21(\beta m+6m+42)(\beta-m-7)^7}{(m+7)(\beta-1)}\Delta_m|_{A_1=7,A_2=A_3=A_4=A_5=A_6=A_7=\frac{m\alpha}{1-\alpha}})|_{m=2}$$
$$=7527240\beta^7+321060600\beta^6+5481032760\beta^5$$
$$+49088406360\beta^4+260748344640\beta^3+2448400464720\beta^2$$
$$+114232430844000\beta+2445586839823680>0,$$

$$\frac{\partial^4}{\partial m^4}(\frac{21(\beta m+6m+42)(\beta-m-7)^7}{(m+7)(\beta-1)}\Delta_m|_{A_1=7,A_2=A_3=A_4=A_5=A_6=A_7=\frac{m\alpha}{1-\alpha}})|_{m=2}$$
$$=9110472\beta^7+458435880\beta^6+9127516608\beta^5$$
$$+94124736720\beta^4+566252507520\beta^3+4085366416992\beta^2$$
$$+115136717756760\beta+1929571639812936>0,$$

$$\frac{\partial^3}{\partial m^3}(\frac{21(\beta m+6m+42)(\beta-m-7)^7}{(m+7)(\beta-1)}\Delta_m|_{A_1=7,A_2=A_3=A_4=A_5=A_6=A_7=\frac{m\alpha}{1-\alpha}})|_{m=2}$$
$$=8067360\beta^7+475749204\beta^6+11019143520\beta^5$$
$$+131028866766\beta^4+899652227544\beta^3+5847046017624\beta^2$$
$$+104828316141528\beta+1384372017151830>0,$$

$$\frac{\partial^2}{\partial m^2}\left(\frac{21(\beta m+6m+42)(\beta-m-7)^7}{(m+7)(\beta-1)}\Delta_m|_{A_1=7,A_2=A_3=A_4=A_5=A_6=A_7=\frac{m\alpha}{1-\alpha}}\right)|_{m=2}$$
$$=5559764\beta^7+382413738\beta^6+10288139526\beta^5$$
$$+141219288000\beta^4+1112178356352\beta^3+7150734728298\beta^2$$
$$+87986423858886\beta+912869425826316>0,$$
$$\frac{\partial^1}{\partial m^1}\left(\frac{21(\beta m+6m+42)(\beta-m-7)^7}{(m+7)(\beta-1)}\Delta_m|_{A_1=7,A_2=A_3=A_4=A_5=A_6=A_7=\frac{m\alpha}{1-\alpha}}\right)|_{m=2}$$
$$=3100276\beta^7+247628535\beta^6+7732998882\beta^5$$
$$+122747067009\beta^4+1113072565248\beta^3+7482752324649\beta^2$$
$$+69146975749194\beta+559812321627327>0,$$
$$\left(\frac{21(\beta m+6m+42)(\beta-m-7)^7}{(m+7)(\beta-1)}\Delta_m|_{A_1=7,A_2=A_3=A_4=A_5=A_6=A_7=\frac{m\alpha}{1-\alpha}}\right)|_{m=2}$$
$$=1437744\beta^7+132652296\beta^6+4806617760\beta^5$$
$$+88486387416\beta^4+927526260528\beta^3+6737216440248\beta^2$$
$$+51263632159488\beta+323987644473192>0,$$
for $\beta\in(0,1)$.

Thus by Lemma 3.1, $\Delta_m|_{A_1=7,A_2=\frac{m\alpha}{1-\alpha},A_3=\frac{m\alpha}{1-\alpha},A_4=\frac{m\alpha}{1-\alpha},A_5=\frac{m\alpha}{1-\alpha},A_6=\frac{m\alpha}{1-\alpha},A_7=\frac{m\alpha}{1-\alpha}}>0$, for $\alpha\in(\frac{6}{m+6},\frac{7}{m+7})$, $m\geq 2$, m integer. By Lemma 3.2, we know that $\Delta_m>0$, for $A_1\geq 7, A_2\geq\frac{m\alpha}{1-\alpha}, A_3\geq\frac{m\alpha}{1-\alpha}, A_4\geq\frac{m\alpha}{1-\alpha}, A_5\geq\frac{m\alpha}{1-\alpha}, A_6\geq\frac{m\alpha}{1-\alpha}, A_7\geq\frac{m\alpha}{1-\alpha}$, $\alpha\in(\frac{6}{m+6},\frac{7}{m+7})$, $m\geq 2$, m integer.

(ii) $\Delta_m>0$, for $m=1$, $a_8\in(7,8)$, $\alpha=1-\frac{1}{a_8}\in(\frac{6}{7},\frac{7}{8})$. By Proposition 3.1, $\frac{\partial\Delta_1}{\partial A_i}>0$, for $1\leq i\leq 6$, $A_1\geq\cdots\geq A_7\geq\frac{m\alpha}{1-\alpha}$, $\alpha\in(0,1)$.
By Proposition 3.4, $\frac{\partial\Delta_1}{\partial A_7}>0$, for $A_1\geq 7, A_2\geq 6, A_3\geq 5, A_4\geq 4, A_5\geq 3, A_6\geq 2, A_7\geq 1$, $\alpha\in(0,1)$. Since $\frac{\alpha}{1-\alpha}\in(6,7)$ here, $\frac{\partial\Delta_1}{\partial A_7}>0$, for $A_1\geq 7, A_2\geq\frac{\alpha}{1-\alpha}, A_3\geq\frac{\alpha}{1-\alpha}, A_4\geq\frac{\alpha}{1-\alpha}, A_5\geq\frac{\alpha}{1-\alpha}, A_6\geq\frac{\alpha}{1-\alpha}, A_7\geq\frac{\alpha}{1-\alpha}$, $\alpha\in(0,1)$.
And

$$\Delta_1|_{A_1=7,A_2=A_3=A_4=A_5=A_6=A_7=\frac{\alpha}{1-\alpha}}$$

$$=-\frac{8\alpha-7}{(1-\alpha)^7}(4528\alpha^6-15637\alpha^5+21471\alpha^4-14600\alpha^3$$
$$+4884\alpha^2-624\alpha-8>0,$$

for $\alpha \in (\frac{6}{7}, \frac{7}{8})$.

Thus, $\Delta_1 > 0$, for $A_1 \geq 7, A_2 \geq \frac{m\alpha}{1-\alpha}, A_3 \geq \frac{m\alpha}{1-\alpha}, A_4 \geq \frac{m\alpha}{1-\alpha}, A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}, A_7 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{6}{7}, \frac{7}{8})$.

Therefore, $\Delta_m > 0$, for $A_1 \geq 7, A_2 \geq \frac{m\alpha}{1-\alpha}, A_3 \geq \frac{m\alpha}{1-\alpha}, A_4 \geq \frac{m\alpha}{1-\alpha}, A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}, A_7 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{6}{m+6}, \frac{7}{m+7})$, $m \geq 1$, m integer.

3.8. Case 8: $a_8 \geq m+7$

For $a_8 \geq m+7$, $\alpha \in [\frac{7}{m+7}, 1)$.

In this case, $\frac{m\alpha}{1-\alpha} \geq 7$, so $A_1 \geq A_2 \geq A_3 \geq A_4 \geq A_5 \geq A_6 \geq A_7 \geq \frac{m\alpha}{1-\alpha}$. By Proposition 3.1, $\frac{\partial \Delta_m}{\partial A_i} > 0, \frac{\partial^2 \Delta_m}{\partial A_j \partial A_7} > 0$, for $1 \leq i, j \leq 6$, $A_1 \geq A_2 \geq A_3 \geq A_4 \geq A_5 \geq A_6 \geq A_7 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (0,1)$, $m \geq 1$, m integer.

By Proposition 3.5, we know that $\frac{\partial \Delta_m}{\partial A_7} > 0$, for $A_1 \geq 7, A_2 \geq 6, A_3 \geq 5, A_4 \geq 4, A_5 \geq 3, A_6 \geq 2, A_7 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in (\frac{1}{m+1}, 1]$, $m \geq 2$, m integer. Since $\frac{m\alpha}{1-\alpha} \geq 7$ here, $\frac{\partial \Delta_m}{\partial A_7} > 0$, for $A_1 \geq A_2 \geq A_3 \geq A_4 \geq A_5 \geq A_6 \geq A_7 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in [\frac{7}{m+7} 1)$, $m \geq 2$, m integer.

(i) $\Delta_m \geq 0$, for $A_1 \geq \frac{m\alpha}{1-\alpha}, A_2 \geq \frac{m\alpha}{1-\alpha}, A_3 \geq \frac{m\alpha}{1-\alpha}, A_4 \geq \frac{m\alpha}{1-\alpha}, A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}, A_7 \geq \frac{m\alpha}{1-\alpha}, \alpha \in [\frac{7}{m+7}, 1)$, $m \geq 2$, m integer.

$$\Delta_m|_{A_1=A_2=A_3=A_4=A_5=A_6=A_7=\frac{m\alpha}{1-\alpha}}$$
$$=\frac{m\alpha}{1-\alpha}(\frac{m\alpha}{1-\alpha}-1)(\frac{m\alpha}{1-\alpha}-2)(\frac{m\alpha}{1-\alpha}-3)(\frac{m\alpha}{1-\alpha}-4)(\frac{m\alpha}{1-\alpha}-5)$$
$$(\frac{m\alpha}{1-\alpha}-6)(\frac{m\alpha}{1-\alpha}-7).$$

Thus $\Delta_m|_{A_1=A_2=A_3=A_4=A_5=A_6=A_7=\frac{m\alpha}{1-\alpha}} \geq 0$, for $\alpha \in [\frac{7}{m+7}, 1)$, $m \geq 2$, m integer. By Lemma 3.2, $\Delta_m \geq 0$, for $A_1 \geq \frac{m\alpha}{1-\alpha}, A_2 \geq \frac{m\alpha}{1-\alpha}, A_3 \geq \frac{m\alpha}{1-\alpha}, A_4 \geq \frac{m\alpha}{1-\alpha}, A_5 \geq \frac{m\alpha}{1-\alpha}, A_6 \geq \frac{m\alpha}{1-\alpha}, A_7 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in [\frac{7}{m+7}, 1)$, $m \geq 2$, m integer. Equality holds if and only if $A_1 = A_2 = A_3 = A_4 = A_5 = A_6 = A_7 = \frac{m\alpha}{1-\alpha}$ and $\alpha = \frac{7}{m+7}$, or equivalently, $a_1 = a_2 = \cdots = a_7 = a_8 = m+7$.

(ii) $\Delta_m \geq 0$, for $m = 1$, $a_8 \geq 8$, $\alpha = 1 - \frac{1}{a_8} \in [\frac{7}{8}, 1)$. By Proposition 3.1, $\frac{\partial \Delta_1}{\partial A_i} > 0$, for $1 \leq i \leq 6$, $A_1 \geq \cdots \geq A_7 \geq \frac{\alpha}{1-\alpha}$, $\alpha \in [\frac{7}{8}, 1)$.
By Proposition 3.4, $\frac{\partial \Delta_1}{\partial A_7} > 0$, for $A_1 \geq 7, A_2 \geq 6, A_3 \geq 5, A_4 \geq 4, A_5 \geq 3, A_6 \geq 2, A_7 \geq 1$, $\alpha \in [\frac{7}{8}, 1)$. Since $\frac{\alpha}{1-\alpha} \geq 7$ here, $\frac{\partial \Delta_1}{\partial A_6} > 0$, for $A_1 \geq \cdots \geq A_7 \geq \frac{\alpha}{1-\alpha}$, $\alpha \in [\frac{7}{8}, 1)$.

And

$$\Delta_1|_{A_1=A_2=A_3=A_4=A_5=A_6=A_7=\frac{\alpha}{1-\alpha}} = \frac{\alpha}{(1-\alpha)^8}(2\alpha-1)(3\alpha-2)(4\alpha-3)(5\alpha-4)(6\alpha-5)(7\alpha-6)(8\alpha-7) \geq 0,$$

for $\alpha \in [\frac{7}{8}, 1)$. Thus, $\Delta_1 \geq 0$, for $A_1 \geq A_2 \geq \cdots \geq A_7 \geq \frac{\alpha}{1-\alpha}$, $\alpha \in [\frac{7}{8}, 1)$. Equality holds if and only if $A_1 = A_2 = A_3 = A_4 = A_5 = A_6 = A_7 = \frac{\alpha}{1-\alpha}$ and $\alpha = \frac{7}{8}$, or equivalently, $a_1 = a_2 = \cdots = a_8 = 8$.

Therefore, $\Delta_m \geq 0$, for $A_1 \geq \cdots \geq A_7 \geq \frac{m\alpha}{1-\alpha}$, $\alpha \in [\frac{7}{m+7}, 1)$, $m \geq 1$, m integer. Equality holds if and only if $A_1 = A_2 = A_3 = A_4 = A_5 = A_6 = A_7 = \frac{m\alpha}{1-\alpha}$ and $\alpha = \frac{7}{m+7}$, or equivalently, $a_1 = a_2 = \cdots = a_8 = m+7$.

3.9. Completion of the proof

To complete the proof, we only need to prove the last statement of Theorem 1.4 (i.e., the necessary and sufficient condition for the equality in (7) holds). In above we have proved that $\Delta_m = g_8 - 8\sum_{k=1}^m g_7(k) \geq 0$, with the equality holds only if $a_1 = a_2 = \cdots = a_8 = m+7$. We have

$$8!P_8 = 8!\sum_{k=1}^m P_7(k) = 8\sum_{k=1}^m 7!P_7(k) \leq 8\sum_{k=1}^m g_7(k) \leq g_8. \tag{16}$$

Therefore, in the last "$\leq$", the equality holds only if $a_1 = \cdots = a_8 = m+7$.

Notice that the sufficiency follows directly from the following result.

Theorem 3.1. *([18]) Let P_n and $a_1, \ldots, a_n$ be as the same as in the Yau Number Theoretic Conjecture. If $a_1 = \cdots = a_n =$integer, then the equality in (5) holds.*

Thus the Main Theorem A is proved.

4. Proof of the Main Theorem B

In the introduction section, we have seen that $\psi(x, y) = Q_n$. Therefore we can apply our sharp estimate of P_8 to the function $\psi(x, y)$ in order to obtain an estimate. Let $p_1 < p_2 < p_3 < p_4 < p_5 < p_6 < p_7 < p_8$ be the first seven prime numbers up to y. If $p_1^{l_1} p_2^{l_2} \cdots p_8^{l_8} \leq x$, then $\frac{l_1}{\frac{\log x}{\log p_1}} + \frac{l_2}{\frac{\log x}{\log p_2}} + \cdots + \frac{l_8}{\frac{\log x}{\log p_8}} \leq 1$. It follows that $a_i = \frac{\log x}{\log p_i}$ and $x_i = l_i$, $1 \leq i \leq 8$. Note that $Q_8 = P(a_1(1+$

$a), a_2(1+a), \cdots, a_8(1+a))$, where $a = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_8}$. We devide the estimate into five cases:

(I) $5 \le y < 7$
(II) $7 \le y < 11$
(III) $11 \le y < 13$
(IV) $13 \le y < 17$
(V) $17 \le y < 19$
(VI) $19 \le y < 23$.

Notice that Case (I)-Case (V) have been proven in [22]. We only need to prove the Case (VI) involving the first eight prime numbers: $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11$, $p_6 = 13$, $p_7 = 17$ and $p_8 = 19$. Consequently,

$$a = \frac{\log 2 + \log 3 + \log 5 + \log 7 + \log 11 + \log 13 + \log 17 + \log 19}{\log x}$$

and $e = 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19$.

$$\begin{aligned}
&\psi(x,y) = Q_8 \\
=&P(\frac{\log x}{\log 2}(1+\frac{\log e}{\log x}), \frac{\log x}{\log 3}(1+\frac{\log e}{\log x}), \frac{\log x}{\log 5}(1+\frac{\log e}{\log x}), \frac{\log x}{\log 7}(1+\frac{\log e}{\log x}), \\
&\frac{\log x}{\log 11}(1+\frac{\log e}{\log x}), \frac{\log x}{\log 13}(1+\frac{\log e}{\log x}), \frac{\log x}{\log 17}(1+\frac{\log e}{\log x})\frac{\log x}{\log 19}(1+\frac{\log e}{\log x})) \\
&\le \frac{1}{8!}[(\frac{\log x}{\log 2}+\frac{\log \frac{e}{2}}{\log 2})(\frac{\log x}{\log 3}+\frac{\log \frac{e}{3}}{\log 3})(\frac{\log x}{\log 5}+\frac{\log \frac{e}{5}}{\log 5})(\frac{\log x}{\log 7}+\frac{\log \frac{e}{7}}{\log 7}) \\
&(\frac{\log x}{\log 11}+\frac{\log \frac{e}{11}}{\log 11})(\frac{\log x}{\log 13}+\frac{\log \frac{e}{13}}{\log 13})(\frac{\log x}{\log 17}+\frac{\log \frac{e}{17}}{\log 17})(\frac{\log x}{\log 19}+\frac{\log \frac{e}{19}}{\log 19}) \\
&-\{(\frac{\log x}{\log 19}+\frac{\log \frac{e}{19}}{\log 19})^8 - (\frac{\log x}{\log 19}+\frac{\log \frac{e}{19}}{\log 19}+1)(\frac{\log x}{\log 19}+\frac{\log \frac{e}{19}}{\log 19}) \\
&(\frac{\log x}{\log 19}+\frac{\log \frac{e}{19}}{\log 19}-1)(\frac{\log x}{\log 19}+\frac{\log \frac{e}{19}}{\log 19}-2)(\frac{\log x}{\log 19}+\frac{\log \frac{e}{19}}{\log 19}-3) \\
&(\frac{\log x}{\log 19}+\frac{\log \frac{e}{19}}{\log 19}-4)(\frac{\log x}{\log 19}+\frac{\log \frac{e}{19}}{\log 19}-5)(\frac{\log x}{\log 19}+\frac{\log \frac{e}{19}}{\log 19}-6)\}] \\
&= \frac{1}{40320}\{\frac{1}{\log 2 \log 3 \log 5 \log 7 \log 11 \log 13 \log 17 \log 19} \\
&\quad (\log x + \log 4849845)(\log x + \log 3233230)(\log x + \log 1939938) \\
&\quad (\log x + \log 1385670)(\log x + \log 881790)(\log x + \log 746130) \\
&\quad (\log x + \log 570570)(\log x + \log 510510)
\end{aligned}$$

$$
\begin{aligned}
-\frac{1}{\log^8 19}[&(\log x+\log 570570)^8-(\log x+\log 19+\log 570570)\\
&(\log x+\log 570570)(\log x+\log 570570-\log 19)\\
&(\log x+\log 570570-2\log 19)(\log x+\log 570570-3\log 19)\\
&(\log x+\log 570570-4\log 19)(\log x+\log 570570-5\log 19)\\
&(\log x+\log 570570-6\log 19)]\}.
\end{aligned}
$$

References

[1] I. Chen, K.-P. Lin, S. S.-T. Yau and H. Q. Zuo. Coordinate-free characterization of homogeneous polynomials with isolated singularities. *Commun. Analy. Geo.* **19(4)**(2011), 661–704.

[2] R. Du and S. S.-T. Yau. Local holomorphic De Rham cohomology. *Commun. Analy. Geo.* **18(2)**(2010), 365–374.

[3] A. Durfee. The signature of smoothings of complex surface singularities. *Math. Ann.* **232**(1978), 85–98.

[4] E. Ehrhart. Sur un problème de gèomètrie diophantienne linèaire II. *J. Reine Angew. Math.* **227**(1967), 25–49.

[5] V. Ennola. On numbers with small prime divisors. *Ann. Acad. Sci. Fenn. Ser. Al.* **440**(1969), 1-16.

[6] K.-P. Lin, X. Luo, S. S.-T. Yau, and H. Q. Zuo. On number theoretic conjecture of positive integral points in 5-dimensional tetrahedron and a sharp estimation of Dickman-De Bruijn function. *J. Eur. Math. Soc.* **16**(2014), 90–102.

[7] K.-P. Lin, S. Raghuvanshi, S. S.-T. Yau and H. Q. Zuo. On Yau geometric conjecture for weighted homogeneous singularities. 46pp. *to appear Asian J. Math.*

[8] K.-P. Lin and S. S.-T. Yau. Counting the number of integral points in general n-dimensional tetrahedra and bernoulli polynomials. *Can. Math. Bull.* **46(2)**(2003), 229–241.

[9] X. Luo, S. S.-T. Yau, and H. Q. Zuo. A sharp estimate of Dickman-De Bruijn function and a sharp polynomial estimate of positive integral points in 4-dimension tetrahedron. *Math. Nachr.*

288(1)(2015), 61–75.

[10] A. Liang, S.S.-T. Yau and H. Q. Zuo. A sharp estimate of positive integral points in 6-dimensional polyhedra and a sharp estimate of smooth numbers. *Sci. China Math.*, **59**(2016), 425-444.

[11] J. Milnor and P. Orlik. Isolated singularities defined by weighted homogeneous polynomials. *Topology*, **9(4)**(1970), 385–393.

[12] L. J. Mordell. Lattice points in a tetrahedron and generalize Dedekind sums. *J. Indian Math.* **15**(1951), 41–46.

[13] M. Merle and B.Teissier. Conditions d'adjonction, d'après du val. In *Séminaire sur les Singularités des Surfaces.* Springer, 1980, 229–245.

[14] J. E. Pommersheim. Toric varieties, lattice points and dedekind sums. *Math. Ann.* **295(1)**(1993), 1–24.

[15] C. Pomerance. The Role of Smooth Numbers in Number Theoretic Algorithms. In *ICM Zürich*, (1994), 411–422.

[16] K. Saito. Quasihomogene isolierte singularitäten von hyperflächen. *Invent. Math.* **14(2)**(1971), 123–142.

[17] B. L. van der Waerden. *Modern Algebra*, volume I. Frederick Unger Publishing Co., 1950.

[18] X.-J. Wang and S. S.-T. Yau. On the GLY conjecture of upper estimate of positive integral points in real right-angled simplices. *J. Number Theory*, **122(1)**(2007), 184 – 210.

[19] Y.-J. Xu and S. S.-T. Yau. A sharp estimate of the number of integral points in a 4-dimensional tetrahedra. *J. Reine Angew. Math.* **473**(1996), 69–86.

[20] Y.-J. Xu and S. S.-T. Yau. A sharp estimate of the number of integral points in a tetrahedron. *J. Reine Angew. Math.*, **423**(1992), 199–219.

[21] S. S.-T. Yau and L.-T. Zhang. An upper estimate of integral points in real simplices with an application to singularity theory. *Math. Res. Lett.* **13**(2006), 911–921.

[22] S. S.-T. Yau, B. Yuan and H. Q. Zuo. Biggest sharp polynomial estimate of integral points in right-angled simplices. *J. Number Theory*, **160**(2016), 254 – 286.

[23] S. S.-T. Yau, L. Zhao and H. Q. Zuo. On the polynomial sharp upper estimate conjecture in 7-dimensional simplex. *Topology of algebraic varieties and singularities, Contemp. Math., Amer. Math. Soc., Providence, RI* **538**(2011), 433–467.

[24] S. S.-T. Yau and H. Q. Zuo. Lower estimate of Milnor number and characterization of isolated homogeneous hypersurface singularities. *Pacific J. Math.* **273(1)**(2015), 213–224.

[25] S. S.-T. Yau and H. Q. Zuo. Complete characterization of isolated homogeneous hypersurfacesingularities. *Pacific J. Math.* **260(1)**(2012), 245–255.

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