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Abstract

For any given $sl(2, \mathbf{C})$ action on $\mathbf{C}[[x_1, \ldots, x_n]]$ via derivations preserving the m-adic filtration, we give a necessary and sufficient condition for a gradient space $I(f)$ of a homogeneous polynomial f i.e. a vector space spanned by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}$, to be a $sl(2, \mathbf{C})$-submodule for $n \leq 5$.

Key words and phrases. Invariant polynomial, weight, irreducible submodule, representation, completely reducible.

§0. Introduction

Let M_n^k be the space of homogeneous polynomials of degree k in n variables $x_1, x_2, \ldots, x_n$. Let us fix a non-trivial $sl(2,\mathbf{C})$ action on M_n^k. We shall denote S_n^k the subspace of M_n^k on which $sl(2,\mathbf{C})$ acts trivially. Let $S_n = \oplus_{k\geq 0} S_n^k$ be the graded ring of invariants. The main object of the invariant theory is to give explicit description of S_n. In case $sl(2,\mathbf{C})$ acts on $\oplus_{k\geq 0} M_n^k$ via

$$\begin{aligned}
\tau &= (n-1)x_1\frac{\partial}{\partial x_1} + (n-3)x_2\frac{\partial}{\partial x_2} + \ldots + (-(n-3))x_{n-1}\frac{\partial}{\partial x_{n-1}} \\
&\quad + (-(n-1))x_n\frac{\partial}{\partial x_n} \qquad (0.1)\\
X_+ &= (n-1)x_1\frac{\partial}{\partial x_2} + 2(n-2)x_2\frac{\partial}{\partial x_3} + \ldots + i(n-i)x_i\frac{\partial}{\partial x_{i+1}} + \ldots + (n-1)x_{n-1}\frac{\partial}{\partial x_n} \\
X_- &= x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + \ldots + x_i\frac{\partial}{\partial x_{i-1}} + \ldots + x_n\frac{\partial}{\partial x_{n-1}}.
\end{aligned}$$

This example is identical with the theory of binary quantics, which was diligently studied in 2nd half of nineteenth century. It is an amazingly difficult job to describe S_n explicitly. A complete success was achieved only for $n \leq 6$, the cases $n = 5$ and 6 being one of crowning glories of the theory. Elliott's book [1] has an excellent account on this subject. In 1967 Shioda [4] was able to describe S_8 explicitly. Recently the theory of invariants, pronounced dead at the turn of the century, is once again at the forefront of mathematics because of of combinatorial thrust due to Rota (cf. [2]) and his school.

In [5], we developed a new theory which connects isolated singularities on the one hand, and finite dimensional Lie algebras on the other hand. The natural question arising there is the following. Let f be a homogeneous polynomial of degree $k+1$ in n variables. Consider the vector subspace $I(f)$ spanned by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}, \ldots, \frac{\partial f}{\partial x_n}$.

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Give a necessary and sufficient condition for $I(f)$ to be a $sl(2,\mathbf{C})$-submodule. Here we shall consider all possible $sl(2,\mathbf{C})$ actions on $\mathbf{C}[[x_1,\ldots,x_n]]$ via derivations preserving the m-adic filtration. Notice that (0.1) is just one example of $sl(2,\mathbf{C})$ action only. We first observe that if $f \in S_n^{k+1}$ is an $sl(2,\mathbf{C})$ invariant polynomial, then $I(f)$ is a $sl(2,\mathbf{C})$-submodule. The main purpose of this paper is to determine precisely when $I(f)$ is a $sl(2,\mathbf{C})$-submodule for $f \in M_n^{k+1}$, $n \leq 5$. We establish that the converse of the above statement is essentially true.

Main Theorem. For $n \leq 5$, $I(f) = \langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \rangle$ is a $sl(2,\mathbf{C})$-submodule if and only if $I(f) = \langle \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \ldots, \frac{\partial g}{\partial x_n} \rangle$ for some $sl(2,\mathbf{C})$ invariant polynomial g.

Notice that f is not necessarily a $sl(2,\mathbf{C})$ invariant polynomial even though $I(f)$ is a $sl(2,\mathbf{C})$-submodule. In Section 1, we shall describe our results in great detail. Several examples of this sort certainly appear. These phenomena occur precisely because the variety defined by f is highly singular. It is this observation that allows us to prove that the Lie algebras that we constructed from isolated singularities (cf. [5]) are solvable. This application of the main theorem will appear in [6]. The general case of our main theorem will be treated in a future paper.

Our paper is organized as follows. In Section 1 we state our results for $n = 2, 3, 4$, and 5 in a precise way. The cases $n = 2$ and 3 were treated in our previous article [5]. In order to avoid the repetition, we shall only give the proof for the case $n = 5$. The proof for the case $n = 4$ can be sorted out from there. From Section 2 to Section 7, we shall deal with different actions of $sl(2,\mathbf{C})$ on M_n^k. We shall assign each variable x_i a weight according to the action of $sl(2,\mathbf{C})$. It turns out that each monomial is an eigenvector of τ with eigenvalue equal to its weight. We prove that if $I(f)$ is a $sl(2,\mathbf{C})$

module, then f is essentially of weight 0. In Section 8 we shall prove all the results stated in Section 1. Our main theorem is just a short summary of these results.

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§1. Preliminaries and Statement of Results

In this section, we give the detail of our results in this article. The main theorem stated in the introduction is a consequence of these results. We first recall our result in [6] on classification of $sl(2,\mathbf{C})$ in *Der* $\mathbf{C}[[x_1,\ldots,x_n]]$ preserving the m-adic filtration.

Proposition [6]. Let $L = sl(2,\mathbf{C})$ act on $\mathbf{C}[[x_1,x_2,\ldots,x_n]]$ via derivations preserving the m-adic filtration i.e., $L(m^k) \subseteq m^k$ where m is the maximal ideal in $\mathbf{C}[[x_1,\ldots,x_n]]$. Then there exists a coordinate change $y_1,\ldots,y_n$ with respect to which $sl(2,\mathbf{C})$ is spanned by

$$\tau = \sum_{j=1}^{n} a_{1j}\frac{\partial}{\partial y_j}$$

$$X_+ = \sum_{j=1}^{n} a_{2j}\frac{\partial}{\partial y_j}$$

$$X_- = \sum_{j=1}^{n} a_{3j}\frac{\partial}{\partial y_j}$$

where a_{ij} is linear function in $y_1,\ldots,y_n$ variables for all $1 \le i \le 3$ and $1 \le j \le n$. Here $\{\tau, X_+, X_-\}$ is a standard basis for $sl(2,\mathbf{C})$ i.e., $[\tau, X_+] = 2X_+$, $[\tau, X_-] = 2X_-$ and $[X_+, X_-] = \tau$.

Theorem [6]. Let $sl(2,\mathbf{C})$ act on the formal power series ring $\mathbf{C}[[x_1,\ldots,x_n]]$ preserving the m-adic filtration where m is the maximal ideal in $\mathbf{C}[[x_1,\ldots,x_n]]$. Then there exists a coordinate system

$$x_1, x_2, \ldots, x_{l_1}, x_{l_1+1}, \ldots, x_{l_1+l_2}, \ldots x_{l_1+l_2+\ldots+l_{s-1}+1}, \ldots, x_{l_1+l_2+\ldots+l_s}$$

such that

$$\tau = D_{\tau,1} + \ldots + D_{\tau,j} + \ldots + D_{\tau,r}$$

$$X_+ = D_{X_+,1} + \ldots + D_{X_+,j} + \ldots + D_{X_+,r}$$

$$X_- = D_{X_-,1} + \ldots + D_{X_-,j} + \ldots + D_{X_-,r}$$

where $r \leq s$ and

$$
\begin{aligned}
D_{\tau,j} &= (l_j-1)x_{l_1+\ldots+l_{j-1}+1}\frac{\partial}{\partial x_{l_1+\ldots+l_{j-1}+1}} + (l_j-3)x_{l_1+\ldots+l_{j-1}+2}\frac{\partial}{\partial x_{l_1+\ldots+l_{j-1}+2}} \\
&\quad + \ldots + (-(l_j-3))x_{l_1+\ldots+l_j-1}\frac{\partial}{\partial x_{l_1+\ldots+l_j-1}} \\
&\quad +(-(l_j-1))x_{l_1+\ldots+l_j}\frac{\partial}{\partial x_{l_1+\ldots+l_j}} \\
D_{X_+,j} &= (l_j-1)x_{l_1+\ldots+l_{j-1}+1}\frac{\partial}{\partial x_{l_1+\ldots+l_{j-1}+2}} \\
&\quad + \ldots + i(l_j-i)x_{l_1+\ldots+l_{j-1}+i}\frac{\partial}{\partial x_{l_1+\ldots+l_{j-1}+i+1}} \\
&\quad + \ldots + (-(l_j-1))x_{l_1+\ldots+l_j-1}\frac{\partial}{\partial x_{l_1+\ldots+l_j}} \\
D_{X_-,j} &= x_{l_1+\ldots+l_{j-1}+2}\frac{\partial}{\partial x_{l_1+\ldots+l_{j-1}+1}} + \ldots + x_{l_1+\ldots+l_{j-1}+i+1}\frac{\partial}{\partial x_{l_1+\ldots+l_{j-i}+i}} \\
&\quad + \ldots + x_{l_1+\ldots+l_j}\frac{\partial}{\partial x_{l_1+\ldots+l_j-1}}.
\end{aligned}
$$

Suppose $sl(2,\mathbf{C})$ acts non-trivially on the space M_2^k of homogeneous polynomial of degree k in 2 variables x_1 and x_2. Then the $sl(2,\mathbf{C})$ action is given by

$$
\begin{aligned}
\tau &= x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} \\
X_+ &= x_1\frac{\partial}{\partial x_2} \\
X_- &= x_2\frac{\partial}{\partial x_1}.
\end{aligned}
\tag{1.1}
$$

The following theorem is trivial and can be found in [5].

Theorem 1. Let $I(f)$ be the complex vector subspace of M_2^k spanned by $\frac{\partial f}{\partial x_1}$ and $\frac{\partial f}{\partial x_2}$, where f is a homogeneous polynomial of degree $k+1$ in x_1, x_2 variables and $k \geq 2$. Then $I(f)$ is not a $sl(2,\mathbf{C})$-submodule.

Suppose $sl(2,\mathbf{C})$ acts non-trivially on the space M_3^k of homogeneous polynomials of degree k in 3 variables x_1, x_2 and x_3. Then the $sl(2,\mathbf{C})$ action is given by either (1.1) or

$$\begin{aligned} \tau &= 2x_1\frac{\partial}{\partial x_1} - 2x_3\frac{\partial}{\partial x_3} \\ X_+ &= 2x_1\frac{\partial}{\partial x_2} + 2x_2\frac{\partial}{\partial x_3} \\ X_- &= x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2}. \end{aligned} \tag{1.2}$$

From now on, we shall use the following notation. By (l), we shall mean a l-dimensional irreducible representation of $sl(2,\mathbf{C})$. The following theorem can be found in [5].

Theorem 2. Let $I(f)$ be the complex vector subspace of M_3^k spanned by $\frac{\partial f}{\partial x_1}$, $\frac{\partial f}{\partial x_2}$ and $\frac{\partial f}{\partial x_3}$, where f is a homogeneous polynomial of degree $k+1$ in x_1, x_2, x_3 variables and $k \geq 2$.

(i) If the $sl(2,\mathbf{C})$ action of M_3^k is given by (1.1), then $I(f)$ is a $sl(2,\mathbf{C})$-submodule if and only if $f(x_1,x_2,x_3) = cx_3^{k+1}$ and $I(f) = (1) = \langle x_3^k \rangle$ where c is a nonzero constant and $\langle x_3^k \rangle$ denotes the one-dimensional vector space spanned by x_3^k.

(ii) If the $sl(2,\mathbf{C})$ action on M_3^k is given by (1.2), then $I(f)$ is a $sl(2,\mathbf{C})$-submodule if and only if $k+1 = 2l$ is an even integer and $f(x_1,x_2,x_3) = c(x_2^2 - 2x_1x_3)^{2l}$ and $I(f) = (3) = \langle x_1(x_2^2 - 2x_1x_3),\ x_2(x_2^2 - 2x_1x_3), x_3(x_2^2 - 2x_1x_3)\rangle$.

Suppose $sl(2,\mathbf{C})$ acts nontrivially on the space M_4^k of homogeneous polynomial of degree k in 4 variables x_1, x_2, x_3 and x_4. Then the $sl(2,\mathbf{C})$ action is given by either

(1.1) or (1.2) or

$$\begin{aligned}
\tau &= x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} - x_4\frac{\partial}{\partial x_4} \\
X_+ &= x_1\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_4} \\
X_- &= x_2\frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial x_3}
\end{aligned} \tag{1.3}$$

or

$$\begin{aligned}
\tau &= 3x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} - x_3\frac{\partial}{\partial x_3} - 3x_4\frac{\partial}{\partial x_4} \\
X_+ &= 3x_1\frac{\partial}{\partial x_2} + 4x_2\frac{\partial}{\partial x_3} + 3x_3\frac{\partial}{\partial x_4} \\
X_- &= x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3}.
\end{aligned} \tag{1.4}$$

Theorem 3. Let $I(f)$ be the complex vector subspace of M_4^k spanned by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}$ and $\frac{\partial f}{\partial x_4}$, where f is a homogeneous polynomial of degree $k+1$ in x_1, x_2, x_3, x_4 variables and $k \geq 2$.

(i) If the $sl(2,\mathbb{C})$ action on M_4^k is given by (1.1), then $I(f)$ is a $sl(2,\mathbb{C})$ submodule if and only if either one of the following occurs.

(a) f is a polynomial in x_3, x_4 variables and $I = (1) \oplus (1) = \langle \frac{\partial f}{\partial x_3}(x_3, x_4)\rangle \oplus \langle \frac{\partial f}{\partial x_4}(x_3, x_4)\rangle$.

(b) $f = (c_1x_3 + c_2x_4)^{k+1}$ where c_1 and c_2 are constants not all zero and $I = (1) = \langle (c_1x_3 + c_2x_4)^k \rangle$.

(ii) If the $sl(2,\mathbb{C})$ action on M_4^k is given by (1.2), then $I(f)$ is a $sl(2,\mathbb{C})$ submodule if and only if either one of the following occurs.

(a) f is a $sl(2,\mathbb{C})$ invariant polynomial in x_1, x_2, x_3 and x_4 variables and $I = (3) \oplus (1)$.

(b) f is a $sl(2,\mathbb{C})$ invariant polynomial in x_1, x_2 and x_3 variables and $I = (3)$.

(c) $f = cx_4^{k+1}$ and $I = (1) = \langle x_4^k \rangle$.

(iii) If the $sl(2, \mathbf{C})$ action on M_4^k is given by (1.3), then $I(f)$ is a $sl(2, \mathbf{C})$-submodule if and only if f is a $sl(2, \mathbf{C})$ invariant polynomial in x_1, x_2, x_3 and x_4 variables and $I = (2) \oplus (2)$.

(iv) If the $sl(2, \mathbf{C})$ action on M_4^k is given by (1.4), then $I(f)$ is a $sl(2, \mathbf{C})$ submodule if and only if f is a $sl(2, \mathbf{C})$ invariant polynomial in x_1, x_2, x_3 and x_4 variables and $I = (4)$.

Suppose $sl(2, \mathbf{C})$ acts nontrivially on the space M_5^k of homogeneous polynomials of degree k in 5 variables x_1, x_2, x_3, x_4 and x_5. Then the $sl(2, \mathbf{C})$ action is given by either (1.1), or (1.2), or (1.3), or (1.4), or

$$
\begin{aligned}
\tau &= 2x_1 \frac{\partial}{\partial x_1} - 2x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4} - x_5 \frac{\partial}{\partial x_5} \\
X_+ &= 2x_1 \frac{\partial}{\partial x_2} + 2x_2 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_5} \\
X_- &= x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_5 \frac{\partial}{\partial x_4}
\end{aligned}
\tag{1.5}
$$

or

$$
\begin{aligned}
\tau &= 4x_1 \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_2} - 2x_4 \frac{\partial}{\partial x_4} - 4x_5 \frac{\partial}{\partial x_5} \\
X_+ &= 4x_1 \frac{\partial}{\partial x_2} + 6x_2 \frac{\partial}{\partial x_3} + 6x_3 \frac{\partial}{\partial x_4} + 4x_4 \frac{\partial}{\partial x_5} \\
X_- &= x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} + x_5 \frac{\partial}{\partial x_4}.
\end{aligned}
\tag{1.6}
$$

Theorem 4. Let $I(f)$ be the complex vector subspace of M_5^k spanned by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}$ and $\frac{\partial f}{\partial x_5}$, where f is a homogeneous polynomial of degree $k+1$ in x_1, x_2, x_3, x_4 and x_5 variables and $k \geq 2$.

(i) If the $sl(2, \mathbf{C})$ action on M_5^k is given by (1.1), then $I(f)$ is a $sl(2, \mathbf{C})$-submodule if and only if one of the following occurs.

(a) f is a polynomial in x_3, x_4 and x_5 variables and $I = (1) \oplus (1) \oplus (1) = \langle \frac{\partial f}{\partial x_3}(x_3, x_4, x_5) \rangle \oplus \langle \frac{\partial f}{\partial x_4}(x_3, x_4, x_5) \rangle \oplus \langle \frac{\partial f}{\partial x_5}(x_3, x_4, x_5) \rangle$:

(b) f is a polynomial in x_3, x_4 and x_5 variables and

$$I = (1) \oplus (1) = \langle \frac{\partial f}{\partial x_3}(x_3, x_4, x_5) \rangle \oplus \langle \frac{\partial f}{\partial x_4}(x_3, x_4, x_5) \rangle$$
$$\text{or } \langle \frac{\partial f}{\partial x_4}(x_3, x_4, x_5) \rangle \oplus \langle \frac{\partial f}{\partial x_5}(x_3, x_4, x_5) \rangle$$
$$\text{or } \langle \frac{\partial f}{\partial x_3}(x_3, x_4, x_5) \rangle \oplus \langle \frac{\partial f}{\partial x_5}(x_3, x_4, x_5) \rangle.$$

(c) $f = (c_1x_3 + c_2x_4 + c_3x_5)^{k+1}$ where c_1, c_2 and c_3 are not all zero constants and $I = \langle (c_1x_3 + c_2x_4 + c_3x_3)^k \rangle$.

(ii) If the $sl(2,\mathrm{C})$ action on M_5^k is given by (1.2), then $I(f)$ is a $sl(2,\mathrm{C})$-submodule if and only if one of the following occurs.

(a) (1) f is a $sl(2,\mathrm{C})$ invariant polynomial and $I = (3) \oplus (1) \oplus (1)$.

(2) $f = g(x_1, x_2, x_3, x_4, x_5) + c_1x_1(x_4+rx_5)^k + c_2x_2(x_4+rx_5)^k + c_3x_3(x_4+rx_5)^k$ where $g(x_1, x_2, x_3, x_4, x_5) = d_1(x_2^2 - 2x_1x_3)(x_4 + rx_5)^{k-1} + d_2x_5(x_4 + rx_5)^k + d_3(x_4+rx_5)^{k+1}$ is a $sl(2,\mathrm{C})$ invariant polynomial with $d_1 \neq 0$ and $d_2 \neq 0$. $I = \langle \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_3}, \frac{\partial g}{\partial x_4}, \frac{\partial g}{\partial x_5} \rangle = (3) \oplus (1) \oplus (1) = \langle x_1(x_4+rx_5)^{k-1},\ x_2(x_4+rx_5)^{k-1}, x_3(x_4 + rx_5)^{k-1} \rangle \oplus \langle (x_4 + rx_5)^k \rangle \oplus \langle (k-1)d_1(x_2^2 - 2x_1x_3)(x_4 + rx_5)^{k-2} + kd_2x_5(x_4 + rx_5)^{k-1} \rangle$.

(3) $f = g(x_1, x_2, x_3, x_4, x_5) + c_1x_1(rx_4+x_5)^k + c_2x_2(rx_4+x_5)^k + c_3x_3(rx_4+x_5)^k$ where $g(x_1, x_2, x_3, x_4, x_5) = d_1(x_2^2 - 2x_1x_3)(rx_4 + x_5)^{k-1} + d_2x_4(rx_4 + x_5)^k + d_3(rx_4+x_5)^{k+1}$ is a $sl(2,\mathrm{C})$ invariant polynomial with $d_1 \neq 0$ and $d_2 \neq 0$. $I = \langle \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_3}, \frac{\partial g}{\partial x_4}, \frac{\partial g}{\partial x_5} \rangle = (3) \oplus (1) \oplus (1) = \langle x_1(rx_4+x_5)^{k-1}, x_2(rx_4+x_5)^{k-1}, x_3(rx_4 + x_5)^{k-1} \rangle \oplus \langle (rx_4 + x_5)^k \rangle \oplus \langle (k-1)d_1(x_2^2 - 2x_1x_3)(rx_4 + x_5)^{k-2} + kd_2x_4(rx_4 + x_5)^{k-1} \rangle$.

(b) f is a $sl(2,\mathbf{C})$ invariant polynomial in x_1, x_2, x_3, x_4 and x_5 variables and $I = (3) \oplus (1)$.

(c) f is $sl(2,\mathbf{C})$ invariant polynomial in x_1, x_2 and x_3 variables and $I = (3)$.

(d) f is a $sl(2,\mathbf{C})$ invariant polynomial in x_4 and x_5 variables and $I = (1) \oplus (1) = \langle \frac{\partial f}{\partial x_4}(x_4, x_5) \rangle \oplus \langle \frac{\partial f}{\partial x_5}(x_4, x_5)\rangle$.

(e) $f = (c_1x_4 + c_2x_5)^{k+1}$ where c_1 and c_2 are not all zero constants and $I = (1) = (c_1x_4 + c_2x_5)^k$.

(iii) If the $sl(2,\mathbf{C})$ action on M_5^k is given by (1.3), then $I(f)$ is a $sl(2,\mathbf{C})$ sumbodule if and only if one of the following occurs.

(a) f is a $sl(2,\mathbf{C})$ invariant polynomial in x_1, x_2, x_3, x_4 and x_5 variables and $I = (2) \oplus (2) \oplus (1)$.

(b) f is a $sl(2,\mathbf{C})$ invariant polynomial in x_1, x_2, x_3 and x_4 variables and $I = (2) \oplus (2)$.

(c) $f = cx_5^{k+1}$ where c is a nonzero constant and $I = (1) = \langle x_5^k \rangle$.

(iv) If the $sl(2,\mathbf{C})$ action on M_5^k is given by (1.4), then $I(f)$ is a $sl(2,\mathbf{C})$-submodule if and only if one of the following occurs.

(a) f is a $sl(2,\mathbf{C})$ invariant polynomial in x_1, x_2, x_3, x_4 and x_5 variables and $I = (4) \oplus (1)$.

(b) f is a $sl(2,\mathbf{C})$ invariant polynomial in x_1, x_2, x_3 and x_4 variables and $I = (4)$.

(c) $f = cx_5^{k+1}$ where c is a nonzero constant and $I = (1) = \langle x_5^k \rangle$.

(v) If the $sl(2,\mathbf{C})$ action on M_5^k is given by (1.5), then $I(f)$ is a $sl(2,\mathbf{C})$-submodule if and only if one of the following occurs.

(a) (1) f is a $sl(2,\mathbf{C})$ invariant polynomial in x_1, x_2, x_3, x_4 and x_5 variables and $I = (3) \oplus (2)$.

(2) $f = g(x_1,x_2,x_3,x_4,x_5)+c_1x_4^3+c_2x_4^2x_5+c_3x_4x_5^2+c_4x_5^3$ where $g(x_1,x_2,x_3,x_4,x_5) = 2x_1x_5^2 - 2x_2x_4x_5 + x_3x_4^2$ is a $sl(2,\mathbf{C})$ invariant polynomial and

$$\begin{aligned} I &= \langle \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_3}, \frac{\partial g}{\partial x_4}, \frac{\partial g}{\partial x_5} \rangle = (3) \oplus (2) \\ &= \langle x_4^2, x_4, x_5, x_5^2 \rangle \oplus \langle x_2xx_4 - 2x_1x_5, x_3x_4 - x_2x_5 \rangle. \end{aligned}$$

(b) f is a $sl(2,\mathbf{C})$ invariant polynomial in x_1, x_2 and x_3 variables, and $I = (3)$.

(vi) If the $sl(2,\mathbf{C})$ action on M_5^k is given by (1.6), then $I(f)$ is a $sl(2,\mathbf{C})$-submodule if and only if f is a $sl(2,\mathbf{C})$ invariant polynomial in x_1, x_2, x_3, x_4 and x_5 variables, and $I = (5)$.

§2. $sl(2, \mathbf{C})$ ACTION (1.6) ON M_5^k

Lemma 2.1. Suppose $sl(2, \mathbf{C})$ acts on the space of homogeneous polynomials of degree $k \geq 2$ in x_1, x_2, x_3, x_4, x_5 via (1.6)

$$\tau = 4x_1\frac{\partial}{\partial x_1} + 2x_2\frac{\partial}{\partial x_2} - 2x_4\frac{\partial}{\partial x_4} - 4x_5\frac{\partial}{\partial x_5}$$

$$X_+ = 4x_1\frac{\partial}{\partial x_2} + 6x_2\frac{\partial}{\partial x_3} + 6x_3\frac{\partial}{\partial x_4} + 4x_4\frac{\partial}{\partial x_5}$$

$$X_- = x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3} + x_5\frac{\partial}{\partial x_4}$$

Suppose the weight of x_i is given by the corresponding coefficient in the expression of τ above, i.e.

$$wt(x_1) = 4,\ wt(x_2) = 2,\ wt(x_3) = 0,\ wt(x_4) = -2,\ wt(x_5) = -4.$$

Let I be the complex vector subspace spanned by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}$ and $\frac{\partial f}{\partial x_5}$ where f is a homogeneous polynomial of degree $k+1$. If I is a $sl(2, \mathbf{C})$-submodule and $\dim I = 5$, then f is a homogeneous polynomial of weight 0 and I is an irreducible $sl(2, \mathbf{C})$-submodule.

Proof.

Case 1. $I = (5)$

By the classification theorem of $sl(2, \mathbf{C})$ representations, we know that $\frac{\partial f}{\partial x_i}$, $1 \leq i \leq 5$, is a linear combination of homogeneous polynomials in I of degree k and weights 4, 2, 0, -2 and -4. Since every monomial is of even weight, we can write

$$f = \sum_{i=-\infty}^{\infty} f_{k+1}^{2i}$$

where f_{k+1}^{2i} is a homogeneous polynomial of degree $k+1$ and weight $2i$.

For $|i| \geq 5$

$$\begin{aligned}
&\left| wt\ \frac{\partial f^{2i}_{k+1}}{\partial x_j} \right| \geq 6 && 1 \leq j \leq 5 \\
\Rightarrow\ &\frac{\partial f^{2i}_{k+1}}{\partial x_j} = 0 && 1 \leq j \leq 5 \\
\Rightarrow\ &f^{2i}_{k+1} \equiv 0.
\end{aligned}$$

For $i = -4$

$$\begin{aligned}
&wt\left(\frac{\partial f^{-8}_{k+1}}{\partial x_j}\right) \leq -6 && 1 \leq j \leq 4 \\
\Rightarrow\ &\frac{\partial f^{-8}_{k+1}}{\partial x_j} = 0 && 1 \leq j \leq 4 \\
\Rightarrow\ &f^{-8}_{k+1} && \text{involves only } x_5 \text{ variable i.e. } f^{-8}_{k+1} = cx_5^2
\end{aligned}$$

where c is a constant.

Since f^{-8}_{k+1} is a homogeneous polynomial of degree $k+1 \geq 3$, hence c must be zero.

For $i = -3$

$$\begin{aligned}
&wt\frac{\partial f^{-6}_{k+1}}{\partial x_j} \leq -6 && 1 \leq j \leq 3 \\
\Rightarrow\ &\frac{\partial f^{-6}_{k+1}}{\partial x_j} = 0 && 1 \leq j \leq 3 \\
\Rightarrow\ &f^{-6}_{k+1} && \text{involves only } x_4 \text{ and } x_5 \text{ variables and}
\end{aligned}$$

$$f^{-6}_{k+1} = c_1x_4^3 + c_2x_4x_5 \qquad \text{where } c_1, c_2 \text{ are constants.}$$

Since f^{-6}_{k+1} is a homogeneous polynomial of degree $k+1 \geq 3$, hence $c_2 = 0$ and $f^{-6}_{k+1} = c_1x_4^3$. $\frac{\partial f}{\partial x_4} \in I$ implies $\frac{\partial f^{-6}_{k+1}}{\partial x_4} \in I$. If c_1 were not zero, then x_4^2 would be in I. Then by applying X_+, X_- successively on x_4^2, we have

$$\langle 2x_1x_4 + 9x_2x_3, x_2x_4 + x_3^2, x_3x_4, x_4^2, x_4x_5, x_5^2 \rangle \subseteq I.$$

Therefore $\dim I \geq 6$ which is impossible. Hence $f_{k+1}^{-6} \equiv 0$.

For $i = -2$

$$wt(\tfrac{\partial f_{k+1}^{-4}}{\partial x_j}) \leq -6 \qquad j = 1,2$$

$$\Rightarrow \tfrac{\partial f_{k+1}^{-4}}{\partial x_j} = 0 \qquad j = 1,2$$

$\Rightarrow$ f_{k+1}^{-4} does not involve x_1 and x_2 variables and

$f_{k+1}^{-4} = c_1 x_4^2 x_3^{k-1} + c_2 x_5 x_3^k$ where c_1 and c_2 are constant.

$\frac{\partial f}{\partial x_3} \in I$ implies $\frac{\partial f_{k+1}^{-4}}{\partial x_3} \in I$. Since $X_-(\frac{\partial f_{k+1}^{-4}}{\partial x_3})$ is an element in I of weight -6, hence

$$X_-(\frac{\partial f_{k+1}^{-4}}{\partial x_3}) = 0.$$

$$\begin{aligned} 0 &= X_-(\frac{\partial f_{k+1}^{-4}}{\partial x_3}) \\ &= (x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3} + x_5\frac{\partial}{\partial x_4})[(k-1)c_1 x_4^2 x_3^{k-2} + kc_2 x_5 x_3^{k-1}] \\ &= (k-1)(k-2)c_1 x_3^{k-3} x_4^3 + [2(k-1)c_1 + k(k-1)c_2] x_3^{k-2} x_4 x_5. \end{aligned}$$

If $k \geq 3$, then the above equation implies $c_1 = 0 = c_2$. If $k = 2$, then $c_2 = -c_1$ and $f_3^{-4} = c_1(x_4^2 x_3 - x_5 x_3^2) \cdot \frac{\partial f}{\partial x_5} \in I$ implies $\frac{\partial f_3^{-4}}{\partial x_5} = -c_1 x_3^2 \in I$. If $c_1 \neq 0$, then $x_3^2 \in I$. By successively applying X_+ and X_- on x_3^2, we have

$$\langle x_1x_3, x_2x_3, x_3^2, x_3x_4, x_3x_5, x_4x_5 \rangle \subseteq I.$$

This implies $\dim I \geq 6$ which is impossible. Hence $c_1 = c_2 = 0$. We conclude that $f_{k+1}^{-4} \equiv 0$.

By the similar argument, we an prove that $f_{k+1}^8 \equiv f_{k+1}^6 \equiv f_{k+1}^4 \equiv 0$. Therefore $f = f_{k+1}^2 + f_{k+1}^0 + f_{k+1}^{-2}$

$$wt(\frac{\partial f_{k+1}^2}{\partial x_5}) = 6 \quad \text{and} \quad wt(\frac{\partial f_{k+1}^{-2}}{\partial x_1}) = -6$$

$$\Rightarrow \frac{\partial f_{k+1}^2}{\partial x_5} = 0 = \frac{\partial f_{k+1}^{-2}}{\partial x_1}$$

$\Rightarrow f_{k+1}^2$ does not involve x_5 while f_{k+1}^{-2} does not involve x_1.

We are going to prove that $\frac{\partial f_{k+1}^2}{\partial x_4}=0$. First we observe that $X_+f_{k+1}^2=0$ by the pervious argument because $wt(X_+f_{k+1}^2)=4$. Suppose on the contrary that $\frac{\partial f_{k+1}^2}{\partial x_4}\neq 0$. Then the equation

$$X_+\frac{\partial f_{k+1}^2}{\partial x_3}=\frac{\partial}{\partial x_3}X_+f_{k+1}^2-6\frac{\partial f_{k+1}^2}{\partial x_4}=-6\frac{\partial f_{k+1}^2}{\partial x_4}$$

implies that $\frac{\partial f_{k+1}^2}{\partial x_3}\neq 0$. Since $wt(X_-\frac{\partial f_{k+1}^2}{\partial x_4})=2=wt(\frac{\partial f_{k+1}^2}{\partial x_3})$, there exists a nonzero constant c such that

$$X_-(\frac{\partial f_{k+1}^2}{\partial x_4})=c\frac{\partial f_{k+1}^2}{\partial x_3}.$$

Differentiating the above equation with respect to x_5 variable, it is easy to see that $\frac{\partial^2 f_{k+1}^2}{\partial x_4^2}=0$. Hence $\frac{\partial f_{k+1}^2}{\partial x_4}$ depends only on x_1,x_2 and x_3 variables. As $wt(\frac{\partial f_{k+1}^2}{\partial x_4})=4$, there exist constants c_1 and c_2 such that

$$\frac{\partial f_{k+1}^2}{\partial x_4}=c_1x_1x_3^{k-1}+c_2x_2^2x_3^{k-2}.$$

Easy computations show that

$$\begin{aligned}
X_-^5(x_1x_3^{k-1}) &= 10(k-1)(k-2)^2x_3^{k-3}x_4^3+15(k-1)(2k-3)x_3^{k-2}x_4x_5\\
&+5(k-1)(k-2)(k-3)(k-4)x_2x_3^{k-5}x_4^4+30(k-1)(k-2)(k-3)x_2x_3^{k-4}x_4^2x_5\\
&+15(k-1)(k-2)x_2x_3^{k-3}x_5^2+(k-1)(k-2)(k-3)(k-4)(k-5)x_1x_3^{k-6}x_4^5\\
&+10(k-1)(k-2)(k-3)(k-4)x_1x_3^{k-5}x_4^3x_5+15(k-1)(k-2)(k-3)x_1x_3^{k-4}x_4x_5^2\\
X_-^5(x_2^2x_3^{k-2}) &= 10(k-2)(2k^2-8k+9)x_3^{k-3}x_4^3+20(3k^2-10k+9)x_3^{k-2}x_4x_5\\
&+10(k-2)(k-3)^2(k-4)x_2x_3^{k-5}x_4^4\\
&+20(k-2)(3k^2-17k+24)x_2x_3^{k-4}x_4^2x_5+10(k-2)(3k-7)x_2x_3^{k-3}x_5^2\\
&+(k-2)(k-3)(k-4)(k-5)(k-6)x_2^2x_3^{k-7}x_4^5\\
&+10(k-2)(k-3)(k-4)(k-5)x_2^2x_3^{k-6}x_4^2x_5
\end{aligned}$$

$$+15(k-2)(k-3)(k-4)x_2^2x_3^{k-5}x_4x_5^2.$$

$$X_-\frac{\partial f_{k+1}^2}{\partial x_4}=0$$
$$\Rightarrow c_1X_-(x_1x_3^{k-1})+c_2X_-(x_2^2x_3^{k-2})=0.$$

If $k\geq 3$ then we have

$$\begin{cases}10(k-1)(k-2)^2c_1 & + & 10(k-2)(2k^2-8k+9)c_2 & = & 0\\ 15(k-1)(2k-3)c_1 & + & 20(3k^2-10k+9)c_2 & = & 0\end{cases}$$

$$\begin{cases}(k-1)(k-2)c_1 & + & (2k^2-8k+9)c_2 & = & 0\\ 3(k-1)(2k-3)c_1 & + & 4(3k^2-10k+9)c_2 & = & 0\end{cases}$$

since $\det\begin{pmatrix}(k-1)(k-2) & 2k^2-8k+9\\ 3(k-1)(2k-3) & 4(3k^2-10k+9)\end{pmatrix}=(k-1)(2k^2-10k+9)$ is nonzero, we infer that $c_1=c_2=0$. i.e., $\frac{\partial f_{k+1}^2}{\partial x_4}=0$.

If $k=2$, then we have $15c_1+20c_2=0$. So we can rewrite

$$\frac{\partial f_{k+1}^2}{\partial x_4}\ =c_3(4x_1x_3-3x_2^2)$$

and

$$X_-(\frac{\partial f_{k+1}^2}{\partial x_4})\ =2c_3(2x_1x_4-x_2x_3).$$

Hence

$$\frac{\partial f_{k+1}^2}{\partial x_3}\ =2\frac{c_3}{c}(2x_1x_4-x_2x_3).$$

The fact that $\frac{\partial^2 f_{k+1}^2}{\partial x_3\partial x_4}=4c_3x_1$ and $\frac{\partial^2 f}{\partial x_4\partial x_3}=4\frac{c_3}{c}x_1$ infer easily that $c=1$. If $c_3\neq 0$, there would be a nonzero constant d_1 such that

$$\frac{\partial f_{k+1}^2}{\partial x_2}=d_1X_-(\frac{\partial f_{k+1}^2}{\partial x_3})=d_1X_-[2c_3(2x_1x_4-x_2x_3)]$$
$$=2d_1c_3(x_2x_4+2x_1x_5-x_3^2).$$

By comparing $\frac{\partial^2 f_{k+1}^2}{\partial x_3\partial x_2}$ and $\frac{\partial^2 f_{k+1}^2}{\partial x_2\partial x_3}$, we conclude that $2d_1=1$ and

$$\frac{\partial f_{k+1}^2}{\partial x_2}=c_3(x_2x_4+2x_1x_5-x_3^2).$$

On the one hand $\frac{\partial^2 f_{k+1}^2}{\partial x_4 \partial x_2} = c_3 x_2$, on the other hand $\frac{\partial^2 f_{k+1}^2}{\partial x_2 \partial x_4} = -6c_3 x_2$. Therefore we have $c_3 = 0$. This completes our claim that $\frac{\partial f_{k+1}^2}{\partial x_4} = 0$.

We next claim that $\frac{\partial f_{k+1}^2}{\partial x_3} = 0$. Suppose on the contrary that $\frac{\partial f_{k+1}^2}{\partial x_3} \neq 0$. Then the equation

$$X_+ \frac{\partial f_{k+1}^2}{\partial x_2} = \frac{\partial}{\partial x_2} X_+ f_{k+1}^2 - 4\frac{\partial f_{k+1}^2}{\partial x_3} = -4\frac{\partial f_{k+1}^2}{\partial x_3}$$

implies that $\frac{\partial f_{k+1}^2}{\partial x_2} \neq 0$. Since $wt(X_- \frac{\partial f_{k+1}^2}{\partial x_3}) = 0 = wt(\frac{\partial f_{k+1}^2}{\partial x_2})$, there exists a nonzero constant d such that

$$X_-(\frac{\partial f_{k+1}^2}{\partial x_3}) = d\frac{\partial f_{k+1}^2}{\partial x_2}.$$

Differentiating the above equation with respect to x_4 variable, it is easy to see that $\frac{\partial^2 f_{k+1}^2}{\partial x_3^2} = 0$. Hence $\frac{\partial f_{k+1}^2}{\partial x_3}$ depends only on x_1 and x_2 variables. As $wt(\frac{\partial f_{k+1}^2}{\partial x_3}) = 2$, there exists a constant d_1 such that

$$\frac{\partial f_{k+1}^2}{\partial x_3} = d_1 x_2.$$

This contradicts the fact that $\deg(f_{k+1}^2) = k + 1 \geq 3$. The proof of $\frac{\partial f_{k+1}^2}{\partial x_3} = 0$ is complete. We now see that f_{k+1}^2 depends only on x_1 and x_2 variables. Therefore there exists a constant d_2 such that $f_{k+1}^2 = d_2 x_2$. This again contradicts the fact that $k \geq 2$. We conclude that $f_{k+1}^2 = 0$. Similarly we can show that $f_{k+1}^{-2} = 0$.

Case 2. $I = (4) \oplus (1)$.

By the classification theorem of $sl(2, \mathbf{C})$ representations, we know that every element in (4) is a linear combination of homogeneous polynomials of degree k and weights $3, 1, -1$ and -3. Since every monomial is of even weight, this case cannot happen.

Case 3. $I = (3) \oplus (2)$.

This case cannot occur by the same argument as Case 2.

Case 4. $I = (3) \oplus (1) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weight $2, 0$ and -2. We can show as in Case 1 that

$$f = f_{k+1}^2 + f_{k+1}^0 + f_{k+1}^{-2}.$$

Since $\frac{\partial f_{k+1}^{-2}}{\partial x_1}$ and $\frac{\partial f_{k+1}^{-2}}{\partial x_2}$ are elements in I of weight -6 and -4 respectively, so $\frac{\partial f_{k+1}^{-2}}{\partial x_1} = 0 = \frac{\partial f_{k+1}^{-2}}{\partial x_2}$. This implies that f_{k+1}^{-2} does not involve x_1 and x_2 variables. There exist a constant c such that

$$f_{k+1}^{-2} = c x_4 x_3^k.$$

If c were not zero, then $x_3^k = \frac{1}{c}\frac{f_{k+1}^{-2}}{x_4} \in I$. By applying X_+ and X_- successively on x_3^k, we see that $\dim I \geq 6$, which is a contradiction. Therefore we conclude that $f_{k+1}^{-2} = 0$. Similarly, we can prove that $f_{k+1}^2 = 0$. So f is a homogeneous polynomial of degree $k+1$ and weight 0. It follows that $\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_5} = 0$ by weight consideration. This implies $\dim I \leq 3$, which is impossible. Therefore this case cannot happen.

Case 5. $I = (2) \oplus (2) \oplus (1)$.

Similar argument as Case 2 shows that this case cannot happen.

Case 6. $I = (2) \oplus (1) \oplus (1) \oplus (1)$.

Similar argument as Case 2 shows that this case cannot occur.

Case 7. $I = (1) \oplus (1) \oplus (1) \oplus (1) \oplus (1)$.

Elements of I are linear combinations of invariant polynomials. We can show as in Case 1 that

$$f = f_{k+1}^2 + f_{k+1}^0 + f_{k+1}^{-2}.$$

Since $\frac{\partial f_{k+1}^{-2}}{\partial x_1}$, $\frac{\partial f_{k+1}^{-2}}{\partial x_2}$, $\frac{\partial f_{k+1}^{-2}}{\partial x_3}$ and $\frac{\partial f_{k+1}^{-2}}{\partial x_5}$ are elements in I of weights $-6, -4, -2$ and 2 respectively, so

$$\frac{\partial f_{k+1}^{-2}}{\partial x_1} = \frac{\partial f_{k+1}^{-2}}{\partial x_2} = \frac{\partial f_{k+1}^{-2}}{\partial x_3} = \frac{\partial f_{k+1}^{-2}}{\partial x_5} = 0$$

f_{k+1}^{-2} depends only on x_4 variable. There exists a constant c such that $f_{k+1}^{-2} = cx_4$. Since $k \geq 2$, we have $c = 0$ and $f_{k+1}^{-2} = 0$. Similarly we can prove $f_{k+1}^{2} = 0$. Now f must be a homogeneous polynomial of degree k of weight 0. It follows that $\frac{\partial f}{\partial x_1}$ is a non-zero element of weight -4 in I. This leads to a contradiction. Hence this case cannot happen. Q.E.D.

Lemma 2.2. With the same hypothesis as Lemma 2.1, if $\dim I = 4$, then I cannot be a $sl(2, \mathbf{C})$-submodule.

Proof. We assume on the contrary that I is an $sl(2, \mathbf{C})$-submodule.

Case 1. $I = (4)$.

By the classification theorem of $sl(2, \mathbf{C})$ representations, we know that every element in (4) is a linear combination of homogeneous polynomials in I of degree k and weights $3, 1, -1$ and -3. Since every monomial is of even weight, this case cannot happen.

Case 2. $I = (3) \oplus (1)$.

The same argument as in Case 4 in the proof of Lemma 2.1 will prove that this case cannot occur.

Case 3. $I = (2) \oplus (2)$.

This case cannot happen by the same argument as Case 1.

Case 4. $I = (2) \oplus (1) \oplus (1)$.

This case cannot happen by the same argument as Case 1.

Case 5. $I = (1) \oplus (1) \oplus (1) \oplus (1)$.

The same argument as in Case 6 in the proof of Lemma 2.1 will prove that this case cannot occur. Q.E.D.

Lemma 2.3. With the same hypothesis as Lemma 2.1, if $\dim I = 3$ and I is a $sl(2, \mathbf{C})$-submodule, then f is a homogeneous polynomial of weight 0 in x_2, x_3, x_4 variables and I is an irreducible $sl(2, \mathbf{C})$-submodule.

Proof. Since every monomial is of even weight, we can write

$$f = \sum_{i=-\infty}^{\infty} f_{k+1}^{2i}$$

where f_{k+1}^{2i} is a homogeneous polynomial of degree $k+1$ and weight $2i$.

Case 1. $I = (3)$.

By the classification theorem of $sl(2, \mathbf{C})$ representations, every element in I is a linear combination of homogeneous polynomials of degree k and weights $-2, 0$ and 2.

For $|i| \geq 4$

$$\begin{aligned} & |wt \ \frac{\partial f_{k+1}^{2i}}{\partial x_j}| \geq 4 && 1 \leq j \leq 5 \\ \Rightarrow \quad & \frac{\partial f_{k+1}^{2i}}{\partial x_j} = 0 && 1 \leq j \leq 5 \\ \Rightarrow \quad & f_{k+1}^{2i} = 0 \end{aligned}$$

For $i = -3$

$$wt \ \frac{\partial f_{k+1}^{-6}}{\partial x_j} \leq -4 \qquad 1 \leq j \leq 4$$

$$\Rightarrow \quad \frac{\partial f_{k+1}^{-6}}{\partial x_j} = 0 \qquad\qquad 1 \le j \le 4$$

$$\Rightarrow \quad f_{k+1}^{-6} \text{ involves only } x_5 \text{ variable.}$$

Since 6 is not divisible by 4, this is not possible unless $f_{k+1}^{-6} = 0$.

For $i = -2$

$$wt\ \frac{\partial f_{k+1}^{-4}}{\partial x_j} \le -4 \qquad\qquad 1 \le j \le 3$$

$$\Rightarrow \quad \frac{\partial f_{k+1}^{-4}}{\partial x_j} = 0 \qquad\qquad 1 \le j \le 3$$

$\Rightarrow$ f_{k+1}^{-4} involves only x_4 and x_5 variables and there exist constants c_1 and c_2, such that:

$$f_{k+1}^{-4} = c_1 x_2^2 + c_2 x_5.$$

Since $k \ge 2$ we have $f_{k+1}^{-4} = 0$.

For $i = -1$

$$wt\ \frac{\partial f_{k+1}^{-2}}{\partial x_j} \le -4 \qquad j = 1, 2$$

$$\Rightarrow \frac{\partial f_{k+1}^{-2}}{\partial x_1} = 0 = \frac{\partial f_{k+1}^{-2}}{\partial x_2}$$

$\Rightarrow f_{k+1}^{-2}$ does not involve x_1 and x_2 variables

$\Rightarrow$ there exists a constant c such that $f_{k+1}^{-2} = c x_4 x_3^k$.

$$wt\ X_-(\frac{\partial f_{k+1}^{-2}}{\partial x_3}) = -4$$

$$\Rightarrow X_-(\frac{\partial f_{k+1}^{-2}}{\partial x_3}) = 0$$

$$\Rightarrow (x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} + x_5 \frac{\partial}{\partial x_4})(ckx_4x_3^{k-1}) = 0$$

$$\Rightarrow ck(k-1)x_4^2x_3^{k-2} + ckx_5x_3^{k-1} = 0$$

$$\Rightarrow c = 0 \text{ and } f_{k+1}^{-2} = 0.$$

Similarly we can prove that $f_{k+1}^6 = f_{k+1}^4 = f_{k+1}^2 = 0$. Hence f is a homogeneous polynomial of degree $k+1$ and weight 0. It follows that $wt\ \frac{\partial f}{\partial x_1} = -4$ and $wt\ \frac{\partial f}{\partial x_5} = 4$. Consequently $\frac{\partial f}{\partial x_1} = 0 = \frac{\partial f}{\partial x_5}$ and f involves only x_2, x_3 and x_4 variables.

Case 2. $I = (2) \oplus (1)$.

Elements in I are linear combinations of homogeneous polynomials of degree k and weights $-1, 0$ and 1. The same argument as in Case 1 shows that f is a homogeneous polynomial of degree $k+1$ and weight 0. So $\frac{\partial f}{\partial x_1}$, $\frac{\partial f}{\partial x_2}$, $\frac{\partial f}{\partial x_4}$ and $\frac{\partial f}{\partial x_5}$ are elements in I of weights $-4, -2, 2$ and 4 respectively. It follows that $\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_4} = \frac{\partial f}{\partial x_5} = 0$ and $\dim I \leq 1$. This contradicts to our assumption that $\dim I = 3$. Hence this case cannot happen.

Case 3. $I = (1) \oplus (1) \oplus (1)$.

The same argument as in Case 2 shows that this case cannot happen. Q.E.D.

Lemma 2.4. With the hypothesis as Lemma 2.1, if $\dim I \leq 2$, then I cannot be an $sl(2, \mathbf{C})$-submodule.

Proof. We assume on the contrary that I is an $sl(2, \mathbf{C})$-submodule.

If $\dim I = 2$, then the proof of Lemma 2.2 Case 3 and Case 5 will provide necessary contradiction.

If $\dim I = 1$, then the same argument as in the proof of Lemma 2.1 Case 6 will prove that f is a homogeneous polynomial of degree $k+1$ and weight 0.

$$wt\,\frac{\partial f}{\partial x_1} = -4 \quad wt\,\frac{\partial f}{\partial x_2} = -2 \quad wt\,\frac{\partial f}{\partial x_4} = 2 \quad wt\,\frac{\partial f}{\partial x_5} = 4$$

$$\Rightarrow \quad \frac{\partial f}{\partial x_1} = 0 = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_4} = \frac{\partial f}{\partial x_5}$$

$\Rightarrow$ f does not involve x_1, x_2, x_4, x_5 and there exists a constant c such that $f = cx_3^{k+1}$.

If $c \neq 0$, then $x_3^k = \frac{1}{c}\frac{\partial f}{\partial x_3} \in I$. So $X_-(x_3^k) = kx_3^{k-1}x_4 \in I$. Hence $\dim I \geq 2$, which contradicts to our assumption that $\dim I = 1$. On the other hand if $c = 0$, then $f = 0$ and hence $\dim I = 0$, which again leads to a contradiction. We conclude that this case cannot happen. Q.E.D.

Proposition 2.5. Suppose $sl(2,\mathbf{C})$ acts on the space of homogeneous polynomials of degree $k \geq 2$ in x_1, x_2, x_3, x_4, x_5 via (1.6)

$$\tau = 4x_1\frac{\partial}{\partial x_1} + 2x_2\frac{\partial}{\partial x_2} - 2x_4\frac{\partial}{\partial x_4} - 4x_5\frac{\partial}{\partial x_5}$$

$$X_+ = 4x_1\frac{\partial}{\partial x_2} + 6x_2\frac{\partial}{\partial x_3} + 6x_3\frac{\partial}{\partial x_4} + 4x_4\frac{\partial}{\partial x_5}$$

$$X_- = x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3} + x_5\frac{\partial}{\partial x_4}.$$

Suppose the weight of x_i is given by the corresponding coefficient in the expression of τ above i.e.

$$wt(x_1) = 4,\ wt(x_2) = 2,\ wt(x_3) = 0,\ wt(x_4) = -2,\ wt(x_5) = -4.$$

Let I be the complex vector subspace spanned by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}$ and $\frac{\partial f}{\partial x_5}$ where f is a homogeneous polynomial of degree $k+1$. If I is a $sl(2,\mathbf{C})$-submodule, then f is a homogeneous polynomial of weight 0 and I is an irreducible $sl(2,\mathbf{C})$-submodule of dimension 5 or 3. In the latter case, f is a polynomial in x_2, x_3 and x_4 variables.

Proof. This is an immediate consequences of Lemma 2.1 through Lemma 2.4. Q.E.D.

§3. $sl(2,\mathbf{C})$ ACTION (1.5) IN M_5^k.

Lemma 3.1. Suppose $sl(2,\mathbf{C})$ acts on the space of homogeneous polynomials of degree $k \geq 2$ in x_1, x_2, x_3, x_4 and x_5 via (1.5).

$$\tau = 2x_1\frac{\partial}{\partial x_1} - 2x_3\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_4} - x_5\frac{\partial}{\partial x_5}$$

$$X_+ = 2x_1\frac{\partial}{\partial x_2} + 2x_2\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_5}$$

$$X_- = x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + x_5\frac{\partial}{\partial x_4}.$$

Suppose the weight of x_i is given by the corresponding coefficient in the expression of τ above i.e.

$$wt(x_1) = 2,\ wt(x_2) = 0,\ wt(x_3) = -2,\ wt(x_4) = 1,\ wt(x_5) = -1.$$

Let I be the complex vector subspace of dimension 5 spanned by $\frac{\partial f}{\partial x_1}$, $\frac{\partial f}{\partial x_2}$, $\frac{\partial f}{\partial x_3}$, $\frac{\partial f}{\partial x_4}$ and $\frac{\partial f}{\partial x_5}$ where f is a homogeneous polynomial of degree $k+1$. If I is a $sl(2,\mathbf{C})$-submodule then there exists g, a homogeneous polynomial of degree $k+1$, and weight 0, such that $I = \langle \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_3}, \frac{\partial g}{\partial x_4}, \frac{\partial g}{\partial x_5} \rangle$. Moreover, if f is not a homogeneous polynomial of weight 0, then $k = 2$ and f is of the following form

$$f = 2x_1x_5^2 - 2x_2x_4x_5 + x_3x_4^2 + c_1x_4^3 + c_2x_4^2x_5 + c_3x_4x_5^2 + c_4x_5^3.$$

Proof. Case 1. $I = (5)$.

By the classification theorem of $sl(2,\mathbf{C})$ representations, every element in I is a linear combination of homogeneous polynomial of degree k and weights $4, 2, 0, -2$ and -4. Write

$$f = \sum_{i=-\infty}^{\infty} f_{k+1}^i$$

where f^i_{k+1} is a homogeneous polynomial of degree $k+1$ and weight i.

For $|i| \geq 7$.

$$\begin{aligned} & |wt\,\frac{\partial f^i_{k+1}}{\partial x_j}| \geq 5 \qquad 1 \leq j \leq 5 \\ \Rightarrow\quad & \frac{\partial f^i_{k+1}}{\partial x_j} = 0 \qquad\quad 1 \leq j \leq 5 \\ \Rightarrow\quad & f^i_{k+1} = 0. \end{aligned}$$

For $i = 0,\ \pm 2,\ \pm 4,\ \pm 6$

$$\begin{aligned} & wt\,\tfrac{\partial f^i_{k+1}}{\partial x_j} \text{ are odd integers for } j = 4,5 \\ \Rightarrow\quad & \tfrac{\partial f^i_{k+1}}{\partial x_j} = 0 \quad j = 4,5 \\ \Rightarrow\quad & f^i_{k+1} \text{ involves only } x_1, x_2, x_3 \text{ variables.} \end{aligned}$$

If f^i_{k+1} were not zero, then either $\frac{\partial f^i_{k+1}}{\partial x_1}$ or $\frac{\partial f^i_{k+1}}{\partial x_2}$ or $\frac{\partial f^i_{k+1}}{\partial x_3}$ would generate I because I is an irreducible $sl(2,\mathbf{C})$ module. Hence I would involve only x_1, x_2, x_3 variables. It follows that $\frac{\partial f}{\partial x_j}$, $1 \leq j \leq 5$, involves only x_1, x_2, x_3 variables and hence so does f. This implies that $\frac{\partial f}{\partial x_4} = \frac{\partial f}{\partial x_5} = 0$, which contradicts to the fact that $\dim I = 5$. Thus we have $f^i_{k+1} = 0$.

For $i = \pm 1, \pm 3, \pm 5$.

The weights of $\frac{\partial f^i_{k+1}}{\partial x_1}$, $\frac{\partial f^i_{k+1}}{\partial x_2}$ and $\frac{\partial f^i_{k+1}}{\partial x_3}$ are odd integers. So $\frac{\partial f^i_{k+1}}{\partial x_1} = \frac{\partial f^i_{k+1}}{\partial x_2} = \frac{\partial f^i_{k+1}}{\partial x_3} = 0$ and hence f^i_{k+1} involves only x_4 and x_5. If f^i_{k+1} were not zero, then by applying X_+ and X_- successively on $\frac{\partial f^i_{k+1}}{\partial x_4}$ or $\frac{\partial f^i_{k+1}}{\partial x_5}$, we would have

$$I = \langle x_4^k, x_4^{k-1}x_5, \ldots, x_5^k \rangle \quad \text{and} \quad k = 4.$$

It follows that $\frac{\partial f}{\partial x_j}$, $1 \leq j \leq 5$, involves only x_4 and x_5 variables and hence so does f. This implies that $\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3} = 0$, which contradicts to the fact that $\dim I = 5$.

We conclude that Case 1 cannot occur.

Case 2. $I = (4) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials in I of weights $-3, -1, 1, 3$ and 0.

For $|i| \geq 6$

$$|wt\, \frac{\partial f^i_{k+1}}{\partial x_j}| \geq 4 \qquad 1 \leq j \leq 5$$

$$\Rightarrow \frac{\partial f^i_{k+1}}{\partial x_j} = 0 \qquad 1 \leq j \leq 5$$

$$\Rightarrow f^i_{k+1} = 0.$$

For $i = -5$

$$wt\, \frac{\partial f^{-5}_{k+1}}{\partial x_1} = -7 \quad wt\, \frac{\partial f^{-5}_{k+1}}{\partial x_2} = -5 \quad wt\, \frac{\partial f^{-5}_{k+1}}{\partial x_4} = -6 \quad wt\, \frac{\partial f^{-5}_{k+1}}{\partial x_5} = -4$$

$$\Rightarrow \frac{\partial f^{-5}_{k+1}}{\partial x_1} = \frac{\partial f^{-5}_{k+1}}{\partial x_2} = \frac{\partial f^{-5}_{k+1}}{\partial x_4} = \frac{\partial f^{-5}_{k+1}}{\partial x_5} = 0$$

$\Rightarrow f^{-5}_{k+1}$ involves only x_3 variable

$\Rightarrow f^{-5}_{k+1} = 0$ becasue $wt\, f^{-5}_{k+1} = -5$ is not divisible by $wt\, x_3 = -2$.

For $i = -3$

$$wt\, \frac{\partial f^{-3}_{k+1}}{\partial x_1} = -5 \quad wt\, \frac{\partial f^{-3}_{k+1}}{\partial x_4} = -4 \quad wt\, \frac{\partial f^{-3}_{k+1}}{\partial x_5} = -2$$

$$\Rightarrow \frac{\partial f^{-3}_{k+1}}{\partial x_1} = 0 = \frac{\partial f^{-3}_{k+1}}{\partial x_4} = \frac{\partial f^{-3}_{k+1}}{\partial x_5}$$

$\Rightarrow f^{-3}_{k+1}$ involves only x_2, x_3 variables

$\Rightarrow f^{-3}_{k+1} = 0$ because $wt\, f^{-3}_{k+1} = -3$ is not divisible by $wt\, x_3 = -2$.

Similar argument shows that $f^5_{k+1} = 0 = f^3_{k+1}$ and hence

$$f = f^{-4}_{k+1} + f^{-2}_{k+1} + f^{-1}_{k+1} + f^0_{k+1} + f^1_{k+1} + f^2_{k+1} + f^4_{k+1}.$$

For $i = -4$

$$wt\,\frac{\partial f^{-4}_{k+1}}{\partial x_1} = -6 \quad wt\,\frac{\partial f^{-4}_{k+1}}{\partial x_2} = -4 \quad wt\,\frac{\partial f^{-4}_{k+1}}{\partial x_3} = -2 \quad wt\,\frac{\partial f^{-4}_{k+1}}{\partial x_4} = -5$$

$$\Rightarrow \frac{\partial f^{-4}_{k+1}}{\partial x_1} = \frac{\partial f^{-4}_{k+1}}{\partial x_2} = \frac{\partial f^{-4}_{k+1}}{\partial x_3} = \frac{\partial f^{-4}_{k+1}}{\partial x_4} = 0$$

$\Rightarrow f^{-4}_{k+1}$ involves only x_5 variable and there exists a constant

c such that $f^{-4}_{k+1} = cx_5^4$.

Similar argument shows that f^4_{k+1} involves only x_4 variable. If $c \neq 0$, then $k = 3$ and $(4) = \langle x_4^3, x_4^2x_5, x_4x_5^2, x_5^3 \rangle \subseteq I$. Let ϕ be a homogeneous polynomial of degree 3 and weight 0 such that $\langle \phi \rangle = (1) \subseteq I$.

Write

$$\begin{aligned}\phi = {} & \psi^0_3(x_4,x_5) + \psi^{-2}_2(x_4,x_5)x_1 + \psi^0_2(x_4,x_5)x_2 + \psi^2_2(x_4,x_5)x_3 \\ & + \psi^{-4}_1(x_4,x_5)x_1^2 + \psi^{-2}_1(x_4,x_5)x_1x_2 + \psi^0_1(x_4,x_5)x_2^2 \\ & + \psi^0_1(x_4,x_5)x_1x_3 + \psi^2_1(x_4,x_5)x_2x_3 + \psi^4_1(x_4,x_5)x_3^2 \\ & + \xi^0_3(x_1,x_2,x_3)\end{aligned}$$

where $\psi^k_i(x_4,x_5)$ is a homogeneous polynomial in x_4, x_5 of degree i and weight k, and $\xi^0_3(x_1,x_2,x_3)$ is a homogeneous polynomial in x_1,x_2,x_3 of degree 3 and weigh 0. Since there is no homogeneous polynomial in x_4,x_5 variables of odd degree and even weight, we have

$$
\begin{aligned}
\phi &= \psi_2^{-2}(x_4, x_5)x_1 + \psi_2^0(x_4, x_5)x_2 + \psi_2^2(x_4, x_5)x_3 + \xi_3^0(x_1, x_2, x_3) \\
&= c_1 x_5^2 x_1 + c_2 x_4 x_5 x_2 + c_3 x_4^2 x_3 + c_4 x_1 x_3 x_2 + c_5 x_2^3 \\
\Rightarrow X_-\phi &= (c_1 + c_2)x_2 x_5^2 + (c_2 + 2c_3)x_3 x_4 x_5 + (c_4 + 3c_5)x_2^2 x_3 + c_4 x_1 x_3^2 = 0 \\
&\Rightarrow c_2 = -c_1,\ c_3 = \frac{1}{2}c_1,\ c_5 = -\frac{1}{3}c_4 = 0.
\end{aligned}
$$

Let $c_1 = 2\tilde{c}_1$. Then we have

$$\phi = \tilde{c}_1(2x_1x_5^2 - 2x_2x_4x_5 + x_3x_4^2)$$

and

$$
\begin{aligned}
I &= \langle x_4^3, x_4^2x_5, x_4x_5^2, x_5^3\rangle \oplus \langle 2x_1x_5^2 - 2x_2x_4x_5 + x_3x_4^2 \\
&= (4) \oplus (1) \\
& wt\frac{\partial f_4^{-2}}{\partial x_1} = -4 \quad wt\frac{\partial f_4^{-2}}{\partial x_2} = -2 \quad wt\frac{\partial f_4^{-2}}{\partial x_3} = 0 \\
&\Rightarrow \frac{\partial f_4^{-2}}{\partial x_1} = 0 = \frac{\partial f_4^{-2}}{\partial x_2} \text{ and } \frac{\partial f_4^{-2}}{\partial x_3} = d_1(2x_1x_5^2 - 2x_2x_4x_5 + x_3x_4^2) \\
&\Rightarrow \frac{\partial^2 f_4^{-2}}{\partial x_1 \partial x_3} = 2d_1x_5^2 = 0 \\
&\Rightarrow d_1 = 0.
\end{aligned}
$$

So we have $\frac{\partial f_4^{-2}}{\partial x_1} = 0 = \frac{\partial f_4^{-2}}{\partial x_2} = \frac{\partial f_4^{-2}}{\partial x_3}$, hence f_4^{-2} involves only x_4 and x_5 variables.

Similar argument shows that f_4^2 involves only x_4 and x_5 variables.

$$wt\,\frac{\partial f_4^{-1}}{\partial x_1}=-3\quad wt\,\frac{\partial f_4^{-1}}{\partial x_2}=-1\quad wt\,\frac{\partial f_4^{-1}}{\partial x_3}=1\quad wt\,\frac{\partial f_4^{-1}}{\partial x_4}=2$$

$$wt\,\frac{\partial f_4^{-1}}{\partial x_5}=0$$

$$\Rightarrow\quad \frac{\partial f_4^{-1}}{\partial x_4}=0,\quad \frac{\partial f_4^{-1}}{\partial x_1}=d_2x_5^3,\quad \frac{\partial f_4^{-1}}{\partial x_2}=d_3x_4x_5^2$$

$$\frac{\partial f_4^{-1}}{\partial x_3}=d_4x_4^2x_5,\ \frac{\partial f_4^{-1}}{\partial x_5}=d_5(2x_1x_5^2-2x_2x_4x_5+x_3x_4^2)$$

where d_2,d_3,d_4,d_5,d_6 are constants

$$\Rightarrow\quad \frac{\partial^2 f^{-1}}{\partial x_4\partial x_2}=d_3x_5^2=0$$

$$\frac{\partial^2 f_4^{-1}}{\partial x_4\partial x_3}=2d_4x_4x_5=0$$

$$\frac{\partial^2 f_4^{-1}}{\partial x_4\partial x_5}=d_5(-2x_2x_5+2x_3x_4)$$

$$\Rightarrow d_3=d_4=d_5=0.$$

So we have $\frac{\partial f_4^{-1}}{\partial x_4}=\frac{\partial f_4^{-1}}{\partial x_2}=\frac{\partial f_4^{-1}}{\partial x_3}=\frac{\partial f_4^{-1}}{\partial x_5}=0$, hence f_4^{-1} involves only x_1 variable. It follows that $\frac{\partial f_4^{-1}}{\partial x_1}$ involves only x_1 variable. Consequently $d_2=0$ and $\frac{\partial f_4^{-1}}{\partial x_1}=0$. We conclude that $f_4^{-1}=0$. By the same argument, we have $f_4^1=0$.

$$wt\,\frac{\partial f_4^0}{\partial x_1}=-2\quad wt\,\frac{\partial f_4^0}{\partial x_2}=0\quad wt\,\frac{\partial f_4^0}{\partial x_3}=2$$

$$\Rightarrow\quad \frac{\partial f_4^0}{\partial x_1}=0=\frac{\partial f_4^0}{\partial x_3}\text{ and }\frac{\partial f_4^0}{\partial x_2}=d_6\ (2x_1x_5^2-2x_2x_4x_5+x_3x_4^2)$$

where d_6 is a constant

$$\frac{\partial f_4^0}{\partial x_1\partial x_2}=2d_6x_5^2=0$$

$$\Rightarrow\quad d_6=0.$$

So we have $\frac{\partial f_4^0}{\partial x_1}=0=\frac{\partial f_4^0}{\partial x_2}=\frac{\partial f_4^0}{\partial x_3}$, hence f_4^0 involves only x_4 and x_5. We conclude that

$$f=f_4^{-4}(x_5)+f_4^{-2}(x_4,x_5)+f_4^0(x_4,x_5)+f_4^2(x_4,x_5)+f_4^4(x_4).$$

Therefore $\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3} = 0$. This implies that $\dim I \leq 2$, which contradicts the fact that $\dim I = 5$. To avoid this contradiction, f_{k+1}^{-4} had better equal to zero.

Similar argument shows that $f_{k+1}^{4} = 0$. Thus we have

$$f = f_{k+1}^{-2} + f_{k+1}^{-1} + f_{k+1}^{0} + f_{k+1}^{1} + f_{k+1}^{2}.$$

For $i = -2$

$$\begin{aligned} & wt\,\frac{\partial f_{k+1}^{-2}}{\partial x_1} = -4, \quad wt\,\frac{\partial f_{k+1}^{-2}}{\partial x_2} = -2 \\ \Rightarrow\quad & \frac{\partial f_{k+1}^{-2}}{\partial x_1} = 0 = \frac{\partial f_{k+1}^{-2}}{\partial x_2} \\ \Rightarrow\quad & f_{k+1}^{-2} \text{ involves only } x_3, x_4 \text{ and } x_5. \end{aligned}$$

Since $k \geq 2$, we have

$$f_{k+1}^{-2} = \phi_{k+1}(x_4, x_5) + x_3\phi_k(x_4, x_5) + \ldots + x_3^k\phi_1(x_4, x_5)$$

where $\phi_i(x_4, x_5)$ is a homogeneous polynomial of degree i. We claim that f_{k+1}^{-2} depends only on x_4, x_5. Suppose on the contrary that $\phi_i(x_4, x_5) \neq 0$ for some $1 \leq i \leq k$. Let j be the largest integer such that $x_3^j\phi_{k+1-j}(x_4, x_5) \neq 0$

$$f_{k+1}^{-2} = \phi_{k+1}(x_4, x_5) + x_3\phi_k(x_4, x_5) + \ldots + x_3^j\phi_{k+1-j}(x_4, x_5).$$

Consider

$$\frac{\partial f_{k+1}^{-2}}{\partial x_3} = \phi_k(x_4, x_5) + 2x_3\phi_{k-1}(x_4, x_5) + \ldots + jx_3^{j-1}\phi_{k+1-j}(x_4, x_5).$$

Write

$$\phi_{k+1-j}(x_4, x_5) = \sum_{\beta=0}^{k+1-j} d_\beta x_4 x_5^{k+1-j-\beta}$$

where d_β is a constant. Let α be the biggest integer such that $d_\alpha \neq 0$. Since $\frac{\partial f_{k+1}^{-2}}{\partial x_2}$ is in I, $X_-^\alpha(\frac{\partial f_{k+1}^{-2}}{\partial x_3})$ is also in I and

$$X_-^\alpha(\frac{\partial f_{k+1}^{-2}}{\partial x_3}) = c x_3^{j-1} x_5^{k+1-j} + \sum_{i<j-1} x_3^i \psi_{k-i}(x_4, x_5)$$

where c is a nonzero constant, $\psi_{k-i}(x_4,x_5)$ is a homogeneous polynomial of degree $k-i$. Let

$$X'_- = x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} \qquad X''_- = x_5\frac{\partial}{\partial x_4}$$

$$X'_+ = 2x_1\frac{\partial}{\partial x_2} + 2x_2\frac{\partial}{\partial x_3} \qquad X''_+ = x_4\frac{\partial}{\partial x_5}$$

$$\tau' = 2x_1\frac{\partial}{\partial x_1} - 2x_3\frac{\partial}{\partial x_3} \qquad \tau'' = x_4\frac{\partial}{\partial x_4} - x_5\frac{\partial}{\partial x_5}.$$

Obviously $X'_-X''_- = X''_-X'_-$ and $X'_+X''_+ = X''_+X'_+$. Therefore

$$X_-^l = (X'_- + X''_-)^l = X'^l_- + lX'^{\,l-1}_- X''_- + \ldots + X''^l_- \tag{3.1}$$

$$X_+^l = (X'_+ + X''_+)^l = X'^l_+ + lX'^{\,l-1}_+ X''_+ + \ldots + X''^l_+. \tag{3.2}$$

In view of (3.2) and $k+j-1 > k+i = 2i+k-i$ for $i < j-1$, we have $X_+^{k+j-1}(x_3^i\psi_{k-i}(x_4,x_5)) = 0$. On the other hand $X_+^{k+j-1}(x_3^{j-1}x_5^{k+1-j}) = \tilde{c}x_1^{j-1}x_4^{k+1-j}$ where $\tilde{c}$ is a nonzero constant. So we have

$$x_1^{j-1}x_4^{k+1-j} = \frac{1}{\tilde{c}}X_+^{k+j-1}X_-^\alpha(\frac{\partial f_{k+1}^{-2}}{\partial x_3}) \in I.$$

By applying X_- successively on $x_1^{j-1}x_4^{k+1-j}$, we get an irreducible $sl(2,\mathbf{C})$-submodule of dimension $k+j$, the elements of which are linearly independent bihomogeneous polynomials in I of degree $j-1$ in x_1, x_2, x_3 variables and $k+1-j$ in x_4, x_5 variables.

By our assumption, $\phi_{k+1-j}(x_4,x_5)$ is a nonzero homogeneous polynomial of degree bigger than zero. Either $\frac{\partial\phi_{k+1-i}}{x_4}$ is nonzero or $\frac{\partial\phi_{k+1-i}}{x_5}$ is nonzero. By the same argument as before we can get an irreducible $sl(2,\mathbf{C})$-submodule of dimension $k+j+1$, the elements of which are linearly indpendent bihomogeneous polynomials in I of degree j in x_1, x_2, x_3 variables and $k-j$ in x_4, x_5 variables.

Thus I contains two disjoint $sl(2,\mathbf{C})$-submodules of dimensions $k+j$ and $k+j+1$ respectively. This contradicts the fact that $I=(4)\oplus(1)$. Hence our claim that f_{k+1}^{-2} depends only on x_4 and x_5 variables is established. Now the argument which we used before to prove that $f_{k+1}^{-4}=0$ can be applied here to conclude $f_{k+1}^{-2}=0$.

Similar argument shows that $f_{k+1}^{2}=0$.

For $i=-1$

$$\begin{aligned} & wt\,\frac{\partial f_{k+1}^{-1}}{\partial x_4} = -2 \\ \Rightarrow\quad & \frac{\partial f_{k+1}^{-1}}{\partial x_4} = 0 \\ \Rightarrow\quad & f_{k+1}^{-1} \text{ does not involve } x_4 \text{ variable.} \end{aligned}$$

Since f_{k+1}^{-1} is of weight -1, we can write

$$f_{k+1}^{-1} = \sum_{i=1}^{k} x_5^i \phi_{k-i+1}^{i-1}(x_1, x_2, x_3)$$

where $\phi_{k-i+1}^{i-1}(x_1,x_2,x_3)$ is a homogeneous polynomial of degree $k-i+1$ and weight $i-1$. Let j be the biggest integer such that $\phi_{k-j+1}^{j-1}(x_1,x_2,x_3)\neq 0$. Since $j\leq k$, there exists $1\leq i\leq 3$ such that $\frac{\partial\phi_{k-j+1}}{\partial x_i}(x_1,x_2,x_3)\neq 0$ for some $1\leq i\leq 3$. Consider

$$\begin{aligned} \frac{\partial f_{k+1}^{-1}}{\partial x_i} &= x_5\frac{\partial\phi_k^{-2}}{\partial x_i}(x_1,x_2,x_3) + x_5^2\frac{\partial\phi_{k-1}^{-3}}{\partial x_i}(x_1,x_2,x_3) + \ldots + x_5^j\frac{\partial\phi_{k-j+1}^{j-1}}{\partial x_i}(x_1,x_2,x_3) \\ &= x_5^j\frac{\partial\phi_{k-j+1}^{j-1}}{\partial x_i}(x_1,x_2,x_3) + \text{ bihomogeneous polynomials of total degree} \\ &\qquad \text{in } x_4, x_5 \text{ variables less than } j. \end{aligned}$$

$$X_+\left(\frac{\partial f_{k+1}^{-1}}{\partial x_i}\right) = X_+''(x_5^j)\frac{\partial\phi_{k-j+1}^{j-1}}{\partial x_i} + x_5^j X_+'\left(\frac{\partial\phi_{k-j+1}^{j-1}}{\partial x_i}\right) + \text{ bihomogeneous polynomials}$$

of total degree in x_4 and x_5 variables less than j.

$$\vdots$$

$$X_+^j(\frac{\partial f_{k+1}^{-1}}{\partial x_i}) = [(X_+'')^j(x_5^j)]\frac{\partial \phi_{k-j+1}^{j-1}}{\partial x_i} + \ldots + \binom{j}{l}[(X_+'')^{j-l}(x_5^j)][(X_+')^l\frac{\partial \phi_{k-j+1}^{j-1}}{\partial x_i}]$$

$$+ \ldots + (x_5^j)[X_+')^j\frac{\partial \phi_{k-j+1}^{j-1}}{\partial x_i}] + \text{ bihomogeneous polynomials of total}$$

degree in x_4 and x_5 variables less than j.

The above $j+1$ homogeneous polynomials are linearly independent elements in I. On the other hand, we can consider

$$\frac{\partial f_{k+1}^{-1}}{\partial x_5} = \phi_k^0(x_1,x_2,x_3) + 2x_5\phi_{k-1}^1(x_1,x_2,x_3) + \ldots + jx_5^{j-1}\phi_{k-j+1}^{j-1}(x_1,x_2,x_3)$$

$$= jx_5^{j-1}\phi_{k-j+1}^{j-1}(x_1,x_2,x_3) + \text{ bihomogeneous polynomials of total}$$

degree in x_4, x_5 variables less than $j-1$.

$$X_+(\frac{\partial f_{k+1}^{-1}}{\partial x_5}) = [X_+''(jx_5^{j-1})]\cdot\phi_{k-j+1}^{j-1} + (jx_5^{j-1})\cdot X_+'(\phi_{k-j+1}^{j-1}) + \text{ bihomogeneous}$$

polynomials of total degree in x_4 and x_5 variables less than $j-1$.

$$\vdots$$

$$X_+^{j-1}(\frac{\partial f_{k+1}^{-1}}{\partial x_5}) = [(X_+'')^{j-1}(jx_5^{j-1})]\cdot\phi_{k-j+1}^{j-1} + \ldots + \binom{j-1}{l}[X_+'')^{j-1-l}(jx_5^{j-1})][(X_+')^l\phi_{k-j+1}^{j-1}]$$

$$+ \ldots + (jx_5^{j-1})\cdot[(X_+')^{j-1}\phi_{k-j+1}^{j-1}] + \text{ bihomogeneous polynomials of total}$$

degree in x_4 and x_5 variables less than $j-1$.

The above j homogeneous polynomials are linearly independent elements in I. We have constructed $2j+1$ independent elements in I. Since $\dim I = 5$, we have $1 \le j \le 2$. So

$$f_{k+1}^{-1} = x_5\phi_k^{-2}(x_1,x_2,x_3) + x_5^2\phi_{k-1}^{-3}(x_1,x_2,x_3).$$

Observe that there is no homogeneous polynomial in x_1, x_2 and x_3 of odd weight. In particular, $\phi_{k-1}^{-3}(x_1, x_2, x_3)$ is equal to zero. Hence

$$f_{k+1}^{-1} = x_5 \phi_k^{-2}(x_1, x_2, x_3).$$

If $\phi_k^{-2}(x_1, x_2, x_3) \neq 0$, then $\frac{\partial f_{k+1}^{-1}}{\partial x_5} = \phi_k^{-2}(x_1, x_2, x_3)$ is a nonzero element in I. Since $X_-(\phi_k^{-2}(x_1, x_2, x_3))$ is of weight -4, we have $X_-(\phi_k^{-2}(x_1, x_2, x_3) = 0$. Thus

$$(3) = \langle \phi_k^{-2}(x_1, x_2, x_3),\ X_+(\phi_k^{-2}(x_1, x_2, x_3)),\ X_+^2(\phi_k^{-2}(x_1, x_2, x_3)) \rangle$$

is a 3-dimensional irreducible submodule in I, which is not possible. Therefore $f_{k+1}^{-1} = 0$. Similarly we can prove that $f_{k+1}^1 = 0$.

For $i = 0$.

Since f is a homogeneous polynomial of degree $k+1$ and weight 0, we have $wt\ \frac{\partial f}{\partial x_1} = -2$ and $wt\ \frac{\partial f}{\partial x_3} = 2$. Consequently $\frac{\partial f}{\partial x_1} = 0 = \frac{\partial f}{\partial x_3}$ and f does not involve x_1, x_3 variables. Write

$$f = \sum_{i=0}^{k} x_2^i \phi_{k+1-i}^0(x_4, x_5)$$

where $\phi_{k+1-i}^0(x_4, x_5)$ is a homogeneous polynomial of degree $k+1-i$ and weight 0. Now the argument which we used before to prove that $f_{k+1}^{-2} = 0$ can be applied here also to conclude that $f_{k+1}^0 = 0$. This means that Case 2 cannot occur.

Case 3. $I = (3) \oplus (2)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $2, 0, -2, 1$ and -1.

For $|i| \geq 5$.

$$\begin{aligned} & |wt\ \frac{\partial f_{k+1}^i}{\partial x_j}| \geq 3 && 1 \leq j \leq 5 \\ \Rightarrow\ & \frac{\partial f_{k+1}^i}{\partial x_j} = 0 && 1 \leq j \leq 5 \end{aligned}$$

$$\Rightarrow \quad f^i_{k+1} = 0.$$

For $i = -4$.

$$wt\,\frac{\partial f^{-4}_{k+1}}{\partial x_1} = -6 \quad wt\,\frac{\partial f^{-4}_{k+1}}{\partial x_2} = -4 \quad wt\,\frac{\partial f^{-4}_{k+1}}{\partial x_4} = -5 \quad wt\,\frac{\partial f^{-4}_{k+1}}{\partial x_5} = -3$$

$$\Rightarrow \quad \frac{\partial f^{-4}_{k+1}}{\partial x_1} = \frac{\partial f^{-4}_{k+1}}{\partial x_2} = \frac{\partial f^{-4}_{k+1}}{\partial x_4} = \frac{\partial f^{-4}_{k+1}}{\partial x_5} = 0$$

$$\Rightarrow \quad f^{-4}_{k+1} \text{ involves only } x_3 \text{ and } f^{-4}_{k+1} = cx_3^2 \text{ where } c \text{ is a constant.}$$

If c were not zero, then $k = 1$ which contradicts to our assumption that $k \geq 2$. Hence $f^{-4}_{k+1} = 0$. Similarly, we can prove $f^4_{k+1} = 0$.

For $i = -3$

$$wt\,\frac{\partial f^{-3}_{k+1}}{\partial x_1} = -5 \quad wt\,\frac{\partial f^{-3}_{k+1}}{\partial x_2} = -3 \quad wt\,\frac{\partial f^{-3}_{k+1}}{\partial x_4} = -4$$

$$\Rightarrow \quad \frac{\partial f^{-3}_{k+1}}{\partial x_1} = 0 = \frac{\partial f^{-3}_{k+1}}{\partial x_2} = \frac{\partial f^{-3}_{k+1}}{\partial x_4}$$

$$\Rightarrow \quad f^{-3}_{k+1} \text{ involves only } x_3 \text{ and } x_5 \text{ variables}$$

$$\Rightarrow \quad f^{-3}_{k+1} = c_1x_3x_5 + c_2x_5^3 \text{ where } c_1, c_2 \text{ are constants.}$$

Since $k \geq 2$, we have $c_1 = 0$ and $f^{-3}_{k+1} = c_2x_5^3$. If c_2 is not zero, then $k = 2$ and $x_5^2 = \frac{1}{3c_2}\frac{\partial f^{-3}_{k+1}}{\partial x_5} \in I$. This implies that

$$(3) = \langle x_4^2, x_4x_5, x_5^2\rangle \subseteq I.$$

Let $(2) = \langle \phi_2^1, \phi_2^{-1}\rangle \subseteq I$ where ϕ_3^1 and ϕ_3^{-1} are homogeneous polynomials of degree 2 with weights 1 and -1 respectively. Clearly ϕ_2^{-1} must have the following form

$$\begin{aligned}
\phi_2^{-1} &= \alpha_1x_2x_5 + \alpha_2x_3x_4 \\
X_+\phi_2^{-1} &= 2\alpha_1x_1x_5 + \alpha_1x_2x_4 + 2\alpha_2x_2x_4 = 2\alpha_1x_1x_5 + (\alpha_1 + 2\alpha_2)x_2x_4 \\
X_+^2\phi_2^{-1} &= (4\alpha_1 + 4\alpha_2)x_1x_4.
\end{aligned}$$

Since $X_+^2\phi_2^{-1} = 0$ and ϕ_2^1 is a constant multiple of $X_+\phi_2^{-1}$, we see that

$$\langle \phi_2^1, \phi_2^{-1} \rangle = \langle x_2x_4 - 2x_1x_5, x_3x_4 - x_2x_5 \rangle = (2) \subseteq I$$

$$wt\,\frac{\partial f_3^{-2}}{\partial x_1} = -4 \quad wt\,\frac{\partial f_3^{-2}}{\partial x_2} = -2 \quad wt\,\frac{\partial f_3^{-2}}{\partial x_3} = 0 \quad wt\,\frac{\partial f_3^{-2}}{\partial x_4} = -3 \quad wt\,\frac{\partial f_3^{-2}}{\partial x_5} = -1$$

$$\Rightarrow \quad \frac{\partial f_3^{-2}}{\partial x_1} = 0$$

$$\frac{\partial f_3^{-2}}{\partial x_2} = d_1x_5^2$$

$$\frac{\partial f_3^{-2}}{\partial x_3} = d_2x_4x_5$$

$$\frac{\partial f_3^{-2}}{\partial x_4} = 0$$

$$\frac{\partial f_3^{-2}}{\partial x_5} = d_3(x_3x_4 - x_2x_5) \text{ where } d_1, d_2, d_3 \text{ are constants.}$$

$$d_3x_3 = \frac{\partial^2 f_3^{-2}}{\partial x_4 \partial x_5} = \frac{\partial}{\partial x_5}\left(\frac{\partial f_3^{-2}}{\partial x_4}\right) = 0$$

$$\Rightarrow \quad d_3 = 0 \text{ and } \frac{\partial f_3^{-2}}{\partial x_5} = 0.$$

On the other hand

$$2d_1x_5 = \frac{\partial^2 f_3^{-2}}{\partial x_5 \partial x_2} = \frac{\partial}{\partial x_2}\left(\frac{\partial f_3^{-2}}{\partial x_5}\right) = 0$$

$$2d_2x_4 = \frac{\partial^2 f_3^{-2}}{\partial x_5 \partial x_3} = \frac{\partial}{\partial x_3}\left(\frac{\partial f_3^{-2}}{\partial x_5}\right) = 0$$

$$\Rightarrow d_1 = d_2 = 0 \text{ and } \frac{\partial f_3^{-2}}{\partial x_2} = \frac{\partial f_3^{-2}}{\partial x_3} = 0.$$

Therefore $f_3^{-2} = 0$. Similarly we have $f_3^2 = 0$

$$wt\,\frac{\partial f_3^{-1}}{\partial x_1} = -3 \quad wt\,\frac{\partial f_3^{-1}}{\partial x_2} = -1 \quad wt\,\frac{\partial f_3^{-1}}{\partial x_3} = 1 \quad wt\,\frac{\partial f_3^{-1}}{\partial x_4} = -2 \quad wt\,\frac{\partial f_3^{-1}}{\partial x_5} = 0$$

$$\Rightarrow \quad \frac{\partial f_3^{-1}}{\partial x_1} = 0$$

$$\frac{\partial f_3^{-1}}{\partial x_2} = d_4(x_3x_4 - x_2x_5)$$

$$\frac{\partial f_3^{-1}}{\partial x_3} = d_5(x_2x_4 - 2x_1x_5)$$

$$\frac{\partial f_3^{-1}}{\partial x_4} = d_6x_5^2$$

$$\frac{\partial f_3^{-1}}{\partial x_5} = d_7x_4x_5$$

$$\Rightarrow \quad -2d_1x_5 = \frac{\partial^2 f_3^{-1}}{\partial x_1 \partial x_3} = \frac{\partial}{\partial x_3}\left(\frac{\partial f_3^{-1}}{\partial x_1}\right) = 0$$

$$d_4x_4 = \frac{\partial^2 f_3^{-1}}{\partial x_3 \partial x_2} = \frac{\partial}{\partial x_2}\left(\frac{\partial f_3^{-1}}{\partial x_3}\right)$$

$$\Rightarrow \quad d_1 = 0 \text{ and } \frac{\partial f_3^{-1}}{\partial x_3} = 0.$$

It follows that $d_4 = 0$ and $\frac{\partial f_3^{-1}}{\partial x_3} = 0$. Therefore f_3^{-1} involves only x_4 and x_5 variables. In fact it is easy to see that $f_3^{-1} = c_3x_4x_5^2$. Similarly we can prove that

$$f_3^3 = c_4x_4^3 \text{ and } f_3^1 = c_5x_4^2x_5$$

$$wt\,\frac{\partial f_3^{-0}}{\partial x_1} = -2 \quad wt\,\frac{\partial f_3^{0}}{\partial x_2} = 0 \quad wt\,\frac{\partial f_3^{-0}}{\partial x_3} = 2 \quad wt\,\frac{\partial f_{k+1}^{0}}{\partial x_4} = -1 \quad wt\,\frac{\partial f_{k+1}^{-0}}{\partial x_5} = 1$$

$$\Rightarrow \quad \frac{\partial f_3^0}{\partial x_1} = e_1x_5^2$$

$$\frac{\partial f_3^0}{\partial x_2} = e_2x_4x_5$$

$$\frac{\partial f_3^0}{\partial x_3} = e_3x_4^2$$

$$\frac{\partial f_3^0}{\partial x_4} = e_4(x_3x_4 - x_2x_5)$$

$$\frac{\partial f_3^0}{\partial x_5} = e_5(x_2x_4 - 2x_1x_5) \text{ where } e_1, e_2, e_3, e_4, e_5 \text{ are constant.}$$

$$\Rightarrow \quad f_3^0 = e(2x_1x_5^2 - 2x_2x_4x_5 + x_3x_4^2) \text{ where } e \text{ is a constant.}$$

We conclude that up to constant multiple, f is of the following form

$$f = 2x_1x_5^2 - 2x_2x_4x_5 + x_3x_4^2 + c_4x_4^3 + c_5x_4^2x_5 + c_3x_4x_5^2 + c_2x_5^3$$

where $2x_1x_5^2 - 2x_2x_4x_5 + x_3x_4^2$ is a $sl(2, \mathbf{C})$ invariant polynomial.

For $i = -2$.

In this case we can assume that

$$\begin{aligned}
&f = f_{k+1}^{-2} + f_{k+1}^{-1} + f_{k+1}^{0} + f_{k+1}^{1} + f_{k+1}\\
&wt\,\frac{\partial f_{k+1}^{-2}}{\partial x_1} = -4 \quad wt\,\frac{\partial f_{k+1}^{-2}}{\partial x_4} = -3\\
\Rightarrow\quad &\frac{\partial f_{k+1}^{-2}}{\partial x_1} = 0 = \frac{\partial f_{k+1}^{-2}}{\partial x_4}
\end{aligned}$$

$\Rightarrow\quad f_{k+1}^{-2}$ involves only x_2, x_3 and x_5

$\Rightarrow\quad f_{k+1}^{-2} = c_1x_2^kx_3 + c_2x_2^{k-1}x_5^2$ where c_1, c_2 are constants.

If $c_1 \neq 0$, then $x_2^k = \frac{1}{c_1}\frac{\partial f_{k+1}^{-2}}{\partial x_3} \in I$. By applying X_- and X_+ successively on x_2^k, we have

$$\langle x_2^k, x_2^{k-1}x_3, x_2^{k-2}x_3^2, x_1x_2^{k-1}, \quad x_1^2x_2^{k-2}, x_1^3x_2^{k-3}\rangle \subseteq I.$$

Since $\dim I = 5$, we have $k = 2$ and

$$I = \langle x_2^2, x_2x_3, x_3^2, x_1x_2, x_1^2\rangle.$$

This implies that $I = (5)$, which contradicts to our assumption that $I = (3) \oplus (2)$. Therefore $c_1 = 0$ and

$$f_{k+1}^{-2} = c_2x_2^{k-1}x_5^2.$$

If $c_2 \neq 0$, then $x_2^{k-1}x_5 = \frac{1}{2c_2}\frac{\partial f_{k+1}^{-2}}{\partial x_5} \in I$. Therefore

$$\langle X_+^3(x_2^{k-1}x_5), X_+^2(x_2^{k-1}x_5), X_+(x_2^{k-1}x_5), \qquad x_2^{k-1}x_5, X_-(x_2^{k-1}x_5), X_-^2(x_2^{k-1}x_5)\rangle$$

$= \langle 8(k-1)(k-2)(k-3)x_1^3x_2^{k-4}x_5 + 7(k-1)(k-2)x_1^2x_2^{k-3}x_4,\ 4(k-1)(k-2)x_1^2x_2^{k-3}x_5+$

$+3(k-1)x_1x_2^{k-2}x_4,\ 2(k-1)x_1x_2^{k-2}x_5 + x_2^{k-1}x_4, x_2^{k-1}x_5, x_2^{k-2}x_3x_5, x_2^{k-3}x_3^2x_5\rangle \subseteq I.$

Since $\dim I = 5$, we have $k = 2$ and

$$\langle x_1x_4, 2x_1x_5 + x_2x_4, x_2x_5, x_3x_5\rangle \subseteq I.$$

$X_+(x_3x_5) = 2x_2x_5 + x_3x_4 \in I$. Since x_2x_5 is in I, we have x_3x_4 in I. As $X_+(x_3x_4) = 2x_2x_4 + x_3x_5$ and x_3x_5 are in I, we have $x_2x_4 \in I$. Therefore

$$\langle x_1x_4, x_1x_5, x_2x_5, x_3x_5, x_3x_4, x_2x_4\rangle$$

is a 6 dimensional subspace in I, which is impossible. We conclude that $c_2 = 0$ and $f_{k+1}^{-2} = 0$. Similarly we can prove that $f_{k+1}^{2} = 0$.

For $i = -1$.

In this case we can assume that

$$f = f_{k+1}^{-1} + f_{k+1}^{0} + f_{k+1}^{1}$$

$$wt\,\frac{\partial f_{k+1}^{-1}}{\partial x_1} = -3 \qquad wt\,\frac{\partial f_{k+1}^{1}}{\partial x_3} = 3$$

$$\Rightarrow \frac{\partial f_{k+1}^{-1}}{\partial x_1} = 0 = \frac{\partial f_{k+1}^{1}}{\partial x_3}$$

$\Rightarrow f_{k+1}^{-1}$ does not involve x_1 variable and f_{k+1}^{1} does not

involve x_3 variable.

We claim $\frac{\partial f_{k+1}^{-1}}{\partial x_2} = 0$. Suppose on the contrary that $\frac{\partial f_{k+1}^{-1}}{\partial x_2} \neq 0$. Then observe that $X_- \frac{\partial f_{k+1}^{-1}}{\partial x_2} = 0$ because $wt\, X_- \frac{\partial f_{k+1}^{-1}}{\partial x_2} = -3$. It follows that

$$(2) = \langle \frac{\partial f_{k+1}^{-1}}{\partial x_2},\ X_+ \frac{\partial f_{k+1}^{-1}}{\partial x_2}\rangle \subseteq I.$$

Since $\frac{\partial f_{k+1}^{-1}}{\partial x_3}$ and $X_+\frac{\partial f_{k+1}^{-1}}{\partial x_2}$ are elements in I of same weight 1, there exists a constant c_1 such that

$$\frac{\partial f_{k+1}^{-1}}{\partial x_3} = c_1 X_+ \frac{\partial f_{k+1}^{-1}}{\partial x_2}$$

$$\Rightarrow \frac{\partial^2 f_{k+1}^{-1}}{\partial x_1 \partial x_3} = c_1 \frac{\partial}{\partial x_1} X_+ \frac{\partial f_{k+1}^{-1}}{\partial x_2}$$

$$\Rightarrow 0 = \frac{\partial^2 f_{k+1}^{-1}}{\partial x_3 \partial x_1} = c_1 X_+ \frac{\partial}{\partial_{x_1}}(\frac{\partial f_{k+1}^{-1}}{\partial x_2}) + 2c_1 \frac{\partial}{\partial x_2}(\frac{\partial f_{k+1}^{-1}}{\partial x_2})$$

$$\Rightarrow c_1 \frac{\partial^2 f_{k+1}^{-1}}{\partial x_2^2} = 0$$

$$\Rightarrow \text{ either } c_1 = 0 \text{ or } \frac{\partial^2 f_{k+1}^{-1}}{\partial x_2^2} = 0.$$

If $c_1 = 0$, then $\frac{\partial f_{k+1}^{-1}}{\partial x_3} = 0$. In this case f_{k+1}^{-1} involves only x_2, x_4 and x_5 variables. We can write

$$f_{k+1}^{-1} = \sum_{j=0}^{[\frac{k}{2}]} b_j x_2^{k-2j} x_4^j x_5^{j+1}$$

where $[\frac{k}{2}]$ denotes the largest integer less than $\frac{k}{2}$. Let j_0 be the least integer such that $b_{j_0} \neq 0$. Then

$$\frac{\partial f_{k+1}^{-1}}{\partial x_5} = (j_0+1) b_{j_0} x_2^{k-2j_0} x_4^{j_0} x_5^{j_0} + (j_0+2) b_{j_0+1} x_2^{k-2(j_0+1)} x_4^{j_0+1} x_5^{j_0+1}$$

$$+ \ldots + i b_i x_2^{k-2i} x_4^i x_5^i + \ldots .$$

By formula (3.1), we have

$$X_-^{k-j_0}(\frac{\partial f_{k+1}^{-1}}{\partial x_5}) = \tilde{b} x_3^{k-2j_0} x_5^{2j_0}$$

where $\tilde{b}$ is a non-zero constant. Since $X_-(x_3^{k-2j_0} x_5^{2j_0}) = 0$, we see that there is an irreducible $sl(2,\mathbf{C})$-submodule of dimension $2k-2j_0+1$ of the following form

$$\langle x_3^{k-2j_0} x_5^{2j_0},\ X_+(x_3^{k-2j_0} x_5^{2j_0}), \ldots, X_+^{2k-2j_0}(x_3^{k-2j_0} x_5^{2j_0})\rangle.$$

Since $j_0 \leq [\frac{k}{2}]$, we have $2k-2j_0+1 \geq k+1$. As $I=(3)\oplus(2)$, we have $k+1\leq 3$. This implies that $k=2$ and $j_0 \leq 1$ because of our assumption $k\geq 2$. It follows that $f_3^{-1} = b_0x_2^2x_5 + b_1x_4x_5^2$. Since $\frac{\partial f_3^{-1}}{\partial x_2} \neq 0$ by our hypothesis, we have $b_0 \neq 0$. So $x_2x_5 = \frac{1}{2b_0}\frac{\partial f_3^{-1}}{\partial x_2}$ and $x_3x_5 = X_-(x_2x_5)$ are in I. However, the weight of x_3x_5 is -3. This contradicts to our hypothesis

$$I=(3)\oplus(2).$$

On the other hand if $c_1\neq 0$, then $\frac{\partial^2 f_{k+1}^{-1}}{\partial x_2^2}=0$. This means that the degree of f_{k+1}^{-1} in x_2 variable is at most one.

$$f_{k+1}^{-1} = \sum_{\alpha=0}^{[(k+2)/3]} d_\alpha x_3^\alpha x_4^{\frac{k+\alpha}{2}} x_5^{\frac{k+2-3\alpha}{2}} + \sum_{\beta=0}^{[\frac{k+1}{3}]} e_\beta x_2 x_3^\beta x_4^{\frac{k-1+\beta}{2}} x_5^{\frac{k+1-3\beta}{2}}.$$

Let β_0 be the biggest integer such that $e_{\beta_0}\neq 0$. Then

$$\frac{\partial f_{k+1}^{-1}}{\partial x_2} = e_0 x_4^{\frac{k-1}{2}} x_5^{\frac{k+1}{2}} + \ldots + e_i x_3^i x_4^{\frac{k-1+i}{2}} x_5^{\frac{k+1-3i}{2}} + \ldots + e_{\beta_0} x_3^{\beta_0} x_4^{\frac{k-1+\beta_0}{2}} x_5^{\frac{k+1-3\beta_0}{2}}$$

$$X_-^{\frac{k-1+\beta_0}{2}}\left(\frac{\partial f_{k+1}^{-1}}{\partial x_2}\right) = \tilde{e} x_3^{\beta_0} x_5^{k-\beta_0}$$

where $\tilde{e}$ is a nonzero constant. Since $X_-(x_3^{\beta_0}x_5^{k-\beta_0}) = 0$, we see that there is an irreducible $sl(2,\mathbf{C})$-submodule of dimension $k+2\beta_0+1$ of the following form

$$\langle x_3^{\beta_0}x_5^{k-\beta_0},\ X_+(x_3^{\beta_0}x_5^{k-\beta_0}), \ldots, X_+^{k+2\beta_0}(x_3^{\beta_0}x_5^{k-\beta_0})\rangle.$$

As $I=(3)\oplus(2)$, we have $k+1+2\beta_0\leq 3$. This implies that $k=2$ and $\beta_0=0$ because of our assumption $k\geq 2$. It follows that

$$f_3^{-1} = d_0 x_4 x_5^2.$$

This contradicts to our hypothesis that $\frac{\partial f_{k+1}^{-1}}{\partial x_2}\neq 0$. All these together establish our claim that $\frac{\partial f_{k+1}^{-1}}{\partial x_2}=0$. Hence f_{k+1}^{-1} involves only x_3, x_4 and x_5 variables.

We next claim that $\frac{\partial f_{k+1}^{-1}}{\partial x_3} = 0$. Suppose on the contrary that $\frac{\partial f_{k+1}^{-1}}{\partial x_3} \neq 0$. We can write

$$f_{k+1}^{-1} = \sum_{\gamma=0}^{[\frac{k+2}{3}]} a_\gamma x_3^\gamma x_4^{\frac{k+\gamma}{2}} x_5^{\frac{k+2-3\gamma}{2}}.$$

Let γ_0 be the biggest integer such that $a_{\gamma_0} \neq 0$. Then

$$\begin{aligned} \frac{\partial f_{k+1}^{-1}}{\partial x_3} &= a_1 x_4^{\frac{k+1}{2}} x_5^{\frac{k-1}{2}} + 2a_2 x_3 x_4^{\frac{k+2}{2}} x_5^{\frac{k-4}{2}} + \ldots + i a_i x_3^{i-1} x_4^{\frac{k+i}{2}} x_5^{\frac{k+2-3i}{2}} \\ &\quad + \ldots + \gamma_0 a_{\gamma_0} x_3^{\gamma_0 - 1} x_4^{\frac{k+\gamma_0}{2}} x_5^{\frac{k+2-3\gamma_0}{2}} \\ X_-^{\frac{k+\gamma_0}{2}}\left(\frac{\partial f_{k+1}^{-1}}{\partial x_3}\right) &= \tilde{\gamma} x_3^{\gamma_0-1} x_5^{k+1-\gamma_0} \end{aligned}$$

where $\tilde{\gamma}$ is a nonzero constant. Since $X_-(x_3^{\gamma_0-1} x_5^{k+1-\gamma_0}) = 0$. We see that there is an irreducible $sl(2, \mathbf{C})$-submodule of dimension $k + \gamma_0$ of the following form

$$\langle x_3^{\gamma_0-1} x_5^{k+1-\gamma_0},\ X_+(x_3^{\gamma_0-1} x_5^{k+1-\gamma_0}), \ldots, X_+^{k+\gamma_0-1}(x_3^{\gamma_0-1} x_5^{k+1-\gamma_0})\rangle.$$

Since $I = (3) \oplus (2)$, we have $k + \gamma_0 \leq 3$. Hence (k, γ_0) is either $(3,0)$, $(2,0)$ or $(2,1)$ because $k \geq 2$. In both cases f_{k+1}^{-1} cannot involve x_3 variable, a contradiction to our assumption. This established our claim that $\frac{\partial f_{k+1}^{-1}}{\partial x_3} = 0$.

Now f_{k+1}^{-1} involves only x_4 and x_5 variable. If f_{k+1}^{-1} is not zero, then it is easy to see that there is an irreducible $sl(2, \mathbf{C})$ submodule of dimension $k+1$ of the following form

$$\langle x_4^k, x_4^{k-1} x_5, \ldots, x_5^k \rangle.$$

Since $I = (3) \oplus (2)$ and $k \geq 2$, we have $k = 2$. It follows that $f_3^{-1} = c_2 x_4 x_5^2$ where c_2 is a constant. By using the same argument as before, we have

$$I = \langle x_4^2, x_4 x_5, x_5^2 \rangle \oplus \langle x_2 x_4 - 2x_1 x_5, x_3 x_4 - x_2 x_5 \rangle$$

and $f = 2x_1 x_5^2 - 2x_2 x_4 x_5 + x_3 x_4^2 + c_2 x_4 x_5^2 + c_3 x_4^2 x_5$.

Case 4. $I = (3) \oplus (1) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights 2,0 and -2.

Clearly $f^i_{k+1} = 0$ for $|i| \geq 4$ by the same proof given in Case 3 above.

For $i = -2$

$$\begin{aligned}
&wt\,\frac{\partial f^{-2}_{k+1}}{\partial x_1} = -4 \quad wt\,\frac{\partial f^{-2}_{k+1}}{\partial x_4} = -3 \quad wt\,\frac{\partial f^{-2}_{k+1}}{\partial x_5} = -1\\
&\Rightarrow \frac{\partial f^{-2}_{k+1}}{\partial x_1} = 0 = \frac{\partial f^{-2}_{k+1}}{\partial x_4} = \frac{\partial f^{-2}_{k+1}}{\partial x_5}\\
&\Rightarrow f^{-2}_{k+1} \text{ involves only } x_2 \text{ and } x_3 \text{ variables}\\
&\Rightarrow f^{-2}_{k+1} = c x_2^k x_3 \text{ where } c \text{ is a constant.}
\end{aligned}$$

If $c \neq 0$, then $x_2^k = \frac{1}{c}\frac{\partial f^{-2}_{k+1}}{\partial x_3} \in I$. In particular $x_3^k = \frac{1}{k!}X_-^k x_2^k$ is also in I. It follows that

$$\langle x_3^k, X_+(x_3^k),\ X_+^2(x_3^k), \ldots, X_+^{2k}(x_3^k)\rangle$$

is an irreducible $sl(2,\mathbf{C})$-submodule of dimension $2k+1$. in I. Since $I = (3)\oplus(1)\oplus(1)$, we have $k \leq 1$. This contradicts to our hypothesis that $k \geq 2$. Therefore $f^{-2}_{k+1} = 0$.

Similarly we can prove $f^2_{k+1} = 0$.

We can write

$$f = f^{-3}_{k+1} + f^{-1}_{k+1} + f^0_{k+1} + f^1_{k+1} + f^3_{k+1}.$$

Since weights of $\frac{\partial f^{-3}_{k+1}}{\partial x_i}$, $\frac{\partial f^{-1}_{k+1}}{\partial x_i}$, $\frac{\partial f^{1}_{k+1}}{\partial x_i}$ and $\frac{\partial f^{3}_{k+1}}{\partial x_i}$ are odd integers for $1 \leq i \leq 3$, $\frac{\partial f^{-3}_{k+1}}{\partial x_i}$, $\frac{\partial f^{-1}_{k+1}}{\partial x_i}$, $\frac{\partial f^{1}_{k+1}}{\partial x_i}$, and $\frac{\partial f^{3}_{k+1}}{\partial x_i}$, are zero for $1 \leq i \leq 3$. Therefore f^3_{k+1}, f^{-1}_{k+1}, f^1_{k+1} and f^3_{k+1} are polynomials involving x_4 and x_5 variables only. On the other hand, since weights of $\frac{\partial f^0_{k+1}}{\partial x_4}$ and $\frac{\partial f^0_{k+1}}{\partial x_5}$ are -1 and 1 respectively, we have $\frac{\partial f^0_{k+1}}{\partial x_4} = 0$ and $\frac{\partial f^0_{k+1}}{\partial x_5} = 0$. Hence f^0_{k+1} is a polynomial involving only x_1, x_2 and x_3 variables.

If one of the f_{k+1}^{-3}, f_{k+1}^{-1}, f_{k+1}^{1} and f_{k+1}^{3} is zero, then it is easy to see that there is an irreducible $sl(2, \mathbf{C})$ submodule of dimension $k+1$ in I of the following form

$$\langle x_4^k, x_4^{k-1}x_5, \ldots, x_5^k \rangle.$$

Since $I = (3) \oplus (1) \oplus (1)$ and $k \geq 2$, we have $k = 2$ and

$$(3) = \langle x_4^2, x_4x_5, x_5^2 \rangle \subseteq I.$$

There are constants c_1 and c_2 such that

$$\frac{\partial f_{k+1}^0}{\partial x_1} = c_1 x_5^2 \qquad \frac{\partial f_{k+1}^0}{\partial x_3} = c_2 x_4^2.$$

Since f_{k+1}^0 does not involve x_4, x_5 variables, we conclude that $c_1 = c_2 = 0$. It follows that $\frac{\partial f_{k+1}^0}{\partial x_1} = \frac{\partial f_{k+1}^0}{\partial x_3} = 0$. This implies that $\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_3} = 0$, which contradicts to our hypothesis that $\dim I = 5$.

If all of f_{k+1}^{-3}, f_{k+1}^{-1}, f_{k+1}^{1} and f_{k+1}^{3} are zero, then $f = f_{k+1}^0(x_1, x_2, x_3)$. This implies that $\frac{\partial f}{\partial x_4} = \frac{\partial f}{\partial x_5} = 0$, which contradicts to our hypothesis that $\dim I = 5$.

So Case 4 cannot occur.

Case 5. $I = (2) \oplus (2) \oplus (1)$.

Elements in I are linear combinations of homogeneous polynomials of degree $k+1$ and weight $-1, 0$ and 1.

For $|i| \geq 4$

$$\begin{aligned}
& |wt\, \frac{\partial f_{k+1}^i}{\partial x_j}| \geq 2 && 1 \leq j \leq 5 \\
\Rightarrow\ & \frac{\partial f_{k+1}^i}{\partial x_j}| = 0 && 1 \leq j \leq 5 \\
\Rightarrow\ & f_{k+1}^i = 0 && 1 \leq j \leq 5.
\end{aligned}$$

For $i = -3$

$$wt\,\frac{\partial f_{k+1}^{-3}}{\partial x_1} = -5 \quad wt\,\frac{\partial f_{k+1}^{-3}}{\partial x_2} = -3 \quad wt\,\frac{\partial f_{k+1}^{-3}}{\partial x_4} = -4 \quad wt\,\frac{\partial f_{k+1}^{-3}}{\partial x_5} = -2$$

$$\Rightarrow \quad \frac{\partial f_{k+1}^{-3}}{\partial x_1} = 0 = \frac{\partial f_{k+1}^{-3}}{\partial x_2} = \frac{\partial f_{k+1}^{-3}}{\partial x_4} = \frac{\partial f_{k+1}^{-3}}{\partial x_5}$$

$\Rightarrow$ f_{k+1}^{-3} depends only on x_3 variable

$\Rightarrow$ $f_{k+1}^{-3} = 0$ because -3 is not divisible by $wt(x_3) = -2$.

Similarly $f_{k+1}^{3} = 0$.

For $i = -2$

$$wt\,\frac{\partial f_{k+1}^{-2}}{\partial x_1} = -4 \quad wt\,\frac{\partial f_{k+1}^{-2}}{\partial x_2} = -2 \quad wt\,\frac{\partial f_{k+1}^{-2}}{\partial x_4} = -3$$

$$\Rightarrow \quad \frac{\partial f_{k+1}^{-2}}{\partial x_1} = 0 = \frac{\partial f_{k+1}^{-2}}{\partial x_2} = \frac{\partial f_{k+1}^{-2}}{\partial x_4}$$

$\Rightarrow$ f_{k+1}^{-2} involves only x_3 and x_5 variables

$\Rightarrow$ $f_{k+1}^{-2} = c_1 x_3 + c_2 x_5^2$ where c_1, c_2 are constant

$\Rightarrow$ $f_{k+1}^{-2} = 0$ because $k \geq 2$.

Similarly $f_{k+1}^{2} = 0$.

For $i = -1$

$$wt\,\frac{\partial f_{k+1}^{-1}}{\partial x_1} = -3 \qquad wt\,\frac{\partial f_{k+1}^{-1}}{\partial x_4} = -2$$

$$\Rightarrow \quad \frac{\partial f_{k+1}^{-1}}{\partial x_1} = 0 = \frac{\partial f_{k+1}^{-1}}{\partial x_4}$$

$\Rightarrow$ f_{k+1}^{-1} involves only x_2, x_3 and x_5

$\Rightarrow$ $f_{k+1}^{-1} = c x_2^k x_5$ where c is a constant.

If $c \neq 0$, then $x_2^k = \frac{1}{c}\frac{\partial f_{k+1}^{-1}}{\partial x_5} \in I$. $x_3^k = \frac{1}{k!}X_-^k(x_2^k)$ is in I. It follows that there exists an irreducible $sl(2,\mathbb{C})$-submodule of $\dim 2k+1$ in I of the following form.

$$\langle x_3^k, X_+(x_3^k), \ldots, X_+^{2k}(x_3^k)\rangle.$$

Since $2k+1 \geq 5$, this contradicts our hypothesis $I = (2) \oplus (2) \oplus (1)$. Therefore $c = 0$ and f_{k+1}^{-1}.

Similarly we can prove $f_{k+1}^{1} = 0$.

For $i = 0$,

$$wt\,\frac{\partial f_{k+1}^{-0}}{\partial x_1} = -2 \qquad wt\,\frac{\partial f_{k+1}^{0}}{\partial x_3} = 2$$

$$\Rightarrow \frac{\partial f_{k+1}^{0}}{\partial x_1} = 0 = \frac{\partial f_{k+1}^{0}}{\partial x_3}$$

Since $f = f_{k+1}^{0}$, we see that $\dim I = \dim(\frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_4}, \frac{\partial f}{\partial x_5}) \leq 3$. This contradicts to our hypothesis that $\dim I = 5$.

So Case 5 cannot occur.

Case 6. $I = (2) \oplus (1) \oplus (1) \oplus (1)$.

This case cannot occur. The argument is the same as those shown in Case 5 above.

Case 7. $I = (1) \oplus (1) \oplus (1) \oplus (1) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials of degree $k+1$ and weight 0.

For $|i| \geq 3$

$$|wt\,\frac{\partial f_{k+1}^{i}}{\partial x_j}| \geq 1 \qquad 1 \leq j \leq 5$$

$$\Rightarrow \quad \frac{\partial f_{k+1}^{i}}{\partial x_j}| = 0 \qquad 1 \leq j \leq 5$$

$$\Rightarrow \quad f_{k+1}^{i} = 0 \qquad 1 \leq j \leq 5.$$

For $i = -2$

$$wt\,\frac{\partial f_{k+1}^{-2}}{\partial x_1} = -4 \quad wt\,\frac{\partial f_{k+1}^{-2}}{\partial x_2} = -2 \quad wt\,\frac{\partial f_{k+1}^{-2}}{\partial x_4} = -3 \quad wt\,\frac{\partial f_{k+1}^{-2}}{\partial x_5} = -1$$

$$\Rightarrow \quad \frac{\partial f_{k+1}^{-2}}{\partial x_1} = 0 = \frac{\partial f_{k+1}^{-2}}{\partial x_2} = \frac{\partial f_{k+1}^{-2}}{\partial x_4} = \frac{\partial f_{k+1}^{-2}}{\partial x_5}$$

$\Rightarrow \quad f_{k+1}^{-2}$ depends only on x_3 variable

$\Rightarrow \quad f_{k+1}^{-2} = cx_3$ where c is a constant

$\Rightarrow \quad f_{k+1}^{-2} = 0$ since $k \geq 2$.

Similarly we can prove $f_{k+1}^{2} = 0$.

For $i = -1$

$$wt\,\frac{\partial f_{k+1}^{-1}}{\partial x_1} = -3 \quad wt\,\frac{\partial f_{k+1}^{-1}}{\partial x_2} = -1 \quad wt\,\frac{\partial f_{k+1}^{-1}}{\partial x_3} = 1 \quad wt\,\frac{\partial f_{k+1}^{-1}}{\partial x_4} = -2$$

$$\Rightarrow \quad \frac{\partial f_{k+1}^{-1}}{\partial x_1} = 0 = \frac{\partial f_{k+1}^{-1}}{\partial x_2} = \frac{\partial f_{k+1}^{-1}}{\partial x_3} = \frac{\partial f_{k+1}^{-1}}{\partial x_4}$$

$\Rightarrow \quad f_{k+1}^{-1}$ depends only on x_5 variable

$\Rightarrow \quad f_{k+1}^{-1} = cx_5$ for some constant c

$\Rightarrow \quad f_{k+1}^{-1} = 0$ because $k \geq 2$.

Similarly $f_{k+1}^{1} = 0$.

For $i = 0$

$$wt\,\frac{\partial f_{k+1}^{0}}{\partial x_1} = -2 \quad wt\,\frac{\partial f_{k+1}^{0}}{\partial x_3} = 2 \quad wt\,\frac{\partial f_{k+1}^{0}}{\partial x_4} = -1 \quad wt\,\frac{\partial f_{k+1}^{0}}{\partial x_5} = 1$$

$$\Rightarrow \quad \frac{\partial f_{k+1}^{0}}{\partial x_1} = 0 = \frac{\partial f_{k+1}^{0}}{\partial x_3} = \frac{\partial f_{k+1}^{0}}{\partial x_4} = \frac{\partial f_{k+1}^{0}}{\partial x_5}.$$

Since $f = f_{k+1}^{0}$, we have $\dim I = \dim\langle \frac{\partial f}{\partial x_2}\rangle \leq 1$. This is a contradiction. Hence Case 7 cannot occur. Q.E.D.

Lemma 3.2. With the same hypothesis as Lemma 3.1; if $\dim I = 4$, then I is not a $sl(2, \mathbf{C})$-submodule.

Proof. We shall assume that I is a $sl(2, \mathbf{C})$ module and shall produce contradiction.

Case 1. $I = (4)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $-3, -1, 1$ and 3. Write

$$f = \sum_{i=-\infty}^{\infty} f_{k+1}^{i}$$

where f_{k+1}^{i} is a homogeneous polynomial of degree $k+1$ and weight i.

For $|i| \geq 5$, we have $f_{k+1}^{i} = 0$ by the same argument in Case 2 in the proof of Lemma 3.1.

For $i = 4$

$$wt\,\frac{\partial f_{k+1}^4}{\partial x_1} = 2, \quad wt\,\frac{\partial f_{k+1}^4}{\partial x_2} = 4, \quad wt\,\frac{\partial f_{k+1}^4}{\partial x_3} = 6, \quad wt\,\frac{\partial f_{k+1}^4}{\partial x_5} = 5$$

$$\Rightarrow \quad \frac{\partial f_{k+1}^4}{\partial x_1} = \frac{\partial f_{k+1}^4}{\partial x_2} = \frac{\partial f_{k+1}^4}{\partial x_3} = \frac{\partial f_{k+1}^4}{\partial x_5} = 0$$

$$\Rightarrow \quad f_{k+1}^4 \text{ involves only } x_4 \text{ variable}$$

$$\Rightarrow \quad f_{k+1}^4 = c x_4^4 \text{ where } c \text{ is a constant.}$$

If $c \neq 0$, then $k = 3$ and $x_4^3 = \frac{1}{4c}\frac{\partial f_{k+1}^4}{\partial x_4} \in I$. It follows that

$$I = \langle x_4^3, x_4^2 x_5, x_4 x_5^2, x_5^3 \rangle.$$

As a consequence, f involves only x_4 and x_5 variables. This implies $\frac{\partial f}{\partial x_1} = 0 = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3}$ and $\dim I \leq 2$, which contradicts to our hypothesis that $\dim I = 4$. Hence we conclude that $c = 0$ and $f_{k+1}^4 = 0$.

Similarly we can prove $f_{k+1}^{-4} = 0$.

For $i = 3$

$$wt\,\frac{\partial f_{k+1}^3}{\partial x_3} = 5 \quad wt\,\frac{\partial f_{k+1}^3}{\partial x_4} = 2 \quad wt\,\frac{\partial f_{k+1}^3}{\partial x_5} = 4$$

$$\Rightarrow \quad \frac{\partial f_{k+1}^3}{\partial x_3} = 0 = \frac{\partial f_{k+1}^3}{\partial x_4} = \frac{\partial f_{k+1}^3}{\partial x_5}$$

$$\Rightarrow \quad f^3_{k+1} \text{ involves only } x_1 \text{ and } x_2 \text{ variable}$$

$$\Rightarrow \quad f^3_{k+1} = 0 \text{ because 3 is not divisible by } wt\, x_1 = 2 \text{ and } x_2 \text{ has weight } 0$$

Similarly we can prove $f^{-3}_{k+1} = 0$.

For $i = 2$

$$wt\,\frac{\partial f^2_{k+1}}{\partial x_1} = 0, \quad wt\,\frac{\partial f^2_{k+1}}{\partial x_2} = 2, \quad wt\,\frac{\partial f^2_{k+1}}{\partial x_3} = 4$$

$$\Rightarrow \quad \frac{\partial f^2_{k+1}}{\partial x_1} = 0 = \frac{\partial f^2_{k+1}}{\partial x_2} = \frac{\partial f^2_{k+1}}{\partial x_3}$$

$$\Rightarrow \quad f^2_{k+1} \text{ involves only } x_4 \text{ and } x_5 \text{ variable}$$

If $f^2_{k+1} \neq 0$, then it is easy to see that $k = 2$ and

$$I = \langle x_4^3, x_4^2 x_5, x_4 x_5^2, x_5^3 \rangle$$

As a consequence, f involves only x_4 and x_5 variables. This implies $\frac{\partial f}{\partial x_1} = 0 = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3}$ and $\dim I \leq 2$, which contradicts to our hypothesis that $\dim I = 4$. Hence we conclude that $f^2_{k+1} = 0$. Similarly we can prove $f^{-2}_{k+1} = 0$.

For $i = 1$

$$wt\,\frac{\partial f^1_{k+1}}{\partial x_4} = 0 \quad wt\,\frac{\partial f^1_{k+1}}{\partial x_5} = 2$$

$$\Rightarrow \quad \frac{\partial f^1_{k+1}}{\partial x_4} = \frac{\partial f^1_{k+1}}{\partial x_5} = 0$$

$$\Rightarrow \quad f^1_{k+1} \text{ involves only } x_1, x_2 \text{ and } x_3 \text{ variable}$$

If $f^1_{k+1} \neq 0$, then either $\frac{\partial f^1_{k+1}}{\partial x_1}$ or $\frac{\partial f^1_{k+1}}{\partial x_2}$ or $\frac{\partial f^1_{k+1}}{\partial x_3}$ is a nonzero element in I. Since I is irreducible, we see that elements in I are homogeneous polynomials in x_1, x_2 and x_3 variables. This implies that f_{k+1} is a homogeneous polynomial in x_1, x_2 and x_3

variables. It follows that $\frac{\partial f}{\partial x_4} = \frac{\partial f}{\partial x_5} = 0$ and $\dim I \leq 3$, which contradicts to our hypothesis that $\dim I = 4$. Hence we conclude that $f^1_{k+1} = 0$.

Similarly we can prove that $f^{-1}_{k+1} = 0$.

For $i = 0$

$$wt\,\frac{\partial f^0_{k+1}}{\partial x_1} = -2 \quad wt\,\frac{\partial f^0_{k+1}}{\partial x_2} = 0 \quad wt\,\frac{\partial f^0_{k+1}}{\partial x_3} = 2 \quad wt\,\frac{\partial f^0_{k+1}}{\partial x_4} = -1$$

$$wt\,\frac{\partial f^0_{k+1}}{\partial x_5} = 1$$

$$\Rightarrow \quad \frac{\partial f^0_{k+1}}{\partial x_i} = 0 \text{ for all } 1 \leq i \leq 5$$

$$\Rightarrow \quad f^0_{k+1} = 0.$$

Hence Case 1 cannot occur.

Case 2. $I = (3) \oplus (1)$.

The same argument as in Case 4 in the proof of Lemma 3.1 shows that Case 2 cannot occur.

Case 3. $I = (2) \oplus (2)$.

The same argument as Case 5 in the proof of Lemma 3.1 shows that Case 3 cannot occur.

Case 4. $I = (2) \oplus (1) \oplus (1)$.

This case cannot occur. The proof is the same as those shown in Case 5 in Lemma 3.1.

Case 5. $I = (1) \oplus (1) \oplus (1) \oplus (1)$.

This case cannot occur. The proof is the same as those shown in Case 7 in the proof of Lemma 3.1. Q.E.D.

Lemma 3.3. With the same hypothesis in Lemma 3.1, if I is a $sl(2,\mathbf{C})$-submodule of dimension 3 then f is a homogeneous polynomial in x_1, x_2 and x_3 variables of weight 0 and I is an irreducible $sl(2,\mathbf{C})$-submodule.

Proof. Case 1. $I = (3)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $-2, 0$ and 2. Write

$$f = \sum_{i=-\infty}^{\infty} f_{k+1}^{i}$$

where f_{k+1}^{i} is a homogeneous polynomial of degree $k+1$ and weight i.

By the same argument as in Case 4 of Lemma 3.1 we can write

$$f = f_{k+1}^{-3} + f_{k+1}^{-1} + f_{k+1}^{0} + f_{k+1}^{1} + f_{k+1}^{3}.$$

Since weights of $\frac{\partial f_{k+1}^{-3}}{\partial x_i}$, $\frac{\partial f_{k+1}^{-1}}{\partial x_i}$, $\frac{\partial f_{k+1}^{1}}{\partial x_i}$ and $\frac{\partial f_{k+1}^{3}}{\partial x_i}$ are odd integers for $1 \le i \le 3$, so $\frac{\partial f_{k+1}^{-3}}{\partial x_i}$, $\frac{\partial f_{k+1}^{-1}}{\partial x_i}$, $\frac{\partial f_{k+1}^{1}}{\partial x_i}$ and $\frac{\partial f_{k+1}^{3}}{\partial x_i}$ are zero for $1 \le i \le 3$. Therefore $f_{k+1}^{-3}, f_{k+1}^{-1}, f_{k+1}^{1}$ and f_{k+1}^{3} are polynomials involving x_4 and x_5 variables only. On the other hand, since weights of $\frac{\partial f_{k+1}^{0}}{\partial x_4}$ and $\frac{\partial f_{k+1}^{0}}{\partial x_5}$ are -1 and 1 respectively, we have $\frac{\partial f_{k+1}^{0}}{\partial x_4} = 0$ and $\frac{\partial f_{k+1}^{0}}{\partial x_5} = 0$. Hence f_{k+1}^{0} is a polynomial involving only x_1, x_2 and x_3 variablels.

If one of the $f_{k+1}^{-3}, f_{k+1}^{-1}, f_{k+1}^{1}$ and f_{k+1}^{3} is zero, then it is easy to see that there is an irreducible $sl(2,\mathbf{C})$ sumbodule of dimension $k+1$ in I of the following form.

$$\langle x_4^k, x_4^{k-1}x_5, \ldots, x_5^k \rangle.$$

Since $I = (3)$, we have $k = 2$ and

$$I = \langle x_4^2, x_4x_5, x_5^2 \rangle.$$

There are constants c_1, c_2 and c_3 such that

$$\frac{\partial f_{k+1}^{0}}{\partial x_1} = c_1 x_5^2 \quad \frac{\partial f_{k+1}^{0}}{\partial x_2} = c_2 x_4 x_5 \quad \frac{\partial f_{k+1}^{0}}{\partial x_3} = c_3 x_5^2.$$

Since f^0_{k+1} does not involve x_4, x_5 variables, we conclude that $c_1 = c_2 = c_3 = 0$. It follows that $\frac{\partial f^0_{k+1}}{\partial x_1} = \frac{\partial f^0_{k+1}}{\partial x_2} = \frac{\partial f^0_{k+1}}{\partial x_3} = 0$ and hence $f^0_{k+1} = 0$. This implies that $\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3} = 0$, which contradicts to our hypothesis that $\dim I = 3$.

So all of $f^{-3}_{k+1}, f^{-1}_{k+1}, f^1_{k+1}$ and f^3_{k+1} are zero. Consequently $f = f^0_{k+1}(x_1, x_2, x_3)$.

Case 2. $I = (2) \oplus (1)$.

By using the same argument as Case 5 in the proof of Lemma 3.1, we have $f = f^0_{k+1}(x_2, x_4, x_5)$. By weight consideration, we have

$$\begin{aligned} I &= (2) \oplus (1) \\ &= \langle \frac{\partial f}{\partial x_4}, \frac{\partial f}{\partial x_5} \rangle \oplus \langle \frac{\partial f}{\partial x_2} \rangle. \end{aligned}$$

Write

$$f = f^0_{k+1} = \sum_{\alpha=0}^{[\frac{k+1}{2}]} c_\alpha x_2^{k+1-2\alpha} x_4^\alpha x_5^\alpha.$$

Since $\frac{\partial f}{\partial x_2} = \sum_{\alpha=0}^{[\frac{k+1}{2}]} (k+1-2\alpha) c_\alpha x_2^{k-2\alpha} x_4^\alpha x_5^\alpha$ is nonzero element, clearly $X_-(\frac{\partial f}{\partial x_2}) \neq 0$. This contradicts the fact that $\frac{\partial f}{\partial x_2}$ spans a 1-dimensional $sl(2, \mathbf{C})$-submodule. We conclude that Case 2 cannot occur.

Case 3. $I = (1) \oplus (1) \oplus (1)$.

This case cannot occur. The proof is the same as those given in Case 7 in the proof of Lemma 3.1.

Lemma 3.4. With the same hypothesis as Lemma 3.1; if $\dim I = 2$, then I is not a $sl(2, \mathbf{C})$-submodule.

Proof. We shall assume that I is a $sl(2, \mathbf{C})$ module and shall produce a contradiction.

Case 1. $I = (2)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights -1 and 1. Write

$$f = \sum_{i=-\infty}^{\infty} f_{k+1}^{i}$$

where f_{k+1}^{i} is a homogeneous polynomial of degree $k+1$ and weight i. By the same argument as Case 5 in the proof of Lemma 3.1, we have $f = f_{k+1}^{0}$. Weights of $\frac{\partial f_{k+1}^{0}}{\partial x_1}$, $\frac{\partial f_{k+1}^{0}}{\partial x_2}$ and $\frac{\partial f_{k+1}^{0}}{\partial x_3}$ are even. Hence $\frac{\partial f_{k+1}^{0}}{\partial x_1}$, $\frac{\partial f_{k+1}^{0}}{\partial x_2}$ and $\frac{\partial f_{k+1}^{0}}{\partial x_3}$ are equal to zero and $f = f_{k+1}^{0}$ involves only x_4 and x_5 variables. If f_{k+1}^{0} is nonzero, then it is easy to see that I contains an irreducible submodule of dimension $k+1$ of the following form

$$\langle x_4^k, x_4^{k-1}x_5, \ldots, x_5^k \rangle.$$

This implies that $\dim I \geq k+1 \geq 3$, which contradicts to our hypothesis $\dim I = 2$. Therefore $f = 0$. We conclude that this case cannot occur.

Case 2. $I = (1) \oplus (1)$.

This case cannot occur. The proof is the same as those shown in Case 7 in the proof of Lemma 3.1. Q.E.D.

Lemma 3.5. With the same hypothesis as Lemma 3.1; if $\dim I = 1$, then I is not a $sl(2,\mathbf{C})$-submodule.

Proof. The argument which is used in Case 7 in the proof of Lemma 3.1 shows that $f = cx_2^{k+1}$ where c is a constant. Therefore $I = \langle x_2^k \rangle$ is not a $sl(2,\mathbf{C})$-submodule.Q.E.D.

Proposition 3.6. Suppose $sl(2,\mathbf{C})$ acts on the space of homogeneous polynomials of

degree $k \geq 2$ in x_1, x_2, x_3, x_4 and x_5 variables via

$$\tau = 2x_1\frac{\partial}{\partial x_1} - 2x_3\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_4} - x_5\frac{\partial}{\partial x_5}$$

$$X_+ = 2x_1\frac{\partial}{\partial x_2} + 2x_2\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_5}$$

$$X_- = x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + x_5\frac{\partial}{\partial x_4}.$$

Suppose the weight of x_i is given by the corresponding coefficient in the expression of τ as above, i.e.

$$wt(x_1) = 2 \quad wt(x_2) = 0 \quad wt(x_3) = -2 \quad wt(x_4) = 1 \quad wt(x_5) = -1.$$

Let $I = \langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}, \frac{\partial f}{\partial x_5} \rangle$ be the complex vector subspace spanned by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}$ and $\frac{\partial f}{\partial x_5}$, where f is a homogeneous polynomial of degree $k+1$. If I is a $sl(2, \mathbf{C})$-submodule, then either

(i) f is a homogeneous polynomial in x_1, x_2, x_3, x_4 and x_5 variables of weight 0 and I is $(3) \oplus (2)$ i.e. direct sum of 3-dimensional and 2-dimensional irreducible $sl(2, \mathbf{C})$-submodules, or

(ii) f is a homogeneous polynomial in x_1, x_2 and x_3 variables of weight 0 and I is an irreducible $sl(2, \mathbf{C})$-submodule of dimension 3, or

(iii) f is of the following form

$$f = g + c_1x_4^3 + c_2x_4^2x_5 + c_3x_4x_5^2 + c_4x_5^3$$

where $g = 2x_1x_5^2 - 2x_2x_4x_5 + x_3x_4^2$ is an $sl(2, \mathbf{C})$ invariant polynomial and

$$\begin{aligned} I &= \langle \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_3}, \frac{\partial g}{\partial x_4}, \frac{\partial g}{\partial x_5} \rangle = (3) \oplus (2) \\ &= \langle x_4^2, x_4x_5, x_5^2 \rangle \oplus \langle x_2x_4 - 2x_1x_5, x_3x_4 - x_2x_5 \rangle. \end{aligned}$$

Proof. This is an immediate consequence of Lemma 3.1 through Lemma 3.5.

Remark. g is projectively equivalent to f. In fact

$$g(x_1 + \frac{c_4}{2}x_5,\ x_2 - \frac{c_3}{2}x_5,\ x_3 + c_1x_4 + c_2x_5, x_4, x_5) = f(x_1, x_2, x_3, x_4, x_5).$$

§4. $sl(2,\mathbf{C})$ ACTION (1.4) ON M_5^k.

Lemma 4.1. Suppose $sl(2,\mathbf{C})$ acts on the space of homogeneous polynomials of degree $k \geq 2$ in x_1, x_2, x_3, x_4, and x_5 variables via (1.4)

$$\tau = 3x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} - x_3\frac{\partial}{\partial x_3} - 3x_4\frac{\partial}{\partial x_4}$$

$$X_+ = 3x_1\frac{\partial}{\partial x_2} + 4x_2\frac{\partial}{\partial x_3} + 3x_3\frac{\partial}{\partial x_4}$$

$$X_- = x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3}.$$

Suppose the weight of x_i is given by the corresponding coefficient in the expression of τ above i.e.

$$wt(x_1) = 3, \quad wt(x_2) = 1, \quad wt(x_3) = -1, \quad wt(x_4) = -3, \quad wt(x_5) = 0.$$

Let I be the complex vector subspace of dimension 5 spanned by $\frac{\partial f}{\partial x_1}$, $\frac{\partial f}{\partial x_2}$, $\frac{\partial f}{\partial x_3}$, $\frac{\partial f}{\partial x_4}$ and $\frac{\partial f}{\partial x_5}$ where f is a homogeneous polynomial of degree $k+1$. If I is a $sl(2,\mathbf{C})$-submodule, then $I = (4) \oplus (1)$ and f is a homogeneous polynomial of weight 0.

Proof. Case 1. $I = (5)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $-4, -2, 0, 2$ and 4. Write

$$f = \sum_{i=-\infty}^{\infty} f_{k+1}^i$$

where f_{k+1}^i is a homogeneous polynomial of degree $k+1$ and weight i.

For i is an even integer $wt\frac{\partial f_{k+1}^i}{\partial x_j}$ is an odd integer for all $1 \leq j \leq 4$. Hence $\frac{\partial f_{k+1}^i}{\partial x_j}$ is zero for all $1 \leq j \leq 4$. It follows that f_{k+1}^i depends only on x_5 variable. Since weight of x_5 is zero, we conclude that $f_{k+1}^i = 0$ for i non-zero even integer, and $f_{k+1}^0 = cx_5^{k+1}$. If

$c \neq 0$, then $\langle x_5^k \rangle$ would be a 1-dimensional invariant submodule in I which contradicts to our hypothesis $I = (5)$. Hence $f_{k+1}^0 = 0$ and we have

$$f = \sum_{i=-\infty}^{\infty} f_{k+1}^{2i+1}.$$

For i an odd integer, $wt\, \frac{\partial f_{k+1}^{2i+1}}{\partial x_5} = 2i+1$ is an odd integer 0. Hence $\frac{\partial f_{k+1}^{2i+1}}{\partial x_5} = 0$. It follows that $\frac{\partial f}{\partial x_5} = 0$. This contradicts our hypothesis $I = (5)$. We conclude that Case 1 cannot occur.

Case 2. $I = (4) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $-3, -1, 1, 3$ and 0.

For $|i| \geq 4$ and i odd, $wt\,(\frac{\partial f_{k+1}^i}{\partial x_j})$ is a non-zero even integer for all $1 \leq j \leq 5$. This implies that $\frac{\partial f_{k+1}^i}{\partial x_j} = 0$ for all $1 \leq j \leq 5$. Thus $f_{k+1}^i = 0$.

For $|i| \geq 7$

$$\begin{aligned} & |wt\,(\frac{\partial f_{k+1}^i}{\partial x_j})| \geq 4 && 1 \leq j \leq 5 \\ \Rightarrow\ & \frac{\partial f_{k+1}^i}{\partial x_j} = 0 && 1 \leq j \leq 5 \\ \Rightarrow\ & f_{k+1}^i = 0. \end{aligned}$$

For $i = 6$

$$wt\, \frac{\partial f_{k+1}^6}{\partial x_2} = 5, \quad wt\, \frac{\partial f_{k+1}^6}{\partial x_3} = 7, \quad wt\, \frac{\partial f_{k+1}^6}{\partial x_4} = 9, \quad wt\, \frac{\partial f_{k+1}^6}{\partial x_5} = 6,$$

$$\Rightarrow \quad \frac{\partial f_{k+1}^6}{\partial x_2} = 0 = \frac{\partial f_{k+1}^6}{\partial x_3} = \frac{\partial f_{k+1}^6}{\partial x_4} = \frac{\partial f_{k+1}^6}{\partial x_5}$$

$\Rightarrow \quad f_{k+1}^6$ depends only on x_1 variable

$\Rightarrow \quad f_{k+1}^6 = c x_1^2$

$$\Rightarrow \quad f^6_{k+1} = 0 \text{ because } k \geq 2.$$

Similarly we can prove that $f^{-6}_{k+1} = 0$.

For $i = 4$

$$wt\,\frac{\partial f^4_{k+1}}{\partial x_3} = 5, \quad wt\,\frac{\partial f^4_{k+1}}{\partial x_4} = 7, \quad wt\,\frac{\partial f^4_{k+1}}{\partial x_5} = 4.$$

$$\Rightarrow \quad \frac{\partial f^4_{k+1}}{\partial x_3} = 0 = \frac{\partial f^4_{k+1}}{\partial x_4} = \frac{\partial f^4_{k+1}}{\partial x_5}$$

$$\Rightarrow \quad f^4_{k+1} \text{ depends only on } x_1 \text{ and } x_2 \text{ variables}$$

$$\Rightarrow \quad f^4_{k+1} = cx_1x_2 + dx_2^4 \Rightarrow f^4_{k+1} = dx_2^4 \text{ because } k \geq 2..$$

If $d \neq 0$, then $x_2^3 = \frac{1}{4d}\,\frac{\partial f^4_{k+1}}{\partial x_2} \in I$. It follows that

$$\langle x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_2^2x_3, 2x_2x_3^2 + x_2^2x_4\rangle$$

is a 6-dimensional subspace in I. This contradicts to our hypothesis $\dim I = 5$. Hence $d = 0$ and $f^4_{k+1} = 0$.

Similarly we can prove that $f^{-4}_{k+1} = 0$. So $f = f^{-3}_{k+1} + f^{-2}_{k+1} + f^{-1}_{k+1} + f^0_{k+1} + f^1_{k+1} + f^2_{k+1} + f^3_{k+1}$.

For $i = 3$

$$wt\,\frac{\partial f^3_{k+1}}{\partial x_2} = 2, \quad wt\,\frac{\partial f^3_{k+1}}{\partial x_3} = 4, \quad wt\,\frac{\partial f^3_{k+1}}{\partial x_4} = 6.$$

$$\Rightarrow \quad \frac{\partial f^3_{k+1}}{\partial x_2} = 0 = \frac{\partial f^3_{k+1}}{\partial x_3} = \frac{\partial f^3_{k+1}}{\partial x_4}$$

$$\Rightarrow \quad f^3_{k+1} \text{ depends only on } x_1 \text{ and } x_5 \text{ variables}$$

$$\Rightarrow \quad f^3_{k+1} = cx_1x_5^k$$

Similarly we can prove $f^{-3}_{k+1} = \tilde{c}x_4x_5^k$. If $c \neq 0$ or $\tilde{c} \neq 0$, then it is easy to see that

$$I \;= (4) \oplus (1)$$

$$= \langle x_1x_5^{k-1},\ x_2x_5^{k-1},\ x_3x_5^{k-1},\ x_4x_5^{k-1}\rangle \oplus \langle x_5^k\rangle$$

$$wt\,\frac{\partial f_{k+1}^2}{\partial x_1} = -1,\ wt\,\frac{\partial f_{k+1}^2}{\partial x_2} = 1,\ wt\,\frac{\partial f_{k+1}^2}{\partial x_3} = 3,\ wt\,\frac{\partial f_{k+1}^2}{\partial x_4} = 5,\ wt\,\frac{\partial f_{k+1}^2}{\partial x_5} = 2$$

$\Rightarrow f_{k+1}^2$ depends only on x_1, x_2 and x_3 variables, and there are constants c_1, c_2 and c_3 such that

$$\begin{aligned}
\frac{\partial f_{k+1}^2}{\partial x_1} &= c_1x_3x_5^{k-1},\ \frac{\partial f_{k+1}^2}{\partial x_2} = c_2x_2x_5^{k-1},\ \frac{\partial f_{k+1}^2}{\partial x_3} = c_3x_1x_5^{k-1}\\
\Rightarrow \frac{\partial f_{k+1}^2}{\partial x_1} &= 0\frac{\partial f_{k+1}^2}{\partial x_2} = \frac{\partial f_{k+1}^2}{\partial x_3} \quad \text{since } k \geq 2\\
\Rightarrow f_{k+1}^2 &= 0.
\end{aligned}$$

Similarly we can prove $f_{k+1}^{-2} = 0$.

$$wt\,\frac{\partial f_{k+1}^1}{\partial x_1} = -2,\ wt\,\frac{\partial f_{k+1}^1}{\partial x_2} = 0,\ wt\,\frac{\partial f_{k+1}^1}{\partial x_3} = 2,\ wt\,\frac{\partial f_{k+1}^1}{\partial x_4} = 4,\ wt\,\frac{\partial f_{k+1}^1}{\partial x_5} = 1$$

$\Rightarrow f_{k+1}^1$ depends only on x_2 and x_5 variables and there are constants c_4 and c_5 such that

$$\frac{\partial f_{k+1}^1}{\partial x_2} = c_4x_5^k,\quad \frac{\partial f_{k+1}^1}{\partial x_5} = c_5x_2x_5^{k-1} \Rightarrow f_{k+1}^1 = c_6x_2x_5^k.$$

Similarly we can prove that $f_{k+1}^{-1} = c_7x_3x_5^k$

$$wt\,\frac{\partial f_{k+1}^0}{\partial x_1} = -3,\ wt\,\frac{\partial f_{k+1}^0}{\partial x_2} = -1,\ wt\,\frac{\partial f_{k+1}^0}{\partial x_3} = 1,\ wt\,\frac{\partial f_{k+1}^0}{\partial x_4} = 3,\ wt\,\frac{\partial f_{k+1}^0}{\partial x_5} = 0.$$

$$\begin{aligned}
\Rightarrow\quad & \frac{\partial f_{k+1}^0}{\partial x_1} = c_8x_4x_5^{k-1},\ \frac{\partial f_{k+1}^0}{\partial x_2} = c_9x_3x_5^{k-1},\ \frac{\partial f_{k+1}^0}{\partial x_3} = c_{10}x_2x_5^{k-1}\\
& \frac{\partial f_{k+1}^0}{\partial x_4} = c_{11}x_1x_5^{k-1},\ \frac{\partial f_{k+1}^0}{\partial x_5} = c_{12}x_5^k\\
\Rightarrow\quad & c_8 = c_9 = c_{10} = c_{11} = 0 \text{ by considering } \frac{\partial^2 f_{k+1}^0}{\partial x_i\partial x_5} = \frac{\partial^2 f_{k+1}^0}{\partial x_5\partial x_i}\\
\Rightarrow\quad & f_{k+1}^0 = \frac{c_{12}}{k+1}\quad x_5^{k+1} = \tilde{c}_{12}x_5^{k+1}.
\end{aligned}$$

Therefore we can write

$$f = cx_1x_5^k + c_6x_2x_5^k + \tilde{c}_{12}x_5^{k+1} + c_7x_3x_5^k + \tilde{c}x_4x_5^k.$$

This implies that $\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4} \rangle$ is at most one dimensional, which contradicts to our hypothesis $I = (4) \oplus (1)$. Thus we can conclude that $f^3_{k+1} = f^{-3}_{k+1} = 0$ and we have

$$f = f^{-2}_{k+1} + f^{-1}_{k+1} + f^0_{k+1} + f^1_{k+1} + f^2_{k+1}$$

The same argument as above shows that $f^{-1}_{k+1} = f^1_{k+1} = 0$. Therefore we have

$$f = f^{-2}_{k+1} + f^0_{k+1} + f^2_{k+1}$$

$$wt\, \frac{\partial f^2_{k+1}}{\partial x_4} = 5, \quad wt\, \frac{\partial f^2_{k+1}}{\partial x_5} = 2, \; wt\, \frac{\partial f^{-2}_{k+1}}{\partial x_1} = -5, \; wt\, \frac{\partial f^{-2}_{k+1}}{\partial x_5} = -2$$

$$\Rightarrow \quad \frac{\partial f^2_{k+1}}{\partial x_4} = 0 = \frac{\partial f^2_{k+1}}{\partial x_5} = \frac{\partial f^{-2}_{k+1}}{\partial x_1} = \frac{\partial f^{-2}_{k+1}}{\partial x_5}$$

$\Rightarrow$ f^2_{k+1} depends on x_1, x_2 and x_3 variables while

f^{-2}_{k+1} depends on x_2, x_3, and x_4 variables.

We claim that either $\frac{\partial f^2_{k+1}}{\partial x_1} = 0$ or $\frac{\partial f^2_{k+1}}{\partial x_2} = 0$ or $\frac{\partial f^2_{k+1}}{\partial x_3} = 0$ implies $f^2_{k+1} = 0$.

Suppose $\frac{\partial f^2_{k+1}}{\partial x_1} = 0$. Then f^2_{k+1} depends only on x_2 and x_3 variables. This implies that $k+1$ is even and there exists a constant d_1 such that

$$f^2_{k+1}(x_2, x_3) = d_1 x_2^{\frac{k+3}{2}} x_3^{\frac{k-1}{2}}.$$

If $d_1 \neq 0$, then $x_2^{\frac{k+1}{2}} x_3^{\frac{k-1}{2}} = \frac{2}{d_1(k+3)} \frac{\partial f^2_{k+1}}{\partial x_2}$ is in I. By considering $X_-^{k+1}(x_2^{\frac{k+1}{2}} x_3^{\frac{k-1}{2}})$, we see that x_4^k is in I. Since $X_-(x_4^k) = 0$, it follows that

$$\langle x_4^k, X_+(x_4^k), X_+^2(x_4^k), \ldots, x_+^{3k}(x_4^k)\rangle$$

is an irreducible submodule of dimension $3k+1 \geq 7$ in I. This contradicts to our hypothesis $I = (4) \oplus (1)$. Therefore we vhave $f^2_{k+1} = 0$.

Suppose $\frac{\partial f^2_{k+1}}{\partial x_2} = 0$. Then f^2_{k+1} depends only on x_1 and x_3 variables. This implies that $k+3$ is divisible by 4 and there exists a constant d_2 such that

$$f^2_{k+1}(x_1, x_3) = d_2 x_1^{\frac{k+3}{4}} x_3^{\frac{3k+1}{4}}.$$

If $d_2 \neq 0$, then by the similar argument as before, it is easy to see that

$$\langle x_4^k, X_+(x_4^k), X_+^2(x_4^k), \ldots, X_+^{3k}(x_4^k)\rangle$$

is an irreducible submodule of dimension $3k+1 \geq 7$ in I. This contradicts to our hypothesis $I = (4) \oplus (1)$. Therefore we have $f_{k+1}^2 = 0$.

It is easy to see that $\frac{\partial f_{k+1}^2}{\partial x_3} = 0$ implies $f_{k+1}^2 = 0$.

Similarly we can prove that if either $\frac{\partial f_{k+1}^{-2}}{\partial x_3}$ or $\frac{\partial f_{k+1}^{-2}}{\partial x_4}$ or $\frac{\partial f_{k+1}^{-2}}{\partial x_2}$ is zero, then f_{k+1}^{-2} is zero.

We claim that f_{k+1}^2 and f_{k+1}^{-2} cannot both be nonzero. Suppose on contrary that both f_{k+1}^2 and f_{k+1}^{-2} are nonzero. Since $wt\, \frac{\partial f_{k+1}^2}{\partial x_1} = wt\, \frac{\partial f_{k+1}^{-2}}{\partial x_3} = -1$ and $wt\, \frac{\partial f_{k+1}^2}{\partial x_2} = wt\, \frac{\partial f_{k+1}^{-2}}{\partial x_4} = 1$, there are nonzero constants d_1 and d_2 such that

$$\frac{\partial f_{k+1}^{-2}}{\partial x_3}(x_2,x_3,x_4) = d_1 \frac{\partial f_{k+1}^2}{\partial x_1}(x_1,x_2,x_3),\ \frac{\partial f_{k+1}^{-2}}{\partial x_4}(x_2,x_3,x_4) = d_2 \frac{\partial f_{k+1}^2}{\partial x_2}(x_1,x_2,x_3)$$

$$\Rightarrow\quad f_{k+1}^{-2}(x_2,x_3,x_4) = d_2 x_4 \frac{\partial f_{k+1}^2}{\partial x_2}(x_1,x_2,x_3) + g(x_2,x_3)$$

$$\Rightarrow\quad \frac{\partial f_{k+1}^{-2}}{\partial x_3}(x_2,x_3,x_4) = d_2 x_4 \frac{\partial^2 f_{k+1}^2}{\partial x_3 \partial x_2}(x_1,x_2,x_3) + \frac{\partial g}{\partial x_3}(x_2,x_3)$$

$$\Rightarrow\quad x_4 \frac{\partial^2 f_{k+1}^2}{\partial x_3 \partial x_2}(x_1,x_2,x_3) = \frac{1}{d_2}[d_1 \frac{\partial f_{k+1}^2}{\partial x_1}(x_1,x_2,x_3) - \frac{\partial g}{\partial x_3}(x_2,x_3)]$$

$$\Rightarrow\quad \frac{\partial f_{k+1}^2}{\partial x_2 \partial x_3}(x_1,x_2,x_3) = 0$$

$\Rightarrow\quad f_{k+1}^2(x_1,x_2,x_3)$ does not involve x_2x_3

$\Rightarrow\quad f_{k+1}^2(x_1,x_2,x_3) = d_3 x_2^2 + h(x_1,x_3)$ where d_3 is a constant and $h(x_1,x_3)$ is a

homogeneous polynomial of degree $k+1$ and weight 2

$\Rightarrow\quad f_{k+1}^2(x_1,x_2,x_3) = h(x_1,x_3)$ because $k \geq 2$

$$\Rightarrow \quad \frac{\partial f^2_{k+1}}{\partial x_2} = 0.$$

$$\Rightarrow \quad f^2_{k+1} = 0 \text{ as shown above.}$$

This gives rise to a contradiction. Hence our claim is proved.

We next claim that the case $f^2_{k+1}(x_1, x_2, x_3) \neq 0$ and $f^{-2}_{k+1}(x_2, x_3, x_4) = 0$ cannot occur. Suppose on the contrary that $f^2_{k+1}(x_1, x_2, x_3) \neq 0$ and $f^{-2}_{k+1}(x_2, x_3, x_4) = 0$. Then we have

$$f = f^2_{k+1}(x_1, x_2, x_3) + f^0_{k+1}(x_1, x_2, x_3, x_4, x_5)$$

and $\frac{\partial f^2_{k+1}}{\partial x_1} \neq 0$, $\frac{\partial f^2_{k+1}}{\partial x_2} \neq 0$, $\frac{\partial f^2_{k+1}}{\partial x_3} \neq 0$.

$$wt\, \frac{\partial f^0_{k+1}}{\partial x_4} = -3 = wt\, \frac{\partial f^2_{k+1}}{\partial x_3}, \quad wt\, \frac{\partial f^0_{k+1}}{\partial x_3} = 1 = wt\, \frac{\partial f^2_{k+1}}{\partial x_1}$$

$$\Rightarrow \quad \frac{\partial f^0_{k+1}}{\partial x_4} = d_4 \frac{\partial f^2_{k+1}}{\partial x_3}(x_1, x_2, x_3), \quad \frac{\partial f^0_{k+1}}{\partial x_3} = d_5\, \frac{\partial f^2_{k+1}}{\partial x_1}(x_1, x_2, x_3)$$

$$\Rightarrow \quad f^0_{k+1} = d_4 x_4 \frac{\partial f^2_{k+1}}{\partial x_3}(x_1, x_2, x_3) + h(x_1, x_2, x_3, x_5)$$

$$\Rightarrow \quad \frac{\partial f^0_{k+1}}{\partial x_3} = d_4 x_4 \frac{\partial f^2_{k+1}}{\partial x_3^2}(x_1, x_2, x_3) + \frac{\partial h}{\partial x_3}(x_1, x_2, x_3, x_5)$$

$$\Rightarrow \quad d_4 x_4 \frac{\partial^2 f^2_{k+1}}{\partial x_3^2}(x_1, x_2, x_3) = d_5 \frac{\partial f^2_{k+1}}{\partial x_1}(x_1, x_2, x_3) - \frac{\partial h}{\partial x_3}(x_1, x_2, x_3, x_5)$$

$$\Rightarrow \quad d_4 x_4 \frac{\partial^2 f^2_{k+1}}{\partial x_3^2}(x_1, x_2, x_3) = 0.$$

If $d_4 = 0$, then $\frac{\partial f^0_{k+1}}{\partial x_4} = 0$ and hence $\frac{\partial f}{\partial x_4} = \frac{\partial f^0_{k+1}}{\partial x_4} = 0$. This implies $\dim I \leq 4$, which contradicts to our hypothesis $I = (4) \oplus (1)$. If $d_4 \neq 0$, then $\frac{\partial^2 f^2_{k+1}}{\partial x_3^2}(x_1, x_2, .x_3) = 0$. Hence $f^2_{k+1}(x_1, x_2, x_3)$ does not involve x_3^2 and

$$f^2_{k+1} = d_6 x_1 x_3 + d_7 x_2^3 x_3 \Rightarrow f^2_{k+1} = d_7 x_2^3 x_3$$

because $k \geq 2$. If $d_7 \neq 0$, then $x_2^3 = \frac{1}{d_7}\frac{\partial f_{k+1}^2}{\partial x_3}$ is a nonzero element in I. By similar argument as before, we can see easily that

$$\langle x_1^3, X_-(x_1^3), X_-^2(x_1^3), \ldots, X^9(x_1^3)\rangle$$

is an irreducible submodule of dimension 10 in I. This contradicts to our hypothesis $I = (4) \oplus (1)$.

Similarly we can prove that the case $f_{k+1}^{-2}(x_2, x_3, x_4) \neq 0$ and $f_{k+1}^2(x_1, x_2, x_3) = 0$ cannot occur.

We conclude therefore $f_{k+1}^2 = f_{k+1}^{-2} = 0$ and $f = f_{k+1}^0$.

Case 3. $I = (3) \oplus (2)$

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $-2, -1, 0, 1$ and 2.

For $|i| \geq 6$

$$\begin{aligned}
&|wt\,\frac{\partial f_{k+1}^i}{\partial x_j}| \geq 3 \qquad \text{for } 1 \leq j \leq 5\\
\Rightarrow\quad &\frac{\partial f_{k+1}^i}{\partial x_j} = 0 \qquad \text{for } 1 \leq j \leq 5\\
\Rightarrow\quad &f_{k+1}^i = 0.
\end{aligned}$$

For $i = 5$ or 4

$$\begin{aligned}
&wt\,\frac{\partial f_{k+1}^i}{\partial x_j} \geq 3, \quad \text{for } 2 \leq j \leq 5\\
\Rightarrow\quad &\frac{\partial f_{k+1}^i}{\partial x_2} = 0 = \frac{\partial f_{k+1}^i}{\partial x_3} = \frac{\partial f_{k+1}^i}{\partial x_4} = \frac{\partial f_{k+1}^i}{\partial x_5} = 0\\
\Rightarrow\quad &f_{k+1}^i \text{ depends only on } x_1 \text{ variable}\\
\Rightarrow\quad &f_{k+1}^i = 0 \text{ because both 5 and 4 are not divisible by 3.}
\end{aligned}$$

Similarly we can prove $f_{k+1}^{-5} = 0 = f_{k+1}^{-4}$.

For $i = 3$

$$wt\ \frac{\partial f^3_{k+1}}{\partial x_3} = 4, \quad wt\ \frac{\partial f^3_{k+1}}{\partial x_4} = 6, \quad wt\ \frac{\partial f^3_{k+1}}{\partial x_5} = 3$$

$$\Rightarrow \quad \frac{\partial f^3_{k+1}}{\partial x_3} = 0 = \frac{\partial f^3_{k+1}}{\partial x_4} = \frac{\partial f^3_{k+1}}{\partial x_5}$$

$\Rightarrow$ f^3_{k+1} depends only on x_1 and x_2 variables

$$\Rightarrow \quad f^3_{k+1} = c_1 x_1 + c_2 x_2^3$$

$\Rightarrow$ $f^3_{k+1} = c_2 x_2^3$ because $k \geq 2$

If $c_2 \neq 0$, then $x_2^2 = \frac{1}{3c_2} \frac{\partial f^3_{k+1}}{\partial x_2}$ is in I. It follows that

$$\langle X_+^2(x_2^2),\ X_+(x_2^2), x_2^2,\ X_-(x_2^2), X_-^2(x_2^2), X_-^3(x_2^2)\rangle$$

$$= \langle x_1^2, x_1x_2, x_2^2, x_2x_3, x_3^2 + x_2x_4, x_3x_4\rangle$$

is a subspace of dimension 6 in I. This contradicts to our hypothesis that $I = (3) \oplus (2)$. Hence we have $f^3_{k+1} = 0$.

Similarly we can prove that $f^{-3}_{k+1} = 0$.

For $i = 2$

$$wt\ \frac{\partial f^2_{k+1}}{\partial x_3} = 3, \quad wt\ \frac{\partial f^2_{k+1}}{\partial x_4} = 5$$

$$\Rightarrow \quad \frac{\partial f^2_{k+1}}{\partial x_3} = 0 = \frac{\partial f^2_{k+1}}{\partial x_4}$$

$\Rightarrow$ f^2_{k+1} involves only x_1, x_2 and x_5 variables

$$\Rightarrow \quad f^2_{k+1} = c_3 x_2^2 x_5^{k-1}$$

If $c_3 \neq 0$, then $x_2^2 x_5^{k-2} = \frac{1}{(k-1)c_3} \frac{\partial f^2_{k+1}}{\partial x_5}$ is in I. It follows that

$$\langle X_+^2(x_2^2x_5^{k-2},\ X_+(x_2^2x_5^{k-2}),\ x_2^2x_5^{k-2},\ X_-(x_2^2x_5^{k-2}),\ X_-^2(x_2^2x_5^{k-2}),\ X_-^3(x_2^2x_5^{k-2})\rangle$$

$$= \langle x_1^2x_5^{k-2}, x_1x_2x_5^{k-2}, x_2^2x_5^{k-2}, x_2x_3x_5^{k-2}, (x_3^2 + x_2x_4)x_5^{k-2}, x_3x_4x_5^{k-2}\rangle$$

is a subspace of dimension 6 in I. This contradicts to our hypothesis that $I = (3) \oplus (2)$. Hence we have $f^2_{k+1} = 0$.

Similarly we can prove that $f^{-2}_{k+1} = 0$.

We have proved

$$
\begin{aligned}
& f = f^{-1}_{k+1} + f^0_{k+1} + f^1_{k+1} \\
& wt\,\frac{\partial f^{-1}_{k+1}}{\partial x_1} = -4, \quad wt\,\frac{\partial f^0_{k+1}}{\partial x_1} = -3, \quad wt\,\frac{\partial f^0_{k+1}}{\partial x_4} = 3, \quad wt\,\frac{\partial f^1_{k+1}}{\partial x_4} = 4, \\
\Rightarrow\quad & \frac{\partial f^{-1}_{k+1}}{\partial x_1} = 0 = \frac{\partial f^0_{k+1}}{\partial x_1} = \frac{\partial f^0_{k+1}}{\partial x_4} = \frac{\partial f^1_{k+1}}{\partial x_4} \\
\Rightarrow\quad & f = f^{-1}_{k+1}(x_2,x_3,x_4,x_5) + f^0_{k+1}(x_2,x_3,x_5) + f^1_{k+1}(x_1,x_2,x_3,x_5).
\end{aligned}
$$

Suppose $\frac{\partial f^1_{k+1}}{\partial x_1} = 0$. We are going to prove $f^1_{k+1} = 0$. Clearly f^1_{k+1} depends only on x_2, x_3 and x_5 variables in this case,

$$f^1_{k+1} = \sum_{\alpha=0}^{[\frac{k}{2}]} a_\alpha x_2^{\alpha+1} x_3^\alpha x_5^{k-2\alpha}.$$

Suppose on the contrary that $f^1_{k+1} \neq 0$. Let α_0 be the largest integer such that $a_{\alpha_0} = 0$.

$$\frac{\partial f^1_{k+1}}{\partial x_2} = \sum_{\alpha=0}^{\alpha_0} (\alpha+1) a_\alpha x_2^\alpha x_3^\alpha c_5^{k-2\alpha}$$

is an element in I. By applying $X_+^{3\alpha_0}$ to $\frac{\partial f^1_{k+1}}{\partial x_2}$, we find that $x_1^{2\alpha_0} x_5^{k-2\alpha_0}$ is in I. Since $X_+(x_1^{2\alpha_0} x_5^{k-2\alpha_0}) = 0$, we have an irreducible submodule of dimension $6\alpha_0 + 1$ in I in the following form

$$\langle x_1^{2\alpha_0} x_5^{k-2\alpha_0}, X_+(x_1^{2\alpha_0} x_5^{k-2\alpha_0}), X_+^2(x_1^{2\alpha_0} x_5^{k-2\alpha_0}), \ldots, X_+^{6\alpha_0}(x_1^{2\alpha_0} x_5^{k-2\alpha_0})\rangle.$$

As $I = (3) \oplus (2)$, we have $6\alpha_0 + 1 \leq 3$. This implies $\alpha_0 = 0$. Hence

$$f^1_{k+1} = a_0 x_2 x_5^k.$$

It follows easily that $\langle x_5^k \rangle$ is a one dimensional irreducible submodule of I. This contradicts to our hypothesis $I = (3) \oplus (2)$. Therefore we conclude that $f_{k+1}^1 = 0$.

Suppose $\frac{\partial f_{k+1}^1}{\partial x_2} = 0$. We are going to prove $f_{k+1}^1 = 0$. Clearly f_{k+1}^1 depends only on x_1, x_3 and x_5 variables in this case

$$f_{k+1}^1 = \sum_{\beta=0}^{[\frac{k}{4}]} b_\beta x_1^{\beta+1} x_3^{3\beta} x_5^{k-4\beta}$$

Suppose on the contrary that $f_{k+1}^1 = 0$. Let β_0 be the largest integer such that $b_{\beta_0} \neq 0$.

$$\frac{\partial f_{k+1}^1}{\partial x_1} = \sum_{\beta=0}^{\beta_0} (\beta+1) b_\beta x_1^\beta x_3^{3\beta} x_5^{k-4\beta}$$

is an element in I. By applying $X_+^{6\beta_0}$ to $\frac{\partial f_{k+1}^1}{\partial x_1}$, we find that $x_1^{4\beta_0} x_5^{k-4\beta_0}$ is in I. Since $X_+(x_1^{4\beta_0} x_5^{k-4\beta_0}) = 0$, we have an irreducible submodule of dimension $12\beta_0 + 1$ in I in the following form

$$\langle x_1^{4\beta_0} x_5^{k-4\beta_0}, X_+(x_1^{4\beta_0} x_5^{k-4\beta_0}), \ldots, X_+^{12\beta_0}(x_1^{4\beta_0} x_5^{k-4\beta_0}) \rangle.$$

As in $I = (3) \oplus (2)$, we have $12\beta_0 + 1 \leq 3$ and hence $\beta_0 = 0$. This means

$$f_{k+1}^1 = b_0 x_1 x_5^k.$$

It follows easily $\langle x_5^k \rangle$ is a one dimensional irreducible submodule of I. This contradicts to our hypothesis $I = (3) \oplus (2)$. Therefore we conclude that $f_{k+1}^1 = 0$.

Suppose $\frac{\partial f_{k+1}^1}{\partial x_3} = 0$. We are going to prove $f_{k+1}^1 = 0$. Clearly f_{k+1}^1 depends only on x_1, x_2 and x_5 variables. So

$$f_{k+1}^1 = c_4 x_2 x_5^k.$$

If $f_{k+1}^1 \neq 0$. then $x_5^k = \frac{1}{c_4} \frac{\partial f_{k+1}^1}{\partial x_2}$ is in I. Hence $\langle x_5^k \rangle$ is a 1-dimensional irreducible submodule in I. This contradicts to our hypothesis $I = (3) \oplus (2)$. Thus $f_{k+1}^1 = 0$.

Similarly we can prove that either $\frac{\partial f_{k+1}^{-1}}{\partial x_2} = 0$ or $\frac{\partial f_{k+1}^{-1}}{\partial x_3} = 0$ or $\frac{\partial f_{k+1}^{-1}}{\partial x_4} = 0$ implies $f_{k+1}^{-1} = 0$.

We claim that f_{k+1}^{1} and f_{k+1}^{-1} cannot both be nonzero. Suppose on the contrary that $f_{k+1}^{1} \neq 0$ and $f_{k+1}^{-1} \neq 0$.

$$wt\,\frac{\partial f_{k+1}^{-1}}{\partial x_2}(x_2,x_3,x_4,x_5) = -2 = wt\,\frac{\partial f_{k+1}^{1}}{\partial x_1}(x_1,x_2,x_3,x_5)$$

$$wt\,\frac{\partial f_{k+1}^{-1}}{\partial x_4}(x_2,x_3,x_4,x_5) = 2 = wt\,\frac{\partial f_{k+1}^{1}}{\partial x_3}(x_1,x_2,x_3,x_5)$$

$$wt\,\frac{\partial f_{k+1}^{-1}}{\partial x_3}(x_2,x_3,x_4,x_5) = 0 = wt\frac{\partial f_{k+1}^{1}}{\partial x_2}(x_1,x_2,x_3,x_5)$$

$$\Rightarrow \quad \frac{\partial f_{k+1}^{-1}}{\partial x_2}(x_2,x_3,x_4,x_5) = d_1\frac{\partial f_{k+1}^{1}}{\partial x_1}(x_1,x_2,x_3,x_5)$$

$$\frac{\partial f_{k+1}^{-1}}{\partial x_4}(x_2,x_3,x_4,x_5) = d_2\frac{\partial f_{k+1}^{1}}{\partial x_3}(x_1,x_2,x_3,x_5)$$

$$\frac{\partial f_{k+1}^{-1}}{\partial x_3}(x_2,x_3,x_4,x_5) = d_3\frac{\partial f_{k+1}^{1}}{\partial x_1}(x_1,x_2,x_3,x_5)$$

where d_1, d_2 and d_3 are nonzero constants

$$\Rightarrow \quad f_{k+1}^{-1}(x_2,x_3,x_4,x_5) = d_2x_4\frac{\partial f_{k+1}^{1}}{\partial x_3}(x_1,x_2,x_3,x_5) + g(x_2,x_3,x_5)$$

$$\Rightarrow \quad \frac{\partial f_{k+1}^{-1}}{\partial x_2}(x_2,x_3,x_4,x_5) = d_2x_4\frac{\partial^2 f_{k+1}^{1}}{\partial x_2\partial x_3}(x_1,x_2,x_3,x_5) + \frac{\partial g}{\partial x_2}(x_2,x_3,x_5)$$

$$\frac{\partial f_{k+1}^{-1}}{\partial x_3}(x_2,x_3,x_4,x_5) = d_2x_4\frac{\partial^2 f_{k+1}^{1}}{\partial x_3^2}(x_1,x_2,x_3,x_5) + \frac{\partial g}{\partial x_3}(x_2,x_3,x_5)$$

$$\Rightarrow \quad d_2x_4\frac{\partial^2 f_{k+1}^{1}}{\partial x_2\partial x_3}(x_1,x_2,x_3,x_5) = d_1\frac{\partial f_{k+1}^{1}}{\partial x_1}(x_1,x_2,x_3,x_5) - \frac{\partial g}{\partial x_2}(x_2,x_3,x_5)$$

$$d_2x_4\frac{\partial^2 f_{k+1}^{1}}{\partial x_3^2}(x_1,x_2,x_3,x_5) = d_3\frac{\partial f_{k+1}^{1}}{\partial x_2}(x_1,x_2,x_3,x_5) - \frac{\partial g}{\partial x_3}(x_2,x_3,x_5)$$

$$\Rightarrow \quad \frac{\partial^2 f_{k+1}^{1}}{\partial x_2\partial x_3}(x_1,x_2,x_3,x_5) = \frac{\partial^2 f_{k+1}^{1}}{\partial x_3^2}(x_1,x_2,x_3,x_5) = 0$$

$\Rightarrow \quad f_{k+1}^{1}$ does not involve x_2x_3 and x_3^2

$\Rightarrow \quad f_{k+1}^{1} = d_4x_2x_5^k$ where d_4 is a nonzero constant.

$\Rightarrow \quad \langle x_5^k \rangle$ is a 1-dimensional submodule in I.

This contradicts to our hypothesis that $I = (3) \oplus (2)$.

We next claim that $f_{k+1}^1(x_1, x_2, x_3, x_5) \neq 0$. Suppose that $f_{k+1}^1(x_1, x_2, x_3, x_5)$ were zero. Then

$$f = f_{k+1}^0(x_2, x_3, x_5) + f_{k+1}^{-1}(x_2, x_3, x_4, x_5)$$

would imply $\frac{\partial f}{\partial x_1} = 0$ and hence $\dim I \leq 4$. This contradicts to our hypothesis $I = (3) \oplus (2)$.

Similarly we can prove $f_{k+1}^{-1}(x_2, x_3, x_4, x_5) \neq 0$.

Therefore we conclude that Case 3 cannot occur.

Case 4. $I = (3) \oplus (1) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $-2, 0$ and 2.

By the similar argument as Case 3, we have

$$f = f_{k+1}^{-1} = f_{k+1}^0 + f_{k+1}^1$$

$$wt\,\frac{\partial f_{k+1}^{-1}}{\partial x_1} = -4, \quad wt\,\frac{\partial f_{k+1}^{-1}}{\partial x_5} = -1, \quad wt\,\frac{\partial f_{k+1}^0}{\partial x_1} = -3, \quad wt\,\frac{\partial f_{k+1}^0}{\partial x_2} = -1,$$

$$wt\,\frac{\partial f_{k+1}^0}{\partial x_3} = 1, wt\,\frac{\partial f_{k+1}^0}{\partial x_4} = 3, \quad wt\,\frac{\partial f_{k+1}^1}{\partial x_4} = 4, \quad wt\,\frac{\partial f_{k+1}^1}{\partial x_5} = 1$$

$$\Rightarrow \quad \frac{\partial f_{k+1}^{-1}}{\partial x_1} = 0 = \frac{\partial f_{k+1}^{-1}}{\partial x_5} = \frac{\partial f_{k+1}^0}{\partial x_1} = \frac{\partial f_{k+1}^0}{\partial x_2} = \frac{\partial f_{k+1}^0}{\partial x_3} = \frac{\partial f_{k+1}^0}{\partial x_4}$$

$$= \frac{\partial f_{k+1}^1}{\partial x_4} = \frac{\partial f_{k+1}^1}{\partial x_5}$$

$$\Rightarrow \quad f = f_{k+1}^{-1}(x_2, x_3, x_4) + c_0 x_5^{k+1} + f_{k+1}^1(x_1, x_2, x_3).$$

If $f_{k+1}^{-1}(x_2, x_3, x_4)$ is zero, then $f = c_0 x_5^{k+1} + f_{k+1}^1(x_1, x_2, x_3)$. It follows that

$\frac{\partial f}{\partial x_4} = 0$ and hence $\dim I \leq 4$. This contradicts the hypothesis $I = (3)\oplus(1)\oplus(1)$. Thus $f_{k+1}^{-1}(x_2, x_3, x_4) \neq 0$. Similarly we can prove $f_{k+1}^{-1}(x_1, x_2, x_3) \neq 0$.

We next claim that either $\frac{\partial f_{k+1}^1}{\partial x_1} = 0$ or $\frac{\partial f_{k+1}^1}{\partial x_2} = 0$ or $\frac{\partial f_{k+1}^1}{\partial x_3} = 0$ implies $f_{k+1}^1 = 0$.

Suppose $\frac{\partial f_{k+1}^1}{\partial x_1} = 0$. Then f_{k+1}^1 depends only on x_2 and x_3 variables. There exists constant c_1 such that

$$f_{k+1}^1 = c_1 x_2^{\frac{k+2}{2}} x_3^{\frac{k}{2}}.$$

If $c_1 \neq 0$, then $x_2^{\frac{k}{2}} x_3^{\frac{k}{2}} = \frac{2}{(k+2)c_1} \frac{\partial f_{k+1}^1}{\partial x_2}$ is an element in I. By applying $X_-^{\frac{3k}{2}}$ on $x_2^{\frac{k}{2}} x_3^{\frac{k}{2}}$, we see that x_4^k is an element in I. Since $X_-(x_4^k) = 0$,

$$\langle x_4^k, X_+(x_4^k), X_+^2(x_4^k), \ldots, X_+^{3k}(x_4^k)\rangle$$

is an irreducible submodule of dimension $3k+1 \geq 7$ in I. This contradicts to our hypothesis $I = (3)\oplus(1)\oplus(1)$. Therefore $f_{k+1}^1 = 0$.

Suppose $\frac{\partial f_{k+1}^1}{\partial x_2} = 0$. Then f_{k+1}^1 depends only on x_1 and x_3 variables. There exists constant c_2 such that

$$f_{k+1}^1 = c_2 x_1^{\frac{k+2}{4}} x_3^{\frac{3k+2}{4}}.$$

If $c_2 \neq 0$, then it is easy to see that

$$\langle x_1^k, X_-(x_1^k), X_-^2(x_1^k), \ldots, X_-^{3k}(x_1^k)\rangle$$

is an irreducible submodule of dimension $3k+1 \geq 7$ in I. This contradicts to our hypothesis $I = (3)\oplus(1)\oplus(1)$. Therefore $f_{k+1}^1 = 0$.

Suppose $\frac{\partial f_{k+1}^1}{\partial x_3} = 0$. Then f_{k+1}^1 depends only on x_1 and x_2 variables. There exists a constant c_3 such that

$$f_{k+1}^1 = c_3 x_2.$$

Since $k \geq 2$, we have $f_{k+1}^1 = 0$.

Similarly we can prove that either $\frac{\partial f_{k+1}^{-1}}{\partial x_2} = 0$ or $\frac{\partial f_{k+1}^{-1}}{\partial x_3} = 0$ or $\frac{\partial f_{k+1}^{-1}}{\partial x_4} = 0$ implies $f_{k+1}^{-1} = 0$.

It follows that $\frac{\partial f_{k+1}^{1}}{\partial x_1}(x_1, x_2, x_3)$, $\frac{\partial f_{k+1}^{1}}{\partial x_3}(x_1, x_2, x_3)$, $\frac{\partial f_{k+1}^{-1}}{\partial x_2}(x_2, x_3, x_4)$ and $\frac{\partial f_{k+1}^{-1}}{\partial x_4}(x_2, x_3, x_4)$ are nonzero elements in I.

$$wt\, \frac{\partial f_{k+1}^{-1}}{\partial x_2}(x_2, x_3, x_4) = -2 = wt\, \frac{\partial f_{k+1}^{1}}{\partial x_1}(x_1, x_2, x_3)$$

$$wt\, \frac{\partial f_{k+1}^{-1}}{\partial x_4}(x_2, x_3, x_4) = 2 = wt\, \frac{\partial f_{k+1}^{1}}{\partial x_3}(x_1, x_2, x_3)$$

$$\Rightarrow \quad \frac{\partial f_{k+1}^{-1}}{\partial x_2}(x_2, x_3, x_4) = d_1\, \frac{\partial f_{k+1}^{1}}{\partial x_1}(x_1, x_2, x_3)$$

$$\frac{\partial f_{k+1}^{-1}}{\partial x_4}(x_2, x_3, x_4) = d_2\, \frac{\partial f_{k+1}^{1}}{\partial x_3}(x_1, x_2, x_3)$$

where d_1 and d_2 are nonzero constants

$$\Rightarrow \quad d_2 \frac{\partial^2 f_{k+1}^{1}}{\partial x_1 \partial x_3}(x_1, x_2, x_3) = \frac{\partial^2 f_{k+1}^{-1}}{\partial x_1 \partial x_4}(x_2, x_3, x_4) = 0$$

$$f_{k+1}^{-1}(x_2, x_3, x_4) = d_2 x_4 \frac{\partial f_{k+1}^{1}}{\partial x_3}(x_1, x_2, x_3) + g(x_2, x_3)$$

$$\Rightarrow \quad \frac{\partial^2 f_{k+1}^{1}}{\partial x_1 \partial x_3} = 0$$

$$\frac{\partial f_{k+1}^{-1}}{\partial x_2}(x_2, x_3, x_4) = d_2 x_4\, \frac{\partial^2 f_{k+1}^{1}}{\partial x_2 \partial x_3}(x_1, x_2, x_3) + \frac{\partial g}{\partial x_2}(x_2, x_3)$$

$\Rightarrow$ $f_{k+1}^{1}(x_1, x_2, x_3)$ does not involve $x_1 x_3$ and

$$d_2 x_4 \frac{\partial^2 f_{k+1}^{1}}{\partial x_2 \partial x_3}(x_1, x_2, x_3) = d_1 \frac{\partial f_{k+1}^{1}}{\partial x_1}(x_1, x_2, x_3) - \frac{\partial g}{\partial x_2}(x_2, x_3)$$

$\Rightarrow$ $\frac{\partial^2 f_{k+1}^{1}}{\partial x_2 \partial x_3} = 0$ and hence f_{k+1}^{1} does not involve $x_2 x_3$.

Since $f_{k+1}^{1}(x_1, x_2, x_3)$ does not involve $x_1 x_3$ and $x_2 x_3$, we have $f_{k+1}^{1}(x_1, x_2, x_3) = 0$. This contradicts to what we have proved. We conclude that Case 4 cannot occur.

Case 5. $I = (2) \oplus (2) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $-1, 0$, and 1.

By the same argument as Case 3, we have

$$
\begin{aligned}
& f = f_{k+1}^{-1} + f_{k+1}^{0} + f_{k+1}^{1} \\
& wt\,\frac{\partial f_{k+1}^{-1}}{\partial x_1} = -4,\ wt\,\frac{\partial f_{k+1}^{-1}}{\partial x_2} = -2,\ wt\,\frac{\partial f_{k+1}^{-1}}{\partial x_4} = 2,\ wt\,\frac{\partial f_{k+1}^{0}}{\partial x_1} = -3,\ wt\,\frac{\partial f_{k+1}^{0}}{\partial x_4} = 3 \\
& wt\,\frac{\partial f_{k+1}^{1}}{\partial x_1} = -2,\ wt\,\frac{\partial f_{k+1}^{1}}{\partial x_4} = 4,\ wt\,\frac{\partial f_{k+1}^{1}}{\partial x_3} = 2 \\
\Rightarrow\ & \frac{\partial f_{k+1}^{-1}}{\partial x_1} = 0 = \frac{\partial f_{k+1}^{-1}}{\partial x_4} = \frac{\partial f_{k+1}^{0}}{\partial x_1} = \frac{\partial f_{k+1}^{0}}{\partial x_4} = \frac{\partial f_{k+1}^{1}}{\partial x_1} = \frac{\partial f_{k+1}^{1}}{\partial x_4} = \frac{\partial f_{k+1}^{-1}}{\partial x_2} = \\
& \frac{\partial f_{k+1}^{1}}{\partial x_3} = 0 \\
\Rightarrow\ & f = f_{k+1}^{-1}(x_3, x_5) + f_{k+1}^{0}(x_2, x_3, x_5) + f_{k+1}^{1}(x_2, x_5) \\
\Rightarrow\ & \frac{\partial f}{\partial x_1} = 0 = \frac{\partial f}{\partial x_4} \\
\Rightarrow\ & \dim I \leq 3.
\end{aligned}
$$

This contradicts to our hypothesis $I = (2) \oplus (2) \oplus (1)$. Thus Case 5 cannot occur.

Case 6. $I = (2) \oplus (1) \oplus (1) \oplus (1)$.

Case 6 cannot occur. The proof is the same as Case 5.

Case 7. $I = (1) \oplus (1) \oplus (1) \oplus (1) \oplus (1)$.

Case 7 cannot occur. The proof is the same as Case 5. Q.E.D.

Lemma 4.2. With the same hypothesis as lemma 4.1, if I is a $sl(2,\mathbf{C})$-submodule of dimension 4, then $I = (4)$ and f is a polynomial in x_1, x_2, x_3 and x_4 variables of weight 0.

Proof. Case 1. $I = (4)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $-3, -1, 1$ and 3. The same argument as Case 2 in the proof of lemma 4.1 gives

$$f = f_{k+1}^{-3} + f_{k+1}^{-2} + f_{k+1}^{-1} + f_{k+1}^{0} + f_{k+1}^{1} + f_{k+1}^{2} + f_{k+1}^{3}.$$

For $i = \pm 1, \pm 3$ $wt\,\frac{\partial f_{k+1}^i}{\partial x_j}$ is an even integer for all $1 \le j \le 5$. This implies that $\frac{\partial f_{k+1}^i}{\partial x_j} = 0$ for all $1 \le j \le 5$. Thus $f_{k+1}^i = 0$.

We can write $f = f_{k+1}^{-2} + f_{k+1}^{0} + f_{k+1}^{2}$

$$wt\,\frac{\partial f_{k+1}^2}{\partial x_4} = 5,\ wt\,\frac{\partial f_{k+1}^2}{\partial x_5} = 2,\ wt\,\frac{\partial f_{k+1}^0}{\partial x_5} = 0,\ wt\,\frac{\partial f_{k+1}^{-2}}{\partial x_1} = -5,\ wt\,\frac{\partial f_{k+1}^{-2}}{\partial x_5} = -2$$

$$\Rightarrow \quad \frac{\partial f_{k+1}^2}{\partial x_4} = 0 = \frac{\partial f_{k+1}^2}{\partial x_5} = \frac{\partial f_{k+1}^0}{\partial x_5} = \frac{\partial f_{k+1}^{-2}}{\partial x_1} = \frac{\partial f_{k+1}^{-2}}{\partial x_5}$$

$$\Rightarrow \quad f = f_{k+1}^{-2}(x_2, x_3, x_4) + f_{k+1}^0(x_1, x_2, x_3, x_4) + f_{k+1}^2(x_1, x_2, x_3).$$

By the same argument as in Case 2 in the proof of lemma 4.1, we can conclude that $f_{k+1}^2 = f_{k+1}^{-2} = 0$. Hence $f = f_{k+1}^0(x_1, x_2, x_3, x_4)$.

Case 2. $I = (3) \oplus (1)$.

By the same argument as Case 4 in the proof of lemma 4.1, we deduce that

$$f = f_{k+1}^{-1}(x_2, x_3, x_4) + c_0 x_5^{k+1} + f_{k+1}^1(x_1, x_2, x_3).$$

Weight of $\frac{\partial f_{k+1}^1}{\partial x_1}(x_1, x_2, x_3)$ is -2. So $\frac{\partial f_{k+1}^1}{\partial x_1}(x_1, x_2, x_3)$ is in $(3) \subseteq I$ and $X_-(\frac{\partial f_{k+1}^1}{\partial x_1}(x_1, x_2, x_3)) = 0$. Write

$$\begin{aligned} f_{k+1}^1(x_1, x_2, x_3) &= \phi_k(x_1, x_2)x_3 + \ldots + \phi_{k+1-i}(x_1, x_2)x_3^i \\ &\quad + \ldots + \phi_1(x_1, x_2)x_3^k + \phi_0 x_3^{k+1} \\ \Rightarrow \quad \frac{\partial f_{k+1}^1}{\partial x_1}(x_1, x_2, x_3) &= \frac{\partial \phi_k}{\partial x_1}(x_2, x_2)x_3 + \ldots + \frac{\partial \phi_{k+1-i}(x_1, x_2)}{\partial x_1}x_3^i \end{aligned}$$

$$
\begin{aligned}
&+\ldots+\frac{\partial\phi_1}{\partial x_1}(x_1,x_2)x_3^k \\
\Rightarrow\quad X_-\Big(\frac{\partial f_{k+1}^1}{\partial x_1}(x_1,x_2,x_3)\Big) &= [X_-\Big(\frac{\partial\phi_k}{\partial x_1}(x_1,x_2)\Big)\cdot x_3+\frac{\partial\phi_k}{\partial x_1}(x_1,x_2)\cdot x_4] \\
&+\ldots+[X_-\Big(\frac{\partial\phi_{k+1-i}}{\partial x_1}(x_1,x_2)\Big)\cdot x_3^i+i\frac{\partial\phi_{k+1-i}}{\partial x_1}(x_1,x_2)x_3^{i-1}x_4] \\
&+\ldots+k\frac{\partial\phi_1}{\partial x_1}(x_1,x_2)x_3^{k-1}x_4=0
\end{aligned}
$$

$$
\Rightarrow\quad \frac{\partial\phi_{k+1-i}}{\partial x_1}=0 \qquad \text{for } 1\le i\le k
$$

$\Rightarrow f_{k+1}^1$ depends only on x_2 and x_3 variables.

By the argument as Case 4 in the proof of the previous lemma 4.1, we have $f_{k+1}^1=0$.

Similarly we can prove $f_{k+1}^{-1}=0$. Hence we have $f=c_0x_5^{k+1}$ and hence $\dim I\le 1$. This contradicts to our hypothesis $I=(3)\oplus(1)$. Thus Case 2 cannot occur.

Case 3. $I=(2)\oplus(2)$.

This case cannot occur by the same argument as Case 5 in the proof of lemma 4.1.

Case 4. $I=(2)\oplus(1)\oplus(1)$.

This case cannot occur by the same argument as Case 5 in the proof of lemma 4.1.

Case 5. $I=(1)\oplus(1)\oplus(1)\oplus(1)$.

This case cannot occur by the same argument as Case 5 in the proof of lemma 4.1.

Q.E.D.

Lemma 4.3. With the same hypothesis as lemma 4.1, if dimension I is 3, then I cannot be a $sl(2,\mathbf{C})$-submodule.

Proof. Suppose on the contrary that I is a $sl(2,\mathbf{C})$-submodule.

Case 1. $I = (3)$.

This case cannot occur by the same argument as Case 2 in the proof of lemma 4.2.

Case 2. $I = (2) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $-1, 0$ and 1. By the same arguments as in csae 5 in the proof of lemma 4.1, we have

$$f = f_{k+1}^{-1}(x_3, x_5) + f_{k+1}^{0}(x_2, x_3, x_5) + f_{k+1}^{1}(x_2, x_5)$$

$f_{k+1}^{1}(x_2, x_5) = c_1 x_2 x_5^k$ for some constant c_1. If $c_1 \neq 0$, then $x_2 x_5^{k-1} = \frac{1}{c_1^k} \frac{\partial f_{k+1}^1}{\partial x_1}$ is an element in I. It follows that $\langle x_1 x_5^{k-1}, x_2 x_5^{k-1}, x_3 x_5^{k-1}, x_4 x_5^{k-1} \rangle$ is a 4-dimensional subspace in I. This contradicts to our hypothesis $I = (2) \oplus (1)$. Thus $f_{k+1}^1 = 0$. Similarly we can prove $f_{k+1}^{-1} = 0$. Therefore

$$\begin{aligned}
f &= f_{k+1}^{0}(x_2, x_3, x_5) \\
&= a_0 x_5^{k+1} + a_2 x_2 x_3 x_5^{k-1} + a_4 x_2^2 x_3^2 x_5^{k-3} + \ldots + a_{[\frac{k+1}{2}]} x_2^{[\frac{k+1}{2}]} x_3^{[\frac{k+1}{2}]} x_5^{k+1-[\frac{k+1}{2}]} \\
\Rightarrow \frac{\partial f}{\partial x_2} &= a_2 x_3 x_5^{k-1} + 2a_4 x_2 x_3^2 x_5^{k-3} + \ldots + [\frac{k+1}{2}] a_{[\frac{k+1}{2}]} x_2^{[\frac{k+1}{2}]-1} x_3^{[\frac{k+1}{2}]} x_5^{k+1-[\frac{k+1}{2}]}.
\end{aligned}$$

In view of $wt\, \frac{\partial f}{\partial x_2} = -2$, we hvae $X_- \frac{\partial f}{\partial x_2} = 0$. Observe that

$$\begin{aligned}
X_- \frac{\partial f}{\partial x_2} &= a_2 x_4 x_5^{k-1} + (2a_4 x_3^3 x_5^{k-3} + 4a_4 x_2 x_3 x_4 x_5^{k-3}) \\
&+ \ldots + ([\frac{k+1}{2}] a_{[\frac{k+1}{2}]} x_2^{[\frac{k+1}{2}]-1} x_3^{[\frac{k+1}{2}]+1} x_5^{k+1-[\frac{k+1}{2}]} \\
&+ [\frac{k+1}{2}] a_{[\frac{k+1}{2}]} x_2^{[\frac{k+1}{2}]} x_3^{[\frac{k+1}{2}]-1} x_4 x_5^{k+1-[\frac{k+1}{2}]}).
\end{aligned}$$

Therefore we have $a_2 = 0 = a_3 = \ldots = a_{[\frac{k+1}{2}]}$ and

$$f = a_0 x_5^{k+1}.$$

This implies that $\dim I \leq 1$, which contradicts to our hypothesis $i = (2) \oplus (1)$.

Case 3. $I = (1) \oplus (1) \oplus (1)$.

Elements of I are linear combinatinos of homogeneous polynomials of degree k and weight 0. By the same argument as Case 5 in the proof of lemma 4.1, we have

$$f = f_{k+1}^{-1}(x_3, x_5) + f_{k+1}^{0}(x_2, x_3, x_5) + f_{k+1}^{1}(x_2, x_5).$$

Notice that $wt\, \frac{\partial f_{k+1}^{-1}}{\partial x_5} = -1$ and $wt\, \frac{\partial f_{k+1}^{1}}{\partial x_5} = 1$ imply $\frac{\partial f_{k+1}^{-1}}{\partial x_5} = 0 = \frac{\partial f_{k+1}^{1}}{\partial x_5}$. Hence f_{k+1}^{-1} and f_{k+1}^{1} depends on x_3 and x_2 respectively. There are constants c_1 and c_2 such that $f_{k+1}^{-1} = c_1 x_3$, $f_{k+1}^{1} = c_2 x_2$. As $k \geq 2$, we conclude that $f_{k+1}^{-1} = 0 = f_{k+1}^{1}$ and

$$f = f_{k+1}^{0}(x_2, x_3, x_5)$$

$wt\, \frac{\partial f_{k+1}^{0}}{\partial x_2} = -1$ and $wt\, \frac{\partial f_{k+1}^{0}}{\partial x_3} = 1$ imply $\frac{\partial f_{k+1}^{0}}{\partial x_2} = 0 = \frac{\partial f_{k+1}^{0}}{\partial x_3}$. Thus $f = c_3 x_5^{k+1}$ for some constant c_5. It follows that $\dim I \leq 1$, which contradicts to our hypothesis $I = (1) \oplus (1) \oplus (1)$. Q.E.D.

Lemma 4.4. With the same hypothesis as lemma 4.1, if dimension of I is 2, then I cannot be a $sl(2,\mathbf{C})$-submodule.

Proof. Suppose on the contrary that I is a $sl(2,\mathbf{C})$-submodule.

Case 1. $I = (2)$.

This case cannot occur. The proof is the same as Case 2 in the proof of lemma 4.3.

Case 2. $I = (1) \oplus (1)$.

This case cannot occur. The proof is the same as Case 3 in the proof of lemma 4.3. Q.E.D.

Lemma 4.5. With the same hypothesis as lemma 4.1, if I is a $sl(2, \mathbf{C})$-submodule of dimension one, then $f = cx_5^{k+1}$ where c is a nonzero constant.

Proof. The proof is the same as Case 3 in the proof of lemma 4.3. Q.E.D.

Proposition 4.6. Suppose $sl(2, \mathbf{C})$ acts on the space of homogeneous polynomials of degree $k \geq 2$, in x_1, x_2, x_3, x_4 and x_5 variables via

$$\begin{aligned}
\tau \quad &= 3x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} - x_3\frac{\partial}{\partial x_3} - 3x_4\frac{\partial}{\partial x_4} \\
X_+ \quad &= 3x_1\frac{\partial}{\partial x_2} + 4x_2\frac{\partial}{\partial x_3} + 3x_3\frac{\partial}{\partial x_4} \\
X_- \quad &= x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x}
\end{aligned}$$

Suppose the weight of x_i is given by the corresponding coefficient in the expression of τ as above i.e.

$$wt(x_1) = 3,\ wt(x_2) = 1,\ wt(x_3) = -1,\ wt(x_4) = -3,\ wt(x_5) = 0.$$

Let I be the complex vector subspace spanned by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}$ and $\frac{\partial f}{\partial x_5}$, where f is a homogeneous polynomial of degree $k+1$. If I is a $sl(2, \mathbf{C})$-submodule, then one of the following occurs.

(i) $I = (4) \oplus (1)$ and f is a homogeneous polynomial of weight 0.

(ii) $I = (4)$ and f is a homogeneous polynomial in x_1, x_2, x_3 and x_4 variables of weight 0.

(iii) $I = (1)$ and $f = cx_5^{k+1}$ where c is a nonzero constant.

Proof. This is an immediate consequence of Lemma 4.1 – Lemma 4.5. Q.E.D.

§5. $sl(2,\mathbf{C})$ ACTION (1.3) ON M_5^k.

Lemma 5.1. Let f be a polynomial in x_1, x_2, x_3, x_4 and x_5 variables. Suppose $\frac{\partial f}{\partial x_3} = r_1 \frac{\partial f}{\partial x_1}$ and $\frac{\partial f}{\partial x_4} = r_2 \frac{\partial f}{\partial x_2}$. Then there exists a polynomial g in $\tilde{x}_1, \tilde{x}_2$ and x_5 variables such that

$$f(x_1, x_2, x_3, x_4, x_5) = g(x_1 + r_1 x_3, x_2 + r_2 x_4, x_5).$$

Proof. Introduce independent variables

$$\tilde{x}_1 = x_1 + r_1 x_3$$

$$\tilde{x}_2 = x_2 + r_2 x_4$$

$$\tilde{x}_3 = x_3$$

$$\tilde{x}_4 = x_4$$

$$\tilde{x}_5 = x_5.$$

Let $g(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5) = f(\tilde{x}_1 - r_1\tilde{x}_3, \tilde{x}_2 - r_2\tilde{x}_4, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5)$. We claim that g is independent of $\tilde{x}_3$ and $\tilde{x}_4$ variables.

$$\begin{aligned}
\frac{\partial g}{\partial x_3} &= \frac{\partial f}{\partial x_1}(\tilde{x}_1 - r_1\tilde{x}_3, \tilde{x}_2 - r_2\tilde{x}_4, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5)\frac{\partial(\tilde{x}_1 - r_1\tilde{x}_3)}{\partial \tilde{x}_3} \\
&+ \frac{\partial f}{\partial x_2}(\tilde{x}_1 - r_1\tilde{x}_3, \tilde{x}_2 - r_2\tilde{x}_4, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5)\frac{\partial(\tilde{x}_2 - r_2\tilde{x}_4)}{\partial \tilde{x}_3} \\
&+ \frac{\partial f}{\partial x_3}(\tilde{x}_1 - r_1\tilde{x}_3, \tilde{x}_2 - r_2\tilde{x}_4, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5)\frac{\partial \tilde{x}_3}{\partial \tilde{x}_3} \\
&+ \frac{\partial f}{\partial x_4}(\tilde{x}_1 - r_1\tilde{x}_3, \tilde{x}_2 - r_2\tilde{x}_4, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5)\frac{\partial \tilde{x}_4}{\partial \tilde{x}_3} \\
&+ \frac{\partial f}{\partial x_5}(\tilde{x}_1 - r_1\tilde{x}_3, \tilde{x}_2 - r_2\tilde{x}_4, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5)\frac{\partial \tilde{x}_5}{\partial \tilde{x}_3} \\
&= (-r_1\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_3})(\tilde{x}_1 - r_1\tilde{x}_3, \tilde{x}_2 - r_2\tilde{x}_4, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5) \\
&= 0.
\end{aligned}$$

Similarly $\frac{\partial g}{\partial \tilde{x}_4} = 0$. Therefore g depends only on $\tilde{x}_1, \tilde{x}_2$ and $\tilde{x}_5$ variables and our lemma follows. Q.E.D.

Lemma 5.2. Suppose $sl(2, \mathbf{C})$ acts on the space of homogeneous polynomials of degree $k \geq 2$ in x_1, x_2, x_3, x_4 and x_5 variables via (1.3).

$$\begin{aligned}
\tau \quad &= x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} - x_4 \frac{\partial}{\partial x_4} \\
X_+ \quad &= x_1 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_4} \\
X_- \quad &= x_2 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_3}.
\end{aligned}$$

Suppose the weight of x_i is given by the corresponding coefficient in the expression of τ above i.e.

$$wt(x_1) = 1,\ wt(x_2) = -1,\ wt(x_3) = 1,\ wt(x_4) = -1, wt(x_5) = 0.$$

Let I be the complex vector subspace of dimension 5 spanned by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}$ and $\frac{\partial f}{\partial x_5}$ where f is a homogeneous polynomial of degree $k+1$. If I is a $sl(2, \mathbf{C})$-submodule, then $I = (2) \oplus (2) \oplus (1)$ and f is a homogeneous polynomial of weight 0.

Proof. Case $I = (5)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $-4, -2, 0, 2$ and 4. Write

$$f = \sum_{i=-\infty}^{\infty} f_{k+1}^i$$

where f_{k+1}^i is a homogeneous polynomial of degree $k+1$ and weight i.

For $|i| \geq 6$

$$
\begin{aligned}
& |wt\,\frac{\partial f_{k+1}^i}{\partial x_j}| \geq 5 \quad \text{for } 1 \leq j \leq 5 \\
\Rightarrow\ & \frac{\partial f_{k+1}^i}{\partial x_j} = 0 \qquad \text{for } 1 \leq j \leq 5 \\
\Rightarrow\ & f_{k+1}^i = 0.
\end{aligned}
$$

For $i = \pm 4,\ \pm 2$ and 0

weights of $\frac{\partial f_{k+1}^i}{\partial x_1}$, $\frac{\partial f_{k+1}^i}{\partial x_2}$, $\frac{\partial f_{k+1}^i}{\partial x_3}$ and $\frac{\partial f_{k+1}^i}{x_4}$ are odd integers. So $\frac{\partial f_{k+1}^i}{\partial x_1}$, $\frac{\partial f_{k+1}^i}{\partial x_2}$, $\frac{\partial f_{k+1}^i}{\partial x_3}$ and $\frac{\partial f_{k+1}^i}{\partial x_4}$ are equal to zero. Thus f_{k+1}^i depends only on x_5 variables. It follows that

$$f_{k+1}^4 = f_{k+1}^{-4} = f_{k+1}^2 = f_{k+1}^{-2} = 0$$

and

$$f_{k+1}^0 = c x_5^{k+1}$$

where c is a constant. If $c \neq 0$, then $x_5^k = \frac{1}{(k+1)c}\ \frac{\partial f_{k+1}^0}{\partial x_5}$ is in I. Clearly $\langle x_5^k \rangle$ is an $sl(2,\mathbf{C})$-submodule of dimension one in I. This contradicts to our hypothesis that $I = (5)$. Hence $c = 0$ and $f_{k+1}^0 = 0$.

For $i = \pm 5,\ \pm 3$ and ± 1

$$
\begin{aligned}
& wt\,\frac{\partial f_{k+1}^i}{\partial x_5} = i \\
\Rightarrow & \frac{\partial f_{k+1}^i}{\partial x_5} = 0 \\
\Rightarrow & f_{k+1}^i \text{ does not involve } x_5.
\end{aligned}
$$

Since $f = f_{k+1}^{-5} + f_{k+1}^{-3} + f_{k+1}^{-1} + f_{k+1}^{1} + f_{k+1}^{3} + f_{k+1}^{5}$, therefore f does not involve x_5 variable. If follows that $\frac{\partial f}{\partial x_5} = 0$ and $\dim I \leq 4$. This contradicts to our hypothesis $I = (5)$. So Case 1 cannot occur.

Case 2. $I = (4) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $-3, -1, 1, 3$ and 0.

For $|i| \geq 5$

$$
\begin{aligned}
& |wt\, \frac{\partial f^i_{k+1}}{\partial x_j}| \geq 4 \quad \text{for } 1 \leq j \leq 5 \\
\Rightarrow \quad & \frac{\partial f^i_{k+1}}{\partial x_j} = 0 \quad \text{for } 1 \leq j \leq 5 \\
& \Rightarrow f^i_{k+1} = 0.
\end{aligned}
$$

For $i = 3$.

The weights of $\frac{\partial f^3_{k+1}}{\partial x_1}$, $\frac{\partial f^3_{k+1}}{\partial x_3}$, $\frac{\partial f^3_{k+1}}{\partial x_3}$ and $\frac{\partial f^3_{k+1}}{\partial x_4}$ are nonzero even integers. We have

$$
\frac{\partial f^3_{k+1}}{\partial x_1} = 0 = \frac{\partial f^3_{k+1}}{\partial x_2} = \frac{\partial f^3_{k+1}}{\partial x_3} = \frac{\partial f^3_{k+1}}{\partial x_4}.
$$

Thus f^3_{k+1} depends only on x_5 variables. It follows that $f^3_{k+1} = 0$. Similarly we can prove $f^{-3}_{k+1} = 0$.

For $i = 1$.

$$
\begin{aligned}
& wt\, \frac{\partial f^1_{k+1}}{\partial x_2} = 2 \quad wt\, \frac{\partial f^1_{k+1}}{\partial x_4} = 2 \\
\Rightarrow \quad & \frac{\partial f^1_{k+1}}{\partial x_2} = 0 = \frac{\partial f^1_{k+1}}{\partial x_4} \\
\Rightarrow \quad & f^1_{k+1} \text{ involves only } x_1, x_3 \text{ and } x_5 \text{ variables} \\
\Rightarrow \quad & f^1_{k+1} = c_1 x_1 x_5^k + c_2 x_3 x_5^k \text{ where } c_1, c_2 \text{ are constants.}
\end{aligned}
$$

If $f^1_{k+1} \neq 0$, then $c_1 x_1 x_5^{k-1} + c_2 x_3 x_5^{k-1} = \frac{1}{k} \frac{\partial f^1_{k+1}}{\partial x_5}$ is a nonzero element in I. Since $X_+(c_1 x_1 x_5^{k-1} + c_2 x_3 x_5^{k-1}) = 0$, it follows that

$$
\langle c_1 x_1 x_5^{k-1} + c_2 x_3 x_5^{k-1}, c_1 x_2 x_5^{k-1} + c_2 x_4 x_5^{k-1} \rangle
$$

is an irreducible $sl(2,\mathbf{C})$-submodule of dimension 2 in I. This contradicts to our hypothesis $I=(4)\oplus(1)$. Hence $f^1_{k+1}=0$.

Similarly we can prove that $f^{-1}_{k+1}=0$. We can write

$$f=f^4_{k+1}+f^2_{k+1}+f^0_{k+1}+f^{-2}_{k+1}+f^{-4}_{k+1}.$$

For $i=4$

$$\begin{aligned}
& wt\,\frac{\partial f^4_{k+1}}{\partial x_2}=5=wt\,\frac{\partial f^4_{k+1}}{\partial x_4},\ wt\,\frac{\partial f^4_{k+1}}{\partial x_5}=4\\
\Rightarrow\quad & \frac{\partial f^4_{k+1}}{\partial x_2}=\frac{\partial f^4_{k+1}}{\partial x_4}=0=\frac{\partial f^4_{k+1}}{\partial x_5}\\
\Rightarrow\quad & f^4_{k+1}\text{ involves only } x_1 \text{ and } x_3 \text{ variables.}
\end{aligned}$$

Suppose that $f^4_{k+1}\neq 0$. Without loss of generality, we shall assume that $\frac{\partial f^4_{k+1}}{\partial x_1}\neq 0$. Since $wt\,\frac{\partial f^4_{k+1}}{\partial x_1}=wt\,\frac{\partial f^4_{k+1}}{\partial x_3}=3$, we have $\frac{\partial f^4_{k+1}}{\partial x_3}=r\frac{\partial f^4_{k+1}}{\partial x_1}$. By lemma 5.1, there exists a nonzero constant c_1 such that

$$f^4_{k+1}=c_1(x_1+rx_3)^4.$$

It is easy to see that the irreducible $sl(2,\mathbf{C})$-submodule of dimension 4 in I is one of the following form

$$\langle (x_1+rx_3)^3,\ (x_1+rx_3)^2(x_2+rx_4),(x_1+rx_3)(x_2+rx_4)^2,(x_2+rx_4)^3\rangle.$$

By weight consideration, there are constants $\alpha_1,\alpha_2,\alpha_3$ and α_4 such that

$$\begin{aligned}
\frac{\partial f^2_{k+1}}{\partial x_1}&=\alpha_1(x_1+rx_3)^2(x_2+rx_4)\\
\frac{\partial f^2_{k+1}}{\partial x_2}&=\alpha_2(x_1+rx_3)^3\\
\frac{\partial f^2_{k+1}}{\partial x_3}&=\alpha_3(x_1+rx_3)^2(x_2+rx_4)
\end{aligned}$$

$$\frac{\partial f_{k+1}^2}{\partial x_4} = \alpha_4(x_1 + rx_3)^3.$$

By considering the second partial derivatives of f, we infer

$$\begin{aligned}
\alpha_2 &= \frac{1}{3}\alpha_1 \\
\alpha_3 &= r\alpha_1 \\
\alpha_4 &= \frac{1}{3}r\alpha_1
\end{aligned}$$

and

$$f_{k+1}^2 = \frac{\alpha_1}{3}(x_1 + rx_3)^3(x_2 + rx_4).$$

By weight consideration, there are constants $\alpha_5, \alpha_6.\alpha_7$ and α_8 such that

$$\begin{aligned}
\frac{\partial f_{k+1}^0}{\partial x_1} &= \alpha_5(x_1 + rx_3)(x_2 + rx_4)^2 \\
\frac{\partial f_{k+1}^0}{\partial x_2} &= \alpha_6(x_1 + rx_3)^2(x_2 + rx_4) \\
\frac{\partial f_{k+1}^0}{\partial x_3} &= \alpha_7(x_1 + rx_3)(x_2 + rx_4)^2 \\
\frac{\partial f_{k+1}^0}{\partial x_4} &= \alpha_8(x_1 + rx_3)^2(x_2 + rx_4).
\end{aligned}$$

It follows easily that

$$\begin{aligned}
\alpha_6 &= \alpha_5 \\
\alpha_7 &= r\alpha_5 \\
\alpha_8 &= r\alpha_5
\end{aligned}$$

and

$$f_{k+1}^0 = \frac{\alpha_5}{2}(x_1 + rx_3)^2(x_2 + rx_4)^2 + cx_5^4$$

where c is a constant.

Similarly we can prove that there exist constant β_1 and d_1 such that

$$f_{k+1}^{-2} = \frac{\beta_1}{3}(x_1 + rx_3)(x_2 + rx_4)^3$$

and

$$f_{k+1}^{-4} = d_1(x_2 + rx_4)^4.$$

By renaming the constants, we have

$$f = c_1(x_1 + rx_3)^4 + c_2(x_1 + rx_3)^3(x_2 + rx_4) + c_3(x_1 + rx_3)^2(x_2 + rx_4)^2$$
$$+ c_0x_5^4 + c_4(x_1 + rx_3)(x_2 + rx_4)^3 + c_5(x_2 + rx_4)^4.$$

This implies $\dim I \leq 3$, which contradicts to our hypothesis $I = (3) \oplus (2)$. Thus $f_{k+1}^4 = 0$.

Similarly we can prove $f_{k+1}^{-4} = 0$. Therefore we can write $f = f_{k+1}^2 + f_{k+1}^0 + f_{k+1}^{-2}$.

For $i = 2$.

Since weight of $\frac{\partial f_{k+1}^2}{\partial x_5}$ is 2, so $\frac{\partial f_{k+1}^2}{\partial x_5} = 0$ and f_{k+1}^2 does not depend on x_5 variable. Suppose that $f_{k+1}^2 \neq 0$. It is not hard to see that in order to produce a contradiction, it suffices to prove that the case $\frac{\partial f_{k+1}^2}{\partial x_1} \neq 0 \neq \frac{\partial f_{k+1}^2}{\partial x_2}$ and the case $\frac{\partial f_{k+1}^2}{\partial x_1} \neq 0 = \frac{\partial f_{k+1}^2}{\partial x_2} = \frac{\partial f_{k+1}^2}{\partial x_4}$ both cannot occur.

Let us first consider the case $\frac{\partial f_{k+1}^2}{\partial x_1} \neq 0 \neq \frac{\partial f_{k+1}^2}{\partial x_2}$. Since $wt\,\frac{\partial f_{k+1}^2}{\partial x_1} = wt\,\frac{\partial f_{k+1}^2}{\partial x_3}$ and $wt\,\frac{\partial f_{k+1}^2}{\partial x_2} = wt\,\frac{\partial f_{k+1}^2}{\partial x_4}$, there are constants, r_1 and r_2 such that

$$\frac{\partial f_{k+1}^2}{\partial x_3} = r_1\frac{\partial f_{k+1}^2}{\partial x_1} \qquad \frac{\partial f_{k+1}^2}{\partial x_4} = r_2\frac{\partial f_{k+1}^2}{\partial x_2}.$$

By lemma 5.1, there exists a nonzero constant c_1 such that

$$f_{k+1}^2 = c_1(x_1 + r_1x_3)^{\beta+2}(x_2 + r_2x_4)^{\beta} \text{ where } 2\beta = k - 1.$$

Since $\frac{\partial f^2_{k+1}}{\partial x_1} = c_1(\beta+2)(x_1+r_1x_3)^{\beta+1}(x_2+r_2x_4)^{\beta}$ is a nonzero element in I. $X_-^{\beta+1}\frac{\partial f^2_{k+1}}{\partial x_1} = c_1(\beta+2)!(x_2+r_1x_4)^{\beta+1}(x_2+r_2x_4)^{\beta}$ is also a nonzero element in I. It follows that we have an irreducible submodule of I of the following form

$\langle(x_2+r_1x_4)^{\beta+1}(x_2+r_2x_4)^{\beta}, X_+[(x_2+r_1x_4)^{\beta+1}(x_2+r_2x_4)^{\beta}], \ldots, X_+^k[(x_2+r_1x_4)^{\beta+1}(x_2+r_2x_4)^{\beta}]\rangle$.

This implies that $k=3$, $\beta=1$ and $f^2_{k+1} = c_1(x_1+r_1x_3)^3(x_2+r_2x_4)$. Hence $(4) = \langle(x_2+r_1x_4)^2(x_2+r_2x_4), 2(x_1+r_1x_3)(x_2+r_1x_4)(x_2+r_2x_4)+(x_2+r_1x_4)^2(x_1+r_2x_3), (x_1+r_1x_3)^2(x_2+r_2x_4)+2(x_1+r_1x_3)(x_2+r_1x_4)(x_1+r_2x_3), (x_1+r_1x_3)^2(x_1+r_2x_3)\rangle$.
Since $(x_1+r_1x_3)^2(x_2+r_2x_4) = \frac{1}{3c_1}\frac{\partial f^2_{k+1}}{\partial x_1}$ is an element of weight 1 in I, it is easy to see that $r_1 = r_2$. Therefore

$$f^2_{k+1} = c_1(x_1+r_1x_3)^3(x_2+r_1x_4)$$

and

$$(4) = \langle(x_1+r_1x_3)^3, (x_1+r_1x_3)^2(x_2+r_1x_4), (x_1+r_1x_3)(x_2+r_1x_4)^2, (x_2+r_1x_4)^3\rangle.$$

Similar argument as before will show that

$$f = c_2(x_1+r_1x_3)^3(x_2+r_1x_4) + c_3(x_1+rx_3)^2(x_2+r_1x_4)^2 + c_0x_5^4 + c_4(x_1+r_1x_3)(x_2+r_1x_4)^3.$$

This implies $\dim I \leq 3$, which contradicts to our hypothesis $I = (4)\oplus(1)$.

Now we consider the second case $\frac{\partial f^2_{k+1}}{\partial x_1} \neq 0 = \frac{\partial f^2_{k+1}}{\partial x_2} = \frac{\partial f^2_{k+1}}{\partial x_4}$. In this case, f^2_{k+1} involves only x_1 and x_3 variables. Therefore f^2_{k+1} is a homogeneous polynomial of degree 2. This implies that $k=1$ which contradicts our hypothesis $k \geq 2$.

Therefore we have $f = f^0_{k+1}$.

For $i = 0$.

This is not possible because weight of $\frac{\partial f}{\partial x_i}$ is not equal to 3 for all $1 \le i \le 5$ and I is equal to $(4) \oplus (1)$, so Case 2 cannot occur.

Case 3. $I = (3) \oplus (2)$.

Elements of I are linear combinations of homogeneous polynomials of degree $k+1$ and weights $-2, 0, 2, -1$ and 1.

For $|i| \ge 4$.

$$\begin{aligned} & |wt\,\frac{\partial f^i_{k+1}}{\partial x_j}| \ge 3 && \text{for all } 1 \le j \le 5 \\ \Rightarrow\ & \frac{\partial f^i_{k+1}}{\partial x_j} = 0 && \text{for all } 1 \le j \le 5 \\ \Rightarrow\ & f^i_{k+1} = 0. \end{aligned}$$

For $i = 3$

$$\begin{aligned} & wt\frac{\partial f^3_{k+1}}{\partial x_2} = 4 = wt\frac{\partial f^3_{k+1}}{\partial x_4} \quad wt\frac{\partial f^3_{k+1}}{\partial x_5} = 3 \\ \Rightarrow\ & \frac{\partial f^3_{k+1}}{\partial x_2} = 0 = \frac{\partial f^3_{k+1}}{\partial x_4} = \frac{\partial f^3_{k+1}}{\partial x_5} \\ \Rightarrow\ & f^3_{k+1} \text{ involves only } x_1 \text{ and } x_3 \text{ variables.} \end{aligned}$$

Suppose $f^3_{k+1} \ne 0$. We shall assume without loss of generality that $\frac{\partial f^3_{k+1}}{\partial x_1} \ne 0$. Since $wt\frac{\partial f^3_{k+1}}{\partial x_1} = 2 = wt\frac{\partial f^3_{k+1}}{\partial x_3}$, there exists a constant r_1 such that $\frac{\partial f^3_{k+1}}{\partial x_3} = r_1\frac{\partial f^3_{k+1}}{\partial x_1}$. By Lemma 5.1, we have $k = 2$ and

$$f^3_3 = c_1(x_1 + r_1 x_3)^3$$

where c_1 is a nonzero constant. It is easy to see that the three dimensional submodule of I is of the following form

$$(3) = \langle (x_1 + r_1x_3)^2, (x_1 + r_1x_3)(x_2 + r_1x_4), (x_2 + r_1x_4)^2 \rangle.$$

$$wt\frac{\partial f_3^2}{\partial x_2} = 3 = wt\frac{\partial f_3^2}{\partial x_4}$$

$$\Rightarrow \qquad \frac{\partial f_3^2}{\partial x_2} = 0 = \frac{\partial f_3^2}{\partial x_4}$$

$\Rightarrow$ f_3^2 involves only x_1, x_3 and x_5 variables.

Since $wt\frac{\partial f_3^2}{\partial x_1} = 1 = wt\frac{\partial f_3^2}{\partial x_3}$, in view of Lemma 5.1, we have $f_3^2 = (r_2x_1 + r_3x_3)^2x_5$ where r_2, r_3 are constants. As $\frac{\partial f_3^2}{\partial x_5} = (r_2x_1 + r_3x_3)^2$ is an element in I of weight 2, $(r_2x_1 + r_3x_3)^2$ is a constant multiple of $(x_1 + r_1x_3)^2$. Hence $f_3^2 = c_2(x_1 + r_1x_3)^2x_5$. Similarly because $wt\frac{\partial f_3^1}{\partial x_1} = 0 = wt\frac{\partial f_3^1}{\partial x_3}, wt\frac{\partial f_3^1}{\partial x_2} = 2 = wt\frac{\partial f_3^1}{\partial x_4}$, we have

$$f_3^1 = d_0(r_4x_1 + r_5x_3)^2(r_6x_2 + r_7x_4) + d_1(r_4x_1 + r_5x_3)x_5^2$$

where r_4, r_5, r_6 and r_7 are constants. Since $\frac{\partial f_3^1}{\partial x_2} = d_0r_6(r_4x_1 + r_5x_3)^2$ and $\frac{\partial f_3^1}{\partial x_1} = 2d_0r_4(r_4x_1 + r_5x_3)(r_6x_2 + r_7x_4) + d_1r_4x_5^2$ are constant multiples of $(x_1 + r_1x_3)^2$ and $(x_1 + r_1x_3)(x_2 + r_1x_4)$ respectively, we have $d_1 = 0$ and

$$f_3^1 = c_3(x_1 + r_1x_3)^2(x_2 + r_1x_4).$$

Since $wt\frac{\partial f_3^0}{\partial x_1} = -1 = wt\frac{\partial f_3^0}{\partial x_3}$ and $wt\frac{\partial f_3^0}{\partial x_2} = 1 = wt\frac{\partial f_3^0}{\partial x_4}$, we have

$$f_3^0 = (r_8x_1 + r_9x_3)(r_{10}x_2 + r_{11}x_4)x_5 + c_0x_5^3$$

where r_8, r_9, r_{10}, r_{11} and c_0 are constants. As $\frac{\partial f_3^0}{\partial x_5}$ is an element of weight 0 in I, we have

$$(r_8x_1 + r_9x_3)(r_{10}x_2 + r_{11}x_4) + 3c_0x_5^2 = d(x_1 + r_1x_3)(x_2 + r_1x_4)$$

where d is a constant. It follows that $c_0 = 0$ and $r_8x_1 + r_9x_3$, $r_{10}x_2 + r_{11}x_4$ are constant multiples of $x_1 + r_1x_3$ and $x_2 + r_1x_4$ respectively. Hence

$$f_3^0 \quad = c_4(x_1 + r_1x_3)(x_2 + r_1x_4)x_5$$

and

$$\begin{aligned} f \quad &= f_3^3 + f_3^2 + f_3^1 + f_3^0 + f_3^{-1} + f_3^{-2} + f_3^{-3} \\ &= c_1(x_1 + r_1x_3)^3 + c_2(x_1 + r_1x_3)^2x_5 + c_3(x_1 + r_1x_3)^2(x_2 + r_1x_4) \\ &\quad + c_4(x_1 + r_1x_3)(x_2 + r_1x_4)x_5 + c_5(x_1 + r_1x_3)(x_2 + r_1x_4)^2 \\ &\quad + c_6(x_2 + r_1x_4)^2x_5 + c_7(x_2 + r_1x_4)^3. \end{aligned}$$

This implies that $\dim I \leq 3$, which contradicts to our hypothesis $I = (3)\oplus(2)$. Therefore we have

$$f = f_{k+1}^{-2} + f_{k+1}^{-1} + f_{k+1}^{0} + f_{k+1}^{1} + f_{k+1}^{2}.$$

For $i = 2$.

Suppose $f_{k+1}^2 \neq 0$

$$wt\frac{\partial f_{k+1}^2}{\partial x_1} = 1 = wt\frac{\partial f_{k+1}^2}{\partial x_3}, \quad wt\frac{\partial f_{k+1}^2}{\partial x_4} = 3 = wt\frac{\partial f_{k+1}^2}{\partial x_2}$$

$$\Rightarrow \quad \frac{\partial f_{k+1}^2}{\partial x_4} = 0 = \frac{\partial f_{k+1}^2}{\partial x_2}$$

$$\Rightarrow \quad f_{k+1}^2 \text{ involves only } x_1, x_3 \text{ and } x_5 \text{ variables.}$$

Assume without loss of generality that $\frac{\partial f_{k+1}^2}{\partial x_1} \neq 0$. Then there exists a constant r such that $\frac{\partial f_{k+1}^2}{\partial x_3} = r\frac{\partial f_{k+1}^2}{\partial x_1}$. By Lemma 5.1,

$$f_{k+1}^2 = c_1(x_1 + rx_3)^2x_5^{k-1}$$

where c_1 is a nonzero constant. It follows that the 3-dimensional aned 2-dimensional irreducible $sl(2,\mathrm{C})$ submodules in I are of the following form.

$(3) = \langle (x_1 + rx_3)^2x_5^{k-2}, (x_1 + rx_3)(x_2 + rx_4)x_5^{k-2}, (x_2 + rx_4)^2x_5^{k-2}\rangle$

$(2) = \langle (x_1 + rx_3)x_5^{k-1}, (x_2 + rx_4)x_5^{k-1}\rangle.$

Since $wt\frac{\partial f^1_{k+1}}{\partial x_1} = 0 = wt\frac{\partial f^1_{k+1}}{\partial x_3}$ and $wt\frac{\partial f^1_{k+1}}{\partial x_2}$ and $wt\frac{\partial f^1_{k+1}}{\partial x_2} = 2 = wt\frac{\partial f^1_{k+1}}{\partial x_4}$, by Lemma 5.1, there are constants r_1, r_2, r_3 and r_4 such that

$$f^1_{k+1} = \sum_{\alpha=0}^{[\frac{k}{2}]} d_\alpha (r_1x_1 + r_2x_3)^{\alpha+1}(r_3x_2 + r_4x_4)^\alpha x_5^{k-2\alpha}.$$

If $f^1_{k+1} \neq 0$, then either $r_1 \neq 0$ or $r_2 \neq 0$. Let α_0 be the largest integer such that $d_{\alpha_0}(r_1x_1 + r_2x_3)^{\alpha_0+1}(r_3x_2 + r_4x_4)^{\alpha_0}x_5^{k-2\alpha_0} \neq 0$. It follows that either $r_1(\alpha_0 + 1)d_{\alpha_0}(r_1x_1+r_2x_3)^{\alpha_0}(r_3x_2+r_4x_4)^{\alpha_0}x_5^{k-2\alpha_0} \neq 0$ or $r_2(\alpha_0+1)d_{\alpha_0}(r_1x_1+r_2x_3)^{\alpha_0}(r_3x_2 + r_4x_4)^{\alpha_0}x_5^{k-2\alpha_0} \neq 0$. We claim that $\alpha_0 \leq 1$ and if $\alpha_0 = 1$ then $k = 2$. Observe that $(r_1x_1 + r_2x_3)^{\alpha_0}(r_3x_2 + r_4x_4)^{\alpha_0}x_5^{k-2\alpha_0}+$ polynomials of total degree in x_1, x_3 variables strictly less than α_0, is an element in I. By applying $X_-^{\alpha_0}$ to the above element we get

$$(r_1x_2 + r_2x_4)^{\alpha_0}(r_3x_2 + r_4x_4)^{\alpha_0}x_5^{k-2\alpha_0} \in I.$$

Since $X_-[(r_1x_2+r_2x_4)^{\alpha_0}(r_3x_2+r_4x_4)^{\alpha_0}x_5^{k-2\alpha_0}] = 0$, we have an irreducible submodule $2\alpha_0 + 1$ in I of the following form

$$\langle (r_1x_2 + r_2x_4)^{\alpha_0}(r_3x_2 + r_4x_4)^{\alpha_0}x_5^{k-2\alpha_0}, X_+[(r_1x_2 + r_2x_4)^{\alpha_0}(r_3x_2 + r_4x_4)^{\alpha_0}x_5^{k-2\alpha_0}],$$
$$\ldots, X_+^{2\alpha_0}[(r_1x_2 + r_2x_4)^{\alpha_0}(r_3x_2 + r_4x_4)^{\alpha_0}x_5^{k-2\alpha_0}]\rangle.$$

Since $I = (3) \oplus (2)$, we have $2\alpha_0 + 1 \leq 3$. Hence $\alpha_0 \leq 1$ and

$$f^1_{k+1} = d_0(r_1x_1 + r_2x_3)x_5^k + d_1(r_1x_1 + r_2x_3)^2(r_3x_2 + r_4x_4)x_5^{k-2}.$$

If $\alpha_0 = 0$, then $f^1_{k+1} = d_0(r_1x_1 + r_2x_3)x_5^k$. Hence x_5^k is an element in I because either $\frac{\partial f^1_{k+1}}{\partial x_1}$ or $\frac{\partial f^1_{k+1}}{\partial x_3}$ is nonzero. Thus we get a one dimensional $sl(2, \mathbf{C})$-submodule in I, which is not possible. If $\alpha_0 = 1$, i.e. $d_1(r_1x_1 + r_2x_3)^2(r_3x_2 + r_4x_4)x_5^{k-2} \neq 0$, then $k = 2$ otherwiwse by applying X_+ and X_- sucessively to $\frac{\partial f^1_{k+1}}{\partial x_5}$, there would be an irreducible $sl(2, \mathbf{C})$ submodule of dimension 4 in I which is impossible. Hence,

$$f^1_3 = d_0(r_1x_1 + r_2x_3)x_5^2 + d_1(r_1x_1 + r_2x_3)^2(r_3x_2 + r_4x_4).$$

Then

$$\frac{\partial f_3^1}{\partial x_1} = r_1 d_0 x_5^2 + 2r_1 d_1 (r_1 x_1 + r_2 x_3)(r_3 x_2 + r_4 x_4)$$

$$\frac{\partial f_3^1}{\partial x_3} = r_2 d_0 x_5^2 + 2r_2 d_1 (r_1 x_1 + r_2 x_3)(r_3 x_2 + r_4 x_4)$$

are nonzero elements of weight 0 in I. Therefore both elements are constant multiples of $(x_1 + rx_3)(x_2 + rx_4)$. This implies that $d_0 = 0$ and $(r_1 x_1 + r_2 x_3)(r_3 x_2 + r_4 x_4)$ is a constant multiple of $(x_1 + rx_3)(x_2 + rx_4)$. Therefore

$$f_3^2 = c_1 (x_1 + rx_3)^2 x_5$$

$$f_3^1 = c_2 (x_1 + rx_3)^2 (x_2 + rx_4).$$

Since $wt\frac{\partial f_3^0}{\partial x_1} = wt\frac{\partial f_3^0}{\partial x_3} = -1$ and $wt\frac{\partial f_3^0}{\partial x_2} = wt\frac{\partial f_3^0}{\partial x_4} = 1$, by Lemma 5.1, there are constants r_5, r_6, r_7, r_8 and r_9 such that

$$f_3^0 = (r_5 x_1 + r_6 x_3)(r_7 x_2 + r_8 x_4) x_5 + r_9 x_5^3.$$

$\frac{\partial f_3^0}{\partial x_5} = (r_5 x_1 + r_6 x_3)(r_7 x_2 + r_8 x_4) + 3r_9 x_5^2$ is an element of weight 0 in I. So $\frac{\partial f_3^0}{\partial x_5}$ is a constant multiple of $(x_1 + rx_3)(x_2 + rx_4)$. This implies that $r_9 = 0$ and $(r_5 x_1 + r_6 x_3)(r_7 x_2 + r_8 x_4)$ is a constant multiple of $(x_1 + rx_3)(x_2 + rx_4)$. Hence

$$f_3^0 = c_3 (x_1 + rx_3)(x_2 + rx_4) x_5$$

and

$$\begin{aligned} f &= f_3^{-2} + f_3^{-1} + f_3^0 + f_3^1 + f_3^2 \\ &= c_1 (x_1 + rx_3)^2 x_5 + c_2 (x_1 + rx_3)^2 (x_2 + rx_4) + c_3 (x_1 + rx_3)(x_2 + rx_4) x_5 \\ &\quad + c_4 (x_1 + rx_3)(x_2 + rx_4)^2 + c_5 (x_2 + rx_4)^2 x_5. \end{aligned}$$

This implies that $\dim I \leq 3$, which contradicts to our hypothesis that $I = (3) \oplus (2)$. On the other hand, if $f_{k+1}^1 = 0 = f_{k+1}^{-1}$, then

$$f = f_{k+1}^2 + f_{k+1}^0 + f_{k+1}^{-2}.$$

Since $wt\frac{\partial f^0_{k+1}}{\partial x_1} = wt\frac{\partial f^0_{k+1}}{\partial x_3} = -1$ and $wt\frac{\partial f^0_{k+1}}{\partial x_2} = wt\frac{\partial f^0_{k+1}}{\partial x_4} = 1$, by Lemma 5.1, there are constants $r_{10}, r_{11}, r_{12}, r_{13}$ and d_β such that

$$f^0_{k+1} = \sum_{\beta=0}^{[\frac{k+1}{2}]} d_\beta(r_{10}x_1 + r_{11}x_3)^\beta(r_{12}x_2 + r_{13}x_4)^\beta x_5^{k-2\beta+1}.$$

We observe that f^0_{k+1} cannot be zero, otherwise

$$\begin{aligned} f &= f^2_{k+1} + f^{-2}_{k+1} \\ &= c_1(x_1 + rx_3)^2 x_5^{k-1} + c_3(x_2 + rx_4)^2 x_5^{k-1} \end{aligned}$$

which would imply $\dim I \leq 3$, a contradiction to our hypothesis. Hence we shall assume that $f^0_{k+1} \neq 0$. Let β_0 be the largest integer such that $d_{\beta_0}(r_{10}x_1 + r_{11}x_3)^{\beta_0}(r_{12}x_2 + r_{13}x_4)^{\beta_0} x_5^{k+1-2\beta_0} \neq 0$. Without loss of generality, we shall assume that $r_{10} \neq 0$. We claim that $\beta_0 \leq 1$. If β_0 were strictly bigger than 1, then $(r_{10}x_1 + r_{11}x_3)^{\beta_0-1}(r_{12}x_2 + r_{13}x_4)^{\beta_0} x_5^{k-2\beta_0+1}+$ polynomials of total degree in x_1, x_3 variables strictly less than $\beta_0 - 1$ is an element in I. By applying $X_-^{\beta_0-1}$ to the above element, we get

$$(r_{10}x_2 + r_{11}x_4)^{\beta_0-1}(r_{12}x_2 + r_{13}x_4)^{\beta_0} x_5^{k-2\beta_0+1} \in I.$$

Since $X_-[(r_{10}x_2 + r_{11}x_4)^{\beta_0-1}(r_{12}x_2 + r_{13}x_4)^{\beta_0} x_5^{k-2\beta_0+1}] = 0$, we have an irreducible submodule of dimension $2\beta_0$ in I of the following form.

$\langle (r_{10}x_2+r_{11}x_4)^{\beta_0-1}(r_{12}x_2+r_{13}x_4)^{\beta_0} x_5^{k-2\beta_0+1}, X_+[(r_{10}x_2+r_{11}x_4)^{\beta_0-1}(r_{12}x_2+r_{13}x_4)^{\beta_0} \cdot x_5^{k-2\beta_0+1}], \ldots, X_+^{2\beta_0-1}[(r_{10}x_2 + r_{11}x_4)^{\beta_0-1}(r_{12}x_2 + r_{13}x_4)^{\beta_0} x_5^{k-2\beta_0+1}]\rangle$.

As $I = (3) \oplus (2)$, we have $2\beta_0 \leq 3$. Hence $\beta_0 \leq 1$ and

$$f^0_{k+1} = d_0 x_5^{k+1} + d_1(r_{10}x_1 + r_{11}x_3)(r_{12}x_2 + r_{13}x_4)x_5^{k-1}.$$

Then

$$\frac{\partial f^0_{k+1}}{\partial x_5} = (k+1)d_0 x_5^k + d_1(k-1)(r_{10}x_1 + r_{11}x_3)(r_{12}x_2 + r_{13}x_4)x_5^{k-2}$$

is an element of weight 0 in I. Therefore $\frac{\partial f_{k+1}^0}{\partial x_5}$ is a constant multiple of $(x_1+rx_3)(x_2+rx_4)x_5^{k-2}$. This implies that $d_0 = 0$ and $(r_{10}x_1 + r_{11}x_3)(r_{12}x_2 + r_{13}x_4)$ is a constant multiple of $(x_1 + rx_3)(x_2 + rx_4)$. Therefore

$$f_{k+1}^0 = c_2(x_1 + rx_3)(x_2 + rx_4)x_5^{k-1}$$

and

$$\begin{aligned} f &= f_{k+1}^2 + f_{k+1}^0 + f_{k+1}^{-2} \\ &= c_1(x_1 + rx_3)^2 x_5^{k-1} + c_2(x_1 + rx_3)(x_2 + rx_4)x_5^{k-1} + c_3(x_2 + rx_4)^2 x_5^{k-1}. \end{aligned}$$

This implies that $\dim I \leq 3$, a contradiction to our hypothesis $I = (3) \oplus (2)$. Hence $f_{k+1}^2 = 0$.

Similarly we can prove that $f_{k+1}^{-2} = 0$. Therefore we have proved

$$f = f_{k+1}^1 + f_{k+1}^0 + f_{k+1}^{-1}.$$

For $i = 1$.

By similar argument as before, we can show that there are constants r_1, r_2, r_3, r_4, d_0 and d_1 such that

$$f_3^1 = d_0(r_1x_1 + r_2x_3)x_5^2 + d_1(r_1x_1 + r_2x_3)^2(r_3x_2 + r_4x_4).$$

We shall assume without loss of generality that $r_1 \neq 0$. Then

$$\frac{\partial f_3^1}{\partial x_1} = r_1 d_0 x_5^2 + 2r_1 d_1(r_1x_1 + r_2x_3)(r_3x_2 + r_4x_4)$$

$$X_-(\frac{\partial t f_3^1}{\partial x_1}) = 2r_1 d_1(r_1x_2 + r_2x_4)(r_3x_2 + r_4x_4)$$

and

$$X_+X_-(\frac{\partial f_3^1}{\partial x_1}) = 2r_1 d_1(r_1x_1 + r_2x_3)(r_3x_2 + r_4x_4)$$

$$+2r_1d_1(r_1x_2+r_2x_4)(r_3x_1+r_4x_3)$$

are nonzero elements in I of weights 0, -2 and 0 respectively. There is a constant c such that

$$\begin{aligned}
&r_1d_0x_5^2+2r_1d_1(r_1x_2+r_2x_4)(r_3x_1+r_4x_3)\\
&=cX_+X_-(\frac{\partial f_3^1}{\partial x_1})\\
&=2cr_1d_1(r_1x_2+r_2x_3)(r_3x_2+r_4x_4)\\
&\quad+2cr_1d_1(r_1x_2+r_2x_4)(r_3x_1+r_4x_3).
\end{aligned}$$

This implies that $d_0=0$ and there exists a constant a such that

$$(r_1x_1+r_2x_3)(r_3x_2+r_4x_4)=a(x_1+rx_3)(x_2+rx_4)$$

where $r=\frac{r_2}{r_1}$. Therefore $f_3^1=a_1(x_1+rx_3)^2(x_2+rx_4)$ where a_1 is a nonzero constant and $(3)=\langle(x_1+rx_3)^2,(x_1+rx_3)(x_2+rx_4),(x_2+rx_4)^2\rangle$.

Since $wt\frac{\partial f_3^0}{\partial x_1}=wt\frac{\partial f_3^0}{\partial x_3}=-1$ and $wt\frac{\partial f_3^0}{\partial x_2}=wt\frac{\partial f_3^0}{\partial x_4}=1$, by Lermma 5.1, there are constants r_5,r_6,r_7,r_8 and e_β suc that

$$f_3^0=e_0x_5^3+e_1(r_5x_1+r_6x_3)(r_7x_2+r_8x_4)x_5.$$

We observe that f_{k+1}^0 cannot be zero, otherwise

$$\begin{aligned}
f&=f_{k+1}^1+f_{k+1}^{-1}\\
&=a_1(x_1+rx_3)^2(x_2+rx_4)+a_3(x_1+rx_3)(x_2+rx_4)^2
\end{aligned}$$

which would imply $\dim I\le 2$, a contradiction to our hypothesis. Hence we shall assume that $f_3^0\ne 0$. We claim that $e_1\ne 0$. If e_1 were zero, then $f_3^0=e_0x_5^3$. It follows that $x_5^2=\frac{1}{e_0}\frac{\partial f_3^0}{\partial x_5}$ is in I. We have a one dimensional irreducible $sl(2,\mathbf{C})$-submodule of I, which contradicts to our hypothesis $I = (3) \oplus (2)$. So $e_1 \neq 0$ and

$$\begin{aligned} f_3^0 &= e_0 x_5^3 + e_1(r_5x_1 + r_6x_3)(r_7x_2 + r_8x_4)x_5 \\ \frac{\partial f_3^0}{\partial x_5} &= 3e_0x_5^2 + e_1(r_5x_1 + r_6x_3)(r_7x_2 + r_8x_4) \end{aligned}$$

is an element of weight 0 in I. So $\frac{\partial f_{k+1}^0}{\partial x_5}$ is a constant multiple of $(x_1 + rx_3)(x_2 + rx_4)$. This implies that $e_0 = 0$ and $(r_5x_1 + r_6x_3)(r_7x_2 + r_8x_4)$ is a constant multiple of $(x_1 + rx_3)(x_2 + rx_4)$. Therefore

$$f_3^0 = a_2(x_1 + rx_3)(x_2 + rx_4)x_5$$

and

$$\begin{aligned} f &= f_3^1 + f_3^0 + f_3^{-1} \\ &= a_1(x_1 + rx_3)^2(x_2 + rx_4) + a_2(x_1 + rx_3)(x_2 + rx_4)x_5 \\ &\qquad + a_3(x_1 + rx_3)(x_2 + rx_4)^2. \end{aligned}$$

This implies that $\dim I \leq 3$, a contradiction to our hypothesis $I = (3) \oplus (2)$. Hence $f_3^1 = 0$.

Similarly we can prove that $f_3^{-1} = 0$.

For $i = 0$.

We have proved $f = f_{k+1}^0$. It follows that weight of $\frac{\partial f}{\partial x_i}$, $1 \leq i \leq 5$, is either $-1, 0$ or 1. Therefore there is no irreducible submodule of dimension three in I, which contradicts to our hyypothesis $I = (3) \oplus (2)$.

So Case 3 cannot occur.

Case 4. $I = (3) \oplus (1) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials of degree $k+1$ and weights $-2, 0$, and 2.

For $|i| \geq 4$.

$$\begin{aligned}
&|wt\frac{\partial f_{k+1}^i}{\partial x_j}| \geq 3 &&\text{for all } 1 \leq j \leq e\\
\Rightarrow\quad &\frac{\partial f_{k+1}^i}{\partial x_j} = 0 &&\text{for all } \leq j \leq 5\\
\Rightarrow\quad &f_{k+1}^i = 0.
\end{aligned}$$

For $i = 2$.

$$wt\frac{\partial f_{k+1}^2}{\partial x_1} = 1 = wt\frac{\partial f_{k+1}^2}{\partial x_3}, \quad wt\frac{\partial f_{k+1}^2}{\partial x_2} = 3 = wt\frac{\partial f_{k+1}^2}{\partial x_4}$$

$$\Rightarrow \frac{\partial f_{k+1}^2}{\partial x_1} = 0 = \frac{\partial f_{k+1}^2}{\partial x_2} = \frac{\partial f_{k+1}^2}{\partial x_3} = \frac{\partial f_{k+1}^2}{\partial x_4}$$

$\Rightarrow f_{k+1}^2$ involves only x_5 variable

$\Rightarrow f_{k+1}^2 = 0$.

Similarly we can prove $f_{k+1}^{-2} = 0$.

For $i = 3$.

We have proven $f = f_{k+1}^{-3} + f_{k+1}^{-1} + f_{k+1}^0 + f_{k+1}^1 + f_{k+1}^3$

$$wt\frac{\partial f_{k+1}^3}{\partial x_2} = 4 = wt\frac{\partial f_{k+1}^3}{\partial x_4} \quad wt\frac{\partial f_{k+1}^3}{\partial x_5} = 3$$

$$\Rightarrow \quad \frac{\partial f_{k+1}^3}{\partial x_2} = 0 = \frac{\partial f_{k+1}^3}{\partial x_4} = \frac{\partial f_{k+1}^3}{\partial x_5}$$

$\Rightarrow$ f_{k+1}^3 involves only x_1 and x_3 variables.

Suppose $f_{k+1}^3 \neq 0$. We shall assume without loss of generality that $\frac{\partial f_{k+1}^3}{\partial x_1} \neq 0$. Since $wt\frac{\partial f_{k+1}^3}{\partial x_1} = 2 = wt\frac{\partial f_{k+1}^3}{\partial x_3}$, there exists a constant r_1 such that $\frac{\partial f_{k+1}^3}{\partial x_3} = r_1\frac{\partial f_{k+1}^3}{\partial x_1}$. By Lemma 5.1, we have $k = 2$ and

$$f_3^3 = c_1(x_1 + r_1 x_3)^3$$

where c_1 is a nonzero constant. It is easy to see that the three dimensional submodule of I is of the following form.

$$(3) = \langle (x_1 + r_1x_3)^2, (x_1 + r_1x_3)(x_2 + r_1x_4), (x_2 + r_1x_4)^2 \rangle$$

$wr\frac{\partial f_3^1}{\partial x_5} = 1$ implies $\frac{\partial f_3^1}{\partial x_5} = 0$. This means that f_3^1 does not involve x_5 variabble. As $wt\frac{\partial f_3^1}{\partial x_2} = 2 = wt\frac{\partial f_3^1}{\partial x_4}$, in view of Lemma 5.1, f_3^1 is a polynomial in x_1, x_3 and $r_2x_2 + r_3x_4$, where r_2, r_3 are constants. Then

$$f_3^1 = (b_1x_1^2 + b_2x_1x_3 + b_3x_3^2)(r_2x_2 + r_3x_4)$$

for some constants b_1, b_2 and b_3. If $f_3^1 \neq 0$, then by considering $\frac{\partial f_3^1}{\partial x_2}$ or $\frac{\partial f_3^1}{\partial x_4}$, we see that $b_1x_1^2 + b_2x_1x_3 + b_3x_3^2$ is an element of weight 2 in I. Hence $b_1x_1^2 + b_2x_1x_3 + b_3x_3^2$ is a constant multiple of $(x_1 + rx_3)^2$. So

$$f_3^1 = b_0(x_1 + r_1x_3)^2(r_2x_2 + r_3x_4)$$

where b_0 is a nonzero constant. Observe that $\frac{\partial f_3^1}{\partial x_1} = 2b_0(x_1 + r_1x_3)(r_2x_2 + r_3x_4)$ is an element of weight 0 in I. Obviously $(x_1 + r_1x_3)(r_2x_2 + r_3x_4)$ is not an $sl(2, \mathbf{C})$ invariant polynomial. Hence $(x_1 + r_1x_3)(r_2x_2 + r_3x_4)$ is a constant multiple of $(x_1 + r_1x_3)(x_2 + r_1x_4)+$ invariant polynomial. $X_-[(x_1 + r_1x_3)(r_2x_2 + r_3x_4)] = (x_2 + r_1x_4)(r_2x_2 + r_3x_4)$ is a constant multiple of $(x_1 + r_1x_4)(x_2 + r_1x_4)$. It follows that

$$f_{k+1}^1 = c_2(x_1 + r_1x_3)^2(x_2 + r_1x_4).$$

As $wt\frac{\partial f_3^0}{\partial x_1} = -1 = wt\frac{\partial f_3^0}{\partial x_3}$ and $wt\frac{\partial f_3^0}{\partial x_2} = 1 = wt\frac{\partial f_3^0}{\partial x_4}$, we have $\frac{\partial f_3^0}{\partial x_1} = 0 = \frac{\partial f_3^0}{\partial x_3} = \frac{\partial f_3^0}{\partial x_2} = \frac{\partial f_3^0}{\partial x_4}$. So f_3^0 involves only x_5 variable. It follows that $f_3^0 = c_3x_5^3$ and

$$\begin{aligned} f &= f_3^3 + f_3^1 + f_3^0 + f_3^{-1} + f_3^{-3} \\ &= c_1(x_1 + r_1x_3)^3 + c_2(x_1 + r_1x_3)^2(x_2 + r_1x_4) + c_3x_5^3 + c_4(x_1 + r_1x_3)(x_2 + r_1x_4)^2 \\ &\quad + c_5(x_2 + r_1x_4)^3. \end{aligned}$$

This implies that $\dim I \le 3$, which contradicts to our hypothesis $I = (3) \oplus (1) \oplus (1)$.

For $i = 1$.

Here we have $f = f^1_{k+1} + f^0_{k+1} + f^{-1}_{k+1}$. f^1_{k+1} does not involve x_5 variable because weight of $\frac{\partial f^1_{k+1}}{\partial x_5}$ is one. Suppose on the contrary that $f^1_{k+1} \ne 0$. We shall assume without loss of generality that $\frac{\partial f^1_{k+1}}{\partial x_2} \ne 0$. Since $wt\frac{\partial f^1_{k+1}}{\partial x_2} = 2 = wt\frac{\partial f^1_{k+1}}{\partial x_4}$, we have $\frac{\partial f^1_{k+1}}{\partial x_4} = r_4\frac{\partial f^1_{k+1}}{\partial x_2}$ where r_4 is a constant. In view of Lemma 5.1, f^1_{k+1} is a polynomial in x_1, $(x_2 + r_4x_4)$ and x_3

$$f^1_{k+1} = \sum_{\alpha=0}^{\frac{k}{2}+1} a_\alpha x_1^{\frac{k}{2}+1-\alpha} x_3^\alpha (x_2 + r_4x_4)^{\frac{k}{2}}$$

$$\frac{\partial f^1_{k+1}}{\partial x_2} = \sum_{\alpha=0}^{\frac{k}{2}+1} (\frac{k}{2}) a_\alpha x_1^{\frac{k}{2}-\alpha+1} x_3^\alpha (x_2 + r_4x_4)^{\frac{k}{2}-1}$$

$$X_+(\frac{\partial f^1_{k+1}}{\partial x_2}) = \sum_{\alpha=0}^{\frac{k}{2}+1} \frac{k}{2}(\frac{k}{2}-1) a_\alpha x_1^{\frac{k}{2}-\alpha+1} x_3^\alpha (x_2 + r_4x_4)^{\frac{k}{2}-2}(x_1 + r_4x_3).$$

Since $wt(X_+\frac{\partial f^1_{k+1}}{\partial x_2}) = 4$, we have $X_+\frac{\partial f^1_{k+1}}{\partial x_2} = 0$. It follows that $a_\alpha = 0$ for all $0 \le \alpha \le \frac{k}{2} + 1$. Thus $f^1_{k+1} = 0$.

Similarly we can prove $f^{-1}_{k+1} = 0$.

For $i = 0$.

In this case $f = f^0_{k+1}$. It follows easily as above that $f = cx_5^{k+1}$. This implies $\dim I = 1$, which contradicts to our hypothesis. Hence Case 4 cannot occur.

Case 5. $I = (2) \oplus (2) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials of degree $k+1$ and weights $-1, 0$ and 1.

For $|i| \ge 3$

$$|wt\frac{\partial f^i_{k+1}}{\partial x_j}| \ge 2 \quad \text{for all } 1 \le j \le 5$$

$$\Rightarrow \quad \frac{\partial f^i_{k+1}}{\partial x_j} = 0 \qquad \text{for all } 1 \le j \le 5$$

$$\Rightarrow \quad f^i_{k+1} = 0.$$

For $i = 2$

$$wt\frac{\partial f^2_{k+1}}{\partial x_2} = 3 = wt\frac{\partial f^2_{k+1}}{\partial x_4}, \quad wt\frac{\partial f^2_{k+1}}{\partial x_5} = 2$$

$$\Rightarrow \quad \frac{\partial f^2_{k+1}}{\partial x_2} = \frac{\partial f^2_{k+1}}{\partial x_4} = 0 = \frac{\partial f^2_{k+1}}{\partial x_5}$$

$\Rightarrow$ f^2_{k+1} involves only x_1 and x_3 variables

$\Rightarrow$ $f^2_{k+1} = c_1x_1^2 + c_2x_1x_3 + c_3x_3^2$ where c_1, c_2, c_3 are constants

$\Rightarrow$ $f^2_{k+1} = 0$ because $k \ge 2$.

Similarly we can prove that $f^{-2}_{k+1} = 0$.

For $i = 1$

$$wt\frac{\partial f^1_{k+1}}{\partial x_2} = 2 = wt\frac{\partial f^1_{k+1}}{\partial x_4}$$

$$\Rightarrow \quad \frac{\partial f^1_{k+1}}{\partial x_2} = 0 = \frac{\partial f^1_{k+1}}{\partial x_4}$$

$\Rightarrow$ f^1_{k+1} involves only x_1, x_3 and x_5 variables.

Since $wt\frac{\partial f^1_{k+1}}{\partial x_1} = 0 = wt\frac{\partial f^1_{k+1}}{\partial x_3}$, there exist constants r_1 and r_2 such that

$$f^1_{k+1} = (r_1x_1 + r_2x_3)x_5^k.$$

Suppose $f^1_{k+1} \neq 0$. Without loss of generality, we shall assume $r_1 \neq 0$. Then $f^1_{k+1} = r_1(x_1 + rx3)x_5^k$ where $r = r_2/r_1$. It follows that

(1) $= \langle x_5^k \rangle \subseteq I$

(2) $= \langle (x_1 + rx_3)x_5^{k-1}, (x_2 + rx_4)x_5^{k-1} \rangle \subseteq I.$

If f^0_{k+1} were zero, then

$$\begin{aligned} f &= f^1_{k+1} + f^{-1}_{k+1} \\ &= r_1(x_1 + rx_3)x_5^k + (r_3x_2 + r_4x_4)x_5^k. \end{aligned}$$

This would imply $\dim I \leq 2$, which contradicts to our hypothesis $I = (2) \oplus (2) \oplus (1)$. Hence we shall assume that $f_{k+1}^0 \neq 0$. Write

$$f_{k+1}^0 = \sum_{i=0}^{k+1} \phi_i(x_1, x_2, x_3, x_4) x_5^{k+1-i}$$

where $\phi_i(x_1, x_2, x_3, x_4)$ is a homogeneous polynomial in x_1, x_2, x_3 and x_4 variables of degree i and weight 0. We claim that $\phi_i(x_1, x_2, x_3, x_4) = 0$ for all $1 \leq i \leq k$.

$$\begin{aligned}\frac{\partial f_{k+1}^0}{\partial x_5} = (k+1)x_5^k \quad & +k\phi_1(x_1, x_2, x_3, x_4)x_5^{k-1} \\ & +(k-1)\phi_2(x_1, x_2, x_3, x_4)x_5^{k-2} + \dots \\ & +2\phi_{k-1}(x_1, x_2, x_3, x_4)x_5 + \phi_k(x_1, x_2, x_3, x_4)\end{aligned}$$

is an element of weight 0 in I. Therefore $\frac{\partial f_{k+1}^0}{\partial x_5}$ is a constant multiple of x_5^k. This implies that $\phi_i(x_1, x_2, x_3, x_4) = 0$, for all $1 \leq i \leq k$ and consequently

$$f_{k+1}^0 = c x_5^{k+1} + \phi_{k+1}(x_1, x_2, x_3, x_4)$$

where c is a constant. Observe that

$$\frac{\partial f_{k+1}^0}{\partial x_1} = \frac{\partial \phi_{k+1}}{\partial x_1}(x_1, x_2, x_3, x_4)$$

$$\frac{\partial f_{k+1}^0}{\partial x_2} = \frac{\partial \phi_{k+1}}{\partial x_2}(x_1, x_2, x_3, x_4)$$

$$\frac{\partial f_{k+1}^0}{\partial x_3} = \frac{\partial \phi_{k+1}}{\partial x_3}(x_1, x_2, x_3, x_4)$$

$$\frac{\partial f_{k+1}^0}{\partial x_4} = \frac{\partial \phi_{k+1}}{\partial x_4}(x_1, x_2, x_3, x_4)$$

are elements in I but not in $(2) = \langle (x_1 + rx_3)x_5^{k-1}, (x_2 + rx_4)x_5^{k-1} \rangle$. Since $wt\frac{\partial f_{k+1}^0}{\partial x_1} = -1 = wt\frac{\partial f_{k+1}^0}{\partial x_3}$ and $wt\frac{\partial f_{k+1}^0}{\partial x_2} = 1 = wt\frac{\partial f_{k+1}^0}{\partial x_4}$; and $\frac{\partial f_{k+1}^0}{\partial x_i}$, $1 \leq i \leq 4$, are independent of x_5 variable, there are constants r_5, r_6, r_7 and r_8 such that $r_5\frac{\partial f^0_{k+1}}{\partial x_3} = r_6\frac{\partial f^0_{k+1}}{\partial x_1}$ and $r_7\frac{\partial f^0_{k+1}}{\partial x_4} = r_8\frac{\partial f^0_{k+1}}{\partial x_2}$. In view of Lemma 5.1 we can write

$$f^0_{k+1} = cx_5^{k+1} + (r_5x_1 + r_6x_3)^{\frac{k+1}{2}}(r_7x_2 + r_8x_4)^{\frac{k+1}{2}}.$$

Firstly we claim that $(r_5x_1 + r_6x_3)^{k+1/2}(r_7x_2 + r_8x_4)^{k+1/2} \neq 0$. If this were zero, then $f^0_{k+1} = cx_5^{k+1}$ and

$$\begin{aligned} f &= f^1_{k+1} + f^0_{k+1} + f^{-1}_{k+1} \\ &= (r_1x_1 + r_2x_3)x_5^k + cx_5^{k+1} + (r_3x_2 + r_4x_4)x_5^k. \end{aligned}$$

This would imply $\dim I \leq 3$ which contradicts to our hypothesis $I = (2) \oplus (2) \oplus (1)$. Our first claim is proved. Secondly, by considering $\frac{\partial f^0_{k+1}}{\partial x_1}$ or $\frac{\partial f^0_{k+1}}{\partial x_3}$, we see that

$$(r_5x_1 + r_6x_3)^{\frac{k+1}{2}}(r_7x_2 + r_8x_4)^{\frac{k+1}{2}}$$

is a nonzero element in I. By applying $X_-^{(k-1)/2}$ to the above element, we get

$$(r_5x_2 + r_6x_4)^{\frac{k-1}{2}}(r_7x_2 + r_8x_4)^{\frac{k+1}{2}} \in I.$$

Since $X_-[(r_5x_2+r_6x_4)^{(k-1)/2}(r_7x_2+r_8x_4)^{(k+1)/2}] = 0$, we have an irreducible submodule of dimension $k+1$ of the following form.

$\langle (r_5x_2+r_6x_4)^{\frac{k-1}{2}}(r_7x_2+r_8x_4)^{\frac{k+1}{2}},\ X_+[(r_5x_2+r_6x_4)^{\frac{k-1}{2}}(r_7x_2+r_8x_4)^{\frac{k+1}{2}}], \ldots, , X_+^k[(r_5x_2+r_6x_4)^{\frac{k-1}{2}}(r_7x_2 + r_8x_4)^{\frac{k+1}{2}}]\rangle$.

As $I = (2)_\oplus(2) \oplus (1)$, we have $k+1 \leq 2$. This implies $k \leq 1$, which contradicts to our hypothesis $k \geq 2$. Hence $f^1_{k+1} = 0$.

Similarly we can prove that $f^{-1}_{k+1} = 0$.

Case 6. $I = (2) \oplus (1) \oplus (1) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials of degree $k+1$ and weights $-1, 0$, and 1.

Similar argument as Case 5, we can show that

$$f = f_{k+1}^{-1} + f_{k+1}^{0} + f_{k+1}^{1}.$$

For $i = 1$

$$\begin{aligned} & wt\frac{\partial f_{k+1}^1}{\partial x_2} = 2 = wt\frac{\partial f_{k+1}^1}{\partial x_4} \\ \Rightarrow \quad & \frac{\partial f_{k+1}^1}{\partial x_2} = 0 = \frac{\partial f_{k+1}^1}{\partial x_4} \\ \Rightarrow \quad & f_{k+1}^1 \text{ involves only } x_1, x_3 \text{ and } x_5 \text{ variables.} \\ \Rightarrow \quad & f_{k+1}^1 = (r_1x_1 + r_2x_3)x_5^k \text{ where } r_1, r_2 \text{ are constants.} \end{aligned}$$

Suppose $f_{k+1}^1 \neq 0$. Without loss of generality, we shall assume $r_1 \neq 0$. Then $f_{k+1}^1 = r_1(x_1 + rx_3)x_5^k$ where $r = r_2/r_1$. It follows that

$(1) = \langle x_5^k \rangle \subseteq I$

$(2) = \langle (x_1 + rx_3)x_5^{k-1}, (x_2 + rx_4)x_5^{k-1} \rangle \subseteq I.$

If f_{k+1}^0 were zero, then

$$\begin{aligned} f &= f_{k+1}^1 + f_{k+1}^{-1} \\ &= r_1(x_1 + rx_3)x_5^k + d(x_2 + rx_4)x_5^k, \text{ where } d \text{ is a constant.} \end{aligned}$$

This would imply $\dim I \leq 2$, which contradicts to our hypothesis $I = (2)\oplus(1)\oplus(1)\oplus(1)$. Hence we shall assume that $f_{k+1}^0 \neq 0$. Write

$$\begin{aligned} f_{k+1}^0 = cx_5^{k+1} \quad &+ \phi_1(x_1, x_2, x_3, x_4)x_5^k + \phi_2(x_1, x_2, x_3, x_4)x_5^{k-1} + \dots \\ &+ \phi_k(x_1, x_2, x_3, x_4)x_5 + \phi_{k+1}(x_1, x_2, x_3, x_4) \end{aligned}$$

where $\phi_i(x_1, x_2, x_3, x_4)$ is a homogeneous polynomial of degree i and weight 0 in x_1, x_2, x_3, x_4 variables and c is a constant. We claim that $\phi_i(x_1, x_2, x_3, x_4) = 0$ for $i \neq 2$. This follows from the following observations.

$$\frac{\partial f_{k+1}^0}{\partial x_1} = \frac{\partial \phi_1}{\partial x_1}(x_1,x_2,x_3,x_4)x_5^k + \frac{\partial \phi_2}{\partial x_1}(x_1,x_2,x_3,x_4)x_5^{k-1} + \ldots$$
$$+\frac{\partial \phi_k}{\partial x_1}(x_1,x_2,x_3,x_4)x_5 + \frac{\partial \phi_{k+1}}{\partial x_1}(x_1,x_2,x_3,x_4)$$
$$\frac{\partial f_{k+1}^0}{\partial x_3} = \frac{\partial \phi_1}{\partial x_3}(x_1,x_2,x_3,x_4)x_5^k + \frac{\partial \phi_2}{\partial x_3}(x_1,x_2,x_3,x_4)x_5^{k-1} + \ldots$$
$$+\frac{\partial \phi_k}{\partial x_3}(x_1,x_2,x_3,x_4)x_5 + \frac{\partial \phi_{k+1}}{\partial x_3}(x_1,x_2,x_3,x_4)$$

are elements of weight -1 in I. Hence they are constant multiple of $(x_2 + rx_4)x_5^{k-1}$.

Similarly

$$\frac{\partial f_{k+1}^0}{\partial x_2} = \frac{\partial \phi_1}{\partial x_2}(x_1,x_2,x_3,x_4)x_5^k + \frac{\partial \phi_2}{\partial x_2}(x_1,x_2,x_3,x_4)x_5^{k-1} + \ldots$$
$$+\frac{\partial \phi_k}{\partial x_2}(x_1,x_2,x_3,x_4)x_5 + \frac{\partial \phi_{k+1}}{\partial x_2}(x_1,x_2,x_3,x_4)$$
$$\frac{\partial f_{k+1}^0}{\partial x_4} = \frac{\partial \phi_1}{\partial x_4}(x_1,x_2,x_3,x_4)x_5^k + \frac{\partial \phi_2}{\partial x_4}(x_1,x_2,x_3,x_4)x_5^{k-1} + \ldots$$
$$+\frac{\partial \phi_k}{\partial x_4}(x_1,x_2,x_3,x_4)x_5 + \frac{\partial \phi_{k+1}}{\partial x_4}(x_1,x_2,x_3,x_4)$$

are elements of weight 1 in I. hence they are constant multiple of $(x_1 + rx_3)x_5^{k-1}$. It follows that

$$\frac{\partial \phi_i}{\partial x_j}(x_1,x_2,x_3,x_4) = 0 \text{ for all } i \neq 2 \text{ and } 1 \leq j \leq 4$$
$$\Rightarrow \quad \phi_i(x_1,x_2,x_3,x_4) = 0 \text{ for all } i \neq 2.$$
$$\Rightarrow \quad f_{k+1}^0 = cx_5^{k+1} + \phi_2(x_1,x_2,x_3,x_4)x_5^{k-1}.$$

$\frac{\partial f_{k+1}^0}{\partial x_1} = \frac{\partial \phi_2}{\partial x_1}(x_1,x_2,x_3,x_4)x_5^{k-1}$ and $\frac{\partial f_{k+1}^0}{\partial x_3} = \frac{\partial \phi_2}{\partial x_3}(x_1,x_2,x_3,x_4)x_5^{k-1}$ are constant multiple of $(x_2 + rx_4)x_5^{k-1}$ because they have the same weight -1. Similarly $\frac{\partial f_{k+1}^0}{\partial x_2} = \frac{\partial \phi_2}{\partial x_1}(x_1,x_2,x_3,x_4)x_5^{k-1}$ and $\frac{\partial f_{k+1}^0}{\partial x_4} = \frac{\partial \phi_2}{\partial x_4}(x_1,x_2,x_3,x_4)x_5^{k-1}$ are constant multiple of $(x_1 + rx_3)x_5^{k-1}$. In view of Lemma 5.1, thre are constants r_3, r_4, r_5 and r_6 such that

$$\phi_2 = (r_3x_1 + r_4x_3)(r_5x_2 + r_6x_4)$$

and

$$f^0_{k+1} = cx_5^{k+1} + (r_3x_1 + r_4x_3)(r_5x_2 + r_6x_4)x_5^{k-1}.$$

By using the fact that $\frac{\partial f^0_{k+1}}{\partial x_1}$ and $\frac{\partial f^0_{k+1}}{\partial x_3}$ are constant multiple of $(x_2 + rx_4)x_5^{k-1}$ while $\frac{\partial f^0_{k+1}}{\partial x_2}$ and $\frac{\partial f^0_{k+1}}{\partial x_4}$ are constant multiples of $(x_1 + rx_3)x_5^{k-1}$, we see that there exits a constant b such that

$$\begin{aligned} f^0_{k+1} &= cx_5^{k+1} + b(x_1 + rx_3)(x_2 + rx_4)x_5^{k-1} \\ f &= f^{-1}_{k+1} + f^0_{k+1} + f^1_{k+1} \\ &= d(x_2 + rx_4)x_5^k + cx_5^{k+1} + b(x_1 + rx_3)(x_2 + rx_4)x_5^{k-1} + r_1(x_1 + rx_3)x_5^k. \end{aligned}$$

This implies that $\dim I \leq 3$, which contradicts to our hypothesis $\dim I = 5$. Hence $f^1_{k+1} = 0$.

Similarly we can prove that $f^{-1}_{k+1} = 0$.

For $i = 0$.

In this case $f = f^0_{k+1}$. Since

$$wt\frac{\partial f}{\partial x_1} = -1 = wt\frac{\partial f}{\partial x_3} \quad wt\frac{\partial f}{\partial x_2} = 1 = wt\frac{\partial f}{\partial x_4}$$

$\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_3} \rangle$ and $\langle \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_4} \rangle$ are complex vector spaces of dimension at most one. It follows that $\dim I \leq 3$. This contradicts to our hypothesis $I = (2) \oplus (1) \oplus (1) \oplus (1)$.

Hence Case 6 cannot occur.

Case 7. $I = (1) \oplus (1) \oplus (1) \oplus (1) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials of degree $k+1$ and weight 0.

Similar argument as Case 5, we can show that

$$f = f^{-1}_{k+1} = f^0_{k+1} + f^1_{k+1}.$$

For $i = 1$

$$wt\frac{\partial f^1_{k+1}}{\partial x_2} = 2 = wt\frac{\partial f^1_{k+1}}{\partial x_4} \quad wt\frac{\partial f^1_{k+1}}{\partial x_5} = 1$$

$$\Rightarrow \quad \frac{\partial f^1_{k+1}}{\partial x_2} = 0 = \frac{\partial f^1_{k+1}}{\partial x_4} = \frac{\partial f^1_{k+1}}{\partial x_5}$$

$\Rightarrow$ f^1_{k+1} involves only x_1 and x_3 variables.

$\Rightarrow$ $f^1_{k+1} = c_1x_1 + c_2x_3$ where c_1, c_3 are constant

$\Rightarrow$ $k = 0$.

This contradicts to our hypothesis that $k \geq 2$. Hence $f^1_{k+1} = 0$.

Similarly we can prove $f^{-1}_{k+1} = 0$.

For $i = 0$.

In this case $f = f^0_{k+1}$.

$$wt\frac{\partial f}{\partial x_1} = -1 = wt\frac{\partial f}{\partial x_3}, \quad wt\frac{\partial f}{\partial x_2} = 1 = wt\frac{\partial f}{\partial x_4}.$$

$$\Rightarrow \quad \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_3} = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_4} = 0$$

$$\Rightarrow \quad \dim I \leq 1.$$

This contradicts to our hypothesis. Hence Case 7 cannot occur. Q.E.D.

Lemma 5.3. With the same hypothesis as Lemma 5.3; if I is a $sl(2,\mathbf{C})$-submodule of dimesnion 4, then $I = (2) \oplus (2)$ and f is a homogeneous polynomial of weight 0 in x_1, x_2, x_3 and x_4 variables.

Proof. Case 1. $I = (4)$.

This case canot occur by the same argument as Case 2 in the proof of Lemma 5.2.

Case 2. $I = (3) \oplus (1)$.

Similar argument as Case 4 in the proof of Lemma 5.2 shows that this case cannot occur.

Case 3. $I = (2) \oplus (2)$.

Similar argument as Case 5 in the proof of Lemma 5.2 shows that $f = f_{k+1}^0$. Since $wt\frac{\partial f}{\partial x_5} = 0$, we have $\frac{\partial f}{\partial x_5} = 0$. Hence f is a homogeneous polynomial in x_1, x_2, x_3, x_4 variables.

Case 4. $I = (2) \oplus (1) \oplus (1)$.

Similar argument as Case 6 in the proof of Lemma 5.2 shows that this case cannot occur.

Case 5. $I = (1) \oplus (1) \oplus (1) \oplus (1)$.

Similar argument as in Case 7 in the proof of Lemma 5.2 shows that this case cannot occur.

Lemma 5.4. With the same hypothesis as Lemma 5.2; if dimension of I is 3, then I cannot be a $sl(2, \mathbf{C})$-submodule.

Proof. Assume on th e contrary that I is a $sl(2, \mathbf{C})$-submodule. We shall provide a contradiction.

Case 1. $I = (3)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $-2, 0$ and 2. Write

$$f = \sum_{i=-\infty}^{\infty} f_{k+1}^i$$

where f_{k+1}^i is a homogeneous polynomial of degree $k+1$ and weight i.

Similar argument as Case 4 in the proof of Lemma 5.2 shows that $f^i_{k+1} = 0$ for $i = \pm 2$, and $|i| \geq 4$. The proof of $f^i_{k+1} = 0$ for $i = \pm 3$, ± 1 is the same as those given in the Case 4 in the proof of Lemma 5.2 except that we need to observe $f^0_3 = 0$. f^0_3 is zero because $I = (3)$ does not contain $sl(2,\mathbf{C})$ invariant polynomial. Hence this case cannot occur.

Case 2. $I = (2) \oplus (1)$.

Similar argument as in the Case 5 in the proof of Lemma 5.2 shows that

$$f = f^1_{k+1} + f^0_{k+1} + f^{-1}_{k+1}.$$

We shall follow the notation and the argument in Case 6 in the proof of Lemma 5.2. If $f^1_{k+1} \neq 0$, then $f^1_{k+1} = r_1(x_1 + rx_3)x_5^k$ and

$$\begin{aligned} I &= (2) \oplus (1) \\ &= \langle (x_1 + rx_3)x_5^{k-1}, (x_2 + rx_4)x_5^{k-1} \rangle \oplus \langle x_5^k \rangle. \end{aligned}$$

We also have

$$f^0_{k+1} = cx_5^{k+1} + b(x_1 + rx_3)(x_2 + rx_4)x_5^{k-1}.$$

Since $\frac{\partial f^0_{k+1}}{\partial x_5} = (k+1)cx_5^k = (k-1)b(x_1 + rx_3)(x_2 + rx_4)x_5^{k-2}$ is an element of weight 0 in I, $\frac{\partial f^0_{k+1}}{\partial x_5}$ is a constant multiple of x_5^k. This implies that $b = 0$. Hence $f^0_{k+1} = cx_5^{k+1}$ and

$$\begin{aligned} f &= f^1_{k+1} + f^0_{k+1} + f^{-1}_{k+1} \\ &= d(x_2 + rx_4)x_5^k + cx_5^{k+1} + r_1(x_1 + rx_3)x_5^k. \end{aligned}$$

It follows that $\dim I \leq 2$, which contradicts to our hypothesis $\dim I = 3$. Hence $f^1_{k+1} = 0$. Similarly we can prove $f^{-1}_{k+1} = 0$.

Thus $f = f^0_{k+1}$. Observe that $wt\frac{\partial f}{\partial x_1} = -1 = wt\frac{\partial f}{\partial x_3}$ and $wt\frac{\partial f}{\partial x_2} = 1 = wt\frac{\partial f}{\partial x_4}$. So $\langle\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_3}\rangle$ and $\langle\frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_4}\rangle$ are complex vector spaces of dimension one. Without loss of generality, we shall assume that $\frac{\partial f}{\partial x_1} \neq 0 \neq \frac{\partial f}{\partial x_2}$. By Lemma 5.1, there are constants r_2, and r_3 such that

$$f = \sum_{\alpha=0}^{[\frac{k+1}{2}]} a_\alpha(x_1 + r_2x_3)^\alpha(x_2 + r_3x_4)^\alpha x_5^{k+1-2\alpha}$$

$$\Rightarrow \qquad \frac{\partial f}{\partial x_1} = \sum_{\alpha=1}^{[\frac{k+1}{2}]} \alpha a_\alpha(x_1 + r_2x_3)^\alpha(x_2 + r_3x_4)^\alpha x_5^{k+1-2\alpha}$$

$$\frac{\partial f}{\partial x_5} = \sum_{\alpha=0}^{[\frac{k+1}{2}]-1} (k+1-2\alpha)a_\alpha(x_1 + r_2x_3)^\alpha(x_2 + r_3x_4)^\alpha x_5^{k+1-2\alpha}$$

$$\Rightarrow \quad X_-(\frac{\partial f}{\partial x_1}) = \sum_{\alpha=2}^{[\frac{k+1}{2}]} \alpha(\alpha-1)a_\alpha(x_1 + r_2x_3)^{\alpha-2}(x_2 + r_2x_4)(x_2 + r_3x_4)^\alpha x_5^{k+1-2\alpha}$$

$$X_-(\frac{\partial f}{\partial x_5}) = \sum_{\alpha=2}^{[\frac{k+1}{2}-1]} \alpha(k+1-2\alpha)a_\alpha(x_1 + r_2x_3)^{\alpha-1}(x_2 + r_2x_4)(x_2 + r_3x_4)^\alpha x_5^{k-2\alpha}.$$

Since $wt\, X_-(\frac{\partial f}{\partial x_1}) = -3$, $wt\, X_-(\frac{\partial f}{\partial x_5}) = -2$, we have $X_-(\frac{\partial f}{\partial x_1}) = 0 = X_-(\frac{\partial f}{\partial x_5})$. Thus $a_1 = a_2 = \ldots = a_{[\frac{k+1}{2}]} = 0$, and hence $f = a_0x_5^{k+1}$. It follows that $\dim I \leq 1$. This contradicts to our hypothesis $I = (2) \oplus (1)$. Therefore we conclude that Case 2 cannot occur.

Case 3. $I = (1) \oplus (1) \oplus (1)$.

Similar argument as in Case 7 in the proof of Lemma 5.2 shows that this case cannot occur. Q.E.D.

Lemma 5.5. With the same hypothesis as Lemma 5.2; if dimension of I is 2, then I cannot be a $sl(2, \mathbf{C})$-submodule.

Proof. Assume that I is a $sl(2, \mathbf{C})$-submodule. We shall provide a contradiction.

Case 1. $I = (2)$.

Similar argument as in Case 5 in the proof of Lemma 5.2 shows that

$$f = f_{k+1}^{1} + f_{k+1}^{0} + f_{k+1}^{-1}.$$

We shall follow the notation and the argument in Case 6 in the proof of Lemma 5.2. If $f_{k+1}^{1} \neq 0$, then

$$I \supseteq (2) \oplus (1) = \langle (x_1 + rx_3)x_5^{k-1}, (x_2 + rx_4)x_5^{k-1} \rangle \oplus \langle x_5^k \rangle.$$

This contradicts to our hypothesis $I = (2)$. Thus $f_{k+1} = 0$.

Similarly we can prove that $f_{k+1}^{-1} = 0$.

The same argument of Case 2 in the proof of Lemma 5.4 shows that $f = f_{k+1}^{0} = 0$. This contradicts to our hypothesis $I = (2)$.

Case 2. $I = (1) \oplus (1)$.

The same argument as in Case 7 in the proof of Lemma 5.2 provides a contradiction. Q.E.D.

Lemma 5.6. With the same hypothesis as Lemma 5.2; if I is a $sl(2,\mathbf{C})$-submodule of dimension one, then $f = cx_5^{k+1}$ where c is a nonzero constant.

Proof. This follows from the argument given in Case 7 in the proof of Lemma 5.2. Q.E.D.

Proposition 5.7. Suppose $sl(2,\mathbf{C})$ acts on the space of homogeneous polynomials of

degree $k \geq 2$ in x_1, x_2, x_3, x_4 and x_5 variables via

$$\tau = x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} - x_4 \frac{\partial}{\partial x_4}$$

$$X_+ = x_1 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_4}$$

$$X_- = x_2 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_3}.$$

Suppose the weight of x_i is given by the corresponding coefficient in the expression of τ above i.e.

$$wt(x_1) = 1,\ wt(x_2) = -1,\ wt(x_3) = 1,\ wt(x_4) = -1,\ wt(x_5) = 0.$$

Let $I = \langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}$ and $\frac{\partial f}{\partial x_5} \rangle$, be the complex vector subspace sapnned by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}$ and $\frac{\partial f}{\partial x_5}$, where f is a homogeneous polynomial of degree $k+1$. If I is a $sl(2, \mathbf{C})$-submodule, then either

(i) f is a homogeneous polynomial in x_1, x_2, x_3, x_4 and x_5 variables of weight 0 and $I = (2) \oplus (2) \oplus (1)$, or

(ii) f is a homogeneous polynomial in x_1, x_2, x_3, and x_4 variables of weight 0 and $I = (2) \oplus (2)$, or

(iii) $f = cx_5^{k+1}$ where c is a nonzero constant. In this case, $I = (1)$.

Proof. This is a consequence of Lemmas 5.2, 5.3, 5.4, 5.5 and 5.6. Q.E.D.

§6. $sl(2,\mathbf{C})$-ACTION (1.2) IN M_5^k

Lemma 6.1. Let f be polynomial in x_1, x_2, x_3, x_4 and x_5 variables. Suppose $\frac{\partial f}{\partial x_4} = r_1 \frac{\partial f}{\partial x_2}$ and $\frac{\partial f}{\partial x_5} = r_2 \frac{\partial f}{\partial x_2}$. Then there exists a polynomial $g(y_1, y_2, y_3)$ such that $f(x_1, x_2, x_3, x_4, x_5) = g(x_1, x_2 + r_1 x_4 + r_2 x_5, x_3)$.

Proof. Let

$$y_1 = x_1$$

$$y_2 = x_2 + r_1 x_4 + r_2 x_5$$

$$y_3 = x_3$$

$$y_4 = x_4$$

$$y_5 = x_5.$$

Set $g(y_1, y_2, y_3, y_4, y_5) = f(y_1, y_2 - r_1 y_4 - r_2 y_5, y_3, y_4, y_5)$. We claim that g depends only on y_1, y_2 and y_3 variables.

$$\begin{aligned}
\frac{\partial g}{\partial y_4} &= \frac{\partial f}{\partial x_2}(y_1, y_2 - r_1 y_4 - r_2 y_5, y_3, y_4, y_5) \cdot \frac{\partial x_2}{\partial y_4} + \frac{\partial f}{\partial x_4}(y_1, y_2 - r_1 y_4 - r_2 y_5, y_3, y_4, y_5) \\
&= -r_1 \frac{\partial f}{\partial x_2}(y_1, y_2 - r_1 y_4 - r_2 y_5, y_3, y_4, y_5) + \frac{\partial f}{\partial x_4}(y_1, y_2 - r_1 y_4 - r_2 y_5, y_3, y_4, y_5) \\
&= 0.
\end{aligned}$$

Similarly $\frac{\partial g}{\partial y_5} = 0$. Thus g depends only on y_1, y_2 and y_3 variables. Q.E.D.

Lemma 6.2. Suppose $sl(2,\mathbf{C})$ acts on the space of homogeneous polynomials of degree

$k \geq 2$ in x_1, x_2, x_3, x_4 and x_5 variables via (1.2)

$$\tau = 2x_1 \frac{\partial}{\partial x_1} - 2x_3 \frac{\partial}{\partial x_3}$$

$$X_+ = 2x_1 \frac{\partial}{\partial x_2} + 2x_2 \frac{\partial}{\partial x_3}$$

$$X_- = x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2}.$$

Suppose the weight of x_i is given by the corresponding coefficient in the expression of τ above i.e.

$$wt(x_1) = 2,\ wt(x_2) = 0,\ wt(x_3) = -2,\ wt(x_4) = 0,\ wt(x_5) = 0.$$

Let I be the complex vector subspace of dimension 5 spanned by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}$ and $\frac{\partial f}{\partial x_5}$ where f is a homogeneous polynomial of degree $k+1$. If I is an $sl(2, \mathrm{C})$ module then either

(i) $\quad I = (3) \oplus (1) \oplus (1)$and f is weight 0, or

(ii) $\quad I = (3) \oplus (1) \oplus (1)$

$$= \langle x_1(r_2x_4 + r_3x_5)^{k-1}, x_2(r_2x_4 + r_3x_5)^{k-1}, x_3(r_2x_4 + r_3x_5)^{k-1} \rangle$$
$$\oplus \langle (r_2x_4 + r_3x_5)^k \rangle$$
$$\oplus \langle (k-1)c_3(x_2^2 - 2x_1x_3)(r_2x_4 + r_3x_5)^{k-2} + kc_7(r_2x_4 + r_3x_5)^{k-1} \cdot x_5 \rangle$$

and

$$f = x_1(r_2x_4 + r_3x_5)^k + c_3(x_2^2 - 2x_1x_3)(r_2x_4 + r_3x_5)^{k-1} + c_7(r_2x_4 + r_3x_5)^k \cdot x_5$$
$$+ c_8(r_2x_4 + r_3x_5)^{k+1} + c_4x_2(r_2x_4 + r_3x_5)^k + c_0x_3(r_2x_4 + r_3x_5)^k$$

where $r_2c_3^3c_7 \neq 0$, or

(iii) $\quad I = (3) \oplus (1) \oplus (1)$

$$= \langle x_1(r_2x_4 + r_3x_5)^{k-1}, x_2(r_2x_4 + r_3x_5)^{k-1}, x_3(r_2x_4 + r_3x_5)^{k-1} \rangle$$

$$\oplus\langle (r_2x_4+r_3x_5)^k\rangle$$

$$\oplus\langle (k-1)c_3(x_2^2-2x_1x_3)(r_2x_4+r_3x_5)^{k-2}+kc_7(r_2x_4+r_3x_5)^{k-1}\cdot x_4\rangle$$

and

$$\begin{aligned} f &= x_1(r_2x_4+r_3x_5)^k+c_3(x_2^2-2x_1x_3)(r_2x_4+r_3x_5)^{k-1} \\ &+c_7(r_2x_4+r_3x_5)^k\cdot x_4+c_8(r_2x_4+r_3x_5)^{k+1} \\ &+c_4x_2(r_2x_4+r_3x_5)^k+c_0x_3(r_2x_4+r_3x_5)^k \end{aligned}$$

where $r_3c_3^3c_7\neq 0$.

Proof. Case 1. $I=(5)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $4,2,0,-2$ and -4. Write

$$f=\sum_{i=-\infty}^{\infty} f_{k+1}^{2i}$$

where f_{k+1}^{2i} is a homogeneous polynomial of degree $k+1$ and weight $2i$.

For $|i|\geq 4$

$$\begin{aligned} &|wt\frac{\partial f_{k+1}^{2i}}{\partial x_j}|\geq 6 \qquad \text{for } 1\leq j\leq 5 \\ &\Rightarrow \frac{\partial f_{k+1}^{2i}}{\partial x_j}=0 \qquad \text{for } 1\leq j\leq 5 \\ &\Rightarrow f_{k+1}^{2i}=0. \end{aligned}$$

For $i=3$

$$\begin{aligned} &wt\frac{\partial f_{k+1}^{6}}{\partial x_2}=6=wt\frac{\partial f_{k+1}^{6}}{\partial x_4}=wt\frac{\partial f_{k+1}^{6}}{\partial x_5}, \quad wt\frac{\partial f_{k+1}^{6}}{\partial x_3}=8 \\ \Rightarrow\ &\frac{\partial f_{k+1}^{6}}{\partial x_2}=\frac{\partial f_{k+1}^{6}}{\partial x_3}=\frac{\partial f_{k+1}^{6}}{\partial x_4}=\frac{\partial f_{k+1}^{6}}{\partial x_5}=0 \\ \Rightarrow\ & f_{k+1}^{6} \text{ involves only } x_1 \text{ variable} \end{aligned}$$

$$\Rightarrow \quad f_{k+1}^6 = cx_1^3 \text{ where } c \text{ is a constant.}$$

Suppose $f_{k+1}^6 \neq 0$. Then it is easy to see that $k = 2$ and

$$I = \langle x_1^2, x_1x_2, x_2^2 + x_1x_3, x_2x_3, x_3^2 \rangle.$$

This implies that f does not involve x_4 and x_5 variables. It follows that $\dim I \leq 3$, which contradicts to our hypothesis $I = (5)$. Hence $f_{k+1}^6 = 0$. Similarly we can prove $f_{k+1}^{-6} = 0$.

For $i = 2$.

$wt \frac{\partial f_{k+1}^4}{\partial x_3} = 6$ implies $\frac{\partial f_{k+1}^4}{\partial x_3} = 0$. So f_{k+1}^4 does not involve x_3 variable. Since $wt \frac{\partial f_{k+1}^4}{\partial x_2} = wt \frac{\partial f_{k+1}^4}{\partial x_4} = wt \frac{\partial f_{k+1}^4}{\partial x_5} = 4$, in view of Lemma 6.1, there are constants r_1, r_2 and r_3 such that f_{k+1}^4 is a polynomial in x_1 and $r_1x_2 + r_2x_4 + r_3x_5$

$$f_{k+1}^4 = x_1^2(r_1x_2 + r_2x_4 + r_3x_5)^{k-1}.$$

Suppose $f_{k+1}^4 \neq 0$. Assume first that $r_1 = 0$. Then $f_{k+1}^4 = x_1^2(r_2x_4 + r_3x_5)^{k-1}$ and $\frac{\partial f_{k+1}^4}{\partial x_1} = 2x_1(r_2x_4 + r_3x_5)^{k-1}$. It follows that

$$\langle x_1(r_2x_4 + r_3x_5)^{k-1}, x_2(r_2x_4 + r_3x_5)^{k-1}, x_3(r_2x_4 + r_3x_5)^{k-1} \rangle$$

is an irreducible $sl(2, \mathbf{C})$-submodule of I. This contradicts to our hypothesis $I = (5)$. Hence we shall assume that $r_1 \neq 0$. It is easy to see that $X_-^{k+1}(\frac{\partial f_{k+1}^4}{\partial x_1}) = X_-^{k+1}[2x_1(r_1x_2 + r_2x_4 + r_3x_5)^{k-1}] = c_1x_3^k$ where c_1 is a nonzero constant. Therefore we have an irreducible $sl(2, \mathbf{C})$-submodule of I of the following form.

$$\langle x_3^k, X_+(x_3^k), X_+^2(x_3^k), \ldots, X_+^{2k}(x_3^k) \rangle.$$

Since $I = (5)$, this implies $k = 2$ and

$$I = \langle x_3^2, x_2x_3, x_2^2 + x_1x_3, x_1x_2, x_1^2 \rangle.$$

Consequently f does not involve x_4 and x_5 variables. It follows that $\dim I \leq 3$, which contradicts to our hypothesis $I = (5)$. Hence $f_{k+1}^4 = 0$. Similarly we can prove $f_{k+1}^{-4} = 0$.

For $i = 1$.

Since $wt(\frac{\partial f_{k+1}^2}{\partial x_2}) = 2 = wt(\frac{\partial f_{k+1}^2}{\partial x_4}) = wt(\frac{\partial f_{k+1}^2}{\partial x_5})$, in view of Lemma 6.1, there exist constants r_1, r_2 and r_3 such that f_{k+1}^2 is a polynomial in $x_1, r_1x_2 + r_2x_4 + r_3x_5$ and x_3.

$$f_{k+1}^2 = \sum_{\alpha=0}^{[\frac{k}{2}]} c_\alpha x_1^{\alpha+1} x_3^\alpha (r_1x_2 + r_2x_4 + r_3x_5)^{k-2\alpha}.$$

Similarly we can write

$$f_{k+1}^{-2} = \sum_{\alpha=0}^{[\frac{k}{2}]} a_\alpha x_1^{\alpha} x_3^{\alpha+1} (r_4x_2 + r_5x_4 + r_6x_5)^{k-2\alpha}$$

and

$$f_{k+1}^0 = \sum_{\alpha=0}^{[\frac{k+1}{2}]} b_\alpha x_1^{\alpha} x_3^{\alpha} (r_7x_2 + r_8x_4 + r_9x_5)^{k-2\alpha+1}.$$

Assuming $f_{k+1}^2 \neq 0$. Then

$$\frac{\partial f_{k+1}^2}{\partial x_1} = \sum_{\alpha=0}^{[\frac{k}{2}]} (1+\alpha) c_\alpha x_1^{\alpha} x_3^\alpha (r_1x_2 + r_2x_4 + r_3x_5)^{k-2\alpha}$$

is a nonzero element of weight 0 in I. This implies that

$$\frac{\partial f_{k+1}^{-2}}{\partial x_3} = \sum_{\alpha=0}^{[\frac{k}{2}]} (\alpha+1) a_\alpha x_1^{\alpha} x_3^\alpha (r_4x_2 + r_5x_4 + r_6x_5)^{k-2\alpha}$$

$$\frac{\partial f_{k+1}^0}{\partial x_2} = \sum_{\alpha=0}^{[\frac{k+1}{2}]} r_7(k-2\alpha+1) b_\alpha x_1^{\alpha} x_3^\alpha (r_7x_2 + r_8x_4 + r_9x_5)^{k-2\alpha}$$

$$\frac{\partial f_{k+1}^0}{\partial x_4} = \sum_{\alpha=0}^{[\frac{k+1}{2}]} r_8(k-2\alpha+1) b_\alpha x_1^{\alpha} x_3^\alpha (r_7x_2 + r_8x_4 + r_9x_5)^{k-2\alpha}$$

$$\frac{\partial f_{k+1}^0}{\partial x_5} = \sum_{\alpha=0}^{[\frac{k+1}{2}]} r_9(k-2\alpha+1) b_\alpha x_1^{\alpha} x_3^\alpha (r_7x_2 + r_8x_4 + r_9x_5)^{k-2\alpha}$$

are constant multiples of $\frac{\partial f_{k+1}^2}{\partial x_1}$. It follows that $(r_4x_2 + r_5x_4 + r_6x_5)$ and $(r_7x_2 + r_8x_4 + r_9x_5)$ are constant multiples of $(r_1x_2 + r_2x_4 + r_3x_5)$. Thus

$$\begin{aligned} f &= f_{k+1}^{2} + f_{k+1}^{-1} + f_{k+1}^{0} \\ &= \sum_{\alpha=0}^{[\frac{k}{2}]} c_\alpha x_1^{\alpha+1} x_3^{\alpha} (r_1 x_2 + r_2 x_4 + r_3 x_5)^{k-2\alpha} \\ &+ \sum_{\alpha=0}^{[\frac{k}{2}]} \tilde{a}_\alpha x_1^{\alpha} x_3^{\alpha+1} (r_1 x_2 + r_2 x_4 + r_3 x_5)^{k-2\alpha} \\ &+ \sum_{\alpha=0}^{[\frac{k}{2}]} \tilde{b}_\alpha x_1^{\alpha} x_3^{\alpha} (r_1 x_2 + r_2 x_4 + r_3 x_5)^{k-2\alpha+1}. \end{aligned}$$

This implies $\dim I \leq 3$, which contradicts to our hypothesis $I = (5)$. Hence $f_{k+1}^{2} = 0$.

Similarly we can prove $f_{k+1}^{-2} = 0$.

For $i = 0$.

In this case $f = f_{k+1}^{0}$. The above argument shows that $\dim I$ is at most three. Hence Case 1 cannot occur.

Case 2. $I = (4) \oplus (1)$.

Elements of (4) are linear combinations of homogeneous polynomials of degree k and weights $3, 1, -1$ and -3. Since weights of x_i, $1 \leq i \leq 5$ are even integers, there is no homogeneous polynomial of odd weight. So Case 2 cannot occur.

Case 3. $I = (3) \oplus (2)$.

Case 3 cannot occur. The argument is the same as Case 2.

Case 4. $I = (3) \oplus (1) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $2, 0$ and -2.

For $|i| \geq 3$

$$|wt \frac{\partial f_{k+1}^{2i}}{\partial x_j}| \geq 4 \quad 1 \leq j \leq 5$$

$$\Rightarrow \quad \frac{\partial f_{k+1}^{2i}}{\partial x_j} = 0 \quad 1 \le j \le 5$$

$$\Rightarrow \quad f_{k+1}^{2i} = 0.$$

For $i = 2$

$$wt\frac{\partial f_{k+1}^4}{\partial x_2} = 4 = wt\frac{\partial f_{k+1}^4}{\partial x_4} = wt\frac{\partial f_{k+1}^4}{\partial x_5} \quad wt\frac{\partial f_{k+1}^4}{\partial x_3} = 6$$

$$\Rightarrow \quad \frac{\partial f_{k+1}^4}{\partial x_2} = 0 = \frac{\partial f_{k+1}^4}{\partial x_3} = \frac{\partial f_{k+1}^4}{\partial x_4} = \frac{\partial f_{k+1}^4}{\partial x_5}$$

$\Rightarrow \quad f_{k+1}^4$ involves only x_1 variable

$\Rightarrow \quad f_{k+1}^4 = cx_1^2$ where c is a constant

$\Rightarrow \quad f_{k+1}^4 = 0$ because $k \ge 2$.

Similarly we can show $f_{k+1}^{-4} = 0$.

For $i = 1$

$wt\frac{\partial f_{k+1}^2}{\partial x_3} = 4$ implies $\frac{\partial f_{k+1}^2}{\partial x_3} = 0$. So f_{k+1}^2 does not involve x_3 variable. Since $wt\frac{\partial f_{k+1}^2}{\partial x_2} = wt\frac{\partial f_{k+1}^2}{\partial x_4} = wt\frac{\partial f_{k+1}^2}{\partial x_5} = 2$, in view of Lemma 6.1, there are constants r_1, r_2 and r_3 such that f_{k+1}^2 is a polynomial in x_1 and $(r_1x_2 + r_2x_4 + r_3x_5)$

$$f_{k+1}^2 = x_1(r_1x_2 + r_2x_4 + r_3x_5)^k.$$

Suppose $f_{k+1}^2 \neq 0$. Assume first that $r_1 = 0$. Then $f_{k+1}^2 = x_1(r_2x_4 + r_3x_5)^k$ and either $r_2 \neq 0$ or $r_3 \neq 0$. By considering $\frac{\partial f_{k+1}^2}{\partial x_4}$ or $\frac{\partial f_{k+1}^2}{\partial x_5}$, we see that $x_1(r_2x_4 + r_3x_5)^{k-1}$ is a nonzero element in I. It follows that

$$(3) = \langle x_1(r_2x_4 + r_3x_5)^{k-1},\ x_2(r_2x_4 + r_3x_5)^{k-1},\ x_3(r_2x_4 + r_3x_5)^{k-1}\rangle$$

$$(1) = \langle (r_2x_4 + r_3x_5)^k\rangle$$

are irreducible $sl(2,\mathrm{C})$-submodules of I. Similarly, there are constants r_4, r_5 and r_6 such that f_{k+1}^{-2} is a polynomial in x_3 and $(r_4x_2 + r_5x_4 + r_6x_5)$.

$$f_{k+1}^{-2} = x_3(r_4x_2 + r_5x_4 + r_6x_5)^k.$$

Since $\frac{\partial f_{k+1}^{-2}}{\partial x_2} = r_4x_3(r_4x_2 + r_5x_4 + r_6x_5)^{k-1}$, $\frac{\partial f_{k+1}^{-2}}{\partial x_4} = r_5x_3(r_4x_2 + r_5x_4 + r_6x_5)^{k-1}$ and $\frac{\partial f_{k+1}^{-2}}{\partial x_5} = r_6x_3(r_4x_2 + r_5x_4 + r_6x_5)^{k-1}$ are elements of weight -2 in I, they are constant multiples of $x_3(r_2x_4 + r_3x_5)^{k-1}$. It follows that there is a constant c_0 such that

$$f_{k+1}^{-2} = c_0x_3(r_2x_4 + r_3x_5)^k$$

$\frac{\partial f_{k+1}^0}{\partial x_1}$ and $\frac{\partial f_{k+1}^0}{\partial x_3}$ are elements of weight -1 and 1 respectively in (3). There are constants c_1 and c_2 such that

$$\begin{cases} \frac{\partial f_{k+1}^0}{\partial x_1} = c_1x_3(r_2x_4 + r_3x_5)^{k-1} \\ \frac{\partial f_{k+1}^0}{\partial x_3} = c_2x_1(r_2x_4 + r_3x_5)^{k-1} \end{cases}.$$

From the above two equations, it is not hard to see that $c_1 = c_2$ and there exists a homogeneous polynomial of degree $k+1$ in x_2, x_4 and x_5 variables such that

$$f_{k+1}^0 = c_1x_1x_3(r_2x_4 + r_3x_5)^{k-1} + h(x_2, x_4, x_5).$$

Write $h(x_2, x_4, x_5) = \sum_{\alpha=0}^{k+1} x_2^\alpha \; g_{k+1-\alpha}(x_4, x_5)$ where $g_{k+1-\alpha}(x_4, x_5)$ is a homogeneous polynomial of $k+1-\alpha$ in x_4 and x_5 variables. Let α_0 be the largest integer such that $g_{k+1-\alpha_0}(x_4, x_5) \neq 0$,

$$\frac{\partial f_{k+1}^0}{\partial x_2} = \sum_{\alpha=1}^{\alpha_0} \alpha x_2^{\alpha-1} g_{k+1-\alpha}(x_4, x_5)$$
$$X_-^{\alpha_0-1}\left(\frac{\partial f_{k+1}^0}{\partial x_2}\right) = (\alpha_0)! x_3^{\alpha_0-1} g_{k+1-\alpha_0}(x_4, x_5).$$

Since $X_-[x_3^{\alpha_0-1} g_{k+1-\alpha_0}(x_4, x_5)] = 0$, we have an irreducibe $sl(2,\mathrm{C})$-submodule of I in the following form.

$$\langle x_3^{\alpha_0-1} g_{k+1-\alpha_0}(x_4, x_5), X_+[x_3^{\alpha_0-1} g_{k+1-\alpha_0}(x_4, x_5)], \ldots, X_+^{2\alpha_0-2}[x_3^{\alpha_0-1} g_{k+1-\alpha_0}(x_4, x_5)]\rangle.$$

Thus $2\alpha_0 - 1 \le 3$ because $i = (3) \oplus (1) \oplus (1)$. This implies that $\alpha_0 \le 2$ and

$$f_{k+1}^0 = c_1 x_1 x_3 (r_2 x_4 + r_3 x_5)^{k-1} + g_{k+1}(x_4 x_5) + g_k(x_4 x_5) x_2 + g_{k-1}(x_4 x_5) x_2^2$$
$$\frac{\partial f_{k+1}^0}{\partial x_2} = g_k(x_4, x_5) + 2 g_{k-1}(x_4, x_5) x_2.$$

As $X_-(\frac{\partial f_{k+1}^0}{\partial x_2}) = 2g_{k-1}(x_4, x_5)x_3$ is an element of weight -2 in (3), $g_{k-1}(x_4, x_5)x_3$ is a constant multiple of $x_3(r_2x_4 + r_3x_5)^{k-1}$. Thus there exists a constant c_3 such that

$$g_{k-1}(x_4, x_5) = c_3 (r_2 x_4 + r_3 x_5)^{k-1}$$

and

$$f_{k+1}^0 = c_1 x_1 x_3 (r_2 x_4 + r_3 x_5)^{k-1} + g_{k+1}(x_4, x_5) + g_k(x_4, x_5) x_2 + c_3 (r_2 x_4 + r_3 x_5)^{k-1} x_2^2.$$

It follows that

$$\frac{\partial f_{k+1}^0}{\partial x_4} = r_2(k-1)(c_1 x_1 x_3 + c_3 x_2^2)(r_2 x_4 + r_3 x_5)^{k-2} + \frac{\partial g_{k+1}}{\partial x_4}(x_4, x_5) + x_2 \frac{\partial g_k}{\partial x_4}(x_4, x_5)$$
$$X_-(\frac{\partial f_{k+1}^0}{\partial x_4}) = r_2(k-1)(c_1 + 2c_3) x_2 x_3 (r_2 x_4 + r_3 x_5)^{k-2} + x_3 \frac{\partial g_k}{\partial x_4}(x_4, x_5)$$
$$\frac{\partial f_{k+1}^0}{\partial x_5} = r_3(k-1)(c_1 x_1 x_3 + c_3 x_2^2)(r_2 x_4 + r_3 x_5)^{k-2} + \frac{\partial g_{k+1}}{\partial x_5}(x_4, x_5) + x_2 \frac{\partial g_k}{\partial x_5}(x_4, x_5)$$
$$X_-(\frac{\partial f_{k+1}^0}{\partial x_5}) = r_3(k-1)(c_1 + 2c_3) x_2 x_3 (r_2 x_4 + r_3 x_5)^{k-2} + x_3 \frac{\partial g_k}{\partial x_5}(x_4, x_5).$$

If $c_1 + 2c_3$ were not zero, then by applying X_- to $X_-(\frac{\partial f_{k+1}^0}{\partial x_4})$ or $X_-(\frac{\partial f_{k+1}^0}{\partial x_5})$, we would get a nonzero element $x_3^2(r_2x_4 + r_3x_5)^{k-2}$ of weight -4 in $(3) \subseteq I$. This clearly is impossible. Thus $c_1 = -2c_3$ and $x_3\frac{\partial g_k}{\partial x_4}(x_4, x_5)$, $x_3\frac{\partial g_k}{\partial x_5}(x_4, x_5)$ are both elements of weight -2 in (3). They are constant multiples of $x_3(r_2x_4 + r_3x_5)^{k-1}$. By Euler relation, there are constants r_8 and r_9 such that $g_k = \frac{1}{k}x_4\frac{\partial g_k}{\partial x_4} + \frac{1}{k}x_5\frac{\partial g_k}{\partial x_5} = (r_8x_4 + r_9x_5)(r_2x_4 + r_3x_5)^{k-1}$. Using the fact that both $x_3\frac{\partial g_k}{\partial x_4}(x_4, x_5)$ and $x_3\frac{\partial g_k}{\partial x_5}(x_4, x_5)$ are constant multiples of

$x_3(r_2x_4+r_3x_5)^{k-1}$, we conclude that $r_8x_4+r_9x_5$ is a constant multiple of $r_2x_4+r_3x_5$. Therefore there exists a constant c_4 such that $g_k(x_4,x_5)=c_4(r_2x_4+r_3x_5)^k$. Thus

$$f^0_{k+1}=c_3(x_2^2-2x_1x_3)(r_2x_4+r_3x_5)^{k-1}+g_{k+1}(x_4,x_5)+c_4(r_2x_4+r_3x_5)^k\cdot x_2.$$

If $c_3(x_2^2-2x_1x_3)(r_2x_4+r_3x_5)^{k-1}=0$, then

$$\begin{aligned}f&=f^2_{k+1}+f^0_{k+1}+f^{-2}_{k+1}\\&=x_1(r_2x_4+r_3x_5)^k+g_{k+1}(x_4,x_5)+c_4(r_2x_4+x_3x_5)^k\cdot x_2\\&\quad+c_0x_3(r_2x_4+r_3x_5)^k.\end{aligned}$$

This would imply $\dim I\le 3$ because $\langle\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},\frac{\partial f}{\partial x_3}\rangle$ is one dimensional, which contradicts to our hypothesis $I=(3)\oplus(1)\oplus(1)$.

Suppose then that $c_3(x_2^2-2x_1x_3)(r_2x_4+r_3x_5)^{k-1}\ne 0$. We shall assume without loss of generality that

$$\begin{aligned}&\frac{\partial}{\partial x_4}[c_3(x_2^2-2x_1x_3)(r_2x_4+r_3x_5)^{k-1}]\\&=r_2(k-1)c_3(x_2^2-2x_1x_3)(r_2x_4+r_3x_5)^{k-2}\ne 0.\end{aligned}$$

i.e. $r_2c_3\ne 0$. Observe that

$$r_2(k-1)c_3(x_2^2-2x_1x_3)(r_2x_4+r_3x_5)^{k-2}+\frac{\partial g_{k+1}}{\partial x_4}(x_4,x_5)=\frac{\partial f^0_{k+1}}{\partial x_4}-c_4r_2k(r_2x_4+r_3x_5)^{k-1}\cdot x_2$$

is a nonzero $sl(2,\mathrm{C})$ invariant polynomial in I.

$$\begin{aligned}I&=(3)\oplus(1)\oplus(1)\\&=\langle x_1(r_2x_4+r_3x_5)^{k-1},\ x_2(r_2x_4+r_3x_5)^{k-1},\ x_3(r_2x_4+r_3x_5)^{k-1}\rangle\\&\quad\oplus\langle(r_2x_4+r_3x_5)^k\rangle\\&\quad\oplus\langle r_2(k-1)c_3(x_2^2-2x_1x_3)(r_2x_4+r_3x_5)^{k-2}+\frac{\partial g_{k+1}}{\partial x_4}(x_4,c_5)\rangle.\end{aligned}$$

Similarly $r_3(k-1)c_3(x_2^2-2x_1x_3)(r_2x_4+r_3x_5)^{k-2}+\frac{\partial g_{k+1}}{\partial x_5}(x_4,x_5) = \frac{\partial f^0_{k+1}}{\partial x_5} - kr_3c_4(r_2x_4+r_3x_5)^{k-1}\cdot x_2$ is an $sl(2,\mathbf{C})$ invariant polynomial in I. It is easy to see that there are constants c_5 and c_6 such that

$$\begin{aligned}
& r_3(k-1)c_3(x_2^2-2x_1x_3)(r_2x_4+r_3x_5)^{k-2}+\frac{\partial g_{k+1}}{\partial x_5}(x_4,x_5) \\
& = c_5(r_2x_4+r_3x_5)^k + c_6[r_2(k-1)c_3(x_2^2-2x_1x_3)(r_2x_4+r_3x_5)^{k-2} \\
& +\frac{\partial g_{k+1}}{\partial x_4}(x_4,x_5)] \\
\Rightarrow \quad & (r_3-c_6r_2)(k-1)c_3(x_2^2-2x_1x_3)(r_2x_4+r_3x_5)^{k-2} \\
& = c_5(r_2x_4+r_3x_5)^k + c_6\frac{\partial g_{k+1}}{\partial x_4}(x_4,x_5) - \frac{\partial g_{k+1}}{\partial x_5}(x_4,x_5) \\
\Rightarrow c_6 \; & = \frac{r_3}{r_2} \text{ and } \frac{\partial g_{k+1}}{\partial x_5}(x_4,x_5) - \frac{r_3}{r_2}\frac{\partial g_{k+1}}{\partial x_4}(x_4,x_5) = c_5(r_2x_4+r_3x_5)^k.
\end{aligned}$$

Let $y_4 = x_4 + \frac{r_3}{r_2}x_5$ and $y_5 = x_5$. Let

$$\begin{aligned}
& \tilde{g}_{k+1}(y_4,y_5) = g_{k+1}(y_4-\frac{r_3}{r_2}y_5, y_5) \\
\Rightarrow \quad & \frac{\partial \tilde{g}_{k+1}}{\partial y_5} = \frac{\partial g_{k+1}}{\partial x_4}(y_4-\frac{r_3}{r_2}y_5,y_5)(\frac{-r_3}{r_2}) + \frac{\partial g_{k+1}}{\partial x_5}(y_4-\frac{r_3}{r_2}y_5,y_5) \\
& = c_5[r_2(y_4-\frac{r_3}{r_2}y_5)+r_3y_5]^k \\
& = c_5r_2^ky_4^k \\
\Rightarrow \quad & \tilde{g}_{k+1}(y_4,y_5) = \tilde{c}_7y_4^ky_5 + \tilde{c}_8y_4^{k+1} \\
\Rightarrow \quad & g_{k+1}(x_4,x_5) = c_7(r_2x_4+r_3x_5)^kx_5 + c_8(r_2x_4+r_3x_5)^{k+1}.
\end{aligned}$$

Since $r_2(k-1)c_3(x_2^2-2x_1x_3)(r_2x_4+r_3x_5)^{k-2}+\frac{\partial g_{k+1}}{\partial x_4}(x_4,x_5) = r_2(k-1)c_3(x_2^2-2x_1x_3)(r_2x_4+r_3x_5)^{k-2} + c_7kr_2(r_2x_4+r_3x_5)^{k-1}x_5 + c_8(k+1)(r_2x_4+r_3x_5)^k$ we see that $r_2(k-1)c_3(x_2^2-2x_1x_3)(r_2x_4+r_3x_5)^{k-2}+c_7\cdot k\cdot r_2(r_2x_4+r_3x_5)^{k-1}x_5$ is a nonzero $sl(2,\mathbf{C})$ invariant in I. Hence

$$I \; = \langle x_1(r_2x_4+r_3x_5)^{k-1},\ x_2(r_2x_4+r_3x_5)^{k-1},\ x_3(r_2x_4+r_3x_5)^{k-1}\rangle$$

$$\oplus\langle (r_2x_4+r_3x_5)^k\rangle$$

$$\oplus\langle (k-1)c_3(x_2^2-2x_1x_3)(r_2x_4+r_3x_5)^{k-2}+kc_7(r_2x_4+r_3x_5)^{k-1}\cdot x_5\rangle$$

and

$$\begin{aligned} f &= x_1(r_2x_4+r_3x_5)^k+c_3(x_2^2-2x_1x_3)(r_2x_4+r_3x_5)^{k-1} \\ &+c_7(r_2x_4+r_3x_5)^k\cdot x_5+c_8(r_2x_4+r_3x_5)^{k+1}+c_4x_2(r_2x_4+r_3x_5)^k \\ &+c_0x_3(r_2x_4+r_3x_5)^k. \end{aligned}$$

where

$$\det\begin{bmatrix} 0 & 0 & -2c_3 & 1 & 0 \\ 0 & 2c_3 & 0 & c_4 & 0 \\ -2c_3 & 0 & 0 & c_0 & 0 \\ c_2k & c_4kr_2 & c_0kr_2 & c_8(k+1)r_2 & r_2 \\ r_3k & c_4kr_3 & c_0kr_3 & c_9+(k+1)r_3c_8 & r_3 \end{bmatrix}=8r_2c_7c_3^3\neq 0.$$

Thus we have (iii).

Similarly, if $r_3c_3\neq 0$, then we have (iii).

Suppose next that $r_1\neq 0$. We shall assume without loss of generality that

$$f_{k+1}^2=d_1x_1(x_2+r_2x_4+r_3x_5)^k$$

where d_1 is a nonzero constant. $\frac{\partial f_{k+1}^2}{\partial x_1}=d_1(x_2+r_2x_4+r_3x_5)^k$ is a nonzero element in I. Consider $X_-^k\frac{\partial f_{k+1}^2}{\partial x_1}=d_1k!x_3^k$. Since $X_-[x_3^k]=0$, we have an irreducible $sl(2,\mathrm{C})$-submodule of dimension $2k+1$ in I in the following form.

$$\langle x_3^k, X_+(x_3^k), X_+^2(x_3^k),\ldots,X_+^{2k}(x_3^k)\rangle.$$

As $I=(3)\oplus(1)\oplus(1)$, we conclude $2k+1\leq 3$ i.e. $k\leq 1$. This contradicts to our hypothesis that $k\geq 2$.

Case 5. $I=(2)\oplus(2)\oplus(1)$.

Elements of (2) are linear combinations of homogeneous polynomials of degree k and weights -1, and 1. Since weights of x_i, $1\leq i\leq 5$ are even integers, there is no homogeneous polynomial of odd weight. So Case 5 cannot occur.

Case 6. $I = (2) \oplus (1) \oplus (1) \oplus (1)$.

Case 6 cannot occur. The reasoning is the same as Case 5.

Case 7. $I = (1) \oplus (1) \oplus (1) \oplus (1) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials of degree $k+1$ and weight 0.

For $|i| \geq 2$

$$\begin{aligned} & |wt\frac{\partial f_{k+1}^{2i}}{\partial x_j}| \geq 2 \qquad 1 \leq j \leq 5 \\ \Rightarrow \quad & \frac{\partial f_{k+1}^{2i}}{\partial x_j} = 2 \qquad 1 \leq j \leq 5 \\ \Rightarrow \quad & f_{k+1}^{2i} = 0. \end{aligned}$$

For $i = 1$

$$\begin{aligned} & wt\frac{\partial f_{k+1}^2}{\partial x_2} = 2 = wt\frac{\partial f_{k+1}^2}{\partial x_4} = wt\frac{\partial f_{k+1}^2}{\partial x_5}, \ wt\frac{\partial f_{k+1}^2}{\partial x_3} = 4 \\ \Rightarrow \quad & \frac{\partial f_{k+1}^2}{\partial x_2} = 0 = \frac{\partial f_{k+1}^2}{\partial x_4} = \frac{\partial f_{k+1}^2}{\partial x_5} = \frac{\partial f_{k+1}^2}{\partial x_3} \\ \Rightarrow \quad & f_{k+1}^2 \text{ involves only } x_1 \text{ variable} \\ \Rightarrow \quad & f_{k+1}^2 = cx_1 \text{ where } c \text{ is a constant} \\ \Rightarrow \quad & f_{k+1}^2 = 0 \text{ because } k \geq 2. \end{aligned}$$

Similarly we can show $f_{k+1}^{-2} = 0$.

For $i = 0$.

In this case $f = f_{k+1}^0$.

$$\begin{aligned} & wt\frac{\partial f}{\partial x_1} = -2, \quad wt\frac{\partial f}{\partial x_3} = 2 \\ \Rightarrow \quad & \frac{\partial f}{\partial x_1} = 0 = \frac{\partial f}{\partial x_3} \end{aligned}$$

$$\Rightarrow \quad \dim I \leq 3.$$

This contradicts to our hypothesis $\dim I = 5$. Hence Case 7 cannot occur. Q.E.D.

Lemma 6.3. With the same hypothesis as Lemma 6.2; if I is an $sl(2,\mathbf{C})$ module of dimension 4 then

$$I = (3) \oplus (1)$$

and f is of weight 0.

Proof. Case 1. $I = (4)$.

Elements of (4) are linear combinations of homogeneous polynomials of degree k and weights $3, 1, -1$ and -3. Since weights of x_i, $1 \leq i \leq 5$, are even integers, there is no homogeneous polynomial of odd weight. So Case 1 cannot occur.

Case 2. $I = (3) \oplus (1)$.

We shall follow the argument used in Case 4 in the proof of Lemma 6.2. The argument there gives

$$f = f_{k+1}^2 + f_{k+1}^0 + f_{k+1}^{-2}.$$

If $f_{k+1}^2 \neq 0$. then we can assume that $f_{k+1}^2 = x_1(r_2x_4 + r_3x_5)^k$. It follows that

$$\begin{aligned} I = (3) \oplus (1) &= \langle x_1(r_2x_4 + r_3x_5)^{k-1},\ x_2(r_2x_4 + r_3x_5)^{k-1},\ x_3(r_2x_4 + r_3x_5)^{k-1}\rangle \\ &\oplus \langle (r_2x_4 + r_3x_5)^k\rangle \\ f_{k+1}^{-2} &= c_0x_3(r_2x_4 + r_3x_5)^k \\ f_{k+1}^0 &= c_3(x_2^2 - 2x_1x_3)(r_2x_4 + r_3x_5)^{k-1} + g_{k+1}(x_4, x_5) + c_4(r_2x_4 + r_3x_5)^k \cdot x_2 \end{aligned}$$

If $c_3(x_2^2 - 2x_1x_3)(r_2x_4 + r_3x_5)^{k-1} \neq 0$, then either $r_2(k-1)c_3(x_2^2 - 2x_1x_3)(r_2x_4 + r_3x_5)^{k-2} \neq 0$ or $r_3(k-1)c_3(x_2^2 - 2x_1x_3)(r_2x_4 + r_3x_5)^{k-2} \neq 0$. By considering $\frac{\partial f_{k+1}^0}{\partial x_4}$

and $\frac{\partial f^0_{k+1}}{\partial x_5}$, we see that there is an element in I with nonzero terms of degree 2 in x_1, x_2 and x_3 variables. This contradicts to our previous description of I. Hence

$$f^0_{k+1} = g_{k+1}(x_4, x_5) + c_4(r_2x_4 + r_3x_5)^k \cdot x_2$$

and

$$\begin{aligned} f &= f^2_{k+1} + f^0_{k+1} + f^{-2}_{k+1} \\ &= x_1(r_2x_4 + r_3x_5)^k + g_{k+1}(x_4, x_5) + c_4(r_2x_4 + r_3x_5)^k \cdot x_2 + c_0x_3(r_2x_4 + r_3x_5)^k \end{aligned}$$

This implies that $\dim I \le 3$ because $\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \rangle$ is one dimensional, which contradicts to our hypothesis $i = (3) \oplus (1)$. Therefore $f^2_{k+1} = 0$. Similarly we can prove that $f^{-2}_{k+1} = 0$.

Case 3. $I = (2) \oplus (2)$.

Case 3 cannot occur. The reasoning is the same as Case 1.

Case 4. $I = (2) \oplus (1) \oplus (1)$.

Case 4 cannot occur. The reasoning is the same as Case 1.

Case 5. $I = (1) \oplus (1) \oplus (1) \oplus (1)$.

Case 5 cannot occur. The proof is the same as those given in Case 7 in the proof of Lemma 6.2.

Lemma 6.4. With the same hypothesis as Lemma 6.2; if I is an $sl(2,\mathbf{C})$ module of dimension 3, then $I = (3)$ and f is of weight 0.

Proof. Case 1. $I = (3)$.

We shall follow the argument given in Case 4 in the proof of Lemma 6.2. We get easily $f^{2i}_{k+1} = 0$ for $|i| \ge 2$ and $f^2_{k+1} = x_1(r_1x_2 + r_2x_4 + r_3x_5)^k$. Observe that

$\frac{\partial f^2_{k+1}}{\partial x_1} = (r_1x_2 + r_2x_4 + r_3x_5)^k$ is an invariant $sl(2, \mathbb{C})$ polynomial in I. Since $I = (3)$, we have $(r_1x_2 + r_2x_4 + r_3x_5)^k = 0$ i.e. $f^2_{k+1} = 0$.

Similarly we can prove that $f^{-2}_{k+1} = 0$.

Case 2. $I = (2) \oplus (1)$.

Case 2 cannot occur. The reasoning is the same as in Case 1 in the proof of Lemma 6.3.

Case 3. $I = (1) \oplus (1) \oplus (1)$.

Similar argument as in Case 7 in the proof of Lemma 6.2 shows that $f = f^0_{k+1}$ and $\frac{\partial f}{\partial x_1} = 0 = \frac{\partial f}{\partial x_3}$. We can write

$$f = \sum_{\alpha=0}^{k-1} x_2^\alpha g_{k+1-\alpha}(x_4, x_5)$$

where $g_{k+1-\alpha}(x_4, x_5)$ is a homogeneous polynomial of degree $k + 1 - \alpha$ in x_4 and x_5 variables. Let α_0 be the largest integer such that $g_{k+1-\alpha_0}(x_4, x_5) \neq 0$.

$$\frac{\partial f^0_{k+1}}{\partial x_2} = \sum_{\alpha=0}^{\alpha_0} \alpha x_2^{\alpha-1} g_{k+1-\alpha}(x_4, x_5)$$

$$X_-^{\alpha_0-1}\left(\frac{\partial f^0_{k+1}}{\partial x_2}\right) = (\alpha_0)! x_3^{\alpha_0-1} g_{k+1-\alpha_0}(x_4, x_5).$$

Since $X_-[x_3^{\alpha_0-1} g_{k+1-\alpha_0}(x_4, x_5)] = 0$, we have an irreducible $sl(2, \mathbb{C})$-submodule of I in the following form.

$$\langle x_3^{\alpha_0-1} g_{k+1-\alpha_0}(x_4, x_5), X_+[x_3^{\alpha_0-1} g_{k+1-\alpha_0}(x_4, x_5)], \ldots, X_+^{2\alpha_0-2}[x_3^{\alpha_0-1} g_{k+1-\alpha_0}(x_4, x_5)\rangle.$$

Thus $2\alpha_0 - 1 \leq 1$ because $I = (1) \oplus (1) \oplus (1)$. This implies that $\alpha_0 \leq 1$ and

$$f = g_{k+1}(x_4, x_5) + x_2 g_k(x_4, x_5).$$

Suppose $g_k(x_4,x_5) \neq 0$. Then either $\frac{\partial g_k}{\partial x_4}(x_4,x_5) \neq 0$ or $\frac{\partial g_k}{\partial x_5}(x_4,x_5) \neq 0$. Without loss of generality, we shall assume $\frac{\partial g_k}{\partial x_4}(x_4,x_5) \neq 0$

$$\frac{\partial f}{\partial x_4} = x_2\frac{\partial g_k}{\partial x_4}(x_4,x_5) + \frac{\partial g_{k+1}}{\partial x_4}(x_4,x_5)$$
$$X_-(\frac{\partial f}{\partial x_4}) = x_3\frac{\partial g_k}{\partial x_4}(x_4,x_5).$$

Since $X_-(x_3\frac{\partial g_k}{\partial x_4}(x_4,x_5)) = 0$, we have an irreducible $sl(2,\mathbf{C})$-submodule of I in the following form.

$$\langle x_3\frac{\partial g_k}{\partial x_4}(x_4,x_5),\ x_2\frac{\partial g_k}{\partial x_4}(x_4,x_5),\ x_1\frac{\partial g_k}{\partial x_4}(x_4,x_5)\rangle.$$

This contradicts to our hypothesis $I = (1)\oplus(1)\oplus(1)$. Thus $g_k(x_4,x_5) = 0$ and $f = g_{k+1}(x_4,x_5)$. It follows that $\frac{\partial f}{\partial x_1} = 0 = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3}$. Therefore $\dim I \leq 2$, which again contradicts to our hypothesis $I = (1)\oplus(1)\oplus(1)$. So Case 3 cannot occur.

Lemma 6.5. With the same hypothesis as Lemma 6.2; if I is a $sl(2,\mathbf{C})$-submodule of dimension 2, then f depends only on x_4 and x_5 variables and

$$I = (1)\oplus(1) = \langle\frac{\partial f}{\partial x_4}(x_4,x_5)\rangle \oplus \langle\frac{\partial f}{\partial x_5}(x_4,x_5)\rangle.$$

Proof. Case 1. $I = (2)$.

Case 1 cannot occur. The reasoning is the same as in Case 1 in the proof of Lemma 6.3.

Case 2. $I = (1)\oplus(1)$.

In this Case, we can apply the argument in case 3 in the proof of Lemma 6.4 to conclude our lemma.

Lemma 6.6. With the same hypothesis as Lemma 6.2; if I is a $sl(2, \mathbf{C})$ module, then f depends only x_4 and x_5 variables. In fact, there are constants c_1 and c_2, not all zero, such that

$$f = (c_1x_4 + c_2x_5)^{k+1} \quad \text{and} \quad I = (1) = \langle \frac{\partial f}{\partial x_4} \rangle \quad \text{or} \quad \langle \frac{\partial f}{\partial x_5} \rangle.$$

Proof. This is an easy consequence of Lemma 5.1 and the argument in Case 3 in the proof of Lemma 6.4. Q.E.D.

Proposition 6.7. Suppose $sl(2, \mathbf{C})$ acts on the space of homogeneous polynomials of degree $k \geq 2$ in x_1, x_2, x_3, x_4 and x_5 variables via

$$\tau = 2x_1 \frac{\partial}{\partial x_1} - 2x_3 \frac{\partial}{\partial x_3}$$
$$X_+ = 2x_1 \frac{\partial}{\partial x_2} + 2x_2 \frac{\partial}{\partial x_3}$$
$$X_- = x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2}.$$

Suppose the weight of x_i is given by the corresponding coefficient in the expression of τ above, i.e.

$$wt(x_1) = 2, \; wt(x_2) = 0, \; wt(x_3) = -2, \; wt(x_4) = 0, \; wt(x_5) = 0.$$

Let $I = \langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}, \frac{\partial f}{\partial x_5} \rangle$ be the complex vector subspace spanned by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4},$ and $\frac{\partial f}{\partial x_5}$, where f is a homogeneous polynomial of degree $k+1$. If I is a $sl(2, \mathbf{C})$-submodule, then I is one of the following:

(i) (a) $I = (3) \oplus (1) \oplus (1)$ and f is a homogeneous polynomial of weight 0.

(b) $I = (3) \oplus (1) \oplus (1) = \langle x_1(x_4 + rx_5)^{k-1}, \; x_2(x_4 + rx_5)^{k-1}, \; x_3(x_4 + rx_5)^{k-1} \rangle$
$\oplus \langle (x_4 + rx_5)^k \rangle \oplus \langle (k-1)d_1(x_2^2 - 2x_1x_3)(x_4 + rx_5)^{k-2} + kd_2x_5(x_4 + rx + 5)^{k-1} \rangle$

$$f = g(x_1, x_2, x_3, x_4, x_5) + c_1x_1(x_4 + rx_5)^k + c_2x_2(x_4 + rx_5)^k + c_3x_3(x_4 + rx_5)^k$$

where $g(x_1,x_2,x_3,x_4,x_5) = d_1(x_2^2-2x_1x_3)(x_4+rx_5)^{k-1}+d_2x_5(x_4+rx_5)^k+d_3(x_4+rx_5)^{k+1}$ is a $sl(2,\mathbf{C})$ invariant polynomial with $d_1 \neq 0$ and $d_2 \neq 0$.

(c) $I = (3)\oplus(1)\oplus(1) = \langle x_1(rx_4+x_5)^{k-1}, x_2(rx_4+x_5)^{k-1}, x_3(rx_4+x_5)^{k-1}\rangle \oplus \langle (rx_4+x_5)^k\rangle \oplus \langle (k-1)d_1(x_2^2-2x_1x_3)(rx_4+x_5)^{k-2}+kd_2x_4(rx_4+x_5)^{k-1}\rangle$

$$f = g(x_1,x_2,x_3,x_4,x_5) + c_1x_1(rx_4+x_5)^k + c_2x_2(rx_4+x_5)^k + c_3x_3(rx_4+x_5)^k$$

where $g(x_1,x_2,x_3,x_4,x_5) = d_1(x_2^2-2x_1x_3)(rx_4+x_5)^{k-1} + d_2x_4(rx_4+x_5)^k + d_3(rx_4+x_5)^{k+1}$ is a $sl(2,\mathbf{C})$ invariant polynomial with $d_1 \neq 0$ and $d_2 \neq 0$.

(ii) $I = (3)\oplus(1)$ and f is a homogeneous polynomial of weight 0.

(iii) $I = (3)$ and f is a homogeneous polynomial of weight 0.

(iv) $I = (1)\oplus(1) = \langle \frac{\partial f}{\partial x_4}(x_4,x_5)\rangle \oplus \langle \frac{\partial f}{\partial x_5}(x_4,x_5)\rangle$, f is a homogeneous polynomial of weight 0.

(v) $I = (1) = \langle \frac{\partial f}{\partial x_4}(x_4,x_5)\rangle$ or $\langle \frac{\partial f}{\partial x_5}(x_4,x_5)\rangle$ and $f = (c_1x_4+c_2x_5)^{k+1}$.

§7. $sl(2,\mathbf{C})$ ACTION (1.1) ON M_5^k

Lemma 7.1. Suppose $sl(2,\mathbf{C})$ acts on the space of homogeneous polynomials of degree $k \geq 2$ in x_1, x_2, x_3, x_4 and x_5 variables via (1.1)

$$\tau = x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2}$$
$$X_+ = x_1\frac{\partial}{\partial x_2}$$
$$X_- = x_2\frac{\partial}{\partial x_1}.$$

Suppose the weight of x_i is given by the corresponding coefficient in the expression of τ above, i.e.

$$wt(x_1) = 1,\ wt(x_2) = -1,\ wt(x_3) = 0,\ wt(x_4) = 0,\ wt(x_5) = 0.$$

Let I be the complex vector subspace of dimension 5 spanned by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}$ and $\frac{\partial f}{\partial x_5}$ where f is a homogeneous polynomial of degree $k+1$. Then I is not a $sl(2,\mathbf{C})$ submodule.

Proof. We suppose on the contrary that I is a $sl(2,\mathbf{C})$-submodule.

Case 1. $I = (5)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $-4, -2, 0, 2$ and 4. Write

$$f = \sum_{i=-\infty}^{\infty} f_{k+1}^i$$

where f_{k+1}^i is a homogeneous polynomial of degree $k+1$ and weight i.

For $|i| \geq 6$.

$$|wt(\frac{\partial f_{k+1}^i}{\partial x_j})| \geq 5 \qquad \text{for all } 1 \leq j \leq 5$$

$$\Rightarrow \quad \frac{\partial f^i_{k+1}}{\partial x_j} = 0 \qquad \text{for all } 1 \le j \le 5$$
$$\Rightarrow \quad f^i_{k+1} = 0.$$

For $i = \pm 5, \pm 3$ and ± 1.

$$wt(\frac{\partial f^i_{k+1}}{\partial x_3}) = wt(\frac{\partial f^i_{k+1}}{\partial x_4}) = wt(\frac{\partial f^1_{k+1}}{\partial x_5}) = i$$

$$\Rightarrow \frac{f^i_{k+1}}{\partial x_3} = \frac{\partial f^i_{k+1}}{\partial x_4} = \frac{\partial f^i_{k+1}}{\partial x_5} = 0$$

$\Rightarrow f^i_{k+1}$ involves only x_1 and x_2 variables

$$\Rightarrow f^i_{k+1} = \sum_{\alpha=0}^{\frac{k-i+1}{2}} c_\alpha x_1^{i+\alpha} x_2^\alpha \text{ where } c_\alpha \text{ is a constant.}$$

If $f^i_{k+1} \neq 0$, it is easy to see that

$$I = \langle x_1^k, x_1^{k-1}x_2, x_1^{k-2}x_2^2, \ldots, x_2^k \rangle.$$

This implies that f does not involve x_3, x_4 and x_5 variables. Thus $\dim I \le 2$, which contradicts to our hypothesis $I = (5)$. We conclude that $f^i_{k+1} = 0$.

For $i = \pm 2$ or ± 4

$$wt\frac{\partial f^i_{k+1}}{\partial x_1} = i - 1, \; wt\frac{\partial f^i_{k+1}}{\partial x_2} = i + 1 \text{ are odd integers}$$
$$\Rightarrow \frac{\partial f^i_{k+1}}{\partial x_1} = 0 = \frac{\partial f^i_{k+1}}{\partial x_2}$$

$\Rightarrow f^i_{k+1}$ does not involve x_1, x_2 variables

$\Rightarrow f^i_{k+1} = 0$ because $wt(x_3) = wt(x_4) = wt(x_5) = 0$.

For $i = 0$.

In this case $f = f^0_{k+1}$. By the similar argument as before we have f depends only on x_3, x_4 and x_5 variables. This implies that $\dim I \le 3$, which contradicts to our hypothesis $I = (5)$. Thus Case 1 cannot occur.

Case 2. $I = (4) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $-3, -1, 1, 3$ and 0.

For $|i| \geq 5$

$$\begin{aligned} & |wt(\frac{\partial f^i_{k+1}}{\partial x_j})| \geq 4 && \text{for all } 1 \leq j \leq 5 \\ \Rightarrow\ & \frac{\partial f^i_{k+1}}{\partial x_j} = 0 && \text{for all } 1 \leq j \leq 5 \\ \Rightarrow\ & f^i_{k+1} = 0. \end{aligned}$$

For $i = 3$

$$\begin{aligned} & wt\frac{\partial f^3_{k+1}}{\partial x_1} = 2, \quad wt\frac{\partial f^3_{k+1}}{\partial x_2} = 4 \\ \Rightarrow & \frac{\partial f^3_{k+1}}{\partial x_1} = 0 = \frac{\partial f^3_{k+1}}{\partial x_2} \\ \Rightarrow & f^3_{k+1} \text{ involves only } x_3, x_4 \text{ and } x_5 \text{ variables} \\ \Rightarrow & f^3_{k+1} = 0 \text{ because } wt(x_3) = wt(x_4) = wt(x_5) = 0. \end{aligned}$$

Similarly we can prove $f^{-3}_{k+1} = 0$.

For $i = 1$

$wt\frac{\partial f^1_{k+1}}{\partial x_2} = 2$ implies $\frac{\partial f^1_{k+1}}{\partial x_2} = 0$. So f^1_{k+1} does not involve x_2 variable. Since $wt\frac{\partial f^1_{k+1}}{\partial x_3} = 1 = wt\frac{\partial f^1_{k+1}}{\partial x_4} = wt\frac{\partial f^1_{k+1}}{\partial x_5}$, $\langle\frac{\partial f^1_{k+1}}{\partial x_3}, \frac{\partial f^1_{k+1}}{\partial x_4}, \frac{\partial f^1_{k+1}}{\partial x_5}\rangle$ is a 1-dimensional space. In view of Lemma 6.1, there are constants r_1, r_2, r_3 such that f^1_{k+1} is a polynomial in x_1 and $r_1x_3 + r_2x_4 + r_3x_5$.

$$f^1_{k+1} = x_1(r_1x_3 + r_2x_4 + r_3x_5)^k.$$

If $f^1_{k+1} \neq 0$, then it is easy to see that

$$\langle x_1(r_1x_3 + r_2x_4 + r_3x_5)^{k-1},\ x_2(r_1x_3 + r_2x_4 + r_3x_5)^{k-1}\rangle$$

is an irreducible $sl(2,\mathbb{C})$-submodule of dimension 2 in I. This contradicts to our hypothesis. Hence $f^1_{k+1} = 0$. Similarly we can prove that $f^{-1}_{k+1} = 0$. Thus $f = f^{-4}_{k+1} + f^{-2}_{k+1} + f^0_{k+1} + f^2_{k+1} + f^4_{k+1}$. $wt\frac{\partial f^i_{k+1}}{\partial x_3} = i = wt\frac{\partial f^i_{k+1}}{\partial x_4} = wt\frac{\partial f^i_{k+1}}{\partial x_5}$ is a nonzero even integer. This implies that $\frac{\partial f^i_{k+1}}{\partial x_3} = 0 = \frac{\partial f^i_{k+1}}{\partial x_4} = \frac{\partial f^i_{k+1}}{\partial x_5}$. Thus f^i_{k+1} involves only x_1 and x_2 variables.

Since $wt\frac{\partial f^0_{k+1}}{\partial x_3} = 0 = wt\frac{\partial f^0_{k+1}}{\partial x_4} = wt\frac{\partial f^0_{k+1}}{\partial x_5}$, $\langle \frac{\partial f^0_{k+1}}{\partial x_3}, \frac{\partial f^0_{k+1}}{\partial x_4}, \frac{\partial f^0_{k+1}}{\partial x_5} \rangle$ is at most a 1-dimensional space.

Now observe that $\frac{\partial f}{\partial x_3} = \frac{\partial f^0_{k+1}}{\partial x_3}, \frac{\partial f}{\partial x_4} = \frac{\partial f^0_{k+1}}{\partial x_4}$ and $\frac{\partial f}{\partial x_5} = \frac{\partial f^0_{k+1}}{\partial x_5}$. Hence $\dim I \leq 3$. This contradicts to our hypothesis $I = (4) \oplus (1)$.

Hence Case 2 cannot occur.

Case 3. $I = (3) \oplus (2)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $-2, -1, 0, 1$ and 2.

For $|i| \geq 4$.

$$
\begin{aligned}
& |wt\frac{\partial f^i_{k+1}}{\partial x_j}| \geq 3 && \text{for all } 1 \leq j \leq 5 \\
\Rightarrow\ & f^i_{k+1} = 0 && \text{for all } 1 \leq j \leq 5.
\end{aligned}
$$

For $i = 3$

$$
\begin{aligned}
& wt\frac{\partial f^3_{k+1}}{\partial x_2} = 4,\ wt\frac{\partial f^3_{k+1}}{\partial x_3} = 3 = wt\frac{\partial f^3_{k+1}}{\partial x_4} = wt\frac{\partial f^3_{k+1}}{\partial x_5} \\
\Rightarrow\ & \frac{\partial f^3_{k+1}}{\partial x_2} = 0 = \frac{\partial f^3_{k+1}}{\partial x_3} = \frac{\partial f^3_{k+1}}{\partial x_4} = \frac{\partial f^3_{k+1}}{\partial x_5} \\
\Rightarrow\ & f^3_{k+1} \text{ depends only on } x_1 \text{ variable} \\
\Rightarrow\ & f^3_{k+1} = c_1 x_1^3 \text{ where } c_1 \text{ is a constant.}
\end{aligned}
$$

If $c_1 \neq 0$, then $k = 2$ and

$$
(3) = \langle x_1^2, x_1x_2, x_2^2 \rangle
$$

$wt\frac{\partial f_3^2}{\partial x_2} = 3$ implies $\frac{\partial f_3^2}{\partial x_2} = 0$. Hence f_3^2 does not involve x_2 variable. $wt\frac{\partial f_3^2}{\partial x_3} = 2 = wt\frac{\partial f_3^2}{\partial x_4} = wt\frac{\partial f_3^2}{\partial x_5}$ implies that $\langle\frac{\partial f_3^2}{\partial x_3}, \frac{\partial f_3^2}{\partial x_4}, \frac{\partial f_3^2}{\partial x_5}\rangle$ is a 1-dimensional vector space. In view of Lemma 6.1, there are constants r_1, r_2 and r_3 such that f_3^2 is a polynomial in x_1 and $(r_1x_3 + r_2x_4 + r_3x_5)$.

$$f_3^2 = x_1^2(r_1x_3 + r_2x_4 + r_3x_5).$$

Similarly there are constants $r_4, r_5, r_6, r_7, r_8, r_9$ and c_2 and c_3 such that

$$f_3^1 = x_1(r_4x_3 + r_5x_4 + r_6x_5)^2 + c_2x_1^2x_2$$

and

$$f_3^0 = (r_7x_3 + r_8x_4 + r_9x_5)^3 + c_3x_1x_2(r_7x_3 + r_8x_4 + r_9x_5)$$

$\frac{\partial f_3^1}{\partial x_1} = (r_4x_3+r_5x_4+r_6x_5)^2+2c_2x_1x_2$ is a constant multiple of x_1x_2 because $wt\frac{\partial f_3^1}{\partial x_1} = 0$. This implies that $r_4x_3 + r_5x_4 + r_6x_5 = 0$. $\frac{\partial f_3^0}{\partial x_3} = 3r_7(r_7x_3 + r_8x_4 + r_9x_5)^2 + c_3r_7x_1x_2$ is a constant multiplie of x_1x_2 because $wt\frac{\partial f_3^0}{\partial x_3} = 0$. This implies that $r_7 = 0$. Similarly we have $r_8 = r_9 = 0$. It follows that

$$\begin{aligned} f &= f_3^3 + f_3^2 + f_3^1 + f_3^0 + f_3^{-1} + f_3^{-2} + f_3^{-3} \\ &= c_1x_1^3 + x_1^2(r_1x_3 + r_2x_4 + r_3x_5) + c_2x_1^2x_2 + c_4x_1x_2^2 \\ &\quad + x_2^2(r_{10}x_3 + r_{11}x_4 + r_{12}x_5) + c_6x_2^3. \end{aligned}$$

Observe that $\frac{\partial f_3^2}{\partial x_1} = 2x_1(r_1x_3 + r_2x_4 + r_3x_5)$, $\frac{\partial f_3^{-2}}{\partial x_2} = 2x_2(r_{10}x_3 + r_{11}x_4 + r_{12}x_5)$, and $X_+(\frac{\partial f_3^{-2}}{\partial x_2}) = 2x_1(r_{10}x_3+r_{11}x_4+r_{12}x_5)$. Since $wt(\frac{\partial f_3^2}{\partial x_1}) = 1 = wt(X_+(\frac{\partial f_3^{-2}}{\partial x_2}))$, the vector space $\langle 2x_1(r_1x_3 + r_2x_4 + r_3x_5), 2x_1(r_{10}x_3 + r_{11}x_4 + r_{12}x_5)\rangle$ is at most 1-dimensional. Hence

$$\begin{aligned} f = c_1x_1^3 + x_1^2(r_1x_3 + r_2x_4 + r_3x_5) + c_2x_1^2x_2 + c_4x_1x_2^2 \\ + c_5x_2^2(r_1x_3 + r_2x_4 + r_3x_5) + c_6x_2^3. \end{aligned}$$

This implies that $\dim I \leq 3$, which contradicts to our hypothesis $I = (3) \oplus (2)$. Thus $f^3_{k+1} = 0$. Similarly we can prove $f^{-3}_{k+1} = 0$.

For $i = 2$.

In this case $f = f^2_{k+1} + f^1_{k+1} + f^0_{k+1} + f^{-1}_{k+1} + f^{-2}_{k+1}$, $wt\frac{\partial f^2_{k+1}}{\partial x_2} = 3$ implies $\frac{\partial f^2_{k+1}}{\partial x_2} = 0$. Hence f^2_3 does not involve x_2 variables. $wt\frac{\partial f^2_{k+1}}{\partial x_3} = 2 = wt\frac{\partial f^2_{k+1}}{\partial x_4} = wt\frac{\partial f^2_{k+1}}{\partial x_5}$ impiles that $\langle\frac{\partial f^2_{k+1}}{\partial x_3}, \frac{\partial f^2_{k+1}}{\partial x_4}, \frac{\partial f^2_{k+1}}{\partial x_5}\rangle$ is a 1-dimensional vector space. In view of Lemma 6.1, there are constants r_1, r_2 and r_3 such that f^2_{k+1} is a polynomial in x_1 and $(r_1x_3 + r_2x_4 + r_3x_5)$

$$f^2_{k+1} = x_1^2(r_1x_3 + r_2x_4 + r_3x_5)^{k-1}.$$

If $f^2_{k+1} \neq 0$, then it is easy to see that

$$\begin{aligned} I &= (3) \oplus (2) \\ &= \langle x_1^2(r_1x_3 + r_2x_4 + r_3x_5)^{k-2}, x_1x_2(r_1x_3 + r_2x_4 + r_3x_5)^{k-2}, \\ &\quad x_2^2(r_1x_3 + r_2x_4 + r_3x_5)^{k-2}\rangle \\ &\quad \oplus\langle x_1(r_1x_3 + r_2x_4 + r_3x_5)^{k-1},\ x_2(r_1x_3 + r_2x_4 + r_3x_5)^{k-1}\rangle. \end{aligned}$$

Similarly there are constants $r_4, r_5, r_6, r_7, r_8, r_9, b_\alpha$ and a_α such that

$$f^1_{k+1} = \sum_{\alpha=0}^{[\frac{k}{2}]} b_\alpha x_1^{\alpha+1}x_2^\alpha(r_4x_3 + r_5x_4 + r_6x_5)^{k-2\alpha}$$

$$f^0_{k+1} = \sum_{\alpha=0}^{[\frac{k+1}{2}]} a_\alpha x_1^\alpha x_1^\alpha(r_4x_3 + r_5x_4 + r_6x_5)^{k-2\alpha+1}$$

$\frac{\partial f^1_{k+1}}{\partial x_1} = \sum_{\alpha=0}^{[\frac{k}{2}]}(\alpha+1)b_\alpha x_1^\alpha x_2^\alpha(r_4x_3 + r_5x_4 + r_6x_5)^{k-2\alpha}$ is of weight 0. Hence $\frac{\partial f^1_{k+1}}{\partial x_1}$ is a constant multiple of $x_1x_2(r_1x_3 + r_2x_4 + r_3x_5)^{k-2}$. This implies that $b_\alpha = 0$ for $\alpha \neq 1$ and $(r_4x_3 + r_5x_4 + r_6x_5)$ is a constant multiple of $(r_1x_3 + r_2x_4 + r_3x_5)$. So

$$f^1_{k+1} = \tilde{b}_1x_1^2x_2(r_1x_3 + r_2x_4 + r_3x_5)^{k-2}.$$

On the other hand

$$\frac{\partial f_{k+1}^0}{\partial x_3} = \sum_{\alpha=0}^{[\frac{k}{2}]} r_4(k-2\alpha+1)a_\alpha x_1^\alpha x_2^\alpha (r_4x_3+r_5x_4+r_6x_5)^{k-2\alpha}$$

$$\frac{\partial f_{k+1}^0}{\partial x_4} = \sum_{\alpha=0}^{[\frac{k}{2}]} r_5(k-2\alpha+1)a_\alpha x_1^\alpha x_2^\alpha (r_4x_3+r_5x_4+r_6x_5)^{k-2\alpha}$$

$$\frac{\partial f_{k+1}^0}{\partial x_5} = \sum_{\alpha=0}^{[\frac{k}{2}]} r_6(k-2\alpha+1)a_\alpha x_1^\alpha x_2^\alpha (r_4x_3+r_5x_4+r_6x_5)^{k-2\alpha}$$

are elements of weight 0 in I. So they are constant multiples of $x_1x_2(r_1x_3+r_2x_4+r_3x_5)^{k-2}$.

$$\frac{\partial f_{k+1}^0}{\partial x_1} = \sum_{\alpha=1}^{[\frac{k+1}{2}]} \alpha a_\alpha x_1^{\alpha-1} x_2^\alpha (r_4x_3+r_5x_4+r_6x_5)^{k-2\alpha+1}$$

is an element of weight -1 in I. So it is a constant multiple of $x_2(r_1x_3+r_2x_4+r_3x_5)^{k-1}$. It follows easily that

$$f_{k+1}^0 = \tilde{a}_1 x_1x_2(r_1x_3+r_2x_4+r_3x_5)^{k-1}.$$

Similarly we can prove that

$$\begin{aligned} f_{k+1}^{-1} &= \tilde{b}_2 x_1x_2^2(r_1x_3+r_2x_4+r_3x_5)^{k-2} \\ f_{k+1}^{-2} &= cx_2^2(r_1x_3+r_2x_4+r_3x_5)^{k-1}. \end{aligned}$$

Hence

$$\begin{aligned} f &= f_{k+1}^2 + f_{k+1}^1 + f_{k+1}^0 + f_{k+1}^{-1} + f_{k+1}^{-2} \\ &= x_1^2(r_1x_3+r_2x_4+r_3x_5)^{k-1} + \tilde{b}_1 x_1^2x_2(r_1x_3+r_2x_4+r_3x_5)^{k-2} \\ &\quad + \tilde{a}_1x_1x_2(r_1x_3+r_2x_4+r_3x_5)^{k-1} + \tilde{b}_2x_1x_2^2(r_1x_3+r_2x_4+r_3x_5)^{k-2} \\ &\quad + cx_2^2(r_1x_3+r_2x_4+r_3x_5)^{k-1}. \end{aligned}$$

This implies that $\dim I \leq 3$, which contradicts to our hypothesis $I = (3) \oplus (2)$. Thus $f_{k+1}^2 = 0$. Similarly we can prove $f_{k+1}^{-2} = 0$.

For $i = 1$.

In this case $f = f_{k+1}^{-1} + f_{k+1}^{0} + f_{k+1}^{1}$, $wt\frac{\partial f_{k+1}^1}{\partial x_3} = 1 = wt\frac{\partial f_{k+1}^1}{\partial x_4} = wt\frac{\partial f_{k+1}^1}{\partial x_5}$ implies that $\langle \frac{\partial f_{k+1}^1}{\partial x_3}, \frac{\partial f_{k+1}^1}{\partial x_4}, \frac{\partial f_{k+1}^1}{\partial x_5} \rangle$ is at most a 1-dimensional vector space. In view of Lemma 6.1, there are constants r_1, r_2 and r_3 such that f_{k+1}^1 is a polynomial in x_1, x_2 and $(r_1x_3 + r_2x_4 + r_3x_5)$.

$$f_{k+1}^1 = \sum_{\alpha=0}^{\frac{k}{2}} b_\alpha x_1^{\alpha+1} x_2^{\alpha} (r_1x_3 + r_2x_4 + r_3x_5)^{k-2\alpha}.$$

Suppose $f_{k+1}^1 \neq 0$. Let α_0 be the biggest integer such that $b_{\alpha_0} x_1^{\alpha_0+1} x_2^{\alpha_0}(r_1x_3 + r_2x_4 + r_3x_5)^{k-2\alpha_0} \neq 0$. Consider

$$\frac{\partial f_{k+1}^1}{\partial x_1} = \sum_{\alpha=0}^{\alpha_0} (\alpha+1) b_\alpha x_1^\alpha x_2^\alpha (r_1x_3 + r_2x_4 + r_3x_5)^{k-2\alpha}$$
$$X_-^\alpha \frac{\partial f_{k+1}^1}{\partial x_1} = (\alpha_0+1)! b_{\alpha_0} x_2^{2\alpha_0} (r_1x_3 + r_2x_4 + r_3x_5)^{k-2\alpha_0}.$$

Since $X_-[x_2^{2\alpha_0}(r_1x_3+r_2x_4+r_3x_5)^{k-2\alpha_0}] = 0$, we have an irreducible $sl(2,\mathbf{C})$-submodule of dimension $2\alpha_0 + 1$ in I of the following form.

$$\langle x_2^{2\alpha_0}(r_1x_3 + r_2x_4 + r_3x_5)^{k-2\alpha_0},\ x_1 x_2^{2\alpha_0-1}(r_1x_3 + r_2x_4 + r_3x_5)^{k-2\alpha_0},$$
$$x_1^2 x_2^{2\alpha_0-2}(r_1x_3 + r_2x_4 + r_3x_5)^{k-2\alpha_0}, \ldots, x_1^{2\alpha_0}(r_1x_3 + r_2x_4 + r_3x_5)^{k-2\alpha_0}\rangle.$$

As $I = (3) \oplus (2)$, we have $2\alpha_0 + 1 \leq 3$. Hence $\alpha_0 = 1$. If $\alpha_0 = 1$, then $f_{k+1}^1 = b_0x_1(r_1x_3 + r_2x_4 + r_3x_5)^k + b_1x_1^2x_2(r_1x_3 + r_2x_4 + r_3x_5)^{k-2}$. Without loss of generality, we shall assume that $r_1 \neq 0$. Suppose first that $k \geq 3$. Consider

$$\frac{\partial f_{k+1}^1}{\partial x_3} = kr_1b_0x_1(r_1x_3 + r_2x_4 + r_3x_5)^{k-1}$$
$$+ (k-2)r_1x_1^2x_2(r_1x_3 + r_2x_4 + r_3x_5)^{k-3}$$
$$X^2(\frac{\partial f_{k+1}^1}{\partial x_3}) = 2(k-2)r_1x_2^3(r_1x_3 + r_2x_4 + r_3x_5)^{k-3}.$$

As $X_-[x_2^3(r_1x_3+r_2x_4+r_3x_5)^{k-3}]=0$, we have an irreducible $sl(2,\mathbb{C})$-submodule of dimension 4 in I in the following form.

$$\langle x_2^3(r_1,x_3+r_2x_4+r_3x_5)^{k-3},\ x_1x_2^2(r_1x_3+r_2x_4+r_3x_5)^{k-3},$$
$$x_1^2x_2(r_1x_3+r_2x_4+r_3x_5)^{k-3},\ x_1^3(r_1x_3+r_2x_4+r_3x_5)^{k-3}\rangle.$$

This contradicts to our hypothesis $I=(3)\oplus(2)$. Therefore we shall assume that $k=2$

$$f_3^1=b_0x_1(r_1x_3+r_2x_4+r_3x_5)^2+b_1x_1^2x_2$$
$$\frac{\partial f_3^1}{\partial x_2}=b_1x_1^2.$$

It follows that $(3)=\langle x_1^2,x_1x_2,x_2^2\rangle\subseteq I$. Since $\frac{\partial f_3^1}{\partial x_1}=b_0(r_1x_3+r_2x_4+r_3x_5)^2+2b_1x_1x_2$ is an element of weight 0 in I, $\frac{\partial f_3^1}{\partial x_1}$ is a constant multiple of x_1x_2. Thus $b_0=0$ and

$$f_3^1=b_1x_1^2x_2.$$

Since $wt\frac{\partial f_3^{-1}}{\partial x_3}=-1=wt\frac{\partial f_3^{-1}}{\partial x_4}=wt\frac{\partial f_3^{-1}}{\partial x_5}$, $\langle\frac{\partial f_3^{-1}}{\partial x_3},\frac{\partial f_3^{-1}}{\partial x_4},\frac{\partial f_3^{-1}}{\partial x_5}\rangle$ is at most a 1-dimensional vector space. In view of Lemma 6.1, there are constants r_4,r_5 and r_6 such that

$$f_3^{-1}=c_1x_1x_2^2+c_0x_2(r_4x_3+r_5x_4+r_6x_5)^2.$$

Since $\frac{\partial f_3^{-1}}{\partial x_2}=2c_1x_1x_2+c_0(r_4x_3+r_5x_4+r_6x_5)^2$ is an element of weight 0 in I, $\frac{\partial f_3^{-1}}{\partial x_2}$ is a constant multiple of x_1x_2. Thus $c_0=0$ and

$$f_3^{-1}=c_1x_1x_2^2.$$

Hence $f=b_1x_1^2x_2+f_3^0+c_1x_1x_2^2$. This implies that $\frac{\partial f}{\partial x_3}=\frac{\partial f_3^0}{\partial x_3}$, $\frac{\partial f}{\partial x_4}=\frac{\partial f_3^0}{\partial x_4}$ and $\frac{\partial f}{\partial x_5}=\frac{\partial f_3^0}{\partial x_5}$ are elements of weight 0 in I. Thus $\dim\langle\frac{\partial f}{\partial x_3},\frac{\partial f}{\partial x_4},\frac{\partial f}{\partial x_5}\rangle\leq 1$; in particular, $\dim I\leq 3$. This contradicts to our hypothesis $I=(3)\oplus(2)$.

If $\alpha_0 = 0$, then $f_{k+1}^1 = b_0 x_1 (r_1 x_3 + r_2 x_4 + r_3 x_5)^k$. Since $\frac{\partial f_{k+1}^1}{\partial x_1} = b_0(r_1 x_3 + r_2 x_4 + r_3 x_5)^k$ is a nonzero element of weight 0 in I, we have $(1) = \langle (r_1 x_3 + r_2 x_4 + r_3 x_5)^k \rangle \subseteq I$. This contradicts to our hypothesis $I = (3) \oplus (2)$. Thus we have proved $f_{k+1}^1 = 0$. Similarly we can prove $f_{k+1}^{-1} = 0$.

For $i = 0$.

In this case $f = f_{k+1}^0$. Clearly $|wt \frac{\partial f}{\partial x_j}| \leq 1$ for all $1 \leq j \leq 5$. Thus no element in I is of weight ± 2. This contradicts to our hypothesis $I = (3) \oplus (2)$. Hence Case 3 cannot occur.

Case 4. $I = (3) \oplus (1) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $-2, 0$ and 2.

For $|i| \geq 4$.

$$\begin{aligned} & |wt \tfrac{\partial f_{k+1}^i}{\partial x_j}| \geq 3 && \text{for all } 1 \leq j \leq 5 \\ \Rightarrow\ & f_{k+1}^i = 0 && \text{for all } 1 \leq j \leq 5. \end{aligned}$$

For $i = 2$.

$$wt \frac{\partial f_{k+1}^2}{\partial x_1} = 1 \quad wt \frac{\partial f_{k+1}^2}{\partial x_2} = 3$$

$$\Rightarrow \frac{\partial f_{k+1}^2}{\partial x_1} = 0 = \frac{\partial f_{k+1}^2}{\partial x_2}$$

$\Rightarrow f_{k+1}^2$ involves only x_3, x_4 and x_5 variables

$\Rightarrow f_{k+1}^2 = 0.$

Similarly we can prove $f_{k+1}^{-2} = 0$. Hence

$$f = f_{k+1}^3 + f_{k+1}^1 + f_{k+1}^0 + f_{k+1}^{-1} + f_{k+1}^{-3}.$$

For $i = \pm 3, \pm 1$.

$$wt\frac{\partial f^i_{k+1}}{\partial x_3} = i = wt\frac{\partial f^i_{k+1}}{\partial x_4} = wt\frac{\partial f^i_{k+1}}{\partial x_5}$$

$$\Rightarrow \quad \frac{\partial f^i_{k+1}}{\partial x_3} = \frac{\partial f^i_{k+1}}{\partial x_4} = \frac{\partial f^i_{k+1}}{\partial x_5} = 0$$

$$\Rightarrow \quad f^i_{k+1} \text{ depends only on } x_1 \text{ and } x_2 \text{ variables.}$$

$wt\frac{\partial f^0_{k+1}}{\partial x_1} = -1$ and $wt\frac{\partial f^0_{k+1}}{\partial x_2} = 1$ imply $\frac{\partial f^0_{k+1}}{\partial x_1} = 0 = \frac{\partial f^0_{k+1}}{\partial x_2}$. So f^0_{k+1} depends only on x_3, x_4 and x_5 variables. Hence

$$f = f^3_{k+1}(x_1, x_2) + f^1_{k+1}(x_1, x_2) + f^0_{k+1}(x_3, x_4, x_5) + f^{-1}_{k+1}(x_1, x_2) + f^{-3}_{k+1}(x_1, x_2).$$

Because dimension of I is 5, $\frac{\partial f}{\partial x_3} = \frac{\partial f^0_{k+1}}{\partial x_3}(x_3, x_4, x_5)$, $\frac{\partial f}{\partial x_4} = \frac{\partial f^0_{k+1}}{\partial x_4}(x_3, x_4, x_5)$ and $\frac{\partial f}{\partial x_5} = \frac{\partial f^0_{k+1}}{\partial x_5}(x_3, x_4, x_5)$ are three linear independent invariant $sl(2, \mathbf{C})$ polynomials. This contradicts our hypothesis $I = (3) \oplus (1) \oplus (1)$. Thus Case 4 cannot occur.

Case 5. $I = (2) \oplus (2) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $-1, 0$ and 1.

For $|i| \geq 3$

$$|wt\frac{\partial f^i_{k+1}}{\partial x_j}| \geq 2 \qquad \text{for all } 1 \leq j \leq 5$$

$$\Rightarrow \quad \frac{\partial f^i_{k+1}}{\partial x_j} = 0 \qquad \text{for all } 1 \leq j \leq 5$$

$$\Rightarrow \quad f^i_{k+1} = 0.$$

For $i = 2$.

$$wt\frac{\partial f^2_{k+1}}{\partial x_2} = 3, \quad wt\frac{\partial f^2_{k+1}}{\partial x_3} = 2 = wt\frac{\partial f^2_{k+1}}{\partial x_4} = wt\frac{\partial f^2_{k+1}}{\partial x_5}$$

$$\Rightarrow \frac{\partial f^2_{k+1}}{\partial x_2} = 0 = \frac{\partial f^2_{k+1}}{\partial x_3} = \frac{\partial f^2_{k+1}}{\partial x_4} = \frac{\partial f^2_{k+1}}{\partial x_5}$$

$\Rightarrow f^2_{k+1}$ depends only on x_1 variable

$\Rightarrow f^2_{k+1} = cx_1^2$ where c is a constant.

Since $k \geq 2$, we conclude $f^2_{k+1} = 0$. Similarly we can prove $f^{-2}_{k+1} = 0$. Therefore we can write

$$f = f^{-1}_{k+1} + f^0_{k+1} + f^1_{k+1}$$

$wt\frac{\partial f^1_{k+1}}{\partial x_2} = 2$ implies $\frac{\partial f^1_{k+1}}{\partial x_2} = 0$. So f^1_{k+1} does not depend on x_2 variable. Since weight of $\frac{\partial f^1_{k+1}}{\partial x_1}$ is zero, it follows easily that $\frac{\partial f^1_{k+1}}{\partial x_1}$ is an invariant polynomial. Clearly invariant polynomial depends only on x_3, x_4 and x_5 variables. Hence

$$f^1_{k+1} = x_1 g_k(x_3, x_4, x_5)$$

where $g_k(x_3, x_4, x_5)$ is a homogeneous polynomial of degree k and weight 0.

It is clear that f^0_{k+1} can be written in the following form

$$f^0_{k+1} = \sum_{j=0}^{[\frac{k+1}{2}]} x_1^j x_2^j h_{k-2j+1}(x_3, x_4, x_5)$$

where $h_{k-2j+1}(x_3, x_4, x_5)$ is a homogeneous polynomial of degree $k - 2j + 1$ in x_3, x_4 and x_5 variables. Let j_0 be the largest integer such that $h_{k-2j+1}(x_3, x_4, x_5) \neq 0$. If $j_0 \geq 2$, then we consider the element $\frac{\partial f^0_{k+1}}{\partial x_1} = \sum_{j=0}^{j_0} jx_1^{j-1}x_2^j h_{k-2j+1}(x_3, x_4, x_5)$ in I. $X_-^{j_0-1}\frac{\partial f^0_{k+1}}{\partial x_1} = j_0! x_2^{2j_0-1} h_{k-2j_0+1}(x_3, x_4, x_5)$ is also an element in I. Since

$$X_-[x_2^{2j_0-1} h_{k-2j_0+1}(x_3, x_4, x_5)] = 0,$$

by applying X_+ successively to $x_2^{2j_0-1}h_{k-2j_0+1}(x_3,x_4,x_5)$, we get an irreducible $sl(2,\mathbf{C})$-submodule of dimension $2j_0$ in I in the following form

$$\langle x_2^{2j_0-1}h_{k-2j_0+1}(x_3,x_4,x_5), x_1x_2^{2j_0-2}h_{k-2j_0}(x_3,x_4,x_5),\ldots,x_1^{2j_0-1}h_{k-2j_0+1}(x_3,x_4,x_5)\rangle.$$

Since $2j_0 \geq 4$, this contradicts our hypothesis $I=(2)\oplus(2)\oplus(1)$. We have $j_0 \leq 1$ and

$$f_{k+1}^0 = h_{k+1}(x_3,x_4,x_5) + x_1x_2h_{k-1}(x_3,x_4,x_5).$$

If $j_0 = 1$, then $h_{k-1}(x_3,x_4,x_5) \neq 0$. Without loss of generality, we shall assume $\frac{\partial h_{k-1}}{\partial x_3}(x_3,x_4,x_5)$. Consider the element $\frac{\partial f_{k+1}^0}{\partial x_3} = \frac{\partial h_{k+1}}{\partial x_3}(x_3,x_4,x_5)+x_1x_2\frac{\partial h_{k-1}}{\partial x_3}(x_3,x_4,x_5)$ in I. $X_-(\frac{\partial f_{k+1}^0}{\partial x_3}) = x_2^2\frac{\partial h_{k-1}}{\partial x_3}(x_3,x_4,x_5)$ is also an element in I. So we have an irreducible $sl(2,\mathbf{C})$-submodule of dimension three in I in the following form.

$$\langle x_2^2\frac{\partial h_{k-1}}{\partial x_3}(x_3,x_4,x_5), x_1x_2\frac{\partial h_{k-1}}{\partial x_3}(x_3,x_4,x_5),\ x_1^2\frac{\partial h_{k-1}}{\partial x_3}(x_3,x_4,x_5)\rangle.$$

This contradicts to our hypothesis $I=(2)\oplus(2)\oplus(1)$. Thus $f_{k+1}^0 = h_{k+1}(x_3,x_4,x_5)$. Similarly we can prove that

$$f_{k+1}^{-1} = x_2l_k(x_3,x_4,x_5)$$

where $l_k(x_3,x_4,x_5)$ is a homogeneous polynomial of degree k. $wt(\frac{\partial f_{k+1}^{-1}}{\partial x_2}) = 0 = wt(\frac{\partial f_{k+1}^{1}}{\partial x_1})$ implies the vector space $\langle\frac{\partial f_{k+1}^{-1}}{\partial x_2}, \frac{\partial f_{k+1}^{1}}{\partial x_1}\rangle$ is one dimensional. Observe that

$$\begin{aligned} f &= f_{k+1}^1 + f_{k+1}^0 + f_{k+1}^{-1} \\ &= x_1g_k(x_3,x_4,x_5) + h_{k+1}(x_3,x_4,x_5) + x_2l_k(x_3,x_4,x_5) \\ &\Rightarrow \frac{\partial f}{\partial x_1} = \frac{\partial f_{k+1}^1}{\partial x_1} \text{ and } \frac{\partial f}{\partial x_2} = \frac{\partial f_{k+1}^{-1}}{\partial x_2}. \end{aligned}$$

Thus $\langle\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\rangle$ is a 1-dimensional vector space. It follows that $\dim I \leq 4$, which contradicts to our assumption $I=(2)\oplus(2)\oplus(1)$. Hence Case 5 cannot occur.

Case 6. $I = (2) \oplus (1) \oplus (1) \oplus (1)$.

By the same argument as Case 5, we can prove that

$$f = f^1_{k+1} + f^0_{k+1} + f^{-1}_{k+1}$$

where $f^1_{k+1} = x_1 g_k(x_3, x_4, x_5)$, $f^0_{k+1} = h_{k+1}(x_3, x_4, x_5)$ and $f^{-1}_{k+1} = x_2 l_k(x_3, x_4, x_5)$. First observe that f^1_{k+1} and f^{-1}_{k+1} are not zero, otherwise $\dim I$ would be at most 4, $wt\frac{\partial f^1_{k+1}}{\partial x_3} = wt\frac{\partial f^1_{k+1}}{\partial x_4} = wt\frac{\partial f^1_{k+1}}{\partial x_5} = 1$ implies the vector space $\langle\frac{\partial f^1_{k+1}}{\partial x_3}, \frac{\partial f^1_{k+1}}{\partial x_4}, \frac{\partial f^1_{k+1}}{\partial x_5}\rangle$ is one dimensional. In view of Lemma 6.1, there exist constants r_1, r_2 and r_3 such that

$$f^1_{k+1} = x_1(r_1x_3 + r_2x_4 + r_3x_5)^k.$$

It follows easily that

$$(2) = \langle x_1(r_1x_3 + r_2x_4 + r_3x_5)^{k-1},\ x_2(r_1x_3 + r_2x_4 + r_3x_5)^{k-1}\rangle \subseteq I.$$

Similarly there are constants r_4, r_5 and r_6 such that

$$f^{-1}_{k+1} = x_2(r_4x_3 + r_5x_4 + r_6x_5)^k$$

and

$$(2) = \langle x_1(r_4x_3 + r_5x_4 + r_6x_5)^{k-1}, x_2(r_4x_3 + r_5x_4 + r_6x_5)^{k-1}\rangle \subseteq I.$$

Consequently $r_4x_3 + r_5x_4 + r_6x_5 = c(r_1x_3 + r_2x_4 + r_3x_5)$ for some constant c. Thus

$$\begin{aligned}\langle\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\rangle &= \langle(r_1x_3 + r_2x_4 + r_3x_5)^k,\ c^k(r_1x_3 + r_2x_4 + r_3x_5)^k\rangle \\ &= \langle(r_1x_3 + r_2x_4 + r_3x_5)^k\rangle\end{aligned}$$

and $\dim I \leq 4$, which contradicts to our hypothesis $I = (2) \oplus (1) \oplus (1) \oplus (1)$. So Case 6 cannot occur.

Case 7. $I = (1) \oplus (1) \oplus (1) \oplus (1) \oplus (1)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weight 0.

For $|i| \geq 2$.

$$\begin{aligned} & |wt\frac{\partial f^i_{k+1}}{\partial x_j}| \geq 1 && \text{for all } 1 \leq j \leq 5 \\ \Rightarrow\ & \frac{\partial f^i_{k+1}}{\partial x_j} = 0 && \text{for all } 1 \leq j \leq 5 \\ \Rightarrow\ & f^i_{k+1} = 0. \end{aligned}$$

For $i = 1$.

$$wt\frac{\partial f^1_{k+1}}{\partial x_2} = 2, \quad wt\frac{\partial f^1_{k+1}}{\partial x_3} = 1 = wt\frac{\partial f^1_{k+1}}{\partial x_4} = wt\frac{\partial f^1_{k+1}}{\partial x_5}$$

$$\begin{aligned} \Rightarrow\ & \frac{\partial f^1_{k+1}}{\partial x_2} = 0 = \frac{\partial f^1_{k+1}}{\partial x_3} = \frac{\partial f^1_{k+1}}{\partial x_4} = \frac{\partial f^1_{k+1}}{\partial x_5} \\ \Rightarrow\ & f^1_{k+1} \text{ depends only on } x_1 \text{ variable} \\ \Rightarrow\ & f^1_{k+1} = cx_1 \text{ where } c \text{ is constant} \\ \Rightarrow\ & f^1_{k+1} = 0 \text{ because } k \geq 2. \end{aligned}$$

Similarly we can prove that $f^{-1}_{k+1} = 0$.

For $i = 0$.

In this case, we have $f = f^0_{k+1}$. $wt\frac{\partial f}{\partial x_1} = wt\frac{\partial f^0_{k+1}}{\partial x_1} = -1$ and $wt\frac{\partial f}{\partial x_2} = wt\frac{\partial f^0_{k+1}}{\partial x_2} = 1$ imply $\frac{\partial f}{\partial x_1} = 0 = \frac{\partial f}{\partial x_2}$. Consequently we have $\dim I \leq 3$, which contradicts to our hypothesis $I = (1) \oplus (1) \oplus (1) \oplus (1) \oplus (1)$. So Case 7 cannot occur.

Lemma 7.2. With the same hypothesis as Lemma 7.1; if dimension of I is 4, then I cannot be a $sl(2, \mathbf{C})$-submodule.

Proof. We assume on the contrary that I is a $sl(2, \mathbf{C})$-submodule.

Case 1. $I = (4)$.

This cannot occur. The argument is the same as Case 2 in the proof of Lemma 7.1.

Case 2. $I = (3) \oplus (1)$.

By the same argument as Case 4, in the proof of Lemma 7.1, we have

$$f = f^3_{k+1}(x_1,x_2) + f^1_{k+1}(x_1,x_2) + f^0_{k+1}(x_3,x_4,x_5) + f^{-1}_{k+1}(x_1,x_2) + f^{-3}_{k+1}(x_1,x_2)$$

$\frac{\partial f}{\partial x_3} = \frac{\partial f^0_{k+1}}{\partial x_3}(x_3,x_4,x_5)$, $\frac{\partial f}{\partial x_4} = \frac{\partial f^0_{k+1}}{\partial x_4}(x_3,x_4,x_5)$ and $\frac{\partial f}{\partial x_5} = \frac{\partial f^0_{k+1}}{\partial x_5}(x_3,x_4,x_5)$ are invariant $sl(2,\mathbb{C})$ polynomials. Since $I = (3) \oplus (1)$, we have $\dim\langle \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}, \frac{\partial f}{\partial x_5}\rangle \leq 1$. This implies that $\dim I \leq 3$, which contradicts to our hypothesis $\dim I = 4$. So Case 2 cannot occur.

Case 3. $U = (2) \oplus (2)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights -1 and 1. The same argument as Case 5 in the proof of Lemma 7.1 shows that

$$f = f^1_{k+1} + f^0_{k+1} + f^{-1}_{k+1}.$$

For $i = 1$

$$wt\frac{\partial f^1_{k+1}}{\partial x_1} = 0 \quad wt\frac{\partial f^1_{k+1}}{\partial x_2} = 2$$

$$\Rightarrow \frac{\partial f^1_{k+1}}{\partial x_1} = 0 = \frac{\partial f^1_{k+1}}{\partial x_2}$$

$\Rightarrow f^1_{k+1}$ depends only on x_3, x_4 and x_5 variables

$\Rightarrow f^1_{k+1} = 0$ because $wt(x_3) = wt(x_4) = wt(x_5) = 0$.

Similarly we can prove that $f^{-1}_{k+1} = 0$.

For $i = 0$.

In this case $f = f^0_{k+1}$

$$wt\frac{\partial f}{\partial x_3} = wt\frac{\partial f^0_{k+1}}{\partial x_3} = 0$$

$$wt\frac{\partial f}{\partial x_4} = wt\frac{\partial f^0_{k+1}}{\partial x_4} = 0$$

$$wt\frac{\partial f}{\partial x_5} = wt\frac{\partial f^0_{k+1}}{\partial x_5} = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial x_3} = \frac{\partial f}{\partial x_4} = \frac{\partial f}{\partial x_5} = 0$$

$$\Rightarrow \quad \dim I \leq 2.$$

This contradicts to our hypothesis $I = (2) \oplus (2)$.

Case 4. $I = (2) \oplus (1) \oplus (1)$.

By the same argument as Case 5 in the proof of Lemma 7.1, we can prove that

$$f = f^1_{k+1} + f^0_{k+1} + f^{-1}_{k+1}$$

where $f^1_{k+1} = x_1 g_k(x_3, x_4, x_5)$, $f^0_{k+1} = h_{k+1}(x_3, x_4, x_5)$ and $f^{-1}_{k+1} = x_2 l_k(x_3, x_4, x_5)$. If f^1_{k+1} and f^{-1}_{k+1} were zero, then $\frac{\partial f}{\partial x_1} = \frac{\partial f^0_{k+1}}{\partial x_1} = 0$, $\frac{\partial f}{\partial x_2} = \frac{\partial f^0_{k+1}}{\partial x_2} = 0$ and $\frac{\partial f}{\partial x_3} = \frac{\partial f^0_{k+1}}{\partial x_3}$, $\frac{\partial f}{\partial x_4} = \frac{\partial f^0_{k+1}}{\partial x_4}$, $\frac{\partial f}{\partial x_5} = \frac{\partial f^0_{k+1}}{\partial x_5}$ are $sl(2, \mathbf{C})$ invariant polynomials. This would imply that I cannot contain (2). Without loss of generality we shall assume that $f^1_{k+1} \neq 0$. $wt\frac{\partial f^1_{k+1}}{\partial x_3} = wt\frac{\partial f^1_{k+1}}{\partial x_4} = wt\frac{\partial f^1_{k+1}}{\partial x_5} = 1$ implies the vector space $\langle \frac{\partial f^1_{k+1}}{\partial x_3}, \frac{\partial f^1_{k+1}}{\partial x_4}, \frac{\partial f^1_{k+1}}{\partial x_5} \rangle$ is 1-dimensional. In view of Lemma 6.1, there exist constants r_1, r_2, and r_3 such that

$$f^1_{k+1} = x_1(r_1 x_3 + r_2 x_4 + r_3 x_5)^k.$$

It follows easily that

$(1) = \langle (r_1x_3 + r_2x_4 + r_3x_5)^k \rangle \subseteq I$

$(2) = \langle x_1(r_1x_3 + r_2x_4 + r_3x_5)^{k-1}, x_2(r_1x_3 + r_2x_4 + r_3x_5)^{k-1} \rangle \subseteq I.$

weights of $\frac{\partial f_{k+1}^{-1}}{\partial x_3}$, $\frac{\partial f_{k+1}^{-1}}{\partial x_4}$ and $\frac{\partial f_{k+1}^{-1}}{\partial x_5}$ are equal to -1. Hence $\frac{\partial f_{k+1}^{-1}}{\partial x_3}$, $\frac{\partial f_{k+1}^{-1}}{\partial x_4}$ and $\frac{\partial f_{k+1}^{-1}}{\partial x_5}$ are constant multiples of $x_2(r_1x_3 + r_2x_4 + r_3x_5)^{k-1}$. It follows easily that $l_k(x_3, x_4, x_5) = c(r_1x_3 + r_2x_4 + r_3x_5)^k$ and

$$f_{k+1}^{-1} = cx_2(r_1x_3 + r_2x_4 + r_3x_5)^k.$$

Clearly $\frac{\partial f_{k+1}^{0}}{\partial x_3}(x_3, x_4, x_5)$, $\frac{\partial f_{k+1}^{0}}{\partial x_4}(x_3, x_4, x_5)$ and $\frac{\partial f_{k+1}^{0}}{\partial x_5}(x_3, x_4, x_5)$ are $sl(2,\mathbf{C})$ invariant polynomial in I. Observe that

$$\begin{aligned}
f &= x_1(r_1x_3 + r_2x_4 + r_3x_5)^k + f_{k+1}^0(x_3, x_4, x_5) + cx_2(r_1x_3 + r_2x_4 + r_3x_5)^k \\
\frac{\partial f}{\partial x_1} &= (r_1x_3 + r_2x_4 + r_3x_5)^k \\
\frac{\partial f}{\partial x_2} &= c(r_1x_3 + r_2x_4 + r_3x_5)^k \\
\frac{\partial f}{\partial x_3} &= kr_1[x_1(r_1x_3 + r_2x_4 + r_3x_5)^{k-1} + cx_2(r_1x_3 + r_2x_4 + r_3x_5)^{k-1}] + \frac{\partial f_{k+1}^0}{\partial x_3}(x_3, x_4, x_5) \\
\frac{\partial f}{\partial x_4} &= kr_2[x_1(r_1x_3 + r_2x_4 + r_3x_5)^{k-1} + cx_2(r_1x_3 + r_2x_4 + r_3x_5)^{k-1}] + \frac{\partial f_{k+1}^0}{\partial x_4}(x_3, x_4, x_5) \\
\frac{\partial f}{\partial x_5} &= kr_3[x_1(r_1x_3 + r_2x_4 + r_3x_5)^{k-1} + cx_2(r_1x_3 + r_2x_4 + r_3x_5)^{k-1}] + \frac{\partial f_{k+1}^0}{\partial x_5}(x_3, x_4, x_5).
\end{aligned}$$

It is clear that we have

$$I \subseteq \langle [x_1(r_1x_3 + r_2x_4 + r_3x_5)^{k-1} + cx_2(r_1x_3 + r_2x_4 + r_3x_5)^{k-1}] \rangle \oplus (1) \oplus (1).$$

Since $(2) = \langle x_1(r_1x_3 + r_2x_4 + r_3x_5)^{k-1}, x_2(r_1x_3 + r_2x_4 + r_3x_5)^{k-1} \rangle \subseteq I$, there exist polynomial $p(x_1, x_4, x_5) \in (1) \oplus (1)$ and constant c_1 such that

$$x_2(r_1x_3 + r_2x_4 + r_3x_5)^{k-1} = c_1[x_1(r_1x_3 + r_2x_4 + r_3x_5)^{k-1} + cx_2(r_1x_3 + r_2x_4 + r_3x_5)^{k-1}]$$

$$+ p(x_3, x_4, x_5)$$

$$\Rightarrow p(x_3, x_4, x_5) = -c_1x_1(r_1x_3 + r_2x_4 + r_3x_5)^{k-1} + (1 - c_1c)x_2(r_1x_3 + r_2x_4 + r_3x_5)^{k-1}$$

$$\Rightarrow c_1 = 0 \text{ and } x_2(r_1x_3 + r_2x_4 + r_3x_5)^{k-1} = p(x_3, x_4, x_5)$$

$$\Rightarrow x_2(r_1x_3 + r_2x_4 + r_3x_5)^{k-1} = 0.$$

This contradicts the fact that $(2) = \langle x_1(r_1x_3 + r_2x_4 + r_3x_5)^{k-1}, x_2(r_1x_3 + r_2x_4 + r_3x_5)^{k-1}\rangle$.

Case 5. $I = (1) \oplus (1) \oplus (1) \oplus (1)$.

This case cannot occur. The argument is the same as those given in Case 7 in the proof of Lemma 7.1.

Lemma 7.3. With the same hypothesis as Lemma 7.1; if I is a $sl(2, \mathbf{C})$-submodule of dimension 3, then f is a polynomial in x_3, x_4 and x_5 variables and

$$I = \langle \frac{\partial f}{\partial x_3}(x_3, x_4, x_5)\rangle \oplus \langle \frac{\partial f}{\partial x_4}(x_3, x_4, x_5)\rangle \oplus \frac{\partial f}{\partial x_5}(x_3, x_4, x_5)\rangle.$$

Proof. Case 1. $I = (3)$.

By the same argument as Case 4 in the proof of Lemma 7.1 we have

$$f = f^3_{k+1}(x_1, x_2) + f^1_{k+1}(x_1, x_2) + f^0_{k+1}(x_3, x_4, x_5) + f^{-1}_{k+1}(x_1, x_2) + f^{-3}_{k+1}(x_1, x_2)$$

$\frac{\partial f^0_{k+1}}{\partial x_3}(x_3, x_4, x_5)$, $\frac{\partial f^0_{k+1}}{\partial x_4}(x_3, x_4, x_5)$ and $\frac{\partial f^0_{k+1}}{\partial x_5}(x_3, x_4, x_5)$ are $sl(2, \mathbf{C})$ invariant polynomial in I. Since $I = (3)$, we conclude that $\frac{\partial f^0_{k+1}}{\partial x_3}(x_3, x_4, x_5) = 0 = \frac{\partial f^0_{k+1}}{\partial x_4}(x_3, x_4, x_5) = \frac{\partial f^0_{k+1}}{\partial x_5}(x_3, x_4, x_5)$. It follows that $f^0_{k+1}(x_3, x_4, x_5) = 0$ and f depends only on x_1 and x_2 variables. Hence $\dim I \leq 2$, which contradicts to our hypothesis $I = (3)$. Case 1 cannot occur.

Case 2. $I = (2) \oplus (1)$.

This case cannot occur. The argument is the same as those given in Case 4 in the proof of Lemma 7.2.

Case 3. $I = (1) \oplus (1) \oplus (1)$.

By the same argument in Case 7 in the proof of Lemma 7.1, we have $f = f_{k+1}^0(x_3, x_4, x_5)$. Therefore

$$I = \langle \frac{\partial f}{\partial x_3}(x_3, x_4, x_5) \rangle \oplus \langle \frac{\partial f}{\partial x_4}(x_3, x_4, x_5) \rangle \oplus \langle \frac{\partial f}{\partial x_5}(x_3, x_4, x_5) \rangle.$$

Lemma 7.4. With the same hypothesis as Lemma 7.1; if I is a $sl(2, \mathbf{C})$-submodule of dimension 2, then f is a polynomial in x_3, x_4 and x_5 variables and

$$I = \begin{cases} \langle \frac{\partial f}{\partial x_3}(x_3, x_4, x_5) \rangle \oplus \frac{\partial f}{\partial x_4}(x_3, x_4, x_5) \rangle & \text{or} \\ \langle \frac{\partial f}{\partial x_3}(x_3, x_4, x_5) \rangle \oplus \frac{\partial f}{\partial x_5}(x_3, x_4, x_5) \rangle & \text{or} \\ \langle \frac{\partial f}{\partial x_4}(x_3, x_4, x_5) \rangle \oplus \frac{\partial f}{\partial x_5}(x_3, x_4, x_5) \rangle & \end{cases}$$

Proof. Case 1. $I = (2)$.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights -1 and 1. By the same argument as Case 5 in the proof of Lemma 7.1, we can prove that

$$f = f_{k+1}^1 + f_{k+1}^0 + f_{k+1}^{-1}$$

where $f^1_{k+1} = x_1 g_k(x_3, x_4, x_5), f^0_{k+1} = h_{k+1}(x_3, x_4, x_5)$ and $f^{-1}_{k+1} = x_2 l_k(x_3, x_4, x_5)$

$$wt\frac{\partial f^1_{k+1}}{\partial x_1} = 0 = wt\frac{\partial f^{-1}_{k+1}}{\partial x_2} = wt\frac{\partial f^0_{k+1}}{\partial x_3} = wt\frac{\partial f^0_{k+1}}{\partial x_4} = wt\frac{\partial f^0_{k+1}}{\partial x_5}$$

$$\Rightarrow \frac{\partial f^1_{k+1}}{\partial x_1} = g_k(x_3, x_4, x_5) = 0, \; \frac{\partial f^{-1}_{k+1}}{\partial x_2} = l_k(x_3, x_4, x_5) = 0$$

$$\frac{\partial f^0_{k+1}}{\partial x_3}(x_3, x_4, x_5) = 0 = \frac{\partial f^0_{k+1}}{\partial x_4}(x_3, x_4, x_5) = \frac{\partial f^0_{k+1}}{\partial x_5}(x_3, x_4, x_5)$$

$$\Rightarrow f^1_{k+1} = 0 = f^{-1}_{k+1} = f^0_{k+1}$$

$$\Rightarrow f = 0.$$

Hence Case 1 cannot occur.

Case 2. $I = (1) \oplus (1)$.

By the same argument in Case 7 in the proof of Lemma 7.1 we have $f = f^0_{k+1}(x_3, x_4, x_5)$.

Therefore

$$I = \begin{cases} \langle \frac{\partial f}{\partial x_3}(x_3, x_4, x_5)\rangle \oplus \frac{\partial f}{\partial x_4}(x_3, x_4, x_5)\rangle & \text{or} \\ \langle \frac{\partial f}{\partial x_3}(x_3, x_4, x_5)\rangle \oplus \frac{\partial f}{\partial x_5}(x_3, x_4, x_5)\rangle & \text{or} \\ \langle \frac{\partial f}{\partial x_4}(x_3, x_4, x_5)\rangle \frac{\partial f}{\partial x_5}(x_3, x_4, x_5)\rangle & \end{cases}$$

Q.E.D.

Lemma 7.5. With the same hypothesis as Lemma 7.1; if I is a $sl(2, \mathbf{C})$-submodule of dimension 1, then $f = (r_1x_3 + r_2x_4 + r_3x_5)^{k+1}$ for some constants r_1, r_2, r_3 not all zero and

$$I = \langle (r_1x_3 + r_2x_4 + r_3x_5)^k \rangle.$$

Proof. By the same argument in Case 7 in the proof of Lemma 7.1 we have $f = f^0_{k+1}(x_3, x_4, x_5)$. Since $\langle \frac{\partial f}{\partial x_3}(x_3, x_4, x_5), \frac{\partial f}{\partial x_4}(x_3, x_4, x_5), \frac{\partial f}{\partial x_5}(x_3, x_4, x_5)\rangle$ is 1-dimensional, in view of Lemma 6.1, we have $f = (r_1x_3 + r_2x_4 + r_3x_5)^{k+1}$ for some constants r_1, r_2, r_3 not all zero. Q.E.D.

Proposition 7.6. Suppose $sl(2,\mathbf{C})$ acts on the space of homogeneous polynomials of degree $k \geq 2$ in x_1, x_2, x_3, x_4 and x_5 variables via

$$\begin{aligned} \tau &= x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} \\ X_+ &= x_1 \frac{\partial}{\partial x_2} \\ X_- &= x_2 \frac{\partial}{\partial x_1}. \end{aligned}$$

Suppose the weight of x_i is given by the corresponding coefficient in the expression of τ above i.e.

$$wt(x_1) = 1,\ wt(x_2) = -1,\ wt(x_3) = 0,\ wt(x_4) = 0,\ wt(x_5) = 0.$$

Let I be the complex vector subspace spanned by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}$, and $\frac{\partial f}{\partial x_5}$, where f is a homogeneous polynomial of degree $k+1$. If I is a $sl(2,\mathbf{C})$-submodule, then I is one of the following:

(i). f is a polynomial in x_3, x_4 and x_5 variables and

$$\begin{aligned} I &= (1) \oplus (1) \oplus (1) \\ &= \langle \frac{\partial f}{\partial x_3}(x_3, x_4, x_5) \rangle \oplus \langle \frac{\partial f}{\partial x_4}(x_3, x_4, x_5) \rangle \oplus \frac{\partial f}{\partial x_5}(x_3, x_4, x_5) \rangle. \end{aligned}$$

(ii) f is a polynomial in x_3, x_4 and x_5 variables and

$$I = (1) \oplus (1) = \begin{cases} \langle \frac{\partial f}{\partial x_3}(x_3, x_4, x_5) \rangle \oplus \frac{\partial f}{\partial x_4}(x_3, x_4, x_5) \rangle & \text{or} \\ \langle \frac{\partial f}{\partial x_4}(x_3, x_4, x_5) \rangle \oplus \frac{\partial f}{\partial x_5}(x_3, x_4, x_5) \rangle & \text{or} \\ \langle \frac{\partial f}{\partial x_3}(x_3, x_4, x_5) \rangle \oplus \frac{\partial f}{\partial x_5}(x_3, x_4, x_5) \rangle. \end{cases}$$

(iii) $f = (r_1 x_3 + r_2 x_4 + r_3 x_5)^{k+1}$ where r_1, r_2 and r_3 are constants not all zero and $I = (r_1 x_3 + r_2 x_4 + r_3 x_5)^k$.

§8. Proof of the Main Theorem

In this section, we shall only give a proof of Theorem 4 in Section 1 because the proof of Theorem 1, Theorem 2 and Theorem 3 are similar.

Theorem 8.1. Suppose $sl(2,\mathbf{C})$ acts on the space of homogeneous polynomials of degree $k \geq 2$ in x_1, x_2, x_3, x_4 and x_5 variables via

$$\tau = 4x_1\frac{\partial}{\partial x_1} + 2x_2\frac{\partial}{\partial x_2} - 2x_4\frac{\partial}{\partial x_5} - 4x_5\frac{\partial}{\partial x_5}$$

$$X_+ = 4x_1\frac{\partial}{\partial x_2} + 6x_2\frac{\partial}{\partial x_3} + 6x_3\frac{\partial}{\partial x_4} + 4x_4\frac{\partial}{\partial x_5}$$

$$X_- = x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3} + x_5\frac{\partial}{\partial x_4}.$$

Suppose the weight of x_i is given by the corresponding coefficient in the expression of τ above i.e.

$$wt(x_1) = 4,\ wt(x_2) = 2,\ wt(x_3) = 0,\ wt(x_4) = -2,\ wt(x_5) = -4.$$

Let I be the complex vector subspace spanned by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}$ and $\frac{\partial f}{\partial x_5}$ where f is a homogeneous polynomial of degree $k+1$. If I is a $sl(2,\mathbf{C})$-submodule, then f is an invariant $sl(2,\mathbf{C})$ polynomial in x_1, x_2, x_3, x_4 and x_5 variables, and I is an irreducible $sl(2,\mathbf{C})$-submodule of dimension 5. Moreover, we have

$X_-(\frac{\partial f}{\partial x_1}) = 0, X_-(\frac{\partial f}{\partial x_2}) = -\frac{\partial f}{\partial x_1}, X_-(\frac{\partial f}{\partial x_3}) = -\frac{\partial f}{\partial x_2}, X_-(\frac{\partial f}{\partial x_4}) = -\frac{\partial f}{\partial x_3}, X_-(\frac{\partial f}{\partial x_5}) = -\frac{\partial f}{\partial x_4}$

and

$X_+(\frac{\partial f}{\partial x_1}) = -4\frac{\partial f}{\partial x_2}, X_+(\frac{\partial f}{\partial x_2}) = -6\frac{\partial f}{\partial x_3}, X_+(\frac{\partial f}{\partial x_3}) = -6\frac{\partial f}{\partial x_4}, X_+(\frac{\partial f}{\partial x_4}) = -4\frac{\partial f}{\partial x_5}, X_+(\frac{\partial f}{\partial x_5}) = 0.$

Proof. Let us first observe that

$$[\frac{\partial}{\partial x_1}, X_-] = 0, [\frac{\partial}{\partial x_2}, X_-] = \frac{\partial}{\partial x_1}, [\frac{\partial}{\partial x_3}, X_-] = \frac{\partial}{\partial x_2}, [\frac{\partial}{\partial x_4}, X_-] = \frac{\partial}{\partial x_3}, [\frac{\partial}{\partial x_5}, X_-] = \frac{\partial}{\partial x_4}.$$

In view of Proposition 2.5, we only need to consider the following two cases.

Case 1. f is a homogeneous polynomial of weight 0 and I is an irreducible $sl(2,\mathbf{C})$-submodule of dimension 5.

Elements in I are linear combinations of homogeneous polynomials of degree k and weights $-4,-2,0,2$ and 4. Since f is of weight 0, we have $wt(\frac{\partial f}{\partial x_1}) = -4$, $wt(\frac{\partial f}{\partial x_2}) = -2$, $wt(\frac{\partial f}{\partial x_3}) = 0$, $wt(\frac{\partial f}{\partial x_4}) = 2$ and $wt(\frac{\partial f}{\partial x_5}) = 4$. We are going to prove that $X_- f = 0$. Observe that

$$wt(X_-(\frac{\partial f}{\partial x_1})) = -6,\ wt(X_-(\frac{\partial f}{\partial x_2})) = -4,\ wt(X_-(\frac{\partial f}{\partial x_3})) = -2$$

$$wt(X_-(\frac{\partial f}{\partial x_4})) = 0,\ wt(X_-(\frac{\partial f}{\partial x_5})) = -2.$$

$$\Rightarrow\quad X_-(\frac{\partial f}{\partial x_1}) = 0,\ X_-(\frac{\partial f}{\partial x_2}) = c_1\frac{\partial f}{\partial x_1},\ X_-(\frac{\partial f}{\partial x_3}) = c_2\frac{\partial f}{\partial x_2}$$

$$X_-(\frac{\partial f}{\partial x_4}) = c_3\frac{\partial f}{\partial x_3},\ X_-(\frac{\partial f}{\partial x_5}) = c_4\frac{\partial f}{\partial x_4}$$

$$\Rightarrow\quad \frac{\partial}{\partial x_1}(X_- f) = X_-(\frac{\partial f}{\partial x_1}) = 0$$

$$\frac{\partial}{\partial x_2}(X_- f) = x_-(\frac{\partial f}{\partial x_2}) + \frac{\partial f}{\partial x_1} = (c_1+1)\frac{\partial f}{\partial x_1}$$

$$\frac{\partial}{\partial x_3}(X_- f) = X_-(\frac{\partial f}{\partial x_3}) + \frac{\partial f}{\partial x_2} = (c_1+1)\frac{\partial f}{\partial x_2}$$

$$\frac{\partial}{\partial x_4}(X_- f) = X_-(\frac{\partial f}{\partial x_4}) + \frac{\partial f}{\partial x_3} = (c_3+1)\frac{\partial f}{\partial x_3}$$

$$\frac{\partial}{\partial x_5}(X_- f) = X_-(\frac{\partial f}{\partial x_5}) + \frac{\partial f}{\partial x_4} = (c_4+1)\frac{\partial f}{\partial x_4}$$

$$\Rightarrow\quad \frac{\partial}{\partial x_1}(X_-^2 f) = X_-(\frac{\partial}{\partial x_1}X_- f) = 0$$

$$\frac{\partial}{\partial x_2}(X_-^2 f) = X_-(\frac{\partial}{\partial x_2}X_- f) + \frac{\partial}{\partial x_1}X_- f = (c_1+1)X_-(\frac{\partial f}{\partial x_1}) = 0$$

$$\frac{\partial}{\partial x_3}(X_-^2 f) = X_-(\frac{\partial}{\partial x_3}X_- f) + \frac{\partial}{\partial x_2}(X_- f) = X_-[(c_2+1)\frac{\partial f}{\partial x_2}] + (c_1+1)\frac{\partial f}{\partial x_1}$$

$$= (c_1c_2 + 2c_1 + 1)\frac{\partial f}{\partial x_1}$$

$$\frac{\partial}{\partial x_4}(X_-^2 f) = X_-(\frac{\partial}{\partial x_4}X_- f) + \frac{\partial}{\partial x_3}X_- f = X_-[(c_3+1)\frac{\partial f}{\partial x_3}] + (c_2+1)\frac{\partial f}{\partial x_2}$$

$$= (c_2c_3 + 2c_2 + 1)\frac{\partial f}{\partial x_2}$$

$$\frac{\partial}{\partial x_5}(X_-^2 f) = X_-(\frac{\partial}{\partial x_5}X_- f) + \frac{\partial}{\partial x_4}X_- f = X_-[(c_4+1)\frac{\partial f}{\partial x_4}] + (c_3+1)\frac{\partial f}{\partial x_3}$$

$$= (c_3c_4 + 2c_3 + 1)\frac{\partial f}{\partial x_3}.$$

Therefore $X_-^2 f$ does not depend on x_1 and x_2 variables. Suppose $\frac{\partial}{\partial x_5}(X_-^2 f)$ is not zero. The $\frac{\partial f}{\partial x_3} = \frac{1}{c_3c_4+2c_3+1}\frac{\partial}{\partial x_5}(X_-^2 f)$ is a weight 0 homogeneous polynomial of degree k in x_3, x_4, x_5 variables. There exists a constant $d_1 \neq 0$ such that $\frac{\partial f}{\partial x_3} = d_1 x_3^k$. It follows that $\langle x_3^k, X_-(x_3^k), X_-^2(x_3^k), X_-^3(x_3^{-k}), X_+(x_3^k), X_+^2(x_3^k), X_+^3(x_3^k)\rangle$ is a six dimensional linear subspace in I. This contradicts to our hypothesis $\dim I = 5$. Therefore we conclude that $\frac{\partial}{\partial x_5}(X_-^2 f) = 0$ and $X_-^2 f$ depends only on x_3 and x_4 variables. We next claim that $\frac{\partial}{\partial x_4}(X_-^2 f) = 0$. Suppose on the contrary that $\frac{\partial}{\partial x_4}(X_-^2 f) \neq 0$. Then $\frac{\partial f}{\partial x_2} = \frac{1}{c_2c_3+2c_2+1}\frac{\partial}{\partial x_4}(X_-^2 f)$ is a weight -2 homogeneous polynomial of degree k in x_3 and x_4 variables. There exists a nonzero constant d_2 such that $\frac{\partial f}{\partial x_2} = d_2 x_4 x_3^{k-1}$. It follows that

$$\langle x_4x_3^{k-1}, X_+(x_4x_3^{k-1}), X_+^2(x_4x_3^{k-1}), X_+^3(x_4x_3^{k-1}), X_+^4(x_4x_3^{k-1}), X_-(x_4x_3^{k-1})\rangle$$

is a six dimensional linear subspace in I. This contradicts to our hypothesis $\dim I = 5$. Therefore our claim $\frac{\partial}{\partial x_4}(X_-^2 f) = 0$ is proved. So $X_-^2 f$ depends only on x_3 variable. Suppose on the contrary that $\frac{\partial}{\partial x_3}(X_-^2 f) \neq 0$. Then $\frac{\partial f}{\partial x_1} = \frac{1}{c_1c_2+2c_1+1}\frac{\partial}{\partial x_3}(X_-^2 f)$ is a weight -4 homogeneous polynomial of degree k in x_3 variable. But obviously this is not possible. We conclude that $\frac{\partial}{\partial x_3}(X_-^2 f) = 0$ and hence $X_-^2 f = 0$. It follows that

$$\left.\begin{array}{llll} c_1c_2 & +2c_1 & +1 & =0 \\ c_2c_3 & +2c_2 & +1 & =0 \\ c_3c_4 & +2c_3 & +1 & =0 \end{array}\right\} \tag{8.1}$$

Suppose $c_4+1 \neq 0$. Then equation (8.1) implies $c_3+1 \neq 0$, $c_2+1 \neq 0$ and $c_1+1 \neq 0$. It follows that $\frac{\partial f}{\partial x_4} = \frac{1}{c_4+1}\frac{\partial}{\partial x_5}(X_-f)$, $\frac{\partial f}{\partial x_3} = \frac{1}{c_3+1}\frac{\partial}{\partial x_4}(X_-f)$, $\frac{\partial f}{\partial x_2} = \frac{1}{c_1+1}\frac{\partial}{\partial x_3}(X_-f)$ and $\frac{\partial f}{\partial x_1} = \frac{1}{c_1+1}\frac{\partial}{\partial x_2}(X_-f)$ are homogeneous polynomial of degree k in x_2, x_3, x_4 and x_5 variables. This implies that $\frac{\partial^2 f}{\partial x_1 x_4} = 0 = \frac{\partial^2 f}{\partial x_1 \partial x_3} = \frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_1^2}$. So f does not involve x_1^2, x_1x_2, x_1, x_3 and x_1x_4. We can write f in the following form.

$$f = cx_1^{\frac{k+1}{2}}x_5^{\frac{k+1}{2}} + g_{k+1}(x_2,x_3,x_4,x_5)$$
$$X_-f = c\frac{k+1}{2}x_1^{\frac{k-1}{2}}x_2x_5^{\frac{k+1}{2}} + X_-(g_{k+1}(x_2,x_3,x_4,x_5))$$

where c is a constant and $g_{k+1}(x_2,x_3,x_4,x_5)$ is a homogeneous polynomial of degree $k+1$ and weight 0.

$$\frac{\partial}{\partial x_2}(X_-f) = (c_1+1)\frac{\partial f}{\partial x_1}$$

$$\Rightarrow c\frac{k+1}{2}x_1^{\frac{k-1}{2}}x_5^{\frac{k+1}{2}} + \frac{\partial}{\partial x_2}[X_-g(x_2,x_3,x_4,x_5)] = (c_1+1)c\frac{k+1}{2}x_1^{\frac{k-1}{2}}x_5^{\frac{k+1}{2}}$$
$$\Rightarrow \frac{\partial}{\partial x_2}[X_-g(x_2,x_3,x_4,x_5)] = c_1c\frac{k+1}{2}x_1^{\frac{k-1}{2}}x_5^{\frac{k+1}{2}}.$$

Notice that $\frac{\partial}{\partial x_2}X_-g(x_2,x_3,x_4,x_5)$ involves only x_2, x_3, x_4 and x_5 variables. So by the above equation, we have $c = 0$ because $c_1 \neq 0$ by equation (8.1). This means that f depends only on x_2, x_3, x_4 and x_5 variables. Hence we have $\frac{\partial f}{\partial x_1} = 0$ and $\dim I \leq 4$ which contradicts to our hypothesis $\dim I = 5$.

The remaining case to consider is the case $c_4 + 1 = 0$. In view of equation (8.1), we have also $c_3+1 = 0 = c_2+1 = c_1+1$. Thus we have $\frac{\partial}{\partial x_i}(X_-f) = 0$ for all $1 \leq i \leq 5$. Consequently we have $X_-f = 0$, $X_-(\frac{\partial f}{\partial x_1}) = 0$, $X_-(\frac{\partial f}{\partial x_2}) = -\frac{\partial f}{\partial x_1}$, $X_-(\frac{\partial f}{\partial x_3}) = -\frac{\partial f}{\partial x_2}$, $X_-(\frac{\partial f}{\partial x_4}) = -\frac{\partial f}{\partial x_3}$ and $X_-(\frac{\partial f}{\partial x_5}) = -\frac{\partial f}{\partial x_4}$.

Similarly we can prove $X_+ f = 0$, $X_+(\frac{\partial f}{\partial x_1}) = -4\frac{\partial f}{\partial x_2}$, $X_+(\frac{\partial f}{\partial x_2}) = -6\frac{\partial f}{\partial x_3}$, $X_+(\frac{\partial f}{\partial x_3}) = -6\frac{\partial f}{\partial x_4}$, $X_+(\frac{\partial f}{\partial x_4}) = -4\frac{\partial f}{\partial x_5}$ and $X_+(\frac{\partial f}{\partial x_5}) = 0$.

Case 2. f is a polynomial in x_2, x_3 and x_4 variables of weight 0 and I is an irreducible submodule of dimension 3.

Elements of I are linear combinations of homogeneous polynomials of degree k and weights $-2, 0$ and 2.

$$wt(\frac{\partial f}{\partial x_2}) = -2, \quad wt(\frac{\partial f}{\partial x_3}) = 0, \quad wt(\frac{\partial f}{\partial x_4}) = 2$$

$$\Rightarrow \quad wt[X_-(\frac{\partial f}{\partial x_2})] = -4, \quad wt[X_-(\frac{\partial f}{\partial x_3})] = -2, \quad wt[X_-(\frac{\partial f}{\partial x_4})] = 0$$

$$\Rightarrow \quad X_-(\frac{\partial f}{\partial x_2}) = 0, \quad X_-(\frac{\partial f}{\partial x_3}) = b_1\frac{\partial f}{\partial x_2}, \quad X_-(\frac{\partial f}{\partial x_4}) = b_2\frac{\partial f}{\partial x_3}$$

$$\Rightarrow \quad \frac{\partial}{\partial x_1}(X_- f) = X_-(\frac{\partial f}{\partial x_1}) = 0$$

$$\left.\begin{array}{llll} \frac{\partial}{\partial x_2}(X_- f) & = X_-(\frac{\partial f}{\partial x_2}) & + \frac{\partial f}{\partial x_1} & = 0 \\ \frac{\partial}{\partial x_3}(X_- f) & = X_-(\frac{\partial f}{\partial x_3}) & + \frac{\partial f}{\partial x_2} & = (b_1+1)\frac{\partial f}{\partial x_2} \\ \frac{\partial}{\partial x_4}(X_- f) & = X_-(\frac{\partial f}{\partial x_4}) & + \frac{\partial f}{\partial x_3} & = (b_2+1)\frac{\partial f}{\partial x_3} \\ \frac{\partial}{\partial x_5}(X_- f) & = X_-(\frac{\partial f}{\partial x_5}) & + \frac{\partial f}{\partial x_4} & = \frac{\partial f}{\partial x_4} \end{array}\right\} \tag{8.2}$$

$$\Rightarrow \quad \frac{\partial}{\partial x_1}(X_-^2 f) = X_-^2(\frac{\partial f}{\partial x_1}) = 0$$

$$\frac{\partial}{\partial x_2}(X_-^2 f) = X_-(\frac{\partial}{\partial x_2}X_- f) + \frac{\partial}{\partial x_1}(X_- f) = 0$$

$$\frac{\partial}{\partial x_3}(X_-^2 f) = X_-(\frac{\partial}{\partial x_3}X_- f) + \frac{\partial}{\partial x_2}(X_- f) = 0$$

$$\frac{\partial}{\partial x_4}(X_-^2 f) = X_-(\frac{\partial}{\partial x_4}X_- f) + \frac{\partial}{\partial x_3}(X_- f) = X_-[(b_2+1)\frac{\partial f}{\partial x_3}] + (b_1+1)\frac{\partial f}{\partial x_2}$$

$$= (b_1 b_2 + 2b_1 + 1)\frac{\partial f}{\partial x_2}$$

$$\frac{\partial}{\partial x_5}(X_-^2 f) = X_-(\frac{\partial}{\partial x_5}X_- f) + \frac{\partial}{\partial x_4}(X_- f) = X_-(\frac{\partial f}{\partial x_4}) + (b_2+1)\frac{\partial f}{\partial x_3}$$

$$= (2b_2 + 1)\frac{\partial f}{\partial x_3}$$

$\Rightarrow$ $X_-^2 f$ involves only x_4 and x_5 variables

$\Rightarrow$ $X_-^2 f = d_1 x_4^2 + d_2 x_5$ because $wt(X_-^2 f) = -4$.

$\Rightarrow$ $X_-^2 f = 0$ because degree of $X_-^2 f$ is $k+1 \geq 3$

$$\Rightarrow \begin{cases} b_1 b_2 + 2b_1 + 1 = 0 \\ 2b_2 + 1 = 0 \end{cases} \quad \text{i.e.} \quad \begin{array}{l} b_1 = -\frac{2}{3} \\ b_2 = -\frac{1}{2}. \end{array}$$

From (8.2) we have

$$\begin{cases} \frac{\partial f}{\partial x_2} = 3\frac{\partial}{\partial x_3}(X_- f) \\ \frac{\partial f}{\partial x_3} = 2\frac{\partial}{\partial x_4}(X_- f) \\ \frac{\partial f}{\partial x_4} = \frac{\partial}{\partial x_5}(X_- f) \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\partial^2 f}{\partial x_2^2} = 3\frac{\partial^2}{\partial x_3 \partial x_2}(X_- f) = 0 \\ \frac{\partial^2 f}{\partial x_3 \partial x_2} = 2\frac{\partial^2}{\partial x_4 \partial x_2}(X_- f) = 0 \\ \frac{\partial^2 f}{\partial x_4 \partial x_2} = \frac{\partial^2}{\partial x_5 \partial x_2}(X_- f) = 0 \end{cases}$$

$\Rightarrow$ f does not involve x_1, x_5, x_2^2, x_2x_3 and x_2x_4

$\Rightarrow$ f depends only on x_3 and x_4 variables because degree of f is $k+1 \geq 3$

$\Rightarrow$ $f = bx_3^{k+1}$ because $wt(f) = 0$

$\Rightarrow$ $\dim I \leq 1$.

This contradicts to our hypothesis $\dim I = 3$. So Case 2 cannot occur. Q.E.D.

Theorem 8.2. Suppose $sl(2,\mathbb{C})$ acts on the space of homogeneous polynomials of

degree $k \geq 2$ in x_1, x_2, x_3, x_4 and x_5 variables via

$$\begin{aligned}
\tau &= 2x_1\frac{\partial}{\partial x_1} - 2x_3\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_4} - x_5\frac{\partial}{\partial x_5} \\
X_+ &= 2x_1\frac{\partial}{\partial x_2} + 2x_2\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_5} \\
X_- &= x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + x_5\frac{\partial}{\partial x_4}.
\end{aligned}$$

Suppose the weight of x_i is given by the corresponding coefficient in the expression of τ as above i.e.

$$wt(x_1) = 2,\ wt(x_2) = 0,\ wt(x_3) = -2,\ wt(x_4) = 1,\ wt(x_5) = -1.$$

Let I be the complex vector subspace spanned by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}$ and $\frac{\partial f}{\partial x_5}$, where f is a homogeneous polynomial degree $k+1$. If I is a $sl(2, \mathbf{C})$-submodule, then either

(i) (a) f is an invariant $sl(2,\mathbf{C})$ polynomial in x_1, x_2, x_3, x_4 and x_5 variables and $I = (3) \oplus (2)$. Moreover, we have $X_-(\frac{\partial f}{\partial x_1}) = 0$, $X_-(\frac{\partial f}{\partial x_2}) = -\frac{\partial f}{\partial x_1}$, $X_-(\frac{\partial f}{\partial x_3}) = -\frac{\partial f}{\partial x_2}$, $X_-(\frac{\partial f}{\partial x_4}) = 0$, $X_-(\frac{\partial f}{\partial x_5}) = -\frac{\partial f}{\partial x_4}$ and $X_+(\frac{\partial f}{\partial x_1}) = -2\frac{\partial f}{\partial x_2}$, $X_+(\frac{\partial f}{\partial x_2}) = -2\frac{\partial f}{\partial x_3}$, $X_+(\frac{\partial f}{\partial x_3}) = 0$, $X_+(\frac{\partial f}{\partial x_4}) = -\frac{\partial f}{\partial x_5}$, $X_+(\frac{\partial f}{\partial x_5}) = 0$.

(b) $f = g + c_1x_4^3 + c_2x_4^2x_5 + c_3x_4x_5^2 + c_4x_5^3$ where $g = 2x_1x_5^2 - 2x_2x_4x_5 + x_3x_4^2$ is a $sl(2, \mathbf{C})$ invariant polynomial and

$$\begin{aligned}
I &= \langle\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_3}, \frac{\partial g}{\partial x_4}, \frac{\partial g}{\partial x_5}\rangle = (3) \oplus (2) \\
&= \langle x_4^2, x_4x_5, x_5^2\rangle \oplus \langle x_2x_4 - 2x_1x_5, x_3x_4 - x_2x_5\rangle
\end{aligned}$$

or

(ii) f is a $sl(2, \mathbf{C})$ invariant polynomial in x_1, x_2 and x_3 variables and $I = (3)$. Moreover we have $X_-(\frac{\partial f}{\partial x_1}) = 0$, $X_-(\frac{\partial f}{\partial x_2}) = -\frac{\partial f}{\partial x_1}$, $X_-(\frac{\partial f}{\partial x_3}) = -\frac{\partial f}{\partial x_2}$ and $X_+(\frac{\partial f}{\partial x_1}) = -2\frac{\partial f}{\partial x_2}$, $X_+(\frac{\partial f}{\partial x_2}) = -2\frac{\partial f}{\partial x_3}$, $X_+(\frac{\partial f}{\partial x_3}) = 0$.

Remark. In Case (i)(b), we have

$$g(x_1+\frac{c_4}{2}x_5,\ x_2-\frac{c_3}{2}x_5,\ x_3+c_1x_4+c_2x_5,x_4,x_5)=f(x_1,x_2,x_3,x_4,x_5).$$

Proof. Let us first observe that

$$[\frac{\partial}{\partial x_1},X_-]=0,\ [\frac{\partial}{\partial x_2},X_-]=\frac{\partial}{\partial x_1},\ [\frac{\partial}{\partial x_3},X_-]=\frac{\partial}{\partial x_2},\ [\frac{\partial}{\partial x_4},X_-]=0,\ [\frac{\partial}{\partial x_5},X_-]=\frac{\partial}{\partial x_4}.$$

In view of Proposition 3.6, we only need to consider the following two cases.

Case 1. f is a homogeneous polynomial in x_1,x_2,x_3,x_4, and x_5 variables of weight 0 and $I=(3)\oplus(2)$.

Elements in I are linear combinations of homogeneous polynomials of degree k and weights $-2,0,2$ and $-1,1$. Since f is of weight 0, we have

$$\begin{aligned}
&wt\frac{\partial f}{\partial x_1}=-2\quad wt\frac{\partial f}{\partial x_2}=-0\quad wt\frac{\partial f}{\partial x_3}=2\quad wt\frac{\partial f}{\partial x_4}=-1\quad wt\frac{\partial f}{\partial x_5}=1\\
\Rightarrow\ &wt(X_-\frac{\partial f}{\partial x_1})=-4,\ wt(X_-\frac{\partial f}{\partial x_2})=-2,\ wt(X_-\frac{\partial f}{\partial x_3})=0,\ wt(X_-\frac{\partial f}{\partial x_4})=-3,\\
&wt(X_-\frac{\partial f}{\partial x_5})=-1\\
\Rightarrow\ &X_-\frac{\partial f}{\partial x_1}=0,\ X_-\frac{\partial f}{\partial x_2}=c_1\frac{\partial f}{\partial x_1},\ X_-\frac{\partial f}{\partial x_3}=c_2\frac{\partial f}{\partial x_2},\ X_-\frac{\partial f}{\partial x_4}=0,\ X_-\frac{\partial f}{\partial x_5}=d_1\frac{\partial f}{\partial x_4}
\end{aligned}$$

$$\left.\begin{aligned}
\Rightarrow\ &\tfrac{\partial}{\partial x_1}(X_- f) = X_-(\tfrac{\partial f}{\partial x_1}) = 0\\
&\tfrac{\partial}{\partial x_2}(X_- f) = X_-(\tfrac{\partial f}{\partial x_2}) + \tfrac{\partial f}{\partial x_1} = (c_1+1)\tfrac{\partial f}{\partial x_1}\\
&\tfrac{\partial}{\partial x_3}(X_- f) = X_-(\tfrac{\partial f}{\partial x_3}) + \tfrac{\partial f}{\partial x_2} = (c_2+1)\tfrac{\partial f}{\partial x_2}\\
&\tfrac{\partial}{\partial x_4}(X_- f) = X_- \tfrac{\partial f}{\partial x_4} = 0\\
&\tfrac{\partial}{\partial x_5}(X_- f) = X_- \tfrac{\partial f}{\partial x_5} + \tfrac{\partial f}{\partial x_4} = (d_1+1)\tfrac{\partial f}{\partial x_4}
\end{aligned}\right\} \tag{8.3}$$

$$\Rightarrow \frac{\partial}{\partial x_1}(X_-^2 f) = 0 = \frac{\partial}{\partial x_2}(X_-^2 f) = \frac{\partial}{\partial x_4}(X_-^2 f) = \frac{\partial}{\partial x_5}(X_-^2 f)$$

$$\frac{\partial}{\partial x_3}(X_-^2 f) = (c_1c_2 + 2c_1 + 1)\frac{\partial f}{\partial x_1}.$$

Clearly $X_-^2 f$ is a polynomial in x_3 variable only. If $c_1c_2 + 2c_1 + 1 \neq 0$, then $\frac{\partial f}{\partial x_1} = \frac{1}{c_1c_2+2c_1+1}\frac{\partial}{\partial x_3}(X_-^2 f)$. So $\frac{\partial f}{\partial x_1}$ is a polynomial in x_3 variable also. As weight of $\frac{\partial f}{\partial x_1}$ is -2, we have $\frac{\partial f}{\partial x_1} = dx_3$ where d is a constant. Since $\deg(\frac{\partial f}{\partial x_1}) = k \geq 2$, we have $d = 0$ and $\frac{\partial f}{\partial x_1} = 0$. This implies $\dim I \leq 4$, which contradicts to our hypothesis $I = (3) \oplus (2)$. We conclude that $c_1c_2 + 2c_1 + 1 = 0$ and hence $\frac{\partial}{\partial x_3}(X_-^2 f) = 0$. It follows that $X_-^2 f = 0$.

From (8.3) we know that $X_- f$ is a polynomial in x_2, x_3 and x_5 variables. Since $wt(X_- f) = -2$, there are constants b_1, b_2 such that $X_- f = b_1 x_2^k x_3 + b_2 x_2^{k-1} x_5^2$. The fact that

$$X_-^2 f = kb_1 x_2^{k-1} x_3^2 + (k-1)b_2 x_2^{k-2} x_5^2 x_3 = 0$$

implies $b_2 = b_3 = 0$ and hence $X_- f = 0$. By (8.3) we have $c_1 = -1 = c_2 = d_1$. Thus $X_- \frac{\partial f}{\partial x_2} = -\frac{\partial f}{\partial x_1}$, $X_- \frac{\partial f}{\partial x_3} = -\frac{\partial f}{\partial x_2}$ and $X_- \frac{\partial f}{\partial x_5} = -\frac{\partial f}{\partial x_4}$.

Similarly we can prove $X_+ f = 0$, $X_+(\frac{\partial f}{\partial x_1}) = -2\frac{\partial f}{\partial x_2}$, $X_+(\frac{\partial f}{\partial x_2}) = -2\frac{\partial f}{\partial x_3}$, $X_+(\frac{\partial f}{\partial x_3}) = 0$, $X_+(\frac{\partial f}{\partial x_4}) = -\frac{\partial f}{\partial x_5}$ and $X_+(\frac{\partial f}{\partial x_5}) = 0$.

Case 2. f is a homogeneous polynomial in x_1, x_2 and x_3 variables of weight 0 and $I = (3)$.

Elements in I are linear combinations of homogeneous polynomials of degree k and weights $-2, 0$ and 2. Since f is of weight 0, we have

$$\begin{aligned}
& wt\frac{\partial f}{\partial x_1} = -2,\ wt\frac{\partial f}{\partial x_2} = 0,\ wt\frac{\partial f}{\partial x_3} = 2 \\
\Rightarrow & wt(X_-\frac{\partial f}{\partial x_1}) = -4,\ wt(X_-\frac{\partial f}{\partial x_2}) = -2,\ wt(X_-\frac{\partial f}{\partial x_3}) = 0 \\
\Rightarrow & X_-\frac{\partial f}{\partial x_1} = 0,\ X_-\frac{\partial f}{\partial x_2} = a_1\frac{\partial f}{\partial x_1},\ X_-\frac{\partial f}{\partial x_3} = a_2\frac{\partial f}{\partial x_2}
\end{aligned}$$

$\Rightarrow$

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$$\left.\begin{aligned}
&\tfrac{\partial}{\partial x_1}(X_-f) = X_-(\tfrac{\partial f}{\partial x_1}) = 0 \\
&\tfrac{\partial}{\partial x_2}(X_-f) = X_-(\tfrac{\partial f}{\partial x_2}) + \tfrac{\partial f}{\partial x_1} + (a_1+1)\tfrac{\partial f}{\partial x_1} \\
&\tfrac{\partial}{\partial x_3}(X_-f) = X_-(\tfrac{\partial f}{\partial x_3}) + \tfrac{\partial f}{\partial x_1} + (a_2+1)\tfrac{\partial f}{\partial x_1}
\end{aligned}\right\}. \tag{8.4}$$

So X_-f is a polynomial in x_2 and x_3 variables. Since $wt(X_-f) = -2$, there are constants a_3 such that $X_-(f) = a_3x_2^kx_3$. If $a_1+1=0$, then $\frac{\partial}{\partial x_2}(X_-f) = 0$ by (8.4). On the other hand $\frac{\partial}{\partial x_2}(X_-f) = ka_3x_2^{k-1}x_3$. It follows that $a_3 = 0$ and hence $X_f = 0$. If $a_2+1 \neq 0$, then in view of (8.4), we have $\frac{\partial}{\partial x_3}(X_-f) = a_3x_2^k = 0$. This implies $a_3 = 0$ and $X_-f = 0$.

It remains to consider the case $a_1+1 \neq 0$ and $a_2+1 \neq 0$. The second equation (8.4) implies

$$\begin{aligned}
& \frac{\partial f}{\partial x_1} = \frac{ka_3}{a_1+1}x_2^{k-1}x_3 \\
\Rightarrow & f = \frac{ka_3}{a_1+1}x_1x_2^{k-1}x_3 + g(x_2,x_3) \\
\Rightarrow & \frac{\partial f}{\partial x_2} = \frac{ka_3}{a_1+1}(k-1)x_1x_2^{k-2}x_3 + \frac{\partial g}{\partial x_2}(x_2,x_3).
\end{aligned}$$

In view of the third equation of (8.4) we have

$$k(k-1)a_3\frac{a_2+1}{a_1+1}x_1x_2^{k-2}x_3 = a_3x_2^k - (a_2+1)\frac{\partial g}{\partial x_2}(x_2,x_3).$$

This can happen only when $a_3 = 0$, i.e. $X_- f = 0$. This implies $a_1 = -1 = a_2$. Hence $X_-(\frac{\partial f}{\partial x_2}) = -\frac{\partial f}{\partial x_1}$, $X_-(\frac{\partial f}{\partial x_3}) = -\frac{\partial f}{\partial x_2}$.

Similarly we can prove that $X_+ f = 0$ and $X_+(\frac{\partial f}{\partial x_1}) = -2\frac{\partial f}{\partial x_2}$, $X_+(\frac{\partial f}{\partial x_2}) = -2\frac{\partial f}{\partial x_3}$, $X_+(\frac{\partial f}{\partial x_3}) = 0$.

Theorem 8.3. Suppose $sl(2, \mathbf{C})$ acts on the space of homogeneous polynomials of degree $k \geq 2$ in x_1, x_2, x_3, x_4 and x_5 variables via

$$\begin{aligned} \tau &= 3x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} - x_3\frac{\partial}{\partial x_3} - 3x_4\frac{\partial}{\partial x_4} \\ X_+ &= 3x_1\frac{\partial}{\partial x_2} + 4x_2\frac{\partial}{\partial x_3} + 3x_3\frac{\partial}{\partial x_4} \\ X_- &= x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3}. \end{aligned}$$

Suppose the weight of x_i is given by the corresponding coefficient in the expression of τ as above, i.e.

$$wt(x_1) = 3,\ wt(x_2) = 1,\ wt(x_3) = -1,\ wt(x_4) = -3,\ wt(x_5) = 0.$$

Let I be the complex vector subspace spanned by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}$, and $\frac{\partial f}{\partial x_5}$, where f is a homogeneous polynomial of degree $k+1$. If I is a $sl(2, \mathbf{C})$-submodule, then one of the following occur.

(i) f is a $sl(2, \mathbf{C})$ invariant polynomial in x_1, x_2, x_3, x_4 and x_5 variables and $I = (4) \oplus (1)$. Moreover we have

$X_-\frac{\partial f}{\partial x_1} = 0$, $X_-\frac{\partial f}{\partial x_2} = -\frac{\partial f}{\partial x_1}$, $X_-\frac{\partial f}{\partial x_3} = -\frac{\partial f}{\partial x_2}$, $X_-\frac{\partial f}{\partial x_4} = -\frac{\partial f}{\partial x_3}$, $X_-\frac{\partial f}{\partial x_5} = 0$ and

$X_+\frac{\partial f}{\partial x_1} = -3\frac{\partial f}{\partial x_2}$, $X_+\frac{\partial f}{\partial x_2} = -4\frac{\partial f}{\partial x_3}$, $X_+\frac{\partial f}{\partial x_3} = -3\frac{\partial f}{\partial x_4}$, $X_+\frac{\partial f}{\partial x_4} = 0$, $X_+\frac{\partial f}{\partial x_5} = 0$.

(ii) f is an $sl(2, \mathbf{C})$ invariant polynomial in x_1, x_2, x_3, x_4 and x_5 variables; and $I = (4)$. Moreover we have

$X_-\frac{\partial f}{\partial x_1} = 0$, $X_-\frac{\partial f}{\partial x_2} = -\frac{\partial f}{\partial x_1}$, $X_-\frac{\partial f}{\partial x_3} = -\frac{\partial f}{\partial x_2}$, $X_-\frac{\partial f}{\partial x_4} = -\frac{\partial f}{\partial x_3}$

$X_+\frac{\partial f}{\partial x_1} = -3\frac{\partial f}{\partial x_2}$, $X_+\frac{\partial f}{\partial x_2} = -4\frac{\partial f}{\partial x_3}$, $X_+\frac{\partial f}{\partial x_3} = -3$, $X_+\frac{\partial f}{\partial x_4} = 0$.

(iii) $I = (1)$ and $f = cx_5^{k+1}$ where c is a nonzero constant.

Proof. Let us first observe that

$$[\frac{\partial}{\partial x_1}, X_-] = 0,\ [\frac{\partial}{\partial x_2}, X_-] = \frac{\partial}{\partial x_1},\ [\frac{\partial}{\partial x_3}, X_-] = \frac{\partial}{\partial x_2},\ [\frac{\partial}{\partial x_4}, X_-] = \frac{\partial}{\partial x_3},\ [\frac{\partial}{\partial x_5}, X_-] = 0.$$

In view of Proposition 4.6, we only need to consider Case (i) when f is a homogeneous polynomial of weight 0 in x_1, x_2, x_3, x_4 and x_5 variables and $I = (4) \oplus (1)$. The proof of case (ii) is the same.

Elements in I are linear combinations of homogeneous polynomials of degree k and weights $-3, -1, 1, 3$ and 0. Since f is of weight 0, we have

$$wt\frac{\partial f}{\partial x_1} = -3,\ wt\frac{\partial f}{\partial x_2} = -1,\ wt\frac{\partial f}{\partial x_3} = 1,\ wt\frac{\partial f}{\partial x_4} = 3,\ wt\frac{\partial f}{\partial x_5} = 0$$

$$\Rightarrow\quad wtX_-\frac{\partial f}{\partial x_1} = -5,\ wtX_-\frac{\partial f}{\partial x_2} = -3,\ wtX_-\frac{\partial f}{\partial x_3} = -1,\ wtX_-\frac{\partial f}{\partial x_4} = 1,\ wtX_-\frac{\partial f}{\partial x_5} = -2$$

$$\Rightarrow\quad X_-\frac{\partial f}{\partial x_1} = 0,\ X_-\frac{\partial f}{\partial x_2} = c_1\frac{\partial f}{\partial x_1},\ X_-\frac{\partial f}{\partial x_3} = c_2\frac{\partial f}{\partial x_2},\ X_-\frac{\partial f}{\partial x_4} = c_3\frac{\partial f}{\partial x_3},\ X_-\frac{\partial f}{\partial x_5} = 0$$

$$\Rightarrow\quad \left.\begin{array}{l} \frac{\partial}{\partial x_1}(X_-f) = X_-(\frac{\partial f}{\partial x_1}) = 0 \\ \frac{\partial}{\partial x_2}(X_-f) = X_-(\frac{\partial f}{\partial x_2}) + \frac{\partial f}{\partial x_1} = (c_1+1)\frac{\partial f}{\partial x_1} \\ \frac{\partial}{\partial x_3}(X_-f) = X_(\frac{\partial f}{\partial x_3}) + \frac{\partial f}{\partial x_2} = (c_2+1)\frac{\partial f}{\partial x_2} \\ \frac{\partial}{\partial x_4}(X_-f) = X_-(\frac{\partial f}{\partial x_4}) + \frac{\partial f}{\partial x_3} = (c_3+1)\frac{\partial f}{\partial x_3} \\ \frac{\partial}{\partial x_5}(X_-f) = X_-(\frac{\partial f}{\partial x_5}) = 0 \end{array}\right\} \qquad (8.5a)$$

$$\Rightarrow\quad \begin{array}{l} \frac{\partial}{\partial x_1}(X_-^2 f) = X_-(\frac{\partial}{\partial x_1}(X_-f)) = 0 \\ \frac{\partial}{\partial x_2}(X_-^2 f) = X_-(\frac{\partial}{\partial x_2}(X_-f)) + \frac{\partial}{\partial x_1}X_-f = 0 \\ \frac{\partial}{\partial x_3}(X_-^2 f) = X_-(\frac{\partial}{\partial x_3}(X_-f)) + \frac{\partial}{\partial x_2}X_-f = (c_1c_2 + 2c_1 + 1)\frac{\partial f}{\partial x_1} \end{array}$$

$$\frac{\partial}{\partial x_4}(X_-^2 f) = X_-(\frac{\partial}{\partial x_4}(X_- f)) + \frac{\partial}{\partial x_3}X_- f = (c_2c_3 + 2c_2 + 1)\frac{\partial f}{\partial x_2}$$
$$\frac{\partial}{\partial x_5}(X_-^2 f) = X_-(\frac{\partial}{\partial x_5}(X_- f)) = 0$$

$\Rightarrow$ $X_-^2 f$ is a homogeneous polynomial in x_3, x_4 variables of weight -4

$\Rightarrow$ $X_-^2 f = d_1 x_3^4 + d_2 x_3 x_4$.

Since degree of $X_-^2 f$ is $k+1 \geq 3$, we have $X_-^2 f = d_1 x_3^4$. If $d_1 \neq 0$, then $x_3^3 = \frac{1}{4d_1}\frac{\partial}{\partial x_3}X_-^2 f$ is an element in I. It follows that

$$\langle 3x_1x_3^2 + 8x_2^2x_3, x_2x_3^2, x_3^3, x_3^2x_4, x_3x_4^2, x_4^3\rangle$$

is a six dimensional subspace in I. This contradicts to our hypothesis $\dim I = 5$. Thus we have $X_-^2 f = 0$. Consequently we have

$$c_1c_2 + 2c_1 + 1 = 0 = c_2c_3 + 2c_2 + 1.$$

Therefore either both c_1, c_2 and c_3 are -1 or both c_1, c_2 and c_3 are not -1. We claim the latter case cannot occur. Suppose on the contrary that $c_1 \neq -1$, $c_2 \neq -1$ and $c_3 \neq -1$. From (8.5a), we know that $\frac{\partial f}{\partial x_1}$, $\frac{\partial f}{\partial x_2}$ and $\frac{\partial f}{\partial x_3}$ are polynomials in x_2, x_3 and x_4 variables. Hence we have

$$\frac{\partial^2 f}{\partial x_1^2} = \frac{\partial^2 f}{\partial x_5 \partial x_1} = \frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_5 \partial x_2} = \frac{\partial^2 f}{\partial x_1 \partial x_3} = \frac{\partial^2 f}{\partial x_5 \partial x_3} = 0.$$

Consequently f does not involve $x_1^2, x_1x_5, x_1x_2, x_1x_3, x_2x_5$ and x_3x_5. It follows easily that

$$f = c_4 x_1 x_4 + \phi_{k+1}^0(x_2, x_3, x_4, x_5)$$

where $\phi_{k+1}^0(x_2, x_3, x_4, x_5)$ is a homogeneous polynomial of degree $k+1$ and weight 0. Since f is a homogeneous polynomial of degree $k+1 \geq 3$, we have $c_4 = 0$ and $f = \phi_{k+1}^0(x_2, x_3, x_4, x_5)$. Thus we have $\frac{\partial f}{\partial x_1} = 0$ and $\dim I \leq 4$. This contradicts to our hypothesis $I = (4) \oplus (1)$.

In conclusion, we have $c_1 = -1 = c_2 = c_3$ and $\frac{\partial}{\partial x_i}(X_- f) = 0$ for all $1 \le i \le 5$. So $X_- f$ is necessary zero, and we have

$$X_- \frac{\partial f}{\partial x_1} = 0,\ X_- \frac{\partial f}{\partial x_2} = -\frac{\partial f}{\partial x_1},\ X_- \frac{\partial f}{\partial x_3} = -\frac{\partial f}{\partial x_2},\ X_- \frac{\partial f}{\partial x_4} = -\frac{\partial f}{\partial x_3},\ X_- \frac{\partial f}{\partial x_5} = 0.$$

Similarly we can prove that $X_+ f$ is a zero and we have

$$X_+ \frac{\partial f}{\partial x_1} = -3\frac{\partial f}{\partial x_2},\ X_+ \frac{\partial f}{\partial x_2} = -4\frac{\partial f}{\partial x_3},\ X_+ \frac{\partial f}{\partial x_3} = -3\frac{\partial f}{\partial x_4},\ X_+ \frac{\partial f}{\partial x_4} = 0,\ X_+ \frac{\partial f}{\partial x_5} = 0.$$

Theorem 8.4. Suppose $sl(2,\mathbf{C})$ acts on the space of homgenous polynomials of degree $k \ge 2$, in x_1, x_2, x_3, x_4 and x_5 variables via

$$\begin{aligned} \tau &= x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} - x_4\frac{\partial}{\partial x_4} \\ X_+ &= x_1\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_4} \\ X_- &= x_2\frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial x_3}. \end{aligned}$$

Suppose the weight of x_i is given by the corresponding coefficient in the expression of τ above, i.e.

$$wt(x_1) = 1,\ wt(x_2) = -1,\ wt(x_3) = 1,\ wt(x_4) = -1,\ wt(x_5) = 0.$$

Let I be the complex vector subspace spanned by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}$, and $\frac{\partial f}{\partial x_5}$, where f is a homogeneous polynomial of degree $k+1$. If I is a $sl(2,\mathbf{C})$-submodule, then either

(i) f is a $sl(2,\mathbf{C})$ invariant polynomial in x_1, x_2, x_3, x_4 and x_5 variables and $I = (2) \oplus (2) \oplus (1)$. Moreover we have

$$X_-(\frac{\partial f}{\partial x_1}) = 0,\ X_-(\frac{\partial f}{\partial x_2}) = -\frac{\partial f}{\partial x_1},\ X_-(\frac{\partial f}{\partial x_3}) = 0,\ X_-(\frac{\partial f}{\partial x_4}) = -\frac{\partial f}{\partial x_3},\ X_-(\frac{\partial f}{\partial x_5}) = 0$$

and

$$X_+(\frac{\partial f}{\partial x_1}) = -\frac{\partial f}{\partial x_2},\ X_+(\frac{\partial f}{\partial x_2}) = 0,\ X_+(\frac{\partial f}{\partial x_3}) = -\frac{\partial f}{\partial x_4},\ X_+(\frac{\partial f}{\partial x_4}) = 0,\ X_+(\frac{\partial f}{\partial x_5}) = 0.$$

or

(ii) f is a $sl(2,\mathbf{C})$ invariant polynomial in x_1, x_2, x_3, and x_4 variables and $I = (2)\oplus(2)$.

Moreover we have

$$X_-(\frac{\partial f}{\partial x_4}) = 0,\ X_-(\frac{\partial f}{\partial x_2}) = -\frac{\partial f}{\partial x_1},\ X_-(\frac{\partial f}{\partial x_3}) = 0,\ X_-(\frac{\partial f}{\partial x_4}) = -\frac{\partial f}{\partial x_3},\ X_-(\frac{\partial f}{\partial x_5}) = 0$$

and

$$X_+(\frac{\partial f}{\partial x_1}) = \frac{\partial f}{\partial x_2},\ X_+(\frac{\partial f}{\partial x_2}) = 0,\ X_+(\frac{\partial f}{\partial x_3}) = -\frac{\partial f}{\partial x_4},\ X_+(\frac{\partial f}{\partial x_4}) = 0,\ X_+(\frac{\partial f}{\partial x_5}) = 0$$

or

(iii) $f = cx_5^{k+1}$ where c is a nonzero constant. In this case $I = (1)$.

Proof. Let us first observe that

$$[\frac{\partial}{\partial x_1}, X_-] = 0,\ [\frac{\partial}{\partial x_2}, X_-] = \frac{\partial}{\partial x_1},\ [\frac{\partial}{\partial x_3}, X_-] = 0,\ [\frac{\partial}{\partial x_4}, X_-] = \frac{\partial}{\partial x_3}$$
$$[\frac{\partial}{\partial x_5}, X_-] = 0.$$

In view of Proposition 5.7, we only need to consider the following two cases.

Case 1. f is a homogeneous polynomial of weight 0 in x_1, x_2, x_3, x_4 and x_5 variables and $I = (2)\oplus(2)\oplus(1)$.

Elements in I are linear combination of homogeneous polynomials of degree k and weights $-1, 0$ and 1. Since f is of weight 0, we have

$$wt(\frac{\partial f}{\partial x_1}) = -1,\ wt(\frac{\partial f}{\partial x_2}) = 1,\ wt(\frac{\partial f}{\partial x_3}) = -1,\ wt(\frac{\partial f}{\partial x_4}) = 1,\ wt(\frac{\partial f}{\partial x_5}) = 0$$

$$\Rightarrow\ wt(X_-\frac{\partial f}{\partial x_1}) = -3,\ wt(X_-\frac{\partial f}{\partial x_2}) = -1,\ wt(X_-\frac{\partial f}{\partial x_3}) = -3,\ wt(X_-\frac{\partial f}{\partial x_4}) = -1$$

$$wt(X_-\frac{\partial f}{\partial x_5}) = -2$$

$$\Rightarrow \quad X_-\frac{\partial f}{\partial x_1} = 0$$

$$X_-\frac{\partial f}{\partial x_2} = c_1\frac{\partial f}{\partial x_1} + c_2\frac{\partial f}{\partial x_3}$$

$$X_-\frac{\partial f}{\partial x_3} = 0$$

$$X_-\frac{\partial f}{\partial x_4} = c_3\frac{\partial f}{\partial x_1} + c_4\frac{\partial f}{\partial x_3}$$

$$X_-\frac{\partial f}{\partial x_5} = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial x_1}(X_-f) = X_-(\frac{\partial f}{\partial x_1}) = 0$$

$$\frac{\partial}{\partial x_2}(X_-f) = X_-(\frac{\partial f}{\partial x_2}) + \frac{\partial f}{\partial x_1} = (c_1+1)\frac{\partial f}{\partial x_1} + c_2\frac{\partial f}{\partial x_3}$$

$$\frac{\partial}{\partial x_3}(X_-f) = X_-(\frac{\partial f}{\partial x_3}) = 0$$

$$\frac{\partial}{\partial x_4}(X_-f) = X_-(\frac{\partial f}{\partial x_4}) + \frac{\partial f}{\partial x_3} = c_3\frac{\partial f}{\partial x_1} + (c_4+1)\frac{\partial f}{\partial x_3}$$

$$\frac{\partial}{\partial x_5}(X_-f) = X_-(\frac{\partial f}{\partial x_5}) = 0$$

$\Rightarrow$ X_-f depends only on x_2 and x_4 variables

$\Rightarrow$ $X_-f = c_5x_2^2 + c_6x_2x_4 + c_7x_4^2$ because $wt(x_-f) = -2$

$\Rightarrow$ $X_-f = 0$ because degree of X_-f is greater than 2

It follows that $c_1 = -1 = c_4$ and $c_2 = 0 = c_3$ i.e. $X_-(\frac{\partial f}{\partial x_2}) = -\frac{\partial f}{\partial x_1}$, and $X_-(\frac{\partial f}{\partial x_4}) = -\frac{\partial f}{\partial x_3}$.

Similarly we can prove that $X_+f = 0$, $X_+(\frac{\partial f}{\partial x_1}) = -\frac{\partial f}{\partial x_2}$, $X_+(\frac{\partial f}{\partial x_2}) = 0$, $X_+(\frac{\partial f}{\partial x_3}) = -\frac{\partial f}{\partial x_4}$, $X_+(\frac{\partial f}{\partial x_4}) = 0$ and $X_+(\frac{\partial f}{\partial x_5}) = 0$.

Case 2. f is a homogeneous polynomial of weight 0 in x_1, x_2, x_3, and x_4 variables and $I = (2) \oplus (2)$.

Same argument as above shows that f is an $sl(2,\mathbf{C})$ invariant polynomial with properties listed in (ii) of our theorem. Q.E.D.

Theorem 8.5. Suppose $sl(2,\mathbf{C})$ acts on the space of homogeneous polynomials of degree $k \geq 2$ in x_1, x_2, x_3, x_4, and x_5 variables via

$$\tau = 2x_1\frac{\partial}{\partial x_1} - 2x_3\frac{\partial}{\partial x_3}$$

$$X_+ = 2x_1\frac{\partial}{\partial x_2} + 2x_2\frac{\partial}{\partial x_3}$$

$$X_- = x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2}.$$

Suppose the weight of x_i is given by the corresponding coefficient in the expression of τ above i.e.

$$wt(x_1) = 2,\ wt(x_2) = 0,\ wt(x_3) = -2,\ wt(x_4) = 0,\ wt(x_5) = 0.$$

Let I be the complex vector subspace spanned by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4}$ and $\frac{\partial f}{\partial x_5}$, where f is a homogeneous polynomial of degree $k+1$. If I is a $sl(2,\mathbf{C})$-submodule, then I is one of the following:

(i) (a) $I = (3)\oplus(1)\oplus(1)$ and f is a $sl(2,\mathbf{C})$ invariant polynomial. Moreover we have $X_-(\frac{\partial f}{\partial x_1}) = 0,\ X_-(\frac{\partial f}{\partial x_2}) = -\frac{\partial f}{\partial x_1},\ X_-(\frac{\partial f}{\partial x_3}) = -\frac{\partial f}{\partial x_2},\ X_-(\frac{\partial f}{\partial x_4}) = 0,\ X_-(\frac{\partial f}{\partial x_5}) = 0$ and $X_+(\frac{\partial f}{\partial x_1}) = -2\frac{\partial f}{\partial x_2},\ X_+(\frac{\partial f}{\partial x_2}) = -2\frac{\partial f}{\partial x_3},\ X_+(\frac{\partial f}{\partial x_3}) = 0,\ X_+(\frac{\partial f}{\partial x_4}) = 0,\ X_+(\frac{\partial f}{\partial x_5}) = 0$.

(b)
$$\begin{aligned} I &= (3)\oplus(1)\oplus(1) = \langle\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_3}, \frac{\partial g}{\partial x_4}, \frac{\partial g}{\partial x_5}\rangle \\ &= \langle x_1(x_4+rx_5)^{k-1},\ x_2(x_4+rx_5)^{k-1},\ x_3(x_4+rx_5)^{k-1}\rangle \\ &\oplus\langle(x_4+rx_5)^k \oplus \langle(k-1)d_1(x_2^2-2x_1x_3)(x_4+rx_5)^{k-2} + kd_2x_5(x_4+rx_5)^{k-1}\rangle \end{aligned}$$

where $g(x_1,x_2,x_3,x_4,x_5) = d_1(x_2^2-2x_1x_3)(x_4+rx_5)^{k-1}+d_2x_5(x_4+rx_5)^k+d_3(x_4+rx_5)^{k+1}$ is a $sl(2,\mathrm{C})$ invariant polynomial with $d_1\neq 0$ and $d_2\neq 0$; and $f = g(x_1,x_2,x_3,x_4,x_5)+c_1x_1(x_4+rx_5)^k+c_2x_2(x_4+rx_5)^k+c_3x_3(x_4+rx_5)^k$

(c) $$\begin{aligned} I &= (3)\oplus(1)\oplus(1) = \langle \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_3}, \frac{\partial g}{\partial x_4}, \frac{\partial g}{\partial x_5}\rangle \\ &= \langle x_1(rx_4+x_5)^{k-1},\ x_2(rx_4+x_5)^{k-1},\ x_3(rx_4+x_5)^{k-1}\rangle \\ &\quad \oplus\langle (rx_4+x_5)^k\rangle \oplus \langle (k-1)d_1(x_2^2-2x_1x_3)(rx_4+x_5)^{k-1}+kd_2x_4(rx_4+x_5)^{k-1}\rangle \end{aligned}$$

where $g(x_1,x_2,x_3,x_4,x_5) = d_1(x_2^2-2x_1x_3)(rx_4+x_5)^{k-1}+d_2x_4(rx_4+x_5)^k+d_3(rx_4+x_5)^{k+1}$ is a $sl(2,\mathrm{C})$ invariant polynomial with $d_1\neq 0$ and $d_2\neq 0$; and $f = g(x_1,x_2,x_3,x_4,x_5)+c_1x_1(rx_4+x_5)^k+c_2x_2(rx_4+x_5)^k+c_3x_3(rx_4+x_5)^k$.

(ii) $I = (3)\oplus(1)$ and f is a $sl(2,\mathrm{C})$ invariant polynomial in x_1,x_2,x_3,x_4 and x_5 variables. Moreover we have

$$X_-(\frac{\partial f}{\partial x_1}) = 0,\ X_-(\frac{\partial f}{\partial x_2}) = -\frac{\partial f}{\partial x_1},\ X_-(\frac{\partial f}{\partial x_3}) = -\frac{\partial f}{\partial x_2},\ X_-(\frac{\partial f}{\partial x_4}) = 0,\ X_-(\frac{\partial f}{\partial x_5}) = 0$$

and

$$X_+(\frac{\partial f}{\partial x_1}) = -2\frac{\partial f}{\partial x_2},\ X_+(\frac{\partial f}{\partial x_2}) = -2\frac{\partial f}{\partial x_3},\ X_+(\frac{\partial f}{\partial x_3}) = 0,\ X_+(\frac{\partial f}{\partial x_4}) = 0,\ X_+(\frac{\partial f}{\partial x_5}) = 0.$$

(iii) $I = (3)$ and f is a $sl(2,\mathrm{C})$ invariant polynomial in x_1,x_2 and x_3 variables. Moreover we have

$$X_-(\frac{\partial f}{\partial x_1}) = 0,\ X_-(\frac{\partial f}{\partial x_2}) = -\frac{\partial f}{\partial x_1},\ X_-(\frac{\partial f}{\partial x_3}) = -\frac{\partial f}{\partial x_2}$$

and

$$X_+(\frac{\partial f}{\partial x_1}) = -2\frac{\partial f}{\partial x_3},\ X_+(\frac{\partial f}{\partial x_2}) = -2\frac{\partial f}{\partial x_3},\ X_+(\frac{\partial f}{\partial x_3}) = 0.$$

(iv) $I = (1)\oplus(1) = \langle\frac{\partial f}{\partial x_4}(x_4,x_5)\rangle\oplus\langle\frac{\partial f}{\partial x_5}(x_4,x_5)\rangle$ and f is a $sl(2,\mathrm{C})$ invariant polynomial in x_4 and x_5.

(v) $I = (1) = \langle\frac{\partial f}{\partial x_4}(x_4,x_5)\rangle$ or $\langle\frac{\partial f}{\partial x_5}(x_4,x_5)\rangle$ and $f = (c_1x_4+c_2x_5)^{k+1}$ for some constants c_1 and c_2.

Finally, $sl(2,\mathbf{C})$ invariant polynomial is of the following form

$$f = \sum_{i=0}^{[\frac{k+1}{2}]} (x_2^2 - 2x_1x_3)^j \; q_{k+1-2j}(x_4, x_5)$$

where $q_{k+1-2j}(x_4, x_5)$ is a homogeneous polynomial of degree $k+1-2j$ in x_4 and x_5 variables.

Remark. In (i) (b), after linear change of coordinate, f becomes a polynomial of the form

$$d_1(x_2^2 - 2x_1x_3)(x_5 + rx_5)^{k-1} + d_2x_5(x_4 + rx_5)^k \tag{8.5b}$$

or

$$d_1(x_2^2 - 2x_1x_3)(x_4 + rx_5)^{k-1} + \tilde{d}_3x_4(x_4 + rx_5)^k \tag{8.6}$$

In fact

$$\begin{aligned}
& f(x_1 + \frac{c_3}{2d_1}(x_4 + rx_5),\ x_2 - \frac{c_2}{2d_1}(x_4 + rx_5),\ x_3 + \frac{c_1}{2d_1}(x_4 + rx_5),\ x_4, x_5) \\
= \ & d_1(x_2^2 - 2x_1x_3)(x_4 + rx_5)^{k-1} + d_2x_5(x_4 + rx_5)^k + \tilde{d}_3(x_4 + rx_5)^{k+1} \\
= \ & g(x_1, x_2, x_3, x_4, x_5)
\end{aligned}$$

where $\tilde{d}_3 = d_3 + \frac{c_1c_3}{2d_1} - \frac{c_2^2}{4d_1}$. If $d_2 + r\tilde{d}_3 = 0$, then

$$\begin{aligned}
& g(x_1, x_2, x_3,\ \frac{d_2 + r\tilde{d}_3}{d_2}x_4,\ \frac{d_2 + r\tilde{d}_3}{d_2}[x_5 - \frac{\tilde{d}_3}{d_2 + r\tilde{d}_3}(x_4 + rx_5)]) \\
= & d_1(x_2^2 - 2x_1x_3)(x_4 + rx_5)^{k-1} + (d_2 + r\tilde{d}_3)x_5(x_4 + rx_5)^k.
\end{aligned}$$

If $d_2 + r\tilde{d}_3 = 0$, then

$$\begin{aligned}
g(x_1, x_2, x_3, x_4, x_5) &= d_1(x_2^2 - 2x_1x_3)(x_4 + rx_5)^{k-1} - r\tilde{d}_3x_5(x_4 + rx_5)^k \\
&\quad + \tilde{d}_3(x_4 + rx_5)^{k+1} \\
&= d_1(x_2^2 - 2x_1x_3)(x_4 + rx_5)^{k-1} + \tilde{d}_3x_4(x_4 + rx_5)^{k-1}.
\end{aligned}$$

Similarly in (i) (c), after linear change of coordinate, we can put f in the form of (8.5b) or (8.6).

Proof. Let us first observe that

$$[\frac{\partial}{\partial x_1}, X_-] = 0,\ \frac{\partial}{\partial x_2}, X_-] = \frac{\partial}{\partial x_1}, [\frac{\partial}{\partial x_3}, X_-] = \frac{\partial}{\partial x_2}, [\frac{\partial}{\partial x_4}, X_-] = 0, [\frac{\partial}{\partial x_5}, X_-] = 0.$$

In view of Proposition 6.7, we only need to consider the following three cases.

Case 1. $I = (3) \oplus (1) \oplus (1)$ and f is a homogeneous polynomial of weight 0.

Elements in I are linear combinations of homogeneous polynomials of degree k and weights $-2, 0$ and 2. Since f is of weight 0, we have

$$wt(\frac{\partial f}{\partial x_1}) = -2,\ wt(\frac{\partial f}{\partial x_2}) = 0,\ wt(\frac{\partial f}{\partial x_3}) = 2,\ wt(\frac{\partial f}{\partial x_4}) = 0 = wt(\frac{\partial f}{\partial x_5})$$

$$\Rightarrow\ wt(X_-\frac{\partial f}{\partial x_1}) = -4,\ wt(X_-\frac{\partial f}{\partial x_2}) = -2,\ wt(X_-\frac{\partial f}{\partial x_3}) = 0,$$

$$wt(X_-\frac{\partial f}{\partial x_4}) = -2 = wt(X_-\frac{\partial f}{\partial x_5})$$

$$\Rightarrow\ X_-\frac{\partial f}{\partial x_1} = 0,\ X_-\frac{\partial f}{\partial x_2} = c_1\frac{\partial f}{\partial x_1},\ X_-\frac{\partial f}{\partial x_3} = c_2\frac{\partial f}{\partial x_2} + c_3\frac{\partial f}{\partial x_4} + c_4\frac{\partial f}{\partial x_5}$$

$$X_-\frac{\partial f}{\partial x_4} = c_5\frac{\partial f}{\partial x_1},\ X_-\frac{\partial f}{\partial x_5} = c_6\frac{\partial f}{\partial x_1} \tag{8.7}$$

$$\Rightarrow\ \frac{\partial}{\partial x_1}(X_- f) = X_-(\frac{\partial f}{\partial x_1}) = 0$$

$$\frac{\partial}{\partial x_2}(X_- f) = X_-(\frac{\partial f}{\partial x_2}) + \frac{\partial f}{\partial x_1} = (c_1 + 1)\frac{\partial f}{\partial x_1}$$

$$\frac{\partial}{\partial x_3}(X_- f) = X_-(\frac{\partial f}{\partial x_3}) + \frac{\partial f}{\partial x_2} = (c_2 + 1)\frac{\partial f}{\partial x_2} + c_3\frac{\partial f}{\partial x_4} + c_4\frac{\partial f}{\partial x_5}$$

$$\frac{\partial}{\partial x_4}(X_- f) = X_-(\frac{\partial f}{\partial x_4}) = c_5\frac{\partial f}{\partial x_1}$$

$$\frac{\partial}{\partial x_5}(X_- f) = X_-(\frac{\partial f}{\partial x_5}) = c_6\frac{\partial f}{\partial x_1} \tag{8.8}$$

$\Rightarrow$ $X_- f$ does not involve x_1 variable.

Since the vector space $\langle \frac{\partial}{\partial x_2}(X_- f),\ \frac{\partial}{\partial x_4}(X_- f),\ \frac{\partial}{\partial x_5}(X_- f)\rangle$ is at most one dimension, in view of Lemma 6.1, there are constants r_1, r_2 and r_3 such that $X_- f$ is a polynomial in x_3 and $(r_1x_2 + r_2x_4 + r_3x_5)$. As weight of $X_- f$ is -2, we have

$$X_- f = x_3(r_1x_2 + r_2x_4 + r_3x_5)^k.$$

We claim that $c_1 + 1 = 0$. Suppose on the contrary that $c_1 + 1 \neq 0$. Then

$$\begin{aligned}\frac{\partial f}{\partial x_1} &= \frac{1}{c_1+1}\,\frac{\partial}{\partial x_2}(X_- f)\\ &= \frac{r_1 k}{c_1+1}\,x_3(r_1x_2 + r_2x_4 + r_3x_5)^{k-1}\end{aligned}$$

$$\Rightarrow f = \frac{r_1 k}{c_1+1}\,x_1x_3(r_1x_2 + r_2x_4 + r_3x_5)^{k-1} + g_{k+1}(x_2, x_4, x_5)$$

where g_{k+1} is a homogeneous polynomial of degree $k+1$ and weight 0 in x_2, x_4, x_5 variables because weight of f is zero. It follows that

$$\frac{\partial f}{\partial x_2} = \frac{r_1^2 k(k-1)}{c_1+1}x_1x_3(r_1x_2 + r_2x_4 + r_3x_5)^{k-2} + \frac{\partial g_{k+1}}{\partial x_2}(x_2, x_4, x_5)$$

$$\begin{aligned}X_-(\frac{\partial f}{\partial x_2}) = {}& \frac{r_1^2 k(k-1)}{c_1+1}x_2x_3(r_1x_2 + r_2x_4 + r_3x_5)^{k-2}\\ &+\frac{r_1^3 k(k-1)k-2)}{c_1+1}x_1x_3^2(r_1x_2 + r_2x_4 + r_3x_5)^{k-3} + X_-\frac{\partial g_{k+1}}{\partial x_2}(x_2, x_4, x_5).\end{aligned}$$

Since $wt(\frac{\partial f}{\partial x_1}) = -2$, $\frac{\partial f}{\partial x_1}$ is a nonzero element in (3) $\subseteq I$. Hence we have $r_1 \neq 0$. The equation $X_-(\frac{\partial f}{\partial x_2}) = c_1\frac{\partial f}{\partial x_1}$ implies that

$$\begin{aligned}&-\frac{r_1^3 k(k-1)(k-2)}{c_1+1}x_1x_3^2(r_1x_2 + r_2x_4 + r_3x_5)^{k-3}\\ &= \frac{r_1^2 k(k-1)}{c_1+1}x_2x_3(r_1x_2 + r_2x_4 + r_3x_5)^{k-2} + X_-\frac{\partial g_{k+1}}{\partial x_2}(x_2, x_4, x_5)\\ &\quad-\frac{c_1 r_1 k}{c_1+1}x_3(r_1x_2 + r_2x_4 + r_3x_5)^{k-1}.\end{aligned}$$

The left hand side of the above equation depends on x_1, while the right hand side does not involve x_1 variable. This can happen only if $k = 2$. Therefore

$$f = \frac{2r_1}{c_1+1}x_1x_3(r_1x_2 + r_2x_4 + r_3x_5) + g_3(x_2, x_4, x_5)$$

$$\Rightarrow \quad \frac{\partial f}{\partial x_3} = \frac{2r_1}{c_1+1}x_1(r_1x_2+r_2x_4+r_3x_5)$$

$$\frac{\partial f}{\partial x_2} = \frac{2r_1^2}{c_1+1}x_1x_3 + \frac{\partial g_3}{\partial x_2}(x_2,x_4,x_5)$$

$$\frac{\partial f}{\partial x_4} = \frac{2r_1r_2}{c_1+1}x_1x_3 + \frac{\partial g_3}{\partial x_4}(x_2,x_4,x_5)$$

$$\frac{\partial f}{\partial x_5} = \frac{2r_1r_3}{c_1+1}x_1x_3 + \frac{\partial g_3}{\partial x_5}(x_2,x_4,x_5)$$

$$\Rightarrow \quad X_-(\frac{\partial f}{\partial x_2}) = \frac{2r_1^2}{c_1+1}x_2x_3 + X_-\frac{\partial g_3}{\partial x_2}(x_2,x_4,x_5)$$

$$X_-(\frac{\partial f}{\partial x_3}) = \frac{2r_1}{c_1+1}x_2(r_1x_2+r_2x_4+r_3x_5) + \frac{2r_1^2}{c_1+1}x_1x_3$$

$$X_-(\frac{\partial f}{\partial x_4}) = \frac{2r_1r_2}{c_1+1}x_2x_3 + X_-\frac{\partial g_3}{\partial x_4}(x_2,x_4,x_5)$$

$$X_-(\frac{\partial f}{\partial x_5}) = \frac{2r_1r_3}{c_1+1}x_2x_3 + X_-\frac{\partial g_3}{\partial x_5}(x_2,x_4,x_5)$$

In view of equation (8.7), we have

$$\frac{2r_1^2}{c_1+1}x_2x_3 + X_-\frac{\partial g_3}{\partial x_2}(x_2,x_4,x_5) = \frac{2r_1c_1}{c_1+1}x_3(r_1x_2+r_2x_4+r_3x_5)$$

$$\frac{2r_1}{c_1+1}x_2(r_1x_2+r_2x_4+r_3x_5) + \frac{2r_1}{c_1+1}x_1x_3 = \frac{2r_1^2}{c_1+1}(r_1c_2+r_2c_3+r_3c_4)x_1x_3$$

$$+c_2\frac{\partial g_3}{\partial x_2}(x_2,x_4,x_5) + c_3\frac{\partial g_3}{\partial x_4}(x_2,x_4,x_5) + c_4\frac{\partial g_3}{\partial x_5}(x_2,x_4,x_5)$$

$$\frac{2r_1r_2}{c_1+1}x_2x_3 + X_-\frac{\partial g_3}{\partial x_4}(x_2,x_4,x_5) = \frac{2r_1c_5}{c_1+1}x_3(r_1x_2+r_2x_4+r_3x_5)$$

$$\frac{2r_1r_3}{c_1+1}x_2x_3 + X_-\frac{\partial g_3}{\partial x_5}(x_2,x_4,x_5) = \frac{2r_1c_6}{c_1+1}x_3(r_1x_2+r_2x_4+r_3x_5)$$

$$\Rightarrow X_-\frac{\partial g_3}{\partial x_2}(x_2,x_4,x_5) = \frac{2r_1c_1}{c_1+1}x_3(r_1x_2+r_2x_4+r_3x_5) - \frac{2r_1^2}{c_1+1}x_2x_3$$

$$X_-\frac{\partial g_3}{\partial x_4}(x_2,x_4,x_5) = \frac{2r_1c_5}{c_1+1}x_3(r_1x_2+r_2x_4+r_3x_5) - \frac{2r_1r_2}{c_1+1}x_2x_3$$

$$X_-\frac{\partial g_3}{\partial x_5}(x_2,x_4,x_5) = \frac{2r_1c_6}{c_1+1}x_3(r_1x_2+r_2x_4+r_3x_5) - \frac{2r_1r_3}{c_1+1}x_2x_3$$

$$c_2\frac{\partial g_3}{\partial x_2}(x_2,x_4,x_5)+c_3\frac{\partial g_3}{\partial x_4}(x_2,x_4,x_5)+c_4\frac{\partial g_3}{\partial x_5}(x_2,x_4,x_5)$$

$$=\frac{2r_1}{c_1+1}x_2(r_1x_2+r_2x_4+r_3x_5)$$

and

$$r_1=r_1c_2+r_2c_3+r_3c_4$$

$$\Rightarrow\ \frac{2r_1c_1c_2}{c_1+1}x_3(r_1x_2+r_2x_4+r_3x_5)-\frac{2c_2r_1^2}{c_1+1}x_2x_3$$

$$+\frac{2r_1c_5c_3}{c_1+1}x_3(r_1x_2+r_2x_4+r_3x_5)-\frac{2c_3r_1r_2}{c_1+1}x_2x_3$$

$$+\frac{2r_1c_6c_4}{c_1+1}x_3(r_1x_2+r_2x_4+r_3x_5)-\frac{2c_4r_1r_3}{c_1+1}x_2x_3$$

$$=\frac{2r_1}{c_1+1}X_-[x_2(r_1x_2+r_2x_4+r_3x_5)]$$

$$\Rightarrow\ (c_1c_2+c_3c_5+c_4c_6)x_3(r_1x_2+r_2x_4+r_3x_5)-(c_2r_1+c_3r_2+c_4r_3)x_2x_3$$

$$=x_3(r_1x_2+r_2x_4+r_3x_5)+r_1x_2x_3$$

$$\Rightarrow\ [r_1(c_1c_2+c_3c_5+c_4c_6)-3r_1]x_2x_3$$

$$+[r_2(c_1c_2+c_3c_5+c_4c_6)-r_2]x_3x_4$$

$$+[r_3(c_1c_2+c_3c_5+c_4c_6)-r_3]x_3x_5=0.$$

Since $r_1\neq 0$, we have $c_1c_2+c_3c_5+c_4c_6=3$, $r_2=0=r_3$,

$$f=\frac{2r_1^2}{c_1+1}x_1x_2x_3+g_3(x_2,x_4,x_5).$$

Thus $x_1x_2=\frac{c_1+1}{2r_1^2}\frac{\partial f}{\partial x_3}$ is an element in I. By applying X_+ and X_- successively on x_1x_2, we have an irreducible $sl(2,\mathbf{C})$-submodule of dimension 5 in I of the following form

$$\langle x_1^2, x_1x_2, x_2^2+x_1x_3, x_2x_3, x_3^2\rangle.$$

This contradicts to our hypothesis $I = (3) \oplus (1) \oplus (1)$. Therefore our claim $c_1 + 1 = 0$ is established.

We next consider the case $c_5 \neq 0$. In this case, we can write $\frac{\partial}{\partial x_5}(X_- f) = \frac{c_6}{c_5}\frac{\partial}{\partial x_4}(X_- f)$. It follows that $X_- f$ is a polynomial in x_3 and $x_4 + rx_5$ where $r = \frac{c_6}{c_5}$. In fact, since weight of $X_- f$ is -2, we have

$$X_- f = cx_3(x_4 + rx_5)^k.$$

From (8.8), we have

$$\begin{aligned}\frac{\partial f}{\partial x_1} &= \frac{1}{c_5}\cdot\frac{\partial}{\partial x_4}(X_- f)\\ &= \frac{ck}{c_5}x_3(x_4+rx_5)^{k-1}\\ \Rightarrow f &= \frac{ck}{c_5}x_1x_3(x_4+rx_5)^{k-1} + h_{k+1}(x_2,x_4,x_5)\end{aligned}$$

where h_{k+1} is a homogeneous polynomial of degree $k+1$ and weight 0 in x_2, x_4, x_5 variables because weight of f is zero. It follows that

$$\begin{aligned}\frac{\partial f}{\partial x_2} &= \tfrac{\partial h_{k+1}}{\partial x_2}(x_2,x_4,x_5)\\ \frac{\partial f}{\partial x_3} &= \tfrac{ck}{c_5}x_1(x_4+rx_5)^{k-1}\\ \frac{\partial f}{\partial x_4} &= \tfrac{ck(k-1)}{c_5}x_1x_3(x_4+rx_5)^{k-2} + \tfrac{\partial h_{k+1}}{\partial x_4}(x_2,x_4,x_5)\\ \frac{\partial f}{\partial x_5} &= \tfrac{ck(k-1)}{c_5}rx_1x_3(x_4+rx_5)^{k-2} + \tfrac{\partial h_{k+1}}{\partial x_5}(x_2,x_4,x_5)\end{aligned}$$

$$\begin{aligned}\Rightarrow\quad X_-\frac{\partial f}{\partial x_2} &= X_-\tfrac{\partial h_{k+1}}{\partial x_2}(x_2,x_4,x_5)\\ X_-\frac{\partial f}{\partial x_3} &= \tfrac{ck}{c_5}x_2(x_4+rx_5)^{k-1}\\ X_-\frac{\partial f}{\partial x_4} &= \tfrac{ck(k-1)}{c_5}x_2x_3(x_4+rx_5)^{k-2} + X_-\tfrac{\partial h_{k+1}}{\partial x_4}(x_2,x_4,x_5)\end{aligned}$$

$$X_- \frac{\partial f}{\partial x_5} = \tfrac{ck(k-1)}{c_5} x_2 x_3 (x_4 + r x_5)^{k-2} + X_- \tfrac{\partial h_{k+1}}{\partial x_5}(x_2, x_4, x_5).$$

In view of (8.7), we have

$X_- \frac{\partial h_{k+1}}{\partial x_2}(x_2, x_4, x_5) = \frac{c_1 ck}{c_5} x_3 (x_4 + r x_5)^{k-1}$

$\frac{ck}{c_5} x_2 (x_4 + r x_5)^{k-1} = c_2 \frac{\partial h_{k+1}}{\partial x_2}(x_1, x_4, x_5) + \frac{c_3 ck(k-1)}{c_5} x_1 x_3 (x_4 + r x_5)^{k-2}$

$+ c_3 \frac{\partial h_{k+1}}{\partial x_4}(x_2, x_4, x_5) + \frac{c_4 ck(k-1)}{c_5} r x_1 x_3 (x_4 + r x_5)^{k-2}$

$+ c_4 \frac{\partial h_{k+1}}{\partial x_5}(x_2, x_4, x_5)$

$\frac{ck(k-1)}{c_5} x_2 x_3 (x_4 + r x_5)^{k-2} + X_- \frac{\partial h_{k+1}}{\partial x_4}(x_2, x_4, x_5) = ck x_3 (x_4 + r x_5)^{k-1}$

$\frac{ck(k-1)}{c_5} r x_2 x_3 (x_4 + r x_5)^{k-2} + X_- \frac{\partial h_{k+1}}{\partial x_5}(x_2, x_4, x_5) = \frac{c_6 ck}{c_5} x_3 (x_4 + r x_5)^{k-1}$

$\Rightarrow X_- \frac{\partial h_{k+1}}{\partial x_2}(x_2, x_4, x_5) = \frac{c_1 ck}{c_5} x_3 (x_4 + r x_5)^{k-1}$

$X_- \frac{\partial h_{k+1}}{\partial x_4}(x_2, x_4, x_5) = \frac{c_5 ck}{c_5} x_3 (x_4 + r x_5)^{k-1} - \frac{ck(k-1)}{c_5} x_2 x_3 (x_4 + r x_5)^{k-2}$

$X_- \frac{\partial h_{k+1}}{\partial x_5}(x_2, x_4, x_5) = \frac{c_6 ck}{c_5} x_3 (x_4 + r x_5)^{k-1} - \frac{ck(k-1)}{c_5} r x_2 x_3 (x_4 + r x_5)^{k-2}$

$c_2 \frac{\partial h_{k+1}}{\partial x_2} + c_3 \frac{\partial h_{k+1}}{\partial x_4} + c_4 \frac{\partial h_{k+1}}{\partial x_5} = \frac{ck}{c_5} x_2 (x_4 + r x_5)^{k-1}$

$\Rightarrow \frac{ck(k-1)}{c_5}(c_3 + r c_4) x_1 x_3 (x_4 + r x_5)^{k-2} = 0$

$\frac{ck}{c_5}(c_1 c_2 + c_3 c_5 + c_4 c_6) x_3 (x_4 + r x_5)^{k-1} = X_-[\frac{ck}{c_5} x_2 (x_4 + r x_5)^{k-1}] = \frac{ck}{c_5} x_3 (x_4 + r x_5)^{k-1}$

and $c = 0$ or $c_3 c_5 + c_4 c_6 = 0$ because $r = \frac{c_6}{c_5}$.

Observe that $c \neq 0$, otherwise $X_- f$ would be zero and in view of (8.8) c_5 would be zero also. This contradicts to our assumption that $c_5 \neq 0$. Therefore we have

$$c_3 c_5 + c_4 c_6 = 0 \tag{8.9}$$

and

$$c_1 c_2 + c_3 c_5 + c_4 c_6 = 1 \tag{8.10}$$

Since $c_1 = -1$, the above two equations imply $c_2 - 1$.

Write $h_{k+1}(x_2, x_4, x_5) = \sum\limits_{\alpha=0}^{k+1} x_2^{\alpha} p_{k+1-\alpha}(x_4, x_5)$ where $p_{k+1-\alpha}(x_4, x_5)$ is a homogeneous polynomial of degree $k + 1 - \alpha$ in x_4 and x_5 variables. Let α_0 be the largest integer such that $p_{k+1-\alpha_0}(x_4, x_5) \neq 0$.

$$\begin{aligned}\frac{\partial f}{\partial x_2} &= \frac{\partial h_{k+1}}{\partial x_2}(x_2, x_4, x_5)\\ &= \sum_{\alpha=0}^{\alpha_0} \alpha x_2^{\alpha-1} p_{k+1-\alpha}(x_4, x_5)\\ \Rightarrow X_-^{\alpha_0-1}\left(\frac{\partial f}{\partial x_2}\right) &= (\alpha_0)! x_3^{\alpha_0-1} p_{k+1-\alpha_0}(x_4, x_5).\end{aligned}$$

Since $X_-[x_3^{\alpha_0-1} p_{k+1-\alpha_0}(x_4, x_5)] = 0$, we have an irreducible $sl(2,\mathrm{C})$-submodule of I in the following form

$\langle x_3^{\alpha_0-1} p_{k+1-\alpha_0}(x_4,x_5), X_+[x_3^{\alpha_0-1} p_{k+1-\alpha_0}(x_4,x_5)], \ldots, X_+^{2\alpha_0-1} p_{k+1-\alpha_0}(x_4,x_5)]\rangle$.

Thus $2\alpha_0 - 1 \leq 3$ because $I = (3) \oplus (1) \oplus (1)$. This implies $\alpha_0 \leq 2$ and

$$f = \frac{ck}{c_5} x_1 x_3 (x_4 + r x_5)^{k-1} + p_{k+1}(x_4, x_5) + p_k(x_4, x_5) x_2 + p_{k-1}(x_4, x_5) x_2^2.$$

As $c \neq 0$, it is easy to see that

$$(3) = \langle x_1(x_4 + r x_5)^{k-1}, x_2(x_4 + r x_5)^{k-1}, x_3(x_4 + r x_5)^{k-1}\rangle \subseteq I.$$

$$X_-\left(\frac{\partial f}{\partial x_2}\right) = c_1 \frac{\partial f}{\partial x_1}$$

$$\Rightarrow\ 2x_3 p_{k-1}(x_4, x_5) = \frac{c_1 ck}{c_5} x_3 (x_4 + r x_5)^{k-1}$$

$$\Rightarrow\ p_{k-1}(x_4, x_5) = \frac{c_1 ck}{2c_5}(x_4 + r x_5)^{k-1} = \frac{-ck}{2c_5}(x_4 + r x_5)^{k-1}$$

$$\begin{aligned}\Rightarrow\ f = &\frac{ck}{2c_5}(2x_1x_3 - x_2^2)(x_4 + r x_5)^{k-1} + p_{k+1}(x_4, x_5) +\\ &+ x_2 p_k(x_4, x_5)\end{aligned}$$

$$\Rightarrow\ X_- f = x_3 p_k(x_4, x_5) = c x_3 (x_4 + r x_5)^k$$

$$\Rightarrow\ p_k(x_4, x_5) = c(x_4 + r x_5)^k$$

$$\Rightarrow\ f = \frac{ck}{2c_5}(2x_1x_3 - x_2^2)(x_4 + r x_5)^{k-1} + p_{k+1}(x_4, x_5) + c x_2 (x_4 + r x_5)^k.$$

By (8.8), we have

$$\begin{aligned}
\frac{\partial}{\partial x_3}(X_-f) &= (c_2+1)\frac{\partial f}{\partial x_2} + c_3\frac{\partial f}{\partial x_4} + c_4\frac{\partial f}{\partial x_5} \\
&= c_3\frac{\partial f}{\partial x_4} + c_4\frac{\partial f}{\partial x_5} \\
\Rightarrow c(x_4+rx_5)^k &= c_3[\frac{ck(k-1)}{2c_5}(2x_1x_3-x_2^2)(x_4+rx_5)^{k-2} + \frac{\partial p_{k+1}}{\partial x_4}(x_4,x_5) \\
&\quad +ckx_2(x_4+rx_5)^{k-1}] + c_4[\frac{ck(k-1)}{2c_5}r(2x_1x_3-x_2^2)(x_4+rx_5)^{k-2} \\
&\quad +\frac{\partial p_{k+1}}{\partial x_5}(x_4,x_5) + ckrx_2(x_4+rx_5)^{k-1}]
\end{aligned}$$

$$\Rightarrow c_3\frac{\partial p_{k+1}}{\partial x_4}(x_4,x_5) + c_4\frac{\partial p_{k+1}}{\partial x_5}(x_4,x_5) = c(x_4+rx_5)^k.$$

We claim that $c_4 \neq 0$. Suppose on the contrary that $c_4 = 0$. Then $c_3 \neq 0$ because $c \neq 0$. In particular we have

$$\begin{aligned}
&\frac{\partial p_{k+1}}{\partial x_4}(x_4,x_5) = \frac{c}{c_3}(x_4+rx_5)^k \\
&\Rightarrow p_{k+1}(x_4,x_5) = \frac{c}{c_3(k+1)}(x_4+rx_5)^{k+1} + dx_5^{k+1} \\
&\Rightarrow \qquad f = \frac{ck}{2c_5}(2x_1x_3-x_2^2)(x_4+rx_5)^{k-1} + \frac{c}{c_3(k+1)}(x_4+rx_5)^{k+1} + dx_5^{k+1} \\
&\qquad\qquad + cx_2(x_4+rx_5)^k.
\end{aligned}$$

From (8.8), we have

$$\begin{aligned}
&\frac{\partial f}{\partial x_4} = \frac{1}{c_3}\frac{\partial}{\partial x_3}(X_-f) \\
\Rightarrow\ &\frac{ck(k-1)}{2c_5}(2x_1x_3-x_2^2)(x_4+rx_5)^{k-2} + \frac{c}{c_3}(x_4+rx_5)^k + ckx_2(x_4+rx_5)^{k-1} \\
&= \frac{c}{c_3}(x_4+rx_5)^k \\
\Rightarrow\ &c = 0
\end{aligned}$$

This is impossible as we saw before. Our claim $c_4 \neq 0$ is established. It follows that, by (8.9) we have

$$\frac{\partial p_{k+1}}{\partial x_5}(x_4,x_5) - r\frac{\partial p_{k+1}}{\partial x_4}(x_4,x_5) = \frac{c}{c_4}(x_4+rx_5)^k.$$

Let $y_4 = x_4 + rx_5$ and $y_5 = x_5$. Let

$$\begin{aligned}
\tilde{p}_{k+1}(y_4, y_5) &= p_{k+1}(y_4 - ry_5, y_5) \\
\Rightarrow \frac{\partial \tilde{p}_{k+1}}{\partial y_5}(y_4, y_5) &= \frac{\partial p_{k+1}}{\partial x_4}(y_r - ry_5, y_5)(-r) + \frac{\partial p_{k+1}}{\partial x_5}(y_4 - ry_5, y_5) \\
&= \frac{c}{c_4}[y_4 - ry_5 + ry_5]^k \\
&= \frac{c}{c_4} y_4^k \\
\Rightarrow \tilde{p}_{k+1}(y_4, y_5) &= \frac{c}{c_4} y_4^k y_5 + d y_4^{k+1} \\
\Rightarrow p_{k+1}(x_4, x_5) &= \frac{c}{c_4}(x_4 + rx_5)^k x_5 + d(x_4 + rx_5)^{k+1} \\
\Rightarrow f &= \frac{ck}{2c_5}(2x_1x_3 - x_2^2)(x_4 + rx_5)^{k-1} + \frac{c}{c_4}(x_4 + rx_5)^k x_5 + d(x_4 + rx_5)^{k+1} \\
&\quad + cx_2(x_4 + rx_5)^k.
\end{aligned}$$

So we are in case (i) (b).

Similarly, if $c_6 \neq 0$, then we will be in case (ii) (c).

Finally, we have to consider the case where both c_5 and c_6 are zero. In this case, equation (8.8) becomes

$$\frac{\partial}{\partial x_1}(X_- f) = 0 = \frac{\partial}{\partial x_2}(X_- f) = \frac{\partial}{\partial x_4}(X_- f) = \frac{\partial}{\partial x_5}(X_- f).$$

This implies that $X_- f$ depends only on x_3 variable. Since weight of $X_- f$ is -2, we have $X_- f = ax_3$ for some constant a. Since degree of $X_- f$ is $k+1 \geq 3$, we conclude that $X_- f = 0$. Similarly we can prove that $X_+ f = 0$. Hence f is a $sl(2,\mathbf{C})$ invariant polynomial.

Case 2. $I = (3) \oplus (1)$ and f is a homogeneous polynomial of weight 0. We shall follow the notations in Case 1. The same argument as in Case 1 gives $c_1 = -1$. If $c_5 \neq 0$, then the same argument as in Case 1 shows that $c \neq 0$ and

$$f = \frac{ck}{2c_5}(2x_1x_3 - x_2^2)(x_4 + rx_5)^{k-1} + \frac{c}{c_4}(x_4 + rx_5)^k x_5 + d(x_4 + rx_5)^{k+1} + cx_2(x_4 + rx_5)^k.$$

Observe that $\frac{\partial f}{\partial x_3} = \frac{ck}{c_5}x_1(x_4 + rx_5)^{k-1}$. It follows that

$$(3) = \langle x_1(x_4 + rx_5)^{k-1}, x_2(x_4 + rx_5)^{k-1}, x_3(x_4 + rx_5)^{k-1}\rangle \subseteq I.$$

Hence $(x_4 + rx_5)^k = \frac{1}{c}\frac{\partial f}{\partial x_2} + \frac{k}{c_5}x_2(x_4 + rx_5)^{k-1}$ is in I. Clearly $x_1(x_4 + rx_5)^{k-1}$, $x_2(x_4 + rx_5)^{k-2}$, $x_3(x_4 + rx_5)^{k-1}$, $(x_4 + rx_5)^k$ and

$$\frac{\partial f}{\partial x_4} = \frac{ck(k-1)}{2c_5}(2x_1x_3 - x_2^2)(x_4 + rx_5)^{k-2} + \frac{ck}{c_4}(x_4 + rx_5)^{k-1}x_5 + d(k+1)(x_4 + rx_5)^k + cx_2(x_4 + rx_5)^{k-1}$$

are five linearly independent elements in I. This contradicts to our hypothesis that $\dim I = 4$. Therefore we conclude that $c_5 = 0$. Similarly we have $c_6 = 0$. By applying the same argument as in Case 1, we have $X_- f = 0$. Thus f is a $sl(2, \mathbf{C})$ invariant polynomial.

Case 3. $I = (3)$ and f is a homogeneous polynomial of weight 0. In this case f is a $sl(2, \mathbf{C})$ invariant polynomial. The argument is the same as Case 2.

We finally claim that if f is a $sl(2, \mathbf{C})$ invariant polynomial, then f is of the following form

$$f = \sum_{j=0}^{[\frac{k+1}{2}]} (x_2^2 - 2x_1x_3)^j q_{k+1-2j}(x_4, x_5)$$

where $q_{k+1-2j}(x_4, x_5)$ is a homogeneous polynomial of degree $k + 1 - 2j$ in x_4 and x_5 variables. To see this write

$$f = \sum_{i=0}^{k+1} \sum_{\alpha_1 + \alpha_2 = k+1-i} p_i^{(\alpha_1, \alpha_2)}(x_1, x_2, x_3)x_4^{\alpha_1}x_5^{\alpha_2}$$

where $p_i^{(\alpha_1,\alpha_2)}(x_1,x_2,x_3)$ is a homogeneous polynomial of degree i in x_1, x_2 and x_3 variables.

$X_- f = 0$

$$\Rightarrow \sum_{i=0}^{k+1} \sum_{\alpha_1+\alpha_2=k+1-i} X_-[p_i^{(\alpha_1,\alpha_2)}(x_1,x_2,x_3)]x_4^{\alpha_1}x_5^{\alpha_2} = 0$$

$\Rightarrow$ $X_- p_i^{(\alpha_1,\alpha_2)}(x_1,x_2,x_3) = 0$ for all i and (α_1,α_2).

$\Rightarrow$ $p_i^{(\alpha_1,\alpha_2)}(x_i,x_2,x_3)$ is a $sl(2,\mathbf{C})$ invariant polynomial for all i and (α_1,α_2)

$\Rightarrow$ $i = 2j$ and $p_i^{(\alpha_1,\alpha_2)}(x_1,x_2,x_3) = c_i^{(\alpha_1,\alpha_2)}(x_2^2 - 2x_1x_3)^j$.

Take

$$q_{k+1-2j}(x_4,x_5) = \sum_{\alpha_1+\alpha_2=k+1-2j} c_{2j}^{(\alpha_1,\alpha_2)} x_4^{\alpha_1} x_5^{\alpha_2}.$$

Then we have

$$f = \sum_{j=0}^{[\frac{k+1}{2}]} (x_2^2 - 2x_1x_3)^j q_{k+1-2j}(x_4,x_5)$$

as claimed. Q.E.D.

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