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GENERALIZED CARTAN MATRICES ARISING FROM NEW DERIVATION LIE ALGEBRAS OF ISOLATED HYPERSURFACE SINGULARITIES

NAVEED HUSSAIN, STEPHEN S.-T. YAU AND HUAIQING ZUO

Dedicated to professor Shing Tung Yau on the occasion of his 70th birthday

Let V be a hypersurface with an isolated singularity at the origin defined by the holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. The Yau algebra $L(V)$ is defined to be the Lie algebra of derivations of the moduli algebra $A(V) := \mathcal{O}_n/\big(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\big)$, i.e., $L(V) = \mathrm{Der}(A(V), A(V))$. It is known that $L(V)$ is finite dimensional and its dimension $\lambda(V)$ is called the Yau number. We introduced a new Lie algebra $L^*(V)$ which was defined to be the Lie algebra of derivations of

$$A^*(V) = \mathcal{O}_n \Big/ \left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}, \mathrm{Det}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j=1,\dots,n} \right),$$

i.e., $L^*(V) = \mathrm{Der}(A^*(V), A^*(V))$. $L^*(V)$ is finite dimensional and $\lambda^*(V)$ is the dimension of $L^*(V)$. In this paper we compute the generalized Cartan matrix $C(V)$ and other various invariants arising from the new Lie algebra $L^*(V)$ for simple elliptic singularities and simple hypersurface singularities. We use the generalized Cartan matrix to characterize the ADE singularities.

1. Introduction

Recall that simple (Kleinian, rational double point) singularities, consist of two series $A_k : \{x^2 + y^2 - z^{k+1} = 0\}$, $k \geq 1$, $D_k : \{x^2 + y^2 z + z^{k-1} = 0\}$, $k \geq 4$ and three exceptional singularities E_6, E_7, E_8 defined by polynomials

$$x^2 + y^3 + z^4, \quad x^2 + y^3 + yz^3, \quad x^2 + y^3 + z^5,$$

respectively. $\widetilde{E}_6$ is a simple elliptic singularity defined by

$$\{(x, y, z) \in \mathbb{C}^3 \mid x^3 + y^3 + z^3 = 0\}.$$

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Its (μ, τ)-constant family is given by

$$V_t = \{(x, y, z) \in \mathbb{C}^3 \mid f_t(x, y, z) = x^3 + y^3 + z^3 + txyz = 0\},$$

with $t^3 + 27 \neq 0$ (see [Yau 1983]). $\widetilde{E}_7$ is a simple elliptic singularity defined by $\{(x, y, z) \in \mathbb{C}^3 \mid x^4 + y^4 + z^2 = 0\}$. Seeley and Yau [1990] showed that its (μ, τ)-constant family is given by

$$V_t = \{(x, y, z) \in \mathbb{C}^3 \mid f_t(x, y, z) = x^4 + y^4 + tx^2y^2 + z^2 = 0\},$$

with $t^2 \neq 4$. The simple elliptic singularity $\widetilde{E}_8$ defined by

$$\{(x, y, z) \in \mathbb{C}^3 \mid x^6 + y^3 + z^2 = 0\}.$$

In [Seeley and Yau 1990], the authors had studied the (μ, τ)-constant family of $\widetilde{E}_8$, which is given by

$$V_t = \{(x, y, z) \in \mathbb{C}^3 \mid f_t = x^6 + y^3 + z^2 + tx^4y = 0\},$$

with $4t^3 + 27 \neq 0$.

Finite dimensional Lie algebras are the semidirect product of the semisimple Lie algebras and solvable Lie algebras. Brieskorn gave the connection between simple Lie algebras and simple singularities. The Lie algebra L is called nilpotent if the lower central series of ideals: $L_0 = L,\ L_1 = [L, L],\ L_i = [L, L_{(i-1)}],\ i = 2, 3, \ldots$ terminates. Simple Lie algebras have been well understood, but not the nilpotent Lie algebras.

The problem of classifying nilpotent Lie algebra was studied for the first time by Umlauf [1891], a student of Engle. Umlauf gave the complete list over $\mathbb{C}$ up to dimension 6 and a certain complex family at dimensions 7, 8 and 9. The introduction of the root systems for the nilpotent Lie algebras was given by Bratzlavsky [1974] and Favre [1972; 1973]. The concept of root system constitutes an important step in the classification of nilpotent Lie algebras. By using these root systems, Santharoubane [1983] established a link between the nilpotent Lie algebras and the Kac–Moody Lie algebras (which generalize the semisimple Lie algebras and are of infinite dimension). Yau [1986], introduced many numerical invariants, namely, dimension of the Lie algebra $L(V)$; dimension of the maximal nilpotent subalgebras (i.e., nilradical of nilpotent Lie algebra) $g(V)$ of $L(V)$; dimension of the maximal torus of $g(V)$; generalized Cartan matrix $C(V)$ (see Definition 2.6); type and nilpotency of singularity. Benson and Yau [1987] computed the generalized Cartan matrix $C(V)$ for simple hypersurface singularities by using the Yau algebra. Moreover, Seeley and Yau [1991] computed the generalized Cartan matrix $C(V)$ by using Yau algebras of simple elliptic singularities. Thus it is extremely important to establish connection between singularities and nilpotent Lie algebras.

It is well-known that for any isolated hypersurface singularity $(V, 0) \subset (\mathbb{C}^n, 0)$, where $V = V(f) = \{f = 0\}$, one can consider the finite dimensional moduli algebra

$$A(V) := \mathcal{O}_n \Big/ \left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right).$$

The $\mathcal{O}_n$ is the algebra of convergent power series in n indeterminates and $f \in \mathcal{O}_n$. Mather and Yau [1982] proved that the complex structure of $(V, 0)$ is determined by its moduli algebra. Subsequently, Yau [1986] introduced the Lie algebra $L(V)$ to $(V, 0)$, which is the algebra of derivations of $A(V)$, i.e., $L(V) := \mathrm{Der}(A(V), A(V))$. Yau and his collaborators have been systematically studied the Lie algebras of isolated hypersurface singularities since the 1980s [Yau 1983; 1984; 1986; 1991; Benson and Yau 1987; 1990; Seeley and Yau 1990; 1991, Yau and Zuo 2016a; 2016b; Chen et al. 1995; 2019; ≥ 2020a; ≥ 2020b; Hussain et al. 2018; 2020; 2019a; 2019b; Hussain 2018]. One can construct nilpotent Lie algebras from the Yau algebras, however the Yau algebras can not be used to distinguish the ADE singularities [Elashvili and Khimshiashvili 2006]. Recently, in [Chen et al. ≥ 2020b], a new natural connection between the set of complex analytic isolated hypersurface singularities and the set of finite dimensional nilpotent Lie algebras has been constructed. We introduced a new Lie algebra $L^*(V) := \mathrm{Der}(A^*(V), A^*(V))$, to be the Lie algebra of derivations of the Artinian algebra

$$A^*(V) = \mathcal{O}_n \Big/ \left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}, \mathrm{Det}\left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j=1,\dots,n} \right),$$

and $\lambda^*(V)$ is the dimension of $L^*(V)$. In [Chen et al. ≥ 2020b], we have used it to distinguish ADE singularities and prove Torelli-type theorems for some simple elliptic singularities. This new Lie algebra is a subtle invariant associated to an isolated hypersurface singularity. In this paper we shall study the new Lie algebra $L^*(V)$ for simple hypersurface singularities and simple elliptic singularities. We shall introduce many numerical invariants, namely, dimension of the maximal nilpotent subalgebras (i.e., nilradical of nilpotent Lie algebra) $g(V)$ of $L^*(V)$; dimension of the maximal torus of $g(V)$; generalized Cartan matrix $C(V)$, etc. We shall compute different numerical invariants such as dimension of maximal torus of $g(V)$; type and nilpotency of singularity and generalized Cartan matrix $C(V)$ and so on. We use the generalized Cartan matrix to characterize the ADE singularities with one exceptional case (i.e, A_6 and D_5) and obtain the following result.

Theorem 1.1. *The generalized Cartan matrix characterizes the simple* (*ADE*) *hypersurface singularities except* A_6 *and* D_5 *singularities. I.e.*, *if* X *and* Y *are two simple hypersurface singularities*, *then* $C(X) = C(Y)$ *if and only if* X *and* Y *are analytically isomorphic.*

2. Preliminaries

2A. ***Isolated hypersurface singularities.*** Let $\mathbb{C}[x_1, x_2, \dots, x_n]$ be the algebra of complex polynomials in n indeterminates. Denote by $\mathcal{O}_n$ the algebra of germs of holomorphic functions at the origin of $\mathbb{C}^n$. Obviously, $\mathcal{O}_n$ can be naturally identified with the algebra of convergent power series in n indeterminates with complex coefficients. For a polynomial $f \in \mathbb{C}[x_1, x_2, \dots, x_n]$, denote by $V = V(f)$ the germ at the origin of $\mathbb{C}^n$ of hypersurface $\{f = 0\} \subset \mathbb{C}^n$. We say that V is a germ of isolated hypersurface singularity if the origin is an isolated zero of the gradient of f. The local (function) algebra of V is defined as the (commutative associative) algebra $F(V) \cong \mathcal{O}_n/(f)$, where (f) is the principal ideal generated by the germ of f at the origin. According to Hilbert's Nullstellensatz for an isolated singularity $V = V(f) = \{f = 0\}$ the factor-algebra $A(V) = \mathcal{O}_n/\big(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\big)$ is finite dimensional. This factor-algebra is called the moduli algebra of V and its dimension $\tau(V)$ is called Tjurina number. The well-known Mather–Yau theorem states that

Theorem 2.1 [Mather and Yau 1982]. *The analytic isomorphism type of an isolated hypersurface singularity is determined by the isomorphism class of its moduli algebras i.e.*,

$$(V_1, 0) \cong (V_2, 0) \Leftrightarrow A(V_1) \cong A(V_2).$$

2B. ***Yau algebra.*** Recall that a derivation of commutative associative algebra A is defined as a linear endomorphism D of A satisfying the Leibniz rule: $D(ab) = D(a)b + aD(b)$. Thus for such an algebra A one can consider the Lie algebra of its derivations $\mathrm{Der}(A, A)$ with the bracket defined by the commutator of linear endomorphisms.

Definition 2.2. Let $f(x_1, \dots, x_n)$ be a complex polynomial and $V = \{f = 0\}$ be a germ of an isolated hypersurface singularity at the origin in $\mathbb{C}^n$. Let $A(V)$ be the moduli algebra and $L(V) := \mathrm{Der}(A(V), A(V))$. Yu [1996] calls $L(V)$ the Yau algebra of V. The dimension of $L(V)$ is called the Yau number by Elashvili and Khimshiashvili [2006] and is denoted by $\lambda(V)$.

2C. ***New derivation Lie algebra.*** We recall the following beautiful theorem due to Dimca.

Theorem 2.3 [Dimca 1984]. *Two zero-dimensional isolated complete intersection singularities X and Y are isomorphic if and only if their singular subspaces* $\mathrm{Sing}(X)$ *and* $\mathrm{Sing}(Y)$ *are isomorphic.*

Remark 2.4. Let $V = V(f)$ be an isolated quasihomogeneous hypersurface singularity. Assume that X defined by $\big(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\big)$ is a zero-dimensional isolated complete intersection singularity. Then $\mathrm{Sing}(X)$ is defined by

$$\left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}, \mathrm{Det}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j=1,\dots,n}\right).$$

Theorem 2.3 implies that in order to study the analytic isomorphism type of zero dimensional isolated complete intersection singularity X, we only need to consider the Artinian local algebra $A^*(V)$ which is the coordinate ring of $\mathrm{Sing}(X)$. Thus $A^*(V)$ is defined as the quotient

$$\mathcal{O}_n \Big/ \left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}, \mathrm{Det}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j=1,\dots,n}\right).$$

Combing Theorem 2.3 with the Mather–Yau theorem, we know that the $A^*(V)$ is a complete invariant of quasihomogeneous isolated hypersurface singularities (i.e., $A^*(V)$ determines and is determined by the analytic isomorphism type of the singularity). We call $A^*(V)$ the generalized moduli algebra of V. Based on this important observation, we introduced the following new invariants for isolated hypersurface singularities [Chen et al. $\geq$ 2020b].

Definition 2.5. Let $V = \{f = 0\}$ be a germ of isolated hypersurface singularity at the origin of $\mathbb{C}^n$ defined by $f \in \mathbb{C}[x_1, x_2, \dots, x_n]$. The new Lie algebra arising from the isolated hypersurface singularity V is defined as $L^*(V) := \mathrm{Der}(A^*(V), A^*(V))$ $\big($we simply denote it by $\mathrm{Der}(A^*(V))\big)$. The dimension of this new Lie algebra is denoted by $\lambda^*(V)$.

2D. *Kac–Moody Lie algebras and isolated hypersurface singularities.* Let $(V, 0)$ be an isolated hypersurface singularity. Let $g(V)$ be the maximal ideal of $L^*(V)$ consisting of nilpotent elements. It is follows from [Santharoubane 1983] that a generalized Cartan matrix $C(V)$, constructed from $g(V)$, is an invariant of $(V, 0)$ (see [Yau 1986]).

Definition 2.6. An $l \times l$ matrix with entries in $\mathbb{Z}$, $C = (c_{ij})$ is a generalized Cartan matrix if

(a) $$c_{ii} = 2 \quad \text{for all } i = 1, \dots, l,$$

(b) $$c_{ij} \leq 0 \quad \text{for all } i, j = 1, \dots, l,\ i \neq j,$$

(c) $$c_{ij} = 0 \text{ if and only if } c_{ji} = 0 \quad \text{for all } i, j = 1, \dots, l,\ i \neq j.$$

To each generalized Cartan matrix $C(V)$, one can associate a Lie algebra $\mathrm{KM}(C)$ (called a Kac–Moody Lie algebra) defined by generators

$$\{f_1, \dots, f_l, h_1, \dots, h_l, e_1, \dots, e_l\}$$

and relations:

$$
\begin{aligned}
&[h_i, e_j] = c_{ij} e_j, \quad [h_i, f_j] = -c_{ij} f_j, \quad \text{for all } i, j = 1, \dots, l,\\
&[h_i, h_j] = 0, \quad \text{for all } i, j = 1, \dots, l, \quad [e_i, f_i] = h_i,\\
&[e_i, f_j] = 0, \quad (\operatorname{ad} e_i)^{-c_{ij}+1} e_j = 0 = (\operatorname{ad} f_i)^{-c_{ij}+1} f_j, \quad \text{for all } i \neq j.
\end{aligned}
$$

Let $H = \mathbb{C}h_i + \cdots + \mathbb{C}h_l$; denote $\xi_+(C)$ (resp. $\xi_-(C)$) the subalgebra of $\mathrm{KM}(C)$ generated by $\{e_1, \dots, e_l\}$ (resp. $(f_1, \dots, f_l)$) one shows that:

$$\mathrm{KM}(C) = \xi_+(C) \oplus H \oplus \xi_-(C).$$

One can also define $\xi_+(C)$ by generators $\{e_1, \dots, e_l\}$, and relations

$$(\operatorname{ad} e_i)^{-c_{ij}+1} e_j = 0 \quad \text{for all } i, j = 1, \dots, l, \ i \neq j.$$

We shall construct the generalized Cartan matrix from an isolated hypersurface singularity $(V, 0)$. Let $g(V)$ be the set of all nilpotent elements in $L^*(V)$; then $g(V)$ is the maximal nilpotent Lie subalgebra of $L^*(V)$ and $\mathrm{Der}(g(V))$ is its derivation algebra.

Definition 2.7. A torus on $g(V)$ is a commutative subalgebra of $\mathrm{Der}(g(V))$ whose elements are semisimple endomorphism. A maximal torus is a torus not contained in any other torus. The dimension of the maximal torus is called the generalized Mostow number (GMN). The GMN is an invariant of the isolated singularity $(V, 0)$.

Theorem 2.8 (Mostow's theorem [Santharoubane 1983]). *If T_1 and T_2 are maximal tori of $g(V)$, then there exists $\varphi \in$* $\operatorname{Aut} g(V)$ (automorphism group of $g(V)$) *such that $\varphi T_1 \varphi^{-1} = T_2$.*

Let T be a maximal torus and consider the root space decomposition of $g(V)$ relative to T [Santharoubane 1983]:

$$
\begin{aligned}
g(V) &= \sum_{\beta \in R(T)} g(V)^\beta,\\
g(V)^\beta &= \{x \in g(V) \mid tx = \beta(t)x, \ \forall t \in T\},
\end{aligned}
$$

and

$$
\begin{aligned}
R(T) &= \{\beta \in T^* \mid g(V)^\beta \neq (0)\} \quad \text{(root system)} \ ,\\
R^1(T) &= \bigl\{\beta \in R(T) \mid g(V)^\beta \not\subseteq [g(V), g(V)]\bigr\},\\
l_\beta &= \dim\left(\frac{g(V)^\beta}{[g(V), g(V)] \cap g(V)^\beta}\right), \quad \text{for all } \beta \in R^1(T),\\
d_\beta &= \dim(g(V)^\beta), \quad \beta \in R^1(T).
\end{aligned}
$$

The map $\beta \mapsto d_\beta$, $R^1(T) \to \mathbb{N}^*$ gives the partition

$$R^1(T) = R^1(T)_{p_1} \cup \cdots \cup R^1(T)_{p_q}, \quad p_1 < \cdots < p_q, \quad R^1(T)_{p_i} \neq \varnothing,$$
$$R^1(T)_p = \{\beta \in R^1(T) \mid d_\beta = p\}.$$

Set $s_i = \sharp R^1(T)_{p_i}$ and $s = s_1 + \cdots + s_q$. We let $d_{\beta_i} = d_i$ and $l_{\beta_i} = l_i$.

Let $f : \{1, \ldots, l\} \to \{1, \ldots, s\}$ be defined by

$$f_i = \begin{cases} 1, & 1 \leqslant i \leqslant l_1, \\ 2, & l_1 < i \leqslant l_1 + l_2, \\ \vdots & \\ s, & l_1 + l_2 + \cdots + l_{s-1} < i \leqslant l. \end{cases}$$

Theorem 2.9 [Santharoubane 1983]. *For $i, j \in \{1, \ldots, l\}$ $i \neq j$ let*

$$-c_{ij}(T) = \min\bigl\{-n \in N \mid (\operatorname{ad} v)^{-n+1} w = 0,\ \forall v \in g(V)^{\beta_{f(i)}},\ \forall w \in g(V)^{\beta_{f(j)}}\bigr\},$$

with $(\operatorname{ad} 0)^0 = 0$ *and let* $c_{ii}(T) = 2$ *for* $i = 1, \ldots, l$. *Then*

$$C(T) = (c_{ij}(T))_{1 \le i, j \le l}$$

is a Cartan matrix.

3. Main Results

Now we apply the above theory to study the new Lie algebra $L^*(V)$ of simple hypersurface singularities and simple elliptic singularities. We use the following convention: $g^1 = [g, g], \ldots, g^{p+1} = [g, g^p]$. We use N to denote the set of positive integers.

Proposition 3.1. *Let $V = \{x, y, z \in \mathbb{C}^3 \mid x^2 + y^2 - z^{k+1} = 0\}$ be the A_k singularity, $k \geq 1$. Then*

$$C(A_k) = \begin{cases} \textit{is not defined}, & k=1,2,3,4, \\ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, & k=5, \\ \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, & k=6, \\ \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}, & k=7, \\ \begin{pmatrix} 2 & -(k-5) \\ -(k-5)/2 & 2 \end{pmatrix}, & k \textit{ is odd and } k \geq 9, \\ \begin{pmatrix} 2 & -(k-5) \\ -(k-4)/2 & 2 \end{pmatrix}, & k \textit{ is even and } k \geq 8. \end{cases}$$

Proof. It is easy to see that $A^*(V) = \langle 1, z, z^2, \ldots, z^{k-2} \rangle$ for $k \geq 2$ with the multiplication rule $z^{k-1} = 0$, and $A^*(V) = 0$ for $k = 1$. After a simple calculation we get

$$L^*(V) = \begin{cases} \langle z\partial_z, z^2\partial_z, \dots, z^{k-2}\partial_z\rangle, & k \geq 3, \\ 0, & k = 1, 2, \end{cases}$$

and

$$g(V) = \begin{cases} \langle z^2\partial_z, \dots, z^{k-2}\partial_z\rangle, & k \geq 4, \\ \langle z\partial_z\rangle, & k = 3, \\ 0, & k = 2. \end{cases}$$

It is easy to see the Cartan matrix is not defined when $1 \leq k \leq 4$.

For the A_5 singularity,

$$g(V) = \langle z^2\partial_z, z^3\partial_z\rangle.$$

By setting $x_1 = z^2\partial_z,\ x_2 = z^3\partial_z$, we get the multiplication table $[x_1, x_2] = 0$. The type of A_5 singularity $= \dim g(V)/[g(V), g(V)] = 2$. The nilpotency of A_5 singularity $= \min\{p \in N \cup \{0\} \mid g(V)^{p+1} = 0\} = 0$. It is easy to see from ([Benson and Yau 1987]) that the torus T of $g(V)$ is spanned by

$$\begin{aligned} t_1 : g(V) &\to g(V), & t_2 : g(V) &\to g(V), \\ x_1 &\mapsto x_1, & x_1 &\mapsto 0, \\ x_2 &\mapsto 0, & x_2 &\mapsto x_2. \end{aligned}$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2$. Since $\dim T = 2 =$ the type of A_5, T is the maximal torus of $g(V)$. Let $\beta_i : T \to \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2$;

$$\begin{aligned} g(V) &= g^{\beta_1} \oplus g^{\beta_2} \\ &= \mathbb{C}x_1 \oplus \mathbb{C}x_2 \end{aligned}$$

(x_1, x_2) is a T-minimal system of generators. Therefore the generalized Cartan matrix is

$$C(A_5) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

For the A_6 singularity, we have the following multiplication table $[x_1, x_2] = x_3$. The type of A_6 singularity $= \dim g(V)/[g(V), g(V)] = 2$. The nilpotency of the A_6 singularity $= \min\{p \in N \cup \{0\} \mid g(V)^{p+1} = 0\} = 1$. It is easy to see from [Benson and Yau 1987] that the torus T of $g(V)$ is spanned by

$$\begin{aligned} t_1 : g(V) &\to g(V), & t_2 : g(V) &\to g(V), \\ x_1 &\mapsto x_1, & x_1 &\mapsto 0, \\ x_2 &\mapsto 0, & x_2 &\mapsto x_2, \\ x_3 &\mapsto x_3, & x_3 &\mapsto x_3. \end{aligned}$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2$. Since $\dim T = 2 =$ the type of A_6, T is the maximal torus of $g(V)$. Let $\beta_i : T \to \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2$;

$$\begin{aligned} g(V) &= g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_1+\beta_2} \\ &= \mathbb{C}x_1 \oplus \mathbb{C}x_2 \oplus \mathbb{C}x_3. \end{aligned}$$

Note that (x_1, x_2) is a T-minimal system of generators. The generalized Cartan matrix is

$$C(A_6) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

For the A_7 singularity, we have the following multiplication table:

$$[x_1, x_2] = x_3, \quad [x_1, x_3] = 2x_4.$$

The type of A_7 singularity $= \dim g(V)/[g(V), g(V)] = 2$. The nilpotency of the A_7 singularity $= \min\{p \in N \cup \{0\} \mid g(V)^{p+1} = 0\} = 2$. It is easy to see from [Benson and Yau 1987] that the torus T of $g(V)$ is spanned by

$$\begin{aligned} t_1 : g(V) &\to g(V), & t_2 : g(V) &\to g(V), \\ x_1 &\mapsto x_1, & x_1 &\mapsto 0, \\ x_2 &\mapsto 0, & x_2 &\mapsto x_2, \\ x_3 &\mapsto x_3, & x_3 &\mapsto x_3, \\ x_4 &\mapsto 2x_4, & x_4 &\mapsto x_4. \end{aligned}$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2$. Since $\dim T = 2 =$ the type of A_7, T is the maximal torus of $g(V)$. Let $\beta_i : T \to \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2$;

$$\begin{aligned} g(V) &= g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_1+\beta_2} \oplus g^{2\beta_1+\beta_2} \\ &= \mathbb{C}x_1 \oplus \mathbb{C}x_2 \oplus Cx_3 \oplus \mathbb{C}x_4, \end{aligned}$$

(x_1, x_2) is a T-minimal system of generators. The generalized Cartan matrix is

$$C(A_7) = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}.$$

For the A_k singularity $k \geq 8$,

$$g(V) = \langle z^2\partial_z, z^3\partial_z, \ldots, z^{k-2}\partial_z \rangle.$$

By setting $x_1 = z^2\partial_z, x_2 = z^3\partial_z, \ldots, x_{k-3} = z^{k-2}\partial_z$, we have the following multiplication table:

$$\begin{aligned} &[x_1, x_2] = x_3, \quad [x_1, x_3] = 2x_4, \quad \ldots, \quad [x_1, x_{k-4}] = (k-5)x_{k-3}, \\ &[x_2, x_3] = x_5, \quad [x_2, x_4] = 2x_6, \quad \ldots, \quad [x_2, x_{k-5}] = (k-7)x_{k-3}. \end{aligned}$$

The type of A_k ($k \geq 8$) singularity $= \dim g(V)/[g(V), g(V)] = 2$. The nilpotency of the A_k ($k \geq 8$) singularity is

$$\min\left\{p \in N \cup \{0\} \mid g(V)^{p+1} = 0\right\} = k-5.$$

It is easy to see from [Benson and Yau 1987] that the torus T of $g(V)$ is spanned by

$$\begin{aligned} t : g(V) &\to g(V), \\ x_1 &\mapsto x_1, \\ x_2 &\mapsto 2x_2, \\ x_3 &\mapsto 3x_3, \\ &\vdots \\ x_{k-3} &\mapsto (k-3)x_{k-3}. \end{aligned}$$

It is follows from [Benson and Yau 1987] that T is the maximal torus of $g(V)$. Let $\beta : T \to \mathbb{C}$ be a linear map with $\beta(t) = 1$;

$$\begin{aligned} g(V) &= g^{\beta_1} \oplus g^{2\beta_2} \oplus \cdots \oplus g^{(k-3)\beta_1} \\ &= \mathbb{C}x_1 \oplus \mathbb{C}x_2 \oplus \mathbb{C}x_3 \oplus \cdots \oplus \mathbb{C}x_{k-3}, \end{aligned}$$

(x_1, x_2) is a T-minimal system of generators. It is noted that $(\operatorname{ad} x_1)^{k-4}x_2 = 0$, but $(\operatorname{ad} x_1)^{k-5}x_2 \neq 0$. Therefore $c_{12} = -(k-5)$. In order to compute the c_{21} we divide it into two cases.

Case 1. k is odd and $k = 2l+7 \geq 9$, $l \geq 1$, then

$$(\operatorname{ad} x_2)^{l+1}x_1 = -(2\cdot 2-1)(2\cdot 3-1)\cdots(2l-1)x_{2l+3}, \quad (\operatorname{ad} x_2)^{l+2}x_1 = 0,$$
$$\Rightarrow c_{21} = -(l+1) = -\frac{k-5}{2}.$$

Case 2. k is even and $k = 2l+6 \geq 8$, $l \geq 1$, then

$$(\operatorname{ad} x_2)^{l+1}x_1 = -(2\cdot 2-1)(2\cdot 3-1)\cdots(2l-1)x_{2l+3}, \quad (\operatorname{ad} x_2)^{l+2}x_1 = 0,$$
$$\Rightarrow c_{21} = -(l+1) = -\frac{k-4}{2}.$$

Therefore the generalized Cartan matrix is

$$C(A_k) = \begin{cases} \begin{pmatrix} 2 & -(k-5) \\ -\frac{k-5}{2} & 2 \end{pmatrix}, & k \text{ is odd and } k \geq 9, \\ \begin{pmatrix} 2 & -(k-5) \\ -\frac{k-4}{2} & 2 \end{pmatrix}, & k \text{ is even and } k \geq 8. \end{cases} \qquad \square$$

Proposition 3.2. *Let* $V = \{x, y, z \in \mathbb{C}^3 \mid x^2 + y^2 z + z^{k-1} = 0\}$ *be the* D_k *singularity,* $k \geq 4$. *Then*

$$C(D_k) = \begin{cases} \text{is not defined}, & k=4, \\ \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, & k=5, \\ \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, & k=6, \\ \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, & k=7, \\ \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}, & k=8, \\ \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -(k-6) \\ 0 & 0 & -\frac{k-5}{2} & 2 \end{pmatrix}, & k \text{ is odd and } k \geq 9, \\ \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -(k-6) \\ 0 & 0 & -\frac{k-6}{2} & 2 \end{pmatrix}, & k \text{ is even and } k \geq 10. \end{cases}$$

Proof. It is easy to see from [Arnold et al. 2004, Theorem 13.1],

$$A^*(V) = \langle z^i,\ 0 \leq i \leq k-3,\ y \rangle.$$

After a simple calculation we get

$$L^*(V) = \langle y\partial_y, z^{k-3}\partial_y, y\partial_z, z^i\partial_z,\ 1 \leq i \leq k-3 \rangle,$$
$$g(V) = \langle z^{k-3}\partial_y, y\partial_z, z^i\partial_z,\ 2 \leq i \leq k-3,\ k \geq 5 \rangle.$$

It is easy to see, the Cartan matrix is not defined when $k = 4$. For the D_5 singularity,

$$g(V) = \langle y\partial_z, z^2\partial_y, z^2\partial_z \rangle.$$

We set $x_1 = y\partial_z,\ x_2 = z^2\partial_y$ and $x_3 = z^2\partial_z$. we have following multiplication table: $[x_1, x_2] = -x_3$. The type of D_5 singularity $= \dim g(V)/[g(V), g(V)] = 2$. The nilpotency of the A_5 singularity $= \min\{p \in N \cup \{0\} \mid g(V)^{p+1} = 0\} = 1$. It is easy to see from [Benson and Yau 1987] that the torus T of $g(V)$ is spanned by

$$\begin{aligned} t_1 : g(V) &\to g(V), & t_2 : g(V) &\to g(V), \\ x_1 &\mapsto x_1, & x_1 &\mapsto 0, \\ x_2 &\mapsto 0, & x_2 &\mapsto x_2, \\ x_3 &\mapsto x_3, & x_3 &\mapsto x_3. \end{aligned}$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2$. Since $\dim T = 2 =$ the type of D_5, T is the maximal torus of $g(V)$. Let $\beta_i : T \to \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2$;

$$\begin{aligned} g(V) &= g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_1+\beta_2} \\ &= \mathbb{C}x_1 \oplus \mathbb{C}x_2 \oplus \mathbb{C}x_3, \end{aligned}$$

(x_1, x_2) is a T-minimal system of generators. The generalized Cartan matrix is

$$C(D_5) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

For the D_6 singularity, we have the following multiplication table: $[x_1, x_2] = -x_4$. The type of A_6 singularity $= \dim g(V)/[g(V), g(V)] = 2$. The nilpotency of A_6 singularity $= \min\{p \in N \cup \{0\} \mid g(V)^{p+1} = 0\} = 1$. It is easy to see from [Benson and Yau 1987] that the torus T of $g(V)$ is spanned by

$$\begin{aligned} t_1 : g(V) &\to g(V), & t_2 : g(V) &\to g(V), & t_3 : g(V) &\to g(V), \\ x_1 &\mapsto x_1, & x_1 &\mapsto 0, & x_1 &\mapsto 0, \\ x_2 &\mapsto 0, & x_2 &\mapsto x_2, & x_2 &\mapsto 0, \\ x_3 &\mapsto 0, & x_3 &\mapsto 0, & x_3 &\mapsto x_3, \\ x_4 &\mapsto x_4, & x_4 &\mapsto x_4, & x_4 &\mapsto 0. \end{aligned}$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2 + \mathbb{C}t_3$. Since dim $T = 3 =$ the type of D_6, T is the maximal torus of $g(V)$. Let $\beta_i : T \to \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2, 3$;

$$\begin{aligned} g(V) &= g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_3} \oplus g^{\beta_1+\beta_3} \\ &= \mathbb{C}x_1 \oplus \mathbb{C}x_2 \oplus \mathbb{C}x_3 \oplus \mathbb{C}x_4, \end{aligned}$$

(x_1, x_2, x_3) is a T-minimal system of generators. The generalized Cartan matrix is

$$C(D_6) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

For the D_7 singularity, we have the following multiplication table: $[x_1, x_2] = -x_5$, $[x_3, x_4] = x_5$. The type of D_7 singularity $= \dim g(V)/[g(V), g(V)] = 4$. The nilpotency of the D_7 singularity $= \min\{p \in N \cup \{0\} \mid g(V)^{p+1} = 0\} = 1$. It is easy to see from [Benson and Yau 1987] that the torus T of $g(V)$ is spanned by

$$\begin{array}{ccc} t_1 : g(V) \to g(V), & t_2 : g(V) \to g(V), & t_3 : g(V) \to g(V), \\ x_1 \mapsto x_1, & x_1 \mapsto 0, & x_1 \mapsto 0, \\ x_2 \mapsto 0, & x_2 \mapsto x_2, & x_2 \mapsto 0, \\ x_3 \mapsto 0, & x_3 \mapsto 0, & x_3 \mapsto x_3, \\ x_4 \mapsto x_4, & x_4 \mapsto x_4, & x_4 \mapsto -x_4, \\ x_5 \mapsto x_5, & x_5 \mapsto x_5, & x_5 \mapsto 0. \end{array}$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2 + \mathbb{C}t_3$ is a unique maximal torus on $g(V)$. Let $\beta_i : T \to \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2, 3$;

$$\begin{aligned} g(V) &= g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_3} \oplus g^{\beta_1+\beta_2-\beta_3} \oplus g^{\beta_1+\beta_2} \\ &= \mathbb{C}x_1 \oplus \mathbb{C}x_2 \oplus \mathbb{C}x_3 \oplus \mathbb{C}x_4 \oplus \mathbb{C}x_5, \end{aligned}$$

(x_1, x_2, x_3, x_4) is a T-minimal system of generators. The generalized Cartan matrix is

$$C(D_7) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

For the D_8 singularity, we have the following multiplication table: $[x_1, x_2] = -x_6$, $[x_3, x_4] = x_5$, $[x_3, x_5] = 2x_6$. The type of D_8 singularity $= \dim g(V)/[g(V), g(V)] = 4$. The nilpotency of the D_8 singularity $= \min\{p \in N \cup \{0\} \mid g(V)^{p+1} = 0\} = 2$. It is easy to see from [Benson and Yau 1987] that the torus T of $g(V)$ is spanned by

$$\begin{array}{ccc} t_1 : g(V) \to g(V), & t_2 : g(V) \to g(V), & t_3 : g(V) \to g(V), \\ x_1 \mapsto x_1, & x_1 \mapsto 0, & x_1 \mapsto 0, \\ x_2 \mapsto 0, & x_2 \mapsto x_2, & x_2 \mapsto 0, \\ x_3 \mapsto 0, & x_3 \mapsto 0, & x_3 \mapsto x_3, \\ x_4 \mapsto x_4, & x_4 \mapsto x_4, & x_4 \mapsto -2x_4, \\ x_5 \mapsto x_5, & x_5 \mapsto x_5, & x_5 \mapsto -x_5, \\ x_6 \mapsto x_6, & x_6 \mapsto x_6, & x_6 \mapsto 0. \end{array}$$

Thus $T=\mathbb{C}t_1+\mathbb{C}t_2+\mathbb{C}t_3$ is a unique maximal torus on $g(V)$. Let $\beta_i : T\to\mathbb{C}$ be a linear map with $\beta_i(t_j)=\delta_{ij}$ for $i,\ j=1,2,3$;

$$\begin{aligned}g(V)&=g^{\beta_1}\oplus g^{\beta_2}\oplus g^{\beta_3}\oplus g^{\beta_1+\beta_2-2\beta_3}\oplus g^{\beta_1+\beta_2-\beta_3}\oplus g^{\beta_1+\beta_2}\\&=\mathbb{C}x_1\oplus\mathbb{C}x_2\oplus\mathbb{C}x_3\oplus\mathbb{C}x_4\oplus\mathbb{C}x_5\oplus Cx_6,\end{aligned}$$

(x_1,x_2,x_3,x_4) is a T-minimal system of generators. The generalized Cartan matrix is

$$C(D_8)=\begin{pmatrix}2&-1&0&0\\-1&2&0&0\\0&0&2&-2\\0&0&-1&2\end{pmatrix}.$$

For the D_k singularity $k\geq 9$,

$$g(V)=\langle y\partial_z, z^{k-3}\partial_y, z^2\partial_z, z^3\partial_z, z^4\partial_z,\dots,z^{k-3}\partial_z\rangle.$$

By setting $x_1=z^2\partial_z,\ x_2=z^3\partial_z,\ \dots,\ x_{k-2}=z^{k-3}\partial_z$, we have the following multiplication table:

$$\begin{aligned}&[x_1,x_2]=-x_{k-2},\\&[x_3,x_4]=x_5,\ \ [x_3,x_5]=2x_6,\ \ [x_3,x_6]=3x_7,\ \ \dots,\ \ [x_3,x_{k-3}]=(k-6)x_{k-2},\\&[x_4,x_5]=x_7,\ \ [x_4,x_6]=2x_8,\ \ [x_4,x_7]=3x_9,\ \ \dots,\ \ [x_4,x_{k-4}]=(k-8)x_{k-2},\\&[x_5,x_6]=x_9,\ \ [x_5,x_7]=2x_{10},\ [x_5,x_8]=3x_{11},\ \ \dots,\ \ [x_5,x_{k-5}]=(k-10)x_{k-2}.\end{aligned}$$

The type of D_k singularity (for $k\geq 9$) $=\dim g(V)/[g(V),g(V)]=4$. The nilpotency of the D_k singularity (for $k\geq 9$) $=\min\{p\in N\cup\{0\}\mid g(V)^{p+1}=0\}=k-6$.

Case 1. When k is odd:

It is easy to see from [Benson and Yau 1987] that the torus T of $g(V)$ is spanned by

$$\begin{array}{ll} t_1 : g(V)\to g(V), & t_2 : g(V)\to g(V),\\ \quad x_1\mapsto (k-4)x_1, & \quad x_1\mapsto 0\\ \quad x_2\mapsto 0, & \quad x_2\mapsto (k-4)x_2,\\ \quad x_3\mapsto x_3, & \quad x_3\mapsto x_3,\\ \quad x_4\mapsto 2x_4, & \quad x_4\mapsto 2x_4,\\ \quad x_5\mapsto 3x_5, & \quad x_5\mapsto 3x_5,\\ \quad\vdots & \quad\vdots\\ x_{k-2}\mapsto (k-4)x_{k-2}, & x_{k-2}\mapsto (k-4)x_{k-2}.\end{array}$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2$ is a unique maximal torus on $g(V)$. Let $\beta_i : T \to \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2$;

$$\begin{aligned} g(V) &= g^{(k-4)\beta_1} \oplus g^{(k-4)\beta_2} \oplus g^{\beta_1+\beta_2} \oplus g^{2(\beta_1+\beta_2)} \oplus g^{3(\beta_1+\beta_2)} \oplus \cdots \oplus g^{(k-4)(\beta_1+\beta_2)} \\ &= \mathbb{C}x_1 \oplus \mathbb{C}x_2 \oplus \mathbb{C}x_3 \oplus \mathbb{C}x_4 \oplus \mathbb{C}x_5 \oplus \cdots \oplus \mathbb{C}x_{k-2}, \end{aligned}$$

(x_1, x_2, x_3, x_4) is a T-minimal system of generators. Since $(\operatorname{ad} x_3)^{k-5} x_4 = 0$, and $(\operatorname{ad} x_3)^{k-6} x_4 \neq 0$, so $c_{34} = -(k-6)$. In order to compute the c_{43}, we have $k = 2l + 7 \geq 9$;

$$(\operatorname{ad} x_4)^{l+1} x_3 = -(2\cdot 2 - 1)(2\cdot 3 - 1)\cdots(2l-1)x_{2l+5}, \quad (\operatorname{ad} x_4)^{l+2} x_3 = 0,$$
$$\Rightarrow c_{43} = -(l+1) = -\frac{k-5}{2}.$$

Therefore the generalized Cartan matrix is

$$C(D_k) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -(k-6) \\ 0 & 0 & -\frac{k-5}{2} & 2 \end{pmatrix}.$$

Case 2. When k is even:

It is easy to see from [Benson and Yau 1987] that the torus T of $g(V)$ is spanned by

$$\begin{aligned} t_1 : g(V) &\to g(V), & t_2 : g(V) &\to g(V), \\ x_1 &\mapsto (k-4)^2 x_1, & x_1 &\mapsto 0, \\ x_2 &\mapsto 0, & x_2 &\mapsto (k-4)^2 x_2, \\ x_3 &\mapsto (k-4)x_3, & x_3 &\mapsto (k-4)x_3, \\ x_4 &\mapsto 2(k-4)x_4, & x_4 &\mapsto 2(k-4)x_4, \\ x_5 &\mapsto 3(k-4)x_5, & x_5 &\mapsto 3(k-4)x_5, \\ &\vdots & &\vdots \\ x_{k-2} &\mapsto (k-4)^2 x_{k-2}, & x_{k-2} &\mapsto (k-4)^2 x_{k-2}. \end{aligned}$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2$ is a unique maximal torus on $g(V)$. Let $\beta_i : T \to \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2$;

$$\begin{aligned} g(V) &= g^{(k-4)^2\beta_1} \oplus g^{(k-4)^2\beta_2} \oplus g^{(k-4)(\beta_1+\beta_2)} \oplus g^{2(k-4)(\beta_1+\beta_2)} \oplus g^{3(k-4)(\beta_1+\beta_2)} \oplus \cdots \\ &\qquad \cdots \oplus g^{(k-4)^2(\beta_1+\beta_2)} \\ &= \mathbb{C}x_1 \oplus \mathbb{C}x_2 \oplus \mathbb{C}x_3 \oplus \mathbb{C}x_4 \oplus \mathbb{C}x_5 \oplus \cdots \oplus \mathbb{C}x_{k-2}, \end{aligned}$$

(x_1, x_2, x_3, x_4) is a T-minimal system of generators. It is noted that $(\operatorname{ad} x_3)^{k-5} x_4 = 0$, but $(\operatorname{ad} x_3)^{k-6} x_4 \neq 0$. Therefore $c_{34} = -(k-6)$. In order to compute the c_{43}, we have $k = 2l + 8 \geq 9$;

$$(\operatorname{ad} x_4)^{l+1} x_3 = -(2 \cdot 2 - 1)(2 \cdot 3 - 1) \cdots (2l - 1) x_{2l+5}, \quad (\operatorname{ad} x_4)^{l+2} x_3 = 0,$$

$$\Rightarrow c_{43} = -(l+1) = -\frac{k-6}{2}.$$

Therefore the generalized Cartan matrix is

$$C(D_k) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -(k-6) \\ 0 & 0 & -\frac{k-6}{2} & 2 \end{pmatrix}. \qquad \square$$

Proposition 3.3. *Let* $V = \{x, y, z \in \mathbb{C}^3 \mid x^2 + y^3 + z^4 = 0\}$ *be the* E_6 *singularity. Then*

$$C(E_6) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 2 & -2 & -2 \\ 0 & -2 & 2 & 0 \\ 0 & -2 & 0 & 2 \end{pmatrix}.$$

Proof. It is easy to see that $A^*(V) = \langle 1, y, z, yz, z^2 \rangle$ with the multiplication rules $y^2 = 0 = z^3 = yz^2$. We have the following basis of the new Lie algebra of the E_6 singularity:

$$L^*(V) = \langle y\partial_y, yz\partial_y, z^2\partial_y, y\partial_z, yz\partial_z, z\partial_z, z^2\partial_z \rangle.$$

The nilradical of the new Lie algebra of the E_6 singularity is spanned by

$$g(V) = \langle yz\partial_y, z^2\partial_y, y\partial_z, yz\partial_z, z^2\partial_z \rangle.$$

We set $x_1 = yz\partial_y,\ x_2 = z^2\partial_y,\ \ldots,\ x_5 = z^2\partial_z$. The multiplication table of the nilradical of the Lie algebra is given as

$$[x_1, x_3] = x_4, \quad [x_2, x_3] = -2x_1 + x_5, \quad [x_3, x_5] = 2x_4.$$

The type of E_6 singularity $= \dim g(V)/[g(V), g(V)] = 3$. The nilpotency of the E_6 singularity $= \min\{p \in N \cup \{0\} \mid g(V)^{p+1} = 0\} = 2$. It is easy to see from [Benson and Yau 1987] that the torus T of $g(V)$ is spanned by

$$\begin{array}{lll} t_1 : g(V) \to g(V), & t_2 : g(V) \to g(V), & t_3 : g(V) \to g(V), \\ x_1 \mapsto x_1, & x_1 \mapsto x_5, & x_1 \mapsto 0, \\ x_2 \mapsto 2x_2, & x_2 \mapsto -4x_2, & x_2 \mapsto -x_2, \\ x_3 \mapsto -x_3, & x_3 \mapsto 3x_3, & x_3 \mapsto x_3, \\ x_4 \mapsto 0, & x_4 \mapsto 0, & x_4 \mapsto x_4, \\ x_5 \mapsto x_5, & x_5 \mapsto 4x_1, & x_5 \mapsto 0. \end{array}$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2 + \mathbb{C}t_3$. Since $\dim T = 3 =$ the type of E_6, T is the maximal torus of $g(V)$. Let $\beta_i : T \to \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2, 3$;

$$\begin{aligned} g(V) &= g^{\beta_1+2\beta_2} \oplus g^{\beta_1-2\beta_2} \oplus g^{2\beta_1-4\beta_2-\beta_3} \oplus g^{-\beta_1+3\beta_2+\beta_3} \oplus g^{\beta_3} \\ &= \mathbb{C}\Big(\frac{x_1}{2} + x_5\Big) \oplus \mathbb{C}\Big(-\frac{x_1}{2} + x_5\Big) \oplus \mathbb{C}x_2 \oplus \mathbb{C}x_3 \oplus \mathbb{C}x_4, \end{aligned}$$

$\big(x_2, x_3, \frac{x_1}{2} + x_5, -\frac{x_1}{2} + x_5\big)$ is a T-minimal system of generators. The generalized Cartan matrix is

$$C(E_6) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 2 & -2 & -2 \\ 0 & -2 & 2 & 0 \\ 0 & -2 & 0 & 2 \end{pmatrix}. \qquad \square$$

Proposition 3.4. *Let* $V = \{x, y, z \in \mathbb{C}^3 \mid x^2 + y^3 + yz^3 = 0\}$ *be the* E_7 *singularity. Then*

$$C(E_7) = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -2 & -2 & -3 & 2 \end{pmatrix}.$$

Proof. It is easy to see that $A^*(V) = \langle 1, y, z, yz, y^2, z^2 \rangle$ with the multiplication rules

$$y^2 z = 0 = y^4 = z^3, \quad y^3 + 3z^2 = 0, \quad 6z^2 - 3y^3 = 0.$$

We have the following basis of the new Lie algebra of the E_7 singularity:

$$L^*(V) = \langle z\partial_y, yz\partial_y, y^2\partial_y, z^2\partial_y, yz\partial_z, y^2\partial_z, z^2\partial_z, 3y\partial_y + 2z\partial_z \rangle.$$

The nilradical of the new Lie algebra of E_7 singularity is spanned by

$$g(V) = \langle yz\partial_y, y^2\partial_y, z^2\partial_y, yz\partial_z, y^2\partial_z, z^2\partial_z, z\partial_y \rangle.$$

We set $x_1 = yz\partial_y$, $x_2 = y^2\partial_y$, $\ldots$, $x_7 = z\partial_y$. The multiplication table of the nilradical of the new Lie algebra is given as

$$\begin{aligned} &[x_1, x_7] = -x_3, && [x_1, x_5] = 3x_3, && [x_2, x_5] = -6x_6, \\ &[x_2, x_7] = -2x_1, && [x_4, x_7] = x_1 - x_6, && [x_4, x_5] = 3x_6, \\ &[x_5, x_7] = x_2 - 2x_4, && [x_6, x_7] = x_3. \end{aligned}$$

The type of E_7 singularity $= \dim g(V)/[g(V), g(V)] = 3$. The nilpotency of the E_7 singularity $= \min\{p \in N \cup \{0\} \mid g(V)^{p+1} = 0\} = 3$ It is easy to see from [Benson and Yau 1987] that the torus T of $g(V)$ is spanned by

$$\begin{aligned} t_1 : g(V) &\to g(V), & t_2 : g(V) &\to g(V),\\ x_1 &\mapsto 3x_1, & x_1 &\mapsto 0,\\ x_2 &\mapsto 2x_2, & x_2 &\mapsto -10x_2 - 3x_4,\\ x_3 &\mapsto 4x_3, & x_3 &\mapsto -8x_3,\\ x_4 &\mapsto 2x_4, & x_4 &\mapsto -12x_2 - 10x_4,\\ x_5 &\mapsto x_5, & x_5 &\mapsto -8x_5 + 3x_7,\\ x_6 &\mapsto 3x_6, & x_6 &\mapsto 3x_1 - 12x_6,\\ x_7 &\mapsto x_7, & x_7 &\mapsto 4x_7. \end{aligned}$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2$ is a unique maximal torus of $g(V)$. Let $\beta_i : T \to \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2$;

$$\begin{aligned} g(V) &= g^{3\beta_1} \oplus g^{2\beta_1-4\beta_2} \oplus g^{2\beta_1-16\beta_2} \oplus g^{4\beta_1-8\beta_2} \oplus g^{\beta_1-8\beta_2} \oplus g^{3\beta_1-12\beta_2} \oplus g^{\beta_1+4\beta_2}\\ &= \mathbb{C}(4x_1 + x_6) \oplus \mathbb{C}\Big(-\frac{x_2}{2} + x_4\Big) \oplus \mathbb{C}\Big(\frac{x_2}{2} + x_4\Big) \oplus \mathbb{C}x_3 \oplus \mathbb{C}x_5 \oplus \mathbb{C}x_6\\ &\qquad \oplus \mathbb{C}\Big(\frac{x_5}{4} + x_7\Big), \end{aligned}$$

$\big(x_2/2 + x_4, -x_2/2 + x_4, x_5, x_5/4 + x_7\big)$ is a T-minimal system of generators. The generalized Cartan matrix is

$$C(E_7) = \begin{pmatrix} 2 & 0 & 0 & -1\\ 0 & 2 & -1 & -1\\ 0 & -1 & 2 & -1\\ -2 & -2 & -3 & 2 \end{pmatrix}. \qquad \square$$

Proposition 3.5. *Let* $V = \{x, y, z \in \mathbb{C}^3 \mid x^2 + y^3 + z^5 = 0\}$ *be the* E_8 *singularity. Then*

$$C(E_8) = \begin{pmatrix} 2 & -1 & 0 & 0\\ -2 & 2 & 0 & -1\\ 0 & 0 & 2 & -1\\ 0 & -1 & -1 & 2 \end{pmatrix}.$$

Proof. It is noted that $A^*(V) = \langle 1, y, z, yz, yz^2, z^2, z^3\rangle$ with multiplication rules

$$y^2 = 0 = z^4 = yz^3.$$

We have the following basis of the new Lie algebra of the E_8 singularity,

$$L^*(V) = \langle y\partial_y, yz\partial_y, yz^2\partial_y, z^3\partial_y, y\partial_z, yz\partial_z, yz^2\partial_z, z\partial_z, z^2\partial_z, z^3\partial_z\rangle.$$

The nilradical of the new Lie algebra of the E_8 singularity is spanned by

$$g(V) = \langle yz\partial_y, yz^2\partial_y, z^3\partial_y, y\partial_z, yz\partial_z, yz^2\partial_z, z^2\partial_z, z^3\partial_z\rangle.$$

We set $x_1 = yz\partial_y$, $x_2 = yz^2\partial_y, \dots, x_8 = z^3\partial_z$. The multiplication table of nilradical of new Lie algebra is given as

$$\begin{aligned} &[x_1, x_4] = x_5, && [x_1, x_5] = x_6, && [x_1, x_7] = -x_2, \\ &[x_2, x_4] = x_6, && [x_3, x_4] = -3x_2 + x_8, && [x_4, x_7] = 2x_5, \\ &[x_4, x_8] = 3x_6, && [x_5, x_7] = x_6. \end{aligned}$$

The type of E_8 singularity $= \dim g(V)/[g(V), g(V)] = 4$. The nilpotency of the E_8 singularity $= \min\{p \in N \cup \{0\} \mid g(V)^{p+1} = 0\} = 2$ It is easy to see from [Benson and Yau 1987] that the torus T of $g(V)$ is spanned by

$$\begin{array}{lll} t_1 : g(V) \to g(V), & t_2 : g(V) \to g(V), & t_3 : g(V) \to g(V), \\ \quad x_1 \mapsto -2x_7, & \quad x_1 \mapsto x_1, & \quad x_1 \mapsto 0, \\ \quad x_2 \mapsto -3x_2, & \quad x_2 \mapsto 2x_2, & \quad x_2 \mapsto 0, \\ \quad x_3 \mapsto 0, & \quad x_3 \mapsto 0, & \quad x_3 \mapsto x_3, \\ \quad x_4 \mapsto -3x_4, & \quad x_4 \mapsto 2x_4, & \quad x_4 \mapsto -x_4, \\ \quad x_5 \mapsto -5x_5, & \quad x_5 \mapsto 3x_5, & \quad x_5 \mapsto -x_5, \\ \quad x_6 \mapsto -6x_6, & \quad x_6 \mapsto 4x_6, & \quad x_6 \mapsto -x_6, \\ \quad x_7 \mapsto x_1 - 3x_7, & \quad x_7 \mapsto x_7, & \quad x_7 \mapsto 0, \\ \quad x_8 \mapsto -3x_8, & \quad x_8 \mapsto 2x_8, & \quad x_8 \mapsto 0. \end{array}$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2 + \mathbb{C}t_3$ is a maximal torus of $g(V)$. Let $\beta_i : T \to \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2, 3$;

$$\begin{aligned} g(V) &= g^{-3\beta_1+2\beta_2} \oplus g^{\beta_3} \oplus g^{-3\beta_1+2\beta_2-\beta_3} \oplus g^{-5\beta_1+3\beta_2-\beta_3} \oplus g^{-6\beta_1+4\beta_2-\beta_3} \\ &\qquad \oplus g^{-\beta_1+\beta_2} \oplus g^{-2\beta_1+\beta_2} \oplus g^{-3\beta_1+2\beta_2} \\ &= \mathbb{C}x_2 \oplus \mathbb{C}x_3 \oplus \mathbb{C}x_4 \oplus \mathbb{C}x_5 \oplus \mathbb{C}x_6 \oplus \mathbb{C}(2x_1 + x_7) \oplus \mathbb{C}(x_1 + x_7) \oplus \mathbb{C}x_8, \end{aligned}$$

$(x_3, x_4, 2x_1 + x_7, x_1 + x_7)$ is a T-minimal system of generators. The generalized Cartan matrix is

$$C(E_8) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 2 & 0 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}. \qquad \square$$

From Propositions 3.1–3.5, we have the following conclusion.

Theorem 1.1. *The generalized Cartan matrix characterizes the simple* (*ADE*) *hypersurface singularities except* A_6 *and* D_5 *singularities. I.e.*, *if* X *and* Y *are two simple hypersurface singularities*, *then* $C(X) = C(Y)$ *if and only if* X *and* Y *are analytically isomorphic.*

Remark 3.6. In Theorem 1.1, in order for the generalized Cartan matrix to make sense, the simple singularities there should be $A_k, k \geq 5,\ D_k, k \geq 5,\ E_6, E_7, E_8$.

However, Theorem 1.1 is not true for general singularities. In the following Propositions 3.7, 3.8 and 3.9, we shall see that the generalized Cartan matrix is constant for simple elliptic singularities. Therefore it does not characterize the simple elliptic singularities.

Proposition 3.7. *Let* $V_t = \{(x, y, z) \in \mathbb{C}^3 \mid f_t(x, y, z) = x^3 + y^3 + z^3 + txyz = 0\}$ *with* $t^3 + 27 \neq 0$, *the* $\widetilde{E}_6$ *singularity. Then*

$$C(\widetilde{E}_6) = \begin{pmatrix} 2&0&0&0&0&0&0&0&0\\ 0&2&0&0&0&0&0&0&0\\ 0&0&2&0&0&0&0&0&0\\ 0&0&0&2&0&0&0&0&0\\ 0&0&0&0&2&0&0&0&0\\ 0&0&0&0&0&2&0&0&0\\ 0&0&0&0&0&0&2&0&0\\ 0&0&0&0&0&0&0&2&0\\ 0&0&0&0&0&0&0&0&2 \end{pmatrix}.$$

Proof. It is easy to see that new moduli algebra $A^*(V_t) = \langle 1, x, y, z, xy, xz, yz\rangle$ with multiplication rules, $x^2 = -\frac{t}{3}yz,\ y^2 = -\frac{t}{3}xz,\ z^2 = -\frac{t}{3}xy$. A basis for the new Lie algebra $L^*(V_t)$ (for $t^3 \neq 0, -27, 216$) is

$$e_0 = x\partial_x + y\partial_y + z\partial_z,\quad e_1 = xy\partial_x,\quad e_2 = xy\partial_y,\quad e_3 = xy\partial_z,\quad e_4 = xz\partial_x,$$
$$e_5 = xz\partial_y,\quad e_6 = xz\partial_z,\quad e_7 = yz\partial_x,\quad e_8 = yz\partial_y,\quad e_9 = yz\partial_z.$$

For $t = 0$, $\{e_0\}$ is replaced by $\{e_{10} = x\partial_x, e_{11} = y\partial_y, e_{12} = z\partial_z\}$. For $t = -3$, $\{e_0\}$ is replaced by $\{e_0, z\partial_x + x\partial_y + y\partial_z, y\partial_x + z\partial_y + x\partial_z\}$. For $t = 6$, $\{e_0\}$ is replaced by $\{e_0, -z\partial_x - x\partial_y - y\partial_z, -y\partial_x - z\partial_y - x\partial_z\}$.

The type and nilpotency of $\widetilde{E}_6$ singularity are zero.

The nilradical $g(V)$ of $L^*(V_t)$ is spanned by $\langle e_1, \dots, e_9\rangle$ (for all t such that $t^3 \neq 0, -27, 216$).

It is easy to see that multiplication table for nilradical of the new Lie algebra is zero.

It is follows from [Seeley and Yau 1991], in case of generic t the derivation of the nilradical of $L^*(V_t)$ has a basis of the form

$$f_1 : a_{11} = 1,\quad f_2 : a_{22} = 1,\quad f_3 : a_{33} = 1,\quad f_4 : a_{44} = 1,\quad f_5 : a_{55} = 1,$$
$$f_6 : a_{66} = 1,\quad f_7 : a_{77} = 1,\quad f_8 : a_{88} = 1,\quad f_9 : a_{99} = 1.$$

We have following decomposition of the nilradical of the new Lie algebra with respect to the maximal torus [Seeley and Yau 1991]:

$$g^{\beta_1} = \langle e_1\rangle,\quad g^{\beta_2} = \langle e_2\rangle,\quad g^{\beta_3} = \langle e_3\rangle,\quad g^{\beta_4} = \langle e_4\rangle,\quad g^{\beta_5} = \langle e_5\rangle,$$
$$g^{\beta_6} = \langle e_6\rangle,\quad g^{\beta_7} = \langle e_7\rangle,\quad g^{\beta_8} = \langle e_8\rangle,\quad g^{\beta_9} = \langle e_9\rangle.$$

Therefore we have following Cartan matrix

$$C(\widetilde{E}_6) = \begin{pmatrix} 2&0&0&0&0&0&0&0&0\\ 0&2&0&0&0&0&0&0&0\\ 0&0&2&0&0&0&0&0&0\\ 0&0&0&2&0&0&0&0&0\\ 0&0&0&0&2&0&0&0&0\\ 0&0&0&0&0&2&0&0&0\\ 0&0&0&0&0&0&2&0&0\\ 0&0&0&0&0&0&0&2&0\\ 0&0&0&0&0&0&0&0&2 \end{pmatrix}. \qquad \square$$

Proposition 3.8. *Let $V_t = \{(x, y, z) \in \mathbb{C}^3 \mid f_t(x, y, z) = x^4 + y^4 + tx^2y^2 + z^2 = 0\}$ with $t^2 \neq 4$, be the $\widetilde{E}_7$ singularity. Then*

$$C(\widetilde{E}_7) = \begin{cases} \begin{pmatrix} 2&-1&-1&-1&-1&-1\\ -1&2&-1&-1&-1&-1\\ -1&-1&2&-1&-1&-1\\ -1&-1&-1&2&-1&-1\\ -1&-1&-1&-1&2&-1\\ -1&-1&-1&-1&-1&2 \end{pmatrix}, & t \neq 0,\\ \begin{pmatrix} 2&-1&-1&0&-1&-1\\ -1&2&-1&-1&-1&-1\\ -1&-1&2&-1&-1&-1\\ 0&-1&-1&2&-1&-1\\ -1&-1&-1&-1&2&0\\ -1&-1&-1&-1&0&2 \end{pmatrix}, & t = 0. \end{cases}$$

Proof. In [Seeley and Yau 1990], the authors showed that (μ,τ)-constant family for $\widetilde{E}_7$ is

$$V_t=\{(x,y,z)\in\mathbb{C}^3 \mid f_t(x,y,z)=x^4+y^4+tx^2y^2+z^2=0\},$$

with $t^2\neq 4$. The moduli algebra

$$A^*(V_t)=A(V_t)/(x^2y^2)=\langle 1,x,y,x^2,xy,y^2,x^2y,xy^2\rangle,$$

with the multiplication rules

$$x^3=-\frac{t}{2}xy^2,\quad y^3=-\frac{t}{2}x^2y,\quad x^3y=0=xy^3=x^2y^2.$$

A basis of the new Lie algebra $L^*(V_t)$ (for $t\neq 0,-6,6$) is:

deg0: $e_0=x\partial_x+y\partial_y$,

deg1: $e_1=x^2\partial_x$, $e_2=y^2\partial_y$, $e_3=y^2\partial_x$, $e_4=x^2\partial_y$, $e_5=xy\partial_x$, $e_6=xy\partial_y$,

deg2: $e_7=x^2y\partial_x$, $e_8=xy^2\partial_y$, $e_9=xy^2\partial_x$, $e_{10}=x^2y\partial_y$.

For $t=0$, $\{e_0\}$ is replaced by $\{x\partial_x, y\partial_y\}$. For $t=6$, $\{e_0\}$ is replaced by $\{e_0, y\partial_x+x\partial_y\}$. For $t=-6$, $\{e_0\}$ is replaced by $\{e_0, y\partial_x-x\partial_y\}$.

The type of $\widetilde{E}_7$ singularity $=\dim g(V)/[g(V),g(V)]=4$.

The nilpotency of the $\widetilde{E}_7$ singularity $=\min\{p\in N\cup\{0\}\mid g(V)^{p+1}=0\}=1$

The nilradical $g(V)$ of $L^*(V_t)$ is of dimension 10 and is spanned by $\langle e_1,\dots,e_{10}\rangle$ (for all t such that $t^2\neq 4$). We have the following multiplication table:

$[e_0,e_i]=\deg(e_i)\,e_i$,	$[e_2,e_3]=-te_7$,	$[e_3,e_4]=2e_8-2e_7$,
$[e_1,e_2]=0$,	$[e_2,e_4]=-2e_{10}$,	$[e_3,e_5]=-te_7/2$,
$[e_1,e_3]=-2e_9$,	$[e_2,e_5]=e_9$,	$[e_3,e_6]=-2e_9-te_{10}/2$,
$[e_1,e_4]=-te_8$,	$[e_2,e_6]=-e_8$,	$[e_4,e_5]=-te_9/2-2e_{10}$,
$[e_1,e_5]=-e_7$,	$[e_5,e_6]=e_8-e_7$.	$[e_4,e_6]=-te_8/2$,
$[e_1,e_6]=e_{10}$,		

Other Lie brackets are 0. It follows from [Seeley and Yau 1991], that we can consider derivations which preserve degree to find a maximal torus of derivations on nilradical $g(V)$. Let δ be a such derivation;

$$\delta e_1=\sum_{j=1}^{6}a_{6j}e_j,\quad \delta e_7=\sum_{j=7}^{10}a_{10j}e_j.$$

<u>Case 1</u>. In case of generic t the derivation of the nilradical of $L^*(V_t)$ is spanned by

$$f_1:\frac{a_{11}}{2}=\frac{a_{22}}{2}=\frac{a_{33}}{2}=\frac{a_{44}}{2}=\frac{a_{55}}{2}=\frac{a_{66}}{2}=a_{77}=a_{88}=a_{99}=a_{10,10}=1,$$

and otherwise $a_{ij}=0$. In case of $t=0$ the derivation of the nilradical of $L^*(V_t)$ is spanned by

$$f_1\colon a_{11}=-a_{33}=2a_{44}=a_{66}=a_{77}=a_{88}=2a_{10,10}=1,$$
$$f_2\colon a_{22}=2a_{33}=-a_{44}=a_{55}=a_{77}=a_{88}=2a_{99}=1,$$

and other $a_{ij}=0$. Thus for generic t we have a torus of dimension 1 spanned by $\delta=\mathrm{ad}_{e_0}$:

$$\delta e_i=\begin{cases} e_i, & i=1,2,3,4,5,6,\\ 2e_i, & i=7,8,9,10.\end{cases}$$

For generic t, let $\beta(\delta)=1$. Then we have the following decomposition of the nilpotent Lie algebra with respect to the maximal torus:

$$g^{\beta}=\langle e_1,e_2,e_3,e_4,e_5,e_6\rangle,$$
$$C_{ij}=\begin{cases} 2, & i=j,\\ -1, & i\neq j.\end{cases}$$

We have the following generalized Cartan matrix:

$$C(\widetilde{E}_7)=\begin{pmatrix} 2 & -1 & -1 & -1 & -1 & -1\\ -1 & 2 & -1 & -1 & -1 & -1\\ -1 & -1 & 2 & -1 & -1 & -1\\ -1 & -1 & -1 & 2 & -1 & -1\\ -1 & -1 & -1 & -1 & 2 & -1\\ -1 & -1 & -1 & -1 & -1 & 2\end{pmatrix}.$$

Case 2. For $t=0$ we have a torus of dimension 2, spanned by degree derivations $\delta_1=\mathrm{ad}_{(x\partial_x)}$ and $\delta_2=\mathrm{ad}_{(y\partial_y)}$:

$$\delta_1 e_i=\begin{cases} e_i, & i=1,5,6,8,9,\\ 2e_i, & i=4,7,10,\\ 0, & i=2,3,\end{cases}\qquad \text{and}\qquad \delta_2 e_i=\begin{cases} e_i, & i=5,6,7,10,\\ 2e_i, & i=2,3,8,9,\\ 0, & i=1,4.\end{cases}$$

For $t=0$, we have following decomposition of the nilpotent Lie algebra:

$$\beta_1(\delta_1)=1,\quad \beta_1(\delta_2)=0\quad\Rightarrow\quad g^{\beta_1}=\langle e_1\rangle,$$
$$\beta_2(\delta_1)=1,\quad \beta_2(\delta_2)=1\quad\Rightarrow\quad g^{\beta_2}=\langle e_5,e_6\rangle,$$
$$\beta_3(\delta_1)=2,\quad \beta_3(\delta_2)=0\quad\Rightarrow\quad g^{\beta_3}=\langle e_4\rangle,$$
$$\beta_4(\delta_1)=0,\quad \beta_4(\delta_2)=2\quad\Rightarrow\quad g^{\beta_4}=\langle e_2,e_3\rangle.$$

It is easy to see that

$$f(1)=1,\quad f(2)=2,\quad f(3)=2,\quad f(4)=3,\quad f(5)=4,\quad f(6)=4,$$
$$(\operatorname{ad} e_1)^2 e_5=(\operatorname{ad} e_1)^2 e_6=0 \qquad \Rightarrow C_{12}=C_{13}=C_{21}=C_{31}=-1,$$
$$(\operatorname{ad} e_1)e_4=0 \qquad \Rightarrow C_{14}=C_{41}=0,$$
$$(\operatorname{ad} e_1)^2 e_2=0,\quad (\operatorname{ad} e_1)^2 e_3=[e_1,-2e_9]=0 \quad \Rightarrow C_{15}=C_{51}=C_{16}=C_{61}=-1,$$
$$(\operatorname{ad}(\gamma e_5+e_6))^2 e_5=[\gamma e_5+e_6,\gamma e_8-\gamma e_7]=0,\quad (\operatorname{ad} e_5)^2 e_6=[e_5,e_8-e_7]=0$$
$$\Rightarrow C_{23}=C_{32}=-1,$$
$$(\operatorname{ad} e_5)^2 e_4=(\operatorname{ad} e_6)^2 e_4=0 \qquad \Rightarrow C_{24}=C_{34}=C_{42}=C_{43}=-1,$$
$$(\operatorname{ad} e_5)^2 e_2=0,\quad (\operatorname{ad} e_6)^2 e_2=0,\quad (\operatorname{ad} e_5)^2 e_3=0,\quad (\operatorname{ad} e_6)^2 e_3=0$$
$$\Rightarrow C_{25}=C_{26}=C_{35}=C_{36}=C_{52}=C_{53}=C_{62}=C_{63}=-1,$$
$$(\operatorname{ad} e_4)^2 e_2=[e_4,2e_{10}]=0,\quad (\operatorname{ad} e_4)^2 e_3=[e_4,2e_7-2e_8]=0$$
$$\Rightarrow C_{45}=C_{46}=C_{54}=C_{64}=-1,$$
$$(\operatorname{ad}(\gamma e_2+e_3))e_2=(\operatorname{ad} e_2)e_3=0 \qquad \Rightarrow C_{56}=C_{65}=0.$$

We have the following generalized Cartan matrix:

$$C(\widetilde{E}_7)=\begin{pmatrix} 2 & -1 & -1 & 0 & -1 & -1\\ -1 & 2 & -1 & -1 & -1 & -1\\ -1 & -1 & 2 & -1 & -1 & -1\\ 0 & -1 & -1 & 2 & -1 & -1\\ -1 & -1 & -1 & -1 & 2 & 0\\ -1 & -1 & -1 & -1 & 0 & 2\end{pmatrix}. \qquad \square$$

Proposition 3.9. *Let* $V_t=\{(x,y,z)\in\mathbb{C}^3 \mid f_t=x^6+y^3+z^2+tx^4y=0\}$ *with* $4t^3+27\neq 0$, *be the* $\widetilde{E}_8$ *singularity. Then*

$$C(\widetilde{E}_8)=\begin{cases}\begin{pmatrix} 2 & -3 & -3\\ -3 & 2 & -3\\ -3 & -3 & 2\end{pmatrix}, & t\neq 0,\\[2ex] \begin{pmatrix} 2 & -1 & -1\\ -1 & 2 & -2\\ -1 & -2 & 2\end{pmatrix}, & t=0.\end{cases}$$

Proof. In [Seeley and Yau 1990], the authors had studied the (μ,τ)-constant family of $\widetilde{E}_8$, which is given by

$$V_t=\big\{(x,y,z)\in\mathbb{C}^3 \mid f_t=x^6+y^3+z^2+tx^4y=0\big\},$$

with $4t^3+27\neq 0$. The new moduli algebra

$$A^*(V_t)=\langle 1, x, x^2, y, x^3, xy, x^4, x^2y, x^3y\rangle,$$

with the multiplication rules $y^2=-\frac{t}{3}x^4,\ x^5=-\frac{2t}{3}x^3y,\ x^4y=0$.

By calculation, a basis of the new Lie algebra $L^*(V_t)$ (for $4t^3+27\neq 0$ and $t\neq 0$), is the following:

$$\begin{aligned}
&\deg 0: && e_0=x\partial_x+2y\partial_y,\\
&\deg 1: && e_1=x^2\partial_x+2xy\partial_y,\quad e_2=3y\partial_x-2tx^3\partial_y,\quad e_3=ty\partial_x-3xy\partial_y,\\
&\deg 2: && e_4=x^3\partial_x,\quad e_5=xy\partial_x,\quad e_6=x^2y\partial_y,\quad e_7=x^4\partial_y,\\
&\deg 3: && e_8=x^4\partial_x,\quad e_9=x^2y\partial_x,\quad e_{10}=x^3y,\,\partial_y,\\
&\deg 4: && e_{11}=x^3y\partial_x.
\end{aligned}$$

For $t=0$, $\{e_0\}$ is replaced by $\{x\partial_x, y\partial_y\}$. Let $g(V)$ be the nilradical of $L^*(V_t)$, which is of dimension 11, spanned by $\langle e_1, e_2, \ldots, e_{11}\rangle$ (for all t such that $4t^3+27\neq 0$). By calculation, the multiplication table of $g(V)$ is given as follows:

$$\begin{gathered}
[e_1,e_2]=0,\quad [e_1,e_3]=-3e_6+\frac{2t^2}{3}e_7,\quad [e_2,e_3]=-2t^2e_4+9e_5+6t^2e_6+9te_7,\\
[e_1,e_4]=e_8-2e_{10}\quad [e_2,e_4]=9e_9-4t^2e_{10},\quad [e_1,e_5]=e_9-\frac{4t^2}{9}e_{10},\\
[e_2,e_5]=-3te_8+6te_{10},\quad [e_1,e_6]=2e_{10},\quad [e_2,e_6]=-3e_9+\frac{8t^2}{3}e_{10},\\
[e_1,e_7]=-\frac{4t}{3}e_{10},\quad [e_2,e_7]=-3e_8+12e_{10},\quad [e_1,e_8]=-\frac{4t}{3}e_{11},\\
[e_2,e_8]=12e_{11},\quad [e_1,e_9]=2e_{11},\quad [e_2,e_9]=\frac{8t^2}{3}e_{11},\quad [e_1,e_{10}]=0,\\
[e_2,e_{10}]=-3e_{11},\quad [e_3,e_4]=3te_9+3e_{10},\quad [e_3,e_5]=-\frac{t^2}{3}e_8-3e_9+\frac{2t^2}{3}e_{10},\\
[e_4,e_5]=-2e_{11},\quad [e_3,e_6]=-te_9+\frac{4t^3}{9}e_{10},\quad [e_4,e_6]=0,\\
[e_3,e_7]=-te_8+2te_{10},\quad [e_4,e_7]=0,\quad [e_3,e_8]=4te_{11},\quad [e_5,e_6]=-e_{11},\\
[e_3,e_9]=\Big(\frac{4t^3}{9}-3\Big)e_{11},\quad [e_5,e_7]=\frac{2t}{3}e_{11},\quad [e_3,e_{10}]=-te_{11},\quad [e_6,e_7]=0.
\end{gathered}$$

Other Lie brackets $[e_i, e_j]$ $(i<j)$ are zero. It is follows from [Seeley and Yau 1991], that we can consider derivations which preserve degree to find a maximal torus of derivations on the nilradical $g(V)$. Let δ be a such derivation:

$$\delta e_1=\sum_{j=1}^{3}a_{3j}e_j,\quad \delta e_4=\sum_{j=4}^{7}a_{7j}e_j,\quad \delta e_8=\sum_{j=8}^{10}a_{10j}e_j,\quad \delta e_{11}=a_{11,11}e_{11}.$$

It is follows from the multiplication table that

$$[g(V), g(V)] = \left\langle -3e_6 + \frac{2t^2}{3}e_7, -2t^2e_4 + 9e_5 + 6t^2e_6 + 9te_7, e_8, e_9, e_{10}, e_{11} \right\rangle.$$

Therefore the type of $\widetilde{E}_8$ singularity $= \dim g(V)/[g(V), g(V)] = 5$. The nilpotency of $\widetilde{E}_8$ singularity $= \min\{p \in N \cup \{0\} \mid g(V)^{p+1} = 0\} = 4$

Case 1. It is follows from [Seeley and Yau 1991] in case of generic t the derivation of nilradical of $L^*(V_t)$ is spanned by

$$f_1\colon \frac{3a_{10,10}}{4} = \frac{a_{11}}{4} = a_{11,11} = \frac{a_{22}}{4} = \frac{a_{33}}{4} = \frac{a_{44}}{2} = \frac{a_{55}}{2} = \frac{a_{66}}{2} = \frac{a_{77}}{2} = \frac{3a_{88}}{4} = \frac{3a_{99}}{4} = 1,$$

and other $a_{ij} = 0$. In case of $t = 0$, the derivation of the nilradical of $L^*(V_t)$ is spanned by

$$f_1\colon -a_{11,11} = -a_{22} = -a_{55} = a_{77} = -a_{99} = 1,$$
$$f_2\colon -3a_{10,10} = -a_{11} = 3a_{22} = -a_{33} = -2a_{44} = 2a_{55} = -2a_{66} = -6a_{77} = -3a_{88} = a_{99} = 1,$$

and other $a_{ij} = 0$. Thus for generic t we have a torus of dimension 1 spanned by $\delta = \mathrm{ad}_{e_0}$;

$$\delta e_i = \begin{cases} e_i, & i = 1, 2, 3, \\ 2e_i, & i = 4, 5, 6, 7, \\ 3e_i, & i = 8, 9, 10, \\ 4e_i, & i = 11. \end{cases}$$

Let $\beta(\delta) = 1$, and $g^\beta = \langle e_1, e_2, e_3 \rangle$. Since $(\mathrm{ad}\, e_1)^4 e_2 = 0$, $(\mathrm{ad}\, e_1)^4 e_3 = 0$ and $(\mathrm{ad}\, e_2)^4 e_3 = 0$:

$$C_{ij} = \begin{cases} 2, & i = j, \\ -3, & i \neq j. \end{cases}$$

So we have the following generalized Cartan matrix:

$$C(\widetilde{E}_8) = \begin{pmatrix} 2 & -3 & -3 \\ -3 & 2 & -3 \\ -3 & -3 & 2 \end{pmatrix}.$$

Case 2. For $t = 0$ we have a torus of dimension 2, spanned by degree derivation $\delta_1 = \mathrm{ad}_{(x\partial_x)}$ and $\delta_2 = \mathrm{ad}_{(y\partial_y)}$;

$$\delta_1 e_i = \begin{cases} e_i, & i = 1, 2, 3, 9, \\ 2e_i, & i = 4, 6, 11, \\ 3e_i, & i = 8, 10, \\ 4e_i, & i = 7, \\ 0, & i = 5, \end{cases} \quad \text{and} \quad \delta_2 e_i = \begin{cases} e_i, & i = 2, 5, 7, 9, 11, \\ 0, & i = 1, 3, 4, 6, 8, 10. \end{cases}$$

For $t = 0$, we have the following decomposition of the nilpotent Lie algebra:

$$\beta_1(\delta_1) = 1, \quad \beta_1(\delta_2) = 1 \quad \Rightarrow \quad g^{\beta_1} = \langle e_2, e_9 \rangle,$$
$$\beta_2(\delta_1) = 1, \quad \beta_2(\delta_2) = 0 \quad \Rightarrow \quad g^{\beta_2} = \langle e_1, e_3 \rangle.$$

It is noted that $e_9 \in [g(V), g(V)]$ and we have $f(1) = 1,\ f(2) = f(3) = 2$,

$$(\operatorname{ad} e_2)^2 e_1 = (\operatorname{ad} e_2)^2 e_3 = (\operatorname{ad} e_9)^2 e_1 = (\operatorname{ad} e_9)^2 e_3 = 0$$
$$\Rightarrow C_{12} = C_{21} = C_{31} = C_{13} = -1,$$
$$(\operatorname{ad}(\gamma e_1 + e_3))^3 e_1 = (\operatorname{ad} e_1)^3 e_3 = 0 \qquad \Rightarrow C_{32} = C_{23} = -2.$$

We have the following generalized Cartan matrix:

$$C(\widetilde{E}_8) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -2 \\ -1 & -2 & 2 \end{pmatrix}. \qquad \square$$

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