J. DIFFERENTIAL GEOMETRY
115 (2020) 195-224

THE NONEXISTENCE OF NEGATIVE WEIGHT DERIVATIONS ON POSITIVE DIMENSIONAL ISOLATED SINGULARITIES: GENERALIZED WAHL CONJECTURE

Bingyi Chen, Hao Chen, Stephen S.-T. Yau & Huaiqing Zuo

Abstract

Let $R = \mathbb{C}[x_1, x_2, \cdots, x_n]/(f)$ where f is a weighted homogeneous polynomial defining an isolated singularity at the origin. Then R and $\mathrm{Der}(R, R)$ are graded. It is well-known that $\mathrm{Der}(R, R)$ does not have a negatively graded component. Wahl conjectured that this is still true for $R = \mathbb{C}[x_1, x_2, \cdots, x_n]/(f_1, f_2, \cdots, f_m)$ which defines an isolated, normal and complete intersection singularity and $f_1, f_2, \cdots, f_m$ weighted homogeneous polynomials with the same weight type $(w_1, w_2, \cdots, w_n)$. Here we give a positive answer to the Wahl Conjecture and its generalization (without the condition of complete intersection singularity) for R when the degree of $f_i, 1 \leq i \leq m$ are bounded below by a constant C depending only on the weights $w_1, w_2, \cdots, w_n$. Moreover this bound C is improved when any two of $w_1, w_2, \cdots, w_n$ are coprime. Since there are counter-examples for the Wahl Conjecture and its generalization when f_i are low degree, our theorem is more or less optimal in the sense that only the lower bound constant can be improved.

1. Introduction

On the one hand, in [**YZ2**], we have studied the problems of nonexistence of negative weight derivation on moduli algebras which are zero-dimensional weighted homogeneous singularities. We also gave sharp upper estimates of dimensions of derivation algebras for these moduli algebras [**YZ1**]. The nonexistence of negative weight derivation on zero-dimensional weighted homogeneous complete intersection singularities was also studied in [**PP1**, **PP2**]. On the other hand, the nonexistence of negative weight derivation on positive dimensional weighted homogeneous singularities has also been considered by many mathematicians ([**MS**], [**Wa1**, **Wa2**, **Wa3**]). In [**Ka1**] and [**Ka2**], the nonexistence of

Mathematics Subject Classification. 14B05, 32S05.
Key words and phrases. Negative weight derivations, weighted homogeneous singularities.
Received July 6, 2017.

negative weight derivation was proved for isolated weighted homogeneous hypersurface singularities and weighted homogeneous curve singularities. Kantor proved the following results in detail:

(a) [**Ka1**] If $A = \mathbb{C}[t^{n_1}, \cdots, t^{n_r}]$ is a non-regular monomial curve, then A has no derivations of negative weight.

(b) [**Ka2**] If $A = \mathbb{C}[x_1, \cdots, x_n]/(f)$ is an isolated weighted homogeneous hypersurface singularity and normalized grading, then A has no derivations of negative weight.

Wahl proposed a very general conjecture (cf. Conjecture 1.4, [**Wa2**]) about the nonexistence of negative weight derivation for positive dimensional weighted homogeneous singularities. One special case of his conjecture for singular cones led him to give a beautiful cohomological characterization of complex projective space ([**Wa3**], [**MS**]). As noted in [**GS**], the Wahl Conjecture can be rephrased in the case of the weighted homogeneous isolated complete intersection singularity (ICIS).

Wahl Conjecture (ICIS)**.** Any weighted homogeneous ICIS with dimension ≥ 2 has no negative weight derivations with respect to some positive grading.

The Wahl Conjecture for complete intersections was first solved by Aleksandrov in [**Al**].

Theorem 1.1 ([**AGLV**, pp. 34–35])**.** *Let $(V, 0)$ be a positive dimensional weighted homogeneous ICIS which is defined by $f_1, f_2, \cdots, f_p \in \mathbb{C}[x_1, \cdots, x_n]$. Then*

$$A := \mathbb{C}[x_1, \cdots, x_n]/(f_1, f_2, \cdots, f_p)$$

has no derivations of negative weight except the following two cases: 1). $p = 1$, and f_1 has multiplicity 2; 2). $p \geq 2, n \geq 3p$, dim $V \geq 4$ and f_i has multiplicity 2 for every $i \in \{1, 2, \cdots, p\}$. In the first exceptional case, the grading is not unique and can always be chosen such that the singularity has no derivations of negative weight. In the second case, the grading is defined uniquely, and for such a singularity there may be derivations of negative weight.

Counter-example 1 (Aleksandrov [**Al**])**.** Let $a \geq 3$. If one assigns weights $1, 1, 1, 1, a, a, a$ to the variables $x_1, \cdots, x_7$, the equations

$$f_1 := x_7 x_1 + x_6 x_2 + x_5 x_3 + x_4^{a+1},$$
$$f_2 := x_7 x_4 + x_6 x_1 + x_5 x_2 + x_3^{a+1}$$

define a five-dimensional weighted homogeneous complete intersection

$$A = \mathbb{C}[x_1, \cdots, x_7]/(f_1, f_2)$$

with an isolated singularity. On A there is a derivation

$$D := (x_2 x_4 - x_1^2)\partial/\partial x_5 - (x_3 x_4 - x_1 x_2)\partial/\partial x_6 + (x_1 x_3 - x_2^2)\partial/\partial x_7$$

of negative weight $2 - a$.

Remark 1.1. In the original statement of Theorem 1.1, Aleksandrov mistakely claimed that for embedding dimension 6, all weighted homogeneous ICIS have no negative weight derivations with respect to some positive grading. Recently Granger and Schulze [**GS**] reproved Aleksandrov's theorem and gave the following counter-example for embedding dimension 6.

Counter-example 2 (Granger and Schulze, [**GS**])**.** Let $n \geq 6$ and pick $c_7, \cdots, c_n \in \mathbb{C}\backslash\{1\}$ pairwise different such that $c_i^9 + 1 \neq 0$ for all i. If one assigns weights $8, 8, 5, 2, \cdots, 2$ to the variables $x_1, \cdots, x_n$, the equations

$$f_1 := x_1x_4 + x_2x_5 + x_3^2 - x_4^5 + \sum_{i=7}^{n} x_i^5,$$

$$f_2 := x_1x_5 + x_2x_6 + x_3^2 + x_6^5 + \sum_{i=7}^{n} c_i x_i^5$$

define a weighted homogeneous complete intersection

$$A = \mathbb{C}[x_1, \cdots, x_n]/(f_1, f_2)$$

with an isolated singularity. On A there is a derivation

$$D := 2x_3(x_5 - x_6)\partial/\partial x_1 - 2x_3(x_4 - x_5)\partial/\partial x_2 + (x_4x_6 - x_5^2)\partial/\partial x_3$$

of weight -1.

Both singularities in Counter-examples 1 and 2 are complete intersection. We shall give a non-complete intersection singularity which has a negative weight derivation. This is a Gorenstein singularity obtained by taking quotient of $\mathbb{C}^3$ by finite cyclic group of order 3.

Example 3. Let G be the subgroup of $SL(3, \mathbb{C})$ generated by

$$\begin{pmatrix} \exp(2\pi i/3) & 0 & 0 \\ 0 & \exp(2\pi i/3) & 0 \\ 0 & 0 & \exp(2\pi i/3) \end{pmatrix}.$$

Then a set of minimal generators of $\mathbb{C}[x, y, z]^G$ is

$$x_1 = x^3, \quad x_2 = y^3, \quad x_3 = z^3, \quad x_4 = xyz, \quad x_5 = x^2y,$$
$$x_6 = xy^2, \quad x_7 = x^2z, \quad x_8 = xz^2, \quad x_9 = y^2z, \quad x_{10} = yz^2,$$

whose relations are

$$\begin{array}{cccc} x_5^2 = x_1x_6, & x_5x_6 = x_1x_2, & x_6^2 = x_2x_5, & x_7^2 = x_1x_8, \\ x_7x_8 = x_1x_3, & x_8^2 = x_3x_7, & x_9^2 = x_2x_{10}, & x_9x_{10} = x_2x_3, \\ x_{10}^2 = x_3x_9, & x_1x_9 = x_4x_5, & x_1x_{10} = x_4x_7, & x_2x_7 = x_4x_6, \\ x_2x_8 = x_4x_9, & x_3x_5 = x_4x_8, & x_3x_6 = x_4x_{10}, & x_5x_7 = x_1x_4, \\ x_5x_8 = x_4x_7, & x_5x_9 = x_4x_6, & x_5x_{10} = x_4^2, & x_6x_7 = x_4x_5, \\ x_6x_8 = x_4^2, & x_6x_9 = x_2x_4, & x_6x_{10} = x_4x_9, & x_7x_9 = x_4^2, \\ x_7x_{10} = x_4x_8, & x_8x_9 = x_4x_{10}, & x_8x_{10} = x_3x_4. & \end{array}$$

We assign the following weights

$$wt(x_1) = 1, \quad wt(x_2) = 4, \quad wt(x_3) = 7, \quad wt(x_4) = 4, \quad wt(x_5) = 2,$$
$$wt(x_6) = 3, \quad wt(x_7) = 3, \quad wt(x_8) = 5, \quad wt(x_9) = 5, \quad wt(x_{10}) = 6.$$

Then, it is easy to see that the relation equations are weighted homogeneous under this weight system and define a three-dimensional isolated quotient singularity (cf. [**YY**], Theorem A). We obtain a derivation

$$3x_6\partial/\partial x_2 + x_7\partial/\partial x_4 + x_1\partial/\partial x_5 + 2x_5\partial/\partial x_6 + 2x_4\partial/\partial x_9 + x_8\partial/\partial x_{10}$$

of degree -1.

Based on these examples, it is natural to propose the following conjecture.

Generalized Wahl Conjecture. Let $P = \mathbb{C}[x_1, x_2, \ldots, x_n]$ be the weighted polynomial ring in n weighted variables $x_1, x_2, \ldots, x_n$ ($n \geq 2$) with positive integer weights $w_1 \geq w_2 \geq \cdots \geq w_n$. Let $(V, 0)$ be a positive dimensional variety which is defined by weighted homogeneous polynomials $f_1, f_2, \cdots, f_m \in P$. Suppose $(V, 0)$ is an isolated singularity. Then the graded ring $R = P/(f_1, f_2, \ldots, f_m)$ has no negative weight derivations if the (weighted) degrees of $f_i, 1 \leq i \leq m$, are large.

Remark 1.2 (cf. [**Al**]). If the singularity is a positive dimensional isolated complete intersection singularity, then the derivation algebra is generated by Euler derivations and trivial derivations. Thus, the generators of the derivations are completely known. However, for non-complete intersection singularities, there is no known description of all holomorphic vector fields. Therefore the generalized Wahl Conjecture is substantially more difficult than the Wahl Conjecture for ICIS.

In this paper, we solve the Generalized Wahl Conjecture.

Main Theorem A (Generalized Wahl Conjecture). *Let $P = \mathbb{C}[x_1, x_2, \ldots, x_n]$ be the weighted polynomial ring in n weighted variables $x_1, x_2, \ldots, x_n$ ($n \geq 2$) with positive integer weights $w_1 \geq w_2 \geq \cdots \geq w_n$. Suppose that $f_1, f_2, \ldots, f_m$ are weighted homogeneous polynomials of degrees greater than $(m-1+w_1)(w_1w_2)^{n-1}$ and $f_1, f_2, \ldots, f_m$ define a positive dimensional isolated singularity at the origin. Then there are no non-zero negative weight derivations on $R = P/(f_1, f_2, \ldots, f_m)$.*

Remark 1.3. We claim that our degree condition on $f_1, \cdots, f_m$ implies that $f_1, \cdots, f_m$ cannot contain any quadratic terms when $m \geq 2$. We assume that $wt(x_i) = w_i, 1 \leq i \leq n$ and $w_1 \geq w_2 \geq \cdots \geq w_n \geq 1$ where w_i are integers. Let d_i be the weighted degree of f_i. We have $d_i > (m-1+w_1)(w_1w_2)^{n-1}$, $1 \leq i \leq m$. If $w_1 = 1$, then $w_i = 1$, $2 \leq i \leq n$. Since $d_i > (m-1+w_1)(w_1w_2)^{n-1} \geq 2$, so obviously f_i cannot contain any quadratic terms. If $w_1 > 1$, then $d_i > (m-1+w_1)(w_1w_2)^{n-1} \geq 2w_1$. Thus f_i cannot contain any quadratic terms due to the degree consideration.

From Counter-examples 1 and 2 for the Wahl Conjecture in the complete intersection case ([**Al**], [**GS**]) and Example 3, we know that the nonexistence of negative weight derivation on positive dimensional singularities can be expected only for "large" degree cases. But of course our constant $(m-1+w_1)(w_1w_2)^{n-1}$ here may not be sharp. Main Theorem B below tells us that this bound can be improved under the additional condition that any two of the weights $w_1, w_2, \ldots, w_n$ are coprime.

Main Theorem B. *Let $P = \mathbb{C}[x_1, x_2, \ldots, x_n]$ be the weighted polynomial ring in n weighted variables $x_1, x_2, \ldots, x_n (n \geq 2)$ with positive integer weights $w_1 \geq w_2 \geq \cdots \geq w_n$ and $f_1, f_2, \ldots, f_m$ be m weighted homogeneous polynomials of degrees greater than $(m-1+w_1)w_1w_2$. Suppose that any two of the original weights $w_1, w_2, \ldots, w_n$ are coprime and $f_1, f_2, \ldots, f_m$ define a positive dimensional isolated singularity at the origin. Then there are no non-zero negative weight derivations on $R = P/(f_1, f_2, \ldots, f_m)$.*

Remark 1.4. Notice that the singularities investigated in Main Theorem A and Main Theorem B are not necessarily normal singularities. Indeed, if we take $m = 1, n = 2$, $P = \mathbb{C}[x_1, x_2]$, $f_1 = x_1^8 + x_2^{12}$, $w_1 = 3$, and $w_2 = 2$, then it is easy to check f_1 satisfies the conditions in the main theorems, but the singularity defined by f_1 is not normal.

The main idea of the proofs of the main theorems is as follows. Suppose there exists a non-zero negative weight derivation D on $R = P/I$ with respect to weight type $(w_1, w_2, \ldots, w_n)$ where $w_1 \geq w_2 \cdots \geq w_n \geq 1$. We can regard D as a negative weight derivation on the weighted polynomial ring $P = \mathbb{C}[x_1, x_2, \ldots, x_n]$ which preserves the ideal I. It is well known that D is of the following form

$$D = p_1\partial/\partial x_1 + p_2\partial/\partial x_2 + \cdots + p_n\partial/\partial x_n, \tag{1.1}$$

where p_i are weighted homogeneous polynomials with the degrees $w_i + wt D$, respectively. Let weighted homogeneous polynomials f_1, $f_2, \ldots, f_m$ generate the ideal I and without loss of generality we assume that $\deg f_1 \geq \deg f_2 \geq \cdots \geq \deg f_m$. By the condition $D(f_1, f_2, \ldots, f_m) \subset (f_1, f_2, \ldots, f_m)$ and $\deg f_1 \geq \deg f_2 \geq \cdots \geq \deg f_m$, we have

$$\begin{aligned} Df_1 &= \ell_1^2 f_2 + \ell_1^3 f_3 + \cdots + \ell_1^m f_m, \\ Df_2 &= \ell_2^3 f_3 + \ell_2^4 f_4 + \cdots + \ell_2^m f_m, \\ &\cdots\cdots\cdots\cdots \\ Df_{m-1} &= \ell_{m-1}^m f_m, \\ Df_m &= 0, \end{aligned} \tag{1.2}$$

where ℓ_j^i are weighted homogeneous polynomials.

For any negative weight derivation D as in (1.1) on P we associate families of new weight type $(\ell_1, \ell_2, \ldots, \ell_n)$ controlled by parameters ϵ_i

(see Definition 3.1). In Theorem 4.1, we prove that if we can choose suitable parameters ϵ_i to make the new weight type $(\ell_1, \ell_2, \dots, \ell_n)$ satisfy the three conditions below:

(1) there is only one index $i_0 \in \{1, 2, \dots, n\}$ such that $\ell_{i_0}/w_{i_0} = \max\{\ell_i/w_i : i = 1, 2, \dots, n\}$;
(2) $\epsilon_{i_0} = \epsilon_{\min}$, where

$$\epsilon_{min} = \min\{\epsilon_i \text{ for } i \text{ such that } p_i \text{ is a non-zero polynomial}\};$$

(3) p_{i_0} is a non-zero polynomial,

where p_i is the coefficient of $\partial/\partial x_i$ in D for $i = 1, 2, \dots n$, then the degree of each f_j is low, which contradicts the condition in the main theorems that the degree of each f_j is bounded below by a constant. Thus such D doesn't exist and there are no negative weight derivations on R.

From the argument above, we emphasize here the key point is to choose suitable parameters for a given negative weight derivation D, which preserves the ideal $(f_1, f_2, \dots, f_m)$, to satisfy the above three conditions (1)–(3). First we let

$$\epsilon_i = \begin{cases} \epsilon, & p_i \text{ is a non-zero polynomial} \\ 0, & \text{otherwise} \end{cases},$$

where ϵ is a positive real number. Then we have $\epsilon_{min} = \epsilon$ and $\ell_i = 0$ for i such that p_i is the zero polynomial. Let $I_{\max} = \{e : \ell_e/w_e$ is the maximum among all ℓ_i/w_i for $i = 1, 2, \dots, n\}$. It is easy to see that $\epsilon_i = \epsilon_{\min}$ and p_i is a non-zero polynomial for any $i \in I_{\max}$. Under the additional condition that any two of the weights $w_1, w_2, \dots, w_n$ are coprime, we can prove that $I_{\max}$ has only one element, implying that conditions (1)–(3) in Theorem 4.1 are satisfied. Consequently, Main Theorem B follows immediately using Theorem 4.1. But in general $I_{\max}$ might have more than one element, thus we need to adjust the parameters ϵ_i in order to separate $\{\ell_i/w_i : i \in I_{\max}\}$ such that these numbers have only one maximum. The parameters are adjusted as follows: pick an index $i_1 \notin I_{\max}$ and replace the parameter ϵ_{i_1} with $\epsilon_{i_1} + \epsilon/(w_1 w_2)$, then the new weight type and $I_{\max}$ change accordingly. Then pick an index $i_2 \notin I_{\max}$ and replace the parameter ϵ_{i_2} with $\epsilon_{i_2} + \epsilon/(w_1 w_2)^2$. Repeat this process. Theorem 6.1 guarantees the procedure will be terminated after finite steps, and Main Theorem A is proved. We speculate that this new technique of decomposing equations according to the new weight type might be useful for attacking other problems in singularity theory.

The paper is organized as follows. We recall the definition and properties of derivations in section 2. In section 3 we define and give the necessary properties for the main technical tool—new weight type associated to a negative weight derivation on the weighted polynomial ring.

Some lemmas and theorems which are used in the proof of our main theorems are introduced and are proved in section 4. We shall give the proofs of Main Theorem A and B in section 4 and 5.

Acknowledgements. We thank the anonymous referee for his/her many valuable suggestions and corrections that significantly improved the paper. The first, third and fourth authors are supported by NSFC Grant 11531007 and the start-up fund from Tsinghua University. The second author is supported by NSFC Grants 11371138 and 11531002. The fourth author is supported by NSFC Grants 11771231, 11401335 and Tsinghua University Initiative Scientific Research Program.

2. Derivations

Let $P = \mathbb{C}[x_1, \cdots, x_n]$ be the polynomial ring of n weighted variables $x_1, \ldots, x_n$ with positive integer weights $w_1, w_2, \ldots, w_n$. For a monomial $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ in P, its weighted degree is defined to be $w_1 i_1 + \cdots + w_n i_n$. A polynomial $f \in P$ is called weighted homogeneous with respect to weights $w_1, \cdots, w_n$ if there exists a positive integer d such that $\sum a_i w_i = d$, for each monomial $\prod x_i^{a_i}$ appearing in f with a nonzero coefficient. The number d is called the (weighted) degree of f and denoted by $\deg f$. For an ideal I generated by weighted homogeneous polynomials in P we have a graded quotient algebra $R = P/I = \oplus_{i=0}^{\infty} R_i$. Furthermore, R is called a graded complete intersection algebra if I is generated by a regular sequence $f_1, \cdots, f_m, m \leq n$. When the Krull dimension of R is zero, R is a positively graded Artinian algebra.

Let $R = P/I$ be a positively graded algebra as above. Then the derivations of R are induced by derivations of P sending I to I. Let $Der(R)$ be the R-module of derivations of R. As R is graded, we have a natural grading on $Der(R) = \oplus_{k=-\infty}^{+\infty} Der(R)_k$ where $Der(R)_k = \{D \in Der(R) \colon D(R_i) \subset R_{i+k}$ for any $i\}$. In particular, the Euler derivation $\Delta = \sum w_i x_i \frac{\partial}{\partial x_i}$ has weight 0.

A complete local $\mathbb{C}$-algebra (i.e. singularity) is weighted homogeneous if it is the completion $\hat{R}$ of a graded algebra R. If the singularity is isolated, weighted-homogeneity is equivalent to having a positive grading on the completion. The same singularity may have essentially different graded structures. For example, $R = \mathbb{C}[x, y, z]/(xz - y^2)$ is bigraded, so it has many gradings (e.g., using weights $\{1, k+1, 2k+1\}$). However, Saito [**Sa**] has proved that an isolated weighted homogeneous hypersurface singularity defined by $f \in \mathbb{C}[z_1, \cdots, z_n]$ has unique normalized weights. Saito's choice of weights gives a graded algebra $R = \mathbb{C}[z_1, \cdots, z_n]/(f)$ for which there are no derivations of negative weight. A complete intersection weighted homogeneous isolated singularity $\hat{R}$ uniquely determines a graded algebra R (assuming dim $R > 0$, and excluding the case of multiplicity 2 hypersurfaces). In general, if the maximal reductive automorphism group of $\hat{R}$ has dimension 1, then $\hat{R}$ admits a unique positively graded structure (cf. [**Wa2**]).

3. New weight type

Let $P = \mathbb{C}[x_1, x_2, \dots, x_n]$, $w_1 \geq w_2 \geq \cdots \geq w_n$ be as above and D be a non-zero negative weight derivation on P. It is well known that D is of the following form

$$D = p_1 \partial/\partial x_1 + p_2 \partial/\partial x_2 + \cdots + p_n \partial/\partial x_n, \tag{3.1}$$

where p_i is a weighted homogeneous polynomial of degree $w_i + wt\,D$ with respect to the weight type $(w_1, w_2, \dots, w_n)$ or the zero polynomial for $i = 1, 2, \dots, n$. Since $wt\,D < 0$, we know that p_i is a polynomial in $x_{i+1}, x_{i+2}, \dots, x_n$ for $1 \leq i \leq n$. Thus p_n is a constant polynomial. We define a new weight type associated to D as follows.

Definition 3.1. Let D be a non-zero negative weight derivation on the weighted polynomial ring P as in (3.1). The following weight type $(\ell_1, \ell_2, \dots, \ell_n)$ controlled by the given n parameters $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ are called the new weight type associated to D, where ϵ_i are non-negative real parameters. Set

$$\ell_n = \epsilon_n.$$

If $\ell_n, \ell_{n-1}, \dots, \ell_{q+1}$ are defined, ℓ_q is defined as follows:

(i) if the coefficient $p_q(x_{q+1}, \dots, x_n)$ of $\partial/\partial x_q$ in D is the zero-polynomial

$$\ell_q = \epsilon_q, \tag{3.2}$$

(ii) if the coefficient $p_q(x_{q+1}, \dots, x_n)$ of $\partial/\partial x_q$ in D is a non-zero polynomial

$$\ell_q = \epsilon_q + \max\{\ell_{q+1} i_{q+1} + \ell_{q+2} i_{q+2} + \cdots + \ell_n i_n \mid \text{monomial } x_{q+1}^{i_{q+1}} x_{q+2}^{i_{q+2}} \dots x_n^{i_n} \text{appears in the expansion of } p_q\} \tag{3.3}$$

where p_i is the coefficient of $\partial/\partial x_i$ in D for $i = 1, 2, \dots n$.

It is clear that when

$$\epsilon_i = \begin{cases} -\,wt\,D, & p_i \text{ is a non-zero polynomial} \\ w_i, & \text{otherwise} \end{cases},$$

then the new weight type $(\ell_1, \ell_2, \dots, \ell_n)$ is just the original weight type $(w_1, w_2, \dots, w_n)$.

Definition 3.2. The degree of a monomial $x^\alpha = x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$ is defined to be $w_1 i_1 + w_2 i_2 + \cdots + w_n i_n$. The Q-degree of x^α is defined to be $\ell_1 i_1 + \ell_2 i_2 + \cdots + \ell_n i_n$. And the Q-degree of a polynomial f is defined as follows,

$$\text{Q-deg } f := \max\{\text{Q-degrees of monomials in the expansion of } f\}.$$

Thus $\ell_i = \epsilon_i + \text{Q-deg}\ p_i$ for $i = 1, 2, \ldots, n$ such that p_i is a non-zero polynomial, where p_i is the coefficient of $\partial/\partial x_i$ in D.

Definition 3.3. For any polynomial f in P, we denote by $f_{\max}$ the sum of terms in the expansion of f with maximum Q-degree with respect to $(\ell_1, \ell_2, \ldots, \ell_n)$, i.e., if we write

$$f = \sum_{\alpha \in I} c_\alpha x^\alpha,$$

where I is a finite set, then

$$f_{\max} := \sum_{\alpha \in I \text{ and Q-deg } x^\alpha = \text{Q-deg } f} c_\alpha x^\alpha.$$

Definition 3.4. With the same notation as before, we define

$$d_{\max}(D) := \max\{\text{the Q-degree of } (p_j)_{\max}\partial/\partial x_j \mid p_j \text{ is a non-zero polynomial}\},$$

and

$$D_{\max} := \sum_{\substack{\text{for } j \text{ such that} \\ (p_j)_{\max}\partial/\partial x_j \text{ has Q-degree } d_{\max}(D)}} (p_j)_{\max}\partial/\partial x_j, \tag{3.4}$$

where the Q-degree of $(p_j)_{\max}\partial/\partial x_j$ is defined to be $\text{Q-deg}\ (p_j)_{\max} - \ell_j$.

Proposition 3.1. *With the same notation as above, we have*

$$D_{\max} = \sum_{\substack{\text{for } j \text{ such that} \\ p_j \text{ is a non-zero polynomial and } \epsilon_j = \epsilon_{min}}} (p_j)_{\max}\partial/\partial x_j, \tag{3.5}$$

where

$$\epsilon_{min} = \min\{\epsilon_i \text{ for } i \text{ such that } p_i \text{ is a non-zero polynomial}\}.$$

Then Q-degree $D_{\max} = -\epsilon_{min}$.

Proof. It is clear from the definition of the new weight type and $D_{\max}$. q.e.d.

Proposition 3.2. *Let* $D, D_{\max}, \epsilon_{\min}$ *be as above and* g *be an arbitrary polynomial in* P. *We have either*

(i) $D_{\max} g_{\max} = 0$, *in this case* $\text{Q-deg}\ (Dg)_{\max} < \text{Q-deg}\ g_{\max} - \epsilon_{min}$, *or*

(ii) $D_{\max} g_{\max} = (Dg)_{\max}$.

Proof. Write

$$g = g_{\max} + \text{lower Q-deg terms} = g_{\max} + g_r + g_{r-1} + \ldots,$$

and

$$D = D_{\max} + \text{lower Q-deg terms} = D_{\max} + D_s + D_{s-1} + \ldots,$$

where

$$\cdots < \text{Q-deg } g_{r-1} < \text{Q-deg } g_r < \text{Q-deg } g_{\max},$$

and

$$\cdots < \text{Q-deg } D_{s-1} < \text{Q-deg } D_s < \text{Q-deg } D_{\max}.$$

Then we have

$$Dg = D_{\max}g_{\max} + D_{\max}g_r + D_s g_{\max} + D_s g_r + \dots .$$

If $D_{\max}g_{\max} \neq 0$, then $Dg = D_{\max}g_{\max} + \text{lower Q-deg terms}$. Thus we have

$$D_{\max}g_{\max} = (Dg)_{\max}.$$

If $D_{\max}g_{\max} = 0$, then $Dg = D_{\max}g_r + D_s g_{\max} + D_s g_r + \dots .$ Thus

$$\text{Q-deg } (Dg)_{\max} \le \max\{\text{Q-deg } D_{\max} + \text{Q-deg } g_r, \text{Q-deg } D_s + \text{Q-deg } g_{\max}\}.$$

Since Q-degree $D_{\max} = -\epsilon_{min}$ by Proposition 3.1, we have

$$\begin{aligned}\text{Q-deg } D_{\max} + \text{Q-deg } g_r &< \text{Q-deg } D_{\max} + \text{Q-deg } g_{\max}\\ &= \text{Q-deg } g_{\max} - \epsilon_{min},\end{aligned}$$

and

$$\begin{aligned}\text{Q-deg } D_s + \text{Q-deg } g_{\max} &< \text{Q-deg } D_{\max} + \text{Q-deg } g_{\max}\\ &= \text{Q-deg } g_{\max} - \epsilon_{min}.\end{aligned}$$

Therefore, $\text{Q-deg } (Dg)_{\max} < \text{Q-deg } g_{\max} - \epsilon_{min}$. q.e.d.

Corollary 3.1. *Let D and g as above. If $Dg = 0$, then $D_{\max}g_{\max} = 0$.*

Proof. This is an immediate consequence of Proposition 3.2. q.e.d.

4. Some lemmas for the proof of main theorems

In this section and the next section, $P = \mathbb{C}[x_1, x_2, \dots, x_n]$ is the weighted polynomial ring in n weighted variables $x_1, x_2, \dots, x_n$ with positive integer weights $w_1 \ge w_2 \ge \cdots \ge w_n$. Let

$$D = p_1\partial/\partial x_1 + p_2\partial/\partial x_2 + \cdots + p_n\partial/\partial x_n$$

be a fixed non-zero negative weight derivation on P, and let $(\ell_1, \ell_2, \dots, \ell_n)$ be the new weight type associated to D controlled by non-negative parameters ϵ_i.

The following simple properties of isolated singularities are needed for our proof of the main results in this section as well as in the next section.

Lemma 4.1. *Let I be the ideal generated by weighted homogeneous polynomials $f_1, f_2, \dots, f_m$ with respect to weight type $(w_1, w_2, \dots, w_n)$ as above and P/I is a non-zero Artinian algebra. Let m be the maximal ideal generated by $x_1, x_2, \dots, x_n$, then we have $m^r \subseteq I$ for some integer $r > 0$ and P/I is a local Artinian algebra.*

Proof. Let d_i be the degree of f_i with respect to $(w_1, w_2, \dots, w_n)$ for $i = 1, 2, \dots, m$. Then for any point $(x_1, x_2, \dots, x_n) \in \mathbb{C}^n$, we have

$$f_i(\alpha^{w_1} x_1, \alpha^{w_2} x_2, \dots, \alpha^{w_n} x_n) = \alpha^{d_i} f_i(x_1, x_2, \dots, x_n)$$

for any $i = 1, 2, \dots, m$ and any $\alpha \in \mathbb{C}$. We claim that $Z(I) = \{0\}$, where $Z(I)$ is the zero locus of I in $\mathbb{C}^n$. If $Z(I) \neq \{0\}$, then there is a point $(x_1, x_2, \dots, x_n) \in Z(I)$ and $(x_1, x_2, \dots, x_n) \neq 0$. Thus $\{(\alpha^{w_1} x_1, \alpha^{w_2} x_2, \dots, \alpha^{w_n} x_n), \alpha \in \mathbb{C}\} \subseteq Z(I)$ has dimension one, which contradicts P/I is an Artinian algebra. Thus $Z(I) = \{0\}$, which yields that $m^r \subseteq I$ for some integer $r > 0$. Hence, for any maximal ideal m' in P such that $I \subseteq m'$, we have $m^r \subseteq m'$, which implies $m = m'$. So P/I has only one maximal ideal, thus P/I is a local Artinian algebra. q.e.d.

Lemma 4.2. *Let $f_1, f_2, \dots, f_m \in \mathbb{C}[x_1, x_2, \dots, x_n]$ be weighted homogeneous polynomials. Suppose that $\mathbb{C}[x_1, x_2, \dots, x_n]/(f_1, f_2, \dots, f_m)$ is a non-zero Artinian algebra. Then for any given index $i \in \{1, 2, \dots, n\}$ there exists an index $j \in \{1, 2, \dots, m\}$ such that $f_j(x_1, x_2, \dots, x_n)$ contains a term $x_i^{a_i}$ (with a_i a positive integer) in its expansion.*

Proof. (By contradiction) Assuming the opposite, we see that the ideal $(f_1, f_2, \dots, f_m)$ has to be contained within the ideal $(x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. However, by Lemma 4.1, there exists some integer $r > 0$, such that

$$(x_1, x_2, \dots, x_n)^r \subseteq (f_1, f_2, \dots, f_m).$$

Consequently, it gives

$$(x_1, x_2, \dots, x_n)^r \subseteq (x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n),$$

which yields a contradiction. The lemma is proved. q.e.d.

Lemma 4.3. *Let $f_1, f_2, \dots, f_m \in \mathbb{C}[x_1, x_2, \dots, x_n]$ be weighted homogeneous polynomials defining the germ of a positive dimensional isolated singularity at the origin by $f_1 = f_2 = \cdots = f_m = 0$. Then for any given index $i \in \{1, 2, \dots, n\}$ there are indices $t \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, n\}$ such that $f_t(x_1, x_2, \dots, x_n)$ contains a term of the form $x_i^{a_i}$ or $x_i^{a_i} x_j$ (with a_i a positive integer) in its expansion.*

Proof. We denote $(V, 0)$ as the germ of isolated singularity defined by $f_1 = f_2 = \cdots = f_m = 0$, and let r be the dimension of V. Since no complete intersection condition is imposed here, $\dim V$ might not be equal to $n - m$. From the condition that the origin is the only singularity of V near the origin, we know that the determinants of $(n - r) \times (n - r)$ submatrices of the following matrix

$$\begin{pmatrix} \partial f_1/\partial x_1, \partial f_1/\partial x_2, \dots, \partial f_1/\partial x_n \\ \partial f_2/\partial x_1, \partial f_2/\partial x_2, \dots, \partial f_2/\partial x_n \\ \dots\dots\dots\dots \\ \partial f_m/\partial x_1, \partial f_m/\partial x_2, \dots, \partial f_m/\partial x_n \end{pmatrix}$$

and $f_1, \cdots, f_m$ generate an ideal I such that $\mathbb{C}[x_1, x_2, \ldots, x_n]/I$ is an Artinian Algebra. By Lemma 4.2, for any index $i \in \{1, 2, \ldots, n\}$, one of the following cases occurs,

(i) there exists a $(n-r) \times (n-r)$ submatrix of the above matrix, such that x_i^b with a positive integer b is contained in the expansion of the determinant of this submatrix. Thus one of its entries, i.e. $\partial f_p/\partial x_q$, contains a power of x_i in its expansion,

(ii) there exists $t \in \{1, 2, \ldots, m\}$ such that x_i^b with a positive integer b is contained in the expansion of f_t.

Thus the conclusion is proved. q.e.d.

The following observations based on the assumption that $\{\ell_i/w_i : i = 1, \ldots, n\}$ has the unique maximum are crucial to our proof of the main results.

Lemma 4.4. *Suppose that there is only one index* $i_0 \in \{1, 2, \ldots, n\}$ *such that* $\beta = \ell_{i_0}/w_{i_0} = \max\{\ell_i/w_i : i = 1, 2, \ldots, n\}$. *Let* $f \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ *be a weighted homogeneous polynomial with respect to both the original weight type* $(w_1, w_2, \ldots, w_n)$ *and the new weight type* $(\ell_1, \ell_2, \ldots, \ell_n)$. *Suppose that the degree of* f *and the Q-degree of* f *satisfy*

(i)

$$\deg f > M/(\beta - \gamma), \tag{4.1}$$

and

(ii)

$$\text{Q-deg} f \geq \beta \deg f - M, \tag{4.2}$$

where M *is a fixed constant and*

$$\gamma = \max\{\ell_i/w_i : i = 1, 2, \ldots, i_0 - 1, i_0 + 1, \ldots, n\}.$$

Then x_{i_0} *divides* f.

Proof. Suppose that some monomial x^a in the expansion of $f(x_1, x_2, \ldots, x_n)$ is not divisible by x_{i_0}. Let us denote $x^a = x_1^{a_1} \cdots x_{i_0-1}^{a_{i_0-1}} x_{i_0+1}^{a_{i_0+1}} \cdots x_n^{a_n}$. By the definition of γ, we conclude that

$$\begin{aligned}
\text{Q-deg } f &= \text{Q-deg } x^a \\
&= a_1\ell_1 + \cdots + a_{i_0-1}\ell_{i_0-1} + a_{i_0+1}\ell_{i_0+1} + \cdots + a_n\ell_n \\
&\leq \gamma(a_1 w_1 + \cdots + a_{i_0-1}w_{i_0-1} + a_{i_0+1}w_{i_0+1} + \cdots + a_n w_n) \\
&= \gamma \deg x^a = \gamma \deg f. \qquad (4.3)
\end{aligned}$$

Combining (4.3) with (4.2), we get

$$\beta \deg f - M \leq \text{Q-deg } f \leq \gamma \deg f. \tag{4.4}$$

This implies

$$\deg f \leq M/(\beta - \gamma), \tag{4.5}$$

which contradicts (4.1). Thus the lemma is proved. q.e.d.

Lemma 4.5. *If the coefficient p_{i_0} of $\partial/\partial x_{i_0}$ in D is a non-zero polynomial and f is a polynomial which is divisible by x_{i_0}, then $Df \neq 0$.*

Proof. Let us expand $f(x_1, x_2, \ldots, x_n)$ in powers of x_{i_0}

$$f(x_1, x_2, \ldots, x_n) = b_q x_{i_0}^q + b_{q-1} x_{i_0}^{q-1} + \cdots + b_h x_{i_0}^h, \text{ with } b_h \neq 0, \tag{4.6}$$

where $h \leq q$ and $b_q, b_{q-1}, \ldots, b_h$ are polynomials of $x_1, x_2, \ldots, x_{i_0-1}$, $x_{i_0+1}, \ldots, x_n$. From the condition of Lemma 4.5, we know that $h \geq 1$. It yields that

$$Df = (D_1 + p_{i_0}\partial/\partial x_{i_0})f, \tag{4.7}$$

where $D_1 = D - p_{i_0}\partial/\partial x_{i_0}$. Therefore, $D_1 f = x_{i_0}^h D_1(b_q x_{i_0}^{q-h} + \cdots + b_h)$ and

(4.8)

$$\begin{aligned} Df &= x_{i_0}^h D_1\Big(b_q x_{i_0}^{q-h} + \cdots + b_h\Big) + p_{i_0}\Big(q b_q x_{i_0}^{q-1} + \cdots + h b_h x_{i_0}^{h-1}\Big) \\ &= x_{i_0}^h D_1\Big(b_q x_{i_0}^{q-h} + \cdots + b_h\Big) + x_{i_0}^h p_{i_0}\Big(q b_q x_{i_0}^{q-h-1} + \cdots + (h+1) b_{h+1}\Big) \\ &\quad + h x_{i_0}^{h-1} p_{i_0} b_h. \end{aligned}$$

It is clear that $p_{i_0} b_h$ is a non-zero polynomial in $x_1, x_2, \ldots, x_{i_0-1}, x_{i_0+1}, \ldots, x_n$. Hence the last term on the right hand side of (4.8) is only divisible by $x_{i_0}^{h-1}$. Thus Df is a non-zero polynomial. q.e.d.

Lemma 4.6. *If $\ell_{i_0}/w_{i_0} = \max\{\ell_i/w_i \colon i = 1, 2 \ldots, n\}$ (not necessarily the unique maximum) and the coefficient p_{i_0} of $\partial/\partial x_{i_0}$ in D is a non-zero polynomial, then $\ell_{i_0}/w_{i_0} \leq \epsilon_{i_0}/(-\,wt\,D)$. That is to say, $\ell_i/w_i \leq \epsilon_{i_0}/(-\,wt\,D)$ for $i = 1, 2, \ldots, n$.*

Proof. Assume that $\ell_{i_0}/w_{i_0} > \epsilon_{i_0}/(-\,wt\,D)$, then by the definition of the new weight type and the fact that $wt\,D = \deg p_{i_0} - w_{i_0}$, we have

$$\frac{\text{Q-deg }(p_{i_0})_{\max} + \epsilon_{i_0}}{\deg(p_{i_0})_{\max} - wt\,D} = \frac{\ell_{i_0}}{w_{i_0}}.$$

Combining with the assumption that $\epsilon_{i_0}/(-\,wt\,D) < \ell_{i_0}/w_{i_0}$, we conclude that

$$\frac{\text{Q-deg }(p_{i_0})_{\max}}{\deg(p_{i_0})_{\max}} > \frac{\ell_{i_0}}{w_{i_0}}. \tag{4.9}$$

However, $(p_{i_0})_{\max}$ is a polynomial in x_t for $t > i_0$ and we have $\ell_t/w_t \leq \ell_{i_0}/w_{i_0}$ for $t > i_0$. Thus, we obtain that

$$\frac{\text{Q-deg }(p_{i_0})_{\max}}{\deg(p_{i_0})_{\max}} \leq \frac{\ell_{i_0}}{w_{i_0}},$$

which contradicts (4.9). Thus this lemma is proved. q.e.d.

The following theorem is critical to the proof of Main Theorem A.

Theorem 4.1. *Let $f_1, f_2, \ldots, f_m$ be m weighted homogeneous polynomials in P with respect to the weight type $(w_1, w_2, \ldots, w_n)$. Suppose these polynomials define a positive dimensional isolated singularity at the origin. Suppose that the negative weight derivation D on P preserves the ideal $(f_1, f_2, \ldots, f_m)$. If we can choose suitable parameters ϵ_i to make the new weight type $(\ell_1, \ell_2, \ldots, \ell_n)$ satisfy the three conditions below:*

(1) *there is only one index $i_0 \in \{1, 2, \ldots, n\}$ such that $\beta = \ell_{i_0}/w_{i_0} = \max\{\ell_i/w_i : i = 1, 2, \ldots, n\}$,*
(2) *$\epsilon_{i_0} = \epsilon_{\min}$, where*

$$\epsilon_{min} = \min\{\epsilon_i \text{ for } i \text{ such that } p_i \text{ is a non-zero polynomial}\},$$

(3) *p_{i_0} is a non-zero polynomial,*

where p_i is the coefficient of $\partial/\partial x_i$ in D for $i = 1, 2, \ldots, n$, then there exists $j \in \{1, 2, \ldots, m\}$ such that

$$\deg f_j \leq \frac{(m - 1 + w_1)\epsilon_{\min}}{\beta - \gamma},$$

where $\gamma = \max\{\ell_i/w_i : i = 1, 2, \ldots, i_0 - 1, i_0 + 1, \ldots, n\}$.

Proof. Without loss of generality, we assume that $\deg f_1 \geq \deg f_2 \geq \cdots \geq \deg f_m$. By comparing degrees, we find that

$$\begin{aligned} Df_1 &= \ell_1^2 f_2 + \cdots + \ell_1^m f_m, \\ Df_2 &= \ell_2^3 f_3 + \cdots + \ell_2^m f_m, \\ &\cdots\cdots\cdots \\ Df_{m-1} &= \ell_{m-1}^m f_m, \\ Df_m &= 0, \end{aligned} \tag{4.10}$$

where ℓ_j^i with $i > j$ are weighted homogeneous polynomials with respect to the original weight type $(w_1, w_2, \ldots, w_n)$.

By Lemma 4.3, we can find one f_{j_0} which contains a term of the form $x_{i_0}^a$ or $x_{i_0}^a x_j$ with $j \in \{1, 2 \ldots, n\}$ in its expansion. If the former holds, then Q-deg $(f_{j_0})_{\max} \geq a\ell_{i_0} = \beta \deg f_{j_0} \geq \beta \deg f_{j_0} - w_1\epsilon_{\min}$. If the latter holds, then we have

$$\begin{aligned} \text{Q-deg } (f_{j_0})_{\max} &\geq a\ell_{i_0} + \ell_j = \beta(\deg f_{j_0} - w_j) + \ell_j \\ &\geq \beta \deg f_{j_0} - \beta w_j \geq \beta \deg f_{j_0} - w_1\epsilon_{\min}, \end{aligned}$$

where the last inequality follows from the fact that $w_j \leq w_1$ and $\beta \leq \epsilon_{i_0} = \epsilon_{\min}$ by Lemma 4.6.

We construct a sequence $j_0 < j_1 < \ldots$ as follows. If $j_0, j_1, \cdots, j_i$ are defined, by Proposition 3.2, then we have either $D_{\max}(f_{j_i})_{\max} = 0$ or $D_{\max}(f_{j_i})_{\max} = (Df_{j_i})_{\max}$. If the former holds, let the sequence end.

If the latter holds, by the j_i-th equation in (4.10), there is an index $j_{i+1} \in \{j_i+1, \ldots, m\}$ such that

$$\text{Q-deg}\left(\ell_{j_i}^{j_{i+1}} f_{j_{i+1}}\right)_{\max} = \text{Q-deg}\left(D_{\max}(f_{j_i})_{\max}\right). \tag{4.11}$$

Now we prove by induction that the sequence has the following property

$$\text{Q-deg}(f_{j_i})_{\max} \geq -i(\beta\, wt\, D + \epsilon_{\min}) + \beta \deg f_{j_i} - w_1\epsilon_{\min}. \tag{4.12}$$

We have proven that (4.12) holds for $i = 0$. Suppose the proposition holds for i, we shall validate it for $i+1$. By (4.11) and Proposition 3.1, we obtain

$$\text{Q-deg}\left(\ell_{j_i}^{j_{i+1}}\right)_{\max} + \text{Q-deg}(f_{j_{i+1}})_{\max} = -\epsilon_{\min} + \text{Q-deg}(f_{j_i})_{\max}.$$

Using the fact that $\deg f_{j_i} + wt\, D = \deg \ell_{j_i}^{j_{i+1}} + \deg f_{j_{i+1}}$ and $\beta \deg \ell_{j_i}^{j_{i+1}} \geq \text{Q-deg}(\ell_{j_i}^{j_{i+1}})_{\max}$, we get

$$\begin{aligned} &\text{Q-deg}(f_{j_{i+1}})_{\max} \\ =\ & -\epsilon_{\min} + \text{Q-deg}(f_{j_i})_{\max} - \text{Q-deg}(\ell_{j_i}^{j_{i+1}})_{\max} \\ \geq\ & -\epsilon_{\min} - i(\beta\, wt\, D + \epsilon_{\min}) + \beta \deg f_{j_i} - w_1\epsilon_{\min} - \text{Q-deg}(\ell_{j_i}^{j_{i+1}})_{\max} \\ =\ & -\epsilon_{\min} - i(\beta\, wt\, D + \epsilon_{\min}) + \beta(\deg \ell_{j_i}^{j_{i+1}} + \deg f_{j_{i+1}} - wt\, D) \\ & - w_1\epsilon_{\min} - \text{Q-deg}(\ell_{j_i}^{j_{i+1}})_{\max} \\ \geq\ & -(i+1)(\beta\, wt\, D + \epsilon_{\min}) + \beta \deg f_{j_{i+1}} - w_1\epsilon_{\min}. \end{aligned} \tag{4.13}$$

From (4.10), we have $Df_m = 0$. It follows from Corollary 3.1 that

$$D_{\max}(f_m)_{\max} = 0.$$

From this point of view and the fact that $j_i < j_{i+1}$, we find that the sequence ends within $(m-1)$ steps. That is to say, there is an index $t \in \{1, 2, \ldots, m-1\}$ such that

$$D_{\max}(f_{j_t})_{\max} = 0, \tag{4.14}$$

$$\text{Q-deg}(f_{j_t})_{\max} \geq -t(\epsilon_{\min} + \beta\, wt\, D) + \beta \deg f_{j_t} - w_1\epsilon_{\min}. \tag{4.15}$$

By Lemma 4.6, we have $(\epsilon_{\min} + \beta\, wt\, D) \geq 0$. Notice that $t \leq m-1$, we have

$$\text{Q-deg}(f_{j_t})_{\max} \geq -(m-1)(\epsilon_{\min} + \beta\, wt\, D) + \beta \deg f_{j_t} - w_1\epsilon_{\min}. \tag{4.16}$$

Assume that

$$\deg(f_{j_t})_{\max} > \frac{(m-1+w_1)\epsilon_{\min}}{\beta - \gamma}. \tag{4.17}$$

By the fact that $wt\, D < 0$, we have

$$\deg(f_{j_t})_{\max} > \frac{(m-1)(\epsilon_{\min} + \beta\, wt\, D) + w_1\epsilon_{\min}}{\beta - \gamma}. \tag{4.18}$$

By Lemma 4.4 (here $M = (m-1)(\epsilon_{\min} + \beta\, wt D) + w_1\epsilon_{\min}$ and notice that $\deg f_{j_t} = \deg(f_{j_t})_{\max}$) we know that $(f_{j_t})_{\max}$ is divisible by x_{i_0}. Since $\epsilon_{i_0} = \epsilon_{\min}$, Proposition 3.1 tells us that the coefficient of $\partial/\partial x_{i_0}$ in $D_{\max}$ is $(p_{i_0})_{\max}$. Since p_{i_0} is a non-zero polynomial, so $(p_{i_0})_{\max}$ is a non-zero polynomial. Thus, $D_{\max}(f_{j_t})_{\max} \neq 0$ by Lemma 4.5, which contradicts (4.14). Thus, the assumption (4.17) is false, and

$$\deg f_{j_t} = \deg(f_{j_t})_{\max} \leq \frac{(m-1+w_1)\epsilon_{\min}}{\beta - \gamma}.$$

The conclusion is proved. q.e.d.

Lemma 4.7. *If there exists a positive real number ε such that all parameters ϵ_i are divisible by ε, that is to say, $\epsilon_i = b_i\varepsilon$ where b_i is a non-negative integer for $i = 1, 2, \ldots, n$, then we have*

(i) *$\ell_i = q_i\varepsilon$, where q_i is a non-negative integer for $i = 1, 2, \ldots, n$;*

(ii) *For any $i, j \in \{1, 2, \ldots, n\}$, if $\ell_i/w_i > \ell_j/w_j$, then*

$$\ell_i/w_i - \ell_j/w_j \geq \varepsilon/(w_1 w_2).$$

Proof. (i) (By induction on i) If $i = n$, then the lemma holds, since $\ell_n = \epsilon_n = b_n\varepsilon$ by Definition 3.1. Suppose it also holds for $i = k+1, \ldots, n$, we prove it for $i = k$. If p_k is the zero polynomial, then the lemma holds obviously since $\ell_k = \epsilon_k$. Otherwise, for any term $x_{k+1}^{a_{k+1}} \ldots x_n^{a_n}$ in the expansion of p_k, we have

$$\text{Q-deg}\, x_{k+1}^{a_{k+1}} \ldots x_n^{a_n} = (a_{k+1}q_{k+1} + \cdots + a_n q_n)\varepsilon.$$

By Definition 3.1, we have

(4.19)

$$\begin{aligned}\ell_k &= \epsilon_k + \max\{\text{Q-degrees of monomials } x_{k+1}^{a_{k+1}} \ldots x_n^{a_n} \\ &\qquad\qquad \text{in the expansion of } p_k\} \\ &= \Big(b_k + \max\{a_{k+1}q_{k+1} + \cdots + a_n q_n : \\ &\qquad \text{the monomial } x_{k+1}^{a_{k+1}} \ldots, x_n^{a_n} \text{ appears in the expansion of } p_k\}\Big)\varepsilon.\end{aligned}$$

Thus the lemma for case $i = k$ holds.

(ii) By (i), we have

$$\ell_i/w_i - \ell_j/w_j = (\ell_i w_j - \ell_j w_i)/(w_i w_j) = (q_i w_j - q_j w_i)\varepsilon/(w_i w_j).$$

Notice that $\ell_i/w_i > \ell_j/w_j$, implies $q_i w_j - q_j w_i > 0$. Since $q_i w_j - q_j w_i$ is an integer, so $q_i w_j - q_j w_i \geq 1$. Recall the fact that $w_1 \geq w_2 \geq \cdots \geq w_n \geq 1$, we obtain that

$$\ell_i/w_i - \ell_j/w_j \geq \varepsilon/(w_i w_j) \geq \varepsilon/(w_1 w_2).$$

q.e.d.

We recall the following Lemma, which is a consequence of Corollary 3.4 in [Ro].

Lemma 4.8. *Let $(V,0)$ be a germ of positive dimensional isolated singularity, defined by $f_1, f_2, \dots, f_m \in \mathbb{C}[x_1, x_2, \dots, x_n]$. Let*

$$D = p_1\partial/\partial x_1 + p_2\partial/\partial x_2 + \cdots + p_n\partial/\partial x_n$$

be a holomorphic vector field on $(V,0)$. Then $p_i(0) = 0$ for $1 \leq i \leq n$.

Remark 4.1. Let $f_1, f_2, \dots, f_m$ be weighted homogeneous polynomials in P and they define a positive dimensional isolated singularity at the origin. Suppose that D is a non-zero negative weight derivation on $P/(f_1, f_2, \dots, f_m)$ as in (3.1). Suppose p_k for some k is not identically zero. From Lemma 4.8, we know that $p_k(0) = 0$, so the polynomial p_k cannot be constant. Thus, the (weighted) degree of p_k is positive. Since D is a negative weight derivation, so p_n is a constant polynomial. This implies p_n has to be the zero polynomial.

5. Proof of Main Theorem B

In this section, we first prove Main Theorem B. Main Theorem A is proved in next section.

Theorem 5.1 (Main Theorem B)**.** *Let $P = \mathbb{C}[x_1, x_2, \dots, x_n]$ be the weighted polynomial ring of n weighted variables $x_1, x_2, \dots, x_n$ with positive integer weights $w_1 \geq w_2 \geq \cdots \geq w_n$ $(n \geq 2)$ and $f_1, f_2, \dots, f_m$ be m weighted homogeneous polynomials of degrees greater than $(m-1+w_1)w_1w_2$. Suppose that any two of the original weights $w_1, w_2, \dots, w_n$ are coprime and $f_1, f_2, \dots, f_m$ define a positive dimensional isolated singularity at the origin. Then there are no non-zero negative weight derivations on $R = P/(f_1, f_2, \dots, f_m)$.*

Proof. (By contradiction) Suppose D is a non-zero negative weight derivation on R or equivalently a non-zero negative weight derivation on P which preserves the ideal $(f_1, f_2, \dots, f_m)$ as in (3.1). We take the new weight type $(\ell_1, \dots, \ell_n)$ of D controlled by parameters ϵ_i, where

$$\epsilon_i = \begin{cases} \epsilon, & p_i \text{ is a non-zero polynomial} \\ 0, & \text{otherwise} \end{cases},$$

with ϵ a positive real number and p_i the coefficient of $\partial/\partial x_i$ in D for $i = 1, 2, \dots, n$. Let $I_{\max} = \{e\colon \ell_e/w_e \text{ is the maximum among all } \ell_i/w_i \text{ for } i = 1, 2, \dots, n\}$. It is clear that $\ell_i > 0$ for any i such that p_i is a non-zero polynomial and $\ell_i = 0$ for any i such that p_i is the zero polynomial. Thus $\ell_i > 0$ and p_i is a non-zero polynomial for any $i \in I_{\max}$, which implies that $\epsilon_i = \epsilon$ for any $i \in I_{\max}$.

We claim that $I_{\max}$ has only one element. Since ϵ divides ϵ_i for all $i = 1, 2, \dots, n$, by Lemma 4.7, we have $\ell_i = q_i\epsilon$, where q_i is a non-negative integer for $i = 1, 2, \dots, n$. Now we prove that $q_i < w_i$ for all i by induction on i. If $i = n$, then we know p_n is the zero polynomial (see Remark 4.1), thus $l_n = \epsilon_n = 0$, which shows that $q_n = 0 < w_n$.

Suppose $q_i < w_i$ for $i = k+1, k+2, \ldots, n$, we prove that $q_k < w_k$. If p_k is the zero polynomial, then $l_k = \epsilon_k = 0$, which yields that $q_k = 0 < w_k$. If p_k is a non-zero polynomial, then $\epsilon_k = \epsilon$, thus $\ell_k = \epsilon_k + \text{Q-deg } p_k = \epsilon + \text{Q-deg } p_k$. By Lemma 4.8 we have $p_k(0) = 0$, thus p_k doesn't contain any constant term. Since p_k is a polynomial in $x_{k+1}, \ldots, x_n$ and $\ell_i = q_i\epsilon < w_i\epsilon$ for $i > k$, we have $\text{Q-deg } p_k < \epsilon \deg p_k$. Notice that $w_k = -\operatorname{wt} D + \deg p_k$, hence

$$\ell_k = \epsilon + \text{Q-deg } p_k < (1 + \deg p_k)\epsilon \leq (-\operatorname{wt} D + \deg p_k)\epsilon = w_k\epsilon,$$

which implies that $q_k < w_k$. Thus $q_i < w_i$ for $i = 1, 2, \ldots, n$. Since $\ell_i > 0$ for any $i \in I_{\max}$, we have $q_i > 0$ for any $i \in I_{\max}$. Suppose that $I_{\max}$ has more than one element, then for any $i, j \in I_{\max}$ such that $i \neq j$, $q_{i/w_i \neq q_j/w_j}$ since $0 < q_i < w_i$, $0 < q_j < w_j$ and w_i, w_j are coprime. It follows that $\ell_i/w_i \neq \ell_j/w_j$, which contradicts $i, j \in I_{\max}$. Thus the claim that $I_{\max}$ has only one element is proved.

Write $I = \{i_0\}$. Let $\beta = \ell_{i_0}/w_{i_0}$ and $\gamma = \max\{\ell_i/w_i : i = 1, 2, \ldots, i_0 - 1, i_0 + 1, \ldots, n\}$. Since ϵ divides ϵ_i for all i, by Lemma 4.7, we have $\beta - \gamma \geq \epsilon/(w_1 w_2)$. Let $\epsilon_{min} = \min\{\epsilon_i$ for i such that p_i is a non-zero polynomial$\}$ and it is clear that $\epsilon_{\min} = \epsilon$. Then by Theorem 4.1, we know that there exists $j \in \{1, 2, \ldots, m\}$ such that

$$\deg f_j \leq \frac{(m - 1 + w_1)\epsilon_{\min}}{\beta - \gamma} \leq (m - 1 + w_1)(w_1 w_2),$$

which contradicts the condition that $\deg f_j > (m - 1 + w_1)(w_1 w_2)$ for all j. So the conclusion is proved. q.e.d.

6. Proof of Main Theorem A

In this section we give the proof of Main Theorem A. In order to use Theorem 4.1, we need to choose suitable parameters ϵ_i to make the new weight type $(\ell_1, \ldots, \ell_n)$ satisfy the following conditions in Theorem 4.1:

(1) there is only one index $i_0 \in \{1, 2, \ldots, n\}$ such that $\ell_{i_0}/w_{i_0} = \max\{\ell_i/w_i : i = 1, 2, \ldots, n\}$;

(2) $\epsilon_{i_0} = \epsilon_{\min}$, where

$$\epsilon_{min} = \min\{\epsilon_i \text{ for } i \text{ such that } p_i \text{ is a non-zero polynomial}\};$$

(3) p_{i_0} is a non-zero polynomial.

First we let

$$\epsilon_i = \begin{cases} \epsilon, & p_i \text{ is a non-zero polynomial} \\ 0, & \text{otherwise} \end{cases},$$

where ϵ is a positive real number. Let $(\ell_1, \ldots, \ell_n)$ be the new weight type associated to a non-zero negative weight derivation D and controlled by parameters ϵ_i. Then we have $\epsilon_{min} = \epsilon$ and $\ell_i = 0$ for i such that p_i is the zero polynomial. Let $I_{\max} = \{e : \ell_e/w_e$ is the maximum among all ℓ_i/w_i for $i = 1, 2, \ldots, n\}$. It is easy to see that $\epsilon_i = \epsilon_{\min}$ and p_i is a non-zero polynomial for any $i \in I_{\max}$. Thus if $I_{\max}$ has only one element then the conditions (1)–(3) in Theorem 4.1 are satisfied. But in general $I_{\max}$ might have more than one element. So the key step is to adjust the parameters ϵ_i in order to separate $\{\ell_i/w_i \colon i \in I_{\max}\}$ such that these numbers have only one maximum.

Lemma 6.1. *Let D be a non-zero negative weight derivation such that $p_i(0) = 0$ for $1 \leq i \leq n$, where p_i is the coefficient of $\partial/\partial x_i$ in D. Suppose there exists a positive real number ε such that all parameters ϵ_i are divisible by ε. Fix an index $j_0 \in \{1, 2, \ldots, n\}$, define another group of parameters ϵ'_i as follows,*

$$\epsilon'_i = \begin{cases} \epsilon_i + \varepsilon/(w_1 w_2), & i = j_0 \\ \epsilon_i, & i \neq j_0 \end{cases}.$$

Let $(\ell_1, \ldots, \ell_n)$ and $(\ell'_1, \ldots, \ell'_n)$ be new weight type associated to D and controlled by parameters ϵ_i and ϵ'_i respectively, then we have

(i) *For any $i, j = 1, 2, \ldots, n$ such that both p_i and p_j are non-zero polynomials, we have*

$$\ell_i/w_i < \ell_j/w_j \Rightarrow \ell'_i/w_i < \ell'_j/w_j.$$

(ii) *For any $i, j = 1, 2, \ldots, n$ such that both p_i and p_j are non-zero polynomials, then for any term t_i and t_j in the expansion of p_i and p_j, respectively, we have*

$$(Q\text{-}deg\ t_i + \epsilon_i)/w_i < (Q\text{-}deg\ t_j + \epsilon_j)/w_j$$
$$\Rightarrow (Q'\text{-}deg\ t_i + \epsilon'_i)/w_i < (Q'\text{-}deg\ t_j + \epsilon'_j)/w_j.$$

(iii) *For any $i = 1, 2, \ldots, n$ such that p_i is a non-zero polynomial, then for any terms t_1 and t_2 in the expansion of p_i, we have*

$$Q\text{-}deg\ t_1 < Q\text{-}deg\ t_2 \Rightarrow Q'\text{-}deg\ t_1 < Q'\text{-}deg\ t_2,$$

where Q-deg and Q'-deg denote the degrees with respect to the new weight type $(\ell_1, \ldots, \ell_n)$ and $(\ell'_1, \ldots, \ell'_n)$, respectively.

Proof. We claim that $0 \leq \ell'_i - \ell_i \leq w_i\varepsilon/(w_1w_2)$ for all i, and $0 \leq \ell'_i - \ell_i < w_i\varepsilon/(w_1w_2)$ for i such that p_i is a non-zero polynomial.

(By induction) If $i = n$, then $\ell_n = \epsilon_n$, $\ell'_n = \epsilon'_n$ and p_n is the zero polynomial. By the definition of ϵ'_i, we know that $0 \leq \epsilon'_n - \epsilon_n \leq \varepsilon/(w_1w_2) \leq w_n\varepsilon/(w_1w_2)$, thus the claim holds for $i = n$.

Suppose the claim holds for $i = k+1, \ldots, n$, we prove it holds for $i = k$. There are the following two cases:

(1) p_k is the zero polynomial. The claim holds due to the same argument as in that of $i = n$.

(2) p_k is a non-zero polynomial, since $p_k(0) = 0$, so there is no constant term in the expansion of p_k. There are two subcases:

(2a) $k = j_0$. Pick any $s > k$. If p_s is a non-zero polynomial, then by the assumption, we have $0 \leq \ell'_s - \ell_s < w_s\varepsilon/(w_1w_2)$. Otherwise, $\ell_s = \epsilon_s$

and $\ell'_s = \epsilon'_s$. Notice that $s \neq k = j_0$, we have $\epsilon_s = \epsilon'_s$, which yields that $\ell'_s - \ell_s = 0 < w_s\varepsilon/(w_1w_2)$. Thus $0 \leq \ell'_s - \ell_s < w_s\varepsilon/(w_1w_2)$ for all $s > k$. For any term $t = x_{k+1}^{a_{k+1}} \dots x_n^{a_n}$ in the expansion of p_k ($a_{k+1}, \dots, a_n$ are not all zero), using the fact that $a_{k+1}w_{k+1} + \cdots + a_nw_n = w_k + wt\,D$, we have

$$\begin{aligned}0 \leq \mathrm{Q}\,'\text{-deg } t - \mathrm{Q}\text{-deg } t &= a_{k+1}(\ell'_{k+1} - \ell_{k+1}) + \cdots + a_n(\ell'_n - \ell_n)\\ &< (a_{k+1}w_{k+1} + \cdots + a_nw_n)\varepsilon/(w_1w_2) = (w_k + wt\,D)\varepsilon/(w_1w_2).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{Q}\,'\text{-deg } t <&\mathrm{Q}\text{-deg } t + (w_k + wt\,D)\varepsilon/(w_1w_2)\\ \leq&\mathrm{Q}\text{-deg } p_k + (w_k + wt\,D)\varepsilon/(w_1w_2),\end{aligned}$$

for any term t in the expansion of p_k. Thus, it follows that

$$\mathrm{Q}\,'\text{-deg } p_k < \mathrm{Q}\text{-deg } p_k + (w_k + wt\,D)\varepsilon/(w_1w_2). \tag{6.1}$$

Since

$$\mathrm{Q}\text{-deg } t \leq \mathrm{Q}\,'\text{-deg } t \leq \mathrm{Q}\,'\text{-deg } p_k,$$

for any term t in the expansion of p_k, we have

$$\mathrm{Q}\text{-deg } p_k \leq \mathrm{Q}\,'\text{-deg } p_k. \tag{6.2}$$

Combining (6.1) and (6.2), we obtain that

$$0 \leq \mathrm{Q}\,'\text{-deg } p_k - \mathrm{Q}\text{-deg } p_k < (w_k + wt\,D)\varepsilon/(w_1w_2).$$

By definition, we have $\epsilon'_k - \epsilon_k = \varepsilon/(w_1w_2)$. Thus

$$0 \leq \ell'_k - \ell_k = \epsilon'_k + \mathrm{Q}\,'\text{-deg } p_k - (\epsilon_k + \mathrm{Q}\text{-deg } p_k) < (w_k + wt\,D + 1)\varepsilon/(w_1w_2).$$

Since $wt\,D$ is a negative integer, we have $wt\,D + 1 \leq 0$, and the claim is proved.

(2b) $k \neq j_0$, so $\epsilon'_k = \epsilon_k$. For any term $t = x_{k+1}^{a_{k+1}} \dots x_n^{a_n}$ in the expansion of p_k ($a_{k+1}, \dots, a_n$ are not all zero), using the fact that $a_{k+1}w_{k+1} + \cdots + a_nw_n = w_k + wt\,D$ and the assumption that $0 \leq \ell'_s - \ell_s \leq w_s\varepsilon/(w_1w_2)$ for $s > k$, we have

$$\begin{aligned}0 \leq \mathrm{Q}\,'\text{-deg } t - \mathrm{Q}\text{-deg } t &= a_{k+1}(\ell'_{k+1} - \ell_{k+1}) + \cdots + a_n(\ell'_n - \ell_n)\\ &\leq (a_{k+1}w_{k+1} + \cdots + a_nw_n)\varepsilon/(w_1w_2) = (w_k + wt\,D)\varepsilon/(w_1w_2).\end{aligned}$$

Therefore, it gives

$$\begin{aligned}\mathrm{Q}\,'\text{-deg } t \leq&\mathrm{Q}\text{-deg } t + (w_k + wt\,D)\varepsilon/(w_1w_2)\\ \leq&\mathrm{Q}\text{-deg } p_k + (w_k + wt\,D)\varepsilon/(w_1w_2),\end{aligned}$$

for any term t in the expansion of p_k. Thus, it yields

$$\mathrm{Q}\,'\text{-deg } p_k \leq \mathrm{Q}\text{-deg } p_k + (w_k + wt\,D)\varepsilon/(w_1w_2). \tag{6.3}$$

Since

$$\mathrm{Q}\text{-deg } t \leq \mathrm{Q}\,'\text{-deg } t \leq \mathrm{Q}\,'\text{-deg } p_k$$

for any term t in the expansion of p_k, we get

$$\text{Q-deg } p_k \le \text{Q }'\text{-deg } p_k. \tag{6.4}$$

Combining (6.3) and (6.4), we have

$$0 \le \text{Q }'\text{-deg } p_k - \text{Q-deg } p_k \le (w_k + wt\,D)\varepsilon/(w_1w_2).$$

Since $\epsilon'_k - \epsilon_k = 0$, by the definition of the new weight type, we have $0 \le \ell'_k - \ell_k \le (w_k + wt\,D)\varepsilon/(w_1w_2)$. Since $wt\,D$ is a negative integer, we have $w_k + wt\,D < w_k$. Thus $0 \le \ell'_k - \ell_k < w_k\varepsilon/(w_1w_2)$ and the claim is proved.

From the argument above, we also know that for any i such that p_i is a non-zero polynomial and for any term t in the expansion of p_i, we have

$$0 \le (\text{Q }'\text{-deg } t + \epsilon'_i) - (\text{Q-deg } t + \epsilon_i) < w_i\varepsilon/(w_1w_2). \tag{6.5}$$

(i) For any i, j such that both p_i and p_j are non-zero polynomials, if $\ell_i/w_i < \ell_j/w_j$, by Lemma 4.7, we have $\ell_j/w_j - \ell_i/w_i \ge \varepsilon/(w_1w_2)$. By the claim above, we have $0 \le \ell'_i/w_i - \ell_i/w_i < w_i\varepsilon/(w_1w_2w_i) = \varepsilon/(w_1w_2)$ and $0 \le \ell'_j/w_j - \ell_j/w_j$. Combining these inequalities, we have $\ell'_i/w_i < \ell'_j/w_j$. Thus (i) is proved.

(ii) For any i, j such that both p_i and p_j are non-zero polynomials and for any term t_i and t_j in the expansion of p_i and p_j, respectively, by Lemma 4.7, we know that all ℓ_k, $k = 1, \cdots, n$ are divisible by ε. Thus Q-deg $t_i + \epsilon_i$ and Q-deg $t_j + \epsilon_j$ are divisible by ε. Let us write Q-deg $t_i + \epsilon_i$ and Q-deg $t_j + \epsilon_j$ as the forms $q_i\varepsilon$ and $q_j\varepsilon$ respectively where q_i and q_j are integers. If $(\text{Q-deg } t_i + \epsilon_i)/w_i < (\text{Q-deg } t_j + \epsilon_j)/w_j$, then $q_iw_j < q_jw_i$. Notice that q_iw_j and q_jw_i are integers, so $q_jw_i - q_iw_j \ge 1$. Thus $(\text{Q-deg } t_j + \epsilon_j)/w_j - (\text{Q-deg } t_i + \epsilon_i)/w_i = (q_jw_i - q_iw_j)\varepsilon/(w_iw_j) \ge \varepsilon/(w_1w_2)$. By (6.5), we have $(\text{Q }'\text{-deg } t_i + \epsilon'_i)/w_i - (\text{Q-deg } t_i + \epsilon_i)/w_i < w_i\varepsilon/(w_1w_2w_i) = \varepsilon/(w_1w_2)$. Combining the two previous inequalities and notice that $\text{Q-deg } t_j + \epsilon_j \le \text{Q }'\text{-deg } t_j + \epsilon'_j$, again by (6.5), we have

$$(\text{Q }'\text{-deg } t_i + \epsilon'_i)/w_i < (\text{Q-deg } t_j + \epsilon_j)/w_j \le (\text{Q }'\text{-deg } t_j + \epsilon'_j)/w_j.$$

(iii) Using (ii) for case $i = j$, we obtain that for any i such that p_i is a non-zero polynomial and for any terms t_1 and t_2 in the expansion of p_i, we have

$$\begin{aligned}&(\text{Q-deg } t_1 + \epsilon_i)/w_i < (\text{Q-deg } t_2 + \epsilon_i)/w_i\\ \Rightarrow\ &(\text{Q }'\text{-deg } t_1 + \epsilon'_i)/w_i < (\text{Q }'\text{-deg } t_2 + \epsilon'_i)/w_i.\end{aligned}$$

Thus, we have

$$\text{Q-deg } t_1 < \text{Q-deg } t_2 \Rightarrow \text{Q }'\text{-deg } t_1 < \text{Q }'\text{-deg } t_2.$$

q.e.d.

Theorem 6.1. *Let $f_1, f_2, \ldots, f_m$ be m weighted homogeneous polynomials in P with respect to a weight type $(w_1, w_2, \ldots, w_n)$. Suppose these polynomials define a positive dimensional isolated singularity at the origin. Let D be a non-zero negative weight derivation as in (3.1) on P preserving the ideal $(f_1, \ldots, f_m)$. Let $(\ell_1, \ldots, \ell_n)$ be the new weight type associated to D and controlled by parameters ϵ_i. Fix a subset I of $\{1, 2, \ldots, n\}, (n \geq 2)$ containing more than one element. Suppose the parameters $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ satisfy the conditions that*

(6.6)

$$\epsilon_i = \begin{cases} \epsilon, & i \in I \text{ and } p_i \text{ is a non-zero polynomial} \\ 0, & i \in I \text{ and } p_i \text{ is the zero polynomial} \\ \epsilon + \epsilon/(w_1 w_2)^{b_i}, & i \notin I \text{ and } p_i \text{ is a non-zero polynomial} \\ \epsilon/(w_1 w_2)^{b_i}, & i \notin I \text{ and } p_i \text{ is the zero polynomial} \end{cases},$$

where ϵ is a positive real number, k is the number of elements in I ($k \geq 2$), and $b : i \mapsto b_i$ is an one-to-one map from $\{1, 2, \ldots, n\} \setminus I$ to $\{1, 2, \ldots, n-k\}$. Let $I_{\max} = \{e : \ell_e/w_e$ is the maximum among all ℓ_i/w_i for $i = 1, 2, \ldots, n\}$. If $I_{\max} \subseteq I$ and p_i is a non-zero polynomial for any $i \in I_{\max}$, then there exists $j \in \{1, 2, \ldots, m\}$ such that

$$\deg f_j \leq (m - 1 + w_1)(w_1 w_2)^{n-1}.$$

Proof. Consider the case that $I_{\max} = I$, which can be easily reduced to the case that $I_{\max}$ is a proper subset of I. In fact, assume that $I_{\max} = I$. Let us write $I = I_{\max} = \{i_1, \ldots, i_k\}$, where $i_1 < i_2 < \cdots < i_k$, $k \geq 2$. Since p_i is a non-zero polynomial for any $i \in I_{\max} = I$, by (6.6), we can see that $\epsilon_i = \epsilon$ for any $i \in I_{\max} = I$.

Before proceeding to give the proof of Theorem 6.1, we shall first prove the following proposition.

Proposition 6.1. *For any term t_0 in the expansion of $(p_{i_{k-1}})_{\max}$ and for any term t_1 in the expansion of $(p_{i_k})_{\max}$, we have $t_0 = c x_{i_k}^a t_1$, with non-negative integer a and non-zero constant coefficient c.*

Proof. Let $h : \{1, 2, \ldots, n-k\} \to \{1, \ldots, n\} \setminus I$ be the inverse function of the map $b : i \mapsto b_i$. That is, $b_{h(i)} = i$ for $i = 1, 2, \ldots, n-k$. Define a group of parameters $\epsilon_i^{(0)}, \epsilon_i^{(1)}, \ldots, \epsilon_i^{(n-k)}$, $i = 1, \cdots, n$ recursively:

$$\epsilon_i^{(0)} = \begin{cases} \epsilon, & p_i \text{ is a non-zero polynomial} \\ 0, & \text{otherwise} \end{cases}.$$

Assume that the $(j-1)$-th group of parameters $(\epsilon_1^{(j-1)}, \ldots, \epsilon_n^{(j-1)})$ has been defined, then we define

$$\epsilon_i^{(j)} = \begin{cases} \epsilon_i^{(j-1)} + \epsilon/(w_1 w_2)^j, & i = h(j) \\ \epsilon_i^{(j-1)}, & i \neq h(j) \end{cases}.$$

By this definition, on the one hand it is clear that $\epsilon_i^{(j)} = \epsilon$ for any $i \in I_{\max} = I$ and any $j = 0, 1, \ldots, n-k$. In particular, $\epsilon_i^{(n-k)} = \epsilon = \epsilon_i$

for $i \in I_{\max} = I$. On the other hand, for $i \notin I_{\max}$, there exists a unique $j \in \{1, \cdots, n-k\}$ such that $h(j) = i$, hence $b_i = j$. It follows from definition that

$$\epsilon_i^{(n-k)} = \epsilon_i^j = \epsilon_i^{(j-1)} + \epsilon/(w_1 w_2)^{b_i} = \epsilon_i^{(0)} + \epsilon/(w_1 w_2)^{b_i} = \epsilon_i.$$

Thus $(\epsilon_1^{(n-k)}, \ldots, \epsilon_n^{(n-k)}) = (\epsilon_1, \ldots, \epsilon_n)$. Let $(\ell_1^{(j)}, \ldots, \ell_n^{(j)})$ be the new weight type controlled by parameters $(\epsilon_1^{(j)}, \ldots, \epsilon_n^{(j)})$ for $j = 0, 1, \ldots, n-k$, and Q(j)-deg means the associated degree. For convenience, we write $i_{k-1} = s$ and $i_k = t$, then $s < t$. Since $s, t \in I_{\max} = I$, so p_s and p_t are not zero polynomials. Pick any term t_0 in the expansion of $(p_s)_{\max}$ and pick any term t_1 in the expansion of $(p_t)_{\max}$. Notice that $s, t \in I_{\max}$, we have $\ell_s/w_s = \ell_t/w_t$, thus (Q-deg $t_0 + \epsilon_s)/w_s$ = (Q-deg $t_1 + \epsilon_t)/w_t$. Since $(\epsilon_1, \ldots, \epsilon_n) = (\epsilon_1^{(n-k)}, \ldots, \epsilon_n^{(n-k)})$, we have

$$\ell_s^{(n-k)}/w_s = \ell_t^{(n-k)}/w_t, \tag{6.7}$$

and

$$(\text{Q(n-k)-deg } t_0 + \epsilon_s^{(n-k)})/w_s = (\text{Q(n-k)-deg } t_1 + \epsilon_t^{(n-k)})/w_t. \tag{6.8}$$

We claim that

$$\ell_s^{(j)}/w_s = \ell_t^{(j)}/w_t, \tag{6.9}$$

for $j = 0, 1, \ldots, n-k$. Suppose that there exists e such that $\ell_s^{(e)}/w_s \neq \ell_t^{(e)}/w_t$, notice that both p_s and p_t are not zero polynomials, by Lemma 6.1(i) (here we set $\varepsilon = \epsilon/(w_1 w_2)^e$) we have $\ell_s^{(e+1)}/w_s \neq \ell_t^{(e+1)}/w_t$. Similarly, $\ell_s^{(e+1)}/w_s \neq \ell_t^{(e+1)}/w_t$ implies $\ell_s^{(e+2)}/w_s \neq \ell_t^{(e+2)}/w_t$. Continuing this process, it implies that $\ell_s^{(n-k)}/w_s \neq \ell_t^{(n-k)}/w_t$, which contradicts (6.7). Hence (6.9) is proved. Similarly, using Lemma 6.1(ii) and (6.8), we have

$$(\text{Q(j)-deg } t_0 + \epsilon_s^{(j)})/w_s = (\text{Q(j)-deg } t_1 + \epsilon_t^{(j)})/w_t. \tag{6.10}$$

Since $\epsilon_s^{(j)} = \epsilon_t^{(j)} = \epsilon$ for $j = 0, \ldots, n-k$, (6.10) implies

$$(\text{Q(j)-deg } t_0 + \epsilon)/w_s = (\text{Q(j)-deg } t_1 + \epsilon)/w_t, \tag{6.11}$$

for $j = 0, 1, \ldots, n-k$. We claim that t_0 is independent of x_i for $i = s+1, \ldots, t-1$. Suppose not, then there exists $e \in \{s+1, \ldots, t-1\}$ such that t_0 depends on x_e. Let $j = b_e$, then $h(j) = e$. Thus by definition we have $\epsilon_i^{(j-1)} = \epsilon_i^{(j)}$ for $i \neq e$ and $\epsilon_e^{(j-1)} < \epsilon_e^{(j)}$, which implies that $\ell_i^{(j-1)} = \ell_i^{(j)}$ for $i > e$, $\ell_e^{(j-1)} < \ell_e^{(j)}$, and $\ell_i^{(j-1)} \leq \ell_i^{(j)}$ for $i < e$. Notice that t_1 is a monomial in $x_{t+1}, \ldots, x_n$ only, and $t+1 > e$, we have

$$\text{Q(j-1)-deg } t_1 = \text{Q(j)-deg } t_1. \tag{6.12}$$

Notice that t_0 depends on x_e, we have

$$\text{Q(j-1)-deg } t_0 < \text{Q(j)-deg } t_0. \tag{6.13}$$

Since (6.12) and (6.13) contradict (6.11), the claim that t_0 is independent of x_i for $i = s+1, \dots, t-1$ is proved. So t_0 can be written as the form $cx_t^a t_2$, where t_2 is a monomial in $x_{t+1}, \dots, x_n$, a is a non-negative integer and c is a constant coefficient.

Next, we will prove $t_2 = t_1$ up to a scale by two steps. Let t_1 and t_2 be written as $c_1 x_{t+1}^{a_{t+1}} \dots x_n^{a_n}$ and $c_2 x_{t+1}^{b_{t+1}} \dots x_n^{b_n}$, respectively.

Step 1: We first prove that

$$a_i/b_i = (\deg t_1 - wt\, D)/(\deg t_2 - wt\, D), \tag{6.14}$$

for $i = t+1, \dots, n$.

Since the term t_1 appears in the expansion of $(p_t)_{\max}$, for any term g in the expansion of p_t we have Q-deg $t_1 \geq$ Q-deg g, i.e. Q(n-k)-deg $t_1 \geq$ Q(n-k)-deg g. Using Lemma 6.1(iii), we obtain Q(j)-deg $t_1 \geq$ Q(j)-deg g for any $j = 0, 1, \dots, n-k$ and for any term g in the expansion of p_t. Thus we have

$$\ell_t^{(j)} = \text{Q(j)-deg } t_1 + \epsilon, \tag{6.15}$$

for $j = 0, 1, \dots, n-k$. By (6.11), (6.15) and the facts that $w_s = \deg t_0 - wt\, D$, $w_t = \deg t_1 - wt\, D$ and $t_0 = cx_t^a t_2$, we have

$$\frac{\text{Q(j)-deg } t_0 + \epsilon}{\deg t_0 - wt\, D} = \frac{a\ell_t^{(j)} + \text{Q(j)-deg } t_2 + \epsilon}{aw_t + \deg t_2 - wt\, D} = \frac{\text{Q(j)-deg } t_1 + \epsilon}{\deg t_1 - wt\, D} = \frac{\ell_t^{(j)}}{w_t},$$

for $j = 0, 1, \dots, n-k$. It implies that

$$\frac{\text{Q(j)-deg } t_2 + \epsilon}{\deg t_2 - wt\, D} = \frac{\text{Q(j)-deg } t_1 + \epsilon}{\deg t_1 - wt\, D}, \tag{6.16}$$

for $j = 0, 1, \dots, n-k$. We prove the claim that $a_i/b_i = (\deg t_1 - wt\, D)/(\deg t_2 - wt\, D)$ for $i = t+1, \dots, n$ by induction. If $i = t+1$, let $j = b_{t+1}$, then $h(j) = t+1$. Thus we have $\ell_{t+1}^{(j)} - \ell_{t+1}^{(j-1)} > 0$ and $\ell_{t+2}^{(j)} - \ell_{t+2}^{(j-1)} = \dots = \ell_n^{(j)} - \ell_n^{(j-1)} = 0$. Consequently, we obtain

$$\text{Q(j)-deg } t_1 = \text{Q(j-1)-deg } t_1 + a_{t+1}(\ell_{t+1}^{(j)} - \ell_{t+1}^{(j-1)}),$$

and

$$\text{Q(j)-deg } t_2 = \text{Q(j-1)-deg } t_2 + b_{t+1}(\ell_{t+1}^{(j)} - \ell_{t+1}^{(j-1)}).$$

By (6.16), one gets $a_{t+1}/b_{t+1} = (\deg t_1 - wt\, D)/(\deg t_2 - wt\, D)$, thus the claim holds for $t+1$.

Suppose (6.14) holds for $t+1, t+2, \dots, i-1$, let us verify it for i. Let $j = b_i$, then $h(j) = i$, thus we have $\ell_i^{(j)} - \ell_i^{(j-1)} > 0$ and $\ell_{i+1}^{(j)} - \ell_{i+1}^{(j-1)} = \dots = \ell_n^{(j)} - \ell_n^{(j-1)} = 0$. Consequently, it gives

$$\text{Q(j)-deg } t_1 = \text{Q(j-1)-deg } t_1 + a_{t+1}(\ell_{t+1}^{(j)} - \ell_{t+1}^{(j-1)}) + \dots + a_i(\ell_i^{(j)} - \ell_i^{(j-1)}),$$

and

$$\text{Q(j)-deg } t_2 = \text{Q(j-1)-deg } t_2 + b_{t+1}(\ell_{t+1}^{(j)} - \ell_{t+1}^{(j-1)}) + \dots + b_i(\ell_i^{(j)} - \ell_i^{(j-1)}).$$

By assumption and (6.16), we obtain $a_i/b_i = (\deg t_1 - wt\,D)/(\deg t_2 - wt\,D)$.

Step 2: We shall prove that $\deg t_1 - wt\,D = \deg t_2 - wt\,D$. Assume that $\deg t_1 - wt\,D > \deg t_2 - wt\,D$, then by (6.14), we have $a_i > b_i$ for $i = t+1, \dots, n$. Let $t_3 = x_{t+1}^{a_{t+1}-b_{t+1}} \dots x_n^{a_n-b_n}$, then $t_1 = t_2t_3$ up to a scale. Consequently, one gets

$$\frac{\ell_t}{w_t} = \frac{\text{Q-deg } t_1 + \epsilon}{\deg t_1 - wt\,D} = \frac{\text{Q-deg } t_2 + \text{Q-deg } t_3 + \epsilon}{\deg t_2 + \deg t_3 - wt\,D}. \tag{6.17}$$

By the fact that $(\epsilon_1^{(n-k)}, \dots, \epsilon_n^{(n-k)}) = (\epsilon_1, \dots, \epsilon_n)$ and (6.16) for $j = n-k$, we have

$$\frac{\text{Q-deg } t_1 + \epsilon}{\deg t_1 - wt\,D} = \frac{\text{Q-deg } t_2 + \epsilon}{\deg t_2 - wt\,D}. \tag{6.18}$$

By (6.17) and (6.18), we obtain $\ell_t/w_t = \text{Q-deg } t_3/\deg t_3$. Since $t \in I_{\max}$ and $t+1, \dots, n \notin I_{\max}$, we have $\ell_t/w_t > \ell_{t+1}/w_{t+1}, \dots, \ell_t/w_t > \ell_n/w_n$. Since t_3 is a monomial of $x_{t+1}, \dots, x_n$, we have $\text{Q-deg } t_3/\deg t_3 < \ell_t/w_t$, which contradicts $\ell_t/w_t = \text{Q-deg } t_3/\deg t_3$. Therefore, the assumption $\deg t_1 - wt\,D > \deg t_2 - wt\,D$ is invalid. Similarly we can prove the assumption $\deg t_1 - wt\,D < \deg t_2 - wt\,D$ is invalid. Thus $\deg t_1 - wt\,D = \deg t_2 - wt\,D$. It implies that $a_i = b_i$ for $i = t+1, \dots, n$, thus $t_1 = t_2$ up to a scale. Thus Proposition 6.1 is proved. q.e.d.

Now we come back to the proof of Theorem 6.1.

Fix a term t_0 in the expansion of $(p_{i_{k-1}})_{\max}$. For any two terms t_1, t_2 in the expansion of $(p_{i_k})_{\max}$, by Proposition 6.1, we have $t_0 = c_1 x_{i_k}^{a_1} t_1$ and $t_0 = c_2 x_{i_k}^{a_2} t_2$, where c_1, c_2 are non-zero constant coefficients and a_1, a_2 are non-negative integers. Therefore, $c_1 x_{i_k}^{a_1} t_1 = c_2 x_{i_k}^{a_2} t_2$. Notice that t_1, t_2 are monomials of variables $x_{i_k+1}, \dots, x_n$, so $t_1 = t_2$ up to a scale. Therefore, there is only one term in the expansion of $(p_{i_k})_{\max}$.

Fix a term t_2 in the expansion of $(p_{i_k})_{\max}$. For any two terms t_0, t_1 in the expansion of $(p_{i_{k-1}})_{\max}$, by Proposition 6.1, we have $t_0 = c_0 x_{i_k}^{a_0} t_2$ and $t_1 = c_1 x_{i_k}^{a_1} t_2$, where c_0, c_1 are non-zero constant coefficients and a_0, a_1 are non-negative integers. Since $p_{i_{k-1}}$ is a weighted homogeneous polynomial with respect to the original weight type $(w_1, w_2, \cdots, w_n)$, $\deg t_0 = \deg t_1$, thus $a_0 = a_1$. So $t_0 = t_1$ up to a scale. It follows that there is only one term in the expansion of $(p_{i_{k-1}})_{\max}$. Hence,

$$(p_{i_{k-1}})_{\max} = c x_{i_k}^a (p_{i_k})_{\max},$$

where c is a non-zero constant coefficient and a is a non-negative integer. Notice that $\deg (p_{i_{k-1}})_{\max} = \deg p_{i_{k-1}} = w_{i_{k-1}} + wt\,D$ and $\deg (p_{i_k})_{\max} = \deg p_{i_k} = w_{i_k} + wt\,D$, we have $w_{i_{k-1}} + wt\,D = a w_{i_k} + w_{i_k} + wt\,D$, which yields that

$$w_{i_{k-1}} = (a+1) w_{i_k}. \tag{6.19}$$

Since $i_{k-1}, i_k \in I_{\max}$, $\ell_{i_{k-1}}/w_{i_{k-1}} = \ell_{i_k}/w_{i_k}$. Thus, we obtain

$$\ell_{i_{k-1}} = (a+1)\ell_{i_k}. \tag{6.20}$$

In the sequel, we shall make a coordinate change which preserves the original weight type $(w_1, w_2, \ldots, w_n)$. The coordinate change is of the following form

$$\begin{aligned} x_1 &= x'_1, \\ &\cdots\cdots \\ x_{i_{k-1}} &= x'_{i_{k-1}} + c(x'_{i_k})^{a+1}/(a+1), \\ &\cdots\cdots \\ x_n &= x'_n. \end{aligned} \tag{6.21}$$

We obtain the transformation of derivations in this coordinate change (6.21) as follows:

$$\begin{aligned} \frac{\partial}{\partial x'_1} &= \frac{\partial}{\partial x_1}, \\ &\cdots\cdots \\ \frac{\partial}{\partial x'_{i_{k-1}}} &= \frac{\partial}{\partial x_{i_{k-1}}}, \\ \frac{\partial}{\partial x'_{i_k}} &= \frac{\partial}{\partial x_{i_k}} + c(x'_{i_k})^a \frac{\partial}{\partial x_{i_{k-1}}}, \\ &\cdots\cdots \\ \frac{\partial}{\partial x'_n} &= \frac{\partial}{\partial x_n}. \end{aligned} \tag{6.22}$$

If the expression of the negative weight derivation D is written in the new coordinate system, say

$$D' = p'_1 \frac{\partial}{\partial x'_1} + p'_2 \frac{\partial}{\partial x'_2} + \cdots + p'_n \frac{\partial}{\partial x'_n}.$$

It is clear that $p'_t = p_t$ for $t \neq i_{k-1}$, and

$$p'_{i_{k-1}} = p_{i_{k-1}} - c(x'_{i_k})^a p_{i_k} = p_{i_{k-1}} - c x^a_{i_k} p_{i_k}.$$

Let $(\ell'_1, \ldots, \ell'_n)$ be the new weight type associated to D' in the new coordinate system and controlled by the original parameters $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ and Q $'$-deg means the associated degree. For any $t > i_{k-1}$, we have $p'_t = p_t$ and p_t is independent of $x_{i_{k-1}}$, thus the expression of p_t in the original coordinate system is the same as that of p'_t in the new coordinate system, since the coordinate change only occurs on $x_{i_{k-1}}$, which implies that $\ell'_t = \ell_t$ for all $t > i_{k-1}$. We claim that $\ell'_{i_{k-1}} < \ell_{i_{k-1}}$. Since $(p_{i_{k-1}})_{\max} = c x^a_{i_k} (p_{i_k})_{\max}$, we have either $(p_{i_{k-1}} - c x^a_{i_k} p_{i_k})$ is the zero polynomial or Q-deg $(p_{i_{k-1}} - c x^a_{i_k} p_{i_k}) <$ Q-deg $p_{i_{k-1}}$. If the former holds, then $p'_{i_{k-1}}$ is the zero polynomial and it is clear that $\ell'_{i_{k-1}} < \ell_{i_{k-1}}$. If the latter holds, notice that $p'_{i_{k-1}} = p_{i_{k-1}} - cx^a_{i_k} p_{i_k}$ is a polynomial in x_t for $t > i_{k-1}$ and $\ell'_t = \ell_t$ for $t > i_{k-1}$, we have Q $'$-deg $p'_{i_{k-1}} =$ Q-deg $(p_{i_{k-1}} - cx^a_{i_k} p_{i_k})$, thus Q $'$-deg $p'_{i_{k-1}} <$ Q-deg $p_{i_{k-1}}$, which implies $\ell'_{i_{k-1}} < \ell_{i_{k-1}}$. Now we claim that $\ell'_t \le \ell_t$ for all $t = 1, 2, \dots, n$ and we shall prove it by induction. From the above argument, the inequality holds for $t \ge i_{k-1}$. Assume the claim holds for $t+1, t+2, \dots, n$, and we shall show it holds for $t < i_{k-1}$. For any term $g = x^{a_{t+1}}_{t+1} \dots x^{a_n}_n$ in the expansion of p_t, then

$$g = (x'_{t+1})^{a_{t+1}} \dots \left(x'_{i_{k-1}} + \frac{c}{a+1}(x'_{i_k})^{a+1}\right)^{a_{i_{k-1}}} \dots (x'_n)^{a_n}$$

in the new coordinate system. By the fact that Q $'$-deg $(x'_{i_k})^{a+1} = (a+1)\ell'_{i_k} = (a+1)\ell_{i_k} = \ell_{i_{k-1}}$ due to (6.20), we obtain Q $'$-deg $g \le$ Q-deg g for any term g in the expression of p_t. Since $p'_t = p_t$, $t < i_{k-1}$, we obtain Q $'$-deg $p'_t \le$ Q-deg p_t, which yields that $\ell'_t \le \ell_t$ and the claim is proved.

Let $I'_{\max} = \{e\colon \ell'_e/w_e$ is the maximum among all ℓ'_i/w_i for $i = 1, 2, \dots, n\}$. From the above argument, we know that for any $i \notin I_{\max}$, $\ell'_i/w_i \le \ell_i/w_i < \ell_{i_k}/w_{i_k} = \ell'_{i_k}/w_{i_k}$, which implies that $i \notin I'_{\max}$. Thus $I'_{\max} \subseteq I_{\max}$. Notice that $\ell'_{i_{k-1}}/w_{i_{k-1}} < \ell_{i_{k-1}}/w_{i_{k-1}} = \ell_{i_k}/w_{i_k} = \ell'_{i_k}/w_{i_k}$, we have $I'_{\max} \subseteq I_{\max} \setminus \{i_{k-1}\}$, which yields that $I'_{\max}$ is a proper subset of $I_{\max} = I$. And for any $i \in I'_{\max}$, we have $i \in I_{\max}$ and $i \ne i_{k-1}$, so that p_i is a non-zero polynomial and $p'_i = p_i$, thus the condition that p'_i is a non-zero polynomial for any $i \in I'_{\max}$ is satisfied. Thus the case that $I_{\max} = I$ can be reduced to the case that $I_{\max}$ is a proper subset of I.

In the sequel, we shall prove Theorem 6.1 by induction on k which is the number of elements in I. If $k = 2$, we may assume that $I_{\max}$ is a proper subset of I, thus $I_{\max}$ has only one element. Assume that $I_{\max} = \{i_0\}$. Let $\beta = \ell_{i_0}/w_{i_0}$, $\gamma = \max\{\ell_i/w_i\colon i = 1, 2, \dots, i_0 - 1, i_0 + 1, \dots, n\}$ and $\epsilon_{\min} = \min\{\epsilon_i$ for i such that p_i is a non-zero polynomial$\}$. Since $i_0 \in I_{\max} \subset I$, we know p_{i_0} is a non-zero polynomial and $\epsilon_{i_0} = \epsilon$. Since $\epsilon_i \ge \epsilon = \epsilon_{i_0}$ for any i such that p_i is a non-zero polynomial, we have $\epsilon_{\min} = \epsilon = \epsilon_{i_0}$. Since all ϵ_i are divisible by $\epsilon/(w_1w_2)^{n-k} = \epsilon/(w_1w_2)^{n-2}$, by Lemma 4.7, we have $\beta - \gamma \ge \epsilon/(w_1w_2)^{n-1}$. By Theorem 4.1, there exists $j \in \{1, 2, \dots, m\}$ such that

$$\deg f_j \le \frac{(m-1+w_1)\epsilon_{\min}}{\beta - \gamma} \le (m-1+w_1)(w_1w_2)^{n-1}.$$

Now by induction, we assume that the conclusion holds for $2, \dots, k-1$, we shall prove it for k. If $I_{\max}$ has only one element, then the conclusion is arrived by using the similar argument as above. Hence, we may assume without loss of generality that $I_{\max}$ contains more than one element, and $I_{\max}$ is a proper subset of I, and pick $j_0 \in I \setminus I_{\max}$. Define another parameters ϵ'_i as follows:

$$\epsilon'_i = \begin{cases} \epsilon_i + \epsilon/(w_1 w_2)^{n-k+1}, & i = j_0 \\ \epsilon_i, & i \neq j_0 \end{cases} .$$

Consider the new weight type $(\ell'_1, \ldots, \ell'_n)$ controlled by parameters $(\epsilon'_1, \ldots, \epsilon'_n)$, let $I'_{\max} = \{e \colon \ell'_e/w_e$ is the maximum among all ℓ'_i/w_i for $i = 1, 2, \ldots, n\}$. We claim that $I'_{\max} \subseteq I_{\max}$. For any $i \notin I_{\max}$, we need to consider the following two cases:

(1) p_i is a non-zero polynomial. Fix an index $j \in I_{\max}$, then $\ell_i/w_i < \ell_j/w_j$. By setting $\varepsilon = \epsilon/(w_1 w_2)^{n-k}$ in Lemma 6.1(i), we have $\ell'_i/w_i < \ell'_j/w_j$, which yields that $i \notin I'_{\max}$.

(2) p_i is a zero polynomial, then $\epsilon'_i \leq \epsilon/(w_1 w_2)$, thus $\ell'_i \leq \epsilon/(w_1 w_2)$. For any $t \in I_{\max} \subset I$, p_t is a non-zero polynomial, so $\epsilon_t = \epsilon$. Since $t \neq j_0$, we have $\epsilon'_t = \epsilon_t = \epsilon$, which implies that $\ell'_t = \epsilon'_t + \mathrm{Q}\,'\text{-deg}\, p_t \geq \epsilon$. The equality holds when $\mathrm{Q}\,'\text{-deg}\, p_t = 0$. Assume that $i \in I'_{\max}$, then we have

$$\epsilon/w_t \leq \ell'_t/w_t \leq \ell'_i/w_i \leq \epsilon/(w_1 w_2 w_i),$$

for any $t \in I_{\max}$. Thus, $w_1 w_2 w_i \leq w_t$ for any $t \in I_{\max}$, hence $w_2 = w_i = 1$ and $w_1 = w_t$ for any $t \in I_{\max}$. Since $I_{\max}$ has more than one element, there exists $t_0 \in I_{\max}$ such that $t_0 \geq 2$, so that $w_{t_0} \leq w_2$. Thus, $w_1 = w_{t_0} \leq w_2 = 1$, so that $w_1 = 1$. That is to say, $w_1 = w_2 = \cdots = w_n$. Notice that $\deg p_i < w_i$ and $p_i(0) = 0$ by Lemma 4.8 for i such that p_i is a non-zero polynomial, thus p_i has to be the zero polynomial for $i = 1, 2, \ldots, n$, i.e., $D = 0$. This leads to a contradiction. Hence the assumption $i \in I'_{\max}$ is absurd.

Thus, $i \notin I'_{\max}$ for all $i \notin I_{\max}$, which yields that $I'_{\max} \subseteq I_{\max} \subseteq I \setminus \{j_0\}$. For any $i \in I'_{\max}$, we have $i \in I_{\max}$, thus p_i is a non-zero polynomial. Let $I' = I \setminus \{j_0\}$, then the number of elements of I' is $k - 1$ and $I'_{\max} \subseteq I'$. The conclusion follows immediately from the assumption. q.e.d.

Theorem 6.2 (Main Theorem A)**.** *Let $P = \mathbb{C}[x_1, x_2, \ldots, x_n]$ be the weighted polynomial ring in n weighted variables $x_1, x_2, \ldots, x_n$ ($n \geq 2$) with positive integer weights $w_1 \geq w_2 \geq \cdots \geq w_n$. Suppose that $f_1, f_2, \ldots, f_m$ are weighted homogeneous polynomials with degrees greater than $(m - 1 + w_1)(w_1 w_2)^{n-1}$ and $f_1, f_2, \ldots, f_m$ define a positive dimensional isolated singularity at the origin. Then there are no non-zero negative weight derivations on $R = P/(f_1, f_2, \ldots, f_m)$.*

Proof. (By contradiction) Suppose D is a non-zero negative weight derivation on R or equivalently a non-zero negative weight derivation on P which preserves the ideal $(f_1, f_2, \ldots, f_m)$ as in (3.1). We take the new weight type $(\ell_1, \ldots, \ell_n)$ of D controlled by parameters ϵ_i, where

$$\epsilon_i = \begin{cases} \epsilon, & p_i \text{ is a non-zero polynomial} \\ 0, & \text{otherwise} \end{cases} ,$$

where ϵ is a positive real number. It is clear that $\ell_i > 0$ for any i such that p_i is a non-zero polynomial and $\ell_i = 0$ for any i such that p_i is the zero polynomial. Thus p_i is a non-zero polynomial for any $i \in I_{\max}$. Let $I = \{1, 2, \ldots, n\}$ and it is clear that $I_{\max} \subseteq I$. Then by Theorem 6.1 we know that there exists $j \in \{1, 2, \ldots, m\}$ such that $\deg f_j \leq (m - 1 + w_1)(w_1 w_2)^{n-1}$, which contradicts the condition that $\deg f_j > (m - 1 + w_1)(w_1 w_2)^{n-1}$ for all j. So the conclusion has been arrived at. q.e.d.

References

[Al] A. G. Aleksandrov, *Cohomology of a quasihomogeneous complete intersection*, Math. USSR Izvestiya **26** (1986), no. 3, 437–477, MR0794953, Zbl 0647.14027.

[AGLV] V. I. Arnold, V. V. Goryunov, O. V. Lyashko, and V. A. Vasil'ev, *Singularity theory II, classification and applications*, Dynamical Systems VIII, Encyclopaedia of Mathematical Sciences, 39, Berlin, Springer-Verlag, 1993, MR1218886, Zbl 0778.58001.

[GS] M. Granger and M. Schulze, *Derivations of negative degree on quasi-homogeneous isolated complete intersection singularities*, ArXiv: 1403.3844v2.

[Ka1] J.-M. Kantor, *Dérivation sur les singularitiés quasi-homogènes: cas des courbes*, C. R. Acad. Sci. Paris **287** (1978), 1117–1119, MR0520418, Zbl 0419.14016.

[Ka2] J.-M. Kantor, *Dérivation sur les singularitiés quasi-homogènes: cas des hypersurface*, C. R. Acad. Sci. Paris **288** (1979), 33–34, MR0522013, Zbl 0412.32031.

[MS] S. Mori and H. Sumihiro, *On Hartshorne's conjecture*, J. Math. Kyoto Univ. **18** (1978), 523–533, MR0509496, Zbl 0422.14030.

[PP1] S. Papadima and L. Paunescu, *Reduced weighted complete intersection and derivations*, J. Algebra **183** (1996), 595–604, MR1399041, Zbl 0912.13002.

[PP2] S. Papadima and L. Paunescu, *Isometry-invariant geodesics and nonpositive derivations of the cohomology*, J. Differential Geometry **71** (2005), 159–176, MR2191771, Zbl 1120.53025.

[Ro] H. Rossi, *Vector fields on analytic spaces*, Ann. Math. **78** (1963), no.3, 455–467, MR0162973, Zbl 0129.29701.

[Sa] K. Saito, *Quasihomogene isolierte Singularitäten von Hyperflächen*, Invent. Math. **14** (1971), 123–142, MR0294699, Zbl 0224.32011.

[Wa1] J. M. Wahl, *Derivations of negative weight and non-smoothability of certain singularities*, Math. Ann. **258** (1982), 383–398, MR0650944, Zbl 0507.14029.

[Wa2] J. M. Wahl, *Derivations, automorphisms and deformations of quasi-homogeneous singularities*, Proc. Symposia in Pure Math. **40**, part 2 (1983), 613–624, MR0713285, Zbl 0534.14001.

[Wa3] J. M. Wahl, *A cohomological characterization of P^n*, Invent. Math. **72** (1983), 315–322, MR0700774, Zbl 0544.14013.

[YY] Y. Yu and S. S.-T. Yau, *Gorenstein quotient singularities in dimension three*, Mem. Amer. Math. Soc. **105** (1993), no. 505, MR1169227, Zbl 0799.14001.

[YZ1] S. S.-T. Yau and H. Q. Zuo, *A sharp upper estimate conjecture for the Yau number of weighted homogeneous isolated hypersurface singularity*, Pure Appl. Math. Q. **12** (2016), no. 3, 165–181, MR3613969, Zbl 06711978.

[YZ2] S. S.-T. Yau and H. Q. Zuo, *Derivations of the moduli algebras of weighted homogeneous hypersurface singularities*, J. Algebra **457** (2016), 18–25, MR3490075, Zbl 1343.32021.

Department of Mathematical Sciences
Tsinghua University
Beijing, 100084
P. R. China
E-mail address: chenby16@mails.tsinghua.edu.cn

College of Information Science and Technology/College of Cyber Security
Jinan University
Guangzhou, Guangdong, 510632
P. R. China
E-mail address: haochen@jnu.edu.cn

Department of Mathematical Sciences
Tsinghua University
Beijing, 100084
P. R. China
E-mail address: yau@uic.edu

Department of Mathematical Sciences
Tsinghua University
Beijing, 100084
P. R. China
E-mail address: hqzuo@mail.tsinghua.edu.cn