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VARIATION OF COMPLEX STRUCTURES AND VARIATION OF LIE ALGEBRAS II: NEW LIE ALGEBRAS ARISING FROM SINGULARITIES

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Dedicated to Professor Heisuke Hironaka on the occasion of his 87th birthday

Abstract

Finite dimensional Lie algebras are semi-direct product of the semi-simple Lie algebras and solvable Lie algebras. Brieskorn gave the connection between simple Lie algebras and simple singularities. Simple Lie algebras have been well understood, but not the solvable (nilpotent) Lie algebras. It is extremely important to establish connections between singularities and solvable (nilpotent) Lie algebras. In this article, a new natural connection between the set of complex analytic isolated hypersurface singularities and the set of finite dimensional solvable (nilpotent) Lie algebras has been constructed. We construct finite dimensional solvable (nilpotent) Lie algebras naturally from isolated hypersurface singularities. These constructions help us to understand the solvable (nilpotent) Lie algebras from the geometric point of view. Moreover, it is known that the classification of nilpotent Lie algebras in higher dimensions (> 7) remains to be a vast open area. There are one-parameter families of non-isomorphic nilpotent Lie algebras (but no two-parameter families) in dimension seven. Dimension seven is the watershed of the existence of such families. It is well-known that no such family exists in dimension less than seven, while it is hard to construct one-parameter family in dimension greater than seven. In this article, we construct an one-parameter family of solvable (resp. nilpotent) Lie algebras of dimension 11 (resp. 10) from $\tilde{E}_7$ singularities and show that the weak Torelli-type theorem holds. We shall also construct an one-parameter family of solvable (resp. nilpotent) Lie algebras of dimension 12 (resp. 11) from $\tilde{E}_8$ singularities and show that the Torelli-type theorem holds. Moreover, we investigate the numerical relation between the dimensions of the new Lie algebras and Yau algebras. Finally, the new Lie algebras arising from fewnomial isolated singularities are also computed.

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1. Introduction

Let G be a semi-simple Lie group acting on its Lie algebra $\mathcal{G}$ by the adjoint action and let $\mathcal{G}/G$ be the variety corresponding to the G-invariant polynomials on $\mathcal{G}$. The quotient morphism $\gamma : \mathcal{G} \to \mathcal{G}/G$ was intensively studied by Kostant ([**Ko1**], [**Ko2**]). Let $\mathcal{H} \subset \mathcal{G}$ be a Cartan subalgebra of $\mathcal{G}$ and W be the corresponding Weyl group.

(i) The space $\mathcal{G}/G$ may be identified with the set of semi-simple G classes in $\mathcal{G}$ such that γ maps an element $x \in \mathcal{G}$ to the class of its semi-simple part x_s. Thus $\gamma^{-1}(0) = N(\mathcal{G})$ is the nilpotent variety. An element $x \in N(\mathcal{G})$ is termed "regular" (resp., "subregular") if its centralizer has minimal dimension (resp., minimal dimension + 2).

(ii) By a theorem of Chevalley, the space $\mathcal{G}/G$ is isomorphic to $\mathcal{H}/W$, an affine space of dimension $r = \mathrm{rank}(\mathcal{G})$. The isomorphism is given by the map of a semi-simple class to its intersection with $\mathcal{H}$ (a W orbit).

The following beautiful theorem of Brieskorn [**Br**] conjectured by Grothendieck [**Gr**] establishes connections between the simple singularities and the simple Lie algebras.

Theorem 1.1 ([**Br**]). *Let $\mathcal{G}$ be a simple Lie algebra over $\mathbb{C}$ of type A_r, D_r, E_r. Then*

(i) the intersection of the variety $N(\mathcal{G})$ of the nilpotent elements of $\mathcal{G}$ with a transverse slice S to the subregular orbit, which has codimension 2 in $N(\mathcal{G})$, is a surface $S \cap N(\mathcal{G})$ with an isolated rational double point of the type corresponding to the algebra $\mathcal{G}$.

(ii) the restriction of the quotient $\gamma : \mathcal{G} \to \mathcal{H}/W$ to the slice S is a realization of a semi-universal deformation of the singularity in $S \cap N(\mathcal{G})$.

The details of this Brieskorn's theory can be found in Slodowy's papers ([**Sl1**], [**Sl2**]).

Among many other things, Brieskorn's theory gives the way to construct rational double points from simple Lie algebras. It is known that finite dimensional Lie algebras are semi-direct product of the semi-simple Lie algebras and solvable Lie algebras. Brieskorn gave the connection between simple Lie algebras and simple singularities. Simple Lie algebras and semi-simple Lie algebras have been well understood, but not the solvable (nilpotent) Lie algebras. Historically, a marked difference is noted between the classification theory of semi-simple Lie algebras and the classification theories of solvable or nilpotent Lie algebras. The semi-simple theory can best be described as beautiful, while the others lack anything resembling elegance. For semi-simple Lie algebras over the complex numbers one has the Killing form, Dynkin diagrams, root space decompositions, the Serre presentation, the theory of highest weight representations, the Weyl character formula for finite-dimensional representations, and much more ([**Hu**], [**Ja**]). In the theory of solvable Lie algebras one has the theorems of Lie and Engel along with Malcev's reduction of the classification problem to the same problem for nilpotent algebras [**Ma**]. There does not seem to be any nice way to classify nilpotent Lie algebras (such as a graph or diagram for each algebra). Therefore, it is of great importance to establish connection between singularities and solvable (nilpotent) Lie algebras. In this article, a new natural connection between the set of complex analytic isolated hypersurface singularities and the set of finite dimensional solvable (nilpotent) Lie algebras has been constructed. We construct finite dimensional solvable (nilpotent) Lie algebras naturally from isolated hypersurface singularities. These constructions help us to understand the solvable (nilpotent) Lie algebras from the geometric point of view.

For any isolated hypersurface singularity $(V, 0) \subset (\mathbb{C}^n, 0)$ where $V = V(f) = \{f = 0\}$, one can consider the moduli algebra

$$A(V) := \mathcal{O}_n/(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}),$$

where $\mathcal{O}_n$ is the algebra of convergent power series in n indeterminates and $f \in \mathcal{O}_n$. In [**MY**], Mather and Yau proved that the complex structure of $(V, 0)$ determines and is determined by its moduli algebra. Subsequently, Yau [**Ya2**] introduced the Lie algebra $L(V)$ to $(V, 0)$, which is the Lie algebra of derivations of $A(V)$, i.e., $L(V) = \mathrm{Der}(A(V), A(V))$ (see Definition 2.6). He proved that $L(V)$ is solvable (cf. [**Ya3**]). Yau and his collaborators have systematically studied the Lie algebras of isolated hypersurface singularities since 1980s ([**Ya1**]–[**Ya3**], [**BY**], [**SY**], [**YZ1**, **YZ2**], [**CYZ**], [**CCYZ**], [**HYZ1**]–[**HYZ3**]). We shall denote as $\lambda(V)$ the dimension of $L(V)$. In [**Yu**], $L(V)$ is called Yau algebra while in [**EK**] $\lambda(V)$ is called Yau number.

Motivated by Dimca's beautiful theorem (Theorem 2.2), which states that two zero-dimensional isolated complete intersection singularities X and Y are isomorphic if and only if their singular subspaces $\mathrm{Sing}(X)$ and $\mathrm{Sing}(Y)$ are isomorphic, we introduce a new Lie algebra $L^*(V)$, $\lambda^*(V) :=\mathrm{dim} L^*(V)$ (see section 2.5). Given a family of complex projective hypersurfaces in $\mathbb{C}P^n$, the Torelli problem studied by Griffiths and his school asks whether the period map is injective on that family, i.e., whether the family of complex hypersurfaces can be distinguished by means of their Hodge structures. A complex projective hypersurface in $\mathbb{C}P^n$ can be viewed as a complex hypersurface with isolated singularity in $\mathbb{C}^{n+1}$. Let $V = \{z \in \mathbb{C}^{n+1} : f(z) = O\}$ be a complex hypersurface with isolated singularity at the origin. Seeley and Yau investigated the family of isolated complex hypersurface singularities using Yau algebras (see Definition 2.6) and obtained two deep Torelli-type theorems for simple elliptic singularities $\tilde{E}_7$ and $\tilde{E}_8$ [**SY**]. The natural question arises: whether the family of isolated complex hypersurface singularities can be distinguished by means of their new Lie algebras. The family of hypersurface singularities here is not arbitrary. First of all, as in projective case, we are actually studying the complex structures of an isolated hypersurface singularity. In view of the theorem of Lê and Ramanujam [**LR**], we require that the Milnor number μ is constant along this family. Recall that the dimension of the moduli algebra (denoted by τ) is a complex analytic invariant. So it suffices to consider only a (μ, τ)-constant family of isolated complex hypersurface singularities [**SY**]. The simple elliptic singularities are such families. We shall prove two Torelli-type theorems for simple elliptic singularities $\tilde{E}_7$ and $\tilde{E}_8$ respectively. However, there is no Torelli-type result for $\tilde{E}_6$, since $L^*(V_t)$ is a trivial family (see section 3). Our method for $\tilde{E}_7$ is completely new and can be used to prove Torelli-type theorems for more general singularities. There are several advantages of our approach. First of all, it works for general complex hypersurface singularities without homogeneity assumption. Second, it allows us to construct a continuous invariant explicitly. Third, it gives a general method to produce a continuous family of nilpotent Lie algebras.

Theorem A. *The Torelli-type theorem holds for simple elliptic singularities $\tilde{E}_8$. That is, $L^*(V_{t_1}) \cong L^*(V_{t_2})$ as Lie algebras, for $t_1 \neq t_2$ in $\mathbb{C} - \{t \in \mathbb{C} : 4t^3 + 27 = 0\}$, if and only if V_{t_1} and V_{t_2} are analytically isomorphic (i.e., $t_1^3 = t_2^3$). In particular, $\tilde{E}_8$ give rise to a non-trivial one-parameter family of solvable (resp. nilpotent) Lie algebras of dimension 12 (resp. 11).*

Theorem B. *The weak Torelli-type theorem holds for simple elliptic singularities $\tilde{E}_7$, i.e., $L^*(V_t)$ is a non-trivial one-parameter family. In particular, $\tilde{E}_7$ give rise to a non-trivial one-parameter family of solvable (resp. nilpotent) Lie algebras of dimension 11 (resp. 10).*

The classification of nilpotent Lie algebras in higher dimensions (> 7) remains wide open. It is known that there are one-parameter families of non-isomorphic nilpotent Lie algebras (but no two-parameter families) in dimension seven. There are no such families in dimension less than seven. And the existence of such families is known in dimension greater than seven. However, such examples are hard to construct (cf. [**Se**]). As a corollary of Theorem A and Theorem B, we obtain non-trivial one-parameter families of 11-dimensional and 12-dimensional solvable (resp. 10-dimensional and 11-dimensional nilpotent) Lie algebras associated to $\tilde{E}_7$ and $\tilde{E}_8$ respectively.

Yau and Zuo [**YZ2**] formulated a sharp upper estimate conjecture for the Yau number (see Definition 2.6) of weighted homogeneous isolated hypersurface singularities and validated this conjecture for binomial isolated hypersurface singularities. A natural question is: what is the numerical relation between the new analytic invariant $\lambda^*(V)$ and the Yau number $\lambda(V)$? We propose the following conjecture:

Conjecture 1.1. Let $(V, 0)$ be an isolated hypersurface singularity defined by $f \in \mathcal{O}_n$, $n \geq 2$, and multiplicity greater than or equal to 3. Let $\lambda^*(V)$ be the dimension of $L^*(V) := \mathrm{Der}_{\mathbb{C}}(A^*(V), A^*(V))$, then $\lambda^*(V) = \lambda(V)$.

The above conjecture is obviously true when the isolated singularity $(V, 0)$ is not quasi-homogeneous. Recall the beautiful result of Saito ([**Sa2**], Corollary 3.8): let $f \in \mathcal{O}_n$ be a germ of a holomorphic function which defines a hypersurface with an isolated singularity at 0, then f is not quasi-homogeneous, precisely when

$$\mathrm{Det}(\frac{\partial^2 f}{\partial x_i \partial x_j})_{i,j=1,\cdots,n} \in (f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}).$$

Consequently, for non-quasi-homogeneous isolated hypersurface singularities, $A(V) = A^*(V)$. It follows that $L^*(V) = L(V)$ and $\lambda^*(V) = \lambda(V)$.

In this article, we shall also prove the following results:

Theorem C. *Let $f(x_1, \ldots, x_n)$, $(n \geq 2)$, be a weighted homogeneous polynomial with respect to weight system $(w_1, \ldots, w_n; 1)$ and with mult$(f) \geq 3$. Suppose that f defines an isolated singularity $(V, 0)$, then*

$$\lambda^*(V) \leq \lambda(V).$$

Conjecture 1.1 is verified when $n \leq 4$.

Theorem D. *Let $f(x_1, \ldots, x_n)$, $(2 \leq n \leq 4)$, be a weighted homogeneous polynomial with respect to weight system $(w_1, \ldots, w_n; 1)$ and with mult$(f) \geq 3$. Suppose that f defines an isolated singularity $(V, 0)$, then*

$$\lambda^*(V) = \lambda(V).$$

Remark 1.1. On the one hand, the Conjecture 1.1 is also verified for n-dimensional weighted homogeneous singularities with an additional condition, see Remark 6.1. On the other hand, different method has been used to prove the Conjecture 1.1 for binomial singularities, see Theorem 7.1.

Elashvili and Khimshiashvili [**EK**] proved the following result: if X and Y are two simple singularities except the pair A_6 and D_5, then $L(X) \cong L(Y)$ as Lie algebras, if and only if X and Y are analytically isomorphic. Finally, we shall also show that the simple hypersurface singularities can be characterized completely by the new Lie algebra $L^*(V)$.

Proposition 1.1. *If X and Y are two simple hypersurface singularities, then $L^*(X) \cong L^*(Y)$ as Lie algebras, if and only if X and Y are analytically isomorphic.*

The proof follows directly from the computation performed in section 7 by a straightforward analysis of the new Lie algebras.

The structure of paper is as follows. In section 2, we recall some basic concepts and definitions of isolated hypersurface singularities and introduce a new Lie algebra. The new Lie algebras $L^*(V)$ of $\widetilde{E}_6$ are calculated in section 3. We prove two Torelli-type theorems, i.e., Theorem B in section 4 and Theorem A in section 5. Theorem C and Theorem D are proved in section 6. We also compute the Lie algebras $L^*(V)$ that arising from fewnomial isolated hypersurface singularities in section 7 and give some applications.

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2. Isolated hypersurface singularities and derivation Lie algebras

We present here the necessary definitions and auxiliary results on hypersurface germs with isolated singularities and their derivation Lie algebras.

2.1. Lie algebra. We recall some preliminaries of Lie algebra, which will be used in our proofs.

A Lie algebra is a vector space L over some field k (in this paper $k = \mathbb{C}$) together with the Lie bracket $[\cdot,\ \cdot] : L \times L \to L$ that satisfies rules of bilinearity, alternativity, and Jacobi identity [**Hu**]. If $x \in L$, $y \to [x, y]$ is an endomorphism of L, denoted as ad x.

Let L be a Lie algebra. For two subspaces A, B of L, the symbol $[A, B]$ denotes the linear span of the set of all $[x, y]$ with x in A and y in B. A sub Lie algebra of L is a subspace, say J, that is closed under the bracket operation (i.e., $[J, J] \subset J$); J becomes then a Lie algebra with the linear and Lie bracket operations inherited from L. A sub Lie algebra J is called an ideal of L if $[L, J] \subset J$ (if $x \in L$ and $y \in J$ then $[x, y] \in J$). The centralizer C_S of a subset S of L is the set of those x in L that commute with all y in S (i.e., $[x, y] = 0$). We say that two Lie algebras L, L' over k are isomorphic if there exists a vector space isomorphism $\phi : L \to L'$ satisfying $\phi([x, y]) = [\phi(x), \phi(y)]$.

We shall basically deal with solvable and nilpotent Lie algebras. For completeness, we recall the following definitions.

Definition 2.1. A Lie algebra L is called nilpotent if the lower centeral series $L_{(*)}$ terminates, while L is called solvable if the derived series $L^{(*)}$ terminates. The two series of ideals are $L_{(*)} = \{L_{(i)}\}, L^{(*)} = \{L^{(i)}\}, L_{(0)} = L^{(0)} = L, L_{(1)} = L^{(1)} = [L, L], L_{(i)} = [L, L_{(i-1)}], L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$, $i = 2, 3, \ldots$.

Definition 2.2. Let V be a finite dimensional vector space over some field k. $x \in \mathrm{End}\ V$ is called semi-simple if the roots of its minimal polynomial over k are all distinct. Equivalently (k being algebraically closed), x is semi-simple if and only if x is diagonalizable.

Definition 2.3. A Cartan subalgebra C in Lie algebra L is a nilpotent subalgebra that is self-normalising (i.e., if $[x, y] \in C$ for all $x \in C$, then $y \in C$). Equivalently, Cartan subalgebra is a maximal commutative subalgebra C such that, for each $h \in C$, ad h is semi-simple.

Cartan subalgebras exist for finite-dimensional Lie algebras L whenever the base field k is infinite. If the field k is algebraically closed of characteristic 0 and the algebra is finite-dimensional then all Cartan subalgebras are conjugate under automorphisms of the Lie algebra, and in particular are all isomorphic. Consequently, they are of the same dimension r, called the rank rk$L$ of Lie algebra L [**Bo**].

According to Engel's theorem, a Lie algebra L is nilpotent if and only if all operators $\mathrm{ad}\, a : L \to L$ are nilpotent for $a \in L$ [**Bo**]. Another general result states that a solvable algebraic Lie algebra can be decomposed into a semi-direct sum of a Cartan subalgebra and maximal nilpotent ideal $N(L)$ (the latter is called the nilpotent radical of L).

2.2. Cohomology of Lie algebras. For a general theory of the cohomology of Lie algebra, the interested readers can refer to [**Kos**].

Let L be a Lie algebra. A p-dimensional cochain of L (with values in L) is a p-linear alternating mapping of L^p in L ($p \in \mathbb{N}^*$). A 0-cochain is a constant function from L to L.

We denote by $C^p(L, L)$ the space of the p-cochains and

$$C^*(L, L) = \oplus_{p \geq 0} C^p(L, L).$$

We provide $C^p(L, L)$ a L-module structure by putting

$$(x\Phi)(x_1, ..., x_p) = [x, \Phi(x_1, ..., x_p)] - \sum_{1 \leq i \leq p} \Phi(x_1, ..., [x, x_i], ..., x_p)$$

for all $x_1, ..., x_p \in L$.

On the space $C^*(L, L)$ we define the endomorphism

$$\delta : C^*(L, L) \to C^*(L, L),$$
$$\Phi \mapsto \delta\Phi$$

by letting

$$\delta\Phi(x) = x\Phi, \text{ if } \Phi \in C^0(L, L),$$

$$\delta\Phi(x_1, ..., x_{p+1}) = \sum_{1 \leq s \leq p+1} (-1)^{s+1} [x_s, \Phi(x_1, ...\hat{x}_s, ..., x_{p+1})]$$
$$+ \sum_{1 \leq s < t \leq p+1} (-1)^{s+t} \Phi([x_s, x_t], x_1, .., .\hat{x}_s, ..., \hat{x}_s, ..., x_{p+1}),$$

if $\Phi \in C^p(L, L), p \geq 1$.

From this definition, it is easy to verify that $\delta(C^p(L,L)) \subset C^{p+1}(L,L)$ with

$$\delta \circ \delta = 0.$$

Let us denote

$$\begin{cases} Z^p(L,L) = \mathrm{Ker}\delta|_{C^p(L,L)} & p \geq 1 \\ B^p(L,L) = \mathrm{Im}\delta|_{C^p(L,L)} & p \geq 1 \end{cases}$$

and $H^p(L,L) = Z^p(L,L)/B^p(L,L), p \geq 1$. This quotient space is called the cohomology space of L of degree p (with values in L). For $p = 0$, we put $B^0(L,L) = \{0\}$ and $H^0(L,L) = Z^0(L,L)$. This last space can be identified to the space of all L-invariant elements, that is,

$$\{x \in L \mid \ \mathrm{ad}y(x) = 0, \ \forall y \in L\}.$$

Then $Z^0(L,L) = C_L$ (the center of L).

Definition 2.4. A derivation f of a Lie algebra L is a linear map

$$f : L \to L$$

satisfying

$$[f(x), y] + [x, f(y)] - f[x,y] = 0, \quad \forall (x,y) \in L^2.$$

Let us denote by Der$L$ the set of derivations of L.

For all x in L, the endomorphism ad$x$ is a derivation of L. The ones of type ad$x$ for $x \in L$ are called inner derivations.

We have

$$Z^1(L,L) = \{f : L \to L \mid \delta f = 0\}.$$

But $\delta f(x,y) = [f(x), y] + [x, f(y)] - f[x,y]$. Then $Z^1(L,L)$ is nothing but the algebra of derivation of L:

$$Z^1(L,L) = \mathrm{Der}L.$$

It is the same as:

$$B^1(L,L) = \{\mathrm{ad}x, x \in L\}.$$

Therefore the space $H^1(L,L)$ can be interpreted as the set of the outer derivations of the Lie algebra L.

2.3. Isolated hypersurface singularities. Let $\mathbb{C}[x_1, x_2, ..., x_n]$ be the algebra of complex polynomials in n indeterminates. Denote by $\mathcal{O}_n$ the algebra of germs of holomorphic functions at the origin of $\mathbb{C}^n$. Obviously, $\mathcal{O}_n$ can be naturally identified with the algebra of convergent power series in n indeterminates with complex coefficients. For a polynomial $f \in \mathbb{C}[x_1, x_2, ..., x_n]$, let us denote by $V = V(f)$ the germ at the origin of $\mathbb{C}^n$ of hypersurface $\{f = 0\} \subset \mathbb{C}^n$. We say that V is a germ of isolated hypersurface singularity if the origin is an isolated zero of the gradient of f. The local (function) algebra of V is defined as the (commutative associative) algebra $F(V) \cong \mathcal{O}_n/(f)$, where (f) is the principal ideal generated by the germ of f at the origin. According to Hilbert's Nullstellensatz for an isolated singularity $V = V(f) = \{f = 0\}$ the factor-algebra $A(V) = \mathcal{O}_n/(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$ is finite dimensional. This factor-algebra is called the moduli algebra of V and its dimension $\tau(V)$ is called Tyurina number. The well-known Mather-Yau theorem states that

Theorem 2.1 ([**MY**]). *The analytic isomorphism type of an isolated hypersurface singularity determine and is determined by the isomorphism class of its moduli algebra. i.e.,*

$$(V_1, 0) \cong (V_2, 0) \Longleftrightarrow A(V_1) \cong A(V_2).$$

Definition 2.5. A polynomial $f \in \mathbb{C}[x_1, x_2, ..., x_n]$ is called quasi-homogeneous (or weighted homogeneous) if there exist positive rational numbers $w_1, \dots, w_n$ (called weights of indeterminates x_j) and d such that, for each monomial $\prod x_j^{k_j}$ appearing in f with non-zero coefficient, one has $\sum w_j k_j = d$. The number d is called the quasi-homogeneous degree (w-degree) of f with respect to weights w_j, denoted as $\deg f$. The collection $(w; d) = (w_1, \dots, w_n; d)$ is called the quasi-homogeneity type (qh-type) of f.

It is well known that, for a quasi-homogeneous polynomial f, one has $f = \frac{1}{d} \sum w_j x_j \frac{\partial f}{\partial x_j}$ (Euler formula). Hence $f \in (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$. Therefore if V is a singularity defined by a quasi-homogeneous polynomial f (i.e., V is a quasi-homogeneous singularity), then the Tyurina number $\tau(V)$ coincides with the *Milnor number* $\mu(V)$, which is defined as $\dim_{\mathbb{C}} M(V)$, where $M(V) = \mathcal{O}_n/(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$ is called *Milnor algebra* of V. In the quasi-homogeneous case $A(V) \cong M(V)$. Furthermore, the Milnor number can be computed by the following simple formula.

Proposition 2.1. *For an isolated hypersurface singularity V defined by a quasi-homogeneous polynomial of $(w_1, \dots, w_n; d)$ type, one has*

$$\tau(V) = \mu(V) = \prod_{i=1}^{n} \frac{d - w_i}{w_i}. \tag{1}$$

2.4. Yau algebra. Recall that a derivation of commutative associative algebra A is defined as a linear endomorphism D of A satisfying the Leibniz rule: $D(ab) = D(a)b + aD(b)$. Thus for such an algebra A, one can consider the Lie algebra of its derivations $\mathrm{Der}(A, A)$ with the bracket defined by the commutator of linear endomorphisms.

Definition 2.6. Let $V = \{f = 0\}$ be a germ of isolated hypersurface singularity at the origin of $\mathbb{C}^n$ defined by $f \in \mathbb{C}[x_1, x_2, ..., x_n]$ and $A(V)$ be the moduli algebra. We denote $L(V) := \mathrm{Der}_{\mathbb{C}}(A(V), A(V))$. Yu [**Yu**] call $L(V)$ the Yau algebra of V. Its dimension denoted as $\lambda(V)$ is called the Yau number by Elashvili and Khimshiashvili [**EK**].

In particular, for a singularity V as above, the local functional algebra of V is defined as the algebra $F(V) := \mathcal{O}_n/(f)$. One can consider the Lie algebras $D(F(V)) := \mathrm{Der}(F(V), F(V))$ and $L(V) = \mathrm{Der}_{\mathbb{C}}(A(V), A(V))$. By the classical result of Pursell and Shanks [**PS**], the isomorphism class of the algebra $C^\infty(M)$ of smooth functions on a smooth manifold M is completely determined by the Lie algebra of its derivations. Thus it is natural to ask if the same holds for the algebras $F(V)$ and $A(V)$. Notice that if this is the case, then by the Mather-Yau theorem, the corresponding Lie algebra determines the analytic isomorphism type of a singularity. It was shown by Hauser and Müller, for an isolated hypersurface singularity V, the Lie algebra $D(F(V))$ indeed determines the analytic type of V [**HM**]. Elegant as it is, this result is not so effective because $D(F(V))$ is an infinite dimensional Lie algebra which is difficult to investigate and work with. At the same time, $L(V)$ is typically a finite dimensional Lie algebra and its structural constants may be found in an algorithmic way. Moreover, Yau showed that, for any isolated hypersurface singularity V, $L(V)$ is a solvable Lie algebra [**Ya3**]. Moreover, some natural numerical invariants of such Lie algebras can be effectively computed and it is natural to try to relate them to the numerical invariants of the singularity [**YZ2**].

2.5. New derivation Lie algebra. The following beautiful theorem of Dimca characterizes zero-dimensional isolated complete intersection singularities.

Theorem 2.2 (Dimca [**Di**])**.** *Two zero-dimensional isolated complete intersection singularities X and Y are isomorphic if and only if their singular subspaces Sing(X) and Sing(Y) are isomorphic.*

Remark 2.1. Let $V = V(f)$ be an isolated quasi-homogeneous hypersurface singularity. Assume that X defined by $(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$ is a zero-dimensional isolated complete intersection singularities. Then $\mathrm{Sing}(X)$ is defined by $(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}, \mathrm{Det}(\frac{\partial^2 f}{\partial x_i \partial x_j})_{i,j=1,\cdots,n})$.

Theorem 2.2 implies that in order to study analytic isomorphism type of zero dimensional isolated complete intersection singularity X, we only need to consider the Artinian local algebra $A^*(V)$ which is the coordinate ring of $\mathrm{Sing}(X)$. Thus $A^*(V)$ is defined as the quotient

$$\mathcal{O}_n/(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}, \mathrm{Det}(\frac{\partial^2 f}{\partial x_i \partial x_j})_{i,j=1,\cdots,n}).$$

Combining Theorem 2.2 with Mather-Yau theorem, we know that $A^*(V)$ is a complete invariant of quasi-homogeneous isolated hypersurface singularities (i.e., $A^*(V)$ determines and is determined by the analytic isomorphism type of the singularity). We call $A^*(V)$ the generalized moduli algebra of V. Based on this important observation, we introduce the following new invariants for isolated hypersurface singularities.

Definition 2.7. Let $V = \{f = 0\}$ be a germ of isolated hypersurface singularity at the origin of $\mathbb{C}^n$ defined by $f \in \mathbb{C}[x_1, x_2, ..., x_n]$. The new Lie algebra arising from the isolated hypersurface singularity V is defined as $L^*(V) := \mathrm{Der}(A^*(V), A^*(V))$ (or $\mathrm{Der}(A^*(V))$ for short). Its dimension is denoted as $\lambda^*(V)$.

It is natural to present the following question.

Question 2.1. What type of singularities such that $L^*(V)$ is a complete invariants. That is, if V_1, V_2 are two singularities of such type, then $L^*(V_1) \cong L^*(V_1)$ if and only if $V_1 \cong V_2$.

In this paper, we shall give an affirmative answer to Question 2.1 for simple singularities and simple elliptic singularities.

The following theorem by Saito will be used later.

Theorem 2.3 ([**Sa1**]). *Let $f \in \mathcal{O}_n$ be a germ of a holomorphic function, defining an isolated quasi-homogeneous singularity at 0. Then*

$$Det(\frac{\partial^2 f}{\partial x_i \partial x_j})_{i,j=1,\cdots,n} \notin (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})\mathcal{O}_n$$

and

$$mDet(\frac{\partial^2 f}{\partial x_i \partial x_j})_{i,j=1,\cdots,n} \subseteq (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})\mathcal{O}_n,$$

where m is the maximal ideal of $\mathcal{O}_n$.

We obtain the following result.

Theorem 2.4. *Let V be an isolated singularity defined by a quasi-homogeneous polynomial f. Then*

$$\mu^*(V) = \mu(V) - 1,$$

where $\mu^*(V)$ is the dimension of $A^*(V)$ and $\mu(V)$ is the Milnor number of V.

Proof. Since the Milnor algebra

$$\mathcal{O}_n/(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}),$$

is a Gorenstein local algebra and has a unique socle, it follows from Theorem 2.3 that $\mathrm{Det}(\frac{\partial^2 f}{\partial x_i \partial x_j})$ is the unique socle. Thus we have $\mu^*(V) = \mu(V) - 1$. q.e.d.

Remark 2.2. It follows from Theorem 2.4 that $A^*(V) = 0$ when $\mu(V) = 1$. For this reason, our new Lie algebra $L^*(V)$ is defined only for singularities with Milnor number $\mu(V) \geq 2$.

Yau algebras are solvable. However, the new Lie algebra is not solvable in general. An example is: $x^3 + y^3$, and its new Lie algebra is spanned by $x\partial_x, y\partial_y, x\partial_y, y\partial_x$. Then it is easy to check that the derived series (see Definition 2.1) does not go down to zero. However, we prove that the new Lie algebra is solvable when the multiplicity of the singularity is at least 4. We first recall an important result obtained by Schulze.

Theorem 2.5 ([**Sc**]). *Let S be a zero-dimensional local $\mathbb{C}$-algebra of embedding dimension embdim(S) and order ord(S), and denote its first deviation by $\varepsilon_1(S)$. Then the Lie algebra $Der_{\mathbb{C}}(S,S)$ is solvable if $\varepsilon_1(S)+1 < embdim(S) + ord(S)$.*

Recall that, by definition, $\varepsilon_1(S) = \dim_{\mathbb{C}} H_1(S)$ where $H_\bullet(S)$ is the Koszul algebra of S. More explicitly, when $S = R/I$ in Theorem 2.5, where $R = \mathcal{O}_n$ and $I \subseteq R$ is a zero-dimensional ideal with $I \subseteq \mathsf{m}^m$, $\mathsf{m} = (x_1, \cdots, x_n)$ and $m \geq 2$ is chosen maximal. Then $n = \text{embdim}(S), m = \text{ord}(S)$, and $\varepsilon_1(S) = \dim_{\mathbb{C}}(I/\mathsf{m}I)$ is the minimal number of generators of I ([**BH**], Thm. 2.3.2(b)).

This result applies in particular to the generalized moduli algebra $A^*(V)$. If f is not quasi-homogeneous, then $\text{Der}(A^*(V), A^*(V))$ is the same as Yau algebra, thus it is solvable. Otherwise we have the following result.

Corollary 2.1. *If f is quasi-homogeneous and mult$(f) \geq 4$, then the new Lie algebra Der$(A^*(V), A^*(V))$ is solvable.*

Proof. Since f is quasi-homogeneous and $\text{mult}(f) \geq 4$, then

$$\text{embdim}(A^*(V)) = n, \varepsilon_1(S) = n+1, \text{and}\, \text{ord}(A^*(V)) \geq 3.$$

It follows from Theorem 2.5 that $\text{Der}(A^*(V), A^*(V))$ is solvable. q.e.d.

2.6. Fewnomial singularities. In this subsection we recall the definition of fewnomial isolated singularities [**Kh**].

Definition 2.8. We say that a polynomial $f \in \mathbb{C}[z_1, z_2, \cdots, z_n]$ is fewnomial if the number of monomials in f does not exceed n.

Obviously, the number of monomials in f may depend on the system of coordinates. In order to obtain a rigorous concept we shall only allow linear transformations of coordinates and f (or rather its germ at the origin) is called a k-nomial if k is the smallest natural number such that f becomes a k-nomial after (possibly) a linear transformation of coordinates. An isolated hypersurface singularity V is called k-nomial if there exists an isolated hypersurface singularity Y analytically isomorphic to V which can be defined by a k-nomial and k is the smallest such number. It was shown in [**CYZ**] that a singularity defined by a fewnomial f is isolated only if f is a n-nomial in n variables when its multiplicity is at least 3.

Definition 2.9. We say that an isolated hypersurface singularity V is fewnomial if it is defined by a fewnomial polynomial f. V is called a weighted homogenous fewnomial isolated singularity, if it is defined by a weighted homogenous fewnomial polynomial f. The 2-nomial (resp. 3-nomial) isolated hypersurface singularity is also called binomial (resp. trinomial) singularity.

The following proposition and corollary tell us that each simple singularity belongs to one of the following three types.

Proposition 2.2 ([**YZ2**]). *Let f be a weighted homogeneous fewnomial isolated hypersurface singularity with multiplicity at least 3. Then f is analytically equivalent to a linear combination of the following three series:*
Type A. $x_1^{a_1} + x_2^{a_2} + \cdots + x_{n-1}^{a_{n-1}} + x_n^{a_n}$, $n \geq 1$,
Type B. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}$, $n \geq 2$,
Type C. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1$, $n \geq 2$.

Corollary 2.2 ([**YZ2**]). *Each binomial isolated singularity is analytically equivalent to one of the three series: A) $x_1^{a_1}+x_2^{a_2}$, B) $x_1^{a_1}x_2+x_2^{a_2}$, and C) $x_1^{a_1}x_2+x_2^{a_2}x_1$.*

In many situations it is necessary to have an explicit basis of $A(V)$. It is well known that there always exist monomial bases. Recall that the monomial bases in moduli algebras of simple singularities (A_k, D_k, E_6, E_7, E_8) are given in [**AGV**].

3. Simple elliptic singularity $\widetilde{E_6}$

$\widetilde{E_6}$ is a simple elliptic singularity defined by $\{(x,y,z) \in \mathbb{C}^3 \mid x^3+y^3+z^3 = 0\}$. Its (μ,τ)-constant family is given by

$$V_t = \{(x,y,z) \in \mathbb{C}^3 \mid f_t(x,y,z) = x^3+y^3+z^3+txyz = 0\}$$

with $t^3+27 \neq 0$ (cf. [**Ya1**]). The moduli algebra of V_t, denoted as $A(V_t)$, is given by

$$\begin{aligned} A(V_t) =& \mathbb{C}\{x,y,z\}/(\frac{\partial f_t}{\partial x}, \frac{\partial f_t}{\partial y}, \frac{\partial f_t}{\partial z}) \\ =& < 1, x, y, z, xy, yz, zx, xyz >, \end{aligned}$$

with multiplication rules

$$x^2 = -\frac{t}{3}yz, y^2 = -\frac{t}{3}zx, z^2 = -\frac{t}{3}xy,$$
$$x^2y = xy^2 = y^2z = yz^2 = x^2z = 0.$$

Let $\mathrm{Hess}(f_t)$ be the Hessian matrix of f_t. Then the generalized moduli algebra $A^*(V_t) := \mathcal{O}_n/(\frac{\partial f_t}{\partial x}, \frac{\partial f_t}{\partial y}, \frac{\partial f_t}{\partial z}, \mathrm{Det}(\mathrm{Hess}(f_t))) = A(V_t)/(x^2y^2) = < 1, x, y, z, xy, yz, zx >$ with multiplication rules

$$x^2 = -\frac{t}{3}yz, y^2 = -\frac{t}{3}zx, z^2 = -\frac{t}{3}xy,$$

and

$$x^2y = xy^2 = y^2z = yz^2 = x^2z = xyz = 0.$$

By calculation, a basis for the new Lie algebra

$$L^*(V_t) = \mathrm{Der}(A^*(V_t), A^*(V_t))$$

denoted as L_t^* for short is:

$$x\partial_x + y\partial_y + z\partial_z, yz\partial_x, yz\partial_y, yz\partial_z, xz\partial_x, xz\partial_y, xz\partial_z, xy\partial_x, xy\partial_y, xy\partial_z,$$

for $t \neq 0$ and $216 - \frac{t^6}{27} + 7t^3 \neq 0$. It is easy to see that in this case, L_t^* are isomorphic as Lie algebra. Thus L_t^* is a trivial family.

4. Simple elliptic singularity $\widetilde{E_7}$

$\widetilde{E_7}$ is a simple elliptic singularity defined by $\{(x, y, z) \in \mathbb{C}^3 \mid x^4 + y^4 + z^2 = 0\}$. In [**SY**], the third author of this paper and his co-worker showed that its (μ, τ)-constant family is given by

$$V_t = \{(x, y, z) \in \mathbb{C}^3 \mid f_t(x, y, z) = x^4 + y^4 + tx^2y^2 + z^2 = 0\}$$

with $t^2 \neq 4$. They constructed a family of Lie algebras by associating the singularities $(V_t, 0)$ to a Lie algebra which is defined as the algebra of derivations of the moduli algebra $A(V_t)$ (i.e. the Yau algebra of V_t). Moreover, they proved the following Torelli-type theorem for $\widetilde{E_7}$: V_t is biholomorphic to V_s if and only if their Yau algebras are isomorphic to each other, and if and only if $s \in \{\pm t, \pm(\frac{12-2t}{2+t}), \pm(\frac{12+2t}{2-t})\}$.

In previous section, we have associated an isolated singularity $(V, 0)$ to a finite dimensional Lie algebra $L^*(V)$, the new Lie algebra of $(V, 0)$. Let L_t^* be the new Lie algebra of V_t ($t^2 \neq 4$), then we get a family of Lie algebras. Suggested as in [**SY**], we strongly believe that the following strong version of Torelli-type theorem is valid for this family: $L_t^* \simeq L_s^*$ if and only if V_t is biholomorphic to V_s, but it is hard to prove. In this section, we shall prove a weak version: L_t^* is a non-trivial one-parameter family. The main tool of our proof is the theory of deformation and cohomology of Lie algebra.

Firstly, we introduce some basic definitions about the deformation and cohomology of Lie algebra. Detailed discussions can be found in [**CE**], [**HS**] and [**NR**].

Let $L = (V, \eta)$ be a finite dimensional Lie algebra where η is a Lie algebra multiplication and V is the based vector space. Let $C^n(L, L)$ be the vector space of all alternating n-linear maps of V into itself. We define the coboundary operator $\delta : C^n(L, L) \to C^{n+1}(L, L)$ as follows.

For any $\alpha \in C^n(L, L)$,

$$\begin{aligned}\delta\alpha(x_0, \ldots, x_n) =& \sum_{i=0}^{n}(-1)^i \eta(x_i, \alpha(x_0, \ldots, \hat{x_i}, \ldots, x_n)) \\ &+ \sum_{i<j}(-1)^{i+j}\alpha(\eta(x_i, x_j), x_0, \ldots, \hat{x_i}, \ldots, \hat{x_j}, \ldots, x_n),\end{aligned}$$

where $(\hat{O})$ means taking O out of the expression. Recall that $Z^n(L, L) = \mathrm{Kernel}(\delta : C^n(L, L) \to C^{n+1}(L, L))$, $B^n(L, L) = \mathrm{Image}(\delta : C^{n-1}(L, L) \to C^n(L, L))$, $B^n(L, L) \subseteq Z^n(L, L)$ and $H^n(L, L) = Z^n(L, L)/B^n(L, L)$.

Let $L = (V, \eta)$ be a Lie algebra and $\varphi \in C^2(L, L)$ be an alternating bilinear map of V into itself. Then $\eta' = \eta + \varphi$ is also a Lie algebra multiplication if and only if it satisfies the Jacobian identity

$$\eta'(x, \eta'(y, z)) + \eta'(y, \eta'(z, x)) + \eta'(z, \eta'(x, y)) = 0 \tag{2}$$

for any $x, y, z \in V$. It can be shown that (2) holds if and only if

$$\delta\varphi - [\varphi, \varphi]/2 = 0, \tag{3}$$

where $[\varphi, \varphi]$ is defined as:

$$[\varphi, \varphi](x, y, z) = 2\varphi(\varphi(x, y), z) + 2\varphi(\varphi(y, z), x) + 2\varphi(\varphi(z, x), y).$$

Equation (3) is called the deformation equation. Let $\eta_t = \eta + t\varphi_1 + t^2\varphi_2 + \ldots$ be a one-parameter family of Lie algebra multiplications on V, where $\varphi_i \in C^2(L, L)$. Then $t\varphi_1 + t^2\varphi_2 + \ldots$ satisfies the deformation equation, which implies that $\delta\varphi_1 = 0$. Hence $\varphi_1 \in Z^2(L, L)$, and we call φ_1 an infinitesimal deformation of η.

A one-parameter family of Lie algebra multiplications $\eta_t = \eta + t\varphi_1 + t^2\varphi_2 + \ldots$ is said to be trivial if $(V, \eta_t) \simeq (V, \eta_s)$ for any s, t. Then there exists a one-parameter family of invertible linear maps $I_t = I + t\alpha_1 + t^2\alpha_2 + \ldots$, where $\alpha_i \in C^1(L, L)$ is a linear map from V to itself and I is the identity map, such that

$$\eta_t(x, y) = I_t\eta((I_t)^{-1}x, (I_t)^{-1}y) \tag{4}$$

for any $x, y \in V$. It is easy to verify that (4) implies that $\varphi_1 = -\delta\alpha_1$. Hence $\varphi_1 \in B^2(L, L)$, and we call φ_1 a trivial infinitesimal deformation.

Now back to $\widetilde{E_7}$, its (μ, τ)-constant family is

$$V_t = \{(x, y, z) \in \mathbb{C}^3 \mid f_t(x, y, z) = x^4 + y^4 + tx^2y^2 + z^2 = 0\}$$

with $t^2 \neq 4$.

Proof of Theorem B. The moduli algebra of V_t, denoted as $A(V_t)$, is given by

$$\begin{aligned}A(V_t) =& \mathbb{C}\{x, y, z\}/(\frac{\partial f_t}{\partial x}, \frac{\partial f_t}{\partial y}, \frac{\partial f_t}{\partial z}) \\ &= < 1, x, y, x^2, xy, y^2, x^2y, xy^2, x^2y^2 >,\end{aligned}$$

with multiplication rules

$$x^3 = -\frac{t}{2}xy^2, y^3 = -\frac{t}{2}x^2y, x^3y = xy^3 = 0.$$

Then the generalized moduli algebra

$$A^*(V_t) =< 1, x, y, x^2, xy, y^2, x^2y, xy^2 >$$

with multiplication rules

$$x^3 = -\frac{t}{2}xy^2, y^3 = -\frac{t}{2}x^2y, x^3y = xy^3 = x^2y^2 = 0.$$

By calculation, a basis for the new Lie algebra $L_t^* := L^*(V_t) = \mathrm{Der}(A^*(V_t), A^*(V_t))$ is:

(5)
$$\begin{aligned}
\deg 0: \quad & e_0 = x\partial_x + y\partial_y;\\
\deg 1: \quad & e_1 = x^2\partial_x, \quad e_2 = y^2\partial_y, \quad e_3 = y^2\partial_x, \quad e_4 = x^2\partial_y,\\
& e_5 = xy\partial_x, \quad e_6 = xy\partial_y;\\
\deg 2: \quad & e_7 = x^2y\partial_x, \quad e_8 = xy^2\partial_y, \quad e_9 = xy^2\partial_x, \quad e_{10} = x^2y\partial_y,
\end{aligned}$$

for $t^2 \neq 0, 4, 36$. For $t = 0, 6$ and -6, $\{e_0\}$ is replaced by $\{x\partial_x, y\partial_y\}$, $\{e_0, y\partial_x + x\partial_y\}$ and $\{e_0, y\partial_x - x\partial_y\}$ respectively. And it is easy to see that L_t^* is a solvable Lie algebra.

The nilradical N_t of L_t^* is spanned by $< e_1, \dots, e_{10} >$ (for all t such that $t^2 \neq 4$). Hence we get a family of 10 dimensional nilpotent Lie algebras. It suffices to show that N_t ($t^2 \neq 4$) is a non-trivial family to guarantee the non-trivialness of L_t^*.

By calculation, we display the multiplication of N_t by the following table:

$$\begin{aligned}
&[e_1, e_2] = 0, && \\
&[e_1, e_3] = -2e_9, && [e_2, e_3] = -te_7,\\
&[e_1, e_4] = -te_8, && [e_2, e_4] = -2e_{10},\\
&[e_1, e_5] = -e_7, && [e_2, e_5] = e_9,\\
&[e_1, e_6] = e_{10}, && [e_2, e_6] = -e_8,\\
&[e_3, e_4] = 2e_8 - 2e_7, && \\
&[e_3, e_5] = -te_7/2, && [e_4, e_5] = -te_9/2 - 2e_{10},\\
&[e_3, e_6] = -2e_9 - te_{10}/2, && [e_4, e_6] = -te_8/2,\\
&[e_5, e_6] = e_8 - e_7. &&
\end{aligned}$$

Other Lie brackets $[e_i, e_j]$ for $i < j$ are 0. And e_7, e_8, e_9, e_{10} are in the center of N_t.

Let V be the based vector space of N_t and η_t be the multiplication of N_t. Then we can write

$$\eta_t = \eta_0 + t\varphi_1,$$

where $\eta_0(e_i, e_j)$ for $1 \le i, j \le 6$ is given by the (i, j)-entry of the following matrix

$$\begin{pmatrix} 0 & 0 & -2e_9 & 0 & -e_7 & e_{10} \\ 0 & 0 & 0 & -2e_{10} & e_9 & -e_8 \\ 2e_9 & 0 & 0 & 2e_8 - 2e_7 & 0 & -2e_9 \\ 0 & 2e_{10} & 2e_7 - 2e_8 & 0 & -2e_{10} & 0 \\ e_7 & -e_9 & 0 & 2e_{10} & 0 & e_8 - e_7 \\ -e_{10} & e_8 & 2e_9 & 0 & e_7 - e_8 & 0 \end{pmatrix}$$

and $\eta_0(e_k, e_s) = 0$ for $k, s = 7, 8, 9, 10$. And $\varphi_1(e_i, e_j)$ for $1 \le i, j \le 6$ is given by the (i, j)-entry of the matrix

$$\begin{pmatrix} 0 & 0 & 0 & -e_8 & 0 & 0 \\ 0 & 0 & -e_7 & 0 & 0 & 0 \\ 0 & e_7 & 0 & 0 & -e_7/2 & -e_{10}/2 \\ e_8 & 0 & 0 & 0 & -e_9/2 & -e_8/2 \\ 0 & 0 & e_7/2 & e_9/2 & 0 & 0 \\ 0 & 0 & e_{10}/2 & e_8/2 & 0 & 0 \end{pmatrix},$$

and $\varphi_1(e_k, e_s) = 0$ for $k, s = 7, 8, 9, 10$.

Now we only need to check that φ_1 is a non-trivial infinitesimal deformation, (i.e. $\varphi_1 \notin B^2(L, L)$ where $L = (V, \eta_0)$), so N_t is a non-trivial family. Let $\eta_0(e_i, e_j) = \sum_{s=1}^{10} u_{ij}^s e_s$ and $\varphi_1(e_i, e_j) = \sum_{s=1}^{10} v_{ij}^s e_s$ for $i, j = 1, \ldots, 10$. If there exists a linear map $\alpha : V \to V$ such that $\delta\alpha = \varphi_1$, with $\alpha(e_i) = \sum_{j=1}^{10} a_{ij} e_j$, then we have

$$\begin{aligned} \varphi_1(e_i, e_j) &= \delta\alpha(e_i, e_j) \\ &= \eta_0(e_i, \alpha(e_j)) - \eta_0(e_j, \alpha(e_i)) - \alpha(\eta_0(e_i, e_j)) \\ &= \eta_0(e_i, \sum_{k=1}^{10} a_{jk} e_k) - \eta_0(e_j, \sum_{k=1}^{10} a_{ik} e_k) - \alpha(\sum_{k=1}^{10} u_{ij}^k e_k) \\ &= \sum_{s=1}^{10} \sum_{k=1}^{10} (a_{jk} u_{ik}^s - a_{ik} u_{jk}^s - u_{ij}^k a_{ks}) e_s. \end{aligned}$$

Hence

$$\sum_{k=1}^{n} (a_{jk} u_{ik}^s - a_{ik} u_{jk}^s - u_{ij}^k a_{ks}) = v_{ij}^s,$$

for $i, j, s = 1, \ldots, 10$. 1000 linear equations of 100 variables a_{ij} are derived. This system of linear equations is solved with the help of computer with no solution. Hence $\varphi_1 \notin B^2(L, L)$ and the family is non-trivial. q.e.d.

5. Simple elliptic singularity $\widetilde{E_8}$

$\widetilde{E_8}$ is a simple elliptic singularity defined by $\{(x,y,z)\in\mathbb{C}^3 \mid x^6+y^3+z^2=0\}$. In [**SY**], the third author of this paper and his co-worker studied the (μ,τ)-constant family of $\widetilde{E_8}$

$$V_t=\{(x,y,z)\in\mathbb{C}^3 \mid f_t=x^6+y^3+z^2+tx^4y=0\}$$

with $4t^3+27\neq 0$. They proved the following Torelli-type theorem for $\widetilde{E_8}$: for any t,s in $\mathbb{C}-\{t\in\mathbb{C}:4t^3+27=0\}$, V_t is biholomorphic to V_s if and only if their Yau algebras are isomorphic to each other, and if and only if $t^3=s^3$.

Instead of the Yau algebra, we consider the new Lie algebra of V_t, denoted by L_t^*, then we get a family of finite dimensional Lie algebras. In this section, we shall prove the Torelli-type theorem for this family, i.e. Theorem A.

Definition 5.1. Let $\{(x_1:y_1),(x_2:y_2),(x_3:y_3),(x_4:y_4)\}$ be an ordered set of four distinct points in $\mathbb{C}P^1$, then the cross ratio of this ordered set is defined as

$$\frac{(x_1y_3-x_3y_1)(x_2y_4-x_4y_2)}{(x_1y_4-x_4y_1)(x_2y_3-x_3y_2)}.$$

For example, if $(x_3,y_3)=a(x_1,y_1)+b(x_2,y_2)$ and $(x_4,y_4)=c(x_1,y_1)+d(x_2,y_2)$ for some complex numbers $a,b,c,d\in\mathbb{C}$, then the cross ratio is equal to $\frac{bc}{ad}$.

Proof of Theorem A. The moduli algebra of V_t, denoted by $A(V_t)$, is given by

$$\begin{aligned}A(V_t)=&\mathbb{C}\{x,y,z\}/(\frac{\partial f_t}{\partial x},\frac{\partial f_t}{\partial y},\frac{\partial f_t}{\partial z})\\ =&<1,x,x^2,y,x^3,xy,x^4,x^2y,x^3y,x^4y>\end{aligned}$$

with multiplication rules

$$y^2=-\frac{t}{3}x^4,x^5=-\frac{2t}{3}x^3y \tag{6}$$

and

$$A^*(V_t)=<1,x,x^2,y,x^3,xy,x^4,x^2y,x^3y>,$$

with multiplication rules

$$y^2=-\frac{t}{3}x^4,x^5=-\frac{2t}{3}x^3y,x^4y=0. \tag{7}$$

By calculation, a basis for $L_t^* := L^*(V_t)$ is the following:

$$\begin{aligned}
&\deg 0: && e_0 = x\partial_x + 2y\partial_y;\\
&\deg 1: && e_1 = x^2\partial_x + 2xy\partial_y, \quad e_2 = 3y\partial_x - 2tx^3\partial_y,\\
& && e_3 = ty\partial_x - 3xy\partial_y;\\
&\deg 2: && e_4 = x^3\partial_x, \quad e_5 = xy\partial_x, \quad e_6 = x^2y\partial_y, \quad e_7 = x^4\partial_y;\\
&\deg 3: && e_8 = x^4\partial_x, \quad e_9 = x^2y\partial_x, \quad e_{10} = x^3y\partial_y;\\
&\deg 4: && e_{11} = x^3y\partial_x,
\end{aligned}$$

for $4t^3 + 27 \neq 0$ and $t \neq 0$. For $t = 0$, $\{e_0\}$ is replaced by $\{x\partial_x, y\partial_y\}$. And it is easy to see that L_t^* is a solvable Lie algebra. Let N_t be the nilradical of L_t^* spanned by $< e_1, e_2, \ldots, e_{11} >$, for all t such that $4t^3 + 27 \neq 0$. By calculation, we display the multiplication of N_t by the following table.

$$\begin{aligned}
&[e_1, e_2] = 0, && \\
&[e_1, e_3] = -3e_6 + \frac{2t^2}{3}e_7, && [e_2, e_3] = -2t^2e_4 + 9e_5 + 6t^2e_6 + 9te_7,\\
&[e_1, e_4] = e_8 - 2e_{10}, && [e_2, e_4] = 9e_9 - 4t^2e_{10},\\
&[e_1, e_5] = e_9 - \frac{4t^2}{9}e_{10}, && [e_2, e_5] = -3te_8 + 6te_{10},\\
&[e_1, e_6] = 2e_{10}, && [e_2, e_6] = -3e_9 + \frac{8t^2}{3}e_{10},\\
&[e_1, e_7] = -\frac{4t}{3}e_{10}, && [e_2, e_7] = -3e_8 + 12e_{10},\\
&[e_1, e_8] = -\frac{4t}{3}e_{11}, && [e_2, e_8] = 12e_{11},\\
&[e_1, e_9] = 2e_{11}, && [e_2, e_9] = \frac{8t^2}{3}e_{11},\\
&[e_1, e_{10}] = 0, && [e_2, e_{10}] = -3e_{11},\\
&[e_3, e_4] = 3te_9 + 3e_{10}, && \\
&[e_3, e_5] = -\frac{t^2}{3}e_8 - 3e_9 + \frac{2t^2}{3}e_{10}, && [e_4, e_5] = -2e_{11},\\
&[e_3, e_6] = -te_9 + \frac{4t^3}{9}e_{10}, && [e_4, e_6] = 0,\\
&[e_3, e_7] = -te_8 + 2te_{10}, && [e_4, e_7] = 0,\\
&[e_3, e_8] = 4te_{11}, && [e_5, e_6] = -e_{11},\\
&[e_3, e_9] = (\frac{4t^3}{9} - 3)e_{11}, && [e_5, e_7] = \frac{2t}{3}e_{11},\\
&[e_3, e_{10}] = -te_{11}, && [e_6, e_7] = 0.
\end{aligned}$$

Other Lie brackets $[e_i, e_j]$ $(i < j)$ are zero. There are some invariant spaces of N_t. Each is preserved under all automorphisms of N_t.

$$Z = \text{center}(N_t) =< e_{11} >;$$
$$Z^2 = \{x \in N_t \mid \text{Image}(ad_x) \in Z\} =< e_8, e_9, e_{10}, e_{11} >;$$
$$Z^3 = \{x \in N_t \mid \text{Image}(ad_x) \in Z^2\} =< e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11} >;$$
$$Z^4 = \{x \in N_t \mid \text{Image}(ad_x) \in Z^3\} = N_t;$$
$$D_1 = [N_t, N_t] =< -3e_6 + \frac{2t^2}{3}e_7, -2t^2e_4 + 9e_5 + 6t^2e_6 + 9te_7, e_8, e_9, e_{10}, e_{11} >;$$
$$D_2 = [N_t, D_1] =< e_8, e_9, e_{10}, e_{11} >;$$
$$D_3 = [N_t, D_2] =< e_{11} > .$$

As in ([**SY**]), in the following we shall find out four invariant lines (i.e. they are preserved under all automorphisms of N_t) which lie in a two-dimensional vector space, then their cross-ratio is also invariant under all automorphisms. Then it can be used to distinguish N_t from N_s for $t \neq s$. For this purpose, we construct the following invariant spaces

1) $P_1 = \{x \in Z^4/Z^3 \mid \text{the dimension of Image}(ad_{\bar{x}}) \leq 1\} = \mathbb{C}\bar{e}_1 \oplus \mathbb{C}\bar{e}_2$ where $ad_x : Z^4/Z^3 \to Z^3/Z^2$;
2) $P_2 = D_1/Z^2 = \mathbb{C}(-3\bar{e}_6 + 2t^2\bar{e}_7/3) \oplus \mathbb{C}(-2t^2\bar{e}_4 + 9\bar{e}_5 + 6t^2\bar{e}_6 + 9t\bar{e}_7) \subseteq Z^3/Z^2$;
3) $P_3 = \{x \in Z^3/Z^2 \mid ad_x(Z^3/Z^2) = 0\} = \mathbb{C}(\bar{e}_4 + 2\bar{e}_6) \oplus \mathbb{C}(2t\bar{e}_6/3 + \bar{e}_7)$ where $ad_x : Z^3/Z^2 \to Z$;
4) $P_4 = \{x \in D_1/Z^2 \mid \text{ the dimension of Image}(ad_x) \leq 2\} = \mathbb{C}(-3\bar{e}_6 + 2t^2\bar{e}_7/3) \cup \mathbb{C}\Big(2t^2\bar{e}_4 - 9\bar{e}_5 - 12t^2\bar{e}_6 + t(4t^3 - 27)\bar{e}_7/3\Big) \subseteq Z^3/Z^2$, where $ad_x : Z^4/Z^3 \to Z^2/Z$;
5) $P_5 = \{x \in Z^4/Z^3 \mid \exists y \neq 0 \in P_4 \text{ such that } ad_x(y) = 0\} = \mathbb{C}(t\bar{e}_2 - 3\bar{e}_3) \cup \mathbb{C}(9\bar{e}_1 - t\bar{e}_2 + 3\bar{e}_3)$, where $ad_x : Z^3/Z^2 \to Z^2/Z$;
6) $l_1 = \text{Span}(P_5) \cap P_1 = \mathbb{C}\bar{e}_1 \subseteq Z^4/Z^3$;
7) $l_2 = \{x \in P_3 \mid ad_x(l_1) = 0\} = \mathbb{C}(2t\bar{e}_6/3 + \bar{e}_7) \subseteq Z^3/Z^2$, where $ad_x : Z^4/Z^3 \to Z^2/Z$;
8) $l_3 = [l_1, P_3] = \mathbb{C}(\bar{e}_8 + 2\bar{e}_{10}) \subseteq Z^2/Z$;
9) $l_4 = \text{Image}(ad_{l_1}) = \mathbb{C}(-3\bar{e}_6 + 2t^2\bar{e}_7/3) \subseteq Z^3/Z^2$, where $ad_{l_1} : Z^4/Z^3 \to Z^3/Z^2$;

10) $l_5 = (P_4 \setminus l_4) \cup \{0\} = \mathbb{C}\Big(2t^2\bar{e}_4 - 9\bar{e}_5 - 12t^2\bar{e}_6 + t(4t^3 - 27)\bar{e}_7/3\Big)$;

11) $l_6 = [l_1, l_4] = \mathbb{C}\bar{e}_{10} \subseteq Z^2/Z$;

12) $l_7 = [l_1, l_5] = \mathbb{C}(2t^2\bar{e}_8 - 9\bar{e}_9 - 4t^2(4t^3 + 27)\bar{e}_{10}/9) \subseteq Z^2/Z$;

13) $l_8 = [l_5, P_1] \cap (l_3 \oplus l_6) = \mathbb{C}(\bar{e}_8 - (3 + 4t^3/3)\bar{e}_{10}) \subseteq Z^2/Z$;

14) $l_9 = ([l_2, P_1] \oplus l_7) \cap (l_3 \oplus l_6) = \mathbb{C}(-\bar{e}_8 + (4 + 8t^3/9)\bar{e}_{10}) \subseteq Z^2/Z$.

l_3, l_6, l_8, l_9 are four invariant lines and they span a two dimensional subspace $< \bar{e}_8, \bar{e}_{10} >$ in Z^2/Z. By calculation, the cross ratio of this ordered set is

$$\frac{45 + 12t^3}{8t^3 + 54}.$$

Any isomorphism from N_t to N_s sends $\{l_3(t), l_6(t), l_8(t), l_9(t)\}$ to

$$\{l_3(s), l_6(s), l_8(s), l_9(s)\},$$

hence their cross ratio are the same, i.e.

$$\frac{45 + 12t^3}{8t^3 + 54} = \frac{45 + 12s^3}{8s^3 + 54} \Rightarrow s^3 = t^3.$$

If $L_t^* \simeq L_s^*$, then $N_t \simeq N_s$, which implies $s^3 = t^3$. Conversely, if $s^3 = t^3$, then $s = \zeta t$, $\zeta^3 = 1$, hence

$$f_t(x, \zeta y, z) = x^6 + y^3 + z^2 + \zeta t x^4 y = x^6 + y^3 + z^2 + s x^4 y = f_s(x, y, z).$$

Thus V_t is biholomorphic to V_s, which implies that $L_t^* \simeq L_s^*$. Hence

$$V_t \cong V_s \quad \Leftrightarrow \quad L_t^* \simeq L_s^* \quad \Leftrightarrow \quad t^3 = s^3.$$

This agrees with Seeley and Yau's work [**SY**] that $V_t \cong V_s$ if and only if $t^3 = s^3$. q.e.d.

6. Proof of Theorem C and D

We first recall the following well-known results.

Lemma 6.1. *Let J be an ideal in $P = \mathbb{C}\{x_1, \ldots, x_n\}$. Then there is a natural isomorphism of Lie algebras*

$$(\mathrm{Der}_J P)/(J \cdot \mathrm{Der}_{\mathbb{C}} P) \cong \mathrm{Der}_{\mathbb{C}}(P/J),$$

where $\mathrm{Der}_J P \subseteq \mathrm{Der}_{\mathbb{C}} P$ is the sub Lie algebra consists of all $\sigma \in \mathrm{Der}_{\mathbb{C}} P$ such that $\sigma(J) \subseteq J$.

Proof. There is a natural map $\phi : \mathrm{Der}_J P \to Der_{\mathbb{C}}(P/J)$, whose kernel contains $J \cdot \mathrm{Der}_{\mathbb{C}} P$. Note that $Der_{\mathbb{C}} P$ is a free P-module with basis $\partial/\partial x_1, \ldots, \partial/\partial x_n$ and that the coefficient of $\partial/\partial x_i$ in $\sigma \in Der_{\mathbb{C}} P$ is $\sigma(x_i)$. So if $\sigma \in \mathrm{Ker}\ \phi$, then $\sigma(x_i) \in J$ and hence $\sigma \in J \cdot \mathrm{Der}_{\mathbb{C}} P$. This verifies the injectivity. By a result of Scheja and Wiebe [**SW**], any $\bar{\sigma} \in Der_{\mathbb{C}}(P/J)$ lifts to a $\sigma \in \mathrm{Der}_{\mathbb{C}} P$ which is then necessarily in $Der_J P$. The surjectivity follows. q.e.d.

Lemma 6.2 ([**Sa1**])**.** *If $f \in \mathbb{C}\{x_1, \cdots, x_n\}$ defines an isolated singularity at the origin, for a given $i \in \{1, \cdots, n\}$, then at least one of the monomials of the form $x_i^a x_j$, $a \geq 1, j = 1, \cdots, n$, appears in f with a nonzero coefficient.*

Proof of Theorem C. Let $P = \mathbb{C}\{x_1, x_2, \ldots, x_n\}$, $I = (f_1, f_2, \ldots, f_n)$ and $I' = I + (\triangle)$, where $f_i = \partial f/\partial x_i$ and $\triangle$ is the determinant of the Hessian matrix of f. We need to prove that $\dim\mathrm{Der}_{\mathbb{C}}(P/I') \leq \dim\mathrm{Der}_{\mathbb{C}}(P/I)$. Denote as $\mathrm{Der}_{I'/I}(P/I)$ the sub Lie algebra consists of derivations in $\mathrm{Der}_{\mathbb{C}}(P/I)$ which preserves the ideal I'/I. There is a natural map

$$\begin{aligned} \varphi: \quad & \mathrm{Der}_{I'/I}(P/I) \to \mathrm{Der}_{\mathbb{C}}(P/I') \\ & (D : f \mapsto Df) \mapsto (\overline{D} : \overline{f} \mapsto \overline{Df}). \end{aligned}$$

It is easy to check that φ is well-defined. Moreover, we claim that $\dim \mathrm{Ker}\ \varphi = \dim \mathrm{Coker}\ \varphi = n$, then it follows that

$$\dim\mathrm{Der}_{\mathbb{C}}(P/I') = \dim\mathrm{Der}_{I'/I}(P/I) \leq \dim\mathrm{Der}_{\mathbb{C}}(P/I).$$

This claim will be shown by two steps. By Lemma 6.1, we can think of a derivation in $\mathrm{Der}_{\mathbb{C}}(P/I')$ as a derivation on P which preserves the ideal I', and we can think of a derivation in $\mathrm{Der}_{I'/I}(P/I)$ as a derivation on P which preserves the ideals I and I'.

Step 1. $\dim \mathrm{Ker}\ \varphi = n$. Consider derivations $\triangle\partial_i$, for $i = 1, 2, \ldots, n$. Since $mult(f) \geq 3$, we have $\partial_i f_j \in m$ for $i, j = 1, 2, \ldots, n$, where m is the maximal ideal of P. Hence by Theorem 2.3 we have $\triangle\partial_i f_j \in I$ for $i, j = 1, 2, \ldots, n$, which follows that $\triangle\partial_i$ preserves the ideal I. Obviously $\triangle\partial_i$ preserves I', hence $\triangle\partial_i$ can be viewed as the derivations in $\mathrm{Der}_{I'/I}(P/I)$. A derivation D is contained in the kernel of φ if and only if for each $i = 1, 2, \ldots, n$ the coefficient of ∂_i is contained in I'/I. That is, it can be written as $k\triangle$ in P/I. Thus the kernel of φ is generated by $\triangle\partial_i$ for $i = 1, 2 \ldots, n$, which follows that the dimension of the kernel of φ is equal to n.

Step 2. $\dim \mathrm{Coker}\ \varphi = n$. Let H be the Hessian matrix of f and H^* be the adjoint matrix of H, then $HH^* = \triangle I_n$, where $\triangle$ is the determinant of H and I_n is the unit matrix. Write $H^* = (f_{ij}^*)_{i,j=1}^n$, where f_{ij}^* is a weighted homogeneous polynomial of degree $\deg\triangle - \deg f_{ij}$ in P. Let $D_i = \sum_{k=1}^n f_{ki}^* \partial_k$ for $i = 1, 2, \ldots, n$, then $D_i f_j = \sum_{k=1}^n f_{jk} f_{ki}^* =$ the (j, i)-entry of $HH^* = \triangle I_n$, i.e.

$$D_i f_j = \begin{cases} \triangle & j = i \\ 0 & j \neq i. \end{cases} \tag{8}$$

Since $\deg(f_{ki}^*\partial_k) = \deg\triangle - \deg f_{ki} - w_k = \deg\triangle - \deg f_{ik} - w_k = \deg\triangle - \deg f_i$, then D_i is a weighted homogeneous derivation of degree $\deg\triangle - \deg f_i$, for $i = 1, 2, \ldots, n$.

Next we shall prove that $\deg \triangle \geq \deg f_i$, so that D_i is a non-negative weight derivation. Without loss of generality, we may assume that $w_1 = \max\{w_1, w_2, \ldots, w_n\}$. Since f defines an isolated singularity at the origin, by Lemma 6.2 we know that f contains a term of the form $x_1^a x_j$, $j = 1, \ldots, n$ in its expansion. Since $mult(f) \geq 3$, we have $a \geq 2$ and hence $\deg f \geq 2w_1 + w_j$. So we have

$$\begin{aligned}
\deg \triangle - \deg f_i &= \sum_{k=1}^{n} (\deg f - 2w_k) - (\deg f - w_i) \\
&= (n-1)\deg f - 2\sum_{k=1}^{n} w_k + w_i \\
&\geq (n-1)(2w_1 + w_j) - 2\sum_{k=1}^{n} w_k + w_i \\
&= 2(n-1)w_1 + (n-1)w_j + w_i - 2\sum_{k=1}^{n} w_k \\
&\geq 2(n-1)w_1 + w_j + w_i - 2\sum_{k=1}^{n} w_k \\
&\geq 2(n-1)w_1 - (2\sum_{k=1}^{n} w_k - w_i - w_j) \geq 0.
\end{aligned}$$

Hence D_i is a non-negative weight derivation with degree $\deg \triangle - \deg f_i$, for $i = 1, 2, \ldots, n$. Notice that monomials with degree greater than or equal to $\deg \triangle$ belong to $I' = I + (\triangle)$, so we have $D_i \triangle \in I + (\triangle) = I'$, for $i = 1, 2, \ldots, n$. Also $D_i f_j \in I'$ (by equation (8)), we have D_i preserving the ideal I', thus D_i can be viewed as a derivation on P/I' for $i = 1, 2, \ldots, n$.

Next we shall prove that $D_1, D_2, \ldots, D_n$ generate the cokernel of φ. Fix a derivation D on P preserving I', which can be viewed as an element in $Der_{\mathbb{C}}(P/I')$. Since $Df_i \in I'$ for $i = 1, 2, \ldots, n$, there exist $c_i \in \mathbb{C}$ such that $Df_i - c_i \triangle \in I$, for $i = 1, 2, \ldots, n$. Hence we have

$$(D - \sum_{i=1}^{n} c_i D_i) f_j \in I,$$

for $j = 1, 2, \ldots, n$. Hence $D - \sum_{i=1}^{n} c_i D_i$ preserves I, and it also preserves I', since D and D_i preserve I'. So $D - \sum_{i=1}^{n} c_i D_i$ can be viewed as a derivation in $\mathrm{Der}_{I'/I}(P/I)$, which follows that $D - \sum_{i=1}^{n} c_i D_i$ belongs to the image of φ in $Der_{\mathbb{C}}(P/I')$. Hence Coker(φ) is generated by $D_1, D_2, \ldots, D_n$.

Finally, we shall prove that $D_1, D_2, \ldots, D_n$ are linearly independent in Coker(φ). Suppose $D = \sum_{i=1}^{n} c_i D_i$ is in the image of φ, then there exists a derivation D' on P preserving the ideals I and I', which can be viewed as a derivation in $\mathrm{Der}_{I'/I}(P/I)$, such that $\varphi(D') = D$. Hence $D' - D \in I' \cdot \mathrm{Der}_{\mathbb{C}}P$. Fix an $i \in \{1, 2, \ldots, n\}$, since $mult(f) \geq 3$, we have $\partial_j(f_i) \in m$ for $j = 1, 2, \ldots, n$, where m is the maximal ideal of P. By Theorem 2.3, $mI' \subseteq I$, hence $(D' - D)f_i \in I$. Since $D'(f_i) \in I$ (because D' preserves I) and $D(f_i) = c_i\triangle$ (by Equation (8)), we have $c_i\triangle \in I$, hence $c_i = 0$ for $i = 1, 2, \ldots, n$. Hence $D_1, D_2, \ldots, D_n$ are linearly independent in Coker φ. So the dimension of Coker(φ) is equal to n. q.e.d.

Proof of Theorem D. Let us first recall that the following Yau's conjecture.

Yau Conjecture. Let

$$(V, 0) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n \colon f(x_1, x_2, \ldots, x_n) = 0\}$$

be an isolated singularity defined by the weighted homogeneous polynomial $f(x_1, x_2, \ldots, x_n)$. Then there are no non-zero negative weight derivations on the moduli algebra (= Milnor algebra here) $A(V) = \mathbb{C}[x_1, x_2, \ldots, x_n]/(f_{x_1}, f_{x_2}, \ldots, f_{x_n})$ where $f_{x_i} = \partial f/\partial x_i$, i.e., the Yau algebra $L(V)$ of V is a non-negatively graded algebra.

The Yau Conjecture has only been proven in the low-dimensional case $n \leq 4$ ([**CXY**], [**Ch**]) by explicit calculations. Recently, Yau and Zuo proved the following result.

Theorem 6.1 ([**YZ1**]). *Let*

$$(V, 0) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n \colon f(x_1, x_2, \ldots, x_n) = 0\}$$

be an isolated singularity defined by the weighted homogeneous polynomial $f(x_1, x_2, \ldots, x_n)$ of canonical weight type $(w_1, w_2, \cdots, w_n; d)$ (i.e., $d \geq 2w_1 \geq 2w_2 \geq \cdots \geq 2w_n > 0$). Let

$$A(V) = \mathbb{C}[x_1, x_2, \ldots, x_n]/(f_{x_1}, f_{x_2}, \ldots, f_{x_n})$$

be the moduli algebra. If $w_n \geq w_1/2$, then $Der(A(V))_{<0} = 0$.

In the proof of Theorem C, if there is no negative weight derivation in $Der_{\mathbb{C}}(P/I)$ (i.e., the condition of Yau Conjecture is satisfied), then $Der_{\mathbb{C}}(P/I) = \mathrm{Der}_{I'/I}(P/I)$, since monomials with a degree greater than or equal to deg $\triangle$ belong to $I' = I + (\triangle)$. In this case the dimension of new Lie algebra of $(V, 0)$ defined by a weighted homogeneous polynomial f in $\mathbb{C}[x_1, \ldots, x_n]$ $(n \geq 2)$ with $mult(f) \geq 3$, is equal to the Yau number of $(V, 0)$. Since the Yau Conjecture is true for $n \leq 4$, so is Conjecture 1.1. Thus Theorem D is proved. q.e.d.

REMARK 6.1. As a corollary of Theorem 6.1, we also know that Conjecture 1.1 is true for weighted homogeneous singularities with $w_n \geq w_1/2$.

7. Computing the new Lie algebras

We compute the new Lie algebra for binomial singularities, which includes the simple singularities as special case. As an application, we prove that the simple hypersurface singularities can be characterized completely by the new Lie algebra.

Proposition 7.1. *Let $(V, 0)$ be a weighted homogeneous fewnomial isolated singularity of type A, defined by $f = x_1^{a_1} + x_2^{a_2}$ ($a_1 \geq 2$, $a_2 \geq 3$) with weight type $(\frac{1}{a_1}, \frac{1}{a_2}; 1)$. Then*

$$\lambda^*(V) = \begin{cases} 2a_1a_2 - 3(a_1 + a_2) + 4, & a_1 \geq 3, a_2 \geq 3 \\ a_2 - 3, & a_1 = 2, a_2 \geq 3. \end{cases}$$

Proof. It follows from Proposition 2.1 and Theorem 2.4 that the generalized moduli algebra

$$\begin{aligned} A^*(V) &= \mathbb{C}\{x_1, x_2\}/(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \frac{\partial^2 f}{\partial x_1 \partial x_2}\frac{\partial^2 f}{\partial x_2 \partial x_1}) \\ &= \mathbb{C}\{x_1, x_2\}/(x_1^{a_1-1}, x_2^{a_2-1}, x_1^{a_1-2}x_2^{a_2-2}) \end{aligned}$$

has dimension $(a_1-1)(a_2-1)-1$. It is easy to see that it has a monomial basis of the form:

(9)

($case$ 1) if $a_1 \geq 3$, $\{x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq a_1 - 2; 0 \leq i_2 \leq a_2 - 3; x_1^{i_1}x_2^{a_2-2}, 0 \leq i_1 \leq a_1 - 3\}$,

(10)

($case$ 2) if $a_1 = 2$, $\{x_2^{i_2}, 0 \leq i_2 \leq a_2 - 3\}$,

with

$$x_1^{a_1-1} = 0, \tag{11}$$

$$x_2^{a_2-1} = 0, \tag{12}$$

$$x_1^{a_1-2}x_2^{a_2-2} = 0. \tag{13}$$

In order to compute a derivation D of $A^*(V)$, it suffices to indicate its values on the generators x_1, x_2, written in terms of the basis (9) or (10). Without loss of generality, we write

$$Dx_j = \sum_{i_1=0}^{a_1-2}\sum_{i_2=0}^{a_2-3} c_{i_1,i_2}^j x_1^{i_1}x_2^{i_2} + \sum_{i_1=0}^{a_1-3} c_{i_1,a_2-2}^j x_1^{i_1}x_2^{a_2-2},\ j = 1, 2.$$

Using the relations (11)–(13), one easily finds the necessary and sufficient conditions defining a derivation of $A^*(V)$ as follows:

$$c^1_{0,0} = c^1_{0,1} = \cdots = c^1_{0,a_2-3} = 0; \tag{14}$$

$$c^2_{0,0} = c^2_{1,0} = \cdots = c^2_{a_1-3,0} = 0. \tag{15}$$

Using (14)–(15) we obtain the following description of the new Lie algebras in question.

The following derivations form a basis of $\mathrm{Der} A^*(V)$

$$x_1^{i_1}x_2^{i_2}\partial_1, 1 \le i_1 \le a_1-2, 0 \le i_2 \le a_2-3;\ x_1^{i_1}x_2^{a_2-2}\partial_1, 0 \le i_1 \le a_1-3;$$
$$x_1^{i_1}x_2^{i_2}\partial_2, 0 \le i_1 \le a_1-3, 1 \le i_2 \le a_2-2;\ x_1^{a_1-2}x_2^{i_2}\partial_2, 0 \le i_2 \le a_2-3.$$

Therefore we have

$$\lambda^*(V) = 2a_1a_2 - 3(a_1+a_2) + 4.$$

In the case that $a_1 = 2, a_2 \ge 3$, we obtain the conditions to define a derivation of $A^*(V)$

$$c^1_{0,0} = c^1_{0,1} = \cdots = c^1_{0,a_2-3} = 0; \tag{16}$$

$$c^2_{0,0} = 0. \tag{17}$$

Using (16)–(17), one obtains the derivations

$$x_2^{i_2}\partial_2, 1 \le i_2 \le a_2-3, \tag{18}$$

which form a basis of $L^*(V)$. Therefore, we get

$$\lambda^*(V) = a_2 - 3.$$

q.e.d.

Remark 7.1. Since our new Lie algebra is not defined for the Milnor number $\mu(f) = 1$. The restriction $a_1 \ge 2,\ a_2 \ge 3$ in Proposition 7.1 follows from $\mu(f) \ge 2$. The similar restrictions also appear in Proposition 7.2 and Proposition 7.3 below.

Proposition 7.2. *Let $(V,0)$ be a binomial isolated singularity of type B defined by $f = x_1^{a_1}x_2 + x_2^{a_2}$ ($a_1 \ge 2,\ a_2 \ge 2$) with weight type $(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}; 1)$. Then*

$$\lambda^*(V) = \begin{cases} 2a_1a_2 - 2a_1 - 3a_2 + 5, & a_1 \ge 2, a_2 \ge 3 \\ 2a_1 - 3, & a_1 \ge 2, a_2 = 2. \end{cases}$$

Proof. It is easy to see that the generalized moduli algebra

$$\begin{aligned} A^*(V) = &\mathbb{C}\{x_1,x_2\}/(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \frac{\partial^2 f}{\partial x_1 \partial x_2}\frac{\partial^2 f}{\partial x_2 \partial x_1}) \\ =&\mathbb{C}\{x_1,x_2\}/(x_1^{a_1-1}x_2, x_1^{a_1} + a_2x_2^{a_2-1}, \\ &a_2(a_1-1)(a_2-1)x_1^{a_1-2}x_2^{a_2-1} - a_1x_1^{2a_1-2}) \end{aligned}$$

has dimension $a_2(a_1-1)$ and has a monomial basis of the form (cf. [**AGV**], Theorem 13.1):

(19)
$$(case\ 1)\text{ if } a_1\geq 3,\ \{x_1^{i_1}x_2^{i_2}, 0\leq i_1\leq a_1-2; 0\leq i_2\leq a_2-2;\\ x_1^{i_1}x_2^{a_2-1}, 0\leq i_1\leq a_1-3; x_1^{a_1-1}\},$$

(20)
$$(case\ 2)\text{ if } a_1=2,\ \{x_2^{i_2}, 0\leq i_2\leq a_2-2; x_1^{a_1-1}\},$$

(21)
$$(case\ 3)\text{ if } a_2=2,\ \{x_1^{i_1}, 0\leq i_1\leq a_1-2; x_1^{i_1}x_2, 0\leq i_1\leq a_1-3; x_1^{a_1-1}\},$$

with

$$x_1^{a_1-1}x_2=0, \tag{22}$$

$$x_1^{a_1}+a_2x_2^{a_2-1}=0, \tag{23}$$

$$a_2(a_1-1)(a_2-1)x_1^{a_1-2}x_2^{a_2-1}-a_1x_1^{2a_1-2}=0. \tag{24}$$

Using the relations (22)–(24), we get

$$x_1^i=0,\ i\geq 2a_1-2, \tag{25}$$

$$x_2^i=0,\ i\geq a_2. \tag{26}$$

In order to compute a derivation D of $A^*(V)$, it suffices to indicate its values on the generators x_1,x_2 which can be written in terms of the basis (19) (or (20), (21) resp.). Thus we can write

$$Dx_j=\sum_{i_1=0}^{a_1-2}\sum_{i_2=0}^{a_2-2}c^j_{i_1,i_2}x_1^{i_1}x_2^{i_2}+\sum_{i_1=0}^{a_1-3}c^j_{i_1,a_2-1}x_1^{i_1}x_2^{a_2-1}+c^j_{a_1-1,0}x_1^{a_1-1},\ j=1,2.$$

Using the relations (22)–(26), one finds the conditions to define a derivation of $A^*(V)$

$$c^1_{0,0}=c^1_{0,1}=\cdots=c^1_{0,a_2-3}=0; \tag{27}$$

$$c^2_{0,0}=c^2_{1,0}=\cdots=c^2_{a_1-2,0}=0; \tag{28}$$

$$a_1c^1_{1,0}=(a_2-1)c^2_{0,1}, a_1c^1_{2,0}=(a_2-1)c^2_{1,1},\cdots,\\ a_1c^1_{a_1-2,0}=(a_2-1)c^2_{a_1-3,1}. \tag{29}$$

Using (27)–(29), we obtain the following description of the new Lie algebra. The following derivations form a basis of $\mathrm{Der}A^*(V)$:

$$x_1^{i_1}x_2^{i_2}\partial_1, 1\leq i_1\leq a_1-2, 1\leq i_2\leq a_2-2; x_1^{a_1-1}\partial_1;\\ x_1^{i_1}x_2^{a_2-1}\partial_1, 0\leq i_1\leq a_1-3; x_2^{a_2-2}\partial_1;$$

$$x_1^{a_1-2}x_2^{i_2}\partial_2, 1 \leq i_2 \leq a_2-2; x_1^{a_1-1}\partial_2;$$
$$a_1x_1^{i_1}\partial_1 + (a_2-1)x_1^{i_1-1}x_2\partial_2, 1 \leq i_1 \leq a_1-2;$$
$$x_1^{i_1}x_2^{i_2}\partial_2, 0 \leq i_1 \leq a_1-3, 2 \leq i_2 \leq a_2-1.$$

Therefore, we have

$$\lambda^*(V) = 2a_1a_2 - 2a_1 - 3a_2 + 5.$$

In the case that $a_1 = 2, a_2 \geq 3$, we obtain the conditions to define a derivation of $A^*(V)$ as follows:

$$c^1_{0,0} = c^1_{0,1} = \cdots = c^1_{0,a_2-3} = 0; \tag{30}$$
$$c^2_{0,0} = 0. \tag{31}$$

Using (30)–(31), we obtain the following description of the new Lie algebra. The following derivations form a basis of $\mathrm{Der}A^*(V)$

$$x_1\partial_1, x_2^{a_2-2}\partial_1, x_1\partial_2;$$
$$x_2^{i_2}\partial_2, 1 \leq i_2 \leq a_2-2.$$

Therefore, we have

$$\lambda^*(V) = a_2 + 1. \tag{32}$$

In the case that $a_1 \geq 2, a_2 = 2$, we obtain the conditions to define a derivation of $A^*(V)$

$$c^1_{0,0} = 0. \tag{33}$$
$$c^2_{0,0} = c^2_{1,0} = \cdots = c^2_{a_1-1,0} = 0; \tag{34}$$
$$a_1c^1_{1,0} = c^2_{0,1}, a_1c^1_{2,0} = c^2_{1,1}, \cdots, a_1c^1_{a_1-2,0} = c^2_{a_1-3,1}. \tag{35}$$

Using (33)–(35), we obtain the following description of the new Lie algebra. The following derivations form a basis of $\mathrm{Der}A^*(V)$

$$x_1^{a_1-1}\partial_1, x_1^{i_1}x_2\partial_1, 0 \leq i_1 \leq a_1-3;$$
$$a_1x_1^{i_1}\partial_1 + x_1^{i_1-1}x_2\partial_2, 1 \leq i_1 \leq a_1-2.$$

Therefore, we have

$$\lambda^*(V) = 2a_1 - 3. \tag{36}$$

q.e.d.

Proposition 7.3. *Let* $(V, 0)$ *be a binomial isolated singularity of type C, defined by* $f = x_1^{a_1}x_2 + x_2^{a_2}x_1$ $(a_1 \geq a_2 \geq 2)$ *with weight type* $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}; 1)$. *Then*

$$\lambda^*(V) = \begin{cases} 2a_1a_2 - 2a_1 - 2a_2 + 6, & a_1 \geq a_2 \geq 3, \\ 2a_1, & a_1 \geq a_2 = 2. \end{cases}$$

Proof. It is easy to see that the generalized moduli algebra

$$A^*(V) = \mathbb{C}\{x_1, x_2\}/(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \frac{\partial^2 f}{\partial x_1 \partial x_2}\frac{\partial^2 f}{\partial x_2 \partial x_1})$$
$$= \mathbb{C}\{x_1, x_2\}/(a_1 x_1^{a_1-1}x_2 + x_2^{a_2}, x_1^{a_1} + a_2 x_2^{a_2-1}x_2,$$
$$a_1a_2(a_1a_2 - a_1 - a_2 - 1)x_1^{a_1-1}x_2^{a_2-1} - a_1^2 x_1^{2a_1-2} - a_2^2 x_2^{2a_2-2})$$

has dimension a_1a_2-1 and has monomial basis of the form (cf. [**AGV**], Theorem 13.1):

$$\text{(37)}\quad (1) \text{ if } a_1 \geq a_2 \geq 3, \quad \{x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq a_1 - 2; 0 \leq i_2 \leq a_2 - 1; x_2^{i_2}, a_2 \leq i_2 \leq 2a_2 - 3; x_1^{a_1-1}\},$$

$$\text{(38)}\quad (2) \text{ if } a_1 \geq a_2 = 2, \quad \{x_1^{i_1}, 0 \leq i_1 \leq a_1 - 1; x_2^{i_2}, 0 \leq i_2 \leq a_1 - 2\},$$

with

$$a_1 x_1^{a_1-1}x_2 + x_2^{a_2} = 0, \tag{39}$$
$$x_1^{a_1} + a_2 x_1 x_2^{a_2-1} = 0, \tag{40}$$
$$a_1a_2(a_1a_2 - a_1 - a_2 - 1)x_1^{a_1-1}x_2^{a_2-1} - a_1^2 x_1^{2a_1-2} - a_2^2 x_2^{2a_2-2} = 0. \tag{41}$$

Using the relations (39)–(41), we get

$$x_1^i = 0,\ i \geq 2a_1 - 2, \tag{42}$$
$$x_2^i = 0,\ i \geq 2a_2 - 2. \tag{43}$$

In order to compute a derivation D of $A^*(V)$, it suffices to indicate its values on the generators x_1, x_2, written in terms of the basis (37) (resp. (38)). Thus we can write

$$Dx_j = \sum_{i_1=0}^{a_1-2}\sum_{i_2=0}^{a_2-1} c^j_{i_1,i_2} x_1^{i_1}x_2^{i_2} + \sum_{i_2=a_2}^{2a_2-3} c^j_{0,i_2}x_2^{i_2} + c^j_{a_1-1,0}x_1^{a_1-1},\ j = 1, 2.$$

Using the relations (39)–(43), one easily finds the necessary and sufficient conditions to define a derivation of $A^*(V)$

$$c^1_{0,0} = c^1_{0,1} = \cdots = c^1_{0,a_2-1} = 0; \tag{44}$$
$$c^2_{0,0} = c^2_{1,0} = \cdots = c^2_{a_1-1,0} = 0;\ (a_1 - 1)c^1_{0,a_2-2} = (a_2 - 1)c^2_{a_1-2,0}; \tag{45}$$
$$(a_1 - 1)c^1_{2,0} = (a_2 - 1)c^2_{1,1}, \cdots, a_1 c^1_{a_1-2,0} = a_2 c^2_{a_1-3,1}, \tag{46}$$
$$(a_1 - 1)c^1_{1,1} = (a_2 - 1)c^2_{0,2}, \cdots, (a_1 - 1)c^1_{1,a_2-3} = (a_2 - 1)c^2_{0,a_2-2}. \tag{47}$$

Using (44)–(47), we obtain the following description of the new Lie algebra. The following derivations form a basis of $\mathrm{Der} A^*(V)$

$$x_1^{i_1}x_2^{i_2}\partial_1,\ 2 \le i_1 \le a_1-2, 1 \le i_2 \le a_2-1;$$
$$x_1x_2^{i_2}\partial_1,\ a_2-2 \le i_2 \le a_2-1; x_1^{a_1-1}\partial_1;$$
$$x_2^{i_2}\partial_1,\ a_2-1 \le i_2 \le 2a_2-3;$$
$$x_1^{i_1}x_2^{i_2}\partial_2,\ 1 \le i_1 \le a_1-2, 2 \le i_2 \le a_2-1;$$
$$x_2^{i_2}\partial_2,\ a_2-1 \le i_2 \le 2a_2-3;$$
$$x_1^{a_1-2}x_2\partial_2;\ x_1^{a_1-1}\partial_2;$$
$$a_1x_2^{a_2-2}\partial_1 + a_2x_1^{a_1-2}\partial_2;$$
$$(a_1-1)x_1^{i_1}\partial_1 + (a_2-1)x_1^{i_1-1}x_2\partial_2,\ 1 \le i_1 \le a_1-2;$$
$$(a_1-1)x_1x_2^{i_2-1}\partial_1 + (a_2-1)x_2^{i_2}\partial_2,\ 2 \le i_2 \le a_2-2.$$

Therefore, we have

$$\lambda^*(V) = 2a_1a_2 - 2(a_1+a_2) + 6. \tag{48}$$

In the case that $a_1 \ge a_2 = 2$, similarly, one easily finds the necessary and sufficient conditions to define a derivation of $A^*(V)$

$$c^1_{0,0} = 0; \tag{49}$$

$$c^2_{0,0} = c^2_{1,0} = \cdots = c^2_{a_1-2,0} = 0; \tag{50}$$

$$c^2_{0,1} = 2c^2_{a_1-1,0} + (a_1-1)c^1_{1,0}; \tag{51}$$

(52)
$$(a_1-1)c^1_{2,0} = c^2_{1,1}; (a_1-1)c^1_{3,0} = c^2_{2,1} = \cdots = (a_1-1)c^1_{a_1-2,0} = c^2_{a_1-3,1}.$$

Then similarly, one obtains the basis of derivation represented by following derivations which form a basis of $\mathrm{Der} A^*(V)$

$$x_1^{a_1-1}\partial_1; x_1^{i_1}x_2\partial_1;\ 0 \le i_1 \le a_1-2; x_1^{a_1-2}x_2\partial_2;$$
$$(a_1-1)x_1^{i_1}\partial_1 + x_1^{i_1-1}x_2\partial_2,\ 1 \le i_1 \le a_1-2; (a_1-1)x_1\partial_1 + 2x_1^{a_1-1}\partial_2.$$

Therefore, we get

$$\lambda^*(V) = 2a_1.$$

q.e.d.

In order to prove Proposition 1.1, we need the following proposition.

Proposition 7.4. *The following three pairs of new Lie algebras arising from simple hypersurface singularities are not isomorphic:*

$$L^*(D_7) \ncong L^*(E_6), L^*(A_{10}) \ncong L^*(E_7), \text{and } L^*(D_{10}) \ncong L^*(E_8).$$

Proof. Case (1). $L^*(D_7) \ncong L^*(E_6)$.

It is easy to see, from Proposition 7.2, that $L^*(D_7)$ is a 7-dimensional complex new Lie algebra spanned by

$$< x_1\partial_1, x_2^4\partial_1, x_2\partial_2, x_2^2\partial_2, x_2^3\partial_2, x_2^4\partial_2, x_1\partial_2 > .$$

By setting $d_1 = x_1\partial_1, \cdots, d_7 = x_1\partial_2$, we obtain following multiplication table.

Table 1.

$\diagdown$	d_1	d_2	d_3	d_4	d_5	d_6	d_7
d_1	0	$-d_2$	0	0	0	0	d_7
d_2	d_2	0	$-4d_2$	0	0	0	d_6
d_3	0	$4d_2$	0	d_4	$2d_5$	$3d_6$	$-d_7$
d_4	0	0	$-d_4$	0	d_6	0	0
d_5	0	0	$-2d_5$	$-d_6$	0	0	0
d_6	0	0	$-3d_6$	0	0	0	0
d_7	$-d_7$	$-d_6$	d_7	0	0	0	0

The Cartan subalgebra is generated by $< d_1, d_3 >$. Therefore, the rank of $L^*(D_7)$ is 2. It follows from Table 1 that the sequence of derived series are $L^{(1)} = [L^{(0)}, L^{(0)}] = \{d_2, d_4, d_5, d_6, d_7\}$ and $L^{(2)} = [L^{(1)}, L^{(1)}] = \{d_6\}$, where $L^{(0)} = \{d_1, d_2, d_3, d_4, d_5, d_6, d_7\}$. Therefore, $(\dim(L^{(2)}), \dim(L^{(1)})) = (1, 5)$ is a sequence of dimensions of derived series.

By Proposition 7.1, one obtains the following basis of $L^*(E_6)$

$$L^*(E_6) =< x_1\partial_1, x_1x_2\partial_1, x_2^2\partial_1, x_1\partial_2, x_1x_2\partial_2, x_2\partial_2, x_2^2\partial_2 > .$$

By setting $e_1 = x_1\partial_1, \cdots, e_7 = x_2^2\partial_2$, one obtains the following multiplication table.

Table 2.

$\diagdown$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	0	$-e_3$	e_4	e_5	0	0
e_2	0	0	0	e_5	0	$-e_2$	0
e_3	e_3	0	0	$e_7 - 2e_2$	0	$-2e_3$	0
e_4	$-e_4$	$-e_5$	$2e_2 - e_7$	0	0	e_4	$2e_5$
e_5	$-e_5$	0	0	0	0	0	0
e_6	0	e_2	$2e_3$	$-e_4$	0	0	e_7
e_7	0	0	0	$-2e_5$	0	$-e_7$	0

The Cartan subalgebra is generated by $< e_1, e_6 >$. Therefore the rank of $L^*(E_6)$ is 2. It follows from Table 2 that the sequence of derived series are $L^{(1)} = [L^{(0)}, L^{(0)}] = \{e_2, e_3, e_4, e_5, e_7\}$ and $L^{(2)} = [L^{(1)}, L^{(1)}] = \{e_2, e_5, e_7\}$, where $L^{(0)} = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$. Therefore, $(3, 5)$ is a sequence of dimensions of derived series. It should be noted that both Lie algebras, $L^*(E_6)$ and $L^*(D_7)$, have different sequences of dimensions of derived series. Therefore we conclude that $L^*(E_6)$ and $L^*(D_7)$ are non-isomorphic.

Case (2). $L^*(A_{10}) \ncong L^*(E_7)$.

It is easy to see from Proposition 7.2 that the $L^*(E_7)$ is 8-dimensional complex Lie algebra spanned by

$$< x_2\partial_1, x_1x_2\partial_1, x_1^2\partial_1, x_2^2\partial_1, x_1x_2\partial_2, x_2^2\partial_2, x_2^2\partial_2, 3x_1\partial_1 + 2x_2\partial_2 > .$$

Set $e_1 = x_2\partial_1, \cdots, e_8 = 3x_1\partial_1 + 2x_2\partial_2$. It is easy to see that one obtains the following multiplication table by using Proposition 7.2.

Table 3.

$\diagdown$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	0	e_4	$2e_2$	0	$e_7 - e_2$	$2e_5 - e_3$	$-e_4$	e_1
e_2	$-e_4$	0	0	0	0	0	0	$-2e_2$
e_3	$-2e_2$	0	0	0	0	0	0	$-3e_3$
e_4	0	0	0	0	0	0	0	$-e_4$
e_5	$e_2 - e_7$	0	0	0	0	0	0	$-3e_5$
e_6	$e_3 - 2e_5$	0	0	0	0	0	0	$-4e_6$
e_7	e_4	0	0	0	0	0	0	$-2e_7$
e_8	$-e_1$	$2e_2$	$3e_3$	e_4	$3e_5$	$4e_6$	$2e_7$	0

The Cartan subalgebra is generated by $< e_8 >$. Therefore the rank of $L^*(E_7)$ is 1. It follows from Table 3 that the sequence of derived series are

$$L^{(1)} = [L^{(0)}, L^{(0)}] = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$$

and $L^{(2)} = [L^{(1)}, L^{(1)}] = \{e_2, e_3, e_4, e_5, e_7\}$, where

$$L^{(0)} = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}.$$

Therefore, $(5, 7)$ is a sequence of dimensions of derived series.

It is easy to see, from Proposition 7.2, that the $L^*(A_{10})$ is 8-dimensional complex Lie algebra spanned by

$$\{x_2\partial_2, x_2^2\partial_2, x_2^3\partial_2, x_2^4\partial_2, x_2^5\partial_2, x_2^6\partial_2, x_2^7\partial_2, x_2^8\partial_2\}.$$

Set $a_1 = x_2\partial_2, \cdots, a_8 = x_2^8\partial_2$. It is easy to see that one obtains the following multiplication table by using Proposition 7.2.

Table 4.

$\diagdown$	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
a_1	0	a_2	$2a_3$	$3a_4$	$4a_5$	$5a_6$	$6a_7$	$7a_8$
a_2	$-a_2$	0	a_4	$2a_5$	$3a_6$	$4a_7$	$5a_8$	0
a_3	$-2a_3$	$-a_4$	0	a_6	$2a_7$	$3a_8$	0	0
a_4	$-3a_4$	$-2a_5$	$-a_6$	0	a_8	0	0	0
a_5	$-4a_5$	$-3a_6$	$-2a_7$	$-a_8$	0	0	0	0
a_6	$-5a_6$	$-4a_7$	$-3a_8$	0	0	0	0	0
a_7	$-6a_7$	$-5a_8$	0	0	0	0	0	0
a_8	$-7a_8$	0	0	0	0	0	0	0

The Cartan subalgebra is generated by $< a_1 >$. Therefore the rank of $L^*(A_{10})$ is 1. It follows from Table 4 that the derived series are $L^{(1)} = [L^{(0)}, L^{(0)}] = \{a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$, $L^{(2)} = [L^{(1)}, L^{(1)}] = \{a_4, a_5, a_6, a_7, a_8\}$ and $L^{(3)} = [L^{(2)}, L^{(2)}] = \{a_8\}$, where

$$L^{(0)} = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}.$$

Therefore, $(1, 5, 7)$ is a sequence of dimensions of derived series. It should be noted that both Lie algebras $L^*(E_7)$ and $L^*(A_{10})$ have different sequence of dimensions of derived series. Therefore we conclude that $L^*(E_7)$ and $L^*(A_{10})$ are non-isomorphic.

Case (3). $L^*(D_{10}) \ncong L^*(E_8)$.

It is easy to see from Proposition 7.2, $L^*(D_{10})$ be a 10-dimensional complex new Lie algebra spanned by

$$< x_1\partial_1, x_2^7\partial_1, x_2\partial_2, x_2^2\partial_2, x_2^3\partial_2, x_2^4\partial_2, x_2^5\partial_2, x_2^6\partial_2, x_2^7\partial_2, x_1\partial_2 > .$$

It follows from Proposition 7.1 that the new Lie algebras $L^*(E_8)$ have the following basis:

$$< x_1\partial_1, x_1x_2\partial_1, x_1x_2^2\partial_1, x_2^3\partial_1, x_1\partial_2, x_1x_2\partial_2, x_1x_2^2\partial_2, x_2\partial_2, x_2^2\partial_2, x_2^3\partial_2 > .$$

With the same argument one can easily see that the rank of $L^*(D_{10})$ and $L^*(E_8)$ is 2 and the sequence of dimensions of derived series are $(4, 6, 8)$ and $(4, 8)$ respectively. Therefore both Lie algebras are non-isomorphic. q.e.d.

It is easy to see that, from Propositions 7.1 and 7.2, we get $\dim L^*(A_k) = k - 2$, $\dim L^*(D_k) = k$, $\dim L^*(E_6) = 7, \dim L^*(E_7) = 8$, and $\dim L^*(E_8) = 10$. Cartan subalgebras that from $L^*(A_k)$ and $L^*(D_k)$ are generated by $< x_2\partial_2 >$ and $< x_1\partial_1, x_2\partial_2 >$ respectively. It is then easy to verify that $\mathrm{rk}L^*(A_k) = \mathrm{rk}L^*(E_7) = 1$ while $\mathrm{rk}L^*(E_6) = \mathrm{rk}L^*(E_8) = \mathrm{rk}L^*(D_k) = 2$. When the dimensions or ranks of the new Lie algebras for all simple singularities are different, then they are certainly not isomorphic, so we only need to treat the three pairs of Lie algebras $(L^*(A_{10}), L^*(E_7)), (L^*(E_6), L^*(D_7)), (L^*(E_8), L^*(D_{10}))$ which have the same dimensions and ranks. It follows from the Proposition 7.4 that these three pairs are non-isomorphic. Therefore we have the following proposition.

Proposition 7.5 (i.e. Proposition 1.1)**.** *If X and Y are two simple hypersurface singularities then $L^*(X) \cong L^*(Y)$ as Lie algebras, if and only if X and Y are analytically isomorphic.*

Finally, we prove the following theorem.

Theorem 7.1 (i.e., Conjecture 1.1)**.** *Conjecture 1.1 is true for binomial singularities.*

Proof. In order to prove Conjecture 1.1, i.e., $\lambda^*(V) = \lambda(V)$, we need the following propositions from [**YZ2**].

Proposition 7.6 ([**YZ2**])**.** *Let $(V, 0)$ be a weighted homogeneous fewnomial isolated singularity of type A defined by $f = x_1^{a_1} + x_2^{a_2} + \cdots + x_n^{a_n}$ with weight type $(\frac{1}{a_1}, \frac{1}{a_2}, \cdots, \frac{1}{a_n}; 1)$. Then the Yau number is*

$$\lambda(V) = n\prod_{i=1}^{n}(a_i - 1) - \sum_{i}^{n}(a_1 - 1)(a_2 - 1)\cdots\widehat{(a_i - 1)}\cdots(a_n - 1),$$

where $\widehat{(a_i - 1)}$ means that $a_i - 1$ is omitted.

Proposition 7.7 ([**YZ2**])**.** *Let $(V, 0)$ be a binomial isolated singularity of type B defined by $f = x_1^{a_1}x_2 + x_2^{a_2}$ with weight type $(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}; 1)$. Then the Yau number is*

$$\lambda(V) = 2a_1a_2 - 2a_1 - 3a_2 + 5.$$

Proposition 7.8 ([**YZ2**])**.** *Let $(V, 0)$ be a binomial isolated singularity of type C defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_1$ with weight type $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}; 1)$. If $mult(f) \geq 4$, i.e., $a_1, a_2 \geq 3$, then the Yau number is*

$$\lambda(V) = 2a_1a_2 - 2a_1 - 2a_2 + 6.$$

If $mult(f) = 3$, i.e., $f = x_1^2x_2 + x_2^{a_2}x_1$, then the Yau number is $\lambda(V) = 2a_2$.

Comparing the Yau number $\lambda(V)$ with the new analytic invariant $\lambda^*(V)$ in the case of binomial isolated singularities of type A, type B and type C (see Propositions 7.1–7.3), it is easy to see that the conjecture holds for binomial isolated singularities, i.e.

$$\lambda^*(V) = \lambda(V),$$

and hence Theorem 7.1 is proved. q.e.d.

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