Journal of Algebra 564 (2020) 199–246

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On the non-existence of negative weight derivations of the new moduli algebras of singularities ☆

Guorui Ma, Stephen S.-T. Yau *, Huaiqing Zuo *

Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, PR China

A R T I C L E I N F O

Article history:
Received 4 October 2019
Available online 12 August 2020
Communicated by Steven Dale Cutkosky

MSC:
14B05
32S05

Keywords:
Derivation Lie algebra
Isolated singularity
Weighted homogeneous

A B S T R A C T

We propose a new conjecture about the non-existence of negative weight derivations of the new moduli algebras of weighted homogeneous hypersurface singularities and verify this conjecture up to dimension three.

1. Introduction

For any isolated hypersurface singularity $(V,0) \subset (\mathbb{C}^{n+1},0)$ defined by the holomorphic function $f : (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$, one has the moduli algebra $A(V) := \mathcal{O}_{n+1}/(f, \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n})$ which is finite dimensional. The well-known Mather-Yau theo-

☆ Both Yau and Zuo are supported by NSFC Grants 11961141005 and 11531007. Zuo is supported by NSFC Grant 11771231. Yau is supported by Tsinghua University start-up fund and Tsinghua University Education Foundation fund (042202008).

* Corresponding authors.
E-mail addresses: mgr18@mails.tsinghua.edu.cn (G. Ma), yau@uic.edu (S.S.-T. Yau), hqzuo@mail.tsinghua.edu.cn (H. Zuo).

https://doi.org/10.1016/j.jalgebra.2020.07.023
0021-8693/

rem [15] stated that: Let $(V_1, 0)$ and $(V_2, 0)$ be two isolated hypersurface singularities, $A(V_1)$ and $A(V_2)$ be the moduli algebras, then $(V_1, 0) \cong (V_2, 0) \Longleftrightarrow A(V_1) \cong A(V_2)$. In 1983, motivated by the Mather-Yau theorem, the second author introduced the Lie algebra of derivations of moduli algebra $A(V)$, i.e., $L(V) = \mathrm{Der}(A(V), A(V))$. The finite dimensional Lie algebra $L(V)$ was called Yau algebra and its dimension $\lambda(V)$ was called Yau number in ([13], [24]).

The Yau algebra plays an important role in singularity theory and was used to distinguish complex analytic structure of isolated hypersurface singularities [18]. Yau and his collaborators have been systematically studying the Yau algebras of isolated hypersurface singularities and its generalizations beginning from eighties (cf. [1,2], [5], [6], [4], [7], [10–12], [18], [20], [21–23], [25,26]). In our recent work [6], we introduced a new derivation Lie algebra that is arising from isolated hypersurface singularities. This Lie algebra is a more subtle invariant of singularities compared with previous Lie algebras.

Recall that a holomorphic function $f : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0)$ is said to be quasi-homogeneous if f is in the Jacobian ideal $J(f) := (\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n})$. A polynomial $f(z_0, \ldots, z_n)$ is weighted homogeneous of type $(\alpha_0, \alpha_1, \ldots, \alpha_n; d)$, where $\alpha_0, \alpha_1, \ldots, \alpha_n$ and d are fixed positive integers, if it can be expressed as a linear combination of monomials $z_0^{i_0} z_1^{i_1} \ldots z_n^{i_n}$ for which $\alpha_0 i_0 + \cdots + \alpha_n i_n = d$. According to a beautiful theorem of Saito [17], assume f defines an isolated singularity, then f is a weighted homogeneous polynomial after a biholomorphic change of coordinates if and only if f is quasi-homogeneous. The order of the lowest nonvanishing term in the power series expansion of f at 0 is called the multiplicity (denoted by $mult(f)$) of the singularity $(V, 0)$.

The following beautiful theorem of Dimca characterizes zero-dimensional isolated complete intersection singularities.

Theorem 1.1. *(Dimca [8]) Two zero-dimensional isolated complete intersection singularities X and Y are isomorphic if and only if their singular subspaces $Sing(X)$ and $Sing(Y)$ are isomorphic.*

Remark 1.2. Let $V = V(f)$ be an isolated quasi-homogeneous hypersurface singularity. Let X denote the zero-dimensional isolated complete intersection singularity defined by $(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n})$. Then $\mathrm{Sing}(X)$ is defined by

$$(f, \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}, \det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0 \le i,j \le n}).$$

Theorem 1.1 implies that in order to study analytic isomorphism type of zero-dimensional isolated complete intersection singularity X, we only need to consider the Artinian local algebra $A^*(V)$ which is the coordinate ring of $\mathrm{Sing}(X)$. Thus $A^*(V)$ is defined as the quotient

$$\mathcal{O}_{n+1}/(f, \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}, \det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0 \le i,j \le n}).$$

Combining Theorem 1.1 with Mather-Yau theorem, we know that $A^*(V)$ is a complete invariant of quasi-homogeneous isolated hypersurface singularities (i.e., $A^*(V)$ determines and is determined by the analytic isomorphism type of the singularity). We call $A^*(V)$ the generalized moduli algebra of V. Based on this important observation, we introduced the following new invariant for an isolated hypersurface singularity in [6].

Definition 1.3. Let $V = \{f(z_0, \ldots, z_n) = 0\}$ be a germ of isolated hypersurface singularity at the origin of $\mathbb{C}^{n+1}$ $(n \geq 1)$. The new Lie algebra associated to the isolated hypersurface singularity $(V, 0)$ is defined as $L^*(V) := \mathrm{Der}(A^*(V), A^*(V))$ (or $\mathrm{Der}(A^*(V))$ for short). Its dimension is denoted as $\lambda^*(V)$.

It is known that Yau algebra can not characterize the ADE singularities completely. In fact, Elashvili and Khimshiashvili [9] proved the following result: if X and Y are two simple singularities except the pair A_6 and D_5, then $L(X) \cong L(Y)$ as Lie algebras, if and only if X and Y are analytically isomorphic. However, the ADE singularities can be characterized completely by the new Lie algebra $L^*(V)$. We have reasons to believe that this new Lie algebra $L^*(V)$ and numerical invariant $\lambda^*(V)$ will also play an important role in the study of singularities.

Theorem 1.4. *[6] If X and Y are two ADE singularities, then $L^*(X) \cong L^*(Y)$ as Lie algebras, if and only if X and Y are analytically isomorphic.*

Let A be a weighted zero dimensional complete intersection, i.e., a commutative algebra of the form $A = \mathbb{C}[z_0, \ldots, z_n]/I$ where the ideal I is generated by a regular sequence of length $n+1$, $(f_0, f_1, \ldots, f_n)$. Here the variables have strictly positive integral weights, denoted by $wt(z_i) = \alpha_i, 0 \leq i \leq n$, and the equations are weighted homogeneous with respect to these weights. Consequently the algebra A is graded and one may speak about its homogeneous degree k derivations (k is an integer). Recall that a linear map $D : A \to A$ is a derivation if $D(ab) = D(a)b + aD(b)$, for any $a, b \in A$. D belongs to $\mathrm{Der}^k(A)$ if $D : A^* \to A^{*+k}$.

On the one hand, one of the most prominent open problems in rational homotopy theory is related to the vanishing of the above derivations in strictly negative degrees:

Halperin Conjecture. *[14] If A is as above, then* $\mathrm{Der}^{<0}(A) = 0$.

The Halperin Conjecture has been verified in several particular cases (see [16], [19], [27]). For recent progress, please see [7].

Let $(V, 0) = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \colon f(z_0, \ldots, z_n) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial $f(z_0, \ldots, z_n)$ of type $(\alpha_0, \alpha_1, \ldots, \alpha_n; d)$. Then by a well-known result of Saito [17], we can always assume without loss of generality that $d \geq 2\alpha_i > 0$ for all $0 \leq i \leq n$. We give the variable z_i weight α_i for $0 \leq i \leq n$. Then the moduli algebra $A(V)$ is a graded algebra, i.e., $A(V) = \bigoplus_{i=0}^{\infty} A_i(V)$, and the Lie algebra of derivations $\mathrm{Der}(A(V))$ is also graded. Thus $L(V)$ is graded.

On the other hand, the second author discovered independently the following conjecture on the non-existence of the negative weight derivation which is a special case of Halperin Conjecture.

Yau Conjecture. *(Cf. [3], [4].) Let* $(V,0) = \{(z_0,\dots,z_n) \in \mathbb{C}^{n+1}\colon f(z_0,\dots,z_n) = 0\}$ *be an isolated singularity defined by the weighted homogeneous polynomial* $f(z_0,\dots,z_n)$ *of weight type* $(\alpha_0,\alpha_1,\dots,\alpha_n;d)$. *Assume that* $d \geq 2\alpha_0 \geq 2\alpha_1 \geq \cdots \geq 2\alpha_n > 0$ *without loss of generality. Then there is no non-zero negative weight derivation on the moduli algebra (= Milnor algebra)* $A(V) = \mathbb{C}[z_0,\dots,z_n]/(\frac{\partial f}{\partial z_0},\dots,\frac{\partial f}{\partial z_n})$, *i.e.,* $L(V)$ *is non-negatively graded.*

This conjecture was still open and was only proved in the low-dimensional case $n \leq 3$ ([3], [4]) by explicit calculations. It was also proved for the high-dimensional singularities under certain condition [25] and homogeneous singularities in [20] (cf. Proposition 2.1).

Assume that f is a weighted homogeneous polynomial, then the generalized moduli algebra $A^*(V)$ and $L^*(V)$ are also natural graded. It is natural to propose the following new conjecture:

Conjecture 1.5. *Let* $(V,0) = \{(z_0,\dots,z_n) \in \mathbb{C}^{n+1}\colon f(z_0,\dots,z_n) = 0\}$ *be an isolated singularity defined by the weighted homogeneous polynomial* $f(z_0,\dots,z_n)$ *of weight type* $(\alpha_0,\alpha_1,\dots,\alpha_n;d)$. *Assume that* $d \geq 2\alpha_0 \geq 2\alpha_1 \geq \cdots \geq 2\alpha_n > 0$ *without loss of generality. Then there is no non-zero negative weight derivation on the generalized moduli algebra*

$$\mathbb{C}[z_0,\dots,z_n]/(\frac{\partial f}{\partial z_0},\dots,\frac{\partial f}{\partial z_n},\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq n}),$$

i.e., $L^*(V)$ *is non-negatively graded.*

$n = 0$ is trivial. In this paper, we shall verify Conjecture 1.5 for the case $n = 1, 2, 3$. We obtained the following main result.

Main Theorem. *Let* $(V,0) = \{(z_0,\dots,z_n) \in \mathbb{C}^{n+1}\colon f(z_0,\dots,z_n) = 0\}$ *be an isolated singularity defined by the weighted homogeneous polynomial* $f(z_0,\dots,z_n)$ *of weight type* $(\alpha_0,\alpha_1,\dots,\alpha_n;d)$ *where* $1 \leq n \leq 3$. *Assume that* $d \geq 2\alpha_0 \geq 2\alpha_1 \geq \cdots \geq 2\alpha_n > 0$ *without loss of generality. Let* D *be a derivation of the algebra*

$$\mathbb{C}[z_0,\dots,z_n]/(\frac{\partial f}{\partial z_0},\dots,\frac{\partial f}{\partial z_n},\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq n}).$$

Then $D \equiv 0$, *if* D *is of negative weight, i.e.,* $L^*(V)$ *is non-negatively graded for* $1 \leq n \leq 3$.

2. Proof of the main theorem

Firstly, we recall the following known results which will be used in proof of the main theorem frequently.

Proposition 2.1. *([20], Prop. 2.6) Let $A = \bigoplus_{i=0}^{k} A_i$ be a graded commutative Artinian local algebra with $A_0 = \mathbb{C}$. Suppose the maximal ideal of A is generated by A_j for some $j > 0$. Then $L(A)$ is a graded Lie algebra without negative weight.*

Lemma 2.2. *[22] Let $(A, \mathfrak{m})$ be a commutative local Artinian algebra ($\mathfrak{m}$ is the unique maximal ideal of A) and $D \in L(A)$ be the derivation of A. Then D preserves the m-adic filtration of A, i.e., $D(\mathfrak{m}) \subset \mathfrak{m}$.*

Lemma 2.3. *[4] [Lemma 2.1] Let f be a weighted homogeneous polynomial with isolated singularity in $z_0, \ldots, z_n$ variables of type $(\alpha_0, \ldots, \alpha_n; d)$.*
Assume $wt(z_0) = \alpha_0 \geq wt(z_1) = \alpha_1 \geq \cdots \geq wt(z_n) = \alpha_n$. Then f must be in one of the following two cases:
Case 1.

$$f = z_0^m + a_1(z_1, \ldots, z_n)z_0^{m-1} + \cdots + a_{m-1}(z_1, \ldots, z_n)z_0 + a_m(z_1, \ldots, z_n).$$

Case 2.

$$f = z_0^m z_i + a_1(z_1, \ldots, z_n)z_0^{m-1} + \cdots + a_{m-1}(z_1, \ldots, z_n)z_0 + a_m(z_1, \ldots, z_n).$$

Now we begin to prove the main theorem.

Proof of the Main Theorem. Let

$$A := \mathbb{C}[z_0, z_1, \ldots, z_n]/(\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}, \det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq n}),$$

and

$$B := \mathbb{C}[z_0, z_1, \ldots, z_n]/(\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}).$$

It is clear that $A = B/(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq n})$, thus A is a commutative Artinian local algebra. Let $D \in L(A)$ be a derivation of A, thus D is $A-$linear combination of $\frac{\partial}{\partial z_0}, \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}$. By Lemma 2.2, we know that $D(\mathfrak{m}) \subset \mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal of A, the coefficients of $\frac{\partial}{\partial z_0}, \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}$ do not contain the constant term. Moreover, if D is of negative weight, we have $D = p_0(z_1, \ldots, z_n)\frac{\partial}{\partial z_0} + p_1(z_2, \ldots, z_n)\frac{\partial}{\partial z_1} + \cdots + p_{n-2}(z_{n-1}, z_n)\frac{\partial}{\partial z_{n-2}} + c z_n^k \frac{\partial}{\partial z_{n-1}}$ where $k > 1$ and c is a constant. Observe that $wt(\frac{\partial f}{\partial z_0}) = d - \alpha_0$, $wt(\frac{\partial f}{\partial z_1}) = d - \alpha_1, \ldots, wt(\frac{\partial f}{\partial z_n}) = d - \alpha_n$ and $wt(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq n})) = (n+1)d-$

$2(\alpha_0 + \alpha_1 + \cdots + \alpha_n) \geq 0$, so we have $wt(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0 \leq i,j \leq n})) \geq 0$ and $0 < wt(\frac{\partial}{\partial z_0}) \leq wt(\frac{\partial}{\partial z_1}) \leq \cdots \leq wt(\frac{\partial f}{\partial z_n})$. Since D is a derivation of A, we have $D(J) \subset J$, where $J = (\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}, \det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0 \leq i,j \leq n})$. Because $wt(D(\frac{\partial f}{\partial z_0})) < wt(\frac{\partial f}{\partial z_0})$, we have that $D(\frac{\partial f}{\partial z_0})$ does not contain the multiplication of $\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}$. We have $D(\frac{\partial f}{\partial z_0}) = 0$ or $D(\frac{\partial f}{\partial z_0})$ is a multiplication of $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0 \leq i,j \leq n}$.

Moreover, we will show the following lemma.

Lemma 2.4. *Let f be a weighted homogeneous polynomial with forms in Lemma 2.3. When $n \geq 2$, $m \geq 3$, there is at least one $\frac{\partial f}{\partial z_i}$ for $0 \leq i \leq n$ such that*

$$wt(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0 \leq i,j \leq n}) > wt(\frac{\partial f}{\partial z_i}).$$

Proof. If not, suppose for all $0 \leq i \leq n$, we have

$$wt(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0 \leq i,j \leq n}) \leq wt(\frac{\partial f}{\partial z_i}).$$

So we have the following relations.

$$\begin{cases} (n+1)d - 2(\alpha_0 + \cdots + \alpha_n) \leq d - \alpha_0; \\ (n+1)d - 2(\alpha_0 + \cdots + \alpha_n) \leq d - \alpha_1 \\ \qquad \vdots \\ (n+1)d - 2(\alpha_0 + \cdots + \alpha_n) \leq d - \alpha_{n-1}; \\ (n+1)d - 2(\alpha_0 + \cdots + \alpha_n) \leq d - \alpha_n. \end{cases}$$

We sum up the left hand sides and the right hand sides,

$$n(n+1)d - (2n+1)(\alpha_0 + \cdots + \alpha_n) \leq 0.$$

So $(n+1)\alpha_0 \geq \alpha_0 + \cdots + \alpha_n \geq \frac{n(n+1)d}{2n+1}$, i.e., $\alpha_0 \geq \frac{nd}{2n+1} \geq \frac{2}{5}d$.
For Case 1,

$$f = z_0^m + a_1(z_1, \ldots, z_n)z_0^{m-1} + \cdots + a_{m-1}(z_1, \ldots, z_n)z_0 + a_m(z_1, \ldots, z_n).$$

We have that $\frac{1}{m}d = \alpha_0 \geq \frac{2}{5}d$, i.e., $m \leq 2$, which is absurd.
For Case 2,

$$f = z_0^m z_i + a_1(z_1, \ldots, z_n)z_0^{m-1} + \cdots + a_{m-1}(z_1, \ldots, z_n)z_0 + a_m(z_1, \ldots, z_n).$$

We have that $\frac{1}{m}d > \alpha_0 \geq \frac{2}{5}d$, i.e., $m \leq 2$, which is absurd. □

Remark 2.5. Similarly, we can also show that when $n = 1, m \geq 4$, there is at least one $\frac{\partial f}{\partial z_i}$ for $0 \leq i \leq n$ such that

$$wt(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0 \leq i,j \leq n}) > wt(\frac{\partial f}{\partial z_i}),$$

for Case 1.
When $n = 1, m \geq 3$, there is at least one $\frac{\partial f}{\partial z_i}$ for $0 \leq i \leq n$ such that

$$wt(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0 \leq i,j \leq n}) > wt(\frac{\partial f}{\partial z_i}),$$

for Case 2.

We divide the proof of the main theorem into three propositions.

Proposition 2.6. *Let $f(z_0, z_1)$ be a weighted homogeneous polynomial of type $(\alpha_0, \alpha_1; d)$ with isolated singularity at the origin. Assume that $d \geq 2\alpha_0 \geq 2\alpha_1$. Let D be a derivation of the algebra*

$$\mathbb{C}[z_0, z_1]/(\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0 \leq i,j \leq 1}).$$

Then $D \equiv 0$, if D is of negative weight.

Proof. In the first part of this proof, we consider the case $D(\frac{\partial f}{\partial z_0}) = 0$. We have $D = z_1^k \frac{\partial}{\partial z_0}$ where $k \geq 1$ and $wt(D) = k\alpha_1 - \alpha_0 < 0$. Let

$$f(z_0, z_1) = \sum_{\alpha_0 n_0 + \alpha_1 n_1 = d} c_{(n_0,n_1)} z_0^{n_0} z_1^{n_1}.$$

Then we have

$$\begin{aligned} D(\frac{\partial f}{\partial z_0}) &= z_1^k \frac{\partial}{\partial z_0} \left(\sum_{\alpha_0 n_0 + \alpha_1 n_1 = d} n_0 c_{(n_0,n_1)} z_0^{n_0 - 1} z_1^{n_1} \right) \\ &= \sum_{\alpha_0 n_0 + \alpha_1 n_1 = d} n_0(n_0 - 1) c_{(n_0,n_1)} z_0^{n_0 - 2} z_1^{n_1 + k} \\ &= 0. \end{aligned}$$

Thus, when $n_0 \geq 2$, $c_{(n_0,n_1)} = 0$, i.e.,

$$f(z_0, z_1) = c_{(1,p)} z_0 z_1^p + c_{(0,q)} z_1^q,$$

where $d = \alpha_0 + p\alpha_1 = q\alpha_1$. We have $q > k + p \geq 2$ due to $\alpha_0 > k\alpha_1$. In order that f has an isolated singularity at the origin, we need $p = 1$. Thus

$$f(z_0, z_1) = c_{(1,1)} z_0 z_1 + c_{(0,q)} z_1^q,$$
$$\frac{\partial f}{\partial z_0} = c_{(1,1)} z_1.$$

Hence $D = z_1^k \frac{\partial}{\partial z_0}$ is a zero derivation on $C[z_0, z_1]/(\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1})$.
In the second part of the proof, we consider the case $D(\frac{\partial f}{\partial z_0})$ is a multiple of $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0 \leq i,j \leq 1}$.

Case (1).

$$f = z_0^m + a_1(z_1) z_0^{m-1} + \cdots + a_{m-1}(z_1) z_0 + a_m(z_1).$$

By Remark 2.5, there are three cases.

Case (1.1).
If $m = 1$, then 0 is not an isolated singularity.

Case (1.2).
If $m = 2$, then

$$f = z_0^2 + a_1(z_1) z_0 + a_2(z_1).$$

We have that

$$D(\frac{\partial f}{\partial z_0}) = 2 z_1^k,$$

and

$$\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0 \leq i,j \leq 1} = \begin{vmatrix} 2 & \frac{\partial a_1(z_1)}{\partial z_1} \\ \frac{\partial a_1(z_1)}{\partial z_1} & \frac{\partial^2 a_1(z_1)}{\partial z_1^2} z_0 + \frac{\partial^2 a_2(z_1)}{\partial z_1^2} \end{vmatrix}.$$

Thus $D(\frac{\partial f}{\partial z_0})$ is a multiple of $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0 \leq i,j \leq 1}$ only if $\frac{\partial^2 a_1(z_1)}{\partial z_1^2} = 0$, i.e., $a_1(z_1) = c z_1$ where c is a constant. If $c \neq 0$, then this implies that $\alpha_0 = \alpha_1$, i.e., f is a homogeneous polynomial. By Proposition 2.1, this completes the proof for this case. If $c = 0$, then $f = z_0^2 + d z_1^s$ where d is a non-zero constant and $s \geq 2$. This is a trivial case.

Case (1.3).
If $m = 3$, then

$$f = z_0^3 + a_1(z_1)z_0^2 + a_2(z_1)z_0 + a_3(z_1).$$

We have that

$$D(\frac{\partial f}{\partial z_0}) = (6z_0 + 2a_1(z_1))z_1^k,$$

and

$$\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 1} = \begin{vmatrix} 6z_0 + 2a_1(z_1) & 2z_0\frac{\partial a_1(z_1)}{\partial z_1} + \frac{\partial a_2(z_1)}{\partial z_1} \\ 2z_0\frac{\partial a_1(z_1)}{\partial z_1} + \frac{\partial a_2(z_1)}{\partial z_1} & \frac{\partial^2 a_1(z_1)}{\partial z_1^2}z_0^2 + \frac{\partial^2 a_2(z_1)}{\partial z_1^2}z_0 + \frac{\partial^2 a_3(z_3)}{\partial z_1^2} \end{vmatrix}.$$

Thus $D(\frac{\partial f}{\partial z_0})$ is a multiple of $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 1}$ only if $\frac{\partial^2 a_1(z_1)}{\partial z_1^2} = 0$, i.e., $a_1(z_1) = cz_1$ where c is a constant. If $c \neq 0$, then this implies that $\alpha_0 = \alpha_1$, i.e., f is a homogeneous polynomial. By Proposition 2.1, this completes the proof for this case. If $c = 0$, then $\frac{\partial^2 a_2(z_1)}{\partial z_1^2} = 0$, i.e., $a_2(z_1) = dz_1$ where d is constant which is absurd.

Case (2).

$$f = z_0^m z_1 + a_1(z_1)z_0^{m-1} + \cdots + a_{m-1}(z_1)z_0 + a_m(z_1).$$

By Remark 2.5, there are two cases.

Case (2.1).
If $m = 1$, then

$$f = z_0 z_1 + a_1(z_1).$$

Thus the relation $\alpha_0 + \alpha_1 = d \ge 2\alpha_0 \ge 2\alpha_1$ implies that $\alpha_0 = \alpha_1$, i.e., f is a homogeneous polynomial. By Proposition 2.1, this completes the proof for this case.

Case (2.2).
If $m = 2$, then

$$f = z_0^2 z_1 + a_1(z_1)z_0 + a_2(z_1).$$

We have

$$D(\frac{\partial f}{\partial z_0}) = 2z_1^{k+1},$$

and

$$\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 1} = \begin{vmatrix} 2z_1 & 2z_0 + \frac{\partial a_1(z_1)}{\partial z_1} \\ 2z_0 + \frac{\partial a_1(z_1)}{\partial z_1} & \frac{\partial^2 a_1(z_1)}{\partial z_1^2}z_0 + \frac{\partial^2 a_2(z_1)}{\partial z_1^2} \end{vmatrix}.$$

Thus $D(\frac{\partial f}{\partial z_0})$ is a multiple of $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 1}$ is obviously impossible. $\square$

Proposition 2.7. *Let $f(z_0, z_1, z_2)$ be a weighted homogeneous polynomial of type $(\alpha_0, \alpha_1, \alpha_2; d)$ with isolated singularity at the origin. Assume that $d \ge 2\alpha_0 \ge 2\alpha_1 \ge 2\alpha_2$. Let D be a derivation of the algebra*

$$\mathbb{C}[z_0, z_1, z_2]/(\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 2}).$$

Then $D \equiv 0$, if D is of negative weight.

Proof. In the first part of this proof, we consider the case $D(\frac{\partial f}{\partial z_0}) = 0$. We have $D = p(z_1, z_2)\frac{\partial}{\partial z_0} + cz_2^k \frac{\partial}{\partial z_1}$ where c is a constant. There are two cases $c = 0$ or $c \ne 0$.

Case (1).

$$c = 0.$$

In this case, $D = p(z_1, z_2)\frac{\partial}{\partial z_0}$. By Lemma 2.3, we have to consider two cases.

Case (1.1).

$$f = z_0^m + a_1(z_1, z_2)z_0^{m-1} + \cdots + a_{m-1}(z_1, z_2)z_0 + a_m(z_1, z_2).$$

Then we have that

$$D(\frac{\partial f}{\partial z_0}) = p(z_1, z_2)[m(m-1)z_0^{m-2} + (m-1)(m-2)a_1(z_1, z_2)z_0^{m-3} + \cdots + 2a_{m-2}(z_1, z_2)] = 0,$$

which implies $p(z_1, z_2) = 0$. If not, the above equation holds only if $m = 1$. But when $m = 1$, 0 is not a singular point of f. Thus we must have $p(z_1, z_2) = 0$, i.e. $D \equiv 0$.

Case (1.2).

$$f = z_0^m z_i + a_1(z_1, z_2)z_0^{m-1} + \cdots + a_m(z_1, z_2).$$

Then we have that

$$D(\frac{\partial f}{\partial z_0}) = p(z_1, z_2)[m(m-1)z_0^{m-2}z_i + (m-1)(m-2)a_1(z_1, z_2)z_0^{m-3} + \cdots + 2a_{m-2}(z_1, z_2)],$$

which implies $p(z_1, z_2) = 0$ or $m = 1$. When $p(z_1, z_2) = 0$, we have $D \equiv 0$. When $m = 1$, we have

$$f = z_0 z_i + a_1(z_1, z_2).$$

Case (1.2.1).

$$i = 1.$$

We have $\alpha_0 + \alpha_1 = d \geq 2\alpha_0 \geq 2\alpha_1$, thus $\alpha_0 = \alpha_1$. Since

$$D(\frac{\partial f}{\partial z_1}) = p(z_1, z_2) \cdot \frac{\partial^2 f}{\partial z_0 \partial z_1} = p(z_1, z_2) \in \left(J, det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0 \leq i,j \leq 2}\right).$$

We have $D \equiv 0$ as a derivation of A where $n = 2$.
Case (1.2.2).

$$i = 2.$$

We have $\alpha_0 + \alpha_2 = d \geq 2\alpha_0 \geq 2\alpha_1 \geq 2\alpha_2$, thus $\alpha_0 = \alpha_1 = \alpha_2$ and f is homogeneous. It follows from Proposition 2.1 that $D \equiv 0$ as a derivation of A.

Case (2).

$$c \neq 0.$$

According to Lemma 2.3, we also need to divide it into two cases.

Case (2.1).

$$f = z_0^m + a_1(z_1, z_2) z_0^{m-1} + \cdots + a_{m-1}(z_1, z_2) z_0 + a_m(z_1, z_2).$$

Then we have that

$$\begin{aligned} & D(\frac{\partial f}{\partial z_0}) \\ =& p(z_1, z_2) \left[m(m-1) z_0^{m-2} + (m-1)(m-2) a_1(z_1, z_2) z_0^{m-3} + \cdots + 2a_{m-1}(z_1, z_2) \right] \\ &+ c z_2^k \left[(m-1) \frac{\partial a_1(z_1, z_2)}{\partial z_1} z_0^{m-2} + (m-2) \frac{\partial a_2(z_1, z_2)}{\partial z_1} z_0^{m-3} + \cdots + \frac{\partial a_{m-1}(z_1, z_2)}{\partial z_1} \right]. \end{aligned}$$

Because $D(\frac{\partial f}{\partial z_0}) = 0$ and $m \geq 2$, we have

$$m p(z_1, z_2) = -c z_2^k \frac{\partial a_1(z_1, z_2)}{\partial z_1}.$$

We can do the following coordinate change

$$\begin{cases} z_0 = z_0' - \frac{1}{m} a_1(z_1', z_2') \\ z_1 = z_1' \\ z_2 = z_2'. \end{cases}$$

Then

$$\begin{aligned} D =& -\frac{1}{m} c z_2^k \frac{\partial a_1(z_1,z_2)}{\partial z_1} \frac{\partial}{\partial z_0} + c z_2^k \frac{\partial}{\partial z_1} \\ =& c z_2^k \left(-\frac{1}{m} \frac{\partial a_1(z_1,z_2)}{\partial z_1} \frac{\partial}{\partial z_0} + \frac{\partial}{\partial z_1}\right) \\ =& c(z_2')^k \frac{\partial}{\partial z_1'}. \end{aligned}$$

Let $g(z_0', z_1', z_2') = f(z_0, z_1, z_2)$, we obtain that g is also a weighted homogeneous polynomial and

$$g = (z_0')^m + b_1(z_1', z_2')(z_0')^{m-1} + \cdots + b_m(z_1', z_2').$$

By the same argument as before, we have $D(\frac{\partial g}{\partial z_0'}) = 0$ or $D(\frac{\partial g}{\partial z_0'})$ is a multiple of $\det(\frac{\partial^2 f}{\partial z_i' \partial z_j'})_{0 \le i,j \le 2}$. By Lemma 2.4, we know that $D(\frac{\partial g}{\partial z_0'})$ is a multiplication of $\det(\frac{\partial^2 f}{\partial z_i' \partial z_j'})_{0 \le i,j \le 2}$ only if $m = 2$. In this case we have $\frac{\partial g}{\partial z_0'} = 2z_0'$. This will back to the Proposition 2.6.
Thus we just assume that

$$D(\frac{\partial g}{\partial z_0'}) = c(z_2')^k \frac{\partial}{\partial z_1'}(\frac{\partial g}{\partial z_0'}) = 0.$$

Thus

$$\frac{\partial b_1(z_1', z_2')}{\partial z_1'} = \cdots = \frac{\partial b_{m-1}(z_1', z_2')}{\partial z_1'} = 0.$$

Consider

$$D(\frac{\partial g}{\partial z_1'}) = c(z_2')^k \frac{\partial^2 g}{\partial (z_1')^2} = c(z_2')^k \frac{\partial^2 b_m(z_1', z_2')}{\partial (z_1')^2},$$

since $wt(D(\frac{\partial g}{\partial z_1'})) < wt(\frac{\partial g}{\partial z_1'})$, so $D(\frac{\partial g}{\partial z_1'})$ is a linear combination of $\frac{\partial g}{\partial z_0'}$ and $\det(\frac{\partial^2 g}{\partial z_i' \partial z_j'})_{0 \le i,j \le 2}$.
Case (2.1.1).

$$m = 1.$$

0 is not an isolated singularity.
Case (2.1.2).

$$m = 2.$$

In this case,

$$D(\frac{\partial g}{\partial z_1'}) = c(z_2')^k \frac{\partial^2 b_2(z_1', z_2')}{\partial z_1'^2},$$
$$\frac{\partial g}{\partial z_0'} = 2z_0' + b_1(z_1', z_2')$$

and

$$\det(\frac{\partial^2 g}{\partial z_i' \partial z_j'})_{0\le i,j\le 2} = \begin{vmatrix} 2 & 0 & \frac{\partial b_1(z_1',z_2')}{\partial z_2'} \\ 0 & \frac{\partial^2 b_2(z_1',z_2')}{\partial z_1'^2} & \frac{\partial^2 b_2(z_1',z_2')}{\partial z_1' \partial z_2'} \\ \frac{\partial b_1(z_1',z_2')}{\partial z_2'} & \frac{\partial^2 b_2(z_1',z_2')}{\partial z_1' \partial z_2'} & \frac{\partial^2 b_1(z_1',z_2')}{\partial z_2'^2} z_0 + \frac{\partial^2 b_2(z_1',z_2')}{\partial z_2'^2} \end{vmatrix}.$$

We know that $D(\frac{\partial g}{\partial z_1'})$ can only be expressed by $h\Big[-\frac{\partial^2 b_1(z_1',z_2')}{\partial z_2'^2}\frac{\partial^2 b_2(z_1',z_2')}{\partial z_1'^2}\frac{\partial g}{\partial z_0'} + \det(\frac{\partial^2 g}{\partial z_i'\partial z_j'})_{0\le i,j\le 2}\Big]$ with $\frac{\partial^2 b_1(z_1',z_2')}{\partial z_2'^2}\frac{\partial^2 b_2(z_1',z_2')}{\partial z_1'^2} \neq 0$ (This means $\alpha_0' \geq 2\alpha_2'$). Thus $D(\frac{\partial g}{\partial z_1'})$ is a linear combination of $\frac{\partial g}{\partial z_0'}$ and $\det(\frac{\partial^2 g}{\partial z_i'\partial z_j'})_{0\le i,j\le 2}$ only if

$$k\alpha_2' + 2\alpha_0' - 2\alpha_1' \geq 4\alpha_0' - 2\alpha_1' - 2\alpha_2'.$$

Thus we have $k \geq 2$.
Moreover, by

$$\begin{cases} \alpha_1' > k\alpha_2' \geq 2\alpha_0' - 2\alpha_2' \\ 2\alpha_2' \leq k\alpha_2' < \alpha_1' \end{cases},$$

we obtain that

$$\begin{cases} \alpha_0' < \frac{1}{2}\alpha_1' + \alpha_2' \\ \alpha_2' < \frac{1}{2}\alpha_1' \end{cases}.$$

This implies that $\alpha_0' < \alpha_1'$ which is absurd.
Thus $D(\frac{\partial g}{\partial z_1'})$ is a multiple of $\frac{\partial g}{\partial z_0'}$, i.e., there exists h such that

$$c(z_2')^k \frac{\partial^2 b_m(z_1', z_2')}{\partial (z_1')^2} = h\frac{\partial g(z_1', z_2')}{\partial z_0'} = h\left[m(z_0')^{m-1} + \cdots + b_{m-1}(z_1', z_2')\right].$$

If $\frac{\partial^2 b_m(z_1',z_2')}{\partial (z_1')^2} \neq 0$, then this is possible only if $m = 1$, which is absurd. So we have $\frac{\partial^2 b_m(z_1',z_2')}{\partial (z_1')^2} = 0$. This implies

$$b_m(z_1', z_2') = d_1 z_1'(z_2')^{l_1} + d_2(z_2')^{l_2},$$

where d_1, d_2 are constants. Then g has an isolated singularity at 0 only if $l_1 = 1$. Because $b_m(z_1', z_2')$ is weighted homogeneous, we have $d = \alpha_1 + \alpha_2$. It follows from the assumption

$d \geq 2\alpha_0 \geq 2\alpha_1 \geq 2\alpha_2$ that $\alpha_0 = \alpha_1 = \alpha_2$. So g is a homogeneous polynomial. It follows from Proposition 2.1 that $D \equiv 0$ as a derivation of A.

Case (2.2).

$$f = z_0^m z_i + a_1(z_1, z_2)z_0^{m-1} + \cdots + a_{m-1}(z_1, z_2)z_0 + a_m(z_1, z_2).$$

Case (2.2.1).

$$i = 1.$$

Then we have

$$\begin{aligned}0 =& D(\frac{\partial f}{\partial z_0}) \\ =& p(z_1, z_2)[m(m-1)z_0^{m-2}z_1 + (m-1)(m-2)a_1(z_1, z_2)z_0^{m-3} + \dots] \\ +& cz_2^k[mz_0^{m-1} + (m-1)\frac{\partial a_1(z_1, z_2)}{\partial z_1}z_0^{m-2} + \dots].\end{aligned}$$

This forces $c = 0$ which contradicts to our hypothesis. So this case does not occur.

Case (2.2.2).

$$i = 2.$$

There are two subcases which need to be considered.

Case (2.2.2.1).

$$m = 1.$$

Then we have $\alpha_0 + \alpha_2 = d$. Thus the assumption that $d \geq 2\alpha_0 \geq 2\alpha_1 \geq 2\alpha_2$ implies $\alpha_0 = \alpha_1 = \alpha_2$. So f is a homogeneous polynomial. It follows from Proposition 2.1 that $D \equiv 0$ as a derivation of A.

Case (2.2.2.2).

$$m > 1.$$

Then

$$\begin{aligned}
0 =&D(\frac{\partial f}{\partial z_0})\\
=&p(z_1,z_2)\frac{\partial}{\partial z_0}(\frac{\partial f}{\partial z_0})+cz_2^k\frac{\partial}{\partial z_1}(\frac{\partial f}{\partial z_0})\\
=&\frac{\partial}{\partial z_0}[p(z_1,z_2)\frac{\partial f}{\partial z_0}+cz_2^k\frac{\partial f}{\partial z_1}].
\end{aligned}\tag{1}$$

We know that $D(\frac{\partial f}{\partial z_1})$ is a linear combination of $\frac{\partial f}{\partial z_0}$ and $\det(\frac{\partial^2 f}{\partial z_i\partial z_j})_{0\le i,j\le 2}$. We claim that the coefficient of $\det(\frac{\partial^2 f}{\partial z_i\partial z_j})_{0\le i,j\le 2}$ must be zero.

Case (2.2.2.2.1).

$$m=2,$$

i.e.,

$$f=z_0^2z_2+a_1(z_1,z_2)z_0+a_2(z_1,z_2).$$

In this case,

$$D(\frac{\partial f}{\partial z_1})=p(z_1,z_2)\frac{\partial a_1(z_1,z_2)}{\partial z_1}+cz_2^k\frac{\partial^2 a_2(z_1,z_2)}{\partial z_1^2},$$

and

$$\begin{aligned}
&det(\frac{\partial^2 f}{\partial z_i\partial z_j})_{0\le i,j\le 2}\\
&=\begin{vmatrix}
2z_2 & \frac{\partial a_1(z_1,z_2)}{\partial z_1} & 2z_0+\frac{\partial a_1(z_1,z_2)}{\partial z_2}\\
\frac{\partial a_1(z_1,z_2)}{\partial z_1} & z_0\frac{\partial^2 a_1(z_1,z_2)}{\partial z_1^2}+\frac{\partial^2 a_2(z_1,z_2)}{\partial z_1^2} & z_0\frac{\partial^2 a_1(z_1,z_2)}{\partial z_1\partial z_2}+\frac{\partial^2 a_2(z_1,z_2)}{\partial z_1\partial z_2}\\
2z_0+\frac{\partial a_1(z_1,z_2)}{\partial z_2} & z_0\frac{\partial^2 a_1(z_1,z_2)}{\partial z_1\partial z_2}+\frac{\partial^2 a_2(z_1,z_2)}{\partial z_1\partial z_2} & z_0\frac{\partial^2 a_1(z_1,z_2)}{\partial z_2^2}+\frac{\partial^2 a_2(z_1,z_2)}{\partial z_2^2}
\end{vmatrix}.
\end{aligned}$$

The coefficient of $det(\frac{\partial^2 f}{\partial z_i\partial z_j})_{0\le i,j\le 2}$ is not zero only if

$$\frac{\partial^2 a_1(z_1,z_2)}{\partial z_1^2}=\frac{\partial^2 a_2(z_1,z_2)}{\partial z_1^2}=0,$$

and

$$wt(p)+\alpha_0+\alpha_2-\alpha_1\ge 4\alpha_0+\alpha_2-2\alpha_1.$$

Moreover D is a negative weight derivation, so $\alpha_1>wt(p)\ge 3\alpha_0-\alpha_1$ which is absurd.

Case (2.2.2.2.2).

$$m \geq 3.$$

In this case, $D(\frac{\partial f}{\partial z_1})$ is a linear combination of $\frac{\partial f}{\partial z_0}$ and $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 2}$ only if

$$wt(\frac{\partial f}{\partial z_0}) \leq wt(det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 2}) < wt(\frac{\partial f}{\partial z_1}).$$

Thus

$$2m\alpha_0 < 2d < 2\alpha_0 + \alpha_1 + 2\alpha_2 \leq 5\alpha_0,$$

i.e.,

$$m < \frac{5}{2},$$

which is absurd.
Thus $D(\frac{\partial f}{\partial z_1})$ is a multiple of $\frac{\partial f}{\partial z_0}$.

$$\begin{aligned} & D(\frac{\partial f}{\partial z_1}) \\ =& p(z_1, z_2)\frac{\partial}{\partial z_0}(\frac{\partial f}{\partial z_1}) + cz_2^k \frac{\partial}{\partial z_1}(\frac{\partial f}{\partial z_1}) \\ =& \frac{\partial}{\partial z_1}(p(z_1, z_2)\frac{\partial f}{\partial z_0} + cz_2^k \frac{\partial f}{\partial z_1}) - \frac{\partial p(z_1, z_2)}{\partial z_1} \cdot \frac{\partial f}{\partial z_0} \\ =& h\frac{\partial f}{\partial z_0}. \end{aligned} \tag{2}$$

Equation (2) implies that

$$\frac{\partial}{\partial z_1}\left[p(z_1, z_2)\frac{\partial f}{\partial z_0} + cz_2^k \frac{\partial f}{\partial z_1}\right] = \tilde{h}\frac{\partial f}{\partial z_0}.$$

From equation (1), we know that the left hand side of this equation is independent of z_0 variable. Since $m > 1$, the right hand side of this equation is independent of z_0 variable only if $\tilde{h} = 0$. Thus we have

$$\frac{\partial}{\partial z_1}\left(p(z_1, z_2)\frac{\partial f}{\partial z_0} + cz_2^k \frac{\partial f}{\partial z_1}\right) = 0.$$

Thus we have

$$p(z_1, z_2)\frac{\partial f}{\partial z_0} + cz_2^k \frac{\partial f}{\partial z_1} = uz_2^l,$$

where either $u = 0$ or $u \neq 0$ and $l > k$.
This is

$$\begin{aligned} & p(z_1, z_2)(mz_2 z_0^{m-1} + (m-1)a_1(z_1, z_2)z_0^{m-2} + \cdots + a_{m-1}(z_1, z_2)) \\ & + cz_2^k \left[\frac{\partial a_1(z_1, z_2)}{\partial z_1} z_0^{m-1} + \frac{\partial a_2(z_1, z_2)}{\partial z_1} z_0^{m-2} + \cdots + \frac{\partial z_m(z_1, z_2)}{\partial z_1}\right] \\ & = uz_2^l. \end{aligned} \tag{3}$$

As $m > 1$, Equation (3) implies

$$mp(z_1, z_2)z_2 + cz_2^k \frac{\partial a_1(z_1, z_2)}{\partial z_1} = 0.$$

If $cz_2^k \frac{\partial a_1(z_1,z_2)}{\partial z_1} = 0$, then $p(z_1, z_2) = 0$ and Equation (3) becomes

$$cz_2^k \left(\frac{\partial a_1(z_1, z_2)}{\partial z_1} z_0^{m-1} + \frac{\partial a_2(z_1, z_2)}{\partial z_1} z_0^{m-2} + \cdots + \frac{\partial a_m(z_1, z_2)}{\partial z_1}\right) = uz_2^l.$$

Since $c \neq 0$, we have

$$\frac{\partial a_1(z_1, z_2)}{\partial z_1} = \frac{\partial a_2(z_1, z_2)}{\partial z_1} = \cdots = \frac{\partial a_{m-1}(z_1, z_2)}{\partial z_1} = 0,$$
$$\frac{\partial a_m(z_1, z_2)}{\partial z_1} = ez_2^{l-k},$$

where $e \neq 0$. Hence

$$a_m(z_1, z_2) = ez_1 z_2^{l-k} + z_2^k.$$

We must have $l - k = 1$ and $e \neq 0$, otherwise f would be singular along the z_1-axis, which is a contradiction. Since the tern $z_1 z_2$ appears in f, we conclude that $\alpha_1 + \alpha_2 = d$. The assumption $d \geq 2\alpha_0 \geq 2\alpha_1 \geq 2\alpha_2$ implies $\alpha_0 = \alpha_1 = \alpha_2$. So f is a homogeneous polynomial. By Proposition 2.1, we have that $D \equiv 0$ as a derivation of A where $n = 2$.
If $cz_2^k \frac{\partial a_1(z_1,z_2)}{\partial z_1} \neq 0$. Then

$$p(z_1, z_2) = z_2^{k-1} q(z_1, z_2),$$

where

$$q(z_1, z_2) = -\frac{c}{m} \frac{\partial a_1(z_1, z_2)}{\partial z_1}.$$

So we have

$$q(z_1, z_2)\frac{\partial f}{\partial z_0} + cz_2 \frac{\partial f}{\partial z_1} = uz_2^{l-k+1}.$$

Let us write

$$q(z_1, z_2) = \alpha z_1^s + z_2\gamma(z_1, z_2).$$

We claim that $\alpha = 0$, suppose on the contrary that $\alpha \neq 0$. If we rewrite f in the form

$$f = b_0(z_0, z_1)z_2^n + b_1(z_0, z_1)z_2^{n-1} + \cdots + b_n(z_0, z_1).$$

Then

$$\begin{aligned}
&[\alpha z_1^s + z_2\gamma(z_1, z_2)]\left(\frac{\partial b_0(z_0, z_1)}{\partial z_0}z_2^n + \cdots + \frac{\partial b_n(z_0, z_1)}{\partial z_0}\right)\\
&\quad + cz_2\left(\frac{\partial b_0(z_0, z_1)}{\partial z_1}z_2^n + \cdots + \frac{\partial b_n(z_0, z_1)}{\partial z_1}\right)\\
&= uz_2^{l-k+1}.
\end{aligned} \tag{4}$$

Considering the coefficient of z_1^s, we know that $\frac{\partial b_n(z_0,z_1)}{\partial z_0} = 0$ and hence from Equation (4) again we have

$$\begin{aligned}
&[\alpha z_1^s + z_2\gamma(z_1, z_2)]\left(\frac{\partial b_0(z_0, z_1)}{\partial z_0}z_2^{n-1} + \cdots + \frac{\partial b_{n-1}(z_0, z_1)}{\partial z_0}\right)\\
&\quad + c\left(\frac{\partial b_0(z_0, z_1)}{\partial z_1}z_2^n + \cdots + \frac{\partial b_n(z_0, z_1)}{\partial z_1}\right)\\
&= uz_2^{l-k}.
\end{aligned} \tag{5}$$

Recall that either $u = 0$ or $u \neq 0$ and $l > k$. The Equation (5) implies

$$\alpha z_1^s \frac{\partial b_{n-1}(z_0, z_1)}{\partial z_0} = -c\frac{\partial b_n(z_0, z_1)}{\partial z_1}.$$

Since $\frac{\partial b_n(z_0,z_1)}{\partial z_0} = 0$, we have

$$\frac{\partial b_{n-1}(z_0, z_1)}{\partial z_0} = c' z_1^{s'},$$

where $c' \neq 0$.
If $s' = 0$, then $b_{n-1}(z_0, z_1) = c'z_0 + c''z_1$ and hence z_0z_2 and z_1z_2 occur in f. It follows again that $\alpha_0 = \alpha_1 = \alpha_2$ and we are done.
If $s' > 0$, then

$$b_{n-1}(z_0, z_1) = c' z_1^{s'} z_0 + \tilde{u} z_1^{\tau},$$

where $s' > 0$, $\tau \geq 0$. Now $\frac{\partial b_n(z_0,z_1)}{\partial z_0} = 0$ implies $b_n(z_0, z_1) = wz_1^t$, $t > 1$. Notice that

$$\frac{\partial f}{\partial z_2} = nb_0(z_0, z_1)z_2^{n-1} + (n-1)b_1(z_0, z_1)z_2^{n-2} + \cdots + b_{n-1}(z_0, z_1).$$

It follows that f is singular along the z_0-axis. The contradiction comes from our hypothesis $\alpha \neq 0$. Thus $\alpha = 0$.
Now we have $q(z_1, z_2) = z_2\gamma(z_1, z_2)$ and $\gamma(z_1, z_2)\frac{\partial f}{\partial z_0} + c\frac{\partial f}{\partial z_1} = uz_2^{l-k}$. So we have

$$\begin{aligned}&\gamma(z_1, z_2)\left(\frac{\partial b_0(z_0, z_1)}{\partial z_0}z_2^n + \cdots + \frac{\partial b_n(z_0, z_1)}{\partial z_0}\right) + c\left(\frac{\partial b_0(z_0, z_1)}{\partial z_1}z_2^n + \cdots + \frac{\partial b_n(z_0, z_1)}{\partial z_1}\right)\\ &= uz_2^{l-k}.\end{aligned} \tag{6}$$

It follows that $\gamma(z_1, 0) \neq 0$, otherwise we would have $\frac{\partial b_n(z_0,z_1)}{\partial z_1} = 0$. So $b_n(z_0, z_1) = z_0^m$ for some m. Therefore f would be of the form

$$f = z_0^m + c_1(z_1, z_2)z_0^{m-1} + \cdots + c_m(z_1, z_2),$$

which is a contradiction with our assumption.
Now Let $\gamma(z_1, z_2) = vz_1^h + z_2\bar{\gamma}(z_1, z_2)$, where $h > 0$ and $v \neq 0$. Then we have

$$vz_1^h\frac{\partial b_n(z_0, z_1)}{\partial z_0} + c\frac{\partial b_n(z_0, z_1)}{\partial z_1} = 0,$$

where $v \neq 0$ and $c \neq 0$. Let

$$b_n(z_0, z_1) = d_0z_0^kz_1^{l_0} + d_1z_0^{k-1}z_1^{l_1} + \cdots + d_kz_1^{l_k},$$

where $d_0 \neq 0$. Then

$$\begin{aligned}\frac{\partial b_n(z_0, z_1)}{\partial z_0} &= kd_0z_0^{k-1}z_1^{l_0} + (k-1)d_1z_0^{k-2}z_1^{l_1} + \cdots + d_{k-1}z_1^{l_{k-1}},\\ \frac{\partial b_n(z_0, z_1)}{\partial z_1} &= l_0d_0z_0^kz_1^{l_0-1} + (l_1-1)d_1z_0^{k-1}z_1^{l_1-1} + \cdots + d_kz_1^{l_{k-1}}.\end{aligned}$$

Clearly the equation

$$vz_1^h\frac{\partial b_n(z_0, z_1)}{\partial z_0} + c\frac{\partial b_n(z_0, z_1)}{\partial z_1} = 0,$$

is possible only if $k = 0$ or $l_0 = 0$.
If $k = 0$, then $b_n(z_0, z_1) = d_0z_1^{l_0}$, which is absurd.
If $l_0 = 0$, then

$$b_n(z_0, z_1) = d_0z_0^k + d_1z_0^{k-1}z_1^{l_1} + \cdots + d_kz_1^{l_k}.$$

Hence f is again of the form

$$f = z_0^m + a_1(z_1, z_2)z_0^{m-1} + \cdots + a_m(z_1, z_2),$$

which is absurd.
This completes the proof for Case (2.2).

In the second part of this proof, we consider the case $D(\frac{\partial f}{\partial z_0})$ is a multiple of $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 2}$. By Lemma 2.4, we only need to consider $m = 1, 2$.

Case (1).

$$f = z_0^m + a_1(z_1, z_2)z_0^{m-1} + \cdots + a_{m-1}(z_1, z_2)z_0 + a_m(z_1, z_2).$$

Case (1.1).

$$m = 1.$$

0 is not an isolated singularity.

Case (1.2).

$$m = 2.$$

We have $\frac{\partial f}{\partial z_0} = 2z_2$. This will back to Proposition 2.6.

Case (2).

$$f = z_0^m z_i + a_1(z_1, z_2)z_0^{m-1} + \cdots + a_m(z_1, z_2).$$

There are two subcases.
Case (2.1).

$$m = 1.$$

Case (2.1.1).
If $m = 1$ and $i = 1$, then

$$f = z_0 z_1 + a_1(z_1, z_2).$$

So the relation $\alpha_0 + \alpha_1 = d \geq 2\alpha_0 \geq 2\alpha_1 \geq 2\alpha_2$ implies that $\alpha_0 = \alpha_1$.
In this case,

$$\mathbb{C}[z_0, z_1, z_2]/(\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 2})$$

$$= \mathbb{C}/(\frac{\partial a_1(z_1,z_2)}{\partial z_1}, \frac{\partial a_1(z_1,z_2)}{\partial z_2}, \det(\frac{\partial^2 a_1(z_1,z_2)}{\partial z_i \partial z_j})_{1\le i,j\le 2}).$$

By Proposition 2.6, this completes the proof for this case.

Case (2.1.2).
If $m = 1$ and $i = 2$, then

$$f = z_0 z_2 + a_1(z_1, z_2).$$

Thus the relation $\alpha_0 + \alpha_2 = d \geq 2\alpha_0 \geq 2\alpha_1 \geq 2\alpha_2$ implies that $\alpha_0 = \alpha_1 = \alpha_2$, i.e., f is a homogeneous polynomial. By Proposition 2.1, this completes the proof for this case.

Case (2.2).

$$m = 2.$$

Case (2.2.1).
If $m = 2$ and $i = 1$, then we have

$$f = z_0^2 z_1 + a_1(z_1, z_2) z_0 + a_2(z_1, z_2).$$

Thus

$$D(\frac{\partial f}{\partial z_0}) = 2z_1 p(z_1, z_2) + c z_2^k \left(2z_0 + \frac{\partial a_1(z_1,z_2)}{\partial z_1}\right),$$

and

$$\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 2} = \begin{vmatrix} 2z_1 & 2z_0 + \frac{\partial a_1(z_1,z_2)}{\partial z_1} & \frac{\partial a_1(z_1,z_2)}{\partial z_2} \\ 2z_0 + \frac{\partial a_1(z_1,z_2)}{\partial z_1} & \frac{\partial^2 a_1(z_1,z_2)}{\partial z_1^2} z_0 + \frac{\partial^2 a_2(z_1,z_2)}{\partial z_1^2} & \frac{\partial^2 a_1(z_1,z_2)}{\partial z_1 \partial z_2} z_0 + \frac{\partial^2 a_2(z_1,z_2)}{\partial z_1 \partial z_2} \\ \frac{\partial a_1(z_1,z_2)}{\partial z_2} & \frac{\partial^2 a_1(z_1,z_2)}{\partial z_1 \partial z_2} z_0 + \frac{\partial^2 a_2(z_1,z_2)}{\partial z_1 \partial z_2} & \frac{\partial^2 a_1(z_1,z_2)}{\partial z_2^2} z_0 + \frac{\partial^2 a_2(z_1,z_2)}{\partial z_2^2} \end{vmatrix}.$$

In this case, we know that $wt(D(\frac{\partial f}{\partial z_0})) = \alpha_1 + wt(p)$ and $wt(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 2}) = 4\alpha_0 + \alpha_1 - 2\alpha_2$. Thus $D(\frac{\partial f}{\partial z_0})$ is a multiple of $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 2}$ only if $\alpha_1 + wt(p) \geq 4\alpha_0 + \alpha_1 - 2\alpha_2$. Moreover D is a negative weight derivation, thus $wt(p) < \alpha_0$. We know that

$$\alpha_0 > wt(p) \geq 4\alpha_0 - 2\alpha_2,$$

i.e.,

$$2\alpha_2 > 3\alpha_0,$$

which is absurd.

Case (2.2.2).
If $m = 2$ and $i = 2$, then we have

$$f = z_0^2 z_2 + a_1(z_1, z_2) z_0 + a_2(z_1, z_2).$$

Thus

$$D(\frac{\partial f}{\partial z_0}) = 2z_2 p(z_1, z_2) + c z_2^k \frac{\partial a_1(z_1, z_2)}{\partial z_1},$$

and

$$\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 2}$$
$$= \begin{vmatrix} 2z_2 & \frac{\partial a_1(z_1,z_2)}{\partial z_1} & 2z_0 + \frac{\partial a_1(z_1,z_2)}{\partial z_2} \\ \frac{\partial a_1(z_1,z_2)}{\partial z_1} & \frac{\partial^2 a_1(z_1,z_2)}{\partial z_1^2} z_0 + \frac{\partial^2 a_2(z_1,z_2)}{\partial z_1^2} & \frac{\partial^2 a_1(z_1,z_2)}{\partial z_1 \partial z_2} z_0 + \frac{\partial^2 a_2(z_1,z_2)}{\partial z_1 \partial z_2} \\ 2z_0 + \frac{\partial a_1(z_1,z_2)}{\partial z_2} & \frac{\partial^2 a_1(z_1,z_2)}{\partial z_1 \partial z_2} z_0 + \frac{\partial^2 a_2(z_1,z_2)}{\partial z_1 \partial z_2} & \frac{\partial^2 a_1(z_1,z_2)}{\partial z_2^2} z_0 + \frac{\partial^2 a_2(z_1,z_2)}{\partial z_2^2} \end{vmatrix}.$$

In this case, we know that $wt(D(\frac{\partial f}{\partial z_0})) = \alpha_2 + wt(p)$ and $wt(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 2}) = 4\alpha_0 + \alpha_2 - 2\alpha_1$. So $D(\frac{\partial f}{\partial z_0})$ is a multiple of $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 2}$ only if $\alpha_2 + wt(p) \ge 4\alpha_0 + \alpha_2 - 2\alpha_1$. Moreover D is a negative weight derivation, so $wt(p) < \alpha_0$. We know that

$$\alpha_0 > wt(p) \ge 4\alpha_0 - 2\alpha_1,$$

i.e.,

$$2\alpha_1 > 3\alpha_0,$$

which is absurd. □

Proposition 2.8. *Let $f(z_0, z_1, z_2, z_3)$ be a weighted homogeneous polynomial of type $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ with isolated singularity at the origin. Assume that $d \ge 2\alpha_0 \ge 2\alpha_1 \ge 2\alpha_2 \ge 2\alpha_3$. Let D be a derivation of the algebra*

$$\mathbb{C}[z_0, z_1, z_2, z_3]/(\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \frac{\partial f}{\partial z_3}, \det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 3}).$$

Then $D \equiv 0$, if D is of negative weight.

Proof. When $n = 3$, we have

$$D = p_0(z_1, z_2, z_3)\frac{\partial}{\partial z_0} + p_1(z_2, z_3)\frac{\partial}{\partial z_1} + cz_3^k\frac{\partial}{\partial z_2},$$

where c, k are constants and $k \geq 1$.
The commutator $[\frac{\partial}{\partial z_i}, D]$ is of the following form by a direct computation.

$$[\frac{\partial}{\partial z_0}, D] = 0;$$
$$[\frac{\partial}{\partial z_1}, D] = \frac{\partial p_0}{\partial z_1}\frac{\partial}{\partial z_0};$$
$$[\frac{\partial}{\partial z_2}, D] = \frac{\partial p_0}{\partial z_2}\frac{\partial}{\partial z_0} + \frac{\partial p_1}{\partial z_2}\frac{\partial}{\partial z_1};$$
$$[\frac{\partial}{\partial z_3}, D] = \frac{\partial p_0}{\partial z_3}\frac{\partial}{\partial z_0} + \frac{\partial p_1}{\partial z_3}\frac{\partial}{\partial z_1} + \frac{\partial(cz_3^k)}{\partial z_3}\frac{\partial}{\partial z_2}.$$

By Lemma 2.3, there are also two cases of f.
Case (1).

$$f = z_0^m + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_{m-1}(z_1, z_2, z_3)z_0 + a_m(z_1, z_2, z_3).$$

In this case, we can just assume that $m \geq 3$. If $m = 1$, then 0 is not an isolated singularity. If $m = 2$, then $\frac{\partial f}{\partial z_0} = 2z_0$. So we back to Proposition 2.7.
Firstly, we consider $D(\frac{\partial f}{\partial z_1})$.
By the weighted degrees, we know that $D(\frac{\partial f}{\partial z_1})$ is a linear combination of $\frac{\partial f}{\partial z_0}$ and $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 3}$. By Lemma 2.4, we know that the coefficient of $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 3}$ is not zero only if

$$wt(\frac{\partial f}{\partial z_0}) \leq wt(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 3}) < wt(\frac{\partial f}{\partial z_1}).$$

So

$$3m\alpha_0 = 3d < 2\alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 \leq 7\alpha_0.$$

This implies that $m < \frac{7}{3}$, which is absurd.
So $D(\frac{\partial f}{\partial z_1})$ is a multiple of $\frac{\partial f}{\partial z_0}$.
Secondly, we consider $D(\frac{\partial f}{\partial z_2})$.
By the weighted degrees, we know that $D(\frac{\partial f}{\partial z_2})$ is a linear combination of $\frac{\partial f}{\partial z_0}$, $\frac{\partial f}{\partial z_1}$ and $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 3}$. Similarly, we know that the coefficient of $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 3}$ is not zero only if

$$wt(\frac{\partial f}{\partial z_1}) \leq wt(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 3}) < wt(\frac{\partial f}{\partial z_2}).$$

Thus

$$3m\alpha_0 = 3d < 2\alpha_0 + 2\alpha_1 + \alpha_2 + 2\alpha_3 \leq 7\alpha_0.$$

This implies that $m < \frac{7}{3}$, which is absurd.
Thus $D(\frac{\partial f}{\partial z_2})$ is a multiple of $\frac{\partial f}{\partial z_0}$, $\frac{\partial f}{\partial z_1}$.

Thirdly, we consider $D(\frac{\partial f}{\partial z_3})$.
By the weighted degrees, we know that $D(\frac{\partial f}{\partial z_1})$ is a linear combination of $\frac{\partial f}{\partial z_0}$, $\frac{\partial f}{\partial z_1}$, $\frac{\partial f}{\partial z_2}$ and $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 3}$. Similarly, we know that the coefficient of $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 3}$ is not zero only if

$$wt(\frac{\partial f}{\partial z_2}) \leq wt(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 3}) < wt(\frac{\partial f}{\partial z_3}).$$

Thus

$$3m\alpha_0 = 3d < 2\alpha_0 + 2\alpha_1 + 2\alpha_2 + \alpha_3 \leq 7\alpha_0.$$

This implies that $m < \frac{7}{3}$, which is absurd.
Thus $D(\frac{\partial f}{\partial z_3})$ is a multiple of $\frac{\partial f}{\partial z_0}$, $\frac{\partial f}{\partial z_1}$, $\frac{\partial f}{\partial z_2}$.
We obtain following relations.

$$\begin{aligned}
&D(\frac{\partial f}{\partial z_0}) = 0;\\
&D(\frac{\partial f}{\partial z_1}) = p(z_1, z_2, z_3)\frac{\partial f}{\partial z_0};\\
&D(\frac{\partial f}{\partial z_2}) = q_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + q_1(z_2, z_3)\frac{\partial f}{\partial z_1};\\
&D(\frac{\partial f}{\partial z_3}) = w_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + w_1(z_2, z_3)\frac{\partial f}{\partial z_1} + w_2(z_3)\frac{\partial f}{\partial z_2}.
\end{aligned}$$

Case (2).

$$f = z_0^m z_i + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_{m-1}(z_1, z_2, z_3)z_0 + a_m(z_1, z_2, z_3).$$

Case (2.1).

$$m = 1,$$

i.e.,

$$f = z_0 z_i + a_1(z_1, z_2, z_3).$$

Case (2.1.1).

$$i = 1,$$

i.e.,

$$f = z_0 z_1 + a_1(z_1, z_2, z_3).$$

In this case, let $a_1(z_1, z_2, z_3) = z_1 b_1(z_2, z_3) + b_2(z_2, z_3)$, we can find a new coordinate as follows

$$\begin{cases} z_0' = z_0 + b_1(z_2, z_3) \\ z_1' = z_1 \\ z_2' = z_2 \\ z_3' = z_3 \end{cases}.$$

Let $g(z_2', z_3') = b_2(z_2, z_3)$, then

$$\mathbb{C}[z_0, z_1, z_2, z_3]/(\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0 \le i,j \le 3})$$
$$= \mathbb{C}[z_2', z_3']/(\frac{\partial g}{\partial z_2'}, \frac{\partial g}{\partial z_3'}, \det(\frac{\partial^2 g}{\partial z_i' \partial z_j'})_{2 \le i,j \le 3}).$$

By Proposition 2.7, this completes the proof for this case.
Case (2.1.2).

$$i = 2,$$

i.e.,

$$f = z_0 z_2 + a_1(z_1, z_2, z_3).$$

In this case, let $a_1(z_1, z_2, z_3) = z_2 b_1(z_1, z_3) + b_2(z_1, z_3)$, we can find a new coordinate as follows

$$\begin{cases} z_0' = z_0 + b_1(z_2, z_3) \\ z_1' = z_1 \\ z_2' = z_2 \\ z_3' = z_3 \end{cases}.$$

Let $g(z_1', z_3') = b_2(z_1, z_3)$, then

$$\mathbb{C}[z_0, z_1, z_2, z_3]/(\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 3})$$
$$= \mathbb{C}[z_1', z_3']/(\frac{\partial g}{\partial z_1'}, \frac{\partial g}{\partial z_3'}, \det(\frac{\partial^2 g}{\partial z_i' \partial z_j'})_{i,j\in\{1,3\}}).$$

By Proposition 2.7, this completes the proof for this case.
Case (2.1.3).

$$i = 3,$$

i.e.,

$$f = z_0 z_3 + a_1(z_1, z_2, z_3).$$

In this case, as we know, $\alpha_0 + \alpha_3 = d \ge 2\alpha_0 \ge 2\alpha_1 \ge 2\alpha_2 \ge 2\alpha_3$, so $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3$, i.e., f is a weighted homogeneous polynomial. By Proposition 2.1, this completes the proof for this case.

Case (2.2).

$$m = 2,$$

i.e.,

$$f = z_0^2 z_i + a_1(z_1, z_2, z_3) z_0 + a_2(z_1, z_2, z_3).$$

Case (2.2.1).

$$i = 1,$$

i.e.,

$$f = z_0^2 z_1 + a_1(z_1, z_2, z_3) z_0 + a_2(z_1, z_2, z_3).$$

In this case, we obtain that

$$wt(D(\frac{\partial f}{\partial z_1})) = wt(p_0) + \alpha_0;$$
$$wt(D(\frac{\partial f}{\partial z_2})) = wt(p_0) + \alpha_0 + \alpha_1 - \alpha_2;$$
$$wt(D(\frac{\partial f}{\partial z_3})) = wt(p_0) + \alpha_0 + \alpha_1 - \alpha_3;$$
$$wt(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 3}) = 6\alpha_0 + 2\alpha_1 - 2\alpha_2 - 2\alpha_3.$$

The coefficient of $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 3}$ in $D(\frac{\partial f}{\partial z_1})$ is not zero only if

$$wt(p_0) + \alpha_0 \ge 6\alpha_0 + 2\alpha_1 - 2\alpha_2 - 2\alpha_3.$$

Moreover, D is a negative weight derivation, i.e., $wt(p_0) < \alpha_0$. Thus

$$\alpha_0 > wt(p_0) \ge 5\alpha_0 + 2\alpha_1 - 2\alpha_2 - 2\alpha_3,$$

which is absurd.
The coefficient of $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 3}$ in $D(\frac{\partial f}{\partial z_2})$ is not zero only if

$$wt(p_0) + \alpha_0 + \alpha_1 - \alpha_2 \ge 6\alpha_0 + 2\alpha_1 - 2\alpha_2 - 2\alpha_3.$$

Moreover, D is a negative weight derivation, i.e., $wt(p_0) < \alpha_0$. Thus

$$\alpha_0 > wt(p_0) \ge 5\alpha_0 + \alpha_1 - \alpha_2 - 2\alpha_3$$

which is absurd.
The coefficient of $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 3}$ in $D(\frac{\partial f}{\partial z_3})$ is not zero only if

$$wt(p_0) + \alpha_0 + \alpha_1 - \alpha_3 \ge 6\alpha_0 + 2\alpha_1 - 2\alpha_2 - 2\alpha_3.$$

Moreover, D is a negative weight derivation, i.e., $wt(p_0) < \alpha_0$. Thus

$$\alpha_0 > wt(p_0) \ge 5\alpha_0 + \alpha_1 - 2\alpha_2 - \alpha_3,$$

which is absurd.

Case (2.2.2).

$$i = 2,$$

i.e.,

$$f = z_0^2 z_2 + a_1(z_1, z_2, z_3)z_0 + a_2(z_1, z_2, z_3).$$

In this case, we obtain that

$$\begin{aligned}
&wt(D(\frac{\partial f}{\partial z_1})) = wt(p_0) + \alpha_0 - \alpha_1 + \alpha_2;\\
&wt(D(\frac{\partial f}{\partial z_2})) = wt(p_0) + \alpha_0;\\
&wt(D(\frac{\partial f}{\partial z_3})) = wt(p_0) + \alpha_0 - \alpha_3 + \alpha_2;\\
&wt(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 3}) = 6\alpha_0 - 2\alpha_1 + 2\alpha_2 - 3\alpha_3.
\end{aligned}$$

The coefficient of $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 3}$ in $D(\frac{\partial f}{\partial z_1})$ is not zero only if

$$wt(p_0) + \alpha_0 - \alpha_1 + \alpha_2 \geq 6\alpha_0 - 2\alpha_1 + 2\alpha_2 - 2\alpha_3.$$

Moreover, D is a negative weight derivation, i.e., $wt(p_0) < \alpha_0$. Thus

$$\alpha_0 > wt(p_0) \geq 5\alpha_0 - \alpha_1 + \alpha_2 - 2\alpha_3$$

which is absurd.
The coefficient of $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 3}$ in $D(\frac{\partial f}{\partial z_2})$ is not zero only if

$$wt(p_0) + \alpha_0 \geq 6\alpha_0 - 2\alpha_1 + 2\alpha_2 - 2\alpha_3.$$

Moreover, D is a negative weight derivation, i.e., $wt(p_0) < \alpha_0$. Thus

$$\alpha_0 > wt(p_0) \geq 5\alpha_0 - 2\alpha_1 + 2\alpha_2 - 2\alpha_3,$$

which is absurd.
The coefficient of $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 3}$ in $D(\frac{\partial f}{\partial z_3})$ is not zero only if

$$wt(p_0) + \alpha_0 - \alpha_3 + \alpha_2 \geq 6\alpha_0 - 2\alpha_1 + 2\alpha_2 - 2\alpha_3.$$

Moreover, D is a negative weight derivation, i.e., $wt(p_0) < \alpha_0$. Thus

$$\alpha_0 > wt(p_0) \geq 5\alpha_0 - 2\alpha_1 + \alpha_2 - \alpha_3,$$

which is absurd.

Case (2.2.3).

$$i = 3,$$

i.e.,

$$f = z_0^2 z_3 + a_1(z_1, z_2, z_3) z_0 + a_2(z_1, z_2, z_3).$$

In this case, we obtain that

$$wt(D(\frac{\partial f}{\partial z_1})) = wt(p_0) + \alpha_0 - \alpha_1 + \alpha_3;$$
$$wt(D(\frac{\partial f}{\partial z_2})) = wt(p_0) + \alpha_0 - \alpha_2 + \alpha_3;$$
$$wt(D(\frac{\partial f}{\partial z_3})) = wt(p_0) + \alpha_0;$$
$$wt(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 3}) = 6\alpha_0 - 2\alpha_1 - 2\alpha_2 + 2\alpha_3.$$

The coefficient of $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 3}$ in $D(\frac{\partial f}{\partial z_1})$ is not zero only if

$$wt(p_0) + \alpha_0 - \alpha_1 + \alpha_3 \geq 6\alpha_0 - 2\alpha_1 - 2\alpha_2 + 2\alpha_3.$$

Moreover, D is a negative weight derivation, i.e., $wt(p_0) < \alpha_0$. Thus

$$\alpha_0 > wt(p_0) \geq 5\alpha_0 - \alpha_1 - 2\alpha_2 + \alpha_3,$$

which is absurd.
The coefficient of $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 3}$ in $D(\frac{\partial f}{\partial z_2})$ is not zero only if

$$wt(p_0) + \alpha_0 - \alpha_2 + \alpha_3 \geq 6\alpha_0 - 2\alpha_1 - 2\alpha_2 + 2\alpha_3.$$

Moreover, D is a negative weight derivation, i.e., $wt(p_0) < \alpha_0$. Thus

$$\alpha_0 > wt(p_0) \geq 5\alpha_0 - 2\alpha_1 - \alpha_2 + \alpha_3,$$

which is absurd.
The coefficient of $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 3}$ in $D(\frac{\partial f}{\partial z_3})$ is not zero only if

$$wt(p_0) + \alpha_0 \geq 6\alpha_0 - 2\alpha_1 - 2\alpha_2 + 2\alpha_3.$$

Moreover, D is a negative weight derivation, i.e., $wt(p_0) < \alpha_0$. Thus

$$\alpha_0 > wt(p_0) \geq 5\alpha_0 - 2\alpha_1 - 2\alpha_2 + 2\alpha_3,$$

which is absurd.

Case (2.3).

$$m \geq 3.$$

Case (2.3.1).

$$i = 1.$$

Firstly, we consider $D(\frac{\partial f}{\partial z_1})$.
By the weighted degrees, we know that $D(\frac{\partial f}{\partial z_1})$ is a linear combination of $\frac{\partial f}{\partial z_0}$ and $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 3}$. By Lemma 2.4, we know that the coefficient of $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 3}$ is not zero only if

$$wt(\frac{\partial f}{\partial z_0}) \leq wt(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 3}) < wt(\frac{\partial f}{\partial z_1}).$$

Thus

$$3m\alpha_0 < 3d < 2\alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 \leq 7\alpha_0.$$

This implies that $m < \frac{7}{3}$, which is absurd.
Thus $D(\frac{\partial f}{\partial z_1})$ is a multiple of $\frac{\partial f}{\partial z_0}$.

Secondly, we consider $D(\frac{\partial f}{\partial z_2})$.
By the weighted degrees, we know that $D(\frac{\partial f}{\partial z_2})$ is a linear combination of $\frac{\partial f}{\partial z_0}$, $\frac{\partial f}{\partial z_1}$ and $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 3}$. Similarly, we know that the coefficient of $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 3}$ is not zero only if

$$wt(\frac{\partial f}{\partial z_1}) \leq wt(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 3}) < wt(\frac{\partial f}{\partial z_2}).$$

Thus

$$3m\alpha_0 < 3d < 2\alpha_0 + 2\alpha_1 + \alpha_2 + 2\alpha_3 \leq 7\alpha_0.$$

This implies that $m < \frac{7}{3}$, which is absurd.
So $D(\frac{\partial f}{\partial z_2})$ is a multiple of $\frac{\partial f}{\partial z_0}$, $\frac{\partial f}{\partial z_1}$.

Thirdly, we consider $D(\frac{\partial f}{\partial z_3})$.
By the weighted degrees, we know that $D(\frac{\partial f}{\partial z_1})$ is a linear combination of $\frac{\partial f}{\partial z_0}$, $\frac{\partial f}{\partial z_1}$, $\frac{\partial f}{\partial z_2}$ and $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 3}$. Similarly, we know that the coefficient of $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 3}$ is not zero only if

$$wt(\frac{\partial f}{\partial z_2}) \leq wt(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 3}) < wt(\frac{\partial f}{\partial z_3}).$$

Thus

$$3m\alpha_0 < 3d < 2\alpha_0 + 2\alpha_1 + 2\alpha_2 + \alpha_3 \leq 7\alpha_0.$$

This implies that $m < \frac{7}{3}$, which is absurd.
Thus $D(\frac{\partial f}{\partial z_3})$ is a multiple of $\frac{\partial f}{\partial z_0}$, $\frac{\partial f}{\partial z_1}$, $\frac{\partial f}{\partial z_2}$.
We obtain following relations when $m \geq 2$.

$$\begin{aligned} &D(\frac{\partial f}{\partial z_0}) = 0; \\ &D(\frac{\partial f}{\partial z_1}) = p(z_1, z_2, z_3)\frac{\partial f}{\partial z_0}; \\ &D(\frac{\partial f}{\partial z_2}) = q_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + q_1(z_2, z_3)\frac{\partial f}{\partial z_1}; \end{aligned} \tag{7}$$

$$D(\frac{\partial f}{\partial z_3}) = w_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + w_1(z_2, z_3)\frac{\partial f}{\partial z_1} + w_2(z_3)\frac{\partial f}{\partial z_2}.$$

By Lemma 2.1 in [20], we can assume that

$$L(V) = \bigoplus_k L_k$$

and $D \in L_q$ where L_q is the most negatively graded part.

First Claim. Df does not depend on z_0, z_1.

Since $\frac{\partial(Df)}{\partial z_0} = 0$ it is clear that Df is independent of z_0. Then

$$\frac{\partial(Df)}{\partial z_1} = h(z_1, z_2, z_3)\frac{\partial f}{\partial z_0}$$

is independent of z_0. If $h \not\equiv 0$ we have that $f(z_0, z_1, z_2, z_3)$ is of the form $z_0 a_1(z_1, z_2, z_3) + a_2(z_1, z_2, z_3)$. Because of Lemma 1.2 in [3] $a_1(z_1, z_2, z_3)$ has to be the form $z_j + a_1'(z_1, z_2, z_3)(j \in \{1, 2, 3\})$. Without loss of generality we can assume that $j = 1$. This forces that $a_1(z_1, z_2, z_3) = z_1 + a_1'(z_2, z_3)$ since a_1 is weighted homogeneous. Without loss of generality, we can assume that

$$f(z_0, z_1, z_2, z_3) = z_0 z_1 + g(z_1, z_2, z_3),$$

otherwise we can do a coordinate transformation which preserve type $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$. Let

$$g(z_1, z_2, z_3) = z_1 g_1(z_1, z_2, z_3) + g_2(z_2, z_3),$$

then

$$f(z_0, z_1, z_2, z_3) = z_1(z_0 + g_1(z_1, z_2, z_3)) + g_2(z_2, z_3).$$

Find new coordinates as above so that we can assume without loss of generality that

$$f(z_0, z_1, z_2, z_3) = z_0 z_1 + g_2(z_2, z_3).$$

In this case

$$\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0 \le i,j \le 3} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial^2 g}{\partial z_2^2} & \frac{\partial^2 g}{\partial z_2 \partial z_3} \\ 0 & 0 & \frac{\partial^2 g}{\partial z_3 \partial z_2} & \frac{\partial^2 g}{\partial z_3^2} \end{bmatrix}.$$

Thus

$$\mathbb{C}\{z_0,z_1,z_2,z_3\}/(\frac{\partial f}{\partial z_0},\frac{\partial f}{\partial z_1},\frac{\partial f}{\partial z_2},\frac{\partial f}{\partial z_3},\det(\frac{\partial^2 f}{\partial z_i\partial z_j})_{0\leq i,j\leq 3})$$
$$=\mathbb{C}\{z_2,z_3\}/(\frac{\partial g_2}{\partial z_2},\frac{\partial g_2}{\partial z_3},\det(\frac{\partial^2 g_2}{\partial z_i\partial z_j})_{2\leq i,j\leq 3}).$$

Our statement about D is reduced to the case $n=1$. By the result above, we obtain the conclusion. Hence $h(z_1,z_2,z_3)$ has to be 0 and Df is independent of z_0, z_1.

Second Claim. Df does not depend on z_0, z_1, z_2.
First we know that

$$\frac{\partial(Df)}{\partial z_2}=q_0(z_1,z_2,z_3)\frac{\partial f}{\partial z_0}+q_1(z_2,z_3)\frac{\partial f}{\partial z_1}.$$

Since Df is independent of z_0 and z_1, the left hand side, therefore the right hand side is independent of z_0 and z_1. There are two cases.

Case (i).

$$f=z_0^m+a_1(z_1,z_2,z_3)z_0^{m-1}+\cdots+a_m(z_1,z_2,z_3).$$

Thus

$$\frac{\partial(Df)}{\partial z_2}=z_0^{m-1}(mq_0(z_1,z_2,z_3)+q_1(z_2,z_3)\frac{\partial a_1(z_1,z_2,z_3)}{\partial z_1})+\dots$$

In this case m has to be ≥ 2 and $\frac{\partial(Df)}{\partial z_2}$ is independent of z_0, because $\frac{\partial^2(Df)}{\partial z_0\partial z_2}=0$. Hence we have that $(m-1\geq 1)$

$$q_0(z_1,z_2,z_3)=-\frac{1}{m}q_1(z_2,z_3)\frac{\partial a_1(z_1,z_2,z_3)}{\partial z_1}.$$

Take a new coordinate system which preserves the type $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$

$$\begin{aligned}z_0&=z_0'-\frac{1}{m}a_1(z_1',z_2',z_3');\\ z_1&=z_1';\\ z_2&=z_2';\\ z_3&=z_3';\end{aligned}$$

It is clear that we have the following relations.

$$\begin{aligned}
\frac{\partial}{\partial z_0'} &= \frac{\partial}{\partial z_0};\\
\frac{\partial}{\partial z_1'} &= -\frac{1}{m}\cdot\frac{\partial a_1(z_1',z_2',z_3')}{\partial z_1'}\frac{\partial}{\partial z_0}+\frac{\partial}{\partial z_1};\\
\frac{\partial}{\partial z_2'} &= -\frac{1}{m}\cdot\frac{\partial a_1(z_1',z_2',z_3')}{\partial z_2'}\frac{\partial}{\partial z_0}+\frac{\partial}{\partial z_2};\\
\frac{\partial}{\partial z_3'} &= -\frac{1}{m}\cdot\frac{\partial a_1(z_1',z_2',z_3')}{\partial z_3'}\frac{\partial}{\partial z_0}+\frac{\partial}{\partial z_3}.
\end{aligned}$$

Therefore we have that in the new coordinate system

$$\begin{aligned}
&\frac{\partial(Df)}{\partial z_0'} = 0;\\
&\frac{\partial(Df)}{\partial z_1'} = 0;\\
&\frac{\partial(Df)}{\partial z_2'} = -\frac{1}{m}\frac{\partial a_1(z_1',z_2',z_3')}{\partial z_2'}\cdot\frac{\partial(Df)}{\partial z_0}+\frac{\partial(Df)}{\partial z_2}\\
&= \frac{\partial(Df)}{\partial z_2} = q_1(z_2',z_3')(-\frac{1}{m}\frac{\partial a_1(z_1',z_2',z_3')}{\partial z_1'}\cdot\frac{\partial f}{\partial z_0}+\frac{\partial f}{\partial z_1})\\
&= q_1(z_2',z_3')\frac{\partial f}{\partial z_1'}.
\end{aligned}$$

This means that in the new coordinate system $q_1(z_2',z_3')\frac{\partial f}{\partial z_1'}$ is independent of z_0' and z_1'. By a similar argument of the first claim, we know that $q_1(z_2,z_3) = q_1(z_2',z_3') = 0$ and $\frac{\partial(Df)}{\partial z_2} = 0$.

Case (ii).

$$f = z_0^m z_i + a_1(z_1,z_2,z_3)z_0^{m-1}+\cdots+a_m(z_1,z_2,z_3).$$

If $i = 1$, then

$$\begin{aligned}
\frac{\partial(Df)}{\partial z_2} &= q_0(z_1,z_2,z_3)\frac{\partial f}{\partial z_0}+q_1(z_2,z_3)\frac{\partial f}{\partial z_1}\\
&= q_1(z_2,z_3)z_0^m + \textit{Lower degree terms w.r.t } z_0.
\end{aligned}$$

By a similar argument in the first claim above, we know that

$$q_1 = 0.$$

Therefore we know that

$$\frac{\partial(Df)}{\partial z_2} = q_0(z_1,z_2,z_3)\frac{\partial f}{\partial z_0},$$

is independent of z_0. By a similar argument in the first claim above, we know that

$$q_0 = 0,$$

i.e., $\frac{\partial(Df)}{\partial z_2} = 0$.
If $i = 2$ or $i = 3$, then

$$\begin{aligned}
&\frac{\partial(Df)}{\partial z_2} \\
&= \left(mq_0(z_1, z_2, z_3)z_i + \frac{\partial a_1(z_1, z_2, z_3)}{\partial z_1} \cdot q_1(z_2, z_3) \right) z_0^{m-1} \\
&+ \left[(m-1)q_0(z_1, z_2, z_3)a_1(z_1, z_2, z_3) + \frac{\partial a_2(z_1, z_2, z_3)}{\partial z_1} q_1(z_2, z_3) \right] z_0^{m-2} \\
&+ \cdots + q_0(z_1, z_2, z_3)a_{m-1}(z_1, z_2, z_3) + \frac{\partial a_m(z_1, z_2, z_3)}{\partial z_1} q_1(z_2, z_3).
\end{aligned}$$

If $m \geq 3$, since both sides of above equation have to be independent of z_0, we have that the coefficients of z_0^m and z_0^{m-1} are zero, that is

$$mq_0(z_1, z_2, z_3)z_i = -q_1(z_2, z_3)\frac{\partial a_1(z_1, z_2, z_3)}{\partial z_1},$$

$$(m-1)q_0(z_1, z_2, z_3)a_1(z_1, z_2, z_3) = -q_1(z_2, z_3)\frac{\partial a_2(z_1, z_2, z_3)}{\partial z_1}.$$

Combining these two equations we obtain that

$$\frac{m-1}{m} \cdot a_1(z_1, z_2, z_3)\frac{\partial a_1(z_1, z_2, z_3)}{\partial z_1} = z_i \frac{\partial a_2(z_1, z_2, z_3)}{\partial z_1}.$$

This forces that $\frac{\partial a_1(z_1,z_2,z_3)}{\partial z_1}$ can be divided by z_i. In fact if $a_1(z_1, z_2, z_3)$ can be divided z_i, then $\frac{\partial a_1(z_1,z_2,z_3)}{\partial z_1}$ can be divided by z_i. Otherwise $\frac{\partial a_1(z_1,z_2,z_3)}{\partial z_1}$ has to be divided by z_i. Therefore $q_0(z_1, z_2, z_3)$ can be divided by $q_1(z_2, z_3)$. By an argument as in the case (i) we obtain the conclusion.
If $m = 1$, the argument is trivial by the result of $n = 1, 2$ and the proof of the first claim.
If $m = 2$ we have that

$$f = z_0^2 z_i + a_1(z_1, z_2, z_3)z_0 + a_2(z_1, z_2, z_3).$$

Since $q_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + q_1(z_2, z_3)\frac{\partial f}{\partial z_1}$ is independent of z_0 we have that

$$2q_0(z_1, z_2, z_3)z_i = -q_1(z_2, z_3)\frac{\partial a_1(z_1, z_2, z_3)}{\partial z_1}.$$

Without loss of generality we can assume that

$$q_0(z_1,z_2,z_3) = -\frac{1}{2}\frac{\partial a_1(z_1,z_2,z_3)}{\partial z_1}q_1'(z_2,z_3),$$

and

$$q_1(z_2,z_3) = x_i q_1'(z_2,z_3).$$

Otherwise $\frac{\partial a_1(z_1,z_2,z_3)}{\partial z_1}$ can be divided by z_i and an argument as in the $m \geq 3$ case gives us the conclusion. Therefore

$$\begin{aligned}\frac{\partial(Df)}{\partial z_2} &= q_1'(z_2,z_3)\left(-\frac{1}{2}\frac{\partial a_1(z_1,z_2,z_2)}{\partial z_1}\frac{\partial f}{\partial z_0} + z_i\frac{\partial f}{\partial z_1}\right)\\ &= q_1'(z_2,z_3)\left(-\frac{1}{2}\frac{\partial a_1(z_1,z_2,z_3)}{\partial z_1}a_1(z_1,z_2,z_3) + z_i\frac{\partial a_2(z_1,z_2,z_3)}{\partial z_1}\right)\\ &= q_1'(z_2,z_3)\frac{\partial\left(-\frac{1}{4}a_1^2(z_1,z_2,z_3) + z_i a_2(z_2,z_3)\right)}{\partial z_1},\end{aligned}$$

is independent of z_0 and z_1 by the first claim.
Let

$$a_1(z_1,z_2,z_3) = b_u(z_2,z_3)z_1^u + \cdots + b_0(z_2,z_3).$$

If $\frac{\partial a_1(z_1,z_2,z_3)}{\partial z_1}$ can be divided by z_i then an argument as in $m \geq 3$ case gives us the conclusion. Hence there are some $b_j(z_2,z_3)$ for $j \geq 1$ which cannot be divided by z_i. Suppose that u_0 is the largest index among them. Then

$$a_1^2(z_1,z_2,z_3) = \sum_t \sum_{u+v=t} b_u b_v z_1^t.$$

The coefficient of $z_1^{2u_0}$ is of the form

$$\sum_{u+v=2u_0} b_u b_v.$$

since u_0 is the largest index such that $b_j(z_2,z_3)$ cannot be divided by z_i, it is clear that

$$\sum_{u+v=2u_0} b_u b_v = b_{u_0}^2 + \sum_{u+v=2u_0,\ u\ or\ v>u_0} b_u b_v$$

cannot be divided by z_i. By the fact that

$$\frac{\partial(-\frac{1}{4}a_1^2(z_1,z_2,z_3) + z_i a_2(z_1,z_2,z_3))}{\partial z_1}$$

is independent of z_1 and $2u_0 - 1 \geq 1$ we know that the coefficient of $z_1^{2u_0}$ in the expression of $a_1^2(z_1,z_2,z_3)$ can be divided by z_i. This is a contradiction. Therefore Df is independent of z_0, z_1, z_2.

Third Claim. Df does not depend on z_0, z_1, z_2, z_3.
We already obtain that

$$\frac{\partial(Df)}{\partial z_3} = p_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + p_1(z_2, z_3)\frac{\partial f}{\partial z_1} + c_1 x_3^k \frac{\partial f}{\partial z_2}.$$

The both sides of this equation are independent of z_0, z_1, z_2. What we want to prove here is that

$$p_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + p_1(z_2, z_3)\frac{\partial f}{\partial z_1} + c_1 z_3^k \frac{\partial f}{\partial z_2} = c_2 z_3^l \tag{8}$$

(note that the left hand side has to be weighted homogeneous) implies both sides of it are zero.
If $k = 0$ and $c_1 \neq 0$, we can do a transformation of coordinates which preserves the type $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ such that the equation (8) becomes

$$\frac{\partial f}{\partial z_2} = c_2 z_3^l.$$

By a similar argument of the first claim, we know that $c_2 = 0$.
If $c_1 = 0$, by a similar argument of the second claim, we know that $c_2 = 0$.
If $c_1 \neq 0$ and $k \geq 1$, there are two cases.

Case (i).

$$f = z_0^m + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3).$$

Where $m \geq 2$. We can do a transformation of coordinates which preserves the type $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ such that the equation (8) becomes

$$p_1(z_2, z_3)\frac{\partial f}{\partial z_1} + c_1 z_3^k \frac{\partial f}{\partial z_2} = c_2 z_3^l \tag{9}$$

Lemma 2.9. *Suppose f as in Proposition 2.8 and equation (9) holds then both sides of equation (9) have to be zero or $D \equiv 0$.*

Proof. Without loss of generality we can assume that $p_1(z_2, z_3)$ cannot be divided by z_3.
If $k = 0$, we can do a transformation of coordinates which preserves the type $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ such that the equation (9) becomes

$$\frac{\partial f}{\partial z_2} = c_2 z_3^l.$$

If $c_2 = 0$, we have already finished.
If $c_2 \neq 0$, $f(z_0, z_1, z_2, z_3)$ has to be the form

$$f = z_2 z_3^l + g(z_0, z_1, z_3).$$

By Lemma 1.2 in [3], we know that $l = 1$. By a similar argument of the first claim, we know that $D = 0$.
If $k > 0$, then $\frac{\partial f}{\partial z_1}$ can be divided by z_3. By Lemma 1.2 in [3], f has a term $z_1^t z_3$. If $a(z_0, z_2, z_3)z_1^u$ is a term of f, then $u\alpha_1 + \deg(a) = t\alpha_1 + \alpha_3$. Therefore if $u > t$, then $\deg(a) < \alpha_3$ which is a contradiction with our assumption about the type $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$. Thus we know that f has the form

$$f = z_1^t z_3 + \ldots.$$

By a similar argument of the second claim, we know that $c_2 = 0$. □

Case (ii).

$$f = z_0^m z_i + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3).$$

If $i = 1$ or $i = 2$, then one of p_0, p_1 and c_1 is 0. If $p_0 = 0$, we can obtain the conclusion by $c_1 \neq 0$ and Lemma 2.1 in [3]. If $p_1 = 0$, we can obtain the conclusion similarly by the second claim.
If $m = 1$, we can obtain the conclusion that $D \equiv 0$ similarly by the first claim.
If $m \geq 2$ and $i = 3$, we have the following lemma.

Lemma 2.10. *Suppose f as in case (ii) and*

$$p_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + p_1(z_2, z_3)\frac{\partial f}{\partial z_1} + c_1 z_3^k \frac{\partial f}{\partial z_2} = c_2 z_3^l \neq 0.$$

Then $p_1(z_2, z_3)$ cannot be divided by z_3 or any negative weight derivation D is zero.

Proof. If $p_1(z_2, z_3)$ can be divided by z_3, then $p_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0}$ can be divided by z_3 due to $k \geq 1$.

Case (1). $p_0(z_1, z_2, z_3)$ can be divided by z_3.
In this case $z_3^{l-1} \in J$. Let

$$D' = p_0(z_1, z_2, z_3)\frac{\partial}{\partial z_0} + p_1(z_2, z_3)\frac{\partial}{\partial z_1} + c_1 z_3^k \frac{\partial}{\partial z_2}.$$

Then the following relations hold.

$$D'(\frac{\partial f}{\partial z_0}) = -[\frac{\partial}{\partial z_0}, D']f + \frac{\partial}{\partial z_0}(D'f) = -[\frac{\partial}{\partial z_0}, D']f \in J;$$
$$D'(\frac{\partial f}{\partial z_1}) = -[\frac{\partial}{\partial z_1}, D']f + \frac{\partial}{\partial z_1}(D'f) = -[\frac{\partial}{\partial z_1}, D']f \in J;$$
$$D'(\frac{\partial f}{\partial z_2}) = -[\frac{\partial}{\partial z_2}, D']f + \frac{\partial}{\partial z_2}(D'f) = -[\frac{\partial}{\partial z_2}, D']f \in J;$$
$$D'(\frac{\partial f}{\partial z_3}) = -[\frac{\partial}{\partial z_3}, D']f + \frac{\partial}{\partial z_3}(D'f) = -[\frac{\partial}{\partial z_3}, D']f + lc_2 z_3^{l-1} \in J;$$

This means that D' is a derivation of moduli algebra $\mathbb{C}[z_0, \ldots, z_3]/(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_3})$. Furthermore the moduli algebra is a Gorenstein local algebra and has a unique socle $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 3}$ (cf. [17]). For any z_γ, $\gamma = 0, 1, 2, 3$, we have $z_\gamma \cdot \det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 3} \in J$, thus

$$D'(z_\gamma \cdot \det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 3}) \in J,$$

i.e.,

$$p_\gamma \cdot \det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 3} + z_\gamma \cdot D'(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 3}) \in J,$$

where $p_0 := p_0(z_1, z_2, z_3)$, $p_1 := p_1(z_2, z_3)$, $p_2 := cz_3^k$, $p_3 := 0$. Moreover $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 3}$ is socle of the moduli algebra and $p_i \in \mathfrak{m}$, then $p_\gamma \cdot \det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 3} \in J$. Thus $z_\gamma \cdot D(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 3}) \in J$. This means $\mathfrak{m} \cdot D'(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 3}) \in J$.
Furthermore we claim that $D'(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 3}) \in J$. If $D'(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 3}) \notin J$, for any z_γ $(0 \le i \le 3)$, we already obtain $z_\gamma \cdot D'(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 3}) \in J$. This means $D'(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 3})$ must be a constant multiple of the unique socle $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 3}$, which contradicts to the fact that D' is a negative weight derivation.
Thus

$$D'(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 3}) \in J.$$

This implies that D' is a derivation of A (where $n = 3$). Since $\frac{\partial(Df)}{\partial z_3} = D'f$, thus $\deg D' = \deg D - \alpha_3$. By the assumption of L_q, we know that $L_{q-\alpha_3} = 0$. Thus $D' = 0$ in $L(V)$. Then $D'(z_2) = 0$ in A (where $n = 3$), i.e., $c_1 z_3^k \in J$. On the other hand $\deg(z_3^k) = \deg D - \alpha_3 + \alpha_2$. Thus we have that

$$\deg D - \alpha_3 + \alpha_2 \ge min\ \deg\{\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \frac{\partial f}{\partial z_3}, \det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 3}\}.$$

Thus $\deg D - \alpha_3 + \alpha_2 \ge d - \alpha_0$, i.e., $\alpha_0 + \alpha_2 - \alpha_3 + \deg D = d = m\alpha_0 + \alpha_3$. This implies that $m = 1$, which is a contradiction with our assumption.

Case (2). $p_0(z_1, z_2, z_3)$ cannot be divided by z_3.
In this case $\frac{\partial f}{\partial z_0}$ can be divided by z_3, then $a_1(z_1, z_2, z_3)$ can be divided by z_3. On the other hand the coefficient of z_0^{m-1} in $p_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + p_1(z_2, z_3)\frac{\partial f}{\partial z_1} + c_1 z_3^k \frac{\partial f}{\partial z_2}$,

$$mp_0(z_1, z_2, z_3)z_3 + p_1(z_2, z_3)\frac{\partial a_1(z_1, z_2, z_2)}{\partial z_1} + c_1 z_3^k \frac{\partial a_1(z_1, z_2, z_3)}{\partial z_2}$$

has to be 0 since $p_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + p_1(z_2, z_3)\frac{\partial f}{\partial z_1} + c_1 z_3^k \frac{\partial f}{\partial z_2}$ does not depend on z_0 and $m > 1$. We have that

$$mp_0(z_1, z_2, z_3) = -p_1(z_2, z_3)\frac{\partial a_1'(z_1, z_2, z_2)}{\partial z_1} - c_1 z_3^k \frac{\partial a_1'(z_1, z_2, z_3)}{\partial z_2},$$

where $a_1(z_1, z_2, z_3) = z_3 a_1'(z_1, z_2, z_3)$. By a similar argument of the second claim we can do a coordinate transformation which preserves the type $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ and the equation becomes

$$p_1(z_2, z_3)\frac{\partial f}{\partial z_1} + c_1 z_3^k \frac{\partial f}{\partial z_2} = c_2 z_3^l.$$

By Lemma 2.1 in [3], we obtain the conclusion. □

Since

$$p_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + p_1(z_2, z_3)\frac{\partial f}{\partial z_1} + c_1 z_3^k \frac{\partial f}{\partial z_2} = c_2 z_3^l,$$

and

$$f = z_0^m z_3 + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3),$$

where $m \geq 2$, so the coefficient of z_0^{m-1} on the left hand side of this equation, which is

$$mp_0(z_1, z_2, z_3)z_3 + p_1(z_2, z_3)\frac{\partial a_1(z_1, z_2, z_3)}{\partial z_1} + c_1 z_3^k \frac{\partial a_1(z_1, z_2, z_3)}{\partial z_2},$$

has to be 0. By the lemma above, we know that $\frac{\partial a_1(z_1,z_2,z_3)}{\partial z_1}$ can be divided by z_3, we suppose that

$$a_1(z_1, z_2, z_3) = z_3 a_1'(z_1, z_2, z_3) + c_3 z_2^u,$$

where c_3 is a constant. Then

$$mp_0(z_1, z_2, z_3)z_3 + p_1(z_2, z_3)\frac{\partial a_1(z_1, z_2, z_3)}{\partial z_1} + c_1 z_3^k \frac{\partial a_1(z_1, z_2, z_3)}{\partial z_2} = 0,$$

which is equivalent to the following equation

$$mp_0(z_1,z_2,z_3)z_3 + p_1(z_2,z_3)z_3\frac{\partial a_1'(z_1,z_2,z_3)}{\partial z_1} + c_1z_3^{k+1}\frac{\partial a_1'(z_1,z_2,z_3)}{\partial z_2} + c_1c_3uz_3^kz_2^{u-1} = 0.$$

That is

$$p_0(z_1,z_2,z_3) = -\frac{1}{m}\left(p_1(z_2,z_3)\frac{\partial a_1'(z_1,z_2,z_3)}{\partial z_1} + c_1z_3^k\frac{\partial a_1'(z_1,z_2,z_3)}{\partial z_2} + c_1c_3uz_3^{k-1}z_2^{u-1}\right).$$

Take a new coordinate system (z_0', z_1', z_2', z_3')

$$\begin{aligned} z_0 &= z_0' - \frac{1}{m}a_1'(z_1,z_2,z_3);\\ z_1 &= z_1';\\ z_2 &= z_2';\\ z_3 &= z_3'. \end{aligned}$$

Then we have the following relations

$$\begin{aligned} \frac{\partial}{\partial z_0'} &= \frac{\partial}{\partial z_0};\\ \frac{\partial}{\partial z_1'} &= -\frac{1}{m}\frac{\partial a_1'(z_1',z_2',z_3')}{\partial z_1'}\frac{\partial}{\partial z_0} + \frac{\partial}{\partial z_1};\\ \frac{\partial}{\partial z_2'} &= -\frac{1}{m}\frac{\partial a_1'(z_1',z_2',z_3')}{\partial z_2'}\frac{\partial}{\partial z_0} + \frac{\partial}{\partial z_2};\\ \frac{\partial}{\partial z_3'} &= -\frac{1}{m}\frac{\partial a_1'(z_1',z_2',z_3')}{\partial z_3'}\frac{\partial}{\partial z_0} + \frac{\partial}{\partial z_3}. \end{aligned}$$

Therefore we have that

$$\begin{aligned} &p_0(z_1,z_2,z_3)\frac{\partial f}{\partial z_0} + p_1(z_2,z_3)\frac{\partial f}{\partial z_1} + c_1z_3^k\frac{\partial f}{\partial z_2} = -\frac{1}{m}c_1c_3uz_3^{k-1}z_2^{u-1}\frac{\partial f}{\partial z_0} +\\ &\quad p_1(z_2,z_3)\left(\frac{\partial f}{\partial z_1} - \frac{1}{m}\frac{\partial a_1'(z_1,z_2,z_3)}{\partial z_1}\frac{\partial f}{\partial z_0}\right) + c_1z_3^k\left(\frac{\partial f}{\partial z_2} - \frac{1}{m}\frac{\partial a_1'(z_1,z_2,z_3)}{\partial z_2}\frac{\partial f}{\partial z_0}\right) =\\ &-\frac{1}{m}c_1c_3u(z_3')^{k-1}(z_2')^{u-1}\frac{\partial f(z_0',z_1',z_2',z_3')}{\partial z_0'} + p_1(z_2',z_3')\frac{\partial f(z_0',z_1',z_2',z_3')}{\partial z_1'} +\\ &\quad c_1(z_3')^k\frac{\partial f(z_0',z_1',z_2',z_3')}{\partial z_2'}. \end{aligned}$$

Without loss of generality we can suppose that

$$p_0(z_1,z_2,z_3) = c_4z_3^{k-1}z_2^{u-1}.$$

The condition becomes

$$c_4 z_3^{k-1} z_2^{u-1} \frac{\partial f}{\partial z_0} + p_1(z_2, z_3) \frac{\partial f}{\partial z_1} + c_1 z_3^k \frac{\partial f}{\partial z_2} = c_2 z_3^l,$$

where c_i's are constants. What we want to prove is $c_2 = 0$.

Lemma 2.11. *If c_1 and c_4 are non-zero constants and*

$$c_4 z_3^{k-1} z_2^{u-1} \frac{\partial f}{\partial z_0} + p_1(z_2, z_3) \frac{\partial f}{\partial z_1} + c_1 z_3^k \frac{\partial f}{\partial z_2} = c_2 z_3^l, \tag{10}$$

then we have that the both sides of above equation are 0*.*

Proof. By Lemma 1.2 in [3], there are two cases.

Case (1). $f = z_1^m + b_1(z_0, z_2, z_3) z_1^{m-1} + \dots \ (n \geq 2)$.
It is clear that the coefficient of z_1^{m-1} on the left hand side of the above equation is

$$c_4 z_3^{k-1} z_2^{u-1} \frac{\partial b_1(z_0, z_2, z_3)}{\partial z_0} + m p_1(z_2, z_3) + c_1 z_3^k \frac{\partial b_1(z_0, z_2, z_3)}{\partial z_2} = 0.$$

Since deg $b_1 = \alpha_1 \leq \alpha_0$, if $\frac{\partial b_1(z_0,z_1,z_2,z_3)}{\partial z_0} \neq 0$ we have that $b_1 = c_5 z_0$ and $\alpha_0 = \alpha_1$ after a suitable coordinate change. Therefore $p_1(z_2, z_3) = c_6 z_3^{k-1} z_2^{u-1}$ where c_i's are constants. Then the left hand side of the above equation becomes

$$\begin{aligned} & c_4 z_3^{k-1} z_2^{u-1} \frac{\partial f}{\partial z_0} + c_6 z_3^{k-1} z_2^{u-1} \frac{\partial f}{\partial z_1} + c_1 z_3^k \frac{\partial f}{\partial z_2} \\ &= c_4 z_3^{k-1} z_2^{u-1} \left(\frac{\partial f}{\partial z_0} + \frac{c_6}{c_4} \frac{\partial f}{\partial z_1} \right) + c_1 z_3^k \frac{\partial f}{\partial z_2}. \end{aligned}$$

We can do a coordinate transformation which preserves the type $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$, such that the equation (10) becomes

$$c_4 z_3^{k-1} z_2^{u-1} \frac{\partial f}{\partial z_0} + c_1 z_3^k \frac{\partial f}{\partial z_2} = c_2 z_3^l.$$

By a similar argument of the second claim, we know that $c_2 = 0$ or the negative weight derivation D is 0. Therefore $\frac{\partial b_1(z_0,z_2,z_3)}{\partial z_0} = 0$ and $m p_1(z_2, z_3) + c_1 z_3^k \frac{\partial b_1(z_0,z_2,z_3)}{\partial z_2} = 0$ implies that $p_1(z_2, z_3)$ can be divided by z_3 due to $k \geq 1$, which is absurd.

Case (2). $f = z_1^n z_j + \sum b_t(z_0, z_2, z_3) z_1^t$.
If $n = 1$, then by a similar argument of the first claim, we know that $D \equiv 0$.
If $n \geq 2$, then there are two subcases.

Subcase (2.1). $k \geq 2$. In this case, $\frac{\partial f}{\partial z_1}$ can be divided by z_3, thus $j = 3$ in this subcase and

$$f = z_1^n z_3 + b_1(z_0, z_2, z_3) z_1^{n-1} + \dots .$$

The coefficient of z_1^{n-1} on the left hand side of equation (10),

$$c_4 z_3^{k-1} z_2^{u-1} \frac{\partial b_1(z_0, z_2, z_3)}{\partial z_0} + n p_1(z_2, z_3) z_3 + c_1 z_3^k \frac{\partial b_1(z_0, z_2, z_3)}{\partial z_2},$$

has to be 0 due to $n \geq 2$, i.e.,

$$c_4 z_3^{k-1} z_2^{u-1} \frac{\partial b_1(z_0, z_2, z_3)}{\partial z_0} + n p_1(z_2, z_3) z_3 + c_1 z_3^k \frac{\partial b_1(z_0, z_2, z_3)}{\partial z_2} = 0. \tag{11}$$

Then $k = 2$, otherwise $p_1(z_2, z_3)$ can be divided by z_3, which is absurd. In this subcase we have that $b_1(z_0, z_2, z_3)$ can be divided by z_3 by the fact $n \geq 2$ and that $\frac{\partial f}{\partial z_1}$ can be divided by z_3. Therefore the equation (11) becomes

$$c_4 z_2^{u-1} \frac{\partial b_1(z_0, z_2, z_3)}{\partial z_0} + n p_1(z_2, z_3) + c_1 z_3 \frac{\partial b_1(z_0, z_2, z_3)}{\partial z_2} = 0.$$

Hence $\frac{\partial b_1(z_0, z_2, z_3)}{\partial z_0}$ can be divided by z_3, $p_1(z_2, z_3)$ has to be divisible by z_3 which is absurd.

Subcase (2.2). $k = 1$.
In this subcase, the equation (10) becomes

$$z_2^{u-1} \frac{\partial f}{\partial z_0} + p_1(z_2, z_3) \frac{\partial f}{\partial z_1} + c_1 z_3 \frac{\partial f}{\partial z_2} = c_2 z_3^l,$$

after dividing by a non-zero constant on both sides. We can assume that

$$p_1(z_2, z_3) = z_3 p_1'(z_2, z_3) + c_3 z_2^v,$$

where c_i's are constants. We can do a coordinate transformation which preserves the type $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ as follows.

$$\begin{aligned} z_0 &= z_0'; \\ z_1 &= z_1' + \frac{1}{c_1} \int p_1'(z_2', z_3') dz_2'; \\ z_2 &= z_2'; \\ z_3 &= z_3', \end{aligned}$$

where $\int p_1'(z_2', z_3')dz_2'$ represents a weighted homogeneous polynomial in the variables z_2' and z_3' and satisfying

$$\frac{\partial(\int p_1'(z_2', z_3')dz_2')}{\partial z_2'} = p_1(z_2', z_3').$$

Its existence can be verified directly. Then the equation (10) becomes

$$z_2^{u-1}\frac{\partial f}{\partial z_0} + c_3 z_2^v \frac{\partial f}{\partial z_1} + c_1 z_3 \frac{\partial f}{\partial z_2} = c_2 z_3^l.$$

It is clear that $u-1 \geq v$ by a comparison of the weighted degrees.
If $u-1 = v$, then by a similar argument of the proof of Case (1). The equation becomes

$$z_2^{u-1}\frac{\partial f}{\partial z_0} + c_1 z_3 \frac{\partial f}{\partial z_2} = c_2 z_3^l.$$

By a similar argument of the proof of second claim, after a coordinate transformation, we obtain the conclusion.
If $u-1 > v$, then

$$f = b_m(z_0, z_1, z_3)z_2^m + \cdots + b_0(z_0, z_1, z_3).$$

The coefficient of z_2^v on the left hand side of equation (10) is 0, i.e.,

$$c_3\frac{\partial b_0(z_0, z_1, z_3)}{\partial z_1} + c_1(v+1)z_3 b_{v+1}(z_0, z_1, z_3) = 0.$$

The constant term (relative to powers of z_2) is $c_1 z_3 b_1(z_0, z_1, z_3)$, which is $c_2 z_3^l$ by the equation (10). $b_1(z_0, z_1, z_3)$ can be divided by z_3 due to $l > 1$. On the other hand $c_3\frac{\partial b_0(z_0,z_1,z_3)}{\partial z_1} = -c_1(v+1)z_3 b_{v+1}$ can be divided by z_3, which implies that

$$b_0(z_0, z_1, z_3) = z_3 b_0'(z_0, z_1, z_3) + c_4 z_0^t.$$

But in the case (ii) of the third claim there is no term of the form z_0^t in the expansion of f. Therefore $b_0(z_0, z_1, z_3)$ and $b_1(z_0, z_1, z_3)$ can be divided by z_3. We have that

$$f = z_2^2 f' + z_3 f'',$$

which is singular along a 1-dimensional locus $z_2 = z_3 = f''(z_0, z_1, z_2, z_3) = 0$. This contradicts to the condition of our assumption that f defines an isolated singularity at the origin. □

In the second part of this proof, we consider the case $D(\frac{\partial f}{\partial z_0})$ is a multiple of $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\leq i,j\leq 3}$. By Lemma 2.4, we only need to consider $m = 1$ and $m = 2$.

Case (1).

$$f = z_0^m + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_{m-1}(z_1, z_2, z_3)z_0 + a_m(z_1, z_2, z_3).$$

When $m = 1$, 0 is not an isolated singularity.
When $m = 2$, we have

$$f = z_0^2 + a_1(z_1, z_2, z_3)z_0 + a_2(z_1, z_2, z_3).$$

Thus we have

$$\begin{aligned} D(\frac{\partial f}{\partial z_0}) =& p_0(z_1, z_2, z_3)\frac{\partial^2 f}{\partial z_0^2} + p_1(z_2, z_3)\frac{\partial^2 f}{\partial z_1 \partial z_0} + cz_3^k \frac{\partial^2}{\partial z_0 \partial z_2} \\ =& 2p_0(z_1, z_2, z_3) + p_1(z_2, z_3)\frac{\partial a_1(z_1, z_2, z_3)}{\partial z_1} + cz_3^k \frac{\partial a_1(z_1, z_2, z_3)}{\partial z_2}. \end{aligned}$$

We know that $wt(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 3}) = 6\alpha_0 - 2(\alpha_1 + \alpha_2 + \alpha_3)$ and $wt(D(\frac{\partial f}{\partial z_0})) = k\alpha_3 + \alpha_0 - \alpha_2$. Thus $D(\frac{\partial f}{\partial z_0})$ is a multiple of $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0\le i,j\le 3}$ only if $k\alpha_3 + \alpha_0 - \alpha_3 \ge 6\alpha_0 - 2(\alpha_1 + \alpha_2 + \alpha_3)$. Moreover D is a negative weight derivation, thus $k\alpha_3 - \alpha_2 < 0$. Then we obtain

$$\alpha_0 > k\alpha_3 + \alpha_0 - \alpha_2 \ge 6\alpha_0 - 2(\alpha_1 + \alpha_2 + \alpha_3),$$

i.e.,

$$5\alpha_0 < 2(\alpha_1 + \alpha_2 + \alpha_3) \le 6\alpha_1.$$

Thus $\alpha_1 > \frac{5}{6}\alpha_0$. Because $wt(a_1(z_1, z_2, z_3)) = \alpha_0$, we have the following four subcases.
Case (1.1).

$$a_1(z_1, z_2, z_3) = d_1 z_1 + e_1(z_2, z_3)$$

with $d_1 \neq 0$. In this case, $\alpha_0 = \alpha_1$. Thus

$$3\alpha_0 < 2(\alpha_2 + \alpha_3) \le 4\alpha_2,$$

i.e.,

$$\alpha_2 > \frac{3}{4}\alpha_0, \ \alpha_2 + \alpha_3 > \frac{3}{2}\alpha_0.$$

Thus $e_1(z_2, z_3)$ can only be $e_{11}z_2$, $e_{12}z_2 + e'_{12}z_3$ or $e_{13}z_3$. Each case f is a homogeneous polynomial. By Proposition 2.1, this completes the proof for this case.

Case (1.2).

$$a_1(z_1, z_2, z_3) = d_2 z_1 z_2 + e_2(z_2, z_3)$$

with $d_2 \neq 0$. In this case, $\alpha_0 = \alpha_1 + \alpha_2$. Thus we have $3\alpha_0 < 2\alpha_3$, which is absurd.

Case (1.3).

$$a_1(z_1, z_2, z_3) = d_3 z_1 z_3 + e_3(z_2, z_3)$$

with $d_3 \neq 0$. In this case, $\alpha_0 = \alpha_1 + \alpha_3$. Thus we have $3\alpha_0 < 2\alpha_2$, which is absurd.

Case (1.4).

$$a_1(z_1, z_2, z_3) = e_4(z_2, z_3).$$

By Lemma 1.2 in [3], we know that $a_2(z_1, z_2, z_3)$ must contains one of z_1^2, $z_1^2 z_2$, $z_1^2 z_3$.
If $a_2(z_1, z_2, z_3)$ contains z_1^2, then $\alpha_0 = \alpha_1$. By a similar argument of Case (1.1) we get the conclusion.
If $a_2(z_1, z_2, z_3)$ contains $z_1^2 z_2$, then $2\alpha_0 = 2\alpha_1 + \alpha_2$. By $2\alpha_0 = 2\alpha_1 + \alpha_2$ and $5\alpha_0 < 2(\alpha_1 + \alpha_2 + \alpha_3)$, we obtain $3\alpha_0 < \alpha_2 + 2\alpha_3$ which is absurd.
If $a_2(z_1, z_2, z_3)$ contains $z_1^2 z_3$, then $2\alpha_0 = 2\alpha_1 + \alpha_3$. By $2\alpha_0 = 2\alpha_1 + \alpha_3$ and $5\alpha_0 < 2(\alpha_1 + \alpha_2 + \alpha_3)$, we obtain $3\alpha_0 < 2\alpha_2 + \alpha_3$ which is absurd.

Case (2).

$$f = z_0^m z_i + a_1(z_1, z_2, z_3) z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3).$$

We only need to consider $m = 1$ and $m = 2$. There are two subcases.

Case (2.1).

$$m = 1.$$

Case (2.1.1).
If $m = 1$ and $i = 1$, then we have $f = z_0 z_1 + a_1(z_1, z_2, z_3)$, $\alpha_0 + \alpha_1 = d$. Thus the assumption that $d \geq 2\alpha_0 \geq 2\alpha_1 \geq 2\alpha_2$ implies that $\alpha_0 = \alpha_1$.

$$D(\frac{\partial f}{\partial z_0}) = p_1(z_2, z_3).$$

In this case, we know that $wt(\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0 \leq i,j \leq 3}) = 4\alpha_1 - 2(\alpha_2 + \alpha_3)$ and $wt(D(\frac{\partial f}{\partial z_0})) = wt(p_1)$. Thus $D(\frac{\partial f}{\partial z_0})$ is a multiple of $\det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0 \leq i,j \leq 3}$ only if $wt(p_1) \geq 4\alpha_1 - 2(\alpha_2 + \alpha_3)$. Moreover D is a negative weight derivation, $wt(p_1) - \alpha_1 < 0$. Then we obtain

$$\alpha_1 > wt(p_1) \geq 4\alpha_1 - 2(\alpha_2 + \alpha_3),$$

i.e.,

$$3\alpha_1 < 2(\alpha_2 + \alpha_3) \leq 4\alpha_2.$$

Thus $\alpha_2 > \frac{3}{4}\alpha_1$. By Lemma 1.2 in [3] and $wt(a_1(z_1, z_2, z_3)) = 2\alpha_1$, we obtain that one of z_2^2, $z_2^2 z_1$, $z_2^2 z_3$ must be contained in $a_1(z_1, z_2, z_3)$.
If z_2^2 is contained in $a_1(z_1, z_2, z_3)$, then $\alpha_0 = \alpha_1 = \alpha_2$. By $3\alpha_1 < 2(\alpha_2 + \alpha_3)$, we obtain that $\alpha_1 < 2\alpha_3$.
Moreover, by Lemma 1.2 in [3], we obtain that one of $z_3 z_0$, $z_3 z_1$, $z_3 z_2$, z_3^s must be contained in $a_1(z_1, z_2, z_3)$. If one of $z_3 z_0$, $z_3 z_1$, $z_3 z_2$ is contained in $a_1(z_1, z_2, z_3)$, then f is a homogeneous polynomial. By Proposition 2.1, this completes the proof for this case. Now we just assume that only z_3^s is contained in $a_1(z_1, z_2, z_3)$. This implies that $2\alpha_2 = s\alpha_3$. Hence by the inequality $2\alpha_3 > \alpha_2 = \frac{s}{2}\alpha_3$, we obtain that $s = 3$. (If $s = 2$, then f is a homogeneous polynomial.) Hence we can assume that

$$f = z_0 z_1 + d_1 z_1^2 + d_2 z_1 z_2 + d_3 z_2^2 + d_4 z_3^3$$

where d_1, d_2, d_3, d_4 are constants with $d_3 \neq 0$ and $d_4 \neq 0$. So $det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0 \leq i,j \leq 3} = -12 d_3 d_4 z_3$. This is a trivial case.
If $z_2^2 z_1$ is contained in $a_1(z_1, z_2, z_3)$, then $\alpha_0 = \alpha_1 = \frac{1}{2}\alpha_2$, which contradicts to $3\alpha_1 < 2(\alpha_2 + \alpha_3)$.
If $z_2^2 z_3$ is contained in $a_1(z_1, z_2, z_3)$, then $2\alpha_2 + \alpha_3 = 2\alpha_1$, which contradicts to $\alpha_2 > \frac{3}{4}\alpha_1$.

Case (2.1.2).
If $m = 1$ and $i = 2$, then we have $f = z_0 z_2 + a_1(z_1, z_2, z_3)$, $\alpha_0 + \alpha_2 = d$. Thus the assumption that $d \geq 2\alpha_0 \geq 2\alpha_1 \geq 2\alpha_2$ implies $\alpha_0 = \alpha_1 = \alpha_2$. By the relation $wt(D(\frac{\partial f}{\partial z_0})) \geq wt(det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0 \leq i,j \leq 3})$, we obtain that $\alpha_2 > k\alpha_3 \geq 2(\alpha_2 - \alpha_3)$, i.e., $\alpha_2 < 2\alpha_3$.
Moreover, by Lemma 1.2 in [3], we obtain that one of $z_3 z_0$, $z_3 z_1$, $z_3 z_2$, z_3^s must be contained in $a_1(z_1, z_2, z_3)$. If one of $z_3 z_0$, $z_3 z_1$, $z_3 z_2$ is contained in $a_1(z_1, z_2, z_3)$, then f is a homogeneous polynomial. By Proposition 2.1, this completes the proof for this case. Now we just assume that only z_3^s is contained in $a_1(z_1, z_2, z_3)$. This implies that $2\alpha_2 = s\alpha_3$. Hence by the inequality $2\alpha_3 > \alpha_2 = \frac{s}{2}\alpha_3$, we obtain that $s = 3$. (If $s = 2$, then f is a homogeneous polynomial.) Hence we can assume that

$$f = z_0 z_2 + d_1 z_1^2 + d_2 z_1 z_2 + d_3 z_2^2 + d_4 z_3^3$$

where d_1, d_2, d_3, d_4 are constants with $d_1 \neq 0$ or $d_2 \neq 0$. Moreover we have $d_4 \neq 0$. So $det(\frac{\partial^2 f}{\partial z_i \partial z_j})_{0 \leq i,j \leq 3} = -12 d_1 d_4 z_3$. If $d_1 = 0$, then by the main result in [3], this completes the proof for this case. If $d_1 \neq 0$, then this is a trivial case. □

By Proposition 2.6, Proposition 2.7 and Proposition 2.8, we finish the proof of the Main Theorem. □

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