THE MULTIPLICITY OF ISOLATED TWO-DIMENSIONAL HYPERSURFACE SINGULARITIES: ZARISKI PROBLEM

By Stephen S.-T. Yau

1. Introduction. Let $(V, 0)$ and $(W, 0)$ be two isolated two dimensional hypersurface singularities in $\mathbf{C}^3$. We say that $(V, 0)$ and $(W, 0)$ have the same topological type if $(\mathbf{C}^3, V, 0)$ is homeomorphic equivalent to $(\mathbf{C}^3, W, 0)$. The famous Zariski question [24] asks whether multiplicity of $(V, 0)$ is the same as the multiplicity of $(W, 0)$ if they have the same topological type. In case $(V, 0)$ and $(W, 0)$ are quasi-homogeneous singularities and there is a Milnor number constant family connecting them, the Zariski question was answered affirmatively and independently by Greuel [7] and O'shea [14], (see also Laufer [10]). Recently we have announced in [22] that the topological type of two dimensional isolated quasi-homogeneous singularity determines and is determined by the fundamental group of the link and characteristic polynomial of the monodromy. As a consequence, the Zariski multiplicity question has been solved for quasi-homogeneous singularities. The full detail appeared in [19]. Recall that Wagreich [18] introduced an invariant of singularity called the arithmetic genus p_a which can be computed from a resolution graph. The following theorem answers the special case of the Zariski question affirmatively.

Theorem A. *Let $(V, 0)$ and $(W, 0)$ be two isolated two dimensional hypersurface singularities in $\mathbf{C}^3$ having the same topological type. If $p_a(V, 0) \leq 2$, then $(V, 0)$ and $(W, 0)$ have equal multiplicity.*

As observed by Laufer [8] (see also [19]), one cannot always hope to determine the multiplicity of isolated two dimensional hypersurface singularity $\mathbf{C}^3$ from its non-embedded topology alone, or even from the topology and the Milnor number. However, it is still very desirable to give a sharp upper bound of the multiplicity of $(V, 0)$ in terms of the resolution graph. We have the following result which is an improvement of the result obtained by Laufer [8].

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THEOREM B. *Let $(V, 0)$ be an isolated two dimensional hypersurface singularity with multiplicity ν. Let K be the canonical divisor on the minimal resolution of $(V, 0)$. Then*

$$-K \cdot K \geq 2 + \nu(\nu - 1)(\nu - 3).$$

One of the fundamental questions in the theory of normal two dimensional singularities is the following: What conditions are imposed on the abstract topology of $(V, 0)$ by the hypersurface hypothesis? Recall essentially [12, Theorem 2.10, p. 18], that any isolated singularity is topologically a cone over its link. Moreover, in dimension two, L is a compact real 3-manifold whose oriented homeomorphism type determines and is determined by the weighted dual graph Γ of a canonically determined resolution $\pi : (M, A) \to (V, 0)$ (cf. [13]). So, we may equivalently ask: What conditions does the existence of hypersurface representative $(V, 0)$ put on a weighted dual graph Γ? In other words, we would like to identify the image of the mapping

$$\{\text{isolated hypersurface singularities}\} \to \{\text{weighted dual graphs}\}$$

$$(V, 0) \to \Gamma$$

A hypersurface singularity $(V, 0)$ is Gorenstein [3], [6]. So there exists an integral cycle K on Γ which satisfies the adjunction formula [16].

THEOREM C. *Let Γ be the weighted dual graph of the minimal resolution of an isolated two dimensional hypersurface singularity. Then there is a cycle $K = \Sigma k_i A_i$ on Γ, with integer coefficients, which satisfies the adjunction formula. Let $Z = \Sigma z_i A_i$ be the fundamental cycle. Then*

$$-K \geq (-Z \cdot Z - 2)Z$$

i.e., for all i

$$-k_i \geq (-Z \cdot Z - 2)z_i.$$

Since the fundamental cycle Z can be computed from the graph Γ, the above inequality gives another necessary condition for a graph Γ to have hypersurface surface singularity structure. The weaker form of Theorem C was obtained by Laufer in [8].

Our approach is quite different from those of Laufer [8]. Indeed it is unlikely that his method can be sharpened to produce our results (Theorem B and Theorem C). Finally, we would like to mention that in Theorem 3.3 we prove that arithmetic genus p_a and geometric genus p_g of isolated hypersurface two-dimensional singularity are invariants of topological type. This is exactly the starting point of Theorem A.

2. Known Preliminaries. Consider a resolution $\pi : (M, A) \to (V, 0)$ of the normal two dimensional singularity $(V, 0)$. Throughout this paper $A = \cup A_i$, $1 \leq i \leq n$, will be the decomposition of A into irreducible components. Consider the canonical bundle K on M. The adjunction formula [16] gives, for all i

$$A_i \cdot K = -A_i \cdot A_i + 2g_i - 2. \tag{2.1}$$

Recall [5], that an A_j is an exceptional curve of the first kind, i.e., may be blown down without introducing a singularity, if and only if A_j is a nonsingular rational curve of self-intersection -1. Observe then from (2.1) that

$A_i \cdot K = -1$ if A_i is exceptional of the first kind

$A_i \cdot K \geq 0$ otherwise

π is the minimal resolution if and only if no A_i is exceptional of the first kind.

Since the intersection matrix $(A_i \cdot A_j)$ is nonsingular, there are unique numbers k_i, $1 \leq i \leq n$, such that the rational cycle

$$\tilde{K} = \Sigma\, k_i A_k \qquad 1 \leq i \leq n \tag{2.2}$$

satisfies $A_i \cdot K = A_i \cdot \tilde{K}$ for all i.

PROPOSITION 2.1. *With the above notation, suppose additionally that π is the minimal resolution. Then $K = 0$, i.e., is the trivial bundle, in case $(V, 0)$ is a rational double point. Otherwise, $k_i < 0$ for all i.*

In case $(V, 0)$ is Gorenstein, the $\tilde{K}$ of (2.2) has integral coefficients. Moreover, $\mathcal{O}(\tilde{K})$ is isomorphic to $\mathcal{O}(K)$. Thus in the Gorenstein case, we shall duplicate notation and use K also to denote the Cartier divisor $\tilde{K}$.

From now on in this paper, all "cycles" will be integral combinations of the A_i, i.e., Cartier divisors on M.

In [20], [21], we first introduced the concept of maximal ideal cycle. Let m denote the maximal ideal sheaf at 0 of $(V, 0)$. $\pi^*(m)$, the sheaf on M generated by the pull-back to M of generators of m, need not be locally principal. But there is a unique cycle $Y > 0$, such that $\pi^*(m)/\mathcal{O}(-Y)$ is supported at only a finite number of points, called *embedded* points (for $\pi^*(m)$), on A. Such a cycle Y is called the maximal ideal cycle. In fact

$$Y = \sum_{j=1}^{n} (\min_{f \in M} v_j(f)) A_j$$

where $v_j(f)$ is the vanishing order of $\pi^*(f)$ on A_j for $f \in m$.

3. Main results. We continue with the notation of section 2. Let us first recall the following definition.

Definition 3.1. Let $(V, 0)$ be a normal two dimensional singularity. Let $\pi : (M, A) \to (V, 0)$ be a resolution with exceptional set A. The geometric genus of a normal two dimensional singularity $(V, 0)$ is the integer

$$p_g(V, 0) = \dim_{\mathbf{C}}(R^1 \pi_* \mathcal{O}_M)_0.$$

The arithmetic genus of a normal two dimensional singularity $(V, 0)$ is the integer

$$p_a(V, 0) = \sup_D p_a(D), \qquad \text{where } D \text{ is a positive cycle.}$$

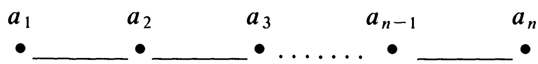
Here, the integer $p_a(D)$ is the virtual genus of the positive cycle on M.

Remark 3.2. Both p_g and p_a are invariants of the singularity, i.e., independent of the choice of the resolutions (see for example, [18]).

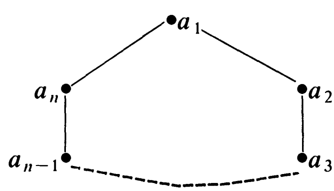
THEOREM 3.3. *Let $(V, 0)$ and $(W, 0)$ be two isolated hypersurface two dimensional singularities. Suppose that they have the same topological type. Then $p_a(V, 0) = p_a(W, 0)$ and $p_g(V, 0) = p_g(W, 0)$.*

Proof. Since $(V, 0)$ and $(W, 0)$ have the same topological type, the fundamental groups of the links of $(V, 0)$ and $(W, 0)$ are isomorphic (see for example [15]). Thus, by the result fo Neumann [13], the minimal resolution graph Γ_V of $(V, 0)$ is the same as the minimal resolution graph Γ_W of $(W, 0)$ except the following two cases:

Case 1. Both Γ_V and Γ_W are exactly those of the form below with all $a_i \leq -2$



Case 2. Both Γ_V and Γ_W are exactly those of the form below with $a_i \leq -2$ and one $a_i \leq -3$.



In Case 1, we have $p_g(V, 0) = p_a(V, 0) = 0 = p_a(W, 0) = p_g(W, 0)$, while in Case 2 we have $p_g(V, 0) = p_a(W, 0) = 1 = p_a(W, 0) = p_g(W, 0)$.

In order to finish the proof of the theorem, we may assume that Γ_V is the same as Γ_W. Since one can compute arithmetic genus from the resolution graph, we conclude that $p_a(V, 0) = p_a(W, 0)$. It is well known that Milnor number μ of an isolated hypersurface singularity is an invariant of topological type (see for example [17]). Therefore, $\mu(V, 0) = \mu(W, 0)$. On the other hand, Laufer's formula [9], says that

$$1 + \mu = K^2 + \chi_T(A) + 12p_g$$

where $\chi_T(A)$ is the topological Euler characteristic of A. Since K^2 and $\chi_T(A)$ can be computed from the resolution graph, it follows that $p_g(V, 0) = p_g(W, 0)$. Q.E.D.

THEOREM 3.4. *Let $(V, 0)$ and $(W, 0)$ be two isolated two dimensional hypersurface singularities in $\mathbf{C}^3$ having the same topological type. If $p_a(V, 0) \leq 2$, then $\nu(V, 0) = \nu(W, 0)$ where $\nu(V, 0)$ and $\nu(W, 0)$ are the multiplicities of $(V, 0)$ and $(W, 0)$ respectively.*

Proof. Let U be a Stein open neighborhood of 0 in $\mathbf{C}^3$. We shall assume without loss of generality that V is a closed subvariety in U with 0 as its only isolated singularity. By a theorem of Levi-Zariski, there exists a proper holomorphic map $\varphi : \tilde{U} \to U$, with $\tilde{U}$ smooth, inducing an isomorphism $\tilde{U} - \varphi^{-1}(0) \to U - \{0\}$, and such that M, the closure in $\tilde{U}$ of $\varphi^{-1}(V - \{0\})$, is smooth. (Whence the induced map $\pi : M \to V$ is a desingularization.) φ is a composition of a sequence of permissible transformations, a permissible transformation being one obtained by blowing up either a point or a smooth curve. Moreover, the centers of the permissible transformations can be chosen to be either a ν-fold point or ν-fold curve ($\nu \geq 2$) in the proper transform of V relative to the previous permissible transformations. (See [25], [26] or lecture 3 of [11a]).

$$\begin{array}{ccc} M & \overset{\iota}{\hookrightarrow} & \tilde{U} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \varphi} \\ V & \subseteq & U \end{array}$$

For any divisor D in $\tilde{U}$, we denote by [D] the associated complex line bundle over $\tilde{U}$. Let K_M and $K_{\tilde{U}}$ be the canonical bundles of M and $\tilde{U}$ respectively. Then

$$K_M = \iota^*(K_{\tilde{U}} \otimes [M]). \tag{3.1}$$

Let E_φ be the degenerate divisor defined by the vanishing of the Jacobian of φ. Then

$$K_{\tilde{U}} = \varphi^* K_U \otimes [E_\varphi]. \tag{3.2}$$

Let $B_1, \ldots, B_m$ be the irreducible components of the degenerate divisor E_φ. Let $A_1, \ldots, A_n$ be the irreducible curves of $\pi^{-1}(0)$, the exceptional set in M. Then

$$\iota^*[B_i] = \prod_{j=1}^{n} [A_j]^{\alpha_{ij}} \tag{3.3}$$

where α_{ij} are nonnegative integers; furthermore

$$[E_\varphi] = \prod_{i-1}^{n} [B_i]^{\rho_i} \tag{3.4}$$

and

$$\varphi^*[V] = [M] \otimes \prod_{j=1}^{m} [B_i]^{\gamma_i} \tag{3.5}$$

where ρ_i and γ_i are positive integers. Thus it follows from (3.1), (3.2), (3.4) and (3.5) that

$$\begin{aligned} K_M &= \iota^*(K_{\tilde{U}} \otimes [M]) \\ &= \iota^*(\varphi^* K_U \otimes [E_\varphi]) \otimes \iota^*[M] \\ &= \iota^*(\varphi^* K_U \otimes [E_\varphi]) \otimes \iota^*\varphi^*[V] \otimes \iota^* \prod_{i=1}^{m} [B_i]^{-\gamma_i} \\ &= \iota^*\varphi^*(K_U \otimes [V]) \otimes \prod_{i=1}^{m} [B_i]^{\rho_i - \gamma_i}. \end{aligned}$$

Because by restricting to a suitable Stein open neighborhood of 0 in $\mathbf{C}^3$, the bundle $K_U \otimes [V]$ is trivial, therefore

$$K_M = \prod_{i=1}^{m} [B_i]^{\rho_i - \gamma_i}. \tag{3.6}$$

We shall examine the behavior of ρ_i and γ_i in the σ-process. We consider a fixed B_i. There are two cases. Case 1: B_i is obtained by σ-process at a point q. Case 2: B_i is obtained by σ-process along a curve C. Let $B_{i_1}, \ldots, B_{i_k}$ be those inserted divisors in the preceding σ-process, which contain q (respectively C). For the proper transform of V relative to the previous σ-process, let q be a point with multiplicity $\nu_i \geq 2$ (respectively C a ν_i-fold curve, $\nu_i \geq 2$). Then

$$\gamma_i = \gamma_{i_1} + \cdots + \gamma_{i_k} + \nu_i. \tag{3.7}$$

Because the σ-process at a point (along a curve respectively) relative to a suitable coordinate is described by the map $(z_1, z_2, z_3) \to (z_1, z_1z_2, z_1z_3)$ ($(z_1, z_2, z_3) \to (z_1, z_1z_2, z_3)$ respectively), and the Jacobian of this map is z_1^2 (z_1 respectively), one obtains

(3.8) $$\rho_i = \rho_{i_1} + \cdots + \rho_{i_k} + 2 \qquad \text{in case 1}$$

(3.9) $$\rho_i = \rho_{i_1} + \cdots + \rho_{i_k} + 1 \qquad \text{in case 2}$$

Let f be a generic holomorphic function on $\mathbf{C}^3$ which vanishes at the origin. Let μ_i be the order to which f vanishes along B_i so that

$$[(f)] = \prod_{i=1}^{m} [B_i]^{\mu_i} \otimes [D]$$

where D is the proper transform of the divisor of f in $\mathbf{C}^3$. By the definition of maximal ideal cycle, we have

(3.10) $$[Y] = \iota^* \prod_{i=1}^{m} [B_i]^{\mu_i}$$

i.e.,

$$Y = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} \mu_i \alpha_{ij} \right) A_j .$$

It follows from (3.7), (3.8) and (3.9) that

(3.11) $$\rho_i - \gamma_i = (\pi_{i_1} - \gamma_{i_1}) + \cdots + (\rho_{i_k} - \gamma_{i_k}) + (2 - \nu_i) \quad \text{in case 1}$$

(3.12) $$\rho_i - \gamma_i = (\rho_{i_1} - \gamma_{i_1}) + \cdots + (\rho_{i_k} - \gamma_{i_k}) + (1 - \nu_i) \quad \text{in case 2}$$

Notice that f is generic. So we can choose f general enough so that the proper transforms of f do not contain the centers of the blow-ups. Hence we have

(3.13) $$\mu_i = \mu_{i_1} + \cdots + \mu_{i_k} .$$

We shall prove by induction that

(3.14) $$\rho_i - \gamma_i \leq (2 - \nu)\mu_i$$

where $\nu = \nu(V, 0)$ is the multiplicity of V at 0. Notice that $\mu_{i_1} = 1$ and

$\nu_{i_1} = \nu$. It follows from (3.11) that $\rho_{i_1} - \gamma_{i_1} = 2 - \nu_{i_1} = (2 - \nu)\mu_{i_1}$. By induction hypothesis and (3.13), we have

$$\sum_{\lambda=1}^{k} (\rho_{i_\lambda} - \gamma_{i_\lambda}) \leq (2 - \nu) \sum_{\lambda=1}^{k} \mu_{i_\lambda} = (2 - \nu)\mu_i.$$

Clearly from (3.11) and (3.12), we have

$$\begin{aligned} \rho_i - \gamma_i &\leq (\rho_{i_1} - \gamma_{i_1}) + \cdots + (\rho_{i_k} - \gamma_{i_k}) + (2 - \nu_i) \\ &\leq (2 - \nu)\mu_i + (2 - \nu_i) \\ &\leq (2 - \nu)\mu_i. \end{aligned} \tag{3.15}$$

(3.6), (3.10) and (3.15) imply that

$$K_M \leq (2 - \nu)Y$$

i.e., there exists a positive cycle D such that

$$-K_M = (\nu - 2)Y + D. \tag{3.16}$$

By the definition of arithmetic genus, we have for any positive integer s

$$\begin{aligned} p_a &\leq p_a(sY) \\ &= \frac{1}{2} sY \cdot (sY + K_M) + 1 \\ &= \frac{1}{2} sY \cdot [sY + (2 - \nu)Y - D] + 1 \quad \text{by} \quad (3.16) \\ &= \frac{1}{2} s(s + 2 - \nu)Y^2 - \frac{1}{2} sY \cdot D + 1 \end{aligned}$$

Since the resolution $\pi : M \to V$ factors through the blowing up of the maximal ideal sheaf m of $(V, 0)$ by construction, we see that $\mathcal{O}_M$ is locally principal and $\mathcal{O}_M = \mathcal{O}_M(-Y)$. By the results of [18] and [21], we have $\nu = -Y^2$ and $Y \cdot D \leq 0$. Therefore we have

$$p_a \geq -\frac{1}{2}s(s+2-\nu)\nu + 1 \tag{3.17}$$

$$= \frac{1}{2}s\nu(\nu - s - 2) + 1$$

There are two cases:

Case 1. ν is odd and $\nu \geq 3$: Take $s = (\nu - 1)/2$ in (3.17). We get

$$p_a \geq \frac{\nu(\nu-1)(\nu-3)}{8} + 1 \tag{3.18}$$

Case 2. ν is even and $\nu \geq 3$: Take $s = (\nu - 2)/2$, we get

$$p_a \geq \frac{\nu(\nu-2)^2}{8} + 1. \tag{3.19}$$

Since $(V, 0)$ and $(W, 0)$ have the same topological type, by Theorem 3.3, we have $p_a(W, 0) = p_a(V, 0) \leq 2$. In view of (3.18) and (3.19), we see that $\nu(V, 0) \leq 3$ and $\nu(W, 0) \leq 3$. Using the deep work of A'Campo [1], Lê and Teisser [11] observed that a surface in $\mathbf{C}^3$ having at 0 a singularity of multiplicity 2 cannot have the same topological type at 0 as another surface of multiplicity different from 2. It follows immediately that $\nu(V, 0) = \nu(W, 0)$. Q.E.D.

Remark 3.5. A similar argument in the proof of Theorem 3.4 was presented by us in an Algebraic Geometry Seminar in March, 1982 at U.I.C. to give a complete topological classification of weakly elliptic two dimensional hypersurface singularities (cf. [23]). However the above results was inspired by the introduction of Laufer's paper [8].

THEOREM 3.6. *Let* $(V, 0)$ *be an isolated two dimensional hypersurface singularity with multiplicity* ν. *Let* K *be the canonical divisor on the minimal resolution of* $(V, 0)$. *Then*

$$-K \cdot K \geq 2 + \nu(\nu - 1)(\nu - 3).$$

Proof. Let m denote the maximal ideal sheaf at 0 of $(V, 0)$. There is a minimal resolution $\pi' : (M', A') \to (V, 0)$ such that $\pi'^*(m)$ has no embedded points. π' is obtained by starting with the minimal resolution π and successively blowing up at all embedded points. Then [18], $\nu = -Y' \cdot Y'$. Consider any two resolutions π_1 and π_2 of $(V, 0)$ with corresponding maximal ideal cycles Y_1 and Y_2, canonical cycles K_1 and K_2 respectively. If π_2 is obtained from π_1 by blowing up at an embedded point q, then

$$Y_2 = \lambda^{-1}(Y_1) + cA_l \qquad c \geq 1 \tag{3.20}$$

and

$$K_2 = \lambda^{-1}(K_1) + J \tag{3.21}$$

where $\pi_2 = \pi_1 \circ \lambda$, $\lambda^{-1}(q) = A_l$, $\lambda^{-1}(Y_1)$ and $\lambda^{-1}(K_1)$ are total transforms of Y_1 and K_1 respectively and J is the divisor of the Jacobian of λ. It follows that

$$-Y_2 \cdot Y_2 \geq 1 - Y_1 \cdot Y_1 \tag{3.22}$$

$$-K_2 \cdot K_2 = K_1 \cdot K_1 + 1. \tag{3.23}$$

Hence at most $\nu - 1$ blowups are required to reach π' from π. Then

$$-K' \cdot K' \leq -K \cdot K + (\nu - 1). \tag{3.24}$$

Let $\pi''(M'', A'') \to (V, 0)$ be the Levi-Zariski resolution. Since the Levi-Zariski resolution is obtained by first flowing up the maximal ideal sheaf of 0 at $(V, 0)$, $\pi''^*(m)$ is locally principal. Then $\pi'' = \pi' \circ \tau$ for a suitable composition of quadratic transformations τ. By (3.16), we have

$$-K'' = (\nu - 2)Y'' + D'' \tag{3.25}$$

where K'' is the canonical cycle, Y'' is the maximal ideal cycle and D'' is a positive cycle on M''. Let $Y' = \Sigma\, y_i' A_i'$ and $K' = \Sigma\, k_i' A_i'$ be the maximal cycle and canonical cycle on M' respectively. From (3.20) and (3.21), we see that for each A_i', we may associate a k_i' and y_i' which remain unchanged under taking proper transforms. In view of (3.25), we have

$$-K' = (\nu - 2)Y' + D' \tag{3.26}$$

where D' is a positive cycle on M'. Since $A_i' \cdot Y' \leq 0$ for all i,

$$K' \cdot K' \leq (\nu - 2)^2 Y' \cdot Y' = -\nu(\nu - 2)^2. \tag{3.27}$$

Notice that the equality in (3.27) holds if and only if $D' = 0$ i.e., $-K' = (\nu - 2)Y'$. (3.24) and (3.27) yield

$$-K \cdot K \geq \nu(\nu - 1)(\nu - 3) + 1. \tag{3.28}$$

Notice that if the equality in (3.28) holds, then exactly $\nu - 1$ blow ups are required to reach π' from π. Since $-Y' \cdot Y' \geq \nu - 1 - Y \cdot Y$ and $-Y' \cdot Y' = \nu$, we have $Y \cdot Y = -1$. For any $f \in m$, the divisor $(f)_M$ on M is of the form

$$(f)_M = \sum_{j=1}^{n} (v_j(f))A_j + D_f$$

where D_f is the proper transform of the divisor $(f)_V$ on V. Then $Y = \Sigma_{j=1}^{n} (\min_{f \in m} v_j(f))A_j$. Let q be an embedded point in $A_1 \cap \cdots \cap A_r$ where $r \geq 1$. There is a $g \in m$ such that $v_1(g) = y_1$, $v_2(g) = Y_2, \ldots,$ $v_r(g) = Y_r$. If D_g does not pass through q, then m is locally principal around q. Thus we conclude that $A_i \cdot \Sigma_{j=1}^{n} (v_j(g))A_j < 0$, for $1 \leq i \leq r$. Now

$$\begin{aligned} Y \cdot A_i &= y_i A_i^2 + \sum_{j \neq i} y_j (A_i \cdot A_j) \\ &\leq y_i A_i^2 + \sum_{j \neq i} v_j(g)(A_i \cdot A_j) \\ &= A_i \cdot \sum_{j=1}^{n} (v_j(g))A_j < 0 \end{aligned}$$

for all $1 \leq i \leq r$. Since $Y \cdot Y = -1$, there is a unique A_1 such that $A_1 \cdot Y = -1$ and the coefficient y_1 of Y in A_1 is one. Therefore there is only one embedded point in the smooth part of $A_1 - \cup_{j \neq 1} A_j$. The equality of (3.28) also implies $-K' = (\nu - 2)Y'$. Let $\pi_1 : M_1 \to M$ be the blowing up of M at q. Then we also have $-K_1 = (\nu - 2)Y_1$ and $-K = (\nu - 2)Y$ since M' dominates both M_1 and M. Denote A_1^1 be the proper transform of A_1 and A_{n+1}^1 be $\pi_1^{-1}(q)$. Let k_{n+1}' and y_{n+1}' be the coefficients of A_{n+1}^1 in K_1

and Y_1 respectively. Observe that $k_1 = -(\nu - 2)$ and $y_1 = 1$. It follows from (3.20) and (3.21) that $k_{n+1}^1 = -(\nu - 2) + 1$ and $y_{n+1}^1 = 2$. Since $-k_{n+1}^1 = (\nu - 2)y_{n+1}^1$, we have $\nu = 1$. This contradicts to the fact that V is singular at 0. Hence our theorem follows. Q.E.D.

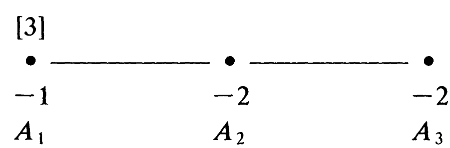
We have the following necessary conditions on a weighted dual graph Γ to come from a hypersurface singularity.

THEOREM 3.7. *Let Γ be the weighted dual graph of the minimal resolution of an isolated two dimensional hypersurface singularity. Then there is a cycle $K = \Sigma k_i A_i$ on Γ, with integer coefficient, which satisfies the adjunction formula. Let $Z = \Sigma z_i A_i$ be the fundamental cycle. Then $-K \geq (-Z \cdot Z - 2)Z$ i.e., for all i, $-k_i \geq (-Z \cdot Z - 2)z_i$.*

Proof. Let Y be the maximal ideal cycle on A. Then $Y \geq Z$. Moreover, the multiplicity ν satisfies $\nu \geq -Z \cdot Z$. Then Theorem 3.7 follows directly from (3.25), (3.20) and (3.21). Q.E.D.

4. Example. In this section we shall demonstrate by an example that our results are sharper than those obtained by Laufer [8]. Consider weighted dual graph Γ as shown below

(4.1)



where genera equal to 0 are omitted from the labeling. Laufer [8] has given some weighted-homogeneous representatives for Γ with their multiplicities and Milnor numbers

(4.2)

Equation	multiplicity	Milnor number
$z^2 + x^7 + y^{42} = 0$	$\nu = 2$	$\mu = 246$
$z^2 + y(x^{12} + y^{18}) = 0$	$\nu = 2$	$\mu = 210$
$z^3 + x^4 + y^{36} = 0$	$\nu = 3$	$\mu = 210$

Laufer knows of no isolated hypersurface singularity $(V, 0)$ having (4.1) as the weighted dual graph of its minimal resolution which does not lie in a μ-constant family which contains one of the three singularities in (4.2). He has shown that the multiplicity ν of isolated hypersurface singularity $(V, 0)$ having (4.1) as the weighted dual graph of its minimal resolution is at most 6. Since $K = -15A_1 - 10A_2 - 5A_3$, we see that $-K^2 = 75$. It follows from Theorem 3.6 that ν is actually at most 5.

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REFERENCES

[1] N. A'Campo, La fonction zêta d'une monodromie, *Comm. Math. H.*, **50** (1975), 233-248.

→ M. Artin, On isolated rational singularities of surfaces, *Am. J. Math.*, **88** (1966), 129-136.

[3] H. Bass, On the ubiquity of Gorenstein surfaces, *Math. Z.*, **82** (1963), 129-136.

[4] E. Brieskorn, Über die Auflösung gewisser Singularitäten von holomorphen Abbildung, *Math. Annal*, **166** (1966), 76-102.

[5] H. Grauert, Über Modifikationen und exzeptionelle analytische Mengen, *Math. Annal*, **146** (1962), 331-368.

[6] ______ and O. Riemenschneider, Verschwindungssatz fur analytische Kohomologie gruppen auf Komplexen Raaumen, *Invent. Math.*, **11** (1970), 263-292.

[7] G.-M. Greuel, Constant Milnor number implies constant multiplicity for quasihomogeneous singularities, *Manusc. Math.*, **56** (1986), 159-166.

→ H. Laufer, The multiplicity of isolated two-dimensional hypersurface singularities, *Trans. A.M.S.*, **302** (1987), 489-496.

[9] ______, On μ for surface singularities, *Proc. Symp. Pure Math.*, **30**, Amer. Math. Soc., Providence, R.I., (1977), 45-49.

[10] ______, Tangent cones for deformations of two-dimensional quasi-homogeneous singularities. (Preprint).

[11] D. T. Lê and B. Tessier, Report on the problem session, *Proc. Symp. Pure Math.*, **40**(2) (1983), 105-116.

[11a] J. Lipman, Introduction to resolution of singularities, *Proc. Symp. Pure Math.*, **29** (1975), 187-230.

[12] J. Milnor, Singular Points of Complex Hypersurfaces, *Ann. Math. Studies*, No. 61, Princeton University Press, Princeton, N.J., (1968).

[→ W. Neumann, A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves, *Trans. A.M.S.*, **268** (1981), 299-344.

[D. O'Shea, Topological trivial deformations of isolated quasihomogeneous hypersurface singularities are equimultiple. *Proc. A.M.S.*, **101** (1987), 260-262.

[15] B. Perron, Conjugaison topologique des germs de fonctions holomorphe a singularite isolee eu dimension trois, *Invent. Math.*, **82** (1985), 27-35.

[16] J. P. Serre, Groupes Algebriques et Corps de Classes. *Actualities Scientifiques.*

[17] B. Teissier, Déformations à type toplogique constant in: Séminaire Douady-Verdier 1971-72, *Astérisque* (*Société Mathématique de France*), No. 16, (1974), 215-249.

[P. Wagreich, Elliptic Singularities of Surfaces, *Am. J. Math.*, **92** (1970), 419-454.

[19] X.-J. Xu and S. S.-T. Yau, Classification of topological types of isolated quasi-homogeneous two dimensional hypersurface singularities, (to appear) Manuscripta Math.

[20] S. S.-T. Yau, Normal singularities of surfaces, *Proc. Symp. Pure Math.*, **32** (1978), 195-198.

[21] ______, On maximally elliptic singularities, *Trans. A.M.S.*, **257** (1980), 269-329.

[22] ______, Topological types and multiplicities of isolated quasi-homogeneous surface singularities. *Bull. A.M.S.*, **9** (1988), 447-454.

[23] ______, Lecture Notes, 1982.

[24] O. Zariski, Some open questions in the theory of singularities, *Bull. A.M.S.*, **77** (1971), 481-491.

[25] ______, Reduction of the singularities of algebraic three dimensional varieties, *Ann. Math.*, **45** (1944), 472-542.

[26] ______, Exceptional singularities of an algebroid surface and their reduction, Accad. Naz. Lincei Rend. Cl. *Sci. Fis. Math. Natur.*, serie III, **43** (1967), 135-146.

[27] M. Tomari, A p_g = formula and elliptic singularities, Publ. RIMS, Kyoto University, **21** (1985), 297-354.