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kTH YAU NUMBER OF ISOLATED HYPERSURFACE SINGULARITIES AND AN INEQUALITY CONJECTURE

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Abstract

Let V be a hypersurface with an isolated singularity at the origin defined by the holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. The Yau algebra $L(V)$ is defined to be the Lie algebra of derivations of the moduli algebra $A(V) := \mathcal{O}_n/(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n)$, that is, $L(V) = \mathrm{Der}(A(V), A(V))$. It is known that $L(V)$ is finite dimensional and its dimension $\lambda(V)$ is called the Yau number. We introduce a new series of Lie algebras, that is, kth Yau algebras $L^k(V)$, $k \geq 0$, which are a generalization of the Yau algebra. The algebra $L^k(V)$ is defined to be the Lie algebra of derivations of the kth moduli algebra $A^k(V) := \mathcal{O}_n/(f, m^k J(f)), k \geq 0$, that is, $L^k(V) = \mathrm{Der}(A^k(V), A^k(V))$, where m is the maximal ideal of $\mathcal{O}_n$. The kth Yau number is the dimension of $L^k(V)$, which we denote by $\lambda^k(V)$. In particular, $L^0(V)$ is exactly the Yau algebra, that is, $L^0(V) = L(V), \lambda^0(V) = \lambda(V)$. These numbers $\lambda^k(V)$ are new numerical analytic invariants of singularities. In this paper we formulate a conjecture that $\lambda^{(k+1)}(V) > \lambda^k(V), k \geq 0$. We prove this conjecture for a large class of singularities.

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1. Introduction

For any isolated hypersurface singularity $(V, 0) \subset (\mathbb{C}^n, 0)$, where $V = V(f) = \{f = 0\}$, one has that the factor algebra $A(V) = \mathcal{O}_n/(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ is finite dimensional. This factor algebra is called the moduli algebra of V and its dimension $\tau(V)$ is called the Tyurina number. The order of the lowest nonvanishing term in the power series expansion of f at 0 is called the multiplicity (denoted by $\mathrm{mult}(f)$) of the singularity $(V, 0)$. It is well known that a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is said to be weighted homogeneous if there exist positive rational numbers $w_1, \ldots, w_n$ (weights of $x_1, \ldots, x_n$) and d such that $\sum a_i w_i = d$ for each monomial $\prod x_i^{a_i}$ appearing in f with nonzero coefficient. The number d is called the weighted homogeneous

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degree (w-degree) of f with respect to the weights w_j. The weight type of f is denoted by $(w_1, \ldots, w_n; d)$. Without loss of generality, we can assume that w-$\deg f = 1$. The Milnor number of the isolated hypersurface singularity is defined by $\mu = \dim \mathbb{C}[x_1, \ldots, x_n]/(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$. In [15], it was shown that the Milnor number of a weighted homogeneous hypersurface singularity of weight type $(w_1, \ldots, w_n; 1)$ is calculated by $\mu = (1/w_1 - 1)(1/w_2 - 1) \cdots (1/w_n - 1)$. According to a beautiful theorem of Saito [16], f is a weighted homogeneous polynomial after a biholomorphic change of coordinates $\Longleftrightarrow \mu = \tau$.

The well-known Mather–Yau theorem [14] states that: let V_1 and V_2 be two isolated hypersurface singularities and $A(V_1)$ and $A(V_2)$ be the moduli algebras; then $(V_1, 0) \cong (V_2, 0) \Longleftrightarrow A(V_1) \cong A(V_2)$. Motivated from the Mather–Yau theorem, Yau considered the Lie algebra of derivations of the moduli algebra $A(V) := O_n/(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n)$, that is, $L(V) = \mathrm{Der}(A(V), A(V))$. The finite-dimensional Lie algebra $L(V)$ is called the Yau algebra and its dimension $\lambda(V)$ is called the Yau number. The Yau algebra plays an important role in singularity theory [17]. Yau and his collaborators have been systematically studying the Yau algebras of isolated hypersurface singularities beginning in the 1980s [2, 3, 5–8, 11–13, 17–26]. In our previous work [13], we introduced a series of new kth Yau algebras that arises from isolated hypersurface singularities. We defined this new kth Yau algebra in the following way.

Recall that we have the following theorem.

Theorem 1.1 [10, Theorem 2.26]. *Let $f, g \in m \subset O_n$. The following are equivalent:*

(1) $(V(f), 0) \cong (V(g), 0)$;
(2) *for all* $k \geq 0$, $O_n/(f, m^k J(f)) \cong O_n/(g, m^k J(g))$ *as* $\mathbb{C}$*-algebra;*
(3) *there is some* $k \geq 0$ *such that* $O_n/(f, m^k J(f)) \cong O_n/(g, m^k J(g))$ *as* $\mathbb{C}$*-algebra,*

where $J(f) = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$.

In particular, if $k = 0$ and $k = 1$ above, then the claim of the equivalence of (1) and (3) is exactly the same as the Mather–Yau theorem [14].

Based on Theorem 1.1, it is natural for us to introduce the new series of kth Yau algebras $L^k(V)$, which are defined to be the Lie algebras of derivations of the kth moduli algebra $A^k(V) = O_n/(f, m^k J(f)), k \geq 0$, that is, $L^k(V) = \mathrm{Der}(A^k(V), A^k(V))$. Its dimension is denoted by $\lambda^k(V)$. This number $\lambda^k(V)$ is a new numerical analytic invariant of a singularity. We call it the kth Yau number. In particular, $L^0(V)$ is exactly the Yau algebra; thus, $L^0(V) = L(V), \lambda^0(V) = \lambda(V)$. We believe that these new Lie algebras $L^k(V)$ and numerical invariants $\lambda^k(V)$ will also play an important role in the study of singularities. A natural question arises: are there any numerical relations between the new analytic invariants $\lambda^k(V), k \geq 0$? We propose the following conjecture.

Conjecture 1.2. With the above notation, let $(V, 0)$ be an isolated hypersurface singularity defined by $f \in O_n$, $n \geq 2$, and $\mathrm{mult}(f) \geq 3$. Then

$$\lambda^{(k+1)}(V) > \lambda^k(V), \quad k \geq 0.$$

In this paper, we shall prove the following main results.

THEOREM A. *Let* $(V, 0)$ *be a binomial singularity defined by the weighted homogeneous polynomial* $f(x_1, x_2)$ *(see Corollary 2.8) with weight type* $(w_1, w_2; 1)$. *Then*

$$\lambda^{(k+1)}(V) > \lambda^k(V), \quad k = 0, 1.$$

THEOREM B. *Let* $(V, 0)$ *be a trinomial singularity defined by the weighted homogeneous polynomial* $f(x_1, x_2, x_3)$ *(see Proposition 2.9) with weight type* $(w_1, w_2, w_3; 1)$. *Then*

$$\lambda^{(k+1)}(V) > \lambda^k(V), \quad k = 0.$$

2. Basic results

The following basic concepts and results will be used to compute the derivation Lie algebras of isolated hypersurface singularities.

Let A, B be associative algebras over $\mathbb{C}$. The subalgebra of endomorphisms of A generated by the identity element and left and right multiplications by elements of A is called the multiplication algebra $M(A)$ of A. The centroid $C(A)$ is defined as the set of endomorphisms of A which commute with all elements of $M(A)$. Obviously, $C(A)$ is a unital subalgebra of $\mathrm{End}(A)$. The following statement is a particular case of a general result from [4, Proposition 1.2]. Let $S = A \otimes B$ be a tensor product of finite-dimensional associative algebras with units. Then

$$\mathrm{Der}\, S \cong (\mathrm{Der}\, A) \otimes C(B) + C(A) \otimes (\mathrm{Der}\, B).$$

We will only use this result for commutative associative algebras with units, in which case the centroid coincides with the algebra itself and one has the following result for commutative associative algebras A, B.

THEOREM 2.1 [4]. *For commutative associative algebras* A, B,

$$\mathrm{Der}\, S \cong (\mathrm{Der}\, A) \otimes B + A \otimes (\mathrm{Der}\, B).$$

We shall use this formula in the following.

DEFINITION 2.2. Let J be an ideal in an analytic algebra S. Then $\mathrm{Der}_J S \subseteq \mathrm{Der}_{\mathbb{C}} S$ is a Lie subalgebra of all $\sigma \in \mathrm{Der}_{\mathbb{C}} S$ for which $\sigma(J) \subset J$.

We shall use the following well-known result to compute the derivations.

THEOREM 2.3 [25]. *Let* J *be an ideal in* $R = \mathbb{C}\{x_1, \ldots, x_n\}$. *Then there is a natural isomorphism of Lie algebras*

$$(\mathrm{Der}_J R)/(J \cdot \mathrm{Der}_{\mathbb{C}} R) \cong \mathrm{Der}_{\mathbb{C}}(R/J).$$

Recall that a derivation of a commutative associative algebra A is defined as a linear endomorphism D of A satisfying the Leibniz rule: $D(ab) = D(a)b + aD(b)$. Thus, for such an algebra A one can consider the Lie algebra of its derivations $\mathrm{Der}(A, A)$ with the bracket defined by the commutator of linear endomorphisms.

Definition 2.4. Let $f(x_1, \ldots, x_n)$ be a complex polynomial and $V = \{f = 0\}$ be a germ of an isolated hypersurface singularity at the origin in $\mathbb{C}^n$. Let $A^k(V) = O_n/(f, m^k J(f))$, $k \geq 0$, be a moduli algebra. Then $\mathrm{Der}(A^k(V), A^k(V))$ defines the derivation Lie algebras $L^k(V)$ and $\lambda^k(V)$ is the dimension of the derivation Lie algebra $L^k(V)$.

It is noted that when $k = 0$, the derivation Lie algebra is called the Yau algebra.

Definition 2.5. A polynomial $f \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ is called quasi-homogeneous (or weighted homogeneous) if there exist positive rational numbers $w_1, \ldots, w_n$ (called weights of indeterminates x_j) and d such that, for each monomial $\prod x_j^{k_j}$ appearing in f with nonzero coefficient, one has $\sum w_j k_j = d$. The number d is called the quasi-homogeneous degree (w-degree) of f with respect to the weights w_j and is denoted by $\deg f$. The collection $(w; d) = (w_1, \ldots, w_n; d)$ is called the quasi-homogeneity type (qh-type) of f.

Definition 2.6. An isolated hypersurface singularity in $\mathbb{C}^n$ is fewnomial if it can be defined by an n-nomial in n variables and it is a weighted homogeneous fewnomial isolated singularity if it can be defined by a weighted homogeneous fewnomial. A 3-nomial isolated hypersurface singularity is also called a trinomial singularity.

Proposition 2.7. *Let f be a weighted homogeneous fewnomial isolated singularity with* $\mathrm{mult}(f) \geq 3$. *Then f is analytically equivalent to a linear combination of the following three series:*

Type A. $x_1^{a_1} + x_2^{a_2} + \cdots + x_{n-1}^{a_{n-1}} + x_n^{a_n}, \quad n \geq 1$;
Type B. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n}, \quad n \geq 2$;
Type C. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_1, \quad n \geq 2$.

Proposition 2.7 has an immediate corollary.

Corollary 2.8. *Each binomial isolated singularity is analytically equivalent to one from the three series: (A) $x_1^{a_1} + x_2^{a_2}$, (B) $x_1^{a_1} x_2 + x_2^{a_2}$, (C) $x_1^{a_1} x_2 + x_2^{a_2} x_1$.*

Ebeling and Takahashi [9] gave the following classification of weighted homogeneous fewnomial singularities in case of three variables.

Proposition 2.9 [9]. *Let $f(x_1, x_2, x_3)$ be a weighted homogeneous fewnomial isolated singularity with* $\mathrm{mult}(f) \geq 3$. *Then f is analytically equivalent to the following five types:*

Type 1. $x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$;
Type 2. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + x_3^{a_3}$;
Type 3. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + x_3^{a_3} x_1$;
Type 4. $x_1^{a_1} + x_2^{a_2} + x_3^{a_3} x_2$;
Type 5. $x_1^{a_1} x_2 + x_2^{a_2} x_1 + x_3^{a_3}$.

In order to prove the main theorems, we need to use the following main results from [25], [12] and [13].

THEOREM 2.10 [25]. *Let $(V_f, 0) \subset (\mathbb{C}^n, 0)$ and $(V_g, 0) \subset (\mathbb{C}^m, 0)$ be defined by weighted homogeneous polynomials $f(x_1, x_2, \ldots, x_n) = 0$ of weight type $(w_1, w_2, \ldots, w_n; 1)$ and $g(y_1, y_2, \ldots, y_m) = 0$ of weight type $(w_{n+1}, w_{n+2}, \ldots, w_{n+m}; 1)$, respectively. Let $\mu(V_f)$, $\mu(V_g)$ and $A(V_f)$, $A(V_g)$ be the Milnor numbers and moduli algebras of $(V_f, 0)$ and $(V_g, 0)$, respectively. Then*

$$\lambda(V_{f+g}) = \mu(V_f)\lambda(V_g) + \mu(V_g)\lambda(V_f).$$

PROPOSITION 2.11 [25]. *Let $(V, 0)$ be a weighted homogeneous fewnomial isolated singularity of type A which is defined by $f = x_1^{a_1} + x_2^{a_2} + \cdots + x_n^{a_n}$ $(a_i \geq 3,\ 1 \leq i \leq n)$ with weight type $(1/a_1, 1/a_2, \ldots, 1/a_n; 1)$. Then the Yau number*

$$\lambda(V) = n\prod_{i=1}^{n}(a_i - 1) - \sum_{i}^{n}(a_1 - 1)(a_2 - 1)\cdots\widehat{(a_i - 1)}\cdots(a_n - 1),$$

where $\widehat{(a_i - 1)}$ means that $a_i - 1$ is omitted.

PROPOSITION 2.12 [25]. *Let $(V, 0)$ be a binomial isolated singularity of type B which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}$ with weight type $((a_2 - 1)/a_1a_2, 1/a_2; 1)$. Then the Yau number*

$$\lambda(V) = 2a_1a_2 - 2a_1 - 3a_2 + 5.$$

PROPOSITION 2.13 [25]. *Let $(V, 0)$ be a binomial isolated singularity of type C which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_1$ with weight type $((a_2 - 1)/(a_1a_2 - 1), (a_1 - 1)/(a_1a_2 - 1); 1)$. If $\mathrm{mult}(f) \geq 4$, that is, $a_1, a_2 \geq 3$, then the Yau number*

$$\lambda(V) = 2a_1a_2 - 2a_1 - 2a_2 + 6.$$

If $\mathrm{mult}(f) = 3$, that is, $f = x_1^2x_2 + x_2^{a_2}x_1$, then the Yau number is $\lambda(V) = 2a_2$.

PROPOSITION 2.14 [13]. *Let $(V, 0)$ be a weighted homogeneous fewnomial isolated singularity of type A which is defined by $f = x_1^{a_1} + x_2^{a_2}$ $(a_1 \geq 2,\ a_2 \geq 2)$ with weight type $(1/a_1, 1/a_2; 1)$. Then*

$$\lambda^1(V) = \begin{cases} 2a_1a_2 - 3(a_1 + a_2) + 10, & a_1 \geq 3,\ a_2 \geq 3, \\ a_1 + 2, & a_1 \geq 2,\ a_2 = 2. \end{cases}$$

PROPOSITION 2.15 [13]. *Let $(V, 0)$ be a binomial isolated singularity of type B which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}$ $(a_1 \geq 1,\ a_2 \geq 2)$ with weight type $((a_2 - 1)/a_1a_2, 1/a_2; 1)$. Then*

$$\lambda^1(V) = \begin{cases} 2a_1a_2 - 2a_1 - 3a_2 + 11, & a_1 \geq 2,\ a_2 \geq 3, \\ 2a_1 + 2, & a_1 \geq 2,\ a_2 = 2, \\ 4, & a_1 = 1,\ a_2 \geq 2. \end{cases}$$

PROPOSITION 2.16 [13]. *Let $(V, 0)$ be a binomial isolated singularity of type C which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_1$ $(a_1 \geq 1, a_2 \geq 1)$ with weight type $((a_2 - 1)/(a_1a_2 - 1), (a_1 - 1)/(a_1a_2 - 1); 1)$. Then*

$$\lambda^1(V) = \begin{cases} 2a_1a_2 - 2a_1 - 2a_2 + 12, & a_1 \geq 3,\ a_2 \geq 3, \\ 2a_1 + 6, & a_1 \geq 2,\ a_2 = 2, \\ 4, & a_1 \geq 1,\ a_2 = 1, \\ 4, & a_1 = 1,\ a_2 \geq 2. \end{cases}$$

Proposition 2.17 [12]. *Let* $(V,0)$ *be a fewnomial isolated singularity of type 2 which is defined by* $f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}$ $(a_1 \geq 2, a_2 \geq 2, a_3 \geq 3)$ *with weight type* $((1-a_3+a_2a_3)/a_1a_2a_3, (a_3-1)/a_2a_3, 1/a_3; 1)$. *Then the Yau number*

$$\lambda(V) = \begin{cases} 3a_1a_2a_3 - 2a_1a_3 - 4a_2a_3 + 6a_3 + 2a_1 - 2a_1a_2 + 2a_2 - 7, & a_1 \geq 2,\ a_2 \geq 3,\ a_3 \geq 3, \\ 4a_1a_3 - 3a_3 - 2a_1 - 1, & a_1 \geq 2,\ a_2 = 2,\ a_3 \geq 3. \end{cases}$$

Proposition 2.18 [12]. *Let* $(V,0)$ *be a fewnomial isolated singularity of type 3 which is defined by* $f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_1$ $(a_1 \geq 2, a_2 \geq 2, a_3 \geq 2)$ *with weight type*

$$\Big(\frac{1-a_3+a_2a_3}{1+a_1a_2a_3}, \frac{1-a_1+a_1a_3}{1+a_1a_2a_3}, \frac{1-a_2+a_1a_2}{1+a_1a_2a_3}; 1\Big).$$

Then the Yau number

$$\lambda(V) = \begin{cases} 12, & a_1 = 2,\ a_2 = 2,\ a_3 = 2, \\ 3a_1a_2a_3 - 2(a_1a_2 + a_1a_3 + a_2a_3) + 2(a_1 + a_2 + a_3) - 1, & \textit{otherwise}. \end{cases}$$

3. Proof of main theorems

In order to prove the main theorems, we need to prove the following propositions.

Proposition 3.1. *Let* $(V,0)$ *be a weighted homogeneous fewnomial isolated singularity of type A which is defined by* $f = x_1^{a_1} + x_2^{a_2}$ $(a_1 \geq 1,\ a_2 \geq 1)$ *with weight type* $(1/a_1, 1/a_2; 1)$. *Then*

$$\lambda^2(V) = \begin{cases} 2a_1a_2 - 3(a_1 + a_2) + 17, & a_1 \geq 5,\ a_2 \geq 5, \\ 3a_2 + 5, & a_1 = 3,\ a_2 \geq 4, \\ 13, & a_1 = 3,\ a_2 = 3, \\ 5a_2 + 4, & a_1 = 4,\ a_2 \geq 5, \\ 23, & a_1 = 4,\ a_2 = 4, \\ a_2 + 5, & a_1 = 2,\ a_2 \geq 3, \\ 6, & a_1 = 2,\ a_2 = 2, \\ 1, & a_1 = 1,\ a_2 \geq 1. \end{cases}$$

Proof. It follows that the generalized moduli algebra

$$A^2(V) = \mathbb{C}\{x_1, x_2\}/(f, m^2.J(f))$$

has dimension $a_1a_2 - (a_1 + a_2) + 6$ and has a monomial basis of the form (cf. [1, Theorem 13.1]):

(1) if $a_1 \geq 3$, $\{x_1^{i_1}x_2^{i_2}, 1 \leq i_1 \leq a_1 - 2; 0 \leq i_2 \leq a_2 - 2; x_1^{a_1-1}; x_1^{a_1-1}x_2; x_1x_2^{a_2-1};$
$x_2^{i_2}, 0 \leq i_2 \leq a_2\}$; (3.1)

(2) if $a_1 = 2, a_2 \geq 3$, $\{x_2^{i_2}, 0 \leq i_2 \leq a_2; x_1x_2; x_1\}$; (3.2)

(3) if $a_1 = 2, a_2 = 2$, $\{1; x_1; x_1x_2; x_2; x_2^2\}$; (3.3)

(4) if $a_1 = 1, a_2 \geq 1$, $\{1; x_2\}$ (3.4)

with the following relations:

$$x_1^{a_1+1} = 0, \tag{3.5}$$
$$x_1^{a_1-1} x_2^2 = 0, \tag{3.6}$$
$$x_1^{a_1} x_2 = 0, \tag{3.7}$$
$$x_1^2 x_2^{a_2-1} = 0, \tag{3.8}$$
$$x_2^{a_2+1} = 0, \tag{3.9}$$
$$x_1 x_2^{a_2} = 0. \tag{3.10}$$

In order to compute a derivation D of $A^2(V)$, it suffices to indicate its values on the generators x_1, x_2, which can be written in terms of the basis (3.1), (3.2), (3.3) or (3.4). Without loss of generality, we write

$$Dx_j = \sum_{i_1=1}^{a_1-2} \sum_{i_2=0}^{a_2-2} c^j_{i_1,i_2} x_1^{i_1} x_2^{i_2} + c^j_{a_1-1,0} x_1^{a_1-1} + c^j_{a_1-1,1} x_1^{a_1-1} x_2 + c^j_{1,a_2-1} x_1 x_2^{a_2-1} + \sum_{i_2=0}^{a_2} c^j_{0,i_2} x_2^{i_2}, \quad j = 1, 2.$$

Using the relations (3.5)–(3.10), one easily finds the necessary and sufficient conditions defining a derivation of $A^2(V)$ as follows:

$$c^1_{0,0} = c^1_{0,1} = \cdots = c^1_{0,a_2-4} = 0, \tag{3.11}$$
$$c^2_{0,0} = c^2_{1,0} = \cdots = c^2_{a_1-4,0} = 0, \tag{3.12}$$
$$a_1 c^1_{1,0} = a_2 c^2_{0,1}. \tag{3.13}$$

Using (3.11)–(3.13), we obtain the following description of the Lie algebras in question. The following derivations form a basis of $\mathrm{Der}A^2(V)$:

$$x_1^{i_1} x_2^{i_2}\partial_1, 1 \le i_1 \le a_1-2, 1 \le i_2 \le a_2-2; x_1^{a_1-1}x_2\partial_1; x_1x_2^{a_2-1}\partial_1; x_2^{i_2}\partial_1, a_2-3 \le i_2 \le a_2;$$
$$x_1^{i_1}\partial_1, 2 \le i_1 \le a_1-1; x_2^{i_2}\partial_2, 2 \le i_2 \le a_2; x_1^{i_1}\partial_2, a_1-3 \le i_1 \le a_1-1;$$
$$x_1^{i_1}x_2^{i_2}\partial_2, 1 \le i_1 \le a_1-2, 1 \le i_2 \le a_2-2; x_1^{a_1-1}x_2\partial_2;$$
$$x_1x_2^{a_2-1}\partial_2; x_1\partial_1 + \frac{a_1}{a_2}x_2\partial_2.$$

Therefore, we have the following formula:

$$\lambda^2(V) = 2a_1a_2 - 3(a_1 + a_2) + 17.$$

In case of $a_1 = 3, a_2 \ge 4$, we have the following derivations which form a basis of $\mathrm{Der}A^2(V)$:

$$x_1x_2^{i_2}\partial_1, 1 \le i_2 \le a_2-1; x_1^2x_2\partial_1; x_1^2\partial_1; x_2^{i_2}\partial_1, a_2-2 \le i_2 \le a_2;$$
$$x_1x_2^{i_2}\partial_2, 1 \le i_2 \le a_2-1; x_1^2x_2\partial_2; x_1^2\partial_2; x_2^{i_2}\partial_2, 2 \le i_2 \le a_2; x_1\partial_1 + \frac{3}{a_2}x_2\partial_2.$$

Therefore, we have the following formula:

$$\lambda^2(V) = 3a_2 + 5.$$

In case of $a_1 = 3, a_2 = 3$, we have the following derivations which form a basis of $\mathrm{Der}A^2(V)$:

$$x_2^2\partial_1;\ x_2^3\partial_1;\ x_1x_2\partial_1;\ x_1x_2^2\partial_1;\ x_1^2\partial_1;\ x_1^2x_2\partial_1;\ x_2^2\partial_2;\ x_2^3\partial_2;\ x_1x_2\partial_2;$$
$$x_1x_2^2\partial_2;\ x_1^2\partial_2;\ x_1^2x_2\partial_2;\ x_1\partial_1 + x_2\partial_2.$$

In case of $a_1 = 4, a_2 \geq 5$, we have the following derivations which form a basis of $\mathrm{Der}A^2(V)$:

$$x_2^{i_2}\partial_1, a_2 - 3 \leq i_2 \leq a_2;\ x_1\partial_1 + \frac{4}{a_2}x_2\partial_2;\ x_1^{i_1}x_2\partial_1, 1 \leq i_1 \leq 2, 1 \leq i_2 \leq a_2 - 2;$$
$$x_1x_2^{a_2-1}\partial_1;\ x_1^2\partial_1;\ x_1^3\partial_1;$$
$$x_1^3x_2\partial_1;\ x_2^{i_2}\partial_2, 2 \leq i_2 \leq a_2;\ x_1^{i_1}x_2^{i_2}\partial_2, 1 \leq i_1 \leq 2, 1 \leq i_2 \leq a_2 - 2;$$
$$x_1x_2^{a_2-1}\partial_2;\ x_1^2\partial_2;\ x_1^3\partial_2;\ x_1^3x_2\partial_2.$$

Therefore, we have the following formula:

$$\lambda^2(V) = 5a_2 + 4.$$

In case of $a_1 = 4, a_2 = 4$, we have the following derivations which form a basis of $\mathrm{Der}A^2(V)$:

$$x_2^2\partial_1;\ x_2^3\partial_1;\ x_2^4\partial_1;\ x_1\partial_1 + x_2\partial_2;\ x_1x_2\partial_1;\ x_1x_2^2\partial_1;\ x_1x_2^3\partial_1;\ x_1^2\partial_1;\ x_1^2x_2\partial_1;$$
$$x_1^2x_2^2\partial_1;\ x_1^3\partial_1;\ x_1^3x_2\partial_1;\ x_2^2\partial_2;\ x_2^3\partial_2;$$
$$x_2^4\partial_2;\ x_1x_2\partial_2;\ x_1x_2^{i_2}\partial_2;\ x_1x_2^3\partial_2;\ x_1^2\partial_2;\ x_1^2x_2\partial_2;\ x_1^2x_2^2\partial_2;\ x_1^3\partial_2;\ x_1^3x_2\partial_1.$$

Therefore, we have the following formula:

$$\lambda^2(V) = 23.$$

In case of $a_1 = 2, a_2 \geq 3$, we have the following derivations which form a basis of $L^2(V)$:

$$x_2^{i_2}\partial_1, a_2 - 1 \leq i_2 \leq a_2;\ x_1\partial_1 + \frac{2}{a_2}x_2\partial_2;\ x_1x_2\partial_1;\ x_2^{i_2}\partial_2, 2 \leq i_2 \leq a_2;\ x_1\partial_2;\ x_1x_2\partial_2.$$

Therefore, we get the following formula:

$$\lambda^2(V) = a_2 + 5.$$

In case of $a_1 = 2, a_2 = 2$, we have the following derivations which form a basis of $L^2(V)$:

$$x_2\partial_1 + x_1\partial_2;\ x_2^2\partial_1;\ x_1\partial_1 + x_2\partial_2;\ x_1x_2\partial_1;\ x_2^2\partial_2;\ x_1x_2\partial_2.$$

Therefore, we get the following formula:

$$\lambda^2(V) = 6.$$

In case of $a_1 = 1, a_2 \geq 1$, we have the following derivation which forms a basis of $L^2(V)$:

$$x_2\partial_2.$$

Therefore, we get the following formula:

$$\lambda^2(V) = 1. \qquad \square$$

Proposition 3.2. *Let* $(V, 0)$ *be a binomial isolated singularity of type B which is defined by* $f = x_1^{a_1}x_2 + x_2^{a_2}$ $(a_1 \geq 1,\ a_2 \geq 1)$ *with weight type* $((a_2 - 1)/a_1a_2, 1/a_2; 1)$*. Then*

$$\lambda^2(V) = \begin{cases} 2a_1a_2 - 2a_1 - 3a_2 + 20, & a_1 \geq 5,\ a_2 \geq 5, \\ 5a_2 + 12, & a_1 = 4,\ a_2 \geq 5, \\ 31, & a_1 = 4,\ a_2 = 4, \\ 4a_1 + 7, & a_1 \geq 3,\ a_2 = 3, \\ 2a_1 + 5, & a_1 \geq 2,\ a_2 = 2, \\ a_2 + 11, & a_1 = 2,\ a_2 \geq 3, \\ 6, & a_1 = 1,\ a_2 \geq 2, \\ 1, & a_1 \geq 1,\ a_2 = 1. \end{cases}$$

Proof. It follows that the generalized moduli algebra

$$A^2(V) = \mathbb{C}\{x_1, x_2\}/(f, m^2.J(f))$$

has dimension $a_1a_2 - a_2 + 6$ and has a monomial basis of the form (cf. [1, Theorem 13.1]):

(1) if $a_1 \geq 2, a_2 \geq 3$, $\{x_1^{i_1}x_2^{i_2}, 1 \leq i_1 \leq a_1 - 2; 1 \leq i_2 \leq a_2 - 2; x_1^{i_1}x_2^{a_2-1}, 1 \leq i_1 \leq a_1 - 2;$
$$x_1^{a_1-1}x_2^{i_2}, 1 \leq i_2 \leq 2; x_2^{i_2}, 0 \leq i_2 \leq a_2; x_1^{i_1}, 1 \leq i_1 \leq a_1 + 1\}; \tag{3.14}$$

(2) if $a_1 \geq 2, a_2 = 2$, $\{x_1^{i_1}x_2^{i_2}, 1 \leq i_1 \leq a_1 - 1; 0 \leq i_2 \leq 1; x_2^{i_2}, 0 \leq i_2 \leq 2;$
$$x_1^{i_1}, a_1 \leq i_1 \leq a_1 + 1\}; \tag{3.15}$$

(3) if $a_1 = 1, a_2 \geq 2$, $\{1; x_1; x_1^2; x_2; x_2^2\}$; (3.16)

(4) if $a_1 \geq 1, a_2 = 1$, $\{x_1^{i_1}, 0 \leq i_1 \leq a_1 + 1\}$ (3.17)

with the following relations:

$$x_1^{a_1+1}x_2 = 0, \tag{3.18}$$
$$x_1^{a_1-1}x_2^3 = 0, \tag{3.19}$$
$$x_1^{a_1}x_2^2 = 0, \tag{3.20}$$
$$x_1^{a_1+2} + a_2x_1^2x_2^{a_2-1} = 0, \tag{3.21}$$
$$x_1^{a_1}x_2^2 + a_2x_2^{a_2+1} = 0, \tag{3.22}$$
$$x_1^{a_1+1}x_2 + a_2x_1x_2^{a_2} = 0. \tag{3.23}$$

In order to compute a derivation D of $A^2(V)$, it suffices to indicate its values on the generators x_1, x_2, which can be written in terms of the basis (3.14), (3.15), (3.16) or (3.17). Without loss of generality, we write

$$Dx_j = \sum_{i_1=1}^{a_1-2}\sum_{i_2=1}^{a_2-2} c^j_{i_1,i_2} x_1^{i_1} x_2^{i_2} + \sum_{i_1=1}^{a_1-2} c^j_{i_1,a_2-1} x_1^{i_1} x_2^{a_2-1} + \sum_{i_2=1}^{2} c^j_{a_1-1,i_2} x_1^{a_1-1} x_2^{i_2} + \sum_{i_2=0}^{a_2} c^j_{0,i_2} x_2^{i_2} + \sum_{i_1=1}^{a_1+1} c^j_{i_1,0} x_1^{i_1}, \quad j = 1, 2.$$

Using the relations (3.18)–(3.23), one easily finds the necessary and sufficient conditions defining a derivation of $A^2(V)$ as follows:

$$c^1_{0,0} = c^1_{0,1} = \cdots = c^1_{0,a_2-4} = 0, \tag{3.24}$$

$$c^2_{0,0} = c^2_{1,0} = \cdots = c^2_{a_1-3,0} = 0, \tag{3.25}$$

$$c^1_{1,0} = c^2_{0,1}, c^1_{2,0} = c^2_{1,1}, c^1_{3,0} = c^2_{2,1}, \ldots, c^1_{a_1-3,0} = c^2_{a_1-4,1}. \tag{3.26}$$

Using (3.24)–(3.26), we obtain the following description of the Lie algebras in question. The following derivations form a basis of $\mathrm{Der}A^2(V)$:

$$x_1^{i_1}x_2^{i_2}\partial_1, 1 \le i_1 \le a_1-2, 1 \le i_2 \le a_2-1; x_1^{i_1}\partial_1 + \frac{a_1}{a_2-1}x_1^{i_1-1}x_2\partial_2, 1 \le i_1 \le a_1-3;$$
$$x_1^{i_1}\partial_1, a_1-2 \le i_1 \le a_1+1; x_1^{a_1-1}x_2^{i_2}\partial_1, 1 \le i_2 \le 2;$$
$$x_1^{i_1}x_2^{i_2}\partial_2, 1 \le i_1 \le a_1-2, 2 \le i_2 \le a_2-1;$$
$$x_2^{i_2}\partial_1, a_2-3 \le i_2 \le a_2; x_2^{i_2}\partial_2, 2 \le i_2 \le a_2; x_1^{i_1}x_2\partial_2, a_1-3 \le i_1 \le a_1-1;$$
$$x_1^{i_1}\partial_2, a_1-2 \le i_1 \le a_1+1.$$

Therefore, we have the following formula:

$$\lambda^2(V) = 2a_1a_2 - 2a_1 - 3a_2 + 20.$$

In case of $a_1 = 4, a_2 \ge 5$, we have the following derivations which form a basis of $L^2(V)$:

$$x_2^{i_2}\partial_1, a_2-3 \le i_2 \le a_2; x_1\partial_1 + \frac{4}{a_2-1}x_2\partial_1; x_1^{i_1}x_2^{i_2}\partial_1, 1 \le i_1 \le 2, 1 \le i_2 \le a_2-1;$$
$$x_1^{i_1}\partial_1, 2 \le i_1 \le 5;$$
$$x_1^3x_2^{i_2}\partial_1, 1 \le i_2 \le 2; x_2^{i_2}\partial_2, 2 \le i_2 \le a_2; x_1^{i_1}x_2^{i_2}\partial_2, 1 \le i_1 \le 2, 1 \le i_2 \le a_2-1;$$
$$x_1^{i_1}\partial_2, 2 \le i_1 \le 5;$$
$$x_1^3x_2^{i_2}\partial_2, 1 \le i_2 \le 2.$$

Therefore, we get the following formula:

$$\lambda^2(V) = 5a_2 + 12.$$

In case of $a_1 = 4, a_2 = 4$, we have the following derivations which form a basis of $L^2(V)$:

$$x_2^2\partial_1; x_2^3\partial_1; x_2^4\partial_1; x_1\partial_1 + \tfrac{4}{3}x_2\partial_1; x_1x_2\partial_1; x_1x_2^2\partial_1; x_1x_2^3\partial_1; x_1^2\partial_1;$$
$$x_1^2x_2\partial_1; x_1^2x_2^2\partial_1; x_1^2x_2^3\partial_1; x_1^3\partial_1; x_1^3x_2\partial_1;$$
$$x_1^3x_2^2\partial_1; x_1^4\partial_1; x_1^5\partial_1; x_2^2\partial_2; x_2^3\partial_2; x_2^4\partial_2; x_1\partial_1 + \tfrac{4}{3}x_2\partial_2; x_1x_2\partial_2; x_1x_2^2\partial_2;$$
$$x_1x_2^3\partial_1; x_1^2\partial_2; x_1^2x_2\partial_1; x_1^2x_2^2\partial_2;$$
$$x_1^2x_2^3\partial_2; x_1^3\partial_2; x_1^3x_2\partial_2; x_1^3x_2^2\partial_2; x_1^4\partial_2; x_1^5\partial_2.$$

Therefore, we get the following formula:

$$\lambda^2(V) = 31.$$

In case of $a_1 \geq 3, a_2 = 3$, we have the following derivations which form a basis of $L^2(V)$:

$$x_2^2\partial_1; x_2^3\partial_1; x_1^{i_1}\partial_1 + \frac{a_1}{2}x_1^{i_1-1}x_2\partial_2, 1 \leq i_1 \leq a_1 - 2; x_1^{i_1}x_2^{i_2}\partial_1, 1 \leq i_1 \leq a_1 - 1, 1 \leq i_2 \leq 2;$$
$$x_1^{i_1}\partial_1, a_1 - 1 \leq i_1 \leq a_1 + 1; x_2^2\partial_2; x_2^3\partial_2; x_1^{i_1}\partial_2, a_1 - 1 \leq i_1 \leq a_1 + 1;$$
$$x_1^{i_1}x_2^2\partial_2, 1 \leq i_1 \leq a_1 - 1;$$
$$x_1^{i_1}x_2\partial_2, a_1 - 2 \leq i_1 \leq a_1 - 1.$$

Therefore, we get the following formula:

$$\lambda^2(V) = 4a_1 + 7.$$

In case of $a_1 \geq 2, a_2 = 2$, we have the following derivations which form a basis of $L^2(V)$:

$$x_2^{i_2}\partial_1, 1 \leq i_2 \leq 2; x_1^{i_1}\partial_1 + a_1x_1^{i_1-1}x_2\partial_2, 1 \leq i_1 \leq a_1 - 1; x_1^{i_1}x_2\partial_1, 1 \leq i_1 \leq a_1 - 1;$$
$$x_1^{i_1}\partial_1, a_1 \leq i_1 \leq a_1 + 1; x_2^2\partial_2; \left(\frac{x_1^{a_1+1}}{2} + x_1x_2\right)\partial_2; x_1^{a_1-1}x_2\partial_2.$$

Therefore, we get the following formula:

$$\lambda^2(V) = 2a_1 + 5.$$

In case of $a_1 = 2, a_2 \geq 3$, it is easy to check that we get the following basis:

$$x_2^{i_2}\partial_1, 2 \leq i_2 \leq a_2; x_1\partial_1 + \frac{2}{a_2 - 1}x_2\partial_2; x_2^{i_2}\partial_1, a_2 - 2 \leq i_2 \leq a_2;$$
$$x_1x_2\partial_1; x_1x_2^2\partial_1; x_1^2\partial_1; x_1^3\partial_1; x_1x_2\partial_2; x_1x_2^2\partial_2; x_1^2\partial_2; x_1^3\partial_2.$$

Therefore,

$$\lambda^2(V) = a_2 + 11.$$

In case of $a_1 = 1, a_2 \geq 2$, we have the following derivations which form a basis of $L^2(V)$:

$$x_2^2\partial_1; x_1\partial_1; x_1^2\partial_1; x_2\partial_2; x_2^2\partial_2; x_1^2\partial_2.$$

Therefore, we get the following formula:

$$\lambda^2(V) = 6.$$

In case of $a_1 \geq 1, a_2 = 1$, we have the following derivation which forms a basis of $L^2(V) = \langle (x_1^{a_1+1} + x_1)\partial_1\rangle$. Therefore, we get the following formula:

$$\lambda^2(V) = 1. \qquad \square$$

PROPOSITION 3.3. *Let $(V, 0)$ be a binomial isolated singularity of type C which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_1$ $(a_1 \geq 1,\ a_2 \geq 1)$ with weight type $((a_2-1)/(a_1a_2-1),$ $(a_1-1)/(a_1a_2-1); 1)$. Then*

$$\lambda^2(V) = \begin{cases} 2a_1a_2 - 2(a_1+a_2) + 21, & a_1 \geq 4,\ a_2 \geq 4, \\ 4a_2 + 13, & a_1 = 3,\ a_2 \geq 3, \\ 2a_2 + 10, & a_1 = 2,\ a_2 \geq 3, \\ 13, & a_1 = 2,\ a_2 = 2, \\ 6, & a_1 = 1,\ a_2 \geq 1. \end{cases}$$

PROOF. It follows that the generalized moduli algebra

$$A^2(V) = \mathbb{C}\{x_1, x_2\}/(f, m^2.J(f))$$

has dimension $a_1a_2 + 5$ and has a monomial basis of the form (cf. [1, Theorem 13.1]):

(1) if $a_1 \geq 2, a_2 \geq 2$, $\{x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq a_1 - 2; 0 \leq i_2 \leq a_2 - 1; x_2^{i_2}, a_2 \leq i_2 \leq 2a_2 - 2;$
$$x_1^{i_1}, a_1 - 1 \leq i_1 \leq a_1 + 1; x_1x_2^{a_2}; x_1^{a_1-1}x_2; x_1^{a_1-1}x_2^2\}; \tag{3.27}$$

(2) if $a_1 = 1, a_2 \geq 1$, $\{x_2^{i_2}, 0 \leq i_2 \leq a_2 + 1; x_1; x_1^2\}$ (3.28)

with the following relations:

$$a_1x_1^{a_1+1} + x_1^2x_2^{a_2} = 0, \tag{3.29}$$
$$a_1x_1^{a_1-1}x_2^3 + x_2^{a_2+2} = 0, \tag{3.30}$$
$$a_1x_1^{a_1}x_2^2 + x_1x_2^{a_2+1} = 0, \tag{3.31}$$
$$x_1^{a_1+2} + a_2x_1^3x_2^{a_2-1} = 0, \tag{3.32}$$
$$x_1^{a_1}x_2^2 + a_2x_1x_2^{a_2+1} = 0, \tag{3.33}$$
$$x_1^{a_1+1}x_2 + a_2x_1^2x_2^{a_2} = 0. \tag{3.34}$$

In order to compute a derivation D of $A^2(V)$, it suffices to indicate its values on the generators x_1, x_2, which can be written in terms of the basis (3.27) or (3.28). Without loss of generality, we write

$$Dx_j = \sum_{i_1=0}^{a_1-2}\sum_{i_2=0}^{a_2-1} c^j_{i_1,i_2}x_1^{i_1}x_2^{i_2} + \sum_{i_2=a_2}^{2a_2-2} c^j_{0,i_2}x_2^{i_2} + \sum_{i_1=a_1-1}^{a_1+1} c^j_{i_1,0}x_1^{i_1} + c^j_{1,a_2}x_1x_2^{a_2}$$
$$+ c^j_{a_1-1,1}x_1^{a_1-1}x_2 + c^j_{a_1-1,2}x_1^{a_1-1}x_2^2, \quad j = 1, 2.$$

Using the relations (3.29)–(3.34), one easily finds the necessary and sufficient conditions defining a derivation of $A^2(V)$ as follows:

$$c^1_{0,0} = c^1_{0,1} = \cdots = c^1_{0,a_2-3} = 0, \tag{3.35}$$

$$c^2_{0,0} = c^2_{1,0} = \cdots = c^2_{a_1-3,0} = 0, \tag{3.36}$$

$$(a_1-1)c^1_{1,0} = (a_2-1)c^2_{0,1}, (a_1-1)c^1_{1,1} = (a_2-1)c^2_{0,2}, \ldots,$$
$$(a_1-1)c^1_{1,a_2-4} = (a_2-1)c^2_{0,a_2-3}, \tag{3.37}$$

$$(a_1-1)c^1_{a_1-3,0} = (a_2-1)c^2_{a_1-4,1}. \tag{3.38}$$

Using (3.35)–(3.38), we obtain the following description of the Lie algebras in question. The following derivations form a basis of Der$A^2(V)$:

$$x_1^{i_1}x_2^{i_2}\partial_1, 2 \le i_1 \le a_1-2, 1 \le i_2 \le a_2-1; x_2^{i_2}\partial_1, a_2-2 \le i_2 \le 2a_2-2;$$
$$x_1^{i_1}\partial_1 + \frac{a_2-1}{a_1}x_1^{i_1-1}x_2\partial_2, 1 \le i_1 \le a_1-3; x_1^{a_1-1}x_2\partial_1; x_1^{a_1-1}x_2^2\partial_1;$$
$$x_1x_2^{i_2}\partial_1 + \frac{a_2-1}{a_1}x_2^{i_2+1}\partial_2, 1 \le i_2 \le a_2-4;$$
$$x_1x_2^{i_2}\partial_1, a_2-3 \le i_2 \le a_2; x_1^{i_1}\partial_1, a_1-2 \le i_1 \le a_1+1;$$
$$x_2^{i_2}\partial_2, a_2-2 \le i_2 \le 2a_2-2; x_1^{i_1}x_2^{i_2}\partial_2, 1 \le i_1 \le a_1-2, 2 \le i_2 \le a_2-1;$$
$$x_1^{i_1}\partial_2, a_1-2 \le i_1 \le a_1+1; x_1^{i_1}x_2\partial_2, a_1-3 \le i_1 \le a_1-1; x_1x_2^{a_2}\partial_2; x_1^{a_1-1}x_2^2\partial_2.$$

Therefore, we have the following formula:

$$\lambda^2(V) = 2a_1a_2 - 2(a_1+a_2) + 21.$$

In case of $a_1 = 3, a_2 \ge 3$, we have the following derivations which form a basis of $L^2(V)$:

$$x_2^{i_2}\partial_1, a_2-2 \le i_2 \le 2a_2-2; x_1\partial_1 + \frac{2}{a_2-1}x_2\partial_2;$$
$$x_1x_2^{i_2}\partial_1 + \frac{2}{a_2-1}x_2^{i_2+1}\partial_2, 1 \le i_2 \le a_2-4;$$
$$x_1x_2^{i_2}\partial_1, a_2-3 \le i_2 \le a_2; x_1^2x_2\partial_1; x_1^2x_2^2\partial_1; x_2^{i_2}\partial_2, a_2-2 \le i_2 \le 2a_2-2; x_1^{i_1}\partial_2, 2 \le i_1 \le 4;$$
$$x_1^{i_1}\partial_1, 2 \le i_1 \le 4; x_1x_2^{i_2}\partial_2, 1 \le i_2 \le a_2; x_1^2x_2\partial_2; x_1^2x_2^2\partial_2.$$

Therefore, we get the following formula:

$$\lambda^2(V) = 4a_2 + 13.$$

In case of $a_1 = 2, a_2 \ge 3$, we have the following derivations which form a basis of $L^2(V)$:

$$x_2^{(a_2-2)-i_2}\partial_1 - \frac{2}{a_2-1}x_2^{i_2}\partial_2, 2 \le i_2 \le a_2-2; x_2^{i_2}\partial_1, 2a_2-3 \le i_2 \le 2a_2-1;$$

$$x_1^2\partial_1; x_1^3\partial_1; x_2^{i_2}\partial_2, a_2-1\le i_2\le 2a_2-1; x_1^2\partial_2; x_1^3\partial_2; x_1x_2\partial_2; x_1x_2^2\partial_2;$$
$$x_1x_2^{i_2}\partial_1+\frac{x_2^{i_2+1}}{a_2-1}\partial_2, 0\le i_2\le 2.$$

Therefore, we get the following formula:

$$\lambda^2(V)=2a_2+10.$$

In case of $a_1=2, a_2=2$, we have the following derivations which form a basis of $L^2(V)$:

$$x_2^2\partial_1; x_2^3\partial_1; x_1\partial_1+x_2\partial_2; x_1x_2\partial_1; x_1x_2^2\partial_1; x_1^2\partial_1; x_1^3\partial_1;$$
$$x_2^2\partial_2; x_2^3\partial_2; x_1x_2\partial_2; x_1x_2^2\partial_2; x_1^2\partial_2; x_1^3\partial_2.$$

Therefore, we get the following formula:

$$\lambda^2(V)=13.$$

In case of $a_1=1, a_2\ge 1$, we have the following derivations which form a basis of $L^2(V)$:

$$(x_2^{a_2+1}+x_2^2)\partial_1; x_1\partial_1; x_1^2\partial_1; (-x_2^{a_2+1}+x_2)\partial_2; (x_2^{a_2+1}+x_2^2)\partial_2; x_1^2\partial_2.$$

Therefore, we get the following formula:

$$\lambda^2(V)=6. \qquad \square$$

PROPOSITION 3.4. *Let $(V,0)$ be a fewnomial surface isolated singularity of type 4 which is defined by $f=x_1^{a_1}+x_2^{a_2}+x_3^{a_3}x_2$ $(a_1\ge 3, a_2\ge 3, a_3\ge 2)$ with weight type $(1/a_1, 1/a_2, (a_2-1)/a_2a_3; 1)$. Then*

$$\lambda(V)=3a_1a_2a_3-4a_1a_2-4a_2a_3-2a_1a_3+6a_1+5a_2+2a_3-7.$$

PROOF. We have $f=x_1^{a_1}+x_2^{a_2}+x_3^{a_3}x_2$. We can also write this as

$$f=f_1+f_2,$$

where $f_1=x_1^{a_1}, f_2=x_2^{a_2}+x_3^{a_3}x_2$. It follows from Theorem 2.10 that

$$\lambda(V_{f_1+f_2})=\mu(V_{f_1})\lambda(V_{f_2})+\mu(V_{f_2})\lambda(V_{f_1}).$$

It is noted that from Propositions 2.11 and 2.12,

$$\lambda(V)=3a_1a_2a_3-4a_1a_2-4a_2a_3-2a_1a_3+6a_1+5a_2+2a_3-7. \qquad \square$$

PROPOSITION 3.5. *Let $(V,0)$ be a fewnomial surface isolated singularity of type 5 which is defined by $f=x_1^{a_1}x_2+x_2^{a_2}x_1+x_3^{a_3}$ $(a_1\ge 2, a_2\ge 2, a_3\ge 3)$ with weight type $((a_2-1)/(a_1a_2-1), (a_1-1)/(a_1a_2-1), 1/a_3; 1)$. Then*

$$\lambda(V)=\begin{cases}3a_1a_2a_3-4a_1a_2-2(a_2a_3+a_1a_3)+2(a_1+a_2)+6a_3-6, & a_1\ge 3,\ a_2\ge 3,\ a_3\ge 3,\\ 4a_2a_3-6a_2, & a_1=2,\ a_2\ge 2,\ a_3\ge 3.\end{cases}$$

Proof. We have $f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}$. We can also write this as

$$f = f_1 + f_2,$$

where $f_1 = x_1^{a_1}x_2 + x_2^{a_2}x_3$, $f_2 = x_3^{a_3}$. It follows from Theorem 2.10 that

$$\lambda(V_{f_1+f_2}) = \mu(V_{f_1})\lambda(V_{f_2}) + \mu(V_{f_2})\lambda(V_{f_1}).$$

It is noted from Propositions 2.11 and 2.13 that when $a_1 \geq 3, a_2 \geq 3, a_3 \geq 3$,

$$\lambda(V) = 3a_1a_2a_3 - 4a_1a_2 - 2(a_2a_3 + a_1a_3) + 2(a_1 + a_2) + 6a_3 - 6.$$

In case of $a_1 = 2, a_2 \geq 2, a_3 \geq 3$,

$$\lambda(V) = 4a_2a_3 - 6a_2.$$ □

Proposition 3.6. *Let* $(V, 0)$ *be a fewnomial surface isolated singularity of type 1 which is defined by* $f = x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$ $(a_1 \geq 3, a_2 \geq 3, a_3 \geq 3)$ *with weight type* $(1/a_1, 1/a_2, 1/a_3; 1)$. *Then*

$$\lambda^1(V) = 3a_1a_2a_3 + 5(a_1 + a_2 + a_3) - 4(a_1a_2 + a_1a_3 + a_2a_3) + 6.$$

Proof. It is easy to see that the moduli algebra $A^1(V) = \mathbb{C}\{x_1, x_2, x_3\}/(f, m.J(f))$ has dimension $(a_1a_2a_3 - a_1a_2 - a_1a_3 - a_2a_3 + a_1 + a_2 + a_3 + 2)$ and has a monomial basis of the form (cf. [1, Theorem 13.1])

$$\{x_1^{i_1}x_2^{i_2}x_3^{i_3}, 1 \leq i_1 \leq a_1 - 2; 1 \leq i_2 \leq a_2 - 2; 0 \leq i_3 \leq a_3 - 2; x_1^{i_1}, 1 \leq i_1 \leq a_1 - 1;$$
$$x_1^{i_1}x_3^{i_3}, 1 \leq i_1 \leq a_1 - 2; 1 \leq i_3 \leq a_3 - 2; x_2^{i_2}x_3^{i_3}, 1 \leq i_2 \leq a_2 - 2; 0 \leq i_3 \leq a_3 - 2; x_2^{a_2-1};$$
$$x_3^{i_3}, 0 \leq i_3 \leq a_3 - 1\}.$$

In order to compute a derivation D of $A^1(V)$, it suffices to indicate its values on the generators x_1, x_2, x_3, which can be written in terms of the basis. Thus, we can write

$$Dx_j = \sum_{i_1=1}^{a_1-2}\sum_{i_2=1}^{a_2-2}\sum_{i_3=0}^{a_3-2} c^j_{i_1,i_2,i_3}x_1^{i_1}x_2^{i_2}x_3^{i_3} + \sum_{i_1=1}^{a_1-1} c^j_{i_1,0,0}x_1^{i_1} + \sum_{i_3=0}^{a_3-1} c^j_{0,0,i_3}x_3^{i_3} + \sum_{i_1=1}^{a_1-2}\sum_{i_3=1}^{a_3-2} c^j_{i_1,0,i_3}x_1^{i_1}x_3^{i_3}$$
$$+ \sum_{i_2=1}^{a_2-2}\sum_{i_3=0}^{a_3-2} c^j_{0,i_2,i_3}x_2^{i_2}x_3^{i_3} + c^j_{0,a_2-1,0}x_2^{a_2-1}, \quad j = 1, 2, 3.$$

Using the above derivations, we obtain the following description of the Lie algebras in question. The derivations represented by the following vector fields form a basis in $\mathrm{Der}A^1(V)$:

$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_1,\ 1 \leq i_1 \leq a_1 - 2, 0 \leq i_2 \leq a_2 - 2, 0 \leq i_3 \leq a_3 - 2;$$
$$x_2^{a_2-2}x_3^{a_3-2}\partial_1; x_1^{a_1-1}\partial_1; x_2^{a_2-1}\partial_1; x_3^{a_3-1}\partial_1;$$
$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2,\ 0 \leq i_1 \leq a_1 - 2, 1 \leq i_2 \leq a_2 - 2, 0 \leq i_3 \leq a_3 - 2;$$
$$x_1^{a_1-2}x_3^{a_3-2}\partial_2; x_1^{a_1-1}\partial_2; x_2^{a_2-1}\partial_2; x_3^{a_3-1}\partial_2;$$

$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_3,\ 0 \le i_1 \le a_1-2, 0 \le i_2 \le a_2-2, 1 \le i_3 \le a_3-2;$$
$$x_1^{a_1-2}x_2^{a_2-2}\partial_3; x_1^{a_1-1}\partial_3; x_2^{a_2-1}\partial_3; x_3^{a_3-1}\partial_3.$$

Therefore,

$$\lambda^1(V) = 3a_1a_2a_3 + 5(a_1+a_2+a_3) - 4(a_1a_2+a_1a_3+a_2a_3) + 6. \qquad \square$$

Proposition 3.7. *Let $(V,0)$ be a fewnomial surface isolated singularity of type 2 which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}$ $(a_1 \ge 2, a_2 \ge 2, a_3 \ge 3)$ with weight type $((1-a_3+a_2a_3)/a_1a_2a_3, (a_3-1)/a_2a_3, 1/a_3; 1)$. Then*

$$\lambda^1(V) = \begin{cases} 4a_1a_3 - 2a_1 - 3a_3 + 11, & a_1 \ge 3,\ a_2 = 2,\ a_3 \ge 3, \\ 5a_3 + 7, & a_1 = 2,\ a_2 = 2,\ a_3 \ge 3, \\ 3a_1a_2a_3 - 2a_1a_2 - 2a_1a_3 - 4a_2a_3 + 2a_1 + 2a_2 + 6a_3 + 5, & \text{otherwise}. \end{cases}$$

Proof. It is easy to see that the moduli algebra $A^1(V) = \mathbb{C}\{x_1,x_2,x_3\}/(f, m.J(f))$ has dimension $(a_1a_2a_3 - a_2a_3 + a_3 + 2)$ and has a monomial basis of the form (cf. [1, Theorem 13.1])

$$\{x_1^{i_1}x_2^{i_2}x_3^{i_3}, 1 \le i_1 \le a_1-2; 1 \le i_2 \le a_2-1; 0 \le i_3 \le a_3-1; x_1^{a_1-1}x_3^{i_3}, 0 \le i_3 \le a_3-2;$$
$$x_2^{i_2}x_3^{i_3}, 0 \le i_2 \le a_2-1; 0 \le i_3 \le a_3-1; x_1^{i_1}x_3^{i_3}, 1 \le i_1 \le a_1-2; 0 \le i_3 \le a_3-1;$$
$$x_2^{a_2}; x_1^{a_1}; x_1^{a_1-1}x_2\}.$$

In order to compute a derivation D of $A^1(V)$, it suffices to indicate its values on the generators x_1, x_2, x_3, which can be written in terms of the basis. Thus, we can write

$$Dx_j = \sum_{i_1=1}^{a_1-2}\sum_{i_2=1}^{a_2-1}\sum_{i_3=0}^{a_3-1} c^j_{i_1,i_2,i_3}x_1^{i_1}x_2^{i_2}x_3^{i_3} + \sum_{i_3=0}^{a_3-2} c^j_{a_1-1,0,i_3}x_1^{a_1-1}x_3^{i_3} + c^j_{a_1-1,1,0}x_1^{a_1-1}x_2 + c^j_{a_1,0,0}x_1^{a_1}$$
$$+ \sum_{i_2=0}^{a_2-1}\sum_{i_3=0}^{a_3-1} c^j_{0,i_2,i_3}x_2^{i_2}x_3^{i_3} + \sum_{i_1=1}^{a_1-2}\sum_{i_3=0}^{a_3-1} c^j_{i_1,0,i_3}x_1^{i_1}x_3^{i_3} + c^j_{0,a_2,0}x_2^{a_2}, \quad j = 1,2,3.$$

Using the above derivations, we obtain the following description of the Lie algebras in question. The derivations represented by the following vector fields form a basis in $\mathrm{Der}A^1(V)$:

$$\left(\frac{1}{a_3}x_2^{a_2} + x_3^{a_3-1}\right)\partial_1; x_2^{a_2-2}x_3^{i_3}\partial_1 + \frac{a_1-1}{a_2}x_1^{a_1-1}x_3^{i_3-1}\partial_2,\ 1 \le i_3 \le a_3-2; x_2^{a_2-2}x_3^{a_3-1}\partial_1;$$
$$(1-a_3+a_2a_3)x_1^{i_1}\partial_1 + a_1(a_3-1)x_1^{i_1-1}x_2\partial_2 + a_1a_2x_1^{i_1-1}x_3\partial_3,\ 1 \le i_1 \le a_1-1;$$
$$x_1^{i_1}x_3^{i_3}\partial_1 + a_1x_1^{i_1-1}x_3^{i_3+1}\partial_3,\ 1 \le i_1 \le a_1-2; 1 \le i_3 \le a_3-2;$$
$$x_1^{a_1-1}x_3^{i_3}\partial_1 - \frac{a_1}{a_3}x_1^{a_1-2}x_3^{i_3-1}\partial_3,\ 1 \le i_3 \le a_3-3;$$
$$x_1^{i_1}x_3^{a_3-1}\partial_1,\ 1 \le i_1 \le a_1-2; x_1^{a_1-1}x_3^{a_3-2}\partial_1; x_1^{a_1-1}x_2\partial_1; x_1^{a_1}\partial_1; x_2^{a_2-1}x_3^{i_3}\partial_1,\ 1 \le i_3 \le a_3-1;$$
$$x_3^{a_3-1}\partial_2 + \frac{1}{a_3(a_3-1)}x_1^{a_1}\partial_3; x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_1,\ 1 \le i_1 \le a_1-2;$$

$$1 \le i_2 \le a_2 - 1; 0 \le i_3 \le a_3 - 1;$$
$$x_2 x_3^{i_3}\partial_2 - a_1 x_3^{i_3+1}\partial_3,\ 1 \le i_3 \le a_3 - 2; x_2 x_3^{a_3-1}\partial_2;$$
$$x_2^{i_2+1}\partial_2 + \frac{a_2}{a_3-1} x_2^{i_2} x_3 \partial_3,\ 1 \le i_2 \le a_2 - 2;$$
$$x_2^{i_2} x_3^{i_3}\partial_2,\ 2 \le i_2 \le a_2 - 1; 1 \le i_3 \le a_3 - 1; x_2^{a_2}\partial_2 - \frac{1}{a_3-1} x_1^{a_1}\partial_3;$$
$$x_1^{i_1} x_3^{a_3-1}\partial_2 - \frac{a_2}{a_3(a_3-1)} x_1^{i_1} x_2^{a_2-1} x_3 \partial_3,\ 1 \le i_1 \le a_1 - 2; x_1^{a_1-1} x_3^{a_3-2}\partial_2; x_1^{a_1-1} x_2 \partial_2; x_1^{a_1}\partial_2;$$
$$x_1^{i_1} x_2^{i_2} x_3^{i_3}\partial_2,\ 1 \le i_1 \le a_1 - 2; 2 \le i_2 \le a_2 - 1; 1 \le i_3 \le a_3 - 1; \left(\frac{1}{a_2} x_2^{a_2} + x_3^{a_3-1}\right)\partial_3;$$
$$x_1^{i_1} x_2^{i_2}\partial_1 + \frac{a_2}{a_3-1} x_1^{i_1} x_2^{i_2-1} x_3 \partial_3,\ 1 \le i_1 \le a_1 - 2; 2 \le i_2 \le a_2 - 1; \frac{1}{a_2} x_1^{a_1}\partial_2 + x_2^{a_2-1} x_3 \partial_3;$$
$$x_2^{i_2} x_3^{i_3}\partial_3,\ 1 \le i_2 \le a_2 - 1; 2 \le i_3 \le a_3 - 1; x_1^{a_1-1} x_3^{i_3}\partial_3,\ 1 \le i_3 \le a_3 - 2; x_1^{a_1-2} x_2^{a_2-1} x_3 \partial_3;$$
$$x_1^{i_1} x_2^{i_2} x_3^{i_3}\partial_3,\ 1 \le i_1 \le a_1 - 2; 1 \le i_2 \le a_2 - 1; 2 \le i_3 \le a_3 - 1; x_1^{a_1-1} x_2 \partial_3;$$
$$x_1^{i_1} x_2 x_3^{i_3}\partial_2 - a_1 x_1^{i_1} x_3^{i_3+1}\partial_3,\ 1 \le i_1 \le a_1 - 2; 1 \le i_3 \le a_3 - 1; x_1^{a_1-2} x_3^{a_3-1}\partial_3.$$

Therefore,

$$\lambda^1(V) = 3a_1a_2a_3 - 2a_1a_2 - 2a_1a_3 - 4a_2a_3 + 2a_1 + 2a_2 + 6a_3 + 5.$$

In case of $a_1 \ge 3, a_2 = 2, a_3 \ge 3$, we obtain the basis of derivation represented by the following derivations which form a basis of $\mathrm{Der}A^1(V)$:

$$(1 + a_3)x_1^{i_1}\partial_1 + a_1(a_3 - 1)x_1^{i_1-1} x_2 \partial_2 + 2a_1 x_1^{i_1-1} x_3 \partial_3,\ 1 \le i_1 \le a_1 - 1;$$
$$x_2 x_3^{i_3}\partial_1,\ 1 \le i_3 \le a_3 - 1; x_1^{a_1}\partial_1;$$
$$x_3^{a_3-1}\partial_1; x_2^2\partial_1; x_1^{i_1} x_3^{i_3}\partial_1 + a_1 x_1^{i_1-1} x_3^{i_3+1}\partial_3,\ 1 \le i_1 \le a_1 - 1; 1 \le i_3 \le a_3 - 2;$$
$$x_1^{i_1} x_3^{a_3-1}\partial_1,\ 1 \le i_1 \le a_1 - 2; x_1^{i_1} x_2 x_3^{i_3}\partial_1,\ 1 \le i_1 \le a_1 - 2;\ 0 \le i_3 \le a_3 - 1; x_1^{a_1-1} x_2 \partial_1;$$
$$x_3^{a_3-1}\partial_2 + \frac{1}{a_3(a_3-1)} x_1^{a_1}\partial_3; x_2 x_3^{i_3}\partial_2 - x_3^{i_3+1}\partial_3,\ 1 \le i_3 \le a_3 - 2; x_2 x_3^{a_3-1}\partial_2;$$
$$x_1^{i_1} x_3^{a_3-1}\partial_2,\ 1 \le i_1 \le a_1 - 2; x_1^{i_1} x_2 x_3^{i_3}\partial_2 - x_1^{i_1} x_3^{i_3+1}\partial_3,\ 1 \le i_1 \le a_1 - 2; 1 \le i_3 \le a_3 - 1;$$
$$x_2^2\partial_2 - \frac{1}{a_1} x_1^{a_1}\partial_3; x_1^{a_1-1} x_3^{a_3-2}\partial_2; x_1^{a_1-1} x_2 \partial_2; x_1^{a_1}\partial_2 + 2x_3^2\partial_3; \left(\frac{1}{a_3} x_2^2 + x_3^{a_3-1}\right)\partial_3;$$
$$x_2 x_3^{i_3}\partial_3,\ 2 \le i_3 \le a_3 - 1; x_1^{i_1} x_2 x_3^{i_3}\partial_3,\ 1 \le i_1 \le a_1 - 2;\ 2 \le i_3 \le a_3 - 1; x_1^{a_1-2} x_3^{a_3-1}\partial_3;$$
$$x_1^{a_1-2} x_2 x_3 \partial_3; x_1^{a_1-1} x_3^{i_3}\partial_3,\ 1 \le i_3 \le a_3 - 2; x_1^{a_1-1} x_2 \partial_3; (\tfrac{1}{2} x_1^{a_1} + x_2 x_3)\partial_3.$$

Therefore,

$$\lambda^1(V) = 4a_1a_3 - 2a_1 - 3a_3 + 11.$$

In case of $a_1 = 2, a_2 = 2, a_3 \ge 3$, we obtain the basis of derivation represented by the following derivations which form a basis of $\mathrm{Der}A^1(V)$:

$$x_3^{a_3-1}\partial_1; x_2 x_3^{i_3}\partial_1,\ 1 \le i_3 \le a_3 - 1; x_2^2\partial_1; (1 + a_3)x_1\partial_1 + 2(a_3 - 1)x_2\partial_2 + 4x_3\partial_3;$$

$$x_1x_3\partial_1 + 2x_3^2\partial_3; x_1x_3^{i_3}\partial_1,\ 2 \le i_3 \le a_3 - 2; x_1x_2\partial_1; x_1^2\partial_1; x_3^{a_3-1}\partial_2; x_2^2\partial_3;$$
$$x_2x_3\partial_2 - x_3^2\partial_3; x_2x_3^{i_3}\partial_2,\ 2 \le i_3 \le a_3 - 1; x_2^2\partial_2; x_1x_3^{a_3-2}\partial_2; x_1x_2\partial_2; x_1^{a_1-2}\partial_3;$$
$$x_1^2\partial_2 + 2x_3^{a_3-2}\partial_3; x_3^{a_3-1}\partial_3; x_2x_3^{i_3}\partial_3,\ 1 \le i_3 \le a_3 - 1; x_1x_3^{i_3}\partial_3,\ 1 \le i_3 \le a_3 - 2; x_1x_2\partial_3.$$

Therefore, we have

$$\lambda^1(V) = 5a_3 + 7. \qquad \square$$

Proposition 3.8. *Let $(V, 0)$ be a fewnomial surface isolated singularity of type 3 which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_1$ ($a_1 \ge 2, a_2 \ge 2, a_3 \ge 2$) with weight type*

$$\Big(\frac{1 - a_3 + a_2a_3}{1 + a_1a_2a_3}, \frac{1 - a_1 + a_1a_3}{1 + a_1a_2a_3}, \frac{1 - a_2 + a_1a_2}{1 + a_1a_2a_3}; 1\Big).$$

Then

$$\lambda^1(V) = \begin{cases} 24, & a_1 = 2,\ a_2 = 2,\ a_3 = 2, \\ 3a_1a_2a_3 + 2(a_1 + a_2 + a_3) - 2(a_1a_2 + a_1a_3 + a_2a_3) + 11, & \text{otherwise.} \end{cases}$$

Proof. It is easy to see that the moduli algebra $A^1(V) = \mathbb{C}\{x_1, x_2, x_3\}/(f, m.J(f))$ has dimension $(a_1a_2a_3 + 3)$ and has a monomial basis of the form (cf. [1, Theorem 13.1])

$$\{x_1^{i_1}x_2^{i_2}x_3^{i_3}, 1 \le i_1 \le a_1 - 2; 1 \le i_2 \le a_2 - 1; 0 \le i_3 \le a_3 - 2; x_1^{i_1}, 1 \le i_1 \le a_1;$$
$$x_1^{i_1}x_3^{i_3}, 1 \le i_1 \le a_1 - 1; 1 \le i_3 \le a_3 - 2; x_2^{i_2}x_3^{i_3}, 0 \le i_2 \le a_2 - 2; 0 \le i_3 \le 2a_3 - 2, x_1x_3^{a_3-1};$$
$$x_1^{i_1}x_2^{i_2}, 0 \le i_1 \le a_1 - 3; a_2 \le i_2 \le 2a_2 - 1; x_1^{a_1-2}x_2^{a_2}; x_1^{a_1-1}x_2; x_2^{a_2-1}x_3^{i_3}, 0 \le i_3 \le a_3 - 1\}.$$

In order to compute a derivation D of $A^1(V)$, it suffices to indicate its values on the generators x_1, x_2, x_3, which can be written in terms of the basis. Thus, we can write

$$\begin{aligned} Dx_j = {} & \sum_{i_1=1}^{a_1-2}\sum_{i_2=1}^{a_2-1}\sum_{i_3=0}^{a_3-2} c^j_{i_1,i_2,i_3}x_1^{i_1}x_2^{i_2}x_3^{i_3} + \sum_{i_1=1}^{a_1} c^j_{i_1,0,0}x_1^{i_1} + \sum_{i_3=0}^{a_3-1} c^j_{0,a_2-1,i_3}x_2^{a_2-1}x_3^{i_3} \\ & + \sum_{i_2=0}^{a_2-2}\sum_{i_3=0}^{2a_3-1} c^j_{0,i_2,i_3}x_2^{i_2}x_3^{i_3} + \sum_{i_1=0}^{a_1-3}\sum_{i_3=a_2}^{2a_2-1} c^j_{i_1,i_2,0}x_1^{i_1}x_2^{i_2} + c^j_{1,0,a_3-1}x_1x_3^{a_3-1} \\ & + c^j_{a_1-2,a_2,0}x_1^{a_1-2}x_2^{a_2} + c^j_{a_1-1,1,0}x_1^{a_1-1}x_2 + \sum_{i_1=1}^{a_1-1}\sum_{i_3=1}^{a_3-2} c^j_{i_1,0,i_3}x_1^{i_1}x_3^{i_3}, \quad j = 1, 2, 3. \end{aligned}$$

Using the above derivations, we obtain the following description of the Lie algebras in question. The derivations represented by the following vector fields form a basis in $\mathrm{Der}A^1(V)$:

$$(1 - a_3 + a_2a_3)x_1^{i_1}\partial_1 + (1 - a_1 + a_1a_3)x_1^{i_1-1}x_2\partial_2 + (1 - a_2 + a_1a_2)x_1^{i_1-1}x_3\partial_3,$$
$$1 \le i_1 \le a_1 - 1;$$
$$(1 - a_3 + a_2a_3)x_1x_3^{i_3}\partial_1 + (1 - a_1 + a_1a_3)x_2x_3^{i_3-1}\partial_2 + (1 - a_2 + a_1a_2)x_3^{i_3+1}\partial_3,$$
$$1 \le i_3 \le a_3 - 2;$$

$$(1-a_3+a_2a_3)x_1x_2^{i_2}\partial_1+(1-a_1+a_1a_3)x_2^{i_2+1}\partial_2+(1-a_2+a_1a_2)x_2^{i_2-1}x_3\partial_3,$$

$$1\le i_2\le a_2-2;$$
$$x_2^{a_2-2}x_3^{a_3-1}\partial_1+\frac{a_1}{a_2}x_1^{a_1-1}x_3^{a_3-2}\partial_2+\frac{a_1}{a_3}x_1^{a_1-2}x_2^{a_2-1}\partial_3;x_2^{i_2}\partial_1,2a_2-2\le i_2\le 2a_2-1;$$
$$x_2^{i_2}x_3^{i_3}\partial_1,\ 0\le i_2\le a_2-2;a_3+1\le i_3\le 2a_3-1;$$
$$x_1^{i_1}x_2^{a_2}\partial_2-\frac{1}{a_3-1}x_1^{i_1+1},\ 0\le i_1\le a_1-3;$$
$$x_1^{i_1}x_2^{i_2}\partial_1,\ 1\le i_1\le a_1-3;a_2+1\le i_2\le 2a_2-1;x_1x_2^{a_2-1}\partial_1+\frac{1}{a_2(a_3-1)}x_1^{a_1}\partial_3;$$
$$x_2^{a_2-2}x_3^{a_3}\partial_1;x_2^{a_2-1}x_3^{a_3-1}\partial_1;x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_1,\ 2\le i_1\le a_1-2;1\le i_2\le a_2-1;1\le i_3\le a_3-2;$$
$$x_1x_2^{a_2-1}x_3^{i_3}\partial_1,\ 1\le i_3\le a_3-2;x_1^{i_1}x_2^{a_2}\partial_1,\ 1\le i_1\le a_1-2;$$
$$x_2^{i_2-1}x_3^{a_3}\partial_1+\frac{a_1}{a_3-1}x_1^{a_1-2}x_2^{i_2}x_3\partial_3,\ 1\le i_2\le a_2-2;$$
$$x_2^{i_2+(a_2-1)}\partial_1+a_1(a_1-1)x_1^{a_1-1}x_2^{i_2}\partial_3;\ 1\le i_2\le a_2-2;$$
$$x_1x_2^{i_2}x_3^{i_3}\partial_1+\frac{a_1-1}{a_3}x_2^{i_2}x_3^{i_3+1}\partial_3,\ 1\le i_2\le a_2-2;1\le i_3\le a_3-2;$$
$$x_1^{i_1}x_3^{i_3}\partial_1+a_1x_1^{i_1-1}x_3^{i_3+1}\partial_3,\ 2\le i_1\le a_1-1;1\le i_3\le a_3-2;$$
$$x_1^{i_1}x_2^{i_2}\partial_1-\frac{1}{a_3-1}x_1^{i_1-1}x_2^{i_2}x_3\partial_3,\ 2\le i_1\le a_1-2;1\le i_2\le a_2-1;$$
$$x_3^{i_3+(a_3-1)}\partial_1+a_1(a_2-1)x_1^{a_1-1}x_3^{i_3}\partial_3,\ 1\le i_3\le a_3-2;x_1^{a_1}\partial_1+a_1x_1^{a_1-1}x_3\partial_3;$$
$$x_3^{i_3}\partial_2,\ 2a_3-2\le i_3\le 2a_3-1;x_2^{i_2}x_3^{i_3}\partial_2,\ 1\le i_2\le a_2-2;\ a_3\le i_3\le 2a_3-1;$$
$$x_2^{i_2}x_3^{i_3}\partial_2+\frac{1}{a_3}x_2^{i_2-1}x_3^{i_3+1}\partial_3,\ 2\le i_2\le a_2-1;\ 1\le i_3\le a_3-2;$$
$$x_2^{i_2}x_3^{a_3-1}\partial_2+\frac{1}{a_3}x_2^{i_2-1}x_3^{a_3},\ 1\le i_2\le a_2-2;x_1^{a_1-1}x_2\partial_1-\frac{1}{a_3-1}x_1^{a_1-2}x_2x_3\partial_3;$$
$$x_1^{i_1}x_2^{i_2}\partial_2,\ 0\le i_1\le a_1-3;\ a_2+1\le i_2\le 2a_2-1;x_2^{a_2-2}x_3^{a_3-1}\partial_2;$$
$$x_1^{i_1}x_2x_3^{i_3}\partial_2-(a_2-1)x_1^{i_1}x_3^{i_3+1}\partial_3,\ 1\le i_1\le a_1-2;\ 1\le i_3\le a_3-2;$$
$$x_1x_3^{a_3-1}\partial_2+\frac{1}{a_3(a_3-1)}x_1^{a_1}\partial_3;x_2^{a_2-1}x_3^{i_3}\partial_1-\frac{a_1}{a_2}x_1^{a_1-1}x_3^{i_3}\partial_3,\ 1\le i_3\le a_3-2;$$
$$x_1^{i_1}x_2^{i_2}\partial_2+\frac{a_2}{a_3-1}x_1^{i_1}x_2^{i_2-1}x_3\partial_3,\ 1\le i_1\le a_1-2;\ 2\le i_2\le a_2-1;$$
$$x_1^{a_1-2}x_2^{a_2}\partial_2;(a_1x_1^{a_1-1}+x_3^{a_3})\partial_3;x_1x_3^{a_3-1}\partial_1-\frac{a_1(a_1-1)}{a_3}x_1^{a_1-1}x_2\partial_3;$$
$$x_1^{a_1-1}x_2\partial_2-(a_2-1)x_1^{a_1-1}x_3\partial_3;x_1^{a_1}\partial_2-\frac{a_2}{a_3}x_2^{a_2-2}x_3^2\partial_3;$$
$$x_2^{i_2}x_3^{i_3}\partial_3,\ 0\le i_2\le a_2-2;\ a_3+1\le i_3\le 2a_3-1;$$
$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2,\ 1\le i_1\le a_1-2;\ 2\le i_2\le a_2-1;1\le i_3\le a_3-2;x_2^{a_2-2}x_3^{a_3}\partial_3;$$

$$\Big(\frac{1}{a_2}x_1^{a_1}+x_2^{a_2-1}x_3\Big)\partial_3;\, x_2^{a_2-1}x_3^{i_3}\partial_3,\ 2\le i_3\le a_3-1;\, x_2^{i_2}\partial_3,\ a_2+1\le i_2\le 2a_2-1;$$
$$x_1^{i_1}x_2^{i_2}\partial_3,\ 1\le i_1\le a_1-3;\ a_2+1\le i_2\le 2a_2-1;\, x_1^{a_1-2}x_2^{a_2}\partial_3;\, x_1^{a_1-2}x_2^{a_2-1}x_3\partial_3;$$
$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_3,\ 1\le i_1\le a_1-2;\ 1\le i_2\le a_2-1; 2\le i_3\le a_3-2;\, (a_3x_1x_3^{a_3-1}+x_2^{a_2})\partial_3.$$

Therefore,

$$\lambda^1(V)=3a_1a_2a_3+2(a_1+a_2+a_3)-2(a_1a_2+a_1a_3+a_2a_3)+11.$$

In case of $a_1=2, a_2=2, a_3=2$, we obtain the basis of derivation represented by the following derivations which form a basis of $\mathrm{Der}A^1(V)$:

$$x_3\partial_1+x_1\partial_2+x_2\partial_3;\, x_3^2\partial_1;\, x_3^3\partial_1;\, x_2\partial_1+x_3\partial_2+x_1\partial_3;\, x_2x_3\partial_1;\, x_2^2\partial_1;\, x_1\partial_1+x_2\partial_2+x_3\partial_3;$$
$$x_1^2\partial_1;\, x_3^2\partial_2;\, x_3^3\partial_2;\, x_2x_3\partial_2;\, x_2^2\partial_2;\, x_1x_3\partial_2;\, x_1x_2\partial_2;\, x_1^2\partial_2;\, x_3^2\partial_3;\, x_3^3\partial_3;\, x_2x_3\partial_3;\, x_2^2\partial_3;\, x_1x_3\partial_3;$$
$$x_1x_3\partial_1;\, x_1x_2\partial_1;\, x_1x_2\partial_3;\, x_1^2\partial_3.$$

Therefore,

$$\lambda^1(V)=24. \qquad \square$$

Proposition 3.9. *Let $(V,0)$ be a fewnomial surface isolated singularity of type 4 which is defined by $f=x_1^{a_1}+x_2^{a_2}+x_3^{a_3}x_2$ $(a_1\ge 3, a_2\ge 3, a_3\ge 2)$ with weight type $(1/a_1, 1/a_2, (a_2-1)/a_2a_3; 1)$. Then*

$$\lambda^1(V)=\begin{cases}5a_1a_2-a_1-7a_2+15, & a_1\ge 3,\ a_2\ge 3,\ a_3=3,\\ 3a_1a_2a_3-4a_1a_2-3a_1a_3-4a_2a_3+8a_1+5a_2+5a_3-1, & \textit{otherwise}.\end{cases}$$

Proof. It is easy to see that the moduli algebra $A^1(V)=\mathbb{C}\{x_1,x_2,x_3\}/(f, m.J(f))$ has dimension $(a_1a_2a_3-a_1a_2-a_2a_3+a_1+a_2+2)$ and has a monomial basis of the form (cf. [1, Theorem 13.1])

$$\{x_1^{i_1}x_2^{i_2}x_3^{i_3}, 0\le i_1\le a_1-2; 0\le i_2\le a_2-2; 0\le i_3\le a_3-2;\, x_3^{i_3}, a_3-1\le i_3\le 2a_3-2;$$
$$x_1^{a_1-1};\, x_2^{a_2-1};\, x_2x_3^{a_3-1};\, x_1^{i_1}x_3^{i_3}, 1\le i_1\le a_1-2; a_3-1\le i_3\le 2a_3-2\}.$$

In order to compute a derivation D of $A^1(V)$, it suffices to indicate its values on the generators x_1, x_2, x_3, which can be written in terms of the basis. Thus, we can write

$$\begin{aligned}Dx_j=&\sum_{i_1=0}^{a_1-2}\sum_{i_2=0}^{a_2-2}\sum_{i_3=0}^{a_3-2}c^j_{i_1,i_2,i_3}x_1^{i_1}x_2^{i_2}x_3^{i_3}+\sum_{i_3=a_3-1}^{2a_3-2}c^j_{0,0,i_3}x_3^{i_3}\\&+\sum_{i_1=1}^{a_1-2}\sum_{i_3=a_3-1}^{2a_3-2}c^j_{i_1,0,i_3}x_1^{i_1}x_3^{i_3}+c^j_{a_1-1,0,0}x_1^{a_1-1}\\&+c^j_{0,a_2-1,0}x_2^{a_2-1}+c^j_{0,1,a_3-1}x_2x_3^{a_3-1},\quad j=1,2,3.\end{aligned}$$

Using the above derivation, we obtain the basis of derivation represented by the following derivations which form a basis of $\mathrm{Der}A^1(V)$:

$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_1,\ 1\le i_1\le a_1-2, 1\le i_2\le a_2-2, 0\le i_3\le a_3-2;$$

$$x_1^{i_1}x_3^{i_3}\partial_1,\ 1 \le i_1 \le a_1-2, 1 \le i_3 \le 2a_3-2; x_2x_3^{a_3-1}\partial_1; x_3^{2a_3-2}\partial_1;$$
$$x_1^{i_1}\partial_1,\ 1 \le i_1 \le a_1-1; (a_2x_2^{a_2-1}+x_3^{a_3})\partial_1;$$
$$x_3^{a_3-1}\partial_2+\frac{a_2}{a_3-1}x_2^{a_2-2}\partial_3; x_3^{i_3}\partial_2,\ a_3 \le i_3 \le 2a_3-2;$$
$$x_2x_3^{i_3-1}\partial_2+\frac{a_1}{a_3}x_3^{i_3}\partial_3,\ 1 \le i_3 \le a_3-1;$$
$$x_2x_3^{a_3-1}\partial_2; x_2^{i_2}x_3^{i_3}\partial_2,\ 2 \le i_2 \le a_2-2,\ 0 \le i_3 \le a_3-2;$$
$$x_2^{a_2-1}\partial_2; x_1^{i_1}x_3^{i_3}\partial_2,\ 1 \le i_1 \le a_1-2,\ a_3-1 \le i_3 \le 2a_3-2;$$
$$x_1x_2x_3^{i_3-1}\partial_2+\frac{a_1}{a_3}x_1x_3^{i_3}\partial_3,\ 1 \le i_3 \le a_3-2;$$
$$x_1x_2x_3^{a_3-2}\partial_2; x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2,\ 1 \le i_1 \le a_1-2, 2 \le i_2 \le a_2-2, 0 \le i_3 \le a_3-2; x_1^{a_1-1}\partial_2;$$
$$x_3^{i_3}\partial_3,\ a_3 \le i_3 \le 2a_3-2; x_2^{i_2}x_3^{i_3}\partial_3,\ 1 \le i_2 \le a_2-2,\ 1 \le i_3 \le a_3-2; x_2x_3^{a_3-1}\partial_3; x_2^{a_2-1}\partial_3;$$
$$x_1^{i_1}x_3^{i_3}\partial_3,\ 1 \le i_1 \le a_1-2,\ a_3-1 \le i_3 \le 2a_3-2; x_1^{a_1-1}\partial_3; x_1^{a_1-2}x_2^{a_2-2}\partial_3;$$
$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_3,\ 1 \le i_1 \le a_1-2, 1 \le i_2 \le a_2-2, 1 \le i_3 \le a_3-2.$$

Therefore,

$$\lambda^1(V) = 3a_1a_2a_3 - 4a_1a_2 - 3a_1a_3 - 4a_2a_3 + 8a_1 + 5a_2 + 5a_3 - 1.$$

In case of $a_1 \ge 3, a_2 \ge 3, a_3 = 3$, we obtain the basis of derivation represented by the following derivations which form a basis of $\mathrm{Der}A^1(V)$:

$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_1,\ 1 \le i_1 \le a_1-2, 1 \le i_2 \le a_2-2, 0 \le i_3 \le 1;$$
$$x_1^{i_1}x_3^{i_3}\partial_1,\ 1 \le i_1 \le a_1-2, 1 \le i_3 \le 4; x_2x_3^2\partial_1; x_3^4\partial_1;$$
$$x_1^{i_1}\partial_1,\ 1 \le i_1 \le a_1-1; (a_2x_2^{a_2-1}+x_3^3)\partial_1; x_3^2\partial_2+\frac{a_2}{2}x_2^{a_2-2}\partial_3; x_3^{i_3}\partial_2,\ 3 \le i_3 \le 4;$$
$$x_2x_3^{i_3-1}\partial_2+\frac{a_1}{a_3}x_3^{i_3}\partial_3,\ 1 \le i_3 \le 2; x_2x_3^2\partial_2; x_2^{i_2}x_3^{i_3}\partial_2,\ 2 \le i_2 \le a_2-2,\ 0 \le i_3 \le 1;$$
$$x_2^{a_2-1}\partial_2; x_1^{i_1}x_3^{i_3}\partial_2,\ 1 \le i_1 \le a_1-2,\ 3 \le i_3 \le 4; x_1x_2\partial_2+\frac{a_1}{3}x_1x_3\partial_3;$$
$$x_1x_2x_3\partial_2; x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2,\ 1 \le i_1 \le a_1-2, 2 \le i_2 \le a_2-2, 0 \le i_3 \le 1; x_1^{a_1-1}\partial_2;$$
$$x_3^{i_3}\partial_3,\ 3 \le i_3 \le 4; x_2^{i_2}x_3\partial_3,\ 1 \le i_2 \le a_2-2,\ x_2x_3^2\partial_3; x_2^{a_2-1}\partial_3;$$
$$x_1^{i_1}x_3^{i_3}\partial_3,\ 1 \le i_1 \le a_1-2,\ 2 \le i_3 \le 4; x_1^{a_1-1}\partial_3; x_1^{a_1-2}x_2^{a_2-2}\partial_3;$$
$$x_1^{i_1}x_2^{i_2}x_3\partial_3,\ 1 \le i_1 \le a_1-2, 1 \le i_2 \le a_2-2.$$

Therefore,

$$\lambda^1(V) = 5a_1a_2 - a_1 - 7a_2 + 15. \qquad \square$$

PROPOSITION 3.10. *Let* $(V, 0)$ *be a fewnomial surface isolated singularity of type 5 which is defined by* $f = x_1^{a_1}x_2 + x_2^{a_2}x_1 + x_3^{a_3}$ $(a_1 \ge 2, a_2 \ge 2, a_3 \ge 3)$ *with weight type* $((a_2-1)/(a_1a_2-1), (a_1-1)/(a_1a_2-1), 1/a_3; 1)$. *Then*

$$\lambda^1(V) = \begin{cases} 4a_2a_3 - 6a_2 + 12, & a_1 = 2,\ a_2 \ge 2,\ a_3 \ge 3, \\ 3a_1a_2a_3 - 4a_1a_2 - 2a_2a_3 - 2a_1a_3 + 2a_1 + 2a_2 + 6a_3 + 6, & \text{otherwise}. \end{cases}$$

Proof. It is easy to see that the moduli algebra $A^1(V) = \mathbb{C}\{x_1, x_2, x_3\}/(f, m.J(f))$ has dimension $(a_1a_2a_3 - a_1a_2 + 3)$ and has a monomial basis of the form [1]

$$\{x_1^{i_1}x_2^{i_2}x_3^{i_3}, 1 \le i_1 \le a_1 - 2; 1 \le i_2 \le a_2 - 1; 0 \le i_3 \le a_3 - 2; x_3^{a_3-1}; x_1^{a_1-1}x_2; x_1^{a_1};$$
$$x_1^{i_1}x_3^{i_3}, 1 \le i_1 \le a_1 - 1; 0 \le i_3 \le a_3 - 2; x_2^{i_2}x_3^{i_3}, 0 \le i_2 \le 2a_2 - 2; 0 \le i_3 \le a_3 - 2\}.$$

In order to compute a derivation D of $A^1(V)$, it suffices to indicate its values on the generators x_1, x_2, x_3, which can be written in terms of the basis. Thus, we can write

$$Dx_j = \sum_{i_1=1}^{a_1-2}\sum_{i_2=1}^{a_2-1}\sum_{i_3=0}^{a_3-2} c^j_{i_1,i_2,i_3}x_1^{i_1}x_2^{i_2}x_3^{i_3} + \sum_{i_1=1}^{a_1-1}\sum_{i_3=0}^{a_3-2} c^j_{i_1,0,i_3}x_1^{i_1}x_3^{i_3} + \sum_{i_2=0}^{2a_2-2}\sum_{i_3=0}^{a_3-2} c^j_{0,i_2,i_3}x_2^{i_2}x_3^{i_3}$$
$$+ c^j_{0,0,a_3-1}x_3^{a_3-1} + c^j_{a_1-1,1,0}x_1^{a_1-1}x_2 + c^j_{a_1,0,0}x_1^{a_1}, \quad j = 1, 2, 3.$$

Using the above derivations, we obtain the following description of the Lie algebras in question. The derivations represented by the following vector fields form a basis in $\mathrm{Der}A^1(V)$:

$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_1,\ 2 \le i_1 \le a_1 - 2, 1 \le i_2 \le a_2 - 1, 0 \le i_3 \le a_3 - 2;$$
$$x_3^{a_3-1}\partial_1; x_2^{a_2-2}x_3^{i_3}\partial_1 + \frac{a_1}{a_2}x_1^{a_1-2}x_3^{i_3}\partial_2,\ 0 \le i_3 \le a_3 - 2;$$
$$x_2^{a_2-1}x_3^{i_3}\partial_1 + \frac{a_1}{a_2}x_1^{a_1-2}x_2x_3^{i_3}\partial_2,\ 0 \le i_3 \le a_3 - 3; x_2^{a_2-1}x_3^{a_3-2}\partial_1;$$
$$x_2^{i_2}x_3^{i_3}\partial_1,\ a_2 \le i_2 \le 2a_2 - 2; 0 \le i_3 \le a_3 - 2;$$
$$x_1^{i_1}x_3^{i_3}\partial_1 + \frac{a_1-1}{a_2-1}x_1^{i_1-1}x_2x_3^{i_3}\partial_2,\ 1 \le i_1 \le a_1 - 2; 0 \le i_3 \le a_3 - 2;$$
$$x_1^{a_1-1}x_3^{i_3}\partial_1 + \frac{a_1-1}{a_2-1}x_1^{a_1-2}x_2x_3^{i_3}\partial_2,\ 0 \le i_3 \le a_3 - 3;$$
$$x_1x_2^{i_2}x_3^{i_3}\partial_1 + \frac{a_1-1}{a_2-1}x_2^{i_2+1}x_3^{i_3}\partial_2,\ 1 \le i_2 \le a_2 - 3; 0 \le i_3 \le a_3 - 2;$$
$$x_1x_2^{a_2-2}x_3^{i_3}\partial_1 + \frac{a_1}{a_2}x_1^{a_1-1}x_3^{i_3}\partial_2,\ 0 \le i_3 \le a_3 - 3; x_1x_2^{a_2-2}x_3^{a_3-2}\partial_1;$$
$$x_1x_2^{a_2-1}x_3^{i_3}\partial_1,\ 0 \le i_3 \le a_3 - 2; x_1^{a_1-1}x_3^{a_3-2}\partial_1; x_1^{a_1-1}x_2\partial_1; x_1^{a_1}\partial_1;$$
$$\Big(-\frac{a_1(a_2-1)}{a_2(a_1-1)}x_1^{a_1-1}x_3^{i_3} + x_2^{a_2-1}x_3^{i_3}\Big)\partial_2,\ 0 \le i_3 \le a_3 - 3; x_2^{a_2-1}x_3^{a_3-2}\partial_2;$$
$$x_2^{i_2}x_3^{i_3}\partial_2,\ a_2 \le i_2 \le 2a_2 - 2; 0 \le i_3 \le a_3 - 2; x_1^{a_1-1}x_3^{a_3-2}\partial_2; x_1^{a_1-1}x_2\partial_2;$$
$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2,\ 1 \le i_1 \le a_1 - 2, 2 \le i_2 \le a_2 - 1, 0 \le i_3 \le a_3 - 2;$$
$$x_1^{a_1}\partial_2; x_1^{a_1-2}x_2x_3^{a_3-2}\partial_2; x_3^{i_3}\partial_3,\ 1 \le i_3 \le a_3 - 1; (a_1x_1^{a_1-1}x_2 + x_2^{a_2})\partial_3;$$
$$\Big(\frac{1}{a_2}x_1^{a_1} + x_1x_2^{a_2-1}\Big)\partial_3; x_1^{a_1-1}x_3^{i_3}\partial_3,\ 1 \le i_3 \le a_3 - 2; x_2^{2a_2-2}x_3^{i_3}\partial_3,\ 0 \le i_3 \le a_3 - 2;$$
$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_3,\ 1 \le i_1 \le a_1 - 2, 0 \le i_2 \le a_2 - 1, 1 \le i_3 \le a_3 - 2;$$
$$x_2^{i_2}x_3^{i_3}\partial_3,\ 1 \le i_2 \le 2a_2 - 3; 1 \le i_3 \le a_3 - 2.$$

Therefore,

$$\lambda^1(V) = 3a_1a_2a_3 - 4a_1a_2 - 2a_2a_3 - 2a_1a_3 + 2a_1 + 2a_2 + 6a_3 + 6.$$

In case of $a_1 = 2, a_2 \geq 2, a_3 \geq 3$, we obtain the basis of derivation represented by the following derivations which form a basis of $\mathrm{Der}A^1(V)$:

$$\begin{gathered}
x_3^{a_3-1}\partial_1; x_2^{a_2-1}x_3^{a_3-2}\partial_1 - \frac{2}{a_2-1}x_2x_3^{a_3-2}\partial_2;\\
x_2^{i_2}x_3^{i_3}\partial_1 - \frac{2}{a_2-1}x_2^{i_2-2}x_3^{i_3}\partial_2,\ a_2 \leq i_2 \leq 2a_2-3, 0 \leq i_3 \leq a_3-2;\\
x_2^{2a_2-2}x_3^{i_3}\partial_1,\ 0 \leq i_3 \leq a_3-2; x_1x_3^{i_3}\partial_1 + \frac{1}{a_2-1}x_2x_3^{i_3}\partial_2, 0 \leq i_3 \leq a_3-2;\\
x_1x_2\partial_1 + \frac{1}{a_2-1}x_1x_2^2\partial_1; x_1^2\partial_1; x_3^{a_3-1}\partial_2; x_2^{a_2-1}x_3^{a_3-2}\partial_2; x_1x_3^{a_3-2}\partial_2;\\
\Big(-\frac{2(a_2-1)}{a_2}x_1x_3^{i_3} + x_2^{a_2-1}x_3^{i_3}\Big)\partial_2,\ 0 \leq i_3 \leq a_3-3; x_1x_2\partial_2; x_1^2\partial_2;\\
x_2^{i_2}x_3^{i_3}\partial_2,\ a_2 \leq i_2 \leq 2a_2-2; 0 \leq i_3 \leq a_3-2; x_3^{i_3}\partial_3,\ 1 \leq i_3 \leq a_3-1;\\
x_2^{i_2}x_3^{i_3}\partial_3,\ 1 \leq i_2 \leq 2a_2-2; 1 \leq i_3 \leq a_3-2; x_2^{2a_2-2}\partial_3; x_1^2\partial_3;\\
x_1x_3^{i_3}\partial_3,\ 1 \leq i_3 \leq a_3-2; (2x_1x_2 + x_2^{a_2})\partial_3.
\end{gathered}$$

Therefore,

$$\lambda^1(V) = 4a_2a_3 - 6a_2 + 12. \qquad \square$$

4. Proof of Theorem A

PROOF. Let $f \in \mathbb{C}\{x_1, x_2\}$ be a weighted homogeneous fewnomial isolated singularity. Then f can be classified into the following three types:

Type A. $x_1^{a_1} + x_2^{a_2}$;
Type B. $x_1^{a_1}x_2 + x_2^{a_2}$;
Type C. $x_1^{a_1}x_2 + x_2^{a_2}x_1$.

It is easy to see from Propositions 2.11, 2.12, 2.13, 2.14, 2.15, 2.16, 3.1, 3.2 and 3.3 that the conjecture $\lambda^{(k+1)}(V) > \lambda^k(V)$, $k = 0, 1$, holds. Hence, Theorem A is proved. $\square$

5. Proof of Theorem B

PROOF. Let $f \in \mathbb{C}\{x_1, x_2, x_3\}$ be a weighted homogeneous fewnomial isolated singularity. Then f can be classified into the following five types:

Type 1. $x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$;
Type 2. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}$;
Type 3. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_1$;
Type 4. $x_1^{a_1} + x_2^{a_2} + x_3^{a_3}x_2$;
Type 5. $x_1^{a_1}x_2 + x_2^{a_2}x_1 + x_3^{a_3}$.

It is easy to see from Propositions 2.11, 2.17, 2.18, 3.4, 3.5, 3.6, 3.7, 3.8, 3.9 and 3.10 that the conjecture $\lambda^{(k+1)}(V) > \lambda^{k}(V)$, $k = 0$, holds. Hence, Theorem B is proved. □

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