Algebras and Representation Theory (2021) 24:1101–1124
https://doi.org/10.1007/s10468-020-09981-x

On the Generalized Cartan Matrices Arising from *k*-th Yau Algebras of Isolated Hypersurface Singularities

Naveed Hussain[1,2] · **Stephen S.-T. Yau**[2] · **Huaiqing Zuo**[2]

Received: 30 July 2019 / Accepted: 7 July 2020 / Published online: 18 July 2020

Abstract

Let $(V, 0)$ be an isolated hypersurface singularity defined by the holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. The k-th Yau algebra $L^k(V)$ is defined to be the Lie algebra of derivations of the k-th moduli algebra $A^k(V) := \mathcal{O}_n/(f, m^k J(f))$, where $k \geq 0$, m is the maximal ideal of $\mathcal{O}_n$. I.e., $L^k(V) := \mathrm{Der}(A^k(V), A^k(V))$. These new series of derivation Lie algebras are quite subtle invariants since they capture enough information about the complexity of singularities. In this paper we formulate a conjecture for the complete characterization of ADE singularities by using generalized Cartan matrix $C^k(V)$ associated to k-th Yau algebras $L^k(V)$, $k \geq 1$. In this paper, we provide evidence for the conjecture and give a new complete characterization for ADE singularities. Furthermore, we compute their other various invariants that arising from the 1-st Yau algebra $L^1(V)$.

Keywords Isolated singularity · Lie algebra · Generalized Cartan matrix

Mathematics Subject Classification (2010) 14B05 · 32S05

Dedicated to Professor Stephen Halperin on the occasion of his 78th birthday

Presented by: Peter Littelmann

Both Yau and Zuo are supported by NSFC Grants 11961141005 and 11531007. Zuo is supported by NSFC Grant 11771231. Yau is supported by Tsinghua University start-up fund and Tsinghua University Education Foundation fund (042202008).

✉ Stephen S.-T. Yau
yau@uic.edu

Naveed Hussain
naveed1573@gmail.com

Huaiqing Zuo
hqzuo@mail.tsinghua.edu.cn

[1] Present address: School of Data Sciences, Huashang College Guangdong University of Finance and Economics, Guangzhou, Guangdong 511300, People's Republic of China

[2] Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, People's Republic of China

1 Introduction

Recall that the class of simple (Du Val, ADE, rational double point) singularities, consist of two series $A_k : \{x_1^{k+1} + x_2^2 = 0\} \subset \mathbb{C}^2, k \geq 1$, $D_k : \{x_1^2 x_2 + x_2^{k-1} = 0\} \subset \mathbb{C}^2, k \geq 4$, and three exceptional singularities E_6, E_7, E_8 defined in $\mathbb{C}^2$ by polynomials $x_1^3 + x_2^4, x_1^3 + x_1 x_2^3, x_1^3 + x_2^5$, respectively. ADE singularities have been studied since antiquity, and there are many number of ways of characterising them (see Durfee [10]). They appear in many areas of geometry, algebraic geometry, singularity theory, group theory, etc.

Recall that the Lie algebras L is called solvable (resp. nilpotent) if the derived series: $L^{(0)} = L, L^{(1)} = [L, L], L^{(i)} = [L^{(i-1)}, L^{(i-1)}], i = 2, 3, \ldots$ (resp. the lower central series: $L_0 = L, L_1 = [L, L], L_i = [L, L_{(i-1)}], i = 2, 3, \ldots$) terminates. Every nilpotent Lie algebra is solvable. It is well-known that finite dimensional Lie algebras are semi-direct product of the semi-simple Lie algebras and solvable Lie algebras. Simple Lie algebras have been well understood, but not the solvable (nilpotent) Lie algebras. The problem of classifying nilpotent Lie algebra was studied first time in 1891 by a student of Engle, Umlauf, who gave the complete list over $\mathbb{C}$ up to the dimension 6 and certain continuous families at the dimensions 7, 8 and 9 [27]. Bratzlavsky [4] and Gabriel [12, 13] introduced the root systems to study nilpotent Lie algebras. The concept of root system constitutes an important step in the classification of nilpotent Lie algebras. By using these root systems, Santharoubane [24] established a link between the nilpotent Lie algebras and the Kac-Mody Lie algebras (which are infinite dimensional version generalization of the semi-simple Lie algebra).

Brieskorn [5] gave a beautiful connection between simple Lie algebras and simple singularities. Thus it is extremely important to establish connection between singularities and solvable (nilpotent) Lie algebras. Recently, in [18, 19] and [7], the authors introduced many new solvable (nilpotent) Lie algebras to isolated hypersurface singularities. There are two different ways to associate Lie algebras to isolated hypersurface singularities.

On the one hand, in [7], a new natural connection between the set of complex analytic isolated hypersurface singularities and the set of finite dimensional solvable (nilpotent) Lie algebras has been constructed. For an isolated hypersurface singularity $(V, 0)$ defined by the holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, the new Lie algebra $L^*(V) := \mathrm{Der}(A^*(V), A^*(V))$, was defined to be the Lie algebra of derivations of the Artinian algebra

$$A^*(V) = \mathcal{O}_n / \left(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}, \mathrm{Det}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j=1,\cdots,n} \right),$$

and $\lambda^*(V)$ is the dimension of $L^*(V)$. In [7], we have used $L^*(V)$ to distinguish ADE singularities. Furthermore, the authors have proven Torelli-type theorems for some simple elliptic singularities. In fact, this new Lie algebra $L^*(V)$ are subtle invariants of isolated hypersurface singularities. It is natural to ask that whether we can distinguish singularities by only using part of the information of $L^*(V)$. In [20], we studied generalized Cartan matrices of the new Lie algebra $L^*(V)$ for simple hypersurface singularities and simple elliptic singularities. We introduced many other numerical invariants, namely, dimension of the maximal nilpotent subalgebras (i.e., nilradical of nilpotent Lie algebra) $g(V)$ of $L^*(V)$; dimension of the maximal torus of $g(V)$, etc. We have proven that the generalized Cartan matrix of $L^*(V)$ can be used to characterize the ADE singularities [20].

On the other hand, for any isolated hypersurface singularity $(V, 0) \subset (\mathbb{C}^n, 0)$ where $V = V(f) = \{f = 0\}$, motivated from the Mather-Yau theorem [22], the second author introduced the Lie algebra of derivations of moduli algebra $A(V) := \mathcal{O}_n / \left(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right)$, i.e., $L(V) := \mathrm{Der}(A(V), A(V))$. It is known that $L(V)$ is a finite dimensional solvable Lie algebra [28, 29]. $L(V)$ is called the Yau algebra of V (its dimension $\lambda(V)$ is called Yau bumber) in [32] and [11] in order to distinguish from Lie algebras of other types appearing in singularity theory [1, 3]. The Yau algerbas play an important role in singularities. Yau and his collaborators have been systematically studying the Lie algebras of isolated hypersurface singularities since early eighties (see, e.g., [2, 3, 6, 8, 9, 16, 17, 21, 23, 25, 26, 28, 31, 33, 34]).

The Mather-Yau theorem was slightly generalized in ([14], Theorem 2.26) (without assuming isolated singularity):

Theorem 1.1 *Let $f, g \in m \subset \mathcal{O}_n$. The following are equivalent:*
1) $(V(f), 0) \cong (V(g), 0)$;
2) For all $k \geq 0$, $\mathcal{O}_n/(f, m^k J(f)) \cong \mathcal{O}_n/(g, m^k J(g))$ as $\mathbb{C}$-algebra;
3) There is some $k \geq 0$ such that $\mathcal{O}_n/(f, m^k J(f)) \cong \mathcal{O}_n/(g, m^k J(g))$ as $\mathbb{C}$-algebra, where $J(f) = \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right)$.

In particular, if $k = 0$ and $k = 1$ above, then the claim of the equivalence of 1) and 3) is exactly the Mather-Yau theorem [22].

Motivated from Theorem 1.1, in [18, 19], we introduced the new series of k-th Yau algebras $L^k(V)$ (or $L^k((V, 0))$) which are defined to be the Lie algebra of derivations of the moduli algebra $A^k(V) = \mathcal{O}_n/(f, m^k J(f))$, $k \geq 0$, where m is the maximal ideal. I.e., $L^k(V) := \text{Der}(A^k(V), A^k(V))$. Its dimension is denoted as $\lambda^k(V)$ (or $\lambda^k((V, 0))$). This series of integers $\lambda^k(V)$ are new numerical analytic invariants of singularities. It is natural to call it k-th Yau number. In particular, when $k = 0$, those are exactly the previous Yau algebra and Yau number. I.e., $L(V) = L^0(V)$, $\lambda^0(V) = \lambda(V)$. In [28], Yau observed that the Yau algebra for the one-parameter family of simple elliptic singularities $\tilde{E}_6$ is constant. It turns out that the 1-st Yau algebra $L^1(V)$ is also constant for the family of simple elliptic singularities $\tilde{E}_6$ (see [15]). However, Torelli-type theorem for $L^k(V)$ for all $k > 1$ do hold on $\tilde{E}_6$([15]). In general, the invariant $L^k(V)$, $k \geq 1$ are more subtle than Yau algebra (i.e., $L^0(V)$). We have many reasons to believe that these new Lie algebras $L^k(V)$ will play an important role in the study of singularities.

Since $L^k(V)$ potentially contains all information of singularities, it is natural to ask what kind of partial information of these k-th Yau algebras can be used to distinguish singularities. In this paper we shall answer this question partially. We introduce many other numerical invariants to k-th Yau algebra, namely, dimension of the maximal nilpotent subalgebras (i.e., nilradical of nilpotent Lie algebra) $g(V)$ of $L(V)$; dimension of maximal torus of $g(V)$; generalized Cartan matrix $C^k(V)$, $k \geq 0$ (see Definition 2.5); type and nilpotency of singularity.

The following proposition is easy to be obtained from results in [2].

Proposition 1.1 *The generalized Cartan matrix $C^0(V)$ that arising from Yau algebra $L(V)$ characterizes the ADE singularities except for the pair A_6 and D_5. I.e., if X and Y are two ADE singularities except for the pair A_6 and D_5, then $C^0(X) = C^0(Y)$ if and only if X and Y are analytically isomorphic.*

Proof It is an immediate corollary of Proposition 2.1 – Proposition 2.5. □

Based on Proposition 1.1, it is natural for us to propose the following conjecture.

Conjecture 1.1 For every $k \geq 1$, the generalized Cartan matrices $C^k(V)$ that arising from k-th Yau algebra $L^k(V)$ characterizes the ADE singularities completely. I.e., if X and Y are two ADE singularities, then $C^k(X) = C^k(Y)$ if and only if X and Y are analytically isomorphic.

Remark 1.1 When $k = 0$, since there exists one pair A_6 and D_5 which have same generalized Cartan matrix (see Proposition 2.1 and Proposition 2.2), so we need $k \geq 1$ in Conjectue 1.1. Moreover, we cannot expect to distinguish more general singularities by using the generalized Cartan matrix. An obvious reason is that the generalized Cartan matrix is basically discrete invariant. It cannot characterize continuous family (e.g. simple elliptic singularities $\tilde{E}_6$). It may be interesting to study the family of singularities which keeps generalized Cartan matrix constant.

In this paper we shall study the 1-st Yau algebra $L^1(V)$ of ADE singularities. We compute different numerical invariants such as the dimension of the maximal torus of $g(V)$; type and nilpotency of singularity and generalized Cartan matrix $C^1(V)$ and so on. Our main result as follow provides evidence for Conjecture 1.1 and gives a new characterization of ADE singularities which extends the results in [10].

Main Theorem *The generalized Cartan matrix $C^1(V)$ that arising from 1-st Yau algebra $L^1(V)$ characterizes the ADE singularities. I.e., if X and Y are two ADE singularities, then $C^1(X) = C^1(Y)$ if and only if X and Y are analytically isomorphic.*

Remark 1.2 The ADE singularities in the above main theorem include: $A_k : \{x_1^{k+1} + x_2^2 = 0\} \subset \mathbb{C}^2, k \geq 1$, $D_k : \{x_1^2 x_2 + x_2^{k-1} = 0\} \subset \mathbb{C}^2, k \geq 4$, and three exceptional singularities E_6, E_7, E_8 defined in $\mathbb{C}^2$ by polynomials $x_1^3 + x_2^4, x_1^3 + x_1 x_2^3, x_1^3 + x_2^5$ respectively.

2 Preliminaries

2.1 Isolated hypersurface singularities

Let $\mathcal{O}_n$ be the algebra of germs of holomorphic functions at the origin of $\mathbb{C}^n$. Obviously, $\mathcal{O}_n$ can be naturally identified with the algebra of convergent power series in n indeterminates with complex coefficients. For $f \in \mathcal{O}_n$, we denote by $V = V(f)$ (or $(V, 0)$) the germ at the origin of $\mathbb{C}^n$ of hypersurface $\{f = 0\} \subset \mathbb{C}^n$. We say that V is a germ of isolated hypersurface singularity if the origin is an isolated zero of the gradient of f. The local (function) algebra of V is defined as the (commutative associative) algebra $F(V) \cong \mathcal{O}_n/(f)$, where (f) is the principal ideal generated by the germ of f at the origin. According to Hilbert's Nullstellensatz for an isolated singularity $V = V(f) = \{f = 0\}$ the factor-algebra $A(V) = \mathcal{O}_n/\left(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right)$ is finite dimensional. This factor-algebra is called the moduli algebra of V and its dimension $\tau(V)$ is called Tjurina number. The well-known Mather-Yau theorem states that

Theorem 2.1 [22] *The analytic isomorphism type of an isolated hypersurface singularity is determined by the isomorphism class of its moduli algebras i.e.,*

$$(V_1, 0) \cong (V_2, 0) \Longleftrightarrow A(V_1) \cong A(V_2).$$

2.2 Yau Algebra

Recall that a derivation of commutative associative algebra A is defined as a linear endomorphism D of A satisfying the Leibniz rule: $D(ab) = D(a)b + aD(b)$. Thus for such an algebra A one can consider the Lie algebra of its derivations $\mathrm{Der}(A, A)$ with the bracket defined by the commutator of linear endomorphisms.

Definition 2.1 Let $V = V(f) = \{f = 0\}$ be a germ of an isolated hypersurface singularity at the origin in $\mathbb{C}^n$. Let $A(V)$ be the moduli algebra and $L(V) := \mathrm{Der}(A(V), A(V))$. Yu [32] call $L(V)$ the Yau algebra of V. The dimension of $L(V)$ is called the Yau number by Elashvili and Khimshiashvili [11] and is denoted by $\lambda(V)$.

The Definition 2.1 was slightly generalized as follows in [18].

Definition 2.2 Let $V = V(f) = \{f = 0\}$ be a germ of an isolated hypersurface singularity at the origin in $\mathbb{C}^n$. Let $A^k(V) = \mathcal{O}_n/(f, m^k J(f)), k \geq 0$, be a k-th moduli algebra. Let $L^k(V) := Der(A^k(V), A^k(V))$, which is called the k-th Yau algebra. The k-th Yau number $\lambda^k(V)$ is the dimension of derivation Lie algebra $L^k(V)$.

It is noted that 0-th Yau algebra is precisely the Yau algebra.

2.3 Kac-Moody Lie Algebras and Isolated Hypersurface Singularities

Let $(V, 0)$ be an isolated hypersurface singularity. Let $g(V)$ be the maximal ideal of $L(V)$ consisting of nilpotent elements. It is follows from [24] a generalized Cartan matrix $C(V)$, constructed from $g(V)$, is an invaraint of $(V, 0)$ (cf. [30]).

Definition 2.3 An $l \times l$ matrix with entries in $\mathbb{Z}$, $C = (c_{ij})$ is a generalized Cartan matrix if
a) $c_{ii} = 2 \quad \forall i = 1, \cdots, l$,
b) $c_{ij} \leq 0 \quad \forall i, j = 1, \cdots, l, i \neq j$,
c) $c_{ij} = 0$ if and only if $c_{ji} = 0\ \forall i, j = 1, \cdots, l, i \neq j$.

To each generalized Cartan matrix $C(V)$, one can associate a Lie algebra $KM(C)$ (called a Kac-Moody Lie algebra) defined by generators:

$$\{f_1, \cdots, f_l, h_1, \cdots, h_l, e_1, \cdots, e_l\}$$

and relations:

$$\begin{aligned}
&[h_i, e_j] = c_{ij}e_j, \quad [h_i, f_j] = -c_{ij} f_j, \quad (\forall i, j = 1, \cdots, l),\\
&[h_i, h_j] = 0, \quad (\forall i, j = 1, \cdots, l), \quad [e_i, f_i] = h_i,\\
&[e_i, f_j] = 0, \quad (ade_i)^{-c_{ij+1}} e_j = 0 = (adf_i)^{-c_{ij+1}} f_j, \quad (\forall i \neq j).
\end{aligned}$$

Let $H = \mathbb{C}h_i + \cdots + \mathbb{C}h_l$; denote $\xi_+(C)$ (resp. $\xi_-(C)$) the subalgebra of $KM(C)$ generated by $\{e_1, \cdots, e_l\}$ (resp. $(f_1, \cdots, f_l)$) one shows that:

$$KM(C) = \xi_+(C) \oplus H \oplus \xi_-(C).$$

One can also define $\xi_+(C)$ by generators:$\{e_1, \cdots, e_l\}$ and relations:

$$(\mathrm{ad}e_i)^{-c_{ij}+1} e_j = 0 \quad \forall i, j = 1, \cdots, l, i \neq j.$$

We shall construct the generalized Cartan matrix from an isolated hypersurface singularity $(V, 0)$. Let $g(V)$ be the set of all nilpotent elements in $L(V)$, then $g(V)$ is the maximal nilpotent Lie subalgebra of $L(V)$ and $Der(g(V))$ be its derivation algebra.

Definition 2.4 A torus on $g(V)$ is a commutative subalgebra of $Der(g(V))$ whose elements are semisimple endomorphism. A maximal torus is a torus not contain in any other torus. The dimension of maximal torus is called generalized Mostow number (GMN). GMN is an invariant of isolated singularity $(V, 0)$.

Theorem 2.2 *(Mostow's theorem, [24]) If T_1 and T_2 are maximal tori of $g(V)$, then there exist $\varphi \in Autg(V)$ (automorphism group of $g(V)$) such that $\varphi T_1 \varphi^{-1} = T_2$.*

Let T be a maximal torus and consider the root space decomposition of $g(V)$ relatively to T [24]:

$$g(V) = \sum_{\beta \in R(T)} g(V)^{\beta},$$

$$g(V)^{\beta} = \{x \in g(V) : tx = \beta(t)x, \forall t \in T\},$$

and

$$R(T) = \{\beta \in T^* : g(V)^{\beta} \neq (0)\}(\text{root system}),$$

$$R^1(T) = \{\beta \in R(T) : g(V)^{\beta} \nsubseteq [g(V), g(V)]\},$$

$$l_{\beta} = dim \left(\frac{g(V)^{\beta}}{[g(V), g(V)] \cap g(V)^{\beta}} \right), \forall \beta \in R^1(T),$$

$$d_{\beta} = dim(g(V)^{\beta}), \beta \in R^1(T).$$

The map: $\beta \mapsto d_{\beta} \quad R^1(T) \to \mathbb{N}^*$ gives the partition:

$$R^1(T) = R^1(T)_{p_1} \cup \cdots \cup R^1(T)_{p_q}, \quad p_1 < \cdots < p_q, \quad R^1(T)_{p_i} \neq \emptyset,$$

$$R^1(T)_p = \{\beta \in R^1(T); d_{\beta} = p\}.$$

Set $s_i = \sharp R^1(T)_{p_i}$ and $s = s_1 + \cdots + s_q$. We let $d_{\beta_i} = d_i$ and $l_{\beta_i} = l_i$.
Let $f : \{1, \cdots, l\} \longrightarrow \{1, \cdots, s\}$ be defined by:

$$f_i = \begin{cases} 1; \ 1 \leqslant i \leqslant l_1, \\ 2; \ l_1 \leqslant i \leqslant l_1 + l_2, \\ \vdots \\ s; \ l_1 + l_2 + \cdots + l_{s-1} \leqslant i \leqslant l. \end{cases}$$

Theorem 2.3 [24] *For $i, j \in \{1, \cdots, l\}$, $i \neq j$, let*

$$-c_{ij}(T) = min\{-n \in N; (\text{ad}v)^{-n+1} w = 0, \quad \forall v \in g^{\beta_{f(i)}}, \quad \forall w \in g^{\beta_{f(j)}}\},$$

with $(\text{ad}0)^0 = 0$ and let $c_{ii}(T) = 2$ for $i = 1, \cdots, l$. Then

$$C(T) = (c_{ij}(T))_{1 \leq i, j \leq l}$$

is a generalized Cartan matrix.

Thus, we define $C(V) := C(T)$, is called the generalized Cartan matrix of isolated hypersurface singularity $(V, 0)$. Similarly we can define a generalized Cartan matrix for k-th Yau algebra $L^k(V)$.

Definition 2.5 The generalized Cartan matrix of the k-th Yau algebra $L^k(V)$ is denoted as $C^k(V)$.

The following propositions tell us that the generalized Cartan matrix $C^0(V)$ that arising from derivation Lie algebra $L^0(V)$ characterizes the ADE singularities except the pair A_6 and D_5.

Proposition 2.1 [2] *Let* $V = \{x_1, x_2 \in \mathbb{C}^2 : x_1^2 + x_2^{k+1} = 0\}$ *be the* A_k *singularity,* $k \geq 1$ *and* $L(V)$ *be a Yau algebra. Then*

$$C^0(A_k) = \begin{cases} \text{is not defined;} & k=1,2,3, \\ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}; & k=4, \\ \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}; & k=5, \\ \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}; & k=6, \\ \begin{pmatrix} 2 & -(k-4) \\ -\frac{k-3}{2} & 2 \end{pmatrix}; & k \text{ is odd and } k \geq 7, \\ \begin{pmatrix} 2 & -(k-4) \\ -\frac{k-4}{2} & 2 \end{pmatrix}; & k \text{ is even and } k \geq 8. \end{cases}$$

Proposition 2.2 [2] *Let* $V = \{x_1, x_2 \in \mathbb{C}^2 : x_1^2 x_2 + x_2^{k-1} = 0\}$ *be the* D_k *singularity,* $k \geq 4$ *and* $L(V)$ *be a Yau algebra. Then*

$$C^0(D_k) = \begin{cases} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}; & k=4, \\ \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}; & k=5, \\ \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}; & k=6, \\ \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}; & k=7, \\ \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -(k-5) \\ 0 & -(k-6) & 2 \end{pmatrix}; & k \geq 8. \end{cases}$$

Proposition 2.3 [2] *Let* $V = \{x_1, x_2 \in \mathbb{C}^2 : x_1^3 + x_2^4 = 0\}$ *be the* E_6 *singularity and* $L(V)$ *be a Yau algebra. Then*

$$C^0(E_6) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Proposition 2.4 [2] *Let* $V = \{x_1, x_2 \in \mathbb{C}^2 : x_1^3 + x_2^3 x_1 = 0\}$ *be the* E_7 *singularity and* $L(V)$ *be a Yau algebra. Then*

$$C^0(E_7) = \begin{pmatrix} 2 & -4 & -3 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Proposition 2.5 [2] *Let* $V = \{x_1, x_2 \in \mathbb{C}^2 : x_1^3 + x_2^5 = 0\}$ *be the* E_8 *singularity and* $L(V)$ *be a Yau algebra. Then*

$$C^0(E_8) = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 \\ -2 & -1 & 2 & -2 \\ -2 & -1 & -2 & 2 \end{pmatrix}.$$

3 Proof of Main Theorem

Now we apply the above theory to study the 1-st Yau algebras $L^1(V)$ of simple hpersurface singularities. We use the following convention: $g^1 = [g, g], \cdots, g^{p+1} = [g, g^p]$. We use N to denote the set of positive integers.

Proposition 3.1 *Let* $V = \{x_1, x_2 \in \mathbb{C}^2 : x_1^2 + x_2^{k+1} = 0\}$ *be the* A_k *singularity,* $k \geq 1$ *and* $L^1(V)$ *be a derivation Lie algebra. Then*

$$C^1(A_k) = \begin{cases} \text{is not defined;} & k=1, \\ \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}; & k=2, \\ \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}; & k=3, \\ \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}; & k=4, \\ \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}; & k=5, \end{cases}$$

$$\begin{cases} \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -(k-3) \\ 0 & 0 & -\frac{k-2}{2} & 2 \end{pmatrix}; \ k \text{ is even and } k \geq 6, \\ \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -(k-3) \\ 0 & 0 & -\frac{k-3}{2} & 2 \end{pmatrix}; \ k \text{ is odd and } k \geq 7. \end{cases}$$

Proof It is easy to see that $A^1(V) =< x_1; x_2^{i_2}, 0 \leq i_2 \leq k >$ with the multiplication rules $x_1x_2 = x_1^2 = x_2^{k+1} = 0$. After simple calculation we get

$$L^1(V) =< x_1\partial_1; x_1\partial_2; x_2^k\partial_1; x_2^{i_2}\partial_2, 1 \leq i_2 \leq k >,$$

and nilradical of Lie algebra $L^1(V)$ defined as

$$g(V) =< x_1\partial_2; x_2^k\partial_1; x_2^{i_2}\partial_2, 2 \leq i_2 \leq k > .$$

It is easy to see that $g(V)$ is not nilradical for $k = 1$.

For A_2 singularity,

$$g(V) =< x_2^2\partial_1; x_1\partial_2; x_2^2\partial_2 > .$$

By setting $e_1 = x_2^2\partial_1, e_2 = x_1\partial_2, e_3 = x_2^2\partial_2$, we get the multiplication table $[e_1, e_2] = e_3$.

The type of A_2 singularity:=dim $g(V)/[g(V), g(V)] = 2$. The nilpotency of A_2 singularity=min$\{p \in N \cup \{0\} : g(V)^{p+1} = 0\} = 1$. It is easy to see from [2] that the torus T of $g(V)$ is spanned by

$$\begin{aligned} t_1 &: g(V) \longrightarrow g(V) & t_2 &: g(V) \longrightarrow g(V) \\ e_1 &\longrightarrow e_1 & e_1 &\longrightarrow 0 \\ e_2 &\longrightarrow 0 & e_2 &\longrightarrow e_2 \\ e_3 &\longrightarrow e_3, & e_3 &\longrightarrow e_3. \end{aligned}$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2$. Since dim $T = 2 =$ the type of A_2, therefore T is the maximal torus of $g(V)$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2$.

$$\begin{aligned} g(V) &= g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_1+\beta_2} \\ &= \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3. \end{aligned}$$

Note that (e_1, e_2) is a T-minimal system of generators. The generalized Cartan matrix is

$$C^1(A_2) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

For A_3 singularity, we have the following multiplication table:

$$[e_1, e_2] = e_4.$$

The type of A_3 singularity =dim $g(V)/[g(V), g(V)] = 3$. The nilpotency of A_3 singularity $= min\{p \in N \cup \{0\} : g(V)^{p+1} = 0\} = 1$. It is easy to see from [2] that the torus T of $g(V)$ is spanned by

$$\begin{array}{lll}
t_1 : g(V) \longrightarrow g(V) & t_2 : g(V) \longrightarrow g(V) & t_3 : g(V) \longrightarrow g(V) \\
e_1 \longrightarrow e_1 & e_1 \longrightarrow 0 & e_1 \longrightarrow 0 \\
e_2 \longrightarrow 0 & e_2 \longrightarrow e_2 & e_2 \longrightarrow 0 \\
e_3 \longrightarrow 0 & e_3 \longrightarrow 0 & e_3 \longrightarrow e_3 \\
e_4 \longrightarrow e_4, & e_4 \longrightarrow e_4, & e_4 \longrightarrow 0.
\end{array}$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2 + \mathbb{C}t_3$. Since dim $T = 3 =$ the type of A_3, therefore T is maximal torus of $g(V)$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2, 3$.

$$\begin{aligned}
g(V) &= g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_3} \oplus g^{\beta_1+\beta_2} \\
&= \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4.
\end{aligned}$$

(e_1, e_2, e_3) is a T-minimal system of generators. The generalized Cartan matrix is

$$C^1(A_3) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

For A_4 singularity, we have the following multiplication table:

$$[e_1, e_2] = e_5, \quad [e_3, e_4] = e_5.$$

The type of A_4 singularity =dim $g(V)/[g(V), g(V)] = 4$. The nilpotency of A_4 singularity $= min\{p \in N \cup \{0\} : g(V)^{p+1} = 0\} = 1$. It is easy to see from [2] that the torus T of $g(V)$ is spanned by

$$\begin{array}{lll}
t_1 : g(V) \longrightarrow g(V) & t_2 : g(V) \longrightarrow g(V) & t_3 : g(V) \longrightarrow g(V) \\
e_1 \longrightarrow e_1 & e_1 \longrightarrow 0 & e_1 \longrightarrow 0 \\
e_2 \longrightarrow 0 & e_2 \longrightarrow e_2 & e_2 \longrightarrow 0 \\
e_3 \longrightarrow 0 & e_3 \longrightarrow 0 & e_3 \longrightarrow e_3 \\
e_4 \longrightarrow e_4 & e_4 \longrightarrow e_4 & e_4 \longrightarrow -e_4 \\
e_5 \longrightarrow e_5, & e_5 \longrightarrow e_5, & e_5 \longrightarrow 0.
\end{array}$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2 + \mathbb{C}t_3$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2, 3$.

$$\begin{aligned}
g(V) &= g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_3} \oplus g^{\beta_1+\beta_2-\beta_3} \oplus g^{\beta_1+\beta_2} \\
&= \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_5.
\end{aligned}$$

(e_1, e_2, e_3, e_4) is a T-minimal system of generators. The generalized Cartan matrix is

$$C^1(A_4) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

For A_5 singularity, we have the following multiplication table:

$$[e_1, e_2] = e_6, \quad [e_3, e_4] = e_5, \quad [e_3, e_5] = 2e_6.$$

The type of A_5 singularity =dim $g(V)/[g(V), g(V)] = 4$. The nilpotency of A_5 singularity $= min\{p \in N \cup \{0\} : g(V)^{p+1} = 0\} = 2$. It is easy to see from [2] that the torus T of $g(V)$ is spanned by

$$\begin{array}{lll}
t_1 : g(V) \longrightarrow g(V) & t_2 : g(V) \longrightarrow g(V) & t_3 : g(V) \longrightarrow g(V) \\
e_1 \longrightarrow e_1 & e_1 \longrightarrow 0 & e_1 \longrightarrow 0 \\
e_2 \longrightarrow 0 & e_2 \longrightarrow e_2 & e_2 \longrightarrow 0 \\
e_3 \longrightarrow 0 & e_3 \longrightarrow 0 & e_3 \longrightarrow e_3 \\
e_4 \longrightarrow e_4 & e_4 \longrightarrow e_4 & e_4 \longrightarrow -2e_4 \\
e_5 \longrightarrow e_5 & e_5 \longrightarrow e_5 & e_5 \longrightarrow -e_5 \\
e_6 \longrightarrow e_6, & e_6 \longrightarrow e_6, & e_6 \longrightarrow 0.
\end{array}$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2 + \mathbb{C}t_3$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2, 3$.

$$\begin{aligned}
g(V) &= g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_3} \oplus g^{\beta_1+\beta_2-2\beta_3} \oplus g^{\beta_1+\beta_2-\beta_3} \oplus g^{\beta_1+\beta_2} \\
&= \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_5 \oplus \mathbb{C}e_6.
\end{aligned}$$

(e_1, e_2, e_3, e_4) is a T-minimal system of generators. The generalized Cartan matrix is

$$C^1(A_5) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

For A_k singularity $k \geq 6$,

$$g(V) = < x_2^k \partial_1; x_1 \partial_2; x_2^{i_2} \partial_2, 2 \leq i_2 \leq k > .$$

By setting $e_1 = x_2^k \partial_1, e_2 = x_1 \partial_2, \cdots, e_{k+1} = x_2^k \partial_2$, we have the following multiplication table:

Case 1. k is even and $k = 2l + 4 \geq 6, l \geq 1$, then

$$\begin{array}{l}
[e_1, e_2] = e_{k+1}, \\
[e_3, e_4] = e_5, \quad [e_3, e_5] = 2e_6, \quad [e_3, e_6] = 3e_7, \quad \cdots \quad , [e_3, e_k] = (k-3)e_{k+1}, \\
[e_4, e_5] = e_7, \quad [e_4, e_6] = 2e_8, \quad [e_4, e_7] = 3e_9, \quad \cdots \quad , [e_4, e_{k-1}] = (k-5)e_{k+1}, \\
[e_5, e_6] = e_9, \quad [e_5, e_7] = 2e_{10}, \quad [e_5, e_8] = 3e_{11}, \quad \cdots \quad , [e_5, e_{k-2}] = (k-7)e_{k+1}, \\
[e_6, e_7] = e_{11}, \quad [e_6, e_8] = 2e_{12}, \quad [e_6, e_9] = 3e_{13}, \quad \cdots \quad , [e_6, e_{k-3}] = (k-9)e_{k+1}, \\
\quad \vdots \\
[e_{l+3}, e_{l+4}] = e_{2l+5}.
\end{array}$$

Case 2. k is odd and $k = 2l + 5 \geq 7, l \geq 1$, then

$$\begin{aligned}
&[e_1, e_2] = e_{k+1}, \\
&[e_3, e_4] = e_5, \quad [e_3, e_5] = 2e_6, \quad [e_3, e_6] = 3e_7, \quad \cdots \quad , [e_3, e_k] = (k-3)e_{k+1}, \\
&[e_4, e_5] = e_7, \quad [e_4, e_6] = 2e_8, \quad [e_4, e_7] = 3e_9, \quad \cdots \quad , [e_4, e_{k-1}] = (k-5)e_{k+1}, \\
&[e_5, e_6] = e_9, \quad [e_5, e_7] = 2e_{10}, \quad [e_5, e_8] = 3e_{11}, \quad \cdots \quad , [e_5, e_{k-2}] = (k-7)e_{k+1}, \\
&[e_6, e_7] = e_{11}, \quad [e_6, e_8] = 2e_{12}, \quad [e_6, e_9] = 3e_{13}, \quad \cdots \quad , [e_6, e_{k-3}] = (k-9)e_{k+1}, \\
&\quad \vdots \\
&[e_{l+3}, e_{l+4}] = e_{2l+5}, \quad [e_{l+3}, e_{l+5}] = 2e_{2l+6}.
\end{aligned}$$

The type of $A_k(k \geq 6)$ singularity =dim $g(V)/[g(V), g(V)] = 4$. The nilpotency of $A_k(k \geq 6) = min\{p \in N \cup \{0\} : g(V)^{p+1} = 0\} = k - 3$. It is easy to see from [2] that the torus T of $g(V)$ is spanned by

$$\begin{array}{ll}
t_1 : g(V) \longrightarrow g(V) & t_2 : g(V) \longrightarrow g(V) \\
e_1 \longrightarrow e_1 & e_1 \longrightarrow 0 \\
e_2 \longrightarrow 0 & e_2 \longrightarrow e_2 \\
e_3 \longrightarrow \dfrac{e_3}{k-1}, & e_3 \longrightarrow \dfrac{e_3}{k-1} \\
e_4 \longrightarrow \dfrac{2e_4}{k-1}, & e_4 \longrightarrow \dfrac{2e_4}{k-1} \\
e_5 \longrightarrow \dfrac{3e_5}{k-1}, & e_5 \longrightarrow \dfrac{3e_5}{k-1} \\
\quad \vdots \qquad \vdots & \\
e_{k+1} \longrightarrow e_{k+1}, & e_{k+1} \longrightarrow e_{k+1}.
\end{array}$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2$.

$$\begin{aligned}
g(V) &= g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\frac{\beta_1+\beta_2}{k-1}} \oplus g^{\frac{2(\beta_1+\beta_2}{k-1})} \oplus g^{\frac{3(\beta_1+\beta_2}{k-1})} \oplus \cdots \oplus g^{(\beta_1+\beta_2)} \\
&= \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_5 \oplus \cdots \oplus \mathbb{C}e_{k+1}.
\end{aligned}$$

(e_1, e_2, e_3, e_4) is a T-minimal system of generators. From multplication table we get

$c_{12} = -1, \quad c_{13} = 0, \quad c_{14} = 0, \quad c_{21} = -1, \quad c_{23} = 0, \quad c_{24} = 0, \quad c_{31} = 0, \quad c_{32} = 0,$
$c_{34} = -(k-3), \quad c_{41} = 0, \quad c_{42} = 0.$

But in case of c_{43} we have following two cases:
Case 1. k is odd and $k = 2l + 5 \geq 7, l \geq 1$, then

$$c_{43} = -\frac{k-3}{2}.$$

Case 2. k is even and $k = 2l + 4 \geq 6, l \geq 1$, then

$$c_{43} = -\frac{k-2}{2}.$$

Therefore the generalized Cartan matrix is

$$C^1(A_k) = \begin{cases} \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -(k-3) \\ 0 & 0 & -\frac{k-3}{2} & 2 \end{pmatrix}; \ k \text{ is odd and } k \geq 7, \\ \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -(k-3) \\ 0 & 0 & -\frac{k-2}{2} & 2 \end{pmatrix}; \ k \text{ is even and } k \geq 6. \end{cases}$$

□

Proposition 3.2 *Let $V = \{x_1, x_2 \in \mathbb{C}^2 : x_1^2 x_2 + x_2^{k-1} = 0\}$ be the D_k singularity, $k \geq 4$ and $L^1(V)$ be a derivation Lie algebra. Then*

$$C^1(D_k) = \begin{cases} \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}; & k{=}4, \\ \begin{pmatrix} 2 & -2 & -1 \\ -3 & 2 & -2 \\ -1 & -1 & 2 \end{pmatrix}; & k{=}5, \\ \begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 2 & -1 \\ -2 & -2 & -1 & 2 \end{pmatrix}; & k{=}6, \\ \begin{pmatrix} 2 & 0 & -1 & -1 & -2 \\ 0 & 2 & -1 & -1 & -2 \\ -1 & -1 & 2 & -1 & 0 \\ -2 & -2 & -1 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{pmatrix}; & k{=}7, \\ \begin{pmatrix} 2 & 0 & -1 & -1 & -(k-5) \\ 0 & 2 & -1 & -1 & -(k-5) \\ -1 & -1 & 2 & -1 & 0 \\ -2 & -2 & -1 & 2 & 0 \\ -\frac{k-4}{2} & -\frac{k-4}{2} & 0 & 0 & 2 \end{pmatrix}; & k \text{ is even and } k \geq 8, \end{cases}$$

$$\begin{cases} \begin{pmatrix} 2 & 0 & -1 & -1 & -(k-5) \\ 0 & 2 & -1 & -1 & -(k-5) \\ -1 & -1 & 2 & -1 & 0 \\ -2 & -2 & -1 & 2 & 0 \\ -\frac{k-5}{2} & -\frac{k-5}{2} & 0 & 0 & 2 \end{pmatrix}; \ k \text{ is odd and } k \geq 9, \end{cases}$$

Proof It is easy to see from [1](cf. Theorem 13.1), that $A^1(V) =< x_2^{i_2}, 0 \leq i_2 \leq k-2, x_1, x_1^2, x_1x_2 >$. After simple calculation we get

$$L^1(V) =< x_1\partial_1; x_1^2\partial_1; x_1x_2\partial_1; x_2^{k-3}\partial_1; x_2^{k-2}\partial_1; x_1\partial_2; x_1^2\partial_2; x_1x_2\partial_2; x_2^{i_2}\partial_2, 1 \leq i_2 \leq k-2 >,$$

$$g(V) =< x_1^2\partial_1; x_1x_2\partial_1; x_2^{k-3}\partial_1; x_2^{k-2}\partial_1; x_1\partial_2; x_1^2\partial_2; x_1x_2\partial_2; x_2^{i_2}\partial_2, 2 \leq i_2 \leq k-2 > .$$

For D_4 singularity,

$$g(V) =< x_1^2\partial_1; x_1x_2\partial_1; x_2^2\partial_1; x_1\partial_2; x_1^2\partial_2; x_1x_2\partial_2; x_2^2\partial_2 > .$$

We set $e_1 = x_1^2\partial_1, e_2 = x_1x_2\partial_1, \cdots, e_7 = x_2^2\partial_2$. We have following multiplication table:

$$[e_1, e_4] = e_5, \quad [e_2, e_4] = -e_1 + e_6, \quad [e_3, e_4] = -2e_2 + e_6, \quad [e_4, e_6] = e_5,$$
$$[e_4, e_7] = 2e_6.$$

The type of D_4 singularity $=\dim g(V)/[g(V), g(V)] = 3$. The nilpotency of D_4 singularity $=\min\{p \in N \cup \{0\} : g(V)^{p+1} = 0\} = 4$. It is easy to see from [2] that the torus T of $g(V)$ is spanned by

$$\begin{array}{lll}
t_1 : g(V) \longrightarrow g(V) & t_2 : g(V) \longrightarrow g(V) & t_3 : g(V) \longrightarrow g(V) \\
e_1 \longrightarrow e_1 & e_1 \longrightarrow e_6 & e_1 \longrightarrow 0 \\
e_2 \longrightarrow 0 & e_2 \longrightarrow 0 & e_2 \longrightarrow e_2 \\
e_3 \longrightarrow -e_3 - \frac{e_7}{2} & e_3 \longrightarrow -e_2 + e_3 - \frac{e_7}{2} & e_3 \longrightarrow 2e_3 + \frac{e_7}{2} \\
e_4 \longrightarrow e_4 & e_4 \longrightarrow -e_4 & e_4 \longrightarrow -e_4 \\
e_5 \longrightarrow 2e_5 & e_5 \longrightarrow -2e_5 & e_5 \longrightarrow -e_5 \\
e_6 \longrightarrow e_6 & e_6 \longrightarrow e_1 & e_6 \longrightarrow 0 \\
e_7 \longrightarrow 0, & e_7 \longrightarrow 2e_7, & e_7 \longrightarrow e_7.
\end{array}$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2 + \mathbb{C}t_3$. Since $\dim T = 3 =$ the type of D_4, therefore T is maximal torus of $g(V)$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2, 3$.

$$\begin{aligned}
g(V) &= g^{-\beta_1+\beta_2+2\beta_3} \oplus g^{\beta_1+\beta_2} \oplus g^{2\beta_2+\beta_3} \oplus g^{2\beta_1-2\beta_2-\beta_3} \oplus g^{\beta_1-\beta_2} \oplus g^{\beta_1-\beta_2-\beta_3} \oplus g^{\beta_1} \oplus g^{\beta_3} \\
&= \mathbb{C}(2e_3 + e_7) \oplus \mathbb{C}(e_1 + e_6) \oplus (\mathbb{C}(e_2 + e_7) \oplus \mathbb{C}e_7) \oplus \mathbb{C}e_5 \oplus \mathbb{C}e_6 \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_2.
\end{aligned}$$

$(2e_3 + e_7, e_7, e_4)$ is a T-minimal system of generators. The generalized cartan matrix is

$$C^1(D_4) = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

For D_5 singularity, we have the following multiplication table:

$$[e_1, e_5] = e_6, \quad [e_2, e_3] = -e_4, \quad [e_2, e_5] = -e_1 + e_7, \quad [e_3, e_5] = -2e_2 + e_8,$$
$$[e_3, e_8] = -2e_4, \quad [e_4, e_5] = e_9, \quad [e_5, e_7] = e_6, \quad [e_5, e_8] = 2e_7.$$

The type of D_5 singularity: =dim $g(V)/[g(V), g(V)] = 3$. The nilpotency of D_5 singularity: =min$\{p \in N \cup \{0\} : g(V)^{p+1} = 0\} = 3$. It is easy to see from [2] that the torus T of $g(V)$ is spanned by

$$\begin{array}{ll} t_1 : g(V) \longrightarrow g(V) & t_2 : g(V) \longrightarrow g(V) \\ e_1 \longrightarrow e_1 & e_1 \longrightarrow 0 \\ e_2 \longrightarrow 0 & e_2 \longrightarrow e_2 \\ e_3 \longrightarrow -e_3 & e_3 \longrightarrow 2e_3 \\ e_4 \longrightarrow -e_4 & e_4 \longrightarrow 3e_4 \\ e_5 \longrightarrow e_5 & e_5 \longrightarrow -e_5 \\ e_6 \longrightarrow 2e_6 & e_6 \longrightarrow -e_6 \\ e_7 \longrightarrow e_7 & e_7 \longrightarrow 0 \\ e_8 \longrightarrow 0 & e_8 \longrightarrow e_8 \\ e_9 \longrightarrow 0, & e_9 \longrightarrow 2e_9. \end{array}$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2$ and maximal torus of $g(V)$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2$.

$$\begin{aligned} g(V) &= g^{\beta_1} \oplus g^{\beta_2} \oplus g^{-\beta_1+2\beta_2} \oplus g^{-\beta_1+3\beta_2} \oplus g^{\beta_1-\beta_2} \oplus g^{2\beta_1-\beta_2} \oplus g^{2\beta_2} \\ &= (\mathbb{C}e_1 \oplus \mathbb{C}e_7) \oplus (\mathbb{C}e_2 \oplus \mathbb{C}e_8) \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_5 \oplus \mathbb{C}e_6 \oplus \mathbb{C}e_9. \end{aligned}$$

(e_3, e_5, e_8) is a T-minimal system of generators. The generalized cartan matrix is

$$C^1(D_5) = \begin{pmatrix} 2 & -2 & -1 \\ -3 & 2 & -2 \\ -1 & -1 & 2 \end{pmatrix}.$$

For D_6 singularity, we have the following multiplication table:

$$\begin{array}{llll} [e_1, e_5] = e_6, & [e_2, e_3] = -e_4, & [e_2, e_5] = -e_1 + e_7, & [e_3, e_5] = e_9, \\ [e_3, e_7] = e_{10}, & [e_3, e_8] = -3e_4, & [e_4, e_5] = e_{10}, & [e_5, e_7] = e_6 \\ [e_5, e_8] = 2e_7, & [e_8, e_9] = e_{10}. & & \end{array}$$

The type of D_6 singularity=dim $g(V)/[g(V), g(V)] = 4$. The nilpotency of D_6 singularity=min$\{p \in N \cup \{0\} : g(V)^{p+1} = 0\} = 2$. It is easy to see from [2] that the torus T of $g(V)$ is spanned by

$$\begin{array}{lll} t_1 : g(V) \longrightarrow g(V) & t_2 : g(V) \longrightarrow g(V) & t_3 : g(V) \longrightarrow g(V) \\ e_1 \longrightarrow e_1 & e_1 \longrightarrow 0 & e_1 \longrightarrow 0 \\ e_2 \longrightarrow 0 & e_2 \longrightarrow e_2 & e_2 \longrightarrow 0 \\ e_3 \longrightarrow 0 & e_3 \longrightarrow 0 & e_3 \longrightarrow e_3 \\ e_4 \longrightarrow 0 & e_4 \longrightarrow e_4 & e_4 \longrightarrow e_4 \\ e_5 \longrightarrow e_5 & e_5 \longrightarrow -e_5 & e_5 \longrightarrow 0 \\ e_6 \longrightarrow 2e_6 & e_6 \longrightarrow -e_6 & e_6 \longrightarrow 0 \\ e_7 \longrightarrow e_7 & e_7 \longrightarrow 0 & e_7 \longrightarrow 0 \\ e_8 \longrightarrow 0 & e_8 \longrightarrow e_8 & e_8 \longrightarrow 0 \\ e_9 \longrightarrow e_9 & e_9 \longrightarrow -e_9 & e_9 \longrightarrow e_9 \\ e_{10} \longrightarrow e_{10}, & e_{10} \longrightarrow 0, & e_{10} \longrightarrow e_{10}. \end{array}$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2 + \mathbb{C}t_3$ is a unique maximal torus on $g(V)$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2, 3$.

$$\begin{aligned} g(V) &= g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_3} \oplus g^{\beta_2+\beta_3} \oplus g^{\beta_1-\beta_2} \oplus g^{2\beta_1-\beta_2} \oplus g^{\beta_1-\beta_2+\beta_3} \oplus g^{\beta_1+\beta_3} \\ &= (\mathbb{C}e_1 \oplus \mathbb{C}e_7) \oplus (\mathbb{C}e_2 \oplus \mathbb{C}e_8) \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_5 \oplus \mathbb{C}e_6 \oplus \mathbb{C}e_9 \oplus \mathbb{C}e_{10}. \end{aligned}$$

(e_2, e_8, e_3, e_5) is a T-minimal system of generators. The generalized Cartan matrix is

$$C^1(D_6) = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 2 & -1 \\ -2 & -2 & -1 & 2 \end{pmatrix}.$$

For D_7 singularity, we have the following multiplication table:

$$\begin{aligned} &[e_1, e_5] = e_6, \quad [e_2, e_3] = -e_4, \quad [e_2, e_5] = -e_1 + e_7, \quad [e_3, e_5] = e_{10}, \\ &[e_3, e_7] = e_{11}, \quad [e_3, e_8] = -4e_4, \quad [e_4, e_5] = e_{11}, \quad [e_5, e_7] = e_6 \\ &[e_5, e_8] = 2e_7, \quad [e_8, e_9] = e_{10}, \quad [e_8, e_{10}] = 2e_{11}. \end{aligned}$$

The type of D_7 singularity=dim $g(V)/[g(V), g(V)] = 5$. The nilpotency of D_7 singularity=min$\{p \in N \cup \{0\} : g(V)^{p+1} = 0\} = 2$. It is easy to see from [2] that the torus T of $g(V)$ is spanned by

$$\begin{array}{lll} t_1 : g(V) \longrightarrow g(V) & t_2 : g(V) \longrightarrow g(V) & t_3 : g(V) \longrightarrow g(V) \\ e_1 \longrightarrow e_1 & e_1 \longrightarrow 0 & e_1 \longrightarrow 0 \\ e_2 \longrightarrow 0 & e_2 \longrightarrow e_2 & e_2 \longrightarrow 0 \\ e_3 \longrightarrow 0 & e_3 \longrightarrow 0 & e_3 \longrightarrow e_3 \\ e_4 \longrightarrow 0 & e_4 \longrightarrow e_4 & e_4 \longrightarrow e_4 \\ e_5 \longrightarrow e_5 & e_5 \longrightarrow -e_5 & e_5 \longrightarrow 0 \\ e_6 \longrightarrow 2e_6 & e_6 \longrightarrow -e_6 & e_6 \longrightarrow 0 \\ e_7 \longrightarrow e_7 & e_7 \longrightarrow e_7 & e_7 \longrightarrow 0 \\ e_8 \longrightarrow 0 & e_8 \longrightarrow e_8 & e_8 \longrightarrow 0 \\ e_9 \longrightarrow e_9 & e_9 \longrightarrow -2e_9 & e_9 \longrightarrow e_9 \\ e_{10} \longrightarrow e_{10} & e_{10} \longrightarrow -e_{10} & e_{10} \longrightarrow e_{10} \\ e_{11} \longrightarrow e_{11}, & e_{11} \longrightarrow e_{11}, & e_{11} \longrightarrow e_{11}. \end{array}$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2 + \mathbb{C}t_3$ is a unique maximal torus on $g(V)$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2, 3$.

$$\begin{aligned} g(V) &= g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_3} \oplus g^{\beta_2+\beta_3} \oplus g^{\beta_1-\beta_2} \oplus g^{2\beta_1-\beta_2} \oplus g^{\beta_1+\beta_2} \oplus g^{\beta_1-2\beta_2+\beta_3} \\ &\quad \oplus g^{\beta_1-\beta_2+\beta_3} \oplus g^{\beta_1+\beta_2+\beta_3} \\ &= \mathbb{C}e_1 \oplus (\mathbb{C}e_2 \oplus \mathbb{C}e_8) \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_5 \oplus \mathbb{C}e_6 \oplus \mathbb{C}e_7 \oplus \mathbb{C}e_9 \oplus \mathbb{C}e_{10} \oplus \mathbb{C}e_{11}. \end{aligned}$$

$(e_2, e_8, e_3, e_5, e_9)$ is a T-minimal system of generators. The generalized Cartan matrix is

$$C^1(D_7) = \begin{pmatrix} 2 & 0 & -1 & -1 & -2 \\ 0 & 2 & -1 & -1 & -2 \\ -1 & -1 & 2 & -1 & 0 \\ -2 & -2 & -1 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{pmatrix}.$$

For D_k singularity $k \geq 8$, we have the following multiplication table:
Case 1. when k is odd and $k = 2l + 7$, $l \geq 1$, then

$$
\begin{aligned}
&[e_1, e_5] = e_6, \quad [e_2, e_3] = -e_4, \quad [e_2, e_5] = -e_1 + e_7, \quad [e_3, e_5] = e_{k+3} \quad [e_3, e_7] = e_{k+4},\\
&[e_3, e_8] = -(k-3)e_4, \quad [e_4, e_5] = e_{k+4}, \quad [e_5, e_7] = e_6, \quad [e_5, e_8] = 2e_7,\\
&[e_8, e_9] = e_{10}, \quad [e_8, e_{10}] = 2e_{11}, \quad [e_8, e_{11}] = 3e_{12}, \quad \cdots \quad , [e_8, e_{k+3}] = (k-5)e_{k+4}\\
&[e_9, e_{10}] = e_{12}, \quad [e_9, e_{11}] = 2e_{13}, \quad [e_9, e_{12}] = 3e_{14}, \quad \cdots \quad , [e_9, e_{k+2}] = (k-7)e_{k+4}\\
&\quad \vdots\\
&[e_{2l+7}, e_{2l+8}] = e_{2l+10}, \quad [e_{2l+7}, e_{2l+9}] = e_{2l+11}.
\end{aligned}
$$

Case 2. when k is even and $k = 2l + 6$, $l \geq 1$, then

$$
\begin{aligned}
&[e_1, e_5] = e_6, \quad [e_2, e_3] = -e_4, \quad [e_2, e_5] = -e_1 + e_7, \quad [e_3, e_5] = e_{k+3} \quad [e_3, e_7] = e_{k+4},\\
&[e_3, e_8] = -(k-3)e_4, \quad [e_4, e_5] = e_{k+4}, \quad [e_5, e_7] = e_6, \quad [e_5, e_8] = 2e_7,\\
&[e_8, e_9] = e_{10}, \quad [e_8, e_{10}] = 2e_{11}, \quad [e_8, e_{11}] = 3e_{12}, \quad \cdots \quad , [e_8, e_{k+3}] = (k-5)e_{k+4}\\
&[e_9, e_{10}] = e_{12}, \quad [e_9, e_{11}] = 2e_{13}, \quad [e_9, e_{12}] = 3e_{14}, \quad \cdots \quad , [e_9, e_{k+2}] = (k-7)e_{k+4}\\
&\quad \vdots\\
&[e_{2l+7}, e_{2l+8}] = e_{2l+10}.
\end{aligned}
$$

The type of D_k singularity (for $k \geq 8$) =dim $g(V)/[g(V), g(V)] = 5$. The nilpotency of D_k singularity (for $k \geq 8$)=min $\{p \in N \cup \{0\} : g(V)^{p+1} = 0\} = k - 5$. It is easy to see from [2] that the torus T of $g(V)$ is spanned by

$$
\begin{aligned}
&t_1 : g(V) \longrightarrow g(V) \quad t_2 : g(V) \longrightarrow g(V)\\
&e_1 \longrightarrow e_1 \quad e_1 \longrightarrow 0\\
&e_2 \longrightarrow 0 \quad e_2 \longrightarrow e_2\\
&e_3 \longrightarrow -e_3 \quad e_3 \longrightarrow (k-3)e_3\\
&e_4 \longrightarrow -e_4 \quad e_4 \longrightarrow (k-2)e_4\\
&e_5 \longrightarrow e_5 \quad e_5 \longrightarrow -e_5\\
&e_6 \longrightarrow 2e_6 \quad e_6 \longrightarrow -e_6\\
&e_7 \longrightarrow e_7 \quad e_7 \longrightarrow 0\\
&e_8 \longrightarrow 0 \quad e_8 \longrightarrow e_8\\
&e_9 \longrightarrow 0 \quad e_9 \longrightarrow 2e_9\\
&e_{10} \longrightarrow 0 \quad e_{10} \longrightarrow 3e_{10}\\
&\qquad \vdots \qquad \quad \vdots\\
&e_{k+4} \longrightarrow 0, \quad e_{k+4} \longrightarrow (k-3)e_{k+4}.
\end{aligned}
$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2$.

$$
\begin{aligned}
g(V) &= g^{\beta_1} \oplus g^{\beta_2} \oplus g^{-\beta_1+(k-3)\beta_2} \oplus g^{-\beta_1+(k-2)\beta_2} \oplus g^{\beta_1-\beta_2} \oplus g^{2\beta_1-\beta_2} \oplus g^{2\beta_2} \oplus g^{3\beta_2}\\
&\quad \oplus g^{4\beta_2} \cdots \oplus g^{(k-3)\beta_1}\\
&= (\mathbb{C}e_1 \oplus \mathbb{C}e_7) \oplus (\mathbb{C}e_2 \oplus \mathbb{C}e_8) \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_5 \oplus \mathbb{C}e_6 \oplus \mathbb{C}e_9 \oplus \mathbb{C}e_{10}\\
&\quad \oplus \mathbb{C}e_{11} \oplus \cdots \oplus \mathbb{C}e_{k+4}.
\end{aligned}
$$

$(e_2, e_8, e_3, e_5, e_9)$ is a T-minimal system of generators. From multplication table we get

$$\begin{array}{llll} c_{12} = 0, & c_{13} = -1, & c_{14} = -1, & c_{15} = -(k-5), \\ c_{21} = 0, & c_{23} = -1, & c_{24} = -1, & c_{25} = -(k-5), \\ c_{31} = -1, & c_{32} = -1, & c_{34} = -1, & c_{35} = 0, \\ c_{41} = -2, & c_{42} = -2, & c_{43} = -1 & c_{45} = 0, \\ c_{53} = 0, & c_{54} = 0. & & \end{array}$$

But in case of c_{51} and c_{52} we have following two cases:

Case 1. k is odd and $k = 2l + 7 \geq 9, l \geq 1$, then

$$c_{51} = c_{52} = -\frac{k-5}{2}.$$

Case 2. k is even and $k = 2l + 6 \geq 8, l \geq 1$, then

$$c_{51} = c_{52} = -\frac{k-4}{2}.$$

Therefore the generalized Cartan matrix is

$$C^1(D_k) = \begin{cases} \begin{pmatrix} 2 & 0 & -1 & -1 & -(k-5) \\ 0 & 2 & -1 & -1 & -(k-5) \\ -1 & -1 & 2 & -1 & 0 \\ -2 & -2 & -1 & 2 & 0 \\ -\frac{k-4}{2} & -\frac{k-4}{2} & 0 & 0 & 2 \end{pmatrix}; & k \text{ is even and } k \geq 8, \\ \begin{pmatrix} 2 & 0 & -1 & -1 & -(k-5) \\ 0 & 2 & -1 & -1 & -(k-5) \\ -1 & -1 & 2 & -1 & 0 \\ -2 & -2 & -1 & 2 & 0 \\ -\frac{k-5}{2} & -\frac{k-5}{2} & 0 & 0 & 2 \end{pmatrix}; & k \text{ is odd and } k \geq 9. \end{cases}$$

□

Proposition 3.3 *Let* $V = \{x_1, x_2 \in \mathbb{C}^2 : x_1^3 + x_2^4 = 0\}$ *be the* E_6 *singularity and* $L^1(V)$ *be a derivation Lie algebra. Then*

$$C^1(E_6) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}.$$

Proof It is easy to see that $A^1(V) =< 1, x_2, x_2^2, x_1, x_1x_2, x_1x_2^2, x_1^2, x_2^3 >$ with the multiplication rules $x_1^3 = 0 = x_2^4 = x_1^2x_2 = x_2^3x_1$. We have the following basis of Lie algebra of E_6 singularity,

$$L^1(V) =< x_1\partial_1, x_1^2\partial_1, x_1x_2\partial_1, x_1x_2^2\partial_1, x_2^2\partial_1, x_2^3\partial_1, x_1\partial_2, x_1^2\partial_2, x_2\partial_2, x_2^2\partial_2, x_1x_2\partial_2, x_1x_2^2\partial_2, x_2^3\partial_2 >.$$

The nilradical of $L^1(V)$ of E_6 singularity is spanned by:

$$g(V) =< x_1^2\partial_1, x_1x_2\partial_1, x_1x_2^2\partial_1, x_2^2\partial_1, x_2^3\partial_1, x_1^2\partial_2, x_2^2\partial_2, x_1x_2\partial_2, x_1x_2^2\partial_2, x_2^3\partial_2 > .$$

We set $e_1 = x_1^2\partial_1, e_2 = x_1x_2\partial_1, \cdots, e_{10} = x_2^3\partial_2$. The multiplication table of nilradical of the Lie algebra is given as:

$$[e_1, e_4] = -2e_3, \quad [e_2, e_4] = -e_5, \quad [e_2, e_7] = -e_3, \quad [e_2, e_8] = e_9, \quad [e_4, e_6] = 2e_9,$$
$$[e_4, e_7] = -2e_5, \quad [e_4, e_8] = -2e_3 + e_{10}, \quad [e_7, e_8] = -e_9.$$

The type of E_6 singularity=dim $g(V)/[g(V), g(V)] = 6$. The nilpotency of E_6 singularity =min$\{p \in N \cup \{0\} : g(V)^{p+1} = 0\} = 1$. It is easy to see from [2] that the torus T of $g(V)$ is spanned by

$$\begin{array}{llll}
t_1 : g(V) \longrightarrow g(V) & t_2 : g(V) \longrightarrow g(V) & t_3 : g(V) \longrightarrow g(V) & t_4 : g(V) \longrightarrow g(V) \\
e_1 \longrightarrow e_1 & e_1 \longrightarrow 0 & e_1 \longrightarrow 0 & e_1 \longrightarrow 0 \\
e_2 \longrightarrow 0 & e_2 \longrightarrow 0 & e_2 \longrightarrow e_2 & e_2 \longrightarrow \frac{-e_7}{2} \\
e_3 \longrightarrow 0 & e_3 \longrightarrow 0 & e_3 \longrightarrow 2e_3 & e_3 \longrightarrow \frac{3e_3}{2} \\
e_4 \longrightarrow -e_4 & e_4 \longrightarrow 0 & e_4 \longrightarrow 2e_4 & e_4 \longrightarrow \frac{3e_4}{2} \\
e_5 \longrightarrow -e_5 & e_5 \longrightarrow 0 & e_5 \longrightarrow 3e_5 & e_5 \longrightarrow \frac{5e_5}{2} \\
e_6 \longrightarrow 2e_6 & e_6 \longrightarrow e_6 & e_6 \longrightarrow -e_6 & e_6 \longrightarrow -e_6 \\
e_7 \longrightarrow 0 & e_7 \longrightarrow 0 & e_7 \longrightarrow e_7 & e_7 \longrightarrow \frac{3e_7}{2} \\
e_8 \longrightarrow e_8 & e_8 \longrightarrow e_1 + e_8 & e_8 \longrightarrow 0 & e_8 \longrightarrow 0 \\
e_9 \longrightarrow e_9 & e_9 \longrightarrow e_9 & e_9 \longrightarrow e_9 & e_9 \longrightarrow \frac{e_9}{2} \\
e_{10} \longrightarrow 0, & e_{10} \longrightarrow e_{10}, & e_{10} \longrightarrow 2e_{10}, & e_{10} \longrightarrow \frac{3e_{10}}{2}.
\end{array}$$

Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2, 3, 4$.

$$\begin{aligned}
g(V) &= g^{\beta_1} \oplus g^{2\beta_3+\frac{3}{2}\beta_4} \oplus g^{-\beta_1+2\beta_3+\frac{3}{2}\beta_4} \oplus g^{-\beta_1+3\beta_3+\frac{5}{2}\beta_4} \oplus g^{2\beta_1+\beta_2-\beta_3-\beta_4} \oplus g^{\beta_2+2\beta_3+\frac{3}{2}\beta_4} \\
&\quad \oplus g^{\beta_1+\beta_2+\beta_3+\frac{1}{2}\beta_4} \oplus g^{\beta_1+\beta_2} \oplus g^{\beta_3+\beta_4} \oplus g^{\beta_3+\frac{1}{2}\beta_4} \\
&= \mathbb{C}e_1 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_5 \oplus \mathbb{C}e_6 \oplus \mathbb{C}e_{10} \oplus \mathbb{C}e_9 \oplus \mathbb{C}(e_1 + e_8) \oplus \mathbb{C}(e_2 + e_7) \\
&\quad \oplus \mathbb{C}(2e_2 + e_7).
\end{aligned}$$

$(e_1, e_4, e_6, e_1 + e_8, e_2 + e_7, 2e_2 + e_7)$ is a T-minimal system of generators. The generalized Cartan matrix is

$$C^1(E_6) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}.$$

□

Proposition 3.4 *Let* $V = \{x_1, x_2 \in \mathbb{C}^2 : x_1^3 x_2 + x_2^3 = 0\}$ *be the* E_7 *singularity and* $L^1(V)$ *be a derivation Lie algebra. Then*

$$C^1(E_7) = \begin{pmatrix} 2 & -3 & -2 \\ -1 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}.$$

Proof It is easy to see that $A^1(V) =< 1, x_1, x_2, x_1x_2, x_1x_2^2, x_1^2x_2, x_1^2, x_1^3, x_2^2 >$ with the multiplication rules:

$$x_1^3x_2 = 0 = x_1^2x_2^2 = x_1^5 = x_2^3, \quad x_1^4 + 3x_1x_2^2 = 0.$$

We have the following basis of Lie algebra $L^1(V)$ of E_7 singularity,

$$L^1(V) = < x_2\partial_1, x_2^2\partial_1, x_1\partial_1 + \frac{3x_2}{2}\partial_2, x_1x_2\partial_1, x_1x_2^2\partial_1, x_1^2\partial_1, x_1^2x_2\partial_1, x_1^3\partial_1, x_2^2\partial_2, x_1x_2\partial_2, x_1x_2^2\partial_2, x_1^2\partial_2, x_1^2x_2\partial_2, x_1^3\partial_2 > .$$

The nilradical of Lie algebra $L^1(V)$ of E_7 singularity is spanned by:

$$g(V) = < x_2\partial_1, x_2^2\partial_1, x_1x_2\partial_1, x_1x_2^2\partial_1, x_1^2\partial_1, x_1^2x_2\partial_1, x_1^3\partial_1, x_2^2\partial_2, x_1x_2\partial_2, x_1x_2^2\partial_2, x_1^2\partial_2, x_1^2x_2\partial_2, x_1^3\partial_2 > .$$

We set $e_1 = x_2\partial_1, e_2 = x_2^2\partial_1, \cdots, e_{13} = x_1^3\partial_2$. The multiplication table of nilradical of new Lie algebra is given as:

$$\begin{aligned} &[e_1, e_3] = e_2, \quad [e_1, e_5] = 2e_3, \quad [e_1, e_6] = 2e_4, \quad [e_1, e_7] = 3e_6, \quad [e_1, e_8] = -e_2, \\ &[e_1, e_9] = e_8 - e_3 \quad [e_1, e_{10}] = -e_4, \quad [e_1, e_{11}] = 2e_9 - e_5, \quad [e_1, e_{12}] = 2e_{10} - e_6, \\ &[e_1, e_{13}] = 3e_{12} - e_7, \quad [e_2, e_5] = 2e_4 \quad [e_2, e_9] = -2e_4, \quad [e_2, e_{11}] = -2e_6 + e_{10}, \quad [e_3, e_5] = 2e_6, \\ &[e_3, e_8] = -e_4, \quad [e_3, e_9] = -e_6 + e_{10} \quad [e_3, e_{11}] = -e_7 + 2e_{12}, \quad [e_3, e_{13}] = 3e_4, \quad [e_5, e_7] = -3e_4, \\ &[e_5, e_9] = e_{12}, \quad [e_5, e_{11}] = 2e_{13} \quad [e_5, e_{13}] = -9e_{11}, \quad [e_7, e_{11}] = -6e_{10}, \quad [e_8, e_9] = -e_{10}, \\ &[e_8, e_{11}] = -2e_{12}, \quad [e_9, e_{11}] = -e_{13} \quad [e_{11}, e_{12}] = -3e_{10}. \end{aligned}$$

The type of E_7 singularity=dim $g(V)/[g(V), g(V)] = 3$. The nilpotency of E_7 singularity=min$\{p \in N \cup \{0\} : g(V)^{p+1} = 0\} = 5$ It is easy to see from [2] that the torus T of $g(V)$ is spanned by

$$\begin{aligned} t : g(V) &\longrightarrow g(V) \\ e_1 &\longrightarrow e_1 \\ e_2 &\longrightarrow 4e_2 \\ e_3 &\longrightarrow 3e_3 \\ e_4 &\longrightarrow 6e_4 \\ e_5 &\longrightarrow 2e_5 \\ e_6 &\longrightarrow 5e_6 \\ e_7 &\longrightarrow 4e_7 \\ e_8 &\longrightarrow 3e_8 \\ e_9 &\longrightarrow 2e_9 \\ e_{10} &\longrightarrow 5e_{10} \\ e_{11} &\longrightarrow e_{11} \\ e_{12} &\longrightarrow 4e_{12} \\ e_{13} &\longrightarrow 3e_{13}. \end{aligned}$$

Thus $T = \mathbb{C}t$ is a unique maximal torus of $g(V)$. Let $\beta : T \longrightarrow \mathbb{C}$ be a linear map with $\beta(t) = 1$.

$$\begin{aligned} g(V) &= g^{\beta} \oplus g^{2\beta} \oplus g^{3\beta} \oplus g^{4\beta} \oplus g^{5\beta} \oplus g^{6\beta} \\ &= (\mathbb{C}e_1 \oplus \mathbb{C}e_{11}) \oplus (\mathbb{C}e_5 \oplus \mathbb{C}e_9) \oplus (\mathbb{C}e_3 \oplus \mathbb{C}e_8 \oplus \mathbb{C}e_{13}) \oplus (\mathbb{C}e_2 \oplus \mathbb{C}e_7 \oplus \mathbb{C}e_{12}) \\ &\quad \oplus (\mathbb{C}e_6 \oplus \mathbb{C}e_{10}) \oplus \mathbb{C}e_4. \end{aligned}$$

(e_1, e_{11}, e_5) is a T-minimal system of generators. The generalized Cartan matrix is

$$C^1(E_7) = \begin{pmatrix} 2 & -3 & -2 \\ -1 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}.$$

□

Proposition 3.5 *Let* $V = \{x_1, x_2 \in \mathbb{C}^2 : x_1^3 + x_2^5 = 0\}$ *be the* E_8 *singularity. Then*

$$C^1(E_8) = \begin{pmatrix} 2 & -1 & -1 & -1 & -1 \\ -1 & 2 & -1 & -3 & -3 \\ -1 & -2 & 2 & -3 & -3 \\ -2 & -1 & -1 & 2 & -1 \\ -2 & -1 & -1 & -1 & 2 \end{pmatrix}.$$

Proof It is noted that $A^1(V) =< 1, x_2, x_2^2, x_2^3, x_2^4, x_1, x_1x_2, x_1x_2^2, x_1x_2^3, x_1^2 >$ with multiplication rules,

$$x_1^3 = 0 = x_2^5 = x_1^2x_2 = x_1x_2^4.$$

We have the following basis of the Lie algebra $L^1(V)$ of E_8 singularity,

$$L^1(V) = < x_1\partial_1, x_2^3\partial_1, x_2^4\partial_1, x_1x_2\partial_1, x_1x_2^2\partial_1, x_1x_2^3\partial_1, x_1^2\partial_1, x_2^2\partial_2, x_2^3\partial_2, x_2^4\partial_2, x_1x_2\partial_2, x_1x_2^2\partial_2, x_1x_2^3\partial_2, x_1^2\partial_2, x_1\partial_2, x_2\partial_2 > .$$

The nilradical of Lie algebra $L^1(V)$ of E_8 singularity is spanned by:

$$g(V) = < x_2^3\partial_1, x_2^4\partial_1, x_1x_2\partial_1, x_1x_2^2\partial_1, x_1x_2^3\partial_1, x_1^2\partial_1, x_2^2\partial_2, x_2^3\partial_2, x_2^4\partial_2, x_1x_2\partial_2, x_1x_2^2\partial_2, x_1x_2^3\partial_2, x_1^2\partial_2, x_1\partial_2 > .$$

We set $e_1 = x_2^3\partial_1, e_2 = x_2^4\partial_1, \cdots, e_{14} = x_1\partial_2$. The multiplication table of nilradical of new Lie algebra is given as:

$$[e_1, e_3] = e_2, \quad [e_1, e_6] = 2e_5, \quad [e_1, e_7] = -3e_2, \quad [e_1, e_{10}] = -3e_5 + e_9, \quad [e_1, e_{13}] = 2e_{12},$$
$$[e_1, e_{14}] = -3e_4 + e_8, \quad [e_3, e_7] = -e_4, \quad [e_3, e_8] = -e_5, \quad [e_3, e_{10}] = e_{11}, \quad [e_3, e_{11}] = e_{12},$$
$$[e_2, e_{14}] = -4e_5 + e_9, \quad [e_4, e_7] = -2e_5, \quad [e_4, e_{10}] = e_{12}, \quad [e_7, e_8] = e_9, \quad [e_7, e_{10}] = -e_{11},$$
$$[e_8, e_{10}] = -e_{11}, \quad [e_3, e_{14}] = e_{10}, \quad [e_4, e_{14}] = e_{11}, \quad [e_5, e_{14}] = e_{12}, \quad [e_6, e_{14}] = e_{13},$$
$$[e_7, e_{14}] = -2e_{10}, \quad [e_8, e_6] = -3e_{11}, \quad [e_9, e_{14}] = -4e_{12}.$$

The type of E_8 singularity=dim $g(V)/[g(V), g(V)] = 5$. The nilpotency of E_8 singularity $= min\{p \in N \cup \{0\} : g(V)^{p+1} = 0\} = 3$. It is easy to see from [2] that the torus T of $g(V)$ is spanned by

$$\begin{aligned} t : g(V) &\longrightarrow g(V) \\ e_1 &\longrightarrow e_1 \\ e_2 &\longrightarrow e_2 \\ e_3 &\longrightarrow 0 \\ e_4 &\longrightarrow 0 \\ e_5 &\longrightarrow 0 \\ e_6 &\longrightarrow -e_6 \\ e_7 &\longrightarrow 0 \\ e_8 &\longrightarrow 0 \\ e_9 &\longrightarrow 0 \\ e_{10} &\longrightarrow -e_{10} \\ e_{11} &\longrightarrow -e_{11} \\ e_{12} &\longrightarrow -e_{12} \\ e_{13} &\longrightarrow -2e_{13} \\ e_{14} &\longrightarrow -e_{14}. \end{aligned}$$

Thus $T = \mathbb{C}t$ is a unique maximal torus of $g(V)$. Let $\beta : T \longrightarrow \mathbb{C}$ be a linear map with $\beta(t) = 1$.

$$\begin{aligned} g(V) &= g^{0\beta} \oplus g^{\beta} \oplus g^{-\beta} \oplus g^{-2\beta} \\ &= (\mathbb{C}e_3 \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_5 \oplus \mathbb{C}e_7 \oplus \mathbb{C}e_8 \oplus \mathbb{C}e_9) \oplus (\mathbb{C}e_1 \oplus \mathbb{C}e_2) \\ &\quad \oplus(\mathbb{C}e_6 \oplus \mathbb{C}e_{10} \oplus \mathbb{C}e_{11} \oplus \mathbb{C}e_{12} \oplus \mathbb{C}e_{14}) \oplus \mathbb{C}e_{13}. \end{aligned}$$

$(e_1, e_3, e_7, e_6, e_{14})$ is a T-minimal system of generators. The generalized Cartan matrix is

$$C^1(E_8) = \begin{pmatrix} 2 & -1 & -1 & -1 & -1 \\ -1 & 2 & -1 & -3 & -3 \\ -1 & -2 & 2 & -3 & -3 \\ -2 & -1 & -1 & 2 & -1 \\ -2 & -1 & -1 & -1 & 2 \end{pmatrix}.$$

□

4 Proof of Main Theorem

Proof The Main Theorem is an immediate corollary of Proposition 3.1, Proposition 3.2, Proposition 3.3, Proposition 3.4, and Proposition 3.5. □

References

1. Arnol'd, V.I., Gusein-Zade, S.M., Varchenko, A.N.: Singularities of differential maps, vol. 1, The classification of critical points, caustics and wave fronts. Translated from the Russian by Ian Porteous and Mark Reynolds. In: Monographs in Mathematics, vol. 82, p. Xi+382. Birkhäauser Boston, Inc., Boston (1985)
2. Benson, M., Yau, S.S.-T.: Lie Algebra and Their Representations Arising from Isolated Singularities: Computer Method in Calculating the Lie Algebras and their Cohomology, Advanced Studies in Pure Mathematics. Complex Analytic Singularities, pp. 3–58. North-Holland, Amsterdam (1987)
3. Benson, M., Yau, S.S.-T.: Equivalence between isolated hypersurface singularities. Math. Ann. **287**, 107–134 (1990)
4. Bratzlavsky, F.: Sur les algèbres admettant un tore d'automorphismes donné. J. Algebra **30**, 305–316 (1974)
5. Brieskorn, E.: Actes congres intern. Math. **2**, 279–284 (1970)
6. Chen, B., Chen, H., Yau, S.S.-T., Zuo, H.: The non-existence of negative weight derivations on positive dimensional isolated singularities: generalized Wahl conjecture. J. Differential Geom. **115**(2), 195–224 (2020)
7. Chen, B., Hussain, N., Yau, S.S.-T., Zuo, H.: Variation of complex structures and variation of Lie algebras II: new Lie algebras arising from singularities. J. Differential Geom. **115**(3), 437–473 (2020)
8. Chen, H., Xu, Y.-J., Yau, S.S.-T.: Nonexistence of negative weight derivation of moduli algebras of weighted homogeneous singularities. J. Algebra **172**, 243–254 (1995)
9. Chen, H., Yau, S.S.-T., Zuo, H.: Non-existence of negative weight derivations on positively graded Artinian algebras. Trans. Amer. Math. Soc. **372**(4), 2493–2535 (2019)
10. Durfee, A.: Fifteen characterizations of rational double points and simple critical points, Enseign. Math. **25**, 131–163 (1979)
11. Elashvili, A., Khimshiashvili, G.: Lie algebras of simple hypersurface singularities. J. Lie Theory **16**, 621–649 (2006)
12. Gabriel, F.: Systeme De Poids Sur Une Algebre De Lie Nilpotente, Thesis 1972, Ecole polytechnique Federale de Lausanne
13. Gabriel, F.: Systeme de poids sur une algebre de Lie nilpotente. Manuscripta Math. **9**(1), 53–90 (1973)
14. Greuel, G.-M., Lossen, C., Shustin, E.: Introduction to Singularities and Deformations, Springer Monographs in Mathematics. Springer, Berlin (2007)
15. Hu, C., Yau, S.S.-T., Zuo, H.: Torelli Theorem for k-th Yau Algebras over Simple Elliptic Singularities, J. Differ. Geom. (submitted)
16. Hussain, N.: Survey on derivation Lie algebras of isolated singularities. Methods Appl. Anal. **25**(4), 307–322 (2018)
17. Hussain, N., Yau, S.S.-T., Zuo, H.: On the derivation Lie algebras of fewnomial singularities. Bull. Aust. Math. Soc. **98**(1), 77–88 (2018)
18. Hussain, N., Yau, S.S.-T., Zuo, H.: On the new k-th Yau algebras of isolated hypersurface singularities. Math. Z. **294**(1-2), 331–358 (2020)

19. Hussain, N., Yau, S.S.-T., Zuo, H.: k-th Yau algebra of isolated hypersurface singularities and an inequality conjecture, J. Aust. Math. Soc. (2019). https://doi.org/10.1017/S1446788719000132
20. Hussain, N., Yau, S.S.-T., Zuo, H.: Generalized Cartan matrices arising from new derivation Lie algebras of isolated hypersurface singularities, Pacific. J. Math. **305**(1), 189–217 (2020)
21. Hussain, N., Yau, S.S.-T., Zuo, H.: On two inequality conjectures for the k-th Yau numbers of isolated hypersurface singularities, Geom. Dedicata (2020). https://doi.org/10.1007/s10711-020-0054
22. Mather, J., Yau, S.S.-T.: Classification of isolated hypersurface singularities by their moduli algebras. Invent. Math. **69**, 243–251 (1982)
23. Ma, G., Yau, S.S.-T., Zuo, H.: On the non-existence of negative weight derivations of the new moduli algebras of singularities, J. algebra (submitted)
24. Santharoubane, L.J.: Kac-Moody Lie algebras and the universal element for the category of nilpotent Lie algebras. Math. Ann. **263**(3), 365–370 (1983)
25. Seeley, C., Yau, S.S.-T.: Variation of complex structure and variation of Lie algebras. Invent. Math. **99**, 545–565 (1990)
26. Seeley, C., Yau, S.S.-T.: Algebraic methods in the study of simple-elliptic singularities, Algebraic Geometry, pp. 216–237. Springer, Berlin (1991)
27. Umlauf, K.A.: Über Die Zusammensetzung Der Endlichen Continuierlichen Transformationsgruppen, Insbesondre Der Gruppen Vom Range Null: Inaugural-Dissertation. Nabu Press, Leipzig (in German (1891)
28. Yau, S.S.-T.: Continuous family of finite-dimensional representations of a solvable Lie algebra arising from singularities. Proc. Natl. Acad. Sci. USA **80**, 7694–7696 (1983)
29. Yau, S.S.-T.: Criterions for right-left equivalence and right equivalence of holomorphic germs with isolated critical points. Proc. Symp. Pure Math. **41**, 291–297 (1984)
30. Yau, S.S.-T.: Solvable Lie algebras and generalized Cartan matrices arising from isolated singularities. Math. Z. **191**, 489–506 (1986)
31. Yau, S.S.-T.: Solvability of Lie algebras arising from isolated singularities and nonisolatedness of singularities defined by $sl(2, \mathbb{C})$ invariant polynomials. Amer. J. Math. **113**, 773–778 (1991)
32. Yu, Y.: On Jacobian ideals invariant by reducible sl(2;C) action. Trans. Amer. Math. Soc. **348**, 2759–2791 (1996)
33. Yau, S.S.-T., Zuo, H.: Derivations of the moduli algebras of weighted homogeneous hypersurface singularities. J. Algebra **457**, 18–25 (2016)
34. Yau, S.S.-T., Zuo, H.: A sharp upper estimate conjecture for the Yau number of weighted homogeneous isolated hypersurface singularity. Pure Appl. Math. Q. **12**(1), 165–181 (2016)