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RIGIDITY OF CR MORPHISMS

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Dedicate to Professor H. Blaine Lawson on the occasion of his 80th birthday

Let X_1 and X_2 be two compact strongly pseudoconvex CR manifolds of dimension $2n-1 \geqslant 5$ which are boundaries of complex varieties V_1 and V_2 with only isolated normal singularities in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$, respectively. We show that any nonconstant CR morphism from X_1 to X_2 can be extended to a finite holomorphic covering map from V_1 to V_2 in the case where V_2 has only isolated complete intersection singularities. In particular, there is no nonconstant CR morphism between the links of two isolated complete intersection singularities with different embedding codimensions. Finally, using Kohn–Rossi cohomology, we obtain some sufficient conditions for the nonexistence of CR morphisms from X_1 to X_2.

1. Introduction

CR manifolds are abstract models of boundaries of complex manifolds. In fact, Boutet de Monvel [1975] proved that any compact strongly pseudoconvex CR manifold of dimension at least five can be CR embedded in some complex Euclidean space. A beautiful theorem of Harvey and Lawson [1975; 2000] says that these CR manifolds are the boundaries of Stein spaces with only isolated normal singularities.

The purpose of this paper is to study the rigidity properties of CR morphisms between strongly pseudoconvex CR manifolds. Our strongly pseudoconvex CR manifolds are always assumed to be compact and embedded in some $\mathbb{C}^N$. Pinčuk [1974] showed that a proper holomorphic mapping between strongly pseudoconvex domains in $\mathbb{C}^n$ is locally biholomorphic. Moreover, he showed that a proper holomorphic self-map of a strongly pseudoconvex domain is biholomorphic. Since CR morphisms between strongly pseudoconvex boundaries extend to holomorphic maps of the

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corresponding domains, Pinčuk's results may be interpreted as rigidity statements about CR morphisms of compact strongly pseudoconvex hypersurfaces in $\mathbb{C}^n$. In [Yau 2011], the second author of this paper took another point of view and investigated the existence of nontrivial CR morphisms between strongly pseudoconvex CR manifolds using the singularity theory. Tu, Yau, and Zuo [Tu et al. 2013] proved the following rigidity property of CR morphisms:

Theorem 1.1 [Tu et al. 2013]. *Let X_1 and X_2 be two $(2n-1)$-dimensional compact strongly pseudoconvex CR manifolds lying in a Stein variety V of dimension n in $\mathbb{C}^N$. Let $V_1 \subset V$, $V_2 \subset V$ such that $\partial V_1 = X_1$ and $\partial V_2 = X_2$. Assume that the singular set S of V is nonempty and is equal to the singular set of V_i, $i = 1, 2$. Then nontrivial CR morphisms from X_1 to X_2 are necessarily CR biholomorphisms.*

In this paper, we continue this investigation and prove the following statement:

Theorem 1.2. *Let X_1 and X_2 be two compact strongly pseudoconvex CR manifolds of dimension $2n-1 \geqslant 5$ which bound complex varieties V_1 and V_2 with only isolated normal singularities in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$, respectively. Let S_1 and S_2 be the singular sets of V_1 and V_2, respectively. Assume that the singular points of V_2 are isolated complete intersection singularities. If $\Phi : X_1 \to X_2$ is a nonconstant CR morphism, then Φ can be extended to a proper holomorphic covering map from V_1 to V_2 and $S_1 = \Phi^{-1}(S_2)$.*

Corollary 1.3. *Let X_1 and X_2 be two compact strongly pseudoconvex CR manifolds of dimension $2n-1 \geqslant 5$ which bound complex varieties V_1 and V_2 with only one isolated normal singularity p and q in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$, respectively. Assume that (V_2, q) is a complete intersection singularity. If there exists a nonconstant CR morphism from X_1 to X_2, then (V_1, p) is biholomorphic to (V_2, q).*

Corollary 1.4. *Let X_1 and X_2 be two compact strongly pseudoconvex CR manifolds of dimension $2n-1 \geqslant 5$ which bound complex varieties V_1 and V_2 in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$, respectively. Assume that V_2 has only one isolated complete intersection singular point q. If there exists an isolated singular point $p \in V_1$ such that the embedding codimensions of (V_1, p) and (V_2, q) are different, then there is no nonconstant CR morphism from X_1 to X_2.*

There are natural CR structures on links of isolated singularities. The above corollary implies that there is no nonconstant CR morphism between the links of two isolated complete intersection singularities with different embedding codimensions.

The following fact is well known and we can give a new proof of it using Theorem 1.2:

Corollary 1.5. *Isolated quotient singularities of dimension at least three cannot be complete intersections unless they are smooth.*

Finally, we show that the nonexistence of nonconstant CR morphism between certain strongly pseudoconvex CR manifolds can be characterized by Kohn–Rossi cohomology groups, which are defined by Kohn and Rossi [1965].

Theorem 1.6. *Let X_1 and X_2 be two compact strongly pseudoconvex CR manifolds of dimension $2n-1 \geqslant 5$ which bound complex varieties V_1 and V_2 with only isolated normal singularities in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$, respectively. Assume that the singular points of V_2 are isolated complete intersection singularities. If there exists a nonconstant CR morphism from X_1 to X_2, then*

$$\dim H^{p,q}(X_1) = d \cdot \dim H^{p,q}(X_2), \quad 1 \leqslant q \leqslant n-2,\ 0 \leqslant p \leqslant n,$$

for some integer $d > 0$. In this case,

$$H^{p,q}(X_1) = 0 \quad \textit{for } p+q \leqslant n-2,\ 1 \leqslant q \leqslant n-2.$$

Corollary 1.7. *Let X_1 and X_2 be two compact strongly pseudoconvex CR manifolds of dimension $2n-1 \geqslant 5$ which bound complex varieties V_1 and V_2 with only isolated normal singularities in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$, respectively. Suppose V_2 has only isolated complete intersection singularities. If one of the following conditions holds*:

(1) $H^{p,q}(X_1) \neq 0$ *for* $p+q \leqslant n-2$, $1 \leqslant q \leqslant n-2$;

(2) *the integer* $\dim H^{p,q}(X_1)$ *is not divisible by* $\dim H^{p,q}(X_2)$ *for* $p+q = n-1$, $1 \leqslant q \leqslant n-2$,

then there is no nonconstant CR morphism from X_1 to X_2.

2. Preliminaries

In this section, we shall recall some basic notations and definitions.

2.1. ***CR manifolds and pseudoconvexity.*** Let X be a connected orientable real manifold of dimension $2n-1$. A CR structure on X is an $(n-1)$-dimensional subbundle S of the complexified tangent bundle $\mathbb{C}T_X$ such that

- $S \cap \bar{S} = \{0\}$;
- if L, L' are local sections of S, then so is $[L, L]$.

A manifold with a CR structure is called a CR manifold.

Let $L_1, \ldots, L_{n-1}$ be a local frame of S. Choose a purely imaginary local section N of $\mathbb{C}T_X$ such that $L_1, \ldots, L_{n-1}, \bar{L}_1, \ldots, \bar{L}_{n-1}, N$ span $\mathbb{C}T_X$. Then the matrix (c_{ij}) defined by

$$[L_i, \bar{L}_j] = \sum a_{ij}^k L_k + \sum b_{ij}^k \bar{L}_k + c_{ij} N$$

is Hermitian, and is called Levi form. The number of nonzero eigenvalues and the absolute value of the signature of (c_{ij}) at each point are independent of the choice of $L_1, \dots, L_{n-1}, N$. Moreover, X is said to be strongly pseudoconvex if the Levi form is definite at each point of X.

Throughout this paper, we always assume that X is a real hypersurface of a complex manifold M. Suppose that X is locally defined by $r = 0$, where r is a real smooth function on M with $|dr| = 1$ on X. For each point $x \in X$, the Levi form at x is the Hermitian form on the $(n-1)$-dimensional space $T^{1,0}_{M,x} \cap \mathbb{C}T_{X,x}$ given by

$$(L_1, L_2) \mapsto 2\langle \partial\bar{\partial} r, L_1 \wedge \bar{L}_2 \rangle,$$

where $T^{1,0}_{M,x}$ is the space of holomorphic vectors at x. Then X is strongly pseudoconvex if the Levi form is positive definite at each point of X.

Let (X_1, S_1) and (X_2, S_2) be two CR manifolds. A smooth map $\Phi : X_1 \to X_2$ is a CR morphism if $d\Phi(S_1) \subset S_2$, where $d\Phi$ is the differential of Φ.

2.2. *Isolated singularities.* We shall often denote by (V, x) the pair of an analytic space V with a point $x \in V$ such that $V \setminus \{x\}$ is smooth and pure dimensional. We call such a pair an isolated singularity (even in the case V is smooth). Two pairs (V, x) and (W, y) are equivalent if there exist a neighborhood $V' \subset V$ of x, a neighborhood $W' \subset W$ of y, and an isomorphism $f : V' \to W'$ such that $f(x) = y$. An equivalent class of such pairs is called a germ of isolated singularities and is also denoted by (V, x).

We next introduce the notation of isolated complete intersection singularities. The conventions followed are those of Looijenga [2013]. Let (U, x) be a complex manifold germ of dimension N, and $(V, x) \subset (U, x)$ an analytic subgerm of dimension n which is given by an ideal $\mathcal{I} \subset \mathscr{O}_{U,x}$. We say that $\mathcal{I}$ defines a complete intersection at x if $\mathcal{I}$ admits $N - n$ generators $f_1, \dots, f_{N-n}$. Moreover, if the common zero set of $f_1, \dots, f_{N-n}$ and the $(N-n)$-form $df_1 \wedge \cdots \wedge df_{N-n}$ is contained in $\{x\}$, then we say that (V, x) is an isolated complete intersection singularity (this includes the case that (V, x) is regular). Given a coordinate $z_1, \dots, z_N$ for (U, x), let $\mathcal{J}$ be the ideal in $\mathscr{O}_{U,x}$ generated by the determinants of the $(N-n) \times (N-n)$ submatrices of the Jacobian matrix $\left(\frac{\partial f_j}{\partial z_i}\right)$. The definition of $\mathcal{J}$ is independent of the choice of generators and the singularity (V, x) is isolated if and only if $\mathcal{J} \supset \mathfrak{m}_x^k$ for some $k \geqslant 1$, where $\mathfrak{m}_x$ is the maximal ideal of $\mathscr{O}_{U,x}$. The number $\dim \mathfrak{m}_x/\mathfrak{m}_x^2$ is called the embedding dimension of (V, x), and $\dim \mathfrak{m}_x/\mathfrak{m}_x^2 - n$ is called the embedding codimension of (V, x).

For the relationship between isolated singularities and strongly pseudoconvex manifolds, one can refer to [Grauert 1962; Grauert et al. 1994; Hironaka 1964].

3. Rigidity of CR morphism

The following proposition can be found in [Yau 2011]. It was proved by using the results of Fornaess [1976] and Pinčuk [1974].

Proposition 3.1. *Let X_1 and X_2 be two compact strongly pseudoconvex CR manifolds of dimension $2n-1 \geqslant 3$ which bound complex varieties V_1 and V_2 in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$, respectively. Suppose the singular set S_i of V_i, $i = 1, 2$, is either an empty set or a set consisting of only isolated normal singularities. If $\Phi : X_1 \to X_2$ is a nonconstant CR morphism, then Φ is surjective and Φ can be extended to a proper holomorphic map from V_1 to V_2 such that $\Phi(S_1) \subset \Phi(S_2)$, $\Phi^{-1}(X_2) = X_1$, and $\Phi : V_1 \setminus \Phi^{-1}(S_2) \to V_2 \setminus S_2$ is a covering map. Moreover, if S_2 does not have quotient singularity, then $\Phi^{-1}(S_2) = S_1$.*

For a subset X of $\mathbb{C}^N$ and a positive real number ε, let

$$X_\varepsilon = \{z \in X;\ |z| \leqslant \varepsilon\}.$$

Let us recall the following theorem of Hamm:

Theorem 3.2 [Hamm 1971; 1981]. *Let X, Y, Z be complex analytic subsets of some neighborhood U of the origin in $\mathbb{C}^N$ such that $Y \subset X$, $Z \subset X$, $Y \cap Z = \{0\}$, and $X-(Y \cup Z)$ is nonsingular. If the dimension of each component of $X-(Z \cup Y)$ is $\geqslant n$ and if Y is defined in X by k holomorphic equations, the pair $(X_\varepsilon - Z_\varepsilon, Y_\varepsilon - \{0\})$ is $(n-k-1)$-connected provided that $\varepsilon > 0$ is small enough.*

Now we can prove the following theorem:

Theorem 3.3. *Let X_1 and X_2 be two compact strongly pseudoconvex CR manifolds of dimension $2n-1 \geqslant 5$ which bound complex varieties V_1 and V_2 with only isolated normal singularities in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$, respectively. Let S_1 and S_2 be the singular sets of V_1 and V_2, respectively. Assume that the singular points of V_2 are isolated complete intersection singularities. If $\Phi : X_1 \to X_2$ is a nonconstant CR morphism, then Φ can be extended to a proper holomorphic covering map from V_1 to V_2 and $S_1 = \Phi^{-1}(S_2)$.*

Proof. The CR morphism $\Phi : X_1 \to X_2$ can be extended to a proper surjective holomorphic map from V_1 to V_2 such that $\Phi(S_1) \subset \Phi(S_2)$ and $\Phi : V_1 \setminus \Phi^{-1}(S_2) \to V_2 \setminus S_2$ is a covering map of degree d. In other words, $\Phi : X_1 \to X_2$ can be extended to a branched covering with ramification locus S_2. This is an immediate consequence of Proposition 3.1. For the convenience of the reader, we give a sketch of the proof. Let $\phi_1, \dots, \phi_{N_2}$ be the component functions of Φ.

Then ϕ_i as a CR holomorphic function on X_1 can be extended onto a one-sided neighborhood of X_1 in V_1. By Andreotti and Grauert [1962, Théorème 15], ϕ_i can be holomorphically extended onto $V_1 - S_1$. Since S_1 is a set consisting of only isolated normal singularities, ϕ_i can be holomorphically extended onto V_1. Then we get a map $\Phi : V_1 \to \mathbb{C}^{N_2}$. It is not hard to show that $\Phi(V_1) \subset V_2$, $\Phi(X_1) \subset X_2$, and $\Phi : V_1 \to V_2$ is proper. By the proper mapping theorem, $\Phi(V_1) \subset V_2$ is an analytic subset. Since V_1 is Stein, it can be shown that $\Phi(V_1) = V_2$, and hence $\Phi(X_1) = X_2$.

By the results of Fornaess [1976] and Pinčuk [1974], Φ is a local biholomorphism near X_1. So we can conclude that $\Phi : V_1 \to V_2$ is a branched covering.

By the theorem of Hamm, if V is a complex analytic subset of some neighborhood of the origin in $\mathbb{C}^N$ which is defined by k holomorphic equations, then the pair

$$(\mathbb{C}^N_\varepsilon - \{0\}, V_\varepsilon - \{0\})$$

is $(N-k-1)$-connected provided that $\varepsilon > 0$ is small enough. The punctured neighborhood $\mathbb{C}^N_\varepsilon - \{0\}$ of the origin is homotopy equivalent to an $(2N-1)$-dimensional sphere, and hence $(2N-2)$-connected. So we can conclude that $V_\varepsilon - \{0\}$ is $(N-k-2)$-connected by the long exact sequence of relative homotopy groups. In particular, the punctured neighborhood of an n-dimensional isolated complete intersection singularity is $n-2$ connected.

Let $q \in S_2$. Since q is an isolated complete intersection singularity of dimension $n \geqslant 3$ by assumption, the punctured neighborhood of q in V_2 is simply connected. Let $p \in \Phi^{-1}(q)$. Then there exists a connected open neighborhood U of p such that Φ map U to a neighborhood W of q as a branch covering with ramification locus $\{q\}$. We may assume that W is small enough such that $W \setminus \{q\}$ is simply connected and $U \cap \Phi^{-1}(q) = p$. Then we can conclude the covering map

$$\Phi : U \setminus \{p\} \to W \setminus \{q\}$$

is a biholomorphism. By the Hartogs extension theorem, $\Phi : U \to W$ is a biholomorphic map. This means the germ (V_1, p) is isomorphic to (V_2, q). So the proper map $\Phi : V_1 \to V_2$ is locally biholomorphic, and hence is a finite covering. □

Corollary 3.4. *Let X_1 and X_2 be two compact strongly pseudoconvex CR manifolds of dimension $2n-1 \geqslant 5$ which bound complex varieties V_1 and V_2 in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$, respectively. Assume that V_2 has only one isolated complete intersection singular point q. If there exists an isolated singular point $p \in V_1$ such that the embedding codimensions of (V_1, p) and (V_2, q) are different, then there is no nonconstant CR morphism from X_1 to X_2.*

Proof. Suppose on the contrary that $\Phi : X_1 \to X_2$ is a nonconstant CR morphism. By Theorem 3.3, Φ can be extended to a proper holomorphic covering map from V_1 to V_2. Then

$$\Phi : (V_1, p) \to (V_2, q)$$

is an isomorphism, and hence the embedding codimensions of (V_1, q_1) and (V_2, q_2) are equal. This leads to a contradiction. □

Let G be a finite group of locally analytic automorphisms of a smooth germ (U, p). Cartan [1957] has shown that this action can be linearized, that is, there exists a coordinate system for (U, p) in terms of which G will act linearly. So we may assume that U is a complex vector space, $p = 0$, and $G \subset \mathrm{GL}(U)$. The orbit space U/G is in a natural manner a normal analytic variety and the quotient map $U \to U/G$ is analytic. The germ of U/G at the image of 0 is called a quotient singularity. The statement below follows from the rigidity of isolated quotient singularities established by Schlessinger [1971] and is well known to specialists.

Corollary 3.5. *Isolated quotient singularities of dimension at least three cannot be complete intersections unless they are smooth.*

Proof. Suppose (V, q) is an n-dimensional isolated quotient singularity which is not smooth. We assume that U is a complex vector space, $p = 0$, $G \subset \mathrm{GL}(U)$, and $V = U/G$. Moreover, by [Prill 1967], it may be assumed that the group G is small, that is, it contains no matrix which has 1 as an eigenvalue of multiplicity $n-1$. Since G is small and the singularity is isolated, it follows that G acts freely on $U \setminus \{0\}$. Thus $\tau : U \to V$ is a finite-sheeted branched covering with ramification locus $\{0\}$ and $q = \tau(0)$. We may assume that V is an analytic subset of $\mathbb{C}^N$. Then the link

$$L_\varepsilon = V \cap \{z \in \mathbb{C}^N;\ |z| = \varepsilon\}, \quad \varepsilon > 0,$$

is a strongly pseudoconvex CR manifold. It is the boundary of the Stein space

$$V_\varepsilon = V \cap \{z \in \mathbb{C}^N;\ |z| < \varepsilon\}.$$

Then $\tau^{-1}(V_\varepsilon)$ is a Stein manifold and its boundary $\tau^{-1}(L_\varepsilon)$ is a strongly pseudoconvex CR manifold. It is obvious that the CR morphism $\tau : \tau^{-1}(L_\varepsilon) \to L_\varepsilon$ is a covering map. If $n \geqslant 3$ and (V, q) is a complete intersection, then

$$\tau : \tau^{-1}(V_\varepsilon) \to V_\varepsilon$$

is also a covering map by Theorem 3.3. In this case, the origin 0 must be a singular point of $\tau^{-1}(V_\varepsilon)$. This leads to a contradiction. □

4. Kohn–Rossi's cohomology

Let M be a complex manifold and $X \subset M$ a real hypersurface. Assume that X is locally defined by $r = 0$, where r is a real smooth function on M with $|dr| = 1$ on X. Let $\mathscr{A}^{p,q}$ be the sheaf of germs of smooth differential forms of type (p, q) on M. Let $\mathcal{A}^{p,q}$ be the space of sections of $\mathscr{A}^{p,q}$ over M and

$$\mathcal{C}^{p,q}(M) = \{\phi \in \mathcal{A}^{p,q}(M);\ \bar\partial r \wedge \phi = 0 \text{ on } X\}.$$

It is easy to show that

$$\bar\partial \mathcal{C}^{p,q}(M) \subset \mathcal{C}^{p,q+1}(M).$$

Let $\mathscr{C}^{p,q}$ denote the sheaf of germs of $\mathcal{C}^{p,q}(M)$ on M. Then there is a natural injection

$$0 \to \mathscr{C}^{p,q} \to \mathscr{A}^{p,q}.$$

The quotient sheaf

$$\mathscr{B}^{p,q} = (\mathscr{A}^{p,q})/\mathscr{C}^{p,q}$$

is a locally free sheaf supported on X. We have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{C}^{p,q} & \longrightarrow & \mathscr{A}^{p,q} & \longrightarrow & \mathscr{B}^{p,q} & \longrightarrow & 0 \\ & & \downarrow \bar\partial & & \downarrow \bar\partial & & \downarrow \bar\partial_b & & \\ 0 & \longrightarrow & \mathscr{C}^{p,q+1} & \longrightarrow & \mathscr{A}^{p,q+1} & \longrightarrow & \mathscr{B}^{p,q+1} & \longrightarrow & 0, \end{array}$$

where $\bar\partial_b$ is the quotient map which is induced by $\bar\partial$. Let $\mathcal{B}^{p,q}(X)$ denote the space of sections of $\mathscr{B}^{p,q}$. Since $\mathscr{C}^{p,q}$ is fine, the induced sequence of global sections

$$0 \to \mathcal{C}^{p,*}(M) \to \mathcal{A}^{p,*}(M) \to \mathcal{B}^{p,*}(X) \to 0$$

is exact. Since $\bar\partial^2 = 0$, it follows that $\bar\partial_b^2 = 0$, so we have the boundary complex

$$0 \to \mathcal{B}^{p,0}(X) \xrightarrow{\bar\partial_b} \mathcal{B}^{p,1}(X) \xrightarrow{\bar\partial_b} \cdots \xrightarrow{\bar\partial_b} \mathcal{B}^{p,n-1}(X) \to 0.$$

The cohomology of the above boundary complex is called the Kohn–Rossi cohomology and is denoted by $H^{p,q}(X)$. For more details, one can refer to [Folland and Kohn 1972; Tanaka 1975].

Theorem 4.1 [Yau 1981]. *Let X be a compact strongly pseudoconvex CR manifold of dimension at least five. If X is the boundary of a complex manifold which is a modification of a Stein space V at the isolated singularities $x_1, \dots, x_m$, then*

$$\dim H^{p,q}(X) = \sum_{i=1}^{m} \dim H^{q+1}_{\{x_i\}}(V, \Omega^p_V) \quad \textit{for } 1 \leqslant q \leqslant n-2.$$

Here, Ω^p_V is the sheaf of holomorphic p-forms on V.

As an application of Theorem 3.3 and Theorem 4.1, we obtain the following theorem:

Theorem 4.2. *Let X_1 and X_2 be two compact strongly pseudoconvex CR manifolds of dimension $2n-1 \geqslant 5$ which bound complex varieties V_1 and V_2 with only isolated normal singularities in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$, respectively. Assume that the singular points of V_2 are isolated complete intersection singularities. If there exists a nonconstant CR morphism from X_1 to X_2, then*

$$\dim H^{p,q}(X_1) = d \cdot \dim H^{p,q}(X_2), \quad 1 \leqslant q \leqslant n-2,\ 0 \leqslant p \leqslant n,$$

for some integer $d > 0$. In this case,

$$H^{p,q}(X_1) = 0 \quad \textit{for } p+q \leqslant n-2,\ 1 \leqslant q \leqslant n-2.$$

Proof. If $\Phi : X_1 \to X_2$ is a nonconstant CR morphism, then, by Theorem 3.3, Φ can be extended to a d-sheeted holomorphic covering map from V_1 to V_2. Let $x_1, \dots, x_m$ be the singular points of V_2. Then, by Theorem 4.1,

$$\dim H^{p,q}(X_2) = \sum_{i=1}^{m} \dim H^{q+1}_{\{x_i\}}(V_2, \Omega^p_{V_2}), \quad 1 \leqslant q \leqslant n-2,\ 0 \leqslant p \leqslant n.$$

The preimage $\Phi^{-1}(x_i)$ has d points for $1 \leqslant i \leqslant m$. These points are singular points of V_1, and for every $y \in \Phi^{-1}(x_i)$, the singularity (V_1, y) is isomorphic to (V_2, x_i). Again by Theorem 4.1,

$$\begin{aligned}
\text{(1)} \qquad \dim H^{p,q}(X_1) &= \sum_{i=1}^{m} \sum_{y \in \Phi^{-1}(x_i)} \dim H^{q+1}_{\{y\}}(V_1, \Omega^p_{V_1}) \\
&= \sum_{i=1}^{m} d \cdot \dim H^{q+1}_{\{x_i\}}(V_2, \Omega^p_{V_2}) \\
&= d \cdot \dim H^{p,q}(X_2)
\end{aligned}$$

for $1 \leqslant q \leqslant n-2$ and $0 \leqslant p \leqslant n$. The last statement follows from [Naruki 1977, Corollary 1.2.1]. □

References

[Andreotti and Grauert 1962] A. Andreotti and H. Grauert, "Théorème de finitude pour la cohomologie des espaces complexes", *Bull. Soc. Math. France* **90** (1962), 193–259. MR Zbl

[Cartan 1957] H. Cartan, "Quotient d'un espace analytique par un groupe d'automorphismes", pp. 90–102 in *Algebraic geometry and topology*, Princeton University Press, 1957. MR Zbl

[Folland and Kohn 1972] G. B. Folland and J. J. Kohn, *The Neumann problem for the Cauchy–Riemann complex*, Annals of Mathematics Studies **75**, Princeton University Press, 1972. MR

[Fornaess 1976] J. E. Fornaess, "Embedding strictly pseudoconvex domains in convex domains", *Amer. J. Math.* **98**:2 (1976), 529–569. MR Zbl

[Grauert 1962] H. Grauert, "Über Modifikationen und exzeptionelle analytische Mengen", *Math. Ann.* **146** (1962), 331–368. MR Zbl

[Grauert et al. 1994] H. Grauert, T. Peternell, and R. Remmert (editors), *Several complex variables. VII*, Encyclopaedia of mathematical sciences: Sheaf-theoretical methods in complex analysis **74**, Springer, Berlin, 1994. MR Zbl

[Hamm 1971] H. Hamm, "Lokale topologische Eigenschaften komplexer Räume", *Math. Ann.* **191** (1971), 235–252. MR Zbl

[Hamm 1981] H. A. Hamm, "On the vanishing of local homotopy groups for isolated singularities of complex spaces", *J. Reine Angew. Math.* **323** (1981), 172–176. MR Zbl

[Harvey and Lawson 1975] F. R. Harvey and H. B. Lawson, Jr., "On boundaries of complex analytic varieties. I", *Ann. of Math.* (2) **102**:2 (1975), 223–290. MR Zbl

[Harvey and Lawson 2000] F. R. Harvey and H. B. Lawson, Jr, "Addendum to Theorem 10.4 in 'Boundaries of analytic varieties'", preprint, 2000. arXiv math/0002195v1

[Hironaka 1964] H. Hironaka, "Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II", *Ann. of Math.* (2) **79** (1964), 109–326. MR Zbl

[Kohn and Rossi 1965] J. J. Kohn and H. Rossi, "On the extension of holomorphic functions from the boundary of a complex manifold", *Ann. of Math.* (2) **81** (1965), 451–472. MR

[Looijenga 2013] E. J. N. Looijenga, *Isolated singular points on complete intersections*, 2nd ed., Surveys of Modern Mathematics **5**, International Press, Somerville, MA, 2013. MR Zbl

[Boutet de Monvel 1975] L. Boutet de Monvel, "Intégration des équations de Cauchy–Riemann induites formelles", pp. art. id. 9 in *Séminaire Goulaouic–Lions–Schwartz* 1974–1975; *Équations aux derivées partielles linéaires et non linéaires*, 1975. MR Zbl

[Naruki 1977] I. Naruki, "Some remarks on isolated singularity and their application to algebraic manifolds", *Publ. Res. Inst. Math. Sci.* **13**:1 (1977), 17–46. MR Zbl

[Pinčuk 1974] S. I. Pinčuk, "Proper holomorphic maps of strictly pseudoconvex domains", *Sibirsk. Mat. Ž.* **15** (1974), 909–917. MR

[Prill 1967] D. Prill, "Local classification of quotients of complex manifolds by discontinuous groups", *Duke Math. J.* **34** (1967), 375–386. MR Zbl

[Schlessinger 1971] M. Schlessinger, "Rigidity of quotient singularities", *Invent. Math.* **14** (1971), 17–26. MR Zbl

[Tanaka 1975] N. Tanaka, *A differential geometric study on strongly pseudo-convex manifolds*, Lectures in Mathematics, Department of Mathematics, Kyoto University **9**, Kinokuniya Book-Store, Tokyo, 1975. MR Zbl

[Tu et al. 2013] Y.-C. Tu, S. S.-T. Yau, and H. Zuo, "Nonconstant CR morphisms between compact strongly pseudoconvex CR manifolds and étale covering between resolutions of isolated singularities", *J. Differential Geom.* **95**:2 (2013), 337–354. MR Zbl

[Yau 1981] S. S. T. Yau, "Kohn–Rossi cohomology and its application to the complex Plateau problem. I", *Ann. of Math.* (2) **113**:1 (1981), 67–110. MR Zbl

[Yau 2011] S. S.-T. Yau, "Rigidity of CR morphisms between compact strongly pseudoconvex CR manifolds", *J. Eur. Math. Soc.* (*JEMS*) **13**:1 (2011), 175–184. MR Zbl

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