Research Article

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Geometric nilpotent Lie algebras and zero-dimensional simple complete intersection singularities

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Abstract: The Levi theorem tells us that every finite-dimensional Lie algebra is the semi-direct product of a semi-simple Lie algebra and a solvable Lie algebra. Brieskorn gave the connection between simple Lie algebras and simple singularities. Simple Lie algebras have been well understood, but not the solvable (nilpotent) Lie algebras. Therefore, it is important to establish connections between singularities and solvable (nilpotent) Lie algebras. In this paper, we give a new connection between nilpotent Lie algebras and nilradicals of derivation Lie algebras of isolated complete intersection singularities. As an application, we obtain the correspondence between the nilpotent Lie algebras of dimension less than or equal to 7 and the nilradicals of derivation Lie algebras of isolated complete intersection singularities with modality less than or equal to 1. Moreover, we give a new characterization theorem for zero-dimensional simple complete intersection singularities.

Keywords: Derivation, nilpotent Lie algebra, isolated singularity, k-th Yau algebras

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Dedicated to Professor William Fulton on the occasion of his 82th birthday

1 Introduction

We use $\mathcal{O}_n$ to denote the local ring of holomorphic function germs $f(x_1, \dots, x_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. It has a unique maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$. For any $f \in \mathcal{O}_n$, we denote by $(V(f), 0)$ (or $(V, 0)$) the germ at the origin of $\mathbb{C}^n$ of the hypersurface $\{f = 0\} \subset \mathbb{C}^n$. If the origin is an isolated zero of $(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$, then $(V, 0)$ is a germ of isolated hypersurface singularity. The algebra

$$A(V) = \mathcal{O}_n\Big/\Big(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\Big)$$

is finite-dimensional. This algebra $A(V)$ is called the Tjurina algebra of $(V, 0)$, and its dimension $\tau(V)$ is called the Tjurina number.

Theorem 1.1 (Mather–Yau theorem [28]). *Let V_1 and V_2 be two isolated hypersurface singularities of the same dimension and let $A(V_1)$ and $A(V_2)$ be the Tjurina algebras. Then $(V_1, 0) \cong (V_2, 0)$ if and only if $A(V_1) \cong A(V_2)$.*

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For any isolated hypersurface singularity $(V(f), 0) \subset (\mathbb{C}^n, 0)$, in the early 1980s, the second author (cf. [33, 36–39]) firstly investigated the Lie algebra of derivations of the Tjurina algebra $A(V)$, i.e.,

$$L(V) := \mathrm{Der}(A(V), A(V)).$$

He firstly proved that $L(V)$ is a solvable Lie algebra (cf. [37, 39]). In order to distinguish it from Lie algebras of other types appearing in singularity theory [2, 3], one calls $L(V)$ the Yau algebra and its dimension $\lambda(V)$ the Yau number of V (cf. [9, 25, 42]). In the recent years, the authors (cf. [15–20, 22]) have introduced several new series of new Lie algebras, which are generalizations of the Yau algebra, for isolated hypersurface singularities. Based on our previous works, we believe that the Yau algebra and its generalizations will play an important role in singularity theory. In the past ten years, Yau, Zuo and their collaborators have been systematically studying various Lie algebras of isolated hypersurface singularities (see, e.g., [5–7, 13–23, 26, 40, 41]).

In [23], we generalized the above construction of $L(V)$ to isolated complete intersection singularities (we abbreviate it to ICIS in the sequel). Let $(X, 0)$ be an ICIS at the origin of $\mathbb{C}^n$ defined by an ideal $I_X = (f_1, \ldots, f_p) \subset \mathfrak{m}^2$. We consider the singular subspace of X, which is the analytic space germ SX defined by the ideal $SI_X \subset \mathfrak{m}$ generated by the f_i and all the $p \times p$ minors in the Jacobian matrix $(\frac{\partial f_i}{\partial x_j})$, $i = 1, \ldots, p$, $j = 1, \ldots, n$. For each ICIS X, it is natural to introduce for X the new derivation Lie algebras $\mathcal{NL}(X)$ as follows.

Definition 1.2. The new derivation Lie algebra $\mathcal{NL}(X)$ is defined to be the Lie algebra of derivations of the local Artinian algebra $\mathcal{O}_n/SI_X$, i.e., $\mathcal{NL}(X) = \mathrm{Der}(\mathcal{O}_n/SI_X)$. Its dimension is denoted by $\upsilon(V)$. This number $\upsilon(V)$ is a new numerical analytic invariant of the ICIS $(X, 0)$.

In [16, 18], we have obtained some new connections between isolated hypersurface singularities and solvable (nilpotent) Lie algebras. In this paper, one of the main purposes is to construct new connections between ICIS and nilpotent Lie algebras. Classification theory of semi-simple Lie algebras over complex numbers includes the killing form, root space decomposition, Dynkin diagrams, the Serre presentation etc. [12, 24]. Seeley [32] gave evidence that graphs and diagrams of nilpotent Lie algebras seem not good enough to classify the nilpotent Lie algebras. Historically, classification theories of semi-simple Lie algebras and solvable or nilpotent Lie algebras have a remarkable difference.

Nilpotent Lie algebras of dimension six were classified by Umlauf [34]. But this classification has several Lie algebras counted more than once because Umlauf was unaware of isomorphisms among them. After that, several attempts have been made to develop some methods that reformulate the classification problem. Morozov [29] introduced an inductive approach by considering the lower bound for the dimension of a maximal abelian ideal I of a nilpotent Lie algebra L. To use the inductive approach, one must know all smaller nilpotent Lie algebras and their finite-dimensional representations. Morozov's method is to consider L as a noncentral extension of L/I, where the abelian ideal I is a nontrivial L/I-module. This method is useful if one is able to classify the irreducible finite-dimensional representation of all known algebras and also familiar with isomorphisms among the resulting new algebras.

In the past decades, the progress towards a complete classification of nilpotent Lie algebras was quite slow. Gauger studied metabelian Lie algebras [10], and developed a method to classify them. But this method is not applicable to study the general nilpotent Lie algebras. Chao [4], Gauger [10] and Santharoubane [31] found that, in dimensions seven, eight and nine, continuous families of non-isomorphic nilpotent algebras occur. Magnin [27] introduced another inductive approach to classify nilpotent Lie algebras. He classified algebras up to dimension six over the real field, and obtained partial information in dimension seven. Safiullina [30] compiled a list of seven-dimensional algebras by using Morozov's approach and knowledge of low-dimensional representations of two- and three-dimensional algebras. Seeley [32] classified the seven-dimensional nilpotent Lie algebras by using a different inductive approach, and his classification consists of 161 tables. These 161 tables were divided into two main parts: the first part has 130 indecomposable algebras and the second part consists of 31 decomposable algebras with six continuous families parametrized by a single complex variable.

To state our main results, we recall that the lower central series of Lie algebras L is a sequence of ideals $L_{(i)}$ defined inductively by $L_{(0)} = L$ and $L_{(i)} = [L, L_{(i-1)}]$, $i = 1, 2, 3, \ldots$. A Lie algebra L is called nilpotent if the lower central series of ideals, i.e., $L_{(0)}, L_{(1)}, L_{(2)}, \dots$, terminates. The nilradical of a finite-dimensional Lie algebra L is its maximal nilpotent ideal. If $h \subset L$ is an ideal, then the generalized center is

$$GC(h) = \{x \in L \mid [x, y] \in h \text{ for all } y \in L\}.$$

The upper central series of a Lie algebra L is a sequence of ideals $C^i(L)$ defined inductively by $C^0(L) = GC(0)$ and $C^{i+1}(L) = GC(C^i(L))$. Note that $C^i(L) \subset C^{i+1}(L)$.

With the same notation as in [27, 32], we shall use the list of central series dimensions (lower central series are used in Theorem 1.3, and upper central series are used in Theorem 1.4) to denote nilpotent Lie algebras. For example, the algebras having upper central series dimensions (resp. lower central series) $\{2, 4, 7\}$ (resp. $\{6, 3, 1\}$) are listed as $2, 4, 7_A$, $2, 4, 7_B$ (resp. $6, 3, 1_A$, $6, 3, 1_B$) and so forth. Note that the subscripts A and B are used to differentiate the two non-isomorphic nilpotent Lie algebras.

Classification of nilpotent Lie algebras with dimension up to 7 is known, but not for dimension greater than 7. In Theorem 1.3, the nilpotent Lie algebras of dimension less than or equal to 6 were classified using lower central series dimensions.

Theorem 1.3 ([27, pp. 122–124]). *The classification of nilpotent Lie algebras of dimension less than or equal to* 6 *have the following list (here we use lower central series dimensions):*

- *Dimension* 1 *(one nilpotent Lie algebra):*

$$\{g_1; \textit{abelian algebra}, 1_A\}.$$

- *Dimension* 2 *(one nilpotent Lie algebra):*

$$\{(g_1)^2 = g_1 \times g_1; 2_A\}.$$

- *Dimension* 3 *(two nilpotent Lie algebras):*

$$\{(g_1)^3; 3_A\}, \quad \{n : [X_1, X_2] = X_3; 3, 1_A\}.$$

- *Dimension* 4 *(three nilpotent Lie algebras):*

$$\{(g_1)^4; 4_A\}, \quad \{n \times g_1; 4, 1_A\}, \quad \{g_4 : [X_1, X_2] = X_3, [X_1, X_3] = X_4; 4, 2, 1_A\}.$$

- *Dimension* 5 *(nine nilpotent Lie algebras):*

$$\begin{aligned}
&\{(g_1)^5; 5_A\},\\
&\{n \times g_1^2; 5, 1_A\},\\
&\{g_{5,1} : [X_1, X_2] = X_5, [X_3, X_4] = X_5; 5, 1_B\},\\
&\{g_4 \times g_1; 5, 2, 1_A\},\\
&\{g_{5,2} : [X_1, X_2] = X_4, [X_1, X_3] = X_5; 5, 2_A\},\\
&\{g_{5,3} : [X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_2, X_5] = X_4; 5, 2, 1_B\},\\
&\{g_{5,4} : [X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_2, X_3] = X_5; 5, 3, 2_A\},\\
&\{g_{5,5} : [X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_1, X_4] = X_5; 5, 3, 2, 1_A\},\\
&\{g_{5,6} : [X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_1, X_4] = X_5, [X_2, X_3] = X_5; 5, 3, 2, 1_B\}.
\end{aligned}$$

- *Dimension* 6*: Direct Product (ten classes):*

$$\{(g_1^6); 6_A\}, \quad n \times n, \quad \{n \times (g_1^3); 6, 1_A\}, \quad g_4 \times (g_1^2), \quad g_1 \times g_{5,i} \quad (1 \le i \le 6),$$

where we use $6, 1_B$ *to denote* $g_1 \times g_{5,1}$.

- *The other* 22 *classes are as follows:*

$$\begin{aligned}
&\{[X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_1, X_5] = X_6; 6, 3, 1_A\},\\
&\{[X_1, X_2] = X_6, [X_1, X_3] = X_4, [X_1, X_4] = X_5, [X_2, X_3] = X_5; 6, 3, 1_B\}, \dots,\\
&\qquad [X_1, X_3] = X_4, [X_1, X_4] = X_6, [X_2, X_5] = X_6; 6, 3, 2, 1_C\}.
\end{aligned}$$

In Theorem 1.4, the nilpotent Lie algebras of dimension 7 were classified using upper central dimensions. The classification of nilpotent Lie algebras consists of indecomposable and decomposable algebras. With similar notation to [32], $2 \oplus 1, 3, 5$ denotes the direct sum of a two-dimensional abelian Lie algebra with the unique algebra $\{1, 3, 5\}$ whose upper central series dimensions are $1, 3, 5$.

Theorem 1.4 ([32, pp. 482–494]). *For the classification of nilpotent Lie algebras of dimension 7, we have the following list (here we use upper central series dimensions):*

- *Decomposable algebras with large centers:*

$$\begin{aligned}
&\{0\ (7\text{-dimensional abelian}); 7\},\\
&\{[a,b]=c; 4\oplus 1,3\},\\
&\{[a,b]=d, [a,c]=e; 2\oplus 2,5\}\\
&\{[a,b]=d, [a,c]=e, [b,c]=f; 1\oplus 3,6\},\\
&\{[a,b]=c, [a,c]=d; 3\oplus 1,2,4\}\\
&\{[a,b]=c, [a,c]=d, [b,c]=e; 2\oplus 2,3,5\}.
\end{aligned}$$

- *Central series dimensions 3,7: Decomposables:*

$$\begin{aligned}
&\{[a,b]=e, [c,d]=e; 2\oplus 1,5\},\\
&\{[a,b]=e, [a,c]=f, [c,d]=e; 1\oplus 2,6\}\\
&\{[a,b]=e, [c,d]=f; 1\oplus 1,3\oplus 1,3\}.
\end{aligned}$$

  Indecomposables:

$$\begin{aligned}
&\{[a,b]=e, [b,c]=f, [b,d]=g; 3,7_A\},\\
&\{[a,b]=e, [b,c]=f, [c,d]=g; 3,7_B\}\\
&\{[a,b]=e, [b,c]=f, [c,d]=e, [b,d]=g; 3,7_C\},\\
&\{[a,b]=e, [a,c]=f, [b,d]=g, [c,d]=e; 3,7_D\}.
\end{aligned}$$

- *Central series dimensions $3, 5, 7$: These lists continue analogously and we omit them until the following ones.*
- *Central series dimensions $1, 2, 3, 4, 5, 7$:*

$$\begin{aligned}
&\{[a,b]=c, [a,c]=d, [a,d]=e, [a,e]=f, [a,f]=g; 1,2,3,4,5,7_A\},\\
&\{[a,b]=c, [a,c]=d, [a,d]=e, [a,e]=f, [a,f]=g, [b,c]=g; 1,2,3,4,5,7_B\},\\
&\quad\vdots\\
&\{[a,b]=c, [a,c]=d, [a,d]=e, [a,e]=f, [a,f]=g, [b,c]=e, [b,d]=f, [b,e]=\xi g,\\
&\qquad [c,d]=(1-\xi)g; 1,2,3,4,5,7_I : \xi, (\xi\neq 1)\}.
\end{aligned}$$

Recall that the classifications of contact simple and unimodal complete intersection singularities were done by Giusti [11] and Aleksandrov [1]. The classification of the contact simple complete intersections (SCI) which are not hypersurface singularities (i.e., with modality 0) is as follows [11].

(1) Zero-dimensional simple complete intersection singularities:

$$\begin{aligned}
&\text{Type } F^{q,r}_{q+r-1}: && (xy,\ x^q+y^r), && q,r\geq 2,\\
&\text{Type } G_5: && (x^2,\ y^3),\\
&\text{Type } G_7: && (x^2,\ y^4),\\
&\text{Type } H_\mu: && (x^2+y^{\mu-3},\ xy^2), && \mu\geq 6,\\
&\text{Type } I_{2q-1}: && (x^2+y^3,\ y^q), && q\geq 4,\\
&\text{Type } I_{2r+2}: && (x^2+y^3,\ xy^r), && r\geq 3.
\end{aligned}$$

(2) Simple complete intersection curve singularities:

$$\begin{aligned}
&\text{Type } S_\mu: && (x^2+y^2+z^{\mu-3},\ yz), \quad \mu \geq 5,\\
&\text{Type } T_7: && (x^2+y^3+z^3,\ yz),\\
&\text{Type } T_8: && (x^2+y^3+z^4,\ yz),\\
&\text{Type } T_9: && (x^2+y^3+z^5,\ yz),\\
&\text{Type } U_7: && (x^2+yz,\ xy+z^3),\\
&\text{Type } U_8: && (x^2+yz+z^3,\ xy),\\
&\text{Type } U_9: && (x^2+yz,\ xy+z^4),\\
&\text{Type } W_8: && (x^2+z^3,\ y^2+xz),\\
&\text{Type } W_9: && (x^2+yz^2,\ y^2+xz),\\
&\text{Type } Z_9: && (x^2+z^3,\ y^2+z^3),\\
&\text{Type } Z_{10}: && (x^2+yz^2,\ y^2+z^3).
\end{aligned}$$

Note that all SCI singularities are weighted homogeneous singularities.

Aleksandrov [1, p. 21] and Wall [35] have obtained the classification in Table 1 for weighted homogeneous unimodal complete intersection singularities (here we use the notations in Aleksandrov's article).

Type(V)	**Equations**
T_{10}	$\{(x^2+y^3+z^6,\ ax+yz),\ 27a^6+4\neq 0\}$
T_k	$\{(x^2+y^3+z^{k-4},\ yz),\ k\geq 11\}$
R_9	$\{(x^2+y^4+z^4,\ ax+yz),\ 4a^4-1\neq 0\}$
R_k	$\{(x^2+y^4+z^{k-5},\ yz),\ k\geq 10\}$
$L_{2,q,r}$	$\{(x^2+y^q+z^r,\ yz),\ q,r\geq 5\}$
U_{11}	$\{(x^2+yz+z^4+axz^2,\ xy),\ a^2-4\neq 0\}$
U_{13}	$(x^2+yz,\ xy+z^6)$
U_{14}	$(x^2+z^5+yz,\ xy)$
U_{15}	$(x^2+yz,\ xy+z^7)$
V_{10}	$\{(x^3+yz+z^3+ax^2z,\ xy),\ 4a^3+27\neq 0\}$
V_{12}	$(x^4+yz+z^3,\ xy)$
V_{13}	$(yz+z^3,\ xy+x^4)$
V_{11}	$(x^5+yz+z^3,\ xy)$
Q_{13}	$(x^3+yz,\ xy+z^4)$
Q_{11}	$(x^3+yz+z^4,\ xy)$
$L_{3,2,4}$	$(x^4+y^2+z^3,\ xy)$
$L_{3,2,5}$	$(x^5+y^2+z^3,\ xy)$
Y_{11}	$\{(x^2+y^3+z^3+ay^2z,\ xy),\ 4a^3+27\neq 0\}$
G_{14}	$(x^2+y^3z+z^3,\ xy)$
H_{13}	$(x^2+y^2z,\ xy+z^3)$
H_{14}	$(x^2+y^3,\ xy+z^3)$
M_{11}	$\{(x^2+z^4,\ y^2+z^3+axz),\ a^2+1\neq 0\}$
M_{12}	$(x^2+yz^3,\ y^2+z^3)$
M_{13}	$(x^2+z^5,\ y^2+z^3)$
M_{14}	$(x^2+yz^4,\ y^2+z^3)$
N_{13}	$(x^2+yz^3,\ y^2+xz)$
N_{14}	$(x^2+z^5,\ y^2+xz)$

Table 1: The classification of weighted homogeneous unimodal complete intersection singularities.

In [21], we obtain an explicit connection between the nilpotent Lie algebras of dimension less than or equal to 7 and the nilradical of several series derivation Lie algebras for isolated hypersurface singularities. In this paper, we investigate the new connection between the nilpotent Lie algebras of dimension less than or equal to 7 and the nilradical of $\mathcal{NL}(V)$ for ICIS. The nilradical of $\mathcal{NL}(V)$ is denoted by $(\mathcal{NL}(V))^*$. We call the nilpotent Lie subalgebra $(\mathcal{NL}(V))^*$ of $\mathcal{NL}(V)$, associated to an ICIS, geometric ICIS nilpotent Lie algebra. The dimension of a nilpotent Lie subalgebra $(\mathcal{NL}(V))^*$ is denoted by $\upsilon^*(V)$. A complete classification of nilpotent Lie algebras up to dimension seven was given in [27, 32]. A natural question is whether these nilpotent Lie algebras in the above lists in Theorem 1.3 and Theorem 1.4 (i.e., the classification nilpotent Lie algebras up to dimension seven) are geometric ICIS nilpotent Lie algebras? In general, this question is very hard to asnwer. In this paper, we answer this question in some sense for the nilradical of ICIS as follows.

Theorem A. *Let*

$$(V,0) = \{(x_1, x_2, \dots, x_m) \in \mathbb{C}^m : f_i(x_1, x_2, \dots, x_m) = 0,\ 1 \le i \le k\}, \quad m \ge 2,$$

be an n-dimensional ($n = m - k$) ICIS with modality less than or equal to 1 *and let the* $(\mathcal{NL}(V))^*$ *be the nilradical of the new derivation Lie algebra* $\mathcal{NL}(V)$. *Assume that* $\dim(\mathcal{NL}(V))^* \le 7$ *and it is isomorphic to one of the nilpotent Lie algebras in the given list of classification of nilpotent Lie algebras of dimension less than or equal to* 7 *(cf. Theorem 1.3 and Theorem 1.4). Then the* $(V,0)$*'s are either zero-dimensional with embedding dimension* 2 *or one-dimensional singularities with embedding dimension* 3*, and their corresponding nilradicals* $(\mathcal{NL}(V))^*$ *are as follows:*

$$\begin{aligned}
&(1): && (xy, x^2+y^2); 1_A,\\
&(2): && (xy, x^2+y^3); 3, 1_A,\\
&(3): && (xy, x^2+y^4); 4, 1_A,\\
&(4): && (xy, x^2+y^5); 5, 1_4,\\
&(5): && (xy, x^2+y^6); 6, 2, 1_A,\\
&(6): && (x^2, y^3); 5, 2, 1_B,\\
&(7): && (x^2+y^3, xy^2); 1, 3, 5, 7_O,\\
&(8): && (x^2+y^2+z^2, yz); 3_A,\\
&(9): && (x^2+y^2+z^3, yz); 5, 1_B,\\
&(10): && (x^2+y^2+z^4, yz); 1, 4, 7_A,\\
&(11): && (xy, x^2+y^7); 1, 4, 5, 7_B.
\end{aligned}$$

Elashvili and Khimshiashvili [9] firstly used the Yau algebra $L(V)$ to distinguish the simplest isolated hypersurface singularities, i.e., the ADE singularities. They proved the following beautiful result: if X and Y are two simple singularities except for the pair A_6 and D_5, then $L(X) \cong L(Y)$ as Lie algebras if and only if X and Y are analytically isomorphic.

Motivated by the result of Elashvili and Khimshiashvili, in this paper, we show that the zero-dimensional simple complete intersection singularities $(X,0)$ can be characterized completely by the new Lie algebra $\mathcal{NL}(X)$. Our result below extends the main results in [8, 9] to ICIS in some sense.

Theorem B. *If X and Y are zero-dimensional simple complete intersection singularities, then* $\mathcal{NL}(X) \cong \mathcal{NL}(Y)$ *as Lie algebras if and only if X and Y are contact equivalent.*

2 Proof of Theorems

In order to prove Theorem A, we need to use the following results.

After simple calculations, we have Tables 2 and 3.

From Table 1 and Table 3, it is easy to see that we get eleven isomorphic pairs of nilpotent Lie algebras in the following way.

Type(*V*)	Equations	*v*(*V*)	*v**(*V*)
$F_{q+r-1}^{q,r}$	$\{(xy, x^q + y^r),\ q, r \geq 2\}$	$q+r$	$q+r-2$
G_5	(x^2, y^3)	7	5
G_7	(x^2, y^4)	10	8
H_6	$(x^2 + y^3, xy^2)$	8	5
H_μ	$\{(x^2 + y^{\mu-3}, xy^2),\ \mu \geq 7\}$	$\mu+2$	$\mu+1$
I_{2q-1}	$\{(x^2 + y^3, y^q),\ q \geq 4\}$	$2q+1$	$2q$
I_{2r+2}	$\{(x^2 + y^3, xy^r),\ r \geq 3\}$	$2r+4$	$2r+3$
S_5	$(x^2 + y^2 + z^2, yz)$	7	3
S_μ	$\{(x^2 + y^2 + z^{\mu-3}, yz),\ \mu \geq 6\}$	$\mu+2$	μ
T_7	$(x^2 + y^3 + z^3, yz)$	9	8
T_8	$(x^2 + y^3 + z^4, yz)$	10	9
T_9	$(x^2 + y^3 + z^5, yz)$	11	10
U_7	$(x^2 + yz, xy + z^3)$	10	8
U_8	$(x^2 + yz + z^3, xy)$	11	10
U_9	$(x^2 + yz, xy + z^4)$	13	11
W_8	$(x^2 + z^3, y^2 + xz)$	12	10
W_9	$(x^2 + yz^2, y^2 + xz)$	13	12
Z_9	$(x^2 + z^3, y^2 + z^3)$	14	12
Z_{10}	$(x^2 + yz^2, y^2 + z^3)$	15	14

Table 2: The v and v^* of zero-dimensional and simple complete intersection singularities.

Type(*V*)	Equations	*v*(*V*)	*v**(*V*)
T_{10}	$\{(x^2 + y^3 + z^6, ax + yz),\ 27a^6 + 4 \neq 0\}$	12	11
T_k	$\{(x^2 + y^3 + z^{k-4}, yz),\ k \geq 11\}$	$k+2$	$k+1$
R_9	$\{(x^2 + y^4 + z^4, ax + yz),\ 4a^4 - 1 \neq 0\}$	11	10
R_k	$\{(x^2 + y^4 + z^{k-5}, yz),\ k \geq 10\}$	$k+2$	$k+1$
$L_{2,q,r}$	$\{(x^2 + y^q + z^r, yz),\ q, r \geq 5\}$	$q+r+3$	$q+r+2$
U_{11}	$\{(x^2 + yz + z^4 + axz^2, xy),\ a^2 - 4 \neq 0\}$	15	14
U_{13}	$(x^2 + yz, xy + z^6)$	19	17
U_{14}	$(x^2 + z^5 + yz, xy)$	19	18
U_{15}	$(x^2 + yz, xy + z^7)$	22	20
V_{10}	$\{(x^3 + yz + z^3 + ax^2z, xy),\ 4a^3 + 27 \neq 0\}$	14	13
V_{12}	$(x^4 + yz + z^3, xy)$	17	16
V_{13}	$(yz + z^3, xy + x^4)$	19	18
V_{11}	$(x^5 + yz + z^3, xy)$	20	19
Q_{13}	$(x^3 + yz, xy + z^4)$	20	18
Q_{11}	$(x^3 + yz + z^4, xy)$	20	19
$L_{3,2,4}$	$(x^4 + y^2 + z^3, xy)$	20	19
$L_{3,2,5}$	$(x^5 + y^2 + z^3, xy)$	23	22
Y_{11}	$\{(x^2 + y^3 + z^3 + ay^2z, xy),\ 4a^3 + 27 \neq 0\}$	17	16
G_{14}	$(x^2 + y^3z + z^3, xy)$	21	20
H_{13}	$(x^2 + y^2z, xy + z^3)$	20	19
H_{14}	$(x^2 + y^3, xy + z^3)$	23	21
M_{11}	$\{(x^2 + z^4, y^2 + z^3 + axz),\ a^2 + 1 \neq 0\}$	16	15
M_{12}	$(x^2 + yz^3, y^2 + z^3)$	17	16
M_{13}	$(x^2 + z^5, y^2 + z^3)$	18	17
M_{14}	$(x^2 + yz^4, y^2 + z^3)$	19	18
N_{13}	$(x^2 + yz^3, y^2 + xz)$	19	18
N_{14}	$(x^2 + z^5, y^2 + xz)$	22	20

Table 3: The v and v^* of weighted homogeneous unimodal complete intersection singularities.

Proposition 2.1. *Let*

$$(V,0) = \{(x_1, x_2, \ldots, x_m) \in \mathbb{C}^m : f_i(x_1, x_2, \ldots, x_m) = 0,\ 1 \le i \le k\}, \quad m \ge 2,$$

be an n-dimensional ($n = m - k$) ICIS with modality ≤ 1 and let $(\mathcal{NL}(V))^*$ be the nilradical of the new derivation Lie algebra $\mathcal{NL}(V)$. Assume that $\dim(\mathcal{NL}(V))^* \le 7$. *Then the following nilpotent Lie algebras with notation* [27, 32] *are pairwise isomorphic:*

$$\begin{aligned}
&(1): && (\mathcal{NL}(xy, x^2+y^2))^* \cong 1_A,\\
&(2): && (\mathcal{NL}(xy, x^2+y^3))^* \cong 3, 1_A,\\
&(3): && (\mathcal{NL}(xy, x^2+y^4))^* \cong 4, 1_A,\\
&(4): && (\mathcal{NL}(xy, x^2+y^5))^* \cong 5, 1_B,\\
&(5): && (\mathcal{NL}(xy, x^2+y^6))^* \cong 6, 2, 1_A,\\
&(6): && (\mathcal{NL}(x^2, y^3))^* \cong 5, 2, 1_B,\\
&(7): && (\mathcal{NL}(x^2+y^3, xy^2))^* \cong 1, 3, 5, 7_O,\\
&(8): && (\mathcal{NL}(x^2+y^2+z^2, yz))^* \cong 3_A,\\
&(9): && (\mathcal{NL}(x^2+y^2+z^3, yz))^* \cong 5, 1_B,\\
&(10): && (\mathcal{NL}(x^2+y^2+z^4, yz))^* \cong 1, 4, 7_A,\\
&(11): && (\mathcal{NL}(xy, x^2+y^7))^* \cong 1, 4, 5, 7_B.
\end{aligned}$$

Proof. After simple calculations, we have Table 4.

Magnin [27] gave the complete classification of nilpotent Lie algebras of dimension less than or equal to 6. Since we deal with the nilradical $(\mathcal{NL}(V))^*$ with dimension less than or equal to 7, it is easy to see from [27, 32] that we have the possible classification of nilpotent Lie algebras shown in Table 4.

It easily follows from Table 4 and Table 5 that we only need to deal with (5), (6), (7), (10) and (11).

It is easy to check that (5) is true under the following map:

$$\begin{aligned}
(5):\ & e_1 \mapsto X_1 + 2X_2 + X_3 + X_4 + X_5 + X_6, \quad e_2 \mapsto 2X_3 + X_4 + X_6, \quad e_3 \mapsto 2X_4 + X_6,\\
& e_4 \mapsto X_6, \quad e_5 \mapsto X_2 - 3X_4 + 2X_5 + X_6, \quad e_6 \mapsto X_2 - X_4 + X_5 + X_6.
\end{aligned}$$

Similarly, (6), (7), (10) and (11) cases are true under the following map:

$$\begin{aligned}
(6):\quad & e_1 \mapsto X_1 + X_2 + X_3 + X_4 + X_5, \quad e_2 \mapsto -X_4, \quad e_3 \mapsto -2X_3 + 4X_4,\\
& e_4 \mapsto X_3 + X_4, \quad e_5 \mapsto 4X_2 + X_3 + X_4 + X_5.\\
(7):\quad & e_1 \mapsto 4b - 4c - 4d + e + f + g, \quad e_2 \mapsto a + b + c + d + e + f + g,\\
& e_3 \mapsto 2c + 4d - 2e - 10f, \quad e_4 \mapsto -e + f + 2g, \quad e_5 \mapsto 4g,\\
& e_6 \mapsto -c + 2d + e + f + g, \quad e_7 \mapsto 4f - 4g,\\
(10):\quad & e_1 \mapsto a + d + e + f + g, \quad e_2 \mapsto \frac{2f}{9} + g, \quad e_3 \mapsto \frac{-2g}{9},\\
& e_4 \mapsto \frac{-\sqrt{6}b}{3} + \frac{\sqrt{6}c}{3} - d + e + f + g, \quad e_5 \mapsto b + c + d + e + f + g,\\
& e_6 \mapsto \frac{-\sqrt{6}d}{18} + \frac{\sqrt{6}e}{18} + \frac{g}{6}, \quad e_7 \mapsto \frac{-d}{9} + \frac{-e}{9} + \frac{g}{9},
\end{aligned}$$

and

$$(11):\quad e_1 \mapsto 6^{\frac{2}{5}}a, \quad e_2 \mapsto -\frac{6^{\frac{4}{5}}b}{6}, \quad e_3 \mapsto 6^{\frac{1}{5}}c, \quad e_4 \mapsto -\frac{6^{\frac{3}{5}}d}{2}, \quad e_5 \mapsto g, \quad e_6 \mapsto e, \quad e_7 \mapsto f.$$

The proposition is proved. □

Remark 2.2. These isomorphisms above can be obtained by using the Maple program which we list in Section A.

$(\mathcal{NL}(\boldsymbol{V}))^*$	Multiplication table	Lower and upper central series dimensions
$(\mathcal{NL}(xy, x^2+y^2))^* = \langle -y\partial_2 - x\partial_1 \rangle$	0	1
$(\mathcal{NL}(xy, x^2+y^3))^* = \langle -x\partial_2, -y^2\partial_1, -y^2\partial_2 \rangle$	$[e_1, e_2] = e_3$	3, 1
$(\mathcal{NL}(xy, x^2+y^4))^* = \langle y^3\partial_1, x\partial_2, y^3\partial_2, y^2\partial_2 \rangle$	$[e_1, e_2] = e_3$	4, 1
$(\mathcal{NL}(xy, x^2+y^5))^* = \langle -y^3\partial_2, -y^2\partial_2, -x\partial_2, -y^4\partial_1, -y^4\partial_2 \rangle$	$[e_1, e_2] = e_5, [e_3, e_4] = e_5$	5, 1
$(\mathcal{NL}(xy, x^2+y^6))^* = \langle y^2\partial_2, y^3\partial_2, y^4\partial_2, y^5\partial_2, x\partial_2, y^5\partial_1 \rangle$	$[e_1, e_2] = e_3, [e_1, e_3] = 2e_4, [e_5, e_6] = -e_4$	6, 2, 1
$(\mathcal{NL}(x^2, y^3))^* = \langle x\partial_2, xy\partial_2, y^2\partial_2, xy\partial_1, y^2\partial_1 \rangle$	$[e_1, e_3] = 2e_2, [e_1, e_4] = -e_2, [e_1, e_5] = -e_3 + 2e_4$	5, 2, 1
$(\mathcal{NL}(x^2+y^3, xy^2))^* = \langle -y^2\partial_1, -y^2\partial_1 - x\partial_2, -y^2\partial_2, -xy\partial_2, -y^3\partial_2, -xy\partial_1, -y^3\partial_1 \rangle$	$[e_1, e_3] = -e_3 + 2e_6, [e_1, e_3] = 2e_7, [e_1, e_4] = -e_5,$ $[e_1, e_6] = -e_7, [e_2, e_3] = -2e_4 + 2e_7, [e_2, e_6] = e_4, [e_2, e_7] = e_5$	1, 3, 5, 7
$(\mathcal{NL}(x^2+y^2+z^2, xy))^* = \langle -z^2\partial_1, -z^2\partial_2, -z^2\partial_3 \rangle$	0	3
$(\mathcal{NL}(x^2+y^2+z^3, yz))^* = \langle z^3\partial_2, \frac{-3z^2\partial_2}{2} + y\partial_3, z^3\partial_1, \frac{5z^2\partial_1}{2} + x\partial_3, z^3\partial_3, \rangle$	$[e_1, e_2] = e_5, [e_3, e_4] = e_5$	5, 1
$(\mathcal{NL}(x^2+y^2+z^4, yz))^* = \langle -z^2\partial_3, -z^3\partial_3, -z^4\partial_3, 2z^3\partial_2 - y\partial_3, -3z^3\partial_1 - x\partial_3, -z^4\partial_2, -z^4\partial_1 \rangle$	$[e_1, e_2] = -e_3, [e_1, e_4] = 6e_6, [e_1, e_5] = -9e_7,$ $[e_4, e_6] = e_3, [e_5, e_7] = e_3$	1, 4, 7
$(\mathcal{NL}(xy, x^2+y^7))^* = \langle -y^2\partial_2, -y^3\partial_2, -y^4\partial_2, -y^5\partial_2, -y^6\partial_2, -x\partial_2, -y^6\partial_1 \rangle$	$[e_1, e_2] = -e_3, [e_1, e_3] = -2e_4, [e_1, e_4] = -3e_5,$ $[e_2, e_3] = -e_5, [e_6, e_7] = e_5$	1, 4, 5, 7

Table 4: The nilradical of ICIS with modality less than or equal to 1.

Dimension of nilpotent Lie algebras	Multiplication table	Lower and upper central series dimensions
1	0	1_A
5	$[X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_2, X_5] = X_4$	$5, 2, 1_B$
3	0	3_A
3	$[X_1, X_2] = X_3$	$3, 1_A$
4	$[X_1, X_2] = X_3$	$4, 1_A$
5	$[X_1, X_2] = X_5, [X_3, X_4] = X_5$	$5, 1_B$
6	$[X_1, X_3] = X_4, [X_1, X_4] = X_6, [X_2, X_5] = X_6$	$6, 2, 1_A$
5	$[X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_2, X_5] = X_4$	$5, 2, 1_B$
7	$[a, b] = c, [a, c] = e, [a, d] = f, [a, f] = g, [b, c] = f, [b, e] = g$	$1, 3, 5, 7_O$
7	$[a, b] = d, [a, c] = e, [a, f] = g, [b, e] = g, [c, d] = g$	$1, 4, 7_A$
7	$[a, b] = c, [a, c] = d, [a, d] = g, [b, c] = g, [e, f] = g$	$1, 4, 5, 7_B$

Table 5: Classification of nilpotent Lie algebras [27, 32].

Proposition 2.3. *Let*

$$V = \{x, y \in \mathbb{C}^2 : (x^2 + y^{\mu-3},\ xy^2),\ \mu \geq 7\}$$

be the H_μ zero-dimensional simple complete intersection singularity. Then

$$\mathcal{NL}(V) = \mu + 2.$$

Proof. It is easy to see that the moduli algebra of H_μ series is given by

$$\mathbb{C}\{x, y\}/(f, g, M_1) = \langle x, xy, 1, y, y^2, y^3, \ldots, y^{(\mu-3)} \rangle.$$

When $\mu \geq 7$, then the Lie algebra $\mathcal{NL}(V)$ has the following basis:

$$\begin{gathered} e_1 = -(\mu-3)x\partial_1 - 2y\partial_2, \quad e_2 = -y^{\mu-4}\partial_1, \quad e_3 = -y^2\partial_2, \ldots, \\ e_{\mu-4} = -y^{(\mu-5)}\partial_2, \quad e_{\mu-3} = -y^{(\mu-4)}\partial_1 - x\partial_2, \quad e_{\mu-2} = -y^{\mu-4}\partial_2, \\ e_{\mu-1} = -xy\partial_2, \quad e_\mu = -y^{\mu-3}\partial_2, \quad e_{\mu+1} = -xy\partial_1, \quad e_{\mu+2} = -y^{\mu-3}\partial_1. \end{gathered}$$

The nilradical $(\mathcal{NL}(V))^*$ of $\mathcal{NL}(V)$ is given by

$$(\mathcal{NL}(V))^* = \langle e_2, e_3, e_4, \ldots, e_{\mu+2} \rangle.$$

When $\mu = 7$, then the nilradical $(\mathcal{NL}(V))^*$ has the following multiplication table:

$$\begin{aligned} &[e_2, e_3] = 3e_9, && [e_2, e_4] = -e_5, && [e_2, e_6] = -e_7, && [e_2, e_8] = -e_9, \\ &[e_3, e_4] = 2e_6 - 3e_9, && [e_3, e_5] = -e_7, && [e_4, e_8] = e_6, && [e_4, e_9] = e_7. \end{aligned}$$

Case 1: When μ is odd and $\mu \geq 9$, then the nilradical $(\mathcal{NL}(V))^*$ has the following multiplication table:

$$\begin{gathered} [e_2, e_3] = (\mu-4)e_{\mu+2}, \quad [e_2, e_{\mu-3}] = -e_{\mu-2}, \quad [e_2, e_{\mu-1}] = -e_\mu, \quad [e_2, e_{\mu+1}] = -e_{\mu+2}, \\ [e_3, e_4] = -e_5, \quad [e_3, e_5] = -2e_6, \quad [e_3, e_6] = -3e_7, \ldots, [e_3, e_{\mu-4}] = -(\mu-7)e_{\mu-3}, \\ [e_3, e_{\mu-3}] = -(\mu-4)e_{\mu+2} + 2e_{\mu-1}, \quad [e_3, e_{\mu-2}] = -(\mu-6)e_\mu, \\ [e_4, e_5] = -e_7, \quad [e_4, e_6] = -2e_8, \quad [e_4, e_7] = -3e_9, \ldots, [e_4, e_{\mu-6}] = -(\mu-10)e_{\mu-4}, \\ [e_4, e_{\mu-5}] = -(\mu-9)e_{\mu-2}, \quad [e_4, e_{\mu-4}] = -(\mu-8)e_\mu, \\ [e_5, e_6] = -e_9, \quad [e_5, e_7] = -2e_{10}, \quad [e_5, e_8] = -3e_{11}, \ldots, [e_5, e_{\mu-7}] = -(\mu-12)e_{\mu-4}, \\ [e_5, e_{\mu-6}] = -(\mu-11)e_{\mu-2}, \quad [e_5, e_{\mu-5}] = -(\mu-10)e_\mu, \\ [e_6, e_7] = -e_{11}, \quad [e_6, e_8] = -2e_{12}, \quad [e_6, e_9] = -3e_{13}, \ldots, [e_6, e_{\mu-8}] = -(\mu-14)e_{\mu-4}, \\ [e_6, e_{\mu-7}] = -(\mu-13)e_{\mu-2}, \quad [e_6, e_{\mu-6}] = -(\mu-12)e_\mu, \\ \vdots \\ [e_{\frac{(\mu-1)}{2}}, e_{\frac{(\mu+1)}{2}}] = -e_\mu, \quad [e_{\mu-3}, e_{\mu+1}] = e_{\mu-1}, \quad [e_{\mu-3}, e_{\mu+2}] = e_\mu. \end{gathered}$$

Case 2: When μ is even and $\mu \geq 8$, then the nilradical $(\mathcal{NL}(V))^*$ has the following multiplication table:

$$\begin{gathered} [e_2, e_3] = (\mu-4)e_{\mu+2}, \quad [e_2, e_{\mu-3}] = -e_{\mu-2}, \quad [e_2, e_{\mu-1}] = -e_\mu, \quad [e_2, e_{\mu+1}] = -e_{\mu+2}, \\ [e_3, e_4] = -e_5, \quad [e_3, e_5] = -2e_6, \quad [e_3, e_6] = -3e_7, \ldots, [e_3, e_{\mu-5}] = -(\mu-8)e_{\mu-4}, \\ [e_3, e_{\mu-4}] = -(\mu-7)e_{\mu-2}, \quad [e_3, e_{\mu-3}] = -(\mu-4)e_{\mu+2} + 2e_{\mu-1}, \quad [e_3, e_{\mu-2}] = -(\mu-6)e_\mu \\ [e_4, e_5] = -e_7, \quad [e_4, e_6] = -2e_8, \quad [e_4, e_7] = -3e_9, \ldots, [e_4, e_{\mu-6}] = -(\mu-10)e_{\mu-4}, \\ [e_4, e_{\mu-5}] = -(\mu-9)e_{\mu-2}, \quad [e_4, e_{\mu-4}] = -(\mu-8)e_\mu, \\ [e_5, e_6] = -e_9, \quad [e_5, e_7] = -2e_{10}, \quad [e_5, e_8] = -3e_{11}, \ldots, [e_5, e_{\mu-7}] = -(\mu-12)e_{\mu-4}, \\ [e_5, e_{\mu-6}] = -(\mu-11)e_{\mu-2}, \quad [e_5, e_{\mu-5}] = -(\mu-10)e_\mu, \\ [e_6, e_7] = -e_{11}, \quad [e_6, e_8] = -2e_{12}, \quad [e_6, e_9] = -3e_{13}, \ldots, [e_6, e_{\mu-8}] = -(\mu-14)e_{\mu-4}, \\ [e_6, e_{\mu-7}] = -(\mu-13)e_{\mu-2}, \quad [e_6, e_{\mu-6}] = -(\mu-12)e_\mu, \\ \vdots \\ [e_{\frac{(\mu-2)}{2}}, e_{\frac{\mu}{2}}] = -e_{\mu-2}, \quad [e_{\frac{(\mu-2)}{2}}, e_{\frac{(\mu+2)}{2}}] = -2e_\mu, \quad [e_{\mu-3}, e_{\mu+1}] = e_{\mu-1}, \quad [e_{\mu-3}, e_{\mu+2}] = e_\mu. \end{gathered}$$

The proposition is proved. □

Proposition 2.4. *Let*

$$V = \{x, y \in \mathbb{C}^2 : (x^2 + y^3,\ y^q),\ q \geq 4\}$$

be the I_{2q-1} zero-dimensional simple complete intersection singularity. Then

$$\mathcal{NL}(V) = 2q + 1.$$

Proof. It is easy to see that the moduli algebra of I_{2q-1} series is given by

$$\mathbb{C}\{x, y\}/(f, g, M_1) = \langle x^{i_1} y^{i_2}, 0 \leq i_1 \leq 1; 0 \leq i_2 \leq q-2; y^{q-1}\rangle.$$

When $q = 4$, then the Lie algebra $\mathcal{NL}(V)$ has the following basis:

$$e_1 = -3x\partial_1 - 2y\partial_2, \quad e_2 = -xy\partial_1, \quad e_3 = -3y^2\partial_1 + 2x\partial_2, \quad e_4 = -xy\partial_2$$
$$e_5 = xy\partial_1 - y^2\partial_2, \quad e_6 = -xy^2\partial_2, \quad e_7 = -y^3\partial_2, \quad e_8 = -xy^2\partial_1, \quad e_9 = -y^3\partial_1.$$

The nilradical $(\mathcal{NL}(V))^*$ is given by

$$(\mathcal{NL}(V))^* = \langle e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\rangle.$$

The nilradical $(\mathcal{NL}(V))^*$ has the following multiplication table:

$$[e_2, e_3] = 2e_4 + 5e_9, \quad [e_2, e_4] = -e_6, \quad [e_2, e_5] = e_8, \quad [e_3, e_4] = -5e_7 + 6e_8,$$
$$[e_3, e_5] = 6e_4 + 11e_9, \quad [e_3, e_7] = 6e_6, \quad [e_3, e_8] = -2e_6, \quad [e_3, e_9] = -2e_7 + 6e_8, \quad [e_4, e_5] = -2e_6.$$

When $q = 5$, then the Lie algebra $\mathcal{NL}(V)$ has the following basis:

$$e_1 = 3x\partial_1 + 2y\partial_2, \quad e_2 = 3xy\partial_1 + 2y^2\partial_2, \quad e_3 = 3y^2\partial_1 - 2x\partial_2, \quad e_4 = xy^2\partial_1,$$
$$e_5 = 3y^3\partial_1 - 2xy\partial_2, \quad e_6 = xy^2\partial_2, \quad e_7 = -xy^2\partial_1 + y^3\partial_2, \quad e_8 = xy^3\partial_2,$$
$$e_9 = y^4\partial_2, \quad e_{10} = xy^3\partial_1, \quad e_{11} = y^4\partial_1.$$

The nilradical $(\mathcal{NL}(V))^*$ of $\mathcal{NL}(V)$ is given by

$$(\mathcal{NL}(V))^* = \langle e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}\rangle.$$

The nilradical $(\mathcal{NL}(V))^*$ has the following multiplication table:

$$[e_2, e_3] = -e_5, \quad [e_2, e_4] = 4e_{10}, \quad [e_2, e_5] = 3e_{11} - 2e_6, \quad [e_2, e_6] = 3e_8,$$
$$[e_2, e_7] = -7e_{10} + 2e_9, \quad [e_3, e_4] = 7e_{11} + 2e_6, \quad [e_3, e_5] = -10e_4 - 4e_7,$$
$$[e_3, e_6] = -6e_{10} + 7e_9, \quad [e_3, e_7] = -13e_{11} - 8e_6, \quad [e_3, e_9] = -8e_8, \quad [e_3, e_{10}] = 2e_8,$$
$$[e_3, e_{11}] = -8e_{10} + 2e_9, \quad [e_4, e_5] = -2e_8, \quad [e_5, e_7] = -6e_8.$$

When $q = 6$, the nilradical $(\mathcal{NL}(V))^*$ of $\mathcal{NL}(V)$ is given by

$$(\mathcal{NL}(V))^* = \langle e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}\rangle.$$

The nilradical $(\mathcal{NL}(V))^*$ has the following multiplication table:

$$[e_2, e_3] = e_5, \quad [e_2, e_4] = -10e_6 - 4e_9, \quad [e_2, e_5] = -e_7, \quad [e_2, e_6] = -6e_{12},$$
$$[e_2, e_7] = -9e_{13} + 6e_8, \quad [e_2, e_8] = -5e_{10}, \quad [e_2, e_9] = -4e_{11} + 9e_{12},$$
$$[e_3, e_4] = -3e_7, \quad [e_3, e_5] = 2e_4, \quad [e_3, e_6] = -9e_{13} - 2e_8, \quad [e_3, e_7] = 20e_6 + 8e_9,$$
$$[e_3, e_8] = -9e_{11} + 6e_{12}, \quad [e_3, e_9] = 15e_{13} + 10e_8, \quad [e_3, e_{11}] = 10e_{10}, \quad [e_3, e_{12}] = -2e_{10},$$
$$[e_3, e_{13}] = -2e_{11} + 10e_{12}, \quad [e_4, e_5] = 3e_{13} - 2e_8, \quad [e_4, e_7] = 2e_{10}, \quad [e_5, e_6] = -2e_{10},$$
$$[e_5, e_7] = 4e_{11} + 6e_{12}, \quad [e_5, e_9] = 8e_{10}.$$

When $q = 7$, the nilradical $(\mathcal{NL}(V))^*$ of $\mathcal{NL}(V)$ is given by

$$(\mathcal{NL}(V))^* = \langle e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}\rangle.$$

The nilradical $(\mathcal{NL}(V))^*$ has the following multiplication table:

$$\begin{aligned}
&[e_2, e_3] = e_5, && [e_2, e_4] = -2e_6, && [e_2, e_5] = -e_7,\\
&[e_2, e_6] = -8e_{11} - 20e_8, && [e_2, e_7] = -3e_9, && [e_2, e_8] = -8e_{14},\\
&[e_2, e_9] = 10e_{10} - 15e_{15}, && [e_2, e_{10}] = -7e_{12}, && [e_2, e_{11}] = -6e_{13} + 11e_{14},\\
&[e_3, e_4] = -3e_7, && [e_3, e_5] = 2e_4, && [e_3, e_6] = -5e_9,\\
&[e_3, e_7] = 4e_6, && [e_3, e_8] = -2e_{10} - 11e_{15}, && [e_3, e_9] = 12e_{11} + 30e_8,\\
&[e_3, e_{10}] = -11e_{13} + 6e_{14}, && [e_3, e_{11}] = 12e_{10} + 17e_{15}, && [e_3, e_{13}] = 12e_{12},\\
&[e_3, e_{14}] = -2e_{12}, && [e_3, e_{15}] = -2e_{13} + 12e_{14}, && [e_4, e_5] = e_9,\\
&[e_4, e_6] = -4e_{13} - 6e_{14}, && [e_4, e_7] = 2e_{10} - 3e_{15}, && [e_4, e_9] = 6e_{12},\\
&[e_5, e_6] = 6e_{10} - 9e_{15}, && [e_5, e_7] = 4e_{11} + 10e_8, && [e_5, e_8] = -2e_{12},\\
&[e_5, e_9] = 8e_{13} + 12e_{14}, && [e_5, e_{11}] = 10e_{12}, && [e_6, e_7] = -2e_{12}.
\end{aligned}$$

When $q \geq 6$, then $\mathcal{NL}(V)$ have the following bases:

$$\begin{aligned}
&e_1 = -3x\partial_1 - 2y\partial_2, \quad e_2 = -3xy\partial_1 - 2y^2\partial_2,\\
&e_3 = -3y^2\partial_1 + 2x\partial_2, \quad e_4 = -3xy^2\partial_1 - 2y^3\partial_2,\\
&e_5 = -3y^3\partial_1 + 2xy\partial_2, \quad e_6 = -3xy^3\partial_1 - 2y^4\partial_2,\\
&e_7 = -3y^4\partial_1 + 2xy^2\partial_2, \quad e_8 = -3xy^4\partial_1 - 2y^5\partial_2,\\
&\vdots\\
&e_{2q-9} = -3y^{q-4}\partial_1 + 2xy^{q-6}\partial_2, \quad e_{2q-8} = -3xy^{q-4}\partial_1 - 2y^{q-3}\partial_2,\\
&e_{2q-7} = -3y^{q-3}\partial_1 + 2xy^{q-5}\partial_2, \quad e_{2q-6} = -xy^{q-3}\partial_1 \quad e_{2q-5} = -3y^{q-2}\partial_1 + 2xy^{q-4}\partial_2,\\
&e_{2q-4} = -xy^{q-3}\partial_2, \quad e_{2q-3} = xy^{q-3}\partial_1 - y^{q-2}\partial_2, \quad e_{2q-2} = -xy^{q-2}\partial_2, \quad e_{2q-1} = -y^{q-1}\partial_2,\\
&e_{2q} = -xy^{q-2}\partial_1, \quad e_{2q+1} = -y^{q-1}\partial_1.
\end{aligned}$$

When $q \geq 6$, the nilradical $(\mathcal{NL}(V))^*$ of $\mathcal{NL}(V)$ is given by

$$(\mathcal{NL}(V))^* = \langle e_2, e_3, e_4, \dots, e_{2q+1}\rangle.$$

Case 1: When q is even and $q \geq 8$, then the nilradical $(\mathcal{NL}(V))^*$ has the following multiplication table:

$$\begin{aligned}
&[e_2, e_3] = e_5, \quad [e_2, e_4] = -2e_6, \quad [e_2, e_5] = -e_7, \quad [e_2, e_6] = -4e_8,\\
&[e_2, e_7] = -3e_9, \dots, [e_2, e_{2q-10}] = (-2q + 12)e_{2q-8}, \quad [e_2, e_{2q-9}] = (-2q + 13)e_{2q-7},\\
&[e_2, e_{2q-8}] = (-10q + 50)e_{2q-6} + (-4q + 20)e_{2q-3}, \quad [e_2, e_{2q-7}] = (-2q + 11)e_{2q-5},\\
&[e_2, e_{2q-6}] = (-2q + 6)e_{2q}, \quad [e_2, e_{2q-5}] = (4q - 18)e_{2q-4} + (-6q + 27)e_{2q+1}\\
&[e_2, e_{2q-4}] = (-2q + 7)e_{2q-2}, \quad [e_2, e_{2q-3}] = (-2q + 8)e_{2q-1} + (2q - 3)e_{2q},\\
&[e_3, e_4] = -3e_7, \quad [e_3, e_5] = 2e_4, \quad [e_3, e_6] = -5e_9, \quad [e_3, e_7] = 4e_6, \dots,\\
&[e_3, e_{2q-8}] = (-2q + 9)e_{2q-5}, \quad [e_3, e_{2q-7}] = (2q - 10)e_{2q-8}, \quad [e_3, e_{2q-6}] = -2e_{2q-4} + (-2q + 3)e_{2q+1},\\
&[e_3, e_{2q-5}] = (10q - 40)e_{2q-6} + (4q - 16)e_{2q-3}, \quad [e_3, e_{2q-4}] = (-2q + 3)e_{2q-1} + 6e_{2q},\\
&[e_3, e_{2q-3}] = (2q - 2)e_{2q-4} + (2q + 3)e_{2q+1}, \quad [e_3, e_{2q-1}] = (2q - 2)e_{2q-2},\\
&[e_3, e_{2q}] = -2e_{2q-2}, \quad [e_3, e_{2q+1}] = -2e_{2q-1} + (2q - 2)e_{2q}, \quad [e_4, e_5] = e_9,\\
&[e_4, e_6] = -2e_{10}, \quad [e_4, e_7] = -e_{11}, \quad [e_4, e_8] = -4e_{12},\\
&[e_4, e_9] = -3e_{13}, \dots, [e_4, e_{2q-12}] = (-2q + 16)e_{2q-8},
\end{aligned}$$

$$[e_4, e_{2q-11}] = (-2q+17)e_{2q-7}, \quad [e_4, e_{2q-10}] = (-10q+70)e_{2q-6} + (-4q+28)e_{2q-3},$$
$$[e_4, e_{2q-9}] = (-2q+15)e_{2q-5}, \quad [e_4, e_{2q-8}] = (-4q+24)e_{2q-1} + (-6q+36)e_{2q},$$
$$[e_4, e_{2q-7}] = (4q-26)e_{2q-4} + (-6q+39)e_{2q+1},$$
$$[e_4, e_{2q-5}] = (4q-22)e_{2q-2}, \quad [e_5, e_6] = -3e_{11}, \quad [e_5, e_7] = 2e_8, \quad [e_5, e_8] = -5e_{13},$$
$$[e_5, e_9] = 4e_{10}, \ldots, [e_5, e_{2q-10}] = (-2q+13)e_{2q-5}, \quad [e_5, e_{2q-9}] = (2q-14)e_{2q-8},$$
$$[e_5, e_{2q-8}] = (4q-22)e_{2q-4} + (-6q+33)e_{2q+1}, \quad [e_5, e_{2q-7}] = (10q-60)e_{2q-6} + (4q-24)e_{2q-3},$$
$$[e_5, e_{2q-6}] = -2e_{2q-2}, \quad [e_5, e_{2q-5}] = (4q-20)e_{2q-1} + (6q-30)e_{2q},$$
$$[e_5, e_{2q-3}] = (2q-4)e_{2q-2}, \quad [e_6, e_7] = e_{13}, \quad [e_6, e_8] = -2e_{14},$$
$$[e_6, e_9] = -e_{15}, \quad [e_6, e_{10}] = -4e_{16}, \quad [e_6, e_{11}] = -3e_{17}, \ldots, [e_6, e_{2q-14}] = (-2q+20)e_{2q-8},$$
$$[e_6, e_{2q-13}] = (-2q+21)e_{2q-7}, \quad [e_6, e_{2q-12}] = (-10q+90)e_{2q-6} + (-4q+36)e_{2q-3},$$
$$[e_6, e_{2q-11}] = (-2q+19)e_{2q-5}, \quad [e_6, e_{2q-10}] = (-4q+32)e_{2q-1} + (-6q+48)e_{2q},$$
$$[e_6, e_{2q-9}] = (4q-34)e_{2q-4} + (-6q+51)e_{2q+1}, \quad [e_6, e_{2q-7}] = (4q-30)e_{2q-2},$$
$$[e_7, e_8] = -3e_{15}, \quad [e_7, e_9] = 2e_{12}, \quad [e_7, e_{10}] = -5e_{17},$$
$$[e_7, e_{11}] = 4e_{14}, \ldots, [e_7, e_{2q-12}] = (-2q+17)e_{2q-5} \quad [e_7, e_{2q-11}] = (2q-18)e_{2q-8},$$
$$[e_7, e_{2q-10}] = (4q-30)e_{2q-4} + (-6q+45)e_{2q+1}, \quad [e_7, e_{2q-9}] = (10q-80)e_{2q-6} + (4q-32)e_{2q-3},$$
$$[e_7, e_{2q-8}] = (4q-26)e_{2q-2}, \quad [e_7, e_{2q-7}] = (4q-28)e_{2q-1} + (6q-42)e_{2q},$$
$$[e_8, e_9] = e_{17}, \quad [e_8, e_{10}] = -2e_{18}, \quad [e_8, e_{11}] = -e_{19},$$
$$[e_8, e_{12}] = -4e_{20}, \quad [e_8, e_{13}] = -3e_{21}, \ldots, [e_8, e_{2q-16}] = (-2q+24)e_{2q-8},$$
$$[e_8, e_{2q-15}] = (-2q+25)e_{2q-7}, \quad [e_8, e_{2q-14}] = (-10q+110)e_{2q-6} + (-4q+44)e_{2q-3},$$
$$[e_8, e_{2q-13}] = (-2q+23)e_{2q-5}, \quad [e_8, e_{2q-12}] = (-4q+40)e_{2q-1} + (-6q+60)e_{2q},$$
$$[e_8, e_{2q-11}] = (4q-42)e_{2q-4} + (-6q+63)e_{2q+1} \quad [e_8, e_{2q-9}] = (4q-38)e_{2q-2}$$
$$[e_9, e_{10}] = -3e_{19}, \quad [e_9, e_{11}] = 2e_{16}, \quad [e_9, e_{12}] = -5e_{21},$$
$$[e_9, e_{13}] = 4e_{18}, \ldots, [e_9, e_{2q-14}] = (-2q+21)e_{2q-5}, \quad [e_9, e_{2q-13}] = (2q-22)e_{2q-8},$$
$$[e_9, e_{2q-12}] = (4q-38)e_{2q-4} + (-6q+57)e_{2q+1}, \quad [e_9, e_{2q-11}] = (10q-100)e_{2q-6} + (4q-40)e_{2q-3},$$
$$[e_9, e_{2q-10}] = (4q-34)e_{2q-2}, \quad [e_9, e_{2q-9}] = (4q-36)e_{2q-1} + (6q-54)e_{2q},$$
$$[e_{10}, e_{11}] = e_{21}, \quad [e_{10}, e_{12}] = -2e_{22}, \quad [e_{10}, e_{13}] = -e_{23}, \quad [e_{10}, e_{14}] = -4e_{24},$$
$$[e_{10}, e_{15}] = -3e_{25}, \ldots, [e_{10}, e_{2q-18}] = (-2q+28)e_{2q-8}, \quad [e_{10}, e_{2q-17}] = (-2q+29)e_{2q-7},$$
$$[e_{10}, e_{2q-16}] = (-10q+130)e_{2q-6} + (-4q+52)e_{2q-3}, \quad [e_{10}, e_{2q-15}] = (-2q+27)e_{2q-5},$$
$$[e_{10}, e_{2q-14}] = (-4q+48)e_{2q-1} + (-6q+72)e_{2q}, \quad [e_{10}, e_{2q-13}] = (4q-50)e_{2q-4} + (-6q+75)e_{2q+1},$$
$$[e_{10}, e_{2q-11}] = (4q-46)e_{2q-2} \quad [e_{11}, e_{12}] = -3e_{24}, \quad [e_{11}, e_{13}] = 2e_{20},$$
$$[e_{11}, e_{14}] = -5e_{25}, \quad [e_{11}, e_{15}] = 4e_{22}, \ldots, [e_{11}, e_{2q-16}] = (-2q+25)e_{2q-5}$$
$$[e_{11}, e_{2q-15}] = (2q-26)e_{2q-8}, \quad [e_{11}, e_{2q-14}] = (4q-46)e_{2q-7} + (-6q+69)e_{2q+1},$$
$$[e_{11}, e_{2q-13}] = (10q-120)e_{2q-6} + (4q-48)e_{2q-3}, \quad [e_{11}, e_{2q-12}] = (4q-42)e_{2q-2},$$
$$[e_{11}, e_{2q-11}] = (4q-44)e_{2q-1} + (6q-66)e_{2q},$$
$$\vdots$$
$$[e_{q-2}, e_{q-1}] = -2e_{2q-4} + 3e_{2q+1}, \quad [e_{q-2}, e_{q+1}] = 2e_{2q-2},$$
$$[e_{q-1}, e_q] = 6e_{2q-2}, \quad [e_{q-1}, e_{q+1}] = 4e_{2q-1} + 6e_{2q}.$$

Case 2: When q is odd and $q \geq 9$, then the nilradical $(\mathcal{NL}(V))^*$ has the following multiplication table:

$$[e_2, e_3] = e_5, \quad [e_2, e_4] = -2e_6, \quad [e_2, e_5] = -e_7, \quad [e_2, e_6] = -4e_8,$$
$$[e_2, e_7] = -3e_9, \ldots, [e_2, e_{2q-10}] = (-2q+12)e_{2q-8}, \quad [e_2, e_{2q-9}] = (-2q+13)e_{2q-7},$$
$$[e_2, e_{2q-8}] = (-10q+50)e_{2q-6} + (-4q+20)e_{2q-3}, \quad [e_2, e_{2q-7}] = (-2q+11)e_{2q-5},$$

$$[e_2,e_{2q-6}]=(-2q+6)e_{2q},\quad [e_2,e_{2q-5}]=(4q-18)e_{2q-4}+(-6q+27)e_{2q+1}$$
$$[e_2,e_{2q-4}]=(-2q+7)e_{2q-2},\quad [e_2,e_{2q-3}]=(-2q+8)e_{2q-1}+(2q-3)e_{2q},$$
$$[e_3,e_4]=-3e_7,\quad [e_3,e_5]=2e_4,\quad [e_3,e_6]=-5e_9,\quad [e_3,e_7]=4e_6,\ldots,$$
$$[e_3,e_{2q-8}]=(-2q+9)e_{2q-5},\quad [e_3,e_{2q-7}]=(2q-10)e_{2q-8},\quad [e_3,e_{2q-6}]=-2e_{2q-4}+(-2q+3)e_{2q+1},$$
$$[e_3,e_{2q-5}]=(10q-40)e_{2q-6}+(4q-16)e_{2q-3},\quad [e_3,e_{2q-4}]=(-2q+3)e_{2q-1}+6e_{2q},$$
$$[e_3,e_{2q-3}]=(2q-2)e_{2q-4}+(2q+3)e_{2q+1},\quad [e_3,e_{2q-1}]=(2q-2)e_{2q-2},$$
$$[e_3,e_{2q}]=-2e_{2q-2},\quad [e_3,e_{2q+1}]=-2e_{2q-1}+(2q-2)e_{2q},\quad [e_4,e_5]=e_9,\quad [e_4,e_6]=-2e_{10},$$
$$[e_4,e_7]=-e_{11},\quad [e_4,e_8]=-4e_{12},\quad ,[e_4,e_9]=-3e_{13},\ldots,[e_4,e_{2q-12}]=(-2q+16)e_{2q-8}$$
$$[e_4,e_{2q-11}]=(-2q+17)e_{2q-7},\quad [e_4,e_{2q-10}]=(-10q+70)e_{2q-6}+(-4q+28)e_{2q-3},$$
$$[e_4,e_{2q-9}]=(-2q+15)e_{2q-5},\quad [e_4,e_{2q-8}]=(-4q+24)e_{2q-1}+(-6q+36)e_{2q},$$
$$[e_4,e_{2q-7}]=(4q-26)e_{2q-4}+(-6q+39)e_{2q+1},\quad [e_4,e_{2q-5}]=(4q-22)e_{2q-2},$$
$$[e_5,e_6]=-3e_{11},\quad [e_5,e_7]=2e_8,\quad [e_5,e_8]=-5e_{13},\quad [e_5,e_9]=4e_{10},\ldots,$$
$$[e_5,e_{2q-10}]=(-2q+13)e_{2q-5},\quad [e_5,e_{2q-9}]=(2q-14)e_{2q-8},$$
$$[e_5,e_{2q-8}]=(4q-22)e_{2q-4}+(-6q+33)e_{2q+1},\quad [e_5,e_{2q-7}]=(10q-60)e_{2q-6}+(4q-24)e_{2q-3},$$
$$[e_5,e_{2q-6}]=-2e_{2q-2},\quad [e_5,e_{2q-5}]=(4q-20)e_{2q-1}+(6q-30)e_{2q},\quad [e_5,e_{2q-3}]=(2q-4)e_{2q-2},$$
$$[e_6,e_7]=e_{13},\quad [e_6,e_8]=-2e_{14},\quad [e_6,e_9]=-e_{15},\quad [e_6,e_{10}]=-4e_{16},$$
$$[e_6,e_{11}]=-3e_{17},\ldots,[e_6,e_{2q-14}]=(-2q+20)e_{2q-8},\quad [e_6,e_{2q-13}]=(-2q+21)e_{2q-7},$$
$$[e_6,e_{2q-12}]=(-10q+90)e_{2q-6}+(-4q+36)e_{2q-3},\quad [e_6,e_{2q-11}]=(-2q+19)e_{2q-5},$$
$$[e_6,e_{2q-10}]=(-4q+32)e_{2q-1}+(-6q+48)e_{2q},$$
$$[e_6,e_{2q-9}]=(4q-34)e_{2q-4}+(-6q+51)e_{2q+1},\quad [e_6,e_{2q-7}]=(4q-30)e_{2q-2},$$
$$[e_7,e_8]=-3e_{15},\quad [e_7,e_9]=2e_{12},\quad [e_7,e_{10}]=-5e_{17},$$
$$[e_7,e_{11}]=4e_{14},\ldots,[e_7,e_{2q-12}]=(-2q+17)e_{2q-5}\quad [e_7,e_{2q-11}]=(2q-18)e_{2q-8},$$
$$[e_7,e_{2q-10}]=(4q-30)e_{2q-4}+(-6q+45)e_{2q+1},\quad [e_7,e_{2q-9}]=(10q-80)e_{2q-6}+(4q-32)e_{2q-3},$$
$$[e_7,e_{2q-8}]=(4q-26)e_{2q-2},\quad [e_7,e_{2q-7}]=(4q-28)e_{2q-1}+(6q-42)e_{2q},$$
$$[e_8,e_9]=e_{17},\quad [e_8,e_{10}]=-2e_{18},\quad [e_8,e_{11}]=-e_{19},$$
$$[e_8,e_{12}]=-4e_{20},\quad [e_8,e_{13}]=-3e_{21},\ldots,[e_8,e_{2q-16}]=(-2q+24)e_{2q-8},$$
$$[e_8,e_{2q-15}]=(-2q+25)e_{2q-7},\quad [e_8,e_{2q-14}]=(-10q+110)e_{2q-6}+(-4q+44)e_{2q-3},$$
$$[e_8,e_{2q-13}]=(-2q+23)e_{2q-5},\quad [e_8,e_{2q-12}]=(-4q+40)e_{2q-1}+(-6q+60)e_{2q},$$
$$[e_8,e_{2q-11}]=(4q-42)e_{2q-4}+(-6q+63)e_{2q+1}\quad [e_8,e_{2q-9}]=(4q-38)e_{2q-2}$$
$$[e_9,e_{10}]=-3e_{19},\quad [e_9,e_{11}]=2e_{16},\quad [e_9,e_{12}]=-5e_{21},\quad [e_9,e_{13}]=4e_{18},\ldots,$$
$$[e_9,e_{2q-14}]=(-2q+21)e_{2q-5},\quad [e_9,e_{2q-13}]=(2q-22)e_{2q-8},$$
$$[e_9,e_{2q-12}]=(4q-38)e_{2q-4}+(-6q+57)e_{2q+1},\quad [e_9,e_{2q-11}]=(10q-100)e_{2q-6}+(4q-40)e_{2q-3},$$
$$[e_9,e_{2q-10}]=(4q-34)e_{2q-2},\quad [e_9,e_{2q-9}]=(4q-36)e_{2q-1}+(6q-54)e_{2q},$$
$$[e_{10},e_{11}]=e_{21},\quad [e_{10},e_{12}]=-2e_{22},\quad [e_{10},e_{13}]=-e_{23},\quad [e_{10},e_{14}]=-4e_{24},$$
$$[e_{10},e_{15}]=-3e_{25},\ldots,[e_{10},e_{2q-18}]=(-2q+28)e_{2q-8},\quad [e_{10},e_{2q-17}]=(-2q+29)e_{2q-7},$$
$$[e_{10},e_{2q-16}]=(-10q+130)e_{2q-6}+(-4q+52)e_{2q-3},\quad [e_{10},e_{2q-15}]=(-2q+27)e_{2q-5},$$
$$[e_{10},e_{2q-14}]=(-4q+48)e_{2q-1}+(-6q+72)e_{2q},\quad [e_{10},e_{2q-13}]=(4q-50)e_{2q-4}+(-6q+75)e_{2q+1},$$
$$[e_{10},e_{2q-11}]=(4q-46)e_{2q-2}\quad [e_{11},e_{12}]=-3e_{24},\quad [e_{11},e_{13}]=2e_{20},$$
$$[e_{11},e_{14}]=-5e_{25},\quad [e_{11},e_{15}]=4e_{22},\ldots,[e_{11},e_{2q-16}]=(-2q+25)e_{2q-5}$$
$$[e_{11},e_{2q-15}]=(2q-26)e_{2q-8},\quad [e_{11},e_{2q-14}]=(4q-46)e_{2q-7}+(-6q+69)e_{2q+1},$$
$$[e_{11},e_{2q-13}]=(10q-120)e_{2q-6}+(4q-48)e_{2q-3},\quad [e_{11},e_{2q-12}]=(4q-42)e_{2q-2},$$

$$[e_{11}, e_{2q-11}] = (4q-44)e_{2q-1} + (6q-66)e_{2q},$$
$$\vdots$$
$$[e_{q-2}, e_{q-1}] = 6e_{2q-4} - 9e_{2q+1}, \quad [e_{q-2}, e_q] = 10e_{2q-6} + 4e_{2q-3}, \quad [e_{q-2}, e_{q+1}] = 10e_{2q-2},$$
$$[e_{q-2}, e_{q+1}] = 8e_{2q-1} + 12e_{2q}, \quad [e_{q-1}, e_q] = -2e_{2q-2}.$$

The proposition is proved. □

Proposition 2.5. *Let*

$$V = \{x, y \in \mathbb{C}^2 : (x^2 + y^3,\ xy^r),\ r \geq 3\}$$

be the I_{2r+2} zero-dimensional simple complete intersection singularity. Then

$$\mathcal{NL}(V) = 2r + 4.$$

Proof. It is easy to see that the moduli algebra of I_{2r+2} series is given by

$$\mathbb{C}\{x, y\}/(f, g, M_1) = \langle x^{i_1} y^{i_2}, 0 \leq i_1 \leq 1; 0 \leq i_2 \leq r-1; y^{r+1}; y^r \rangle.$$

When $r \geq 3$, the $\mathcal{NL}(V)$ have the following bases:

$$e_1 = -3x\partial_1 - 2y\partial_2, \quad e_2 = -3y^2\partial_1 + 2x\partial_2,$$
$$e_3 = -3xy\partial_1 - 2y^2\partial_2, \quad e_4 = -3y^3\partial_1 + 2xy\partial_2,$$
$$e_5 = -3xy^2\partial_1 - 2y^3\partial_2, \quad e_6 = -3y^4\partial_1 + 2xy^2\partial_2,$$
$$\vdots$$
$$e_{2r-5} = -3xy^{r-3}\partial_1 - 2y^{r-2}\partial_2, \quad e_{2r-4} = -3y^{r-1}\partial_1 + 2xy^{r-3}\partial_2,$$
$$e_{2r-3} = -3xy^{r-2}\partial_1 - 2y^{r-1}\partial_2, \quad e_{2r-2} = -y^r\partial_1, \quad e_{2r-1} = -y^r\partial_1 - xy^{r-2}\partial_2,$$
$$e_{2r} = -y^r\partial_2, \quad e_{2r+1} = -xy^{r-1}\partial_2, \quad e_{2r+2} = -y^{r+1}\partial_2, \quad e_{2r+3} = -xy^{r-1}\partial_1,$$
$$e_{2r+4} = -y^{r+1}\partial_1.$$

When $r = 3$, the nilradical $(\mathcal{NL}(V))^*$ of $\mathcal{NL}(V)$ is given by

$$(\mathcal{NL}(V))^* = \langle e_2, e_3, e_4, \ldots, e_{10} \rangle.$$

We have

$$[e_2, e_3] = -5e_4 + 2e_5, \quad [e_2, e_4] = -2e_6 + 6e_9, \quad [e_2, e_5] = -7e_6 + 12e_9,$$
$$[e_2, e_6] = 6e_{10} + 6e_7, \quad [e_2, e_7] = -7e_8, \quad [e_2, e_9] = -7e_{10} - 2e_7 \quad [e_2, e_{10}] = -2e_8,$$
$$[e_3, e_4] = -3e_{10}, \quad [e_3, e_5] = -6e_{10} - e_7, \quad [e_3, e_6] = -2e_8, \quad [e_4, e_5] = -e_8.$$

When $r = 4$, the nilradical $(\mathcal{NL}(V))^*$ of $\mathcal{NL}(V)$ is given by

$$(\mathcal{NL}(V))^* = \langle e_2, e_3, e_4, \ldots, e_{12} \rangle.$$

We have

$$[e_2, e_3] = -e_4, \quad [e_2, e_4] = 2e_5, \quad [e_2, e_5] = -15e_6 + 6e_7,$$
$$[e_2, e_6] = 8e_{11} - 2e_8, \quad [e_2, e_7] = 14e_{11} - 9e_8, \quad [e_2, e_8] = 6e_{12} + 8e_9,$$
$$[e_2, e_9] = -9e_{10} \quad [e_2, e_{11}] = -9e_{12} - 2e_9, \quad [e_2, e_{12}] = -2e_{10},$$
$$[e_3, e_4] = -5e_6 + 2e_7, \quad [e_3, e_5] = -6e_{11} - 4e_8, \quad [e_3, e_6] = -5e_{12},$$
$$[e_3, e_7] = -8e_{12} - 3e_9 \quad [e_3, e_8] = -4e_{10}, \quad [e_4, e_5] = -3e_{12} + 2e_9,$$
$$[e_4, e_6] = -2e_{10}, \quad [e_4, e_7] = -7e_{10}.$$

When $r = 5$, the nilradical $(\mathcal{NL}(V))^*$ of $\mathcal{NL}(V)$ is given by

$$(\mathcal{NL}(V))^* = \langle e_2, e_3, e_4, \dots, e_{14}\rangle.$$

We have

$$\begin{aligned}
&[e_2,e_3]=e_4, && [e_2,e_4]=-2e_5, && [e_2,e_5]=3e_6,\\
&[e_2,e_6]=-4e_7, && [e_2,e_7]=25e_8-10e_9, && [e_2,e_8]=2e_{10}-10e_{13},\\
&[e_2,e_9]=11e_{10}-16e_{13}, && [e_2,e_{10}]=-10e_{11}-6e_{14}, && [e_2,e_{11}]=11e_{12},\\
&[e_2,e_{13}]=2e_{11}+11e_{14}, && [e_2,e_{14}]=2e_{12}, && [e_3,e_4]=e_6,\\
&[e_3,e_5]=2e_7, && [e_3,e_6]=15e_8-6e_9, && [e_3,e_7]=8e_{10}+12e_{13},\\
&[e_3,e_8]=7e_{14}, && [e_3,e_9]=5e_{11}+10e_{14}, && [e_3,e_{10}]=6e_{12},\\
&[e_4,e_5]=5e_8-2e_9, && [e_4,e_6]=-4e_{10}-6e_{13}, && [e_4,e_7]=-6e_{11}+9e_{14},\\
&[e_4,e_8]=2e_{12}, && [e_4,e_9]=9e_{12}, && [e_5,e_6]=-2e_{11}+3e_{14}, \quad [e_5,e_7]=4e_{12}.
\end{aligned}$$

When $r \geq 6$, the nilradical $(\mathcal{NL}(V))^*$ of $\mathcal{NL}(V)$ is given by

$$(\mathcal{NL}(V))^* = \langle e_2, e_3, e_4, \dots, e_{2r+4}\rangle.$$

When $r \geq 6$, then the nilradical $(\mathcal{NL}(V))^*$ has the following multiplication table:

$$\begin{aligned}
&[e_2,e_3]=-e_4, \quad [e_2,e_4]=2e_5, \quad [e_2,e_5]=-3e_6, [e_2,e_6]=4e_7, \quad [e_2,e_7]=-5e_8,\dots,\\
&[e_2,e_{2r-5}]=-(2r-7)e_{2r-4}, \quad [e_2,e_{2r-4}]=(2r-6)e_{2r-3},\\
&[e_2,e_{2r-3}]=-(10r-25)e_{2r-2}+(4r-10)e_{2r-1}, \quad [e_2,e_{2r-2}]=-2e_{2r}+2re_{2r+3},\\
&[e_2,e_{2r-1}]=-(2r+1)e_{2r}+(2r+6)e_{2r+3}, \quad [e_2,e_{2r}]=2re_{2r+1}+6e_{2r+4},\\
&[e_2,e_{2r+1}]=-(2r+1)e_{2r+2}, \quad [e_2,e_{2r+3}]=-2e_{2r+1}-(2r+1)e_{2r+4}, \quad [e_2,e_{2r+4}]=-2e_{2r+2},\\
&[e_3,e_4]=-e_6, \quad [e_3,e_5]=-2e_7, \quad [e_3,e_6]=-3e_8,\dots,\\
&[e_3,e_{2r-5}]=-(2r-8)e_{2r-3}, \quad [e_3,e_{2r-4}]=-(10r-35)e_{2r-2}+(4r-14)e_{2r-1},\\
&[e_3,e_{2r-3}]=-(4r-12)e_{2r}-(6r-18)e_{2r+3}, \quad [e_3,e_{2r-2}]=-(2r-3)e_{2r+4},\\
&[e_3,e_{2r-1}]=-(2r-5)e_{2r+1}-2re_{2r+4}, \quad [e_3,e_{2r}]=-(2r-4)e_{2r+2},\\
&[e_4,e_5]=-e_8, \quad [e_4,e_6]=2e_9, \quad [e_4,e_7]=-3e_{10},\\
&[e_4,e_8]=4e_{11},\dots,[e_4,e_{2r-7}]=-(2r-11)e_{2r-4},\\
&[e_4,e_{2r-6}]=(2r-10)e_{2r-3}, \quad [e_4,e_{2r-5}]=-(10r-45)e_{2r-2}+(4r-18)e_{2r-1},\\
&[e_4,e_{2r-4}]=(4r-16)e_{2r}+(6r-24)e_{2r+3}, \quad [e_4,e_{2r-3}]=(4r-14)e_{2r+1}-(6r-21)e_{2r+4},\\
&[e_4,e_{2r-2}]=-2e_{2r+2}, \quad [e_4,e_{2r-1}]=-(2r-1)e_{2r+2} \quad [e_5,e_6]=-e_{10},\\
&[e_5,e_7]=-2e_{11}, \quad [e_5,e_8]=-3e_{12},\dots,\\
&[e_5,e_{2r-7}]=-(2r-12)e_{2r-3}, \quad [e_5,e_{2r-6}]=-(10r-55)e_{2r-2}+(4r-22)e_{2r-1},\\
&[e_5,e_{2r-5}]=-(4r-20)e_{2r}-(6r-30)e_{2r+3}, \quad [e_5,e_{2r-4}]=(4r-18)e_{2r+1}-(6r-27)e_{2r},\\
&[e_5,e_{2r-3}]=-(4r-16)e_{2r+2}, \quad [e_6,e_7]=-e_{12}, \quad [e_6,e_8]=2e_{13},\\
&[e_6,e_9]=-3e_{14}, \quad [e_6,e_{10}]=4e_{15},\dots,\\
&[e_6,e_{2r-9}]=-(2r-15)e_{2r-4}, \quad [e_6,e_{2r-8}]=(2r-14)e_{2r-3},\\
&[e_6,e_{2r-7}]=-(10r-65)e_{2r-2}+(4r-26)e_{2r-1}, \quad [e_6,e_{2r-6}]=(4r-24)e_{2r}+(6r-36)e_{2r+3},\\
&[e_6,e_{2r-5}]=(4r-22)e_{2r+1}-(6r-33)e_{2r+4}, \quad [e_6,e_{2r-4}]=(4r-20)e_{2r+2},\\
&[e_7,e_8]=-e_{14}, \quad [e_7,e_9]=-2e_{15}, \quad [e_7,e_{10}]=-3e_{16},\dots,\\
&[e_7,e_{2r-9}]=-(2r-16)e_{2r-3}, \quad [e_7,e_{2r-8}]=-(10r-75)e_{2r-2}+(4r-30)e_{2r-1},\\
&[e_7,e_{2r-7}]=-(4r-28)e_{2r}-(6r-42)e_{2r+3}, \quad [e_7,e_{2r-6}]=(4r-26)e_{2r+1}-(6r-39)e_{2r+4},
\end{aligned}$$

$$
\begin{aligned}
&[e_7, e_{2r-5}] = -(4r-24)e_{2r+2}, \quad [e_8, e_9] = -e_{16},\\
&[e_8, e_{10}] = 2e_{17}, \quad [e_8, e_{11}] = -3e_{18}, \quad [e_8, e_{12}] = 4e_{19}, \ldots,\\
&[e_8, e_{2r-11}] = -(2r-19)e_{2r-4}, \quad [e_8, e_{2r-10}] = (2r-18)e_{2r-3},\\
&[e_8, e_{2r-9}] = -(10r-85)e_{2r-2} + (10r-34)e_{2r-1}, \quad [e_8, e_{2r-8}] = (4r-32)e_{2r} + (6r-48)e_{2r+3},\\
&[e_8, e_{2r-7}] = (4r-30)e_{2r+1} - (6r-45)e_{2r+4}, \quad [e_8, e_{2r-6}] = (4r-28)e_{2r+2},\\
&\vdots\\
&[e_{r-1}, e_r] = -5e_{2r-2} + 2e_{2r-1}, \quad [e_{r-1}, e_{r+1}] = -4e_{2r} - 6e_{2r+3}, \quad [e_{r-1}, e_{r+2}] = 6e_{2r+1} - 9e_{2r+4},\\
&[e_{r-1}, e_{r+3}] = -8e_{2r+2}, \quad [e_r, e_{r+1}] = 2e_{2r+1} - 3e_{2r+4}, \quad [e_r, e_{r+2}] = 4e_{2r+2}.
\end{aligned}
$$

The proposition is proved. □

Proposition 2.6. *Let*

$$V = \{x, y \in \mathbb{C}^2 : (xy,\ x^q + y^2),\ q \geq 7\}$$

be the $F_{q+1}^{q,2}$ zero-dimensional simple complete intersection singularity. Then

$$\mathcal{NL}(V) = q + 2.$$

Proof. It is easy to see that the moduli algebra of $F_{q+1}^{q,2}$ series is given by

$$\mathbb{C}\{x, y\}/(f, g, M_1) = \langle x^i, 0 \leq i \leq q-1; y\rangle.$$

When $q \geq 7$, the $\mathcal{NL}(V)$ have the following bases:

$$
\begin{aligned}
&e_1 = -x\partial_1, \quad e_2 = -x^2\partial_1, \quad e_3 = -x^3\partial_1, \ldots,\\
&e_{q-2} = -x^{q-2}\partial_1, \quad e_{q-1} = -x^{q-1}\partial_2, \quad e_q = -y\partial_2, \quad e_{q+1} = -x^{q-1}\partial_1, \quad e_{q+2} = -y\partial_1,
\end{aligned}
$$

The nilradical $(\mathcal{NL}(V))^*$ of $\mathcal{NL}(V)$ is given by

$$(\mathcal{NL}(V))^* = \langle e_2, e_3, e_4, \ldots, e_{q-1}, e_{q+1}, e_{q+2}\rangle.$$

The nilradical $(\mathcal{NL}(V))^*$ has the following multiplication table:

$$
\begin{aligned}
&[e_2, e_3] = -e_4, \quad [e_2, e_4] = -2e_5, \quad [e_2, e_5] = -3e_6, \ldots,\\
&[e_2, e_{q-3}] = -(q-5)e_{q-2}, \quad [e_2, e_{q-2}] = -(q-4)e_{q+1},\\
&[e_3, e_4] = -e_6, \quad [e_3, e_5] = -2e_7, \quad [e_3, e_6] = -3e_8, \ldots,\\
&[e_3, e_{q-4}] = -(q-7)e_{q-2}, \quad [e_3, e_{q-3}] = -(q-6)e_{q+1},\\
&[e_4, e_5] = -e_8, \quad [e_4, e_6] = -2e_9, \quad [e_4, e_7] = -3e_{10}, \ldots,\\
&[e_4, e_{q-5}] = -(q-9)e_{q-2}, \quad [e_4, e_{q-4}] = -(q-8)e_{q+1},\\
&\vdots\\
&[e_{q-1}, e_{q+2}] = -e_{q+1}.
\end{aligned}
$$

The proposition is proved. □

Proof of Theorem A. It is an immediate corollary of Proposition 2.1. □

In the following proof of Theorem B, we shall distinguish the zero-dimensional simple complete intersection singularities by using the corresponding dimension of the Lie algebra $\mathcal{NL}(V)$, dimension of the nilradical $(\mathcal{NL}(V))^*$, minimal number of generators of the nilradical of the Lie algebra $\mathcal{NL}(V)$ and lower central series dimensions of $(\mathcal{NL}(V))^*$.

Proof of Theorem B. After simple calculations and by Propositions 2.3–2.5, we have Table 6.

Type(V)	Equations	$v(V)$	$v^*(V)$
$F^{q,r}_{q+r-1}$	$\{(xy,\ x^q+y^r),\ q,r\geq 2\}$	$q+r$	$q+r-2$
G_5	$(x^2,\ y^3)$	7	5
G_7	$(x^2,\ y^4)$	10	8
H_6	$(x^2+y^3,\ xy^2)$	8	5
H_μ	$\{(x^2+y^{\mu-3},\ xy^2),\ \mu\geq 7\}$	$\mu+2$	$\mu+1$
I_{2q-1}	$\{(x^2+y^3,\ y^q),\ q\geq 4\}$	$2q+1$	$2q$
I_{2r+2}	$\{(x^2+y^3,\ xy^r),\ r\geq 3\}$	$2r+4$	$2r+3$

Table 6: The v and v^* of zero-dimensional simple complete intersection singularities.

Next we need to distinguish the pairs which have the same dimensions of Lie algebra $\mathcal{NL}(V)$. Note that we only need to treat the following eight cases:

$$
\begin{aligned}
&(1):\ (a):\quad \mathcal{NL}(F^{2,8}_9)\ncong\mathcal{NL}(G_7),\quad \mathcal{NL}(F^{3,7}_9)\ncong\mathcal{NL}(G_7),\quad \mathcal{NL}(F^{4,6}_9)\ncong\mathcal{NL}(G_7),\quad \mathcal{NL}(F^{5,5}_9)\ncong\mathcal{NL}(G_7),\\
&\qquad (b):\quad \mathcal{NL}(F^{2,8}_9)\ncong\mathcal{NL}(H_8),\quad \mathcal{NL}(F^{3,7}_9)\ncong\mathcal{NL}(H_8),\quad \mathcal{NL}(F^{4,6}_9)\ncong\mathcal{NL}(H_8),\quad \mathcal{NL}(F^{5,5}_9)\ncong\mathcal{NL}(H_8),\\
&\qquad (c):\quad \mathcal{NL}(F^{2,8}_9)\ncong\mathcal{NL}(I_8),\quad \mathcal{NL}(F^{3,7}_9)\ncong\mathcal{NL}(I_8),\quad \mathcal{NL}(F^{4,6}_9)\ncong\mathcal{NL}(I_8),\quad \mathcal{NL}(F^{5,5}_9)\ncong\mathcal{NL}(I_8),\\
&\qquad (d):\quad \mathcal{NL}(G_7)\ncong\mathcal{NL}(H_8),\quad \mathcal{NL}(G_7)\ncong\mathcal{NL}(I_8),\quad \mathcal{NL}(H_8)\ncong\mathcal{NL}(I_8),\\
&(2):\quad \mathcal{NL}(F^{2,5}_6)\ncong\mathcal{NL}(G_5),\quad \mathcal{NL}(F^{4,3}_6)\ncong\mathcal{NL}(G_5),\\
&(3):\quad \mathcal{NL}(F^{2,6}_7)\ncong\mathcal{NL}(H_6),\quad \mathcal{NL}(F^{3,5}_7)\ncong\mathcal{NL}(H_6),\quad \mathcal{NL}(F^{4,4}_7)\ncong\mathcal{NL}(H_6),\\
&(4):\quad \{\mathcal{NL}(H_\mu):\mu\geq 7\}\ncong\{\mathcal{NL}(F^{q,2}_{q+1}):q\geq 7\},\\
&(5):\quad \{\mathcal{NL}(I_{2q-1}):q\geq 4\}\ncong\{\mathcal{NL}(F^{q,2}_{q+1}):q\text{ is odd and }q\geq 7\},\\
&(6):\quad \{\mathcal{NL}(I_{2r+2}):r\geq 3\}\ncong\{\mathcal{NL}(F^{q,2}_{q+1}):q\text{ is even and }q\geq 8\},\\
&(7):\quad \{\mathcal{NL}(I_{2q-1}):q\geq 4\}\ncong\{\mathcal{NL}(H_\mu):\mu\text{ is odd and }\mu\geq 7\},\\
&(8):\quad \{\mathcal{NL}(I_{2r+2}):r\geq 3\}\ncong\{\mathcal{NL}(H_\mu):\mu\text{ is even and }\mu\geq 8\}.
\end{aligned}
$$

Thus, it is sufficient to prove the following proposition.

Proposition 2.7. *The following eight cases of Lie algebras $\mathcal{NL}(V)$ arising from zero-dimensional simple complete intersection singularities are not isomorphic:*

$$
\begin{aligned}
&(1):\ (a):\quad \mathcal{NL}(F^{2,8}_9)\ncong\mathcal{NL}(G_7),\quad \mathcal{NL}(F^{3,7}_9)\ncong\mathcal{NL}(G_7),\quad \mathcal{NL}(F^{4,6}_9)\ncong\mathcal{NL}(G_7),\quad \mathcal{NL}(F^{5,5}_9)\ncong\mathcal{NL}(G_7),\\
&\qquad (b):\quad \mathcal{NL}(F^{2,8}_9)\ncong\mathcal{NL}(H_8),\quad \mathcal{NL}(F^{3,7}_9)\ncong\mathcal{NL}(H_8),\quad \mathcal{NL}(F^{4,6}_9)\ncong\mathcal{NL}(H_8),\quad \mathcal{NL}(F^{5,5}_9)\ncong\mathcal{NL}(H_8),\\
&\qquad (c):\quad \mathcal{NL}(F^{2,8}_9)\ncong\mathcal{NL}(I_8),\quad \mathcal{NL}(F^{3,7}_9)\ncong\mathcal{NL}(I_8),\quad \mathcal{NL}(F^{4,6}_9)\ncong\mathcal{NL}(I_8),\quad \mathcal{NL}(F^{5,5}_9)\ncong\mathcal{NL}(I_8),\\
&\qquad (d):\quad \mathcal{NL}(G_7)\ncong\mathcal{NL}(H_8),\quad \mathcal{NL}(G_7)\ncong\mathcal{NL}(I_8),\quad \mathcal{NL}(H_8)\ncong\mathcal{NL}(I_8),\\
&(2):\quad \mathcal{NL}(F^{2,5}_6)\ncong\mathcal{NL}(G_5),\quad \mathcal{NL}(F^{4,3}_6)\ncong\mathcal{NL}(G_5),\\
&(3):\quad \mathcal{NL}(F^{2,6}_7)\ncong\mathcal{NL}(H_6),\quad \mathcal{NL}(F^{3,5}_7)\ncong\mathcal{NL}(H_6),\quad \mathcal{NL}(F^{4,4}_7)\ncong\mathcal{NL}(H_6),\\
&(4):\quad \{\mathcal{NL}(H_\mu):\mu\geq 7\}\ncong\{\mathcal{NL}(F^{q,2}_{q+1}):q\geq 7\},\\
&(5):\quad \{\mathcal{NL}(I_{2q-1}):q\geq 4\}\ncong\{\mathcal{NL}(F^{q,2}_{q+1}):q\textit{ is odd and }q\geq 7\},\\
&(6):\quad \{\mathcal{NL}(I_{2r+2}):r\geq 3\}\ncong\{\mathcal{NL}(F^{q,2}_{q+1}):q\textit{ is even and }q\geq 8\},\\
&(7):\quad \{\mathcal{NL}(I_{2q-1}):q\geq 4\}\ncong\{\mathcal{NL}(H_\mu):\mu\textit{ is odd and }\mu\geq 7\},\\
&(8):\quad \{\mathcal{NL}(I_{2r+2}):r\geq 3\}\ncong\{\mathcal{NL}(H_\mu):\mu\textit{ is even and }\mu\geq 8\}.
\end{aligned}
$$

Proof. Case (1): It follows from Propositions 2.3 and 2.5 that the nilradicals $(\mathcal{NL}(V))^*$ of the Lie algebras $\mathcal{NL}(H_8)$ and $\mathcal{NL}(I_8)$ have dimension nine. The nilradicals of $\mathcal{NL}(G_7)$, $\mathcal{NL}(F^{2,8}_9)$, $\mathcal{NL}(F^{3,7}_9)$, $\mathcal{NL}(F^{4,6}_9)$ and $\mathcal{NL}(F^{5,5}_9)$ have dimension eight. So it is easy to see that parts (b) and (c) are done due to different numbers of dimensions of the nilradical. Next we need to deal with parts (a) and (d). To prove parts (a) and (d), we use lower central series dimensions of the nilradical $(\mathcal{NL}(V))^*$ of $\mathcal{NL}(V)$.

Let

$$V = \{x, y \in \mathbb{C}^2 : (x^2, y^4)\}$$

be the G_7 contact simple complete intersection curve singularity. It is easy to see that the moduli algebra of G_7 is given by

$$\mathbb{C}\{x, y\}/(f, g, M_1) = \langle 1, x, y, y^2, y^3, xy, xy^2 \rangle.$$

The Lie algebra $\mathcal{NL}(V)$ has the following basis:

$$\begin{aligned} &e_1 = -x\partial_1 + y\partial_2, \quad e_2 = xy\partial_1, \quad e_3 = 2x\partial_1 - y\partial_2, \quad e_4 = x\partial_2, \quad e_5 = xy\partial_2, \\ &e_6 = -xy\partial_1 + y^2\partial_2, \quad e_7 = xy^2\partial_2, \quad e_8 = y^3\partial_2, \quad e_9 = xy^2\partial_1, \quad e_{10} = y^3\partial_1. \end{aligned}$$

The nilradical $(\mathcal{NL}(V))^*$ of $\mathcal{NL}(V)$ is given by

$$(\mathcal{NL}(V))^* = \langle e_2, e_4, e_5, e_6, e_7, e_8, e_9, e_{10} \rangle.$$

Set $e_2 = e_1, e_4 = e_2, e_5 = e_3, \ldots, e_{10} = e_8$.

The nilradical $(\mathcal{NL}(V))^*$ has the following multiplication table:

$$\begin{aligned} &[e_1, e_2] = e_3, \quad [e_1, e_3] = e_5, \quad [e_1, e_4] = -e_7, \quad [e_2, e_4] = 3e_3, \\ &[e_2, e_6] = 3e_5, \quad [e_2, e_7] = -e_5, \quad [e_2, e_8] = -e_6 + 3e_7, \quad [e_3, e_4] = 2e_5. \end{aligned}$$

The lower central series dimensions of $(\mathcal{NL}(V))^*$ are $[8, 4, 1, 0]$.

Let

$$V = \{x, y \in \mathbb{C}^2 : (xy, x^2 + y^8)\}$$

be the $F_9^{2,8}$ contact simple complete intersection curve singularity. It is easy to see that the moduli algebra of $F_9^{2,8}$ is given by

$$\mathbb{C}\{x, y\}/(f, g, M_1) = \langle 1, x, y, y^2, y^3, y^4, y^5, y^6, y^7 \rangle.$$

The Lie algebra $\mathcal{NL}(V)$ has the following basis:

$$\begin{aligned} &e_1 = y\partial_2, \quad e_2 = y^2\partial_2, \quad e_3 = y^3\partial_2, \quad e_4 = y^4\partial_2, \ldots, \\ &e_7 = y^7\partial_2, \quad e_8 = x\partial_2, \quad e_9 = y^7\partial_1, \quad e_{10} = x\partial_1. \end{aligned}$$

The nilradical $(\mathcal{NL}(V))^*$ of $\mathcal{NL}(V)$ is given by

$$(\mathcal{NL}(V))^* = \langle e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9 \rangle.$$

The nilradical $(\mathcal{NL}(V))^*$ has the following multiplication table:

$$\begin{aligned} &[e_2, e_3] = e_4, \quad [e_2, e_4] = 2e_5, \quad [e_2, e_5] = 3e_6, \quad [e_2, e_6] = 4e_7, \\ &[e_3, e_4] = e_6, \quad [e_3, e_5] = 2e_7, \quad [e_8, e_9] = -e_7. \end{aligned}$$

The lower central series dimensions of $(\mathcal{NL}(V))^*$ are $[8, 4, 3, 2, 1, 0]$.

Let

$$V = \{x, y \in \mathbb{C}^2 : (xy, x^3 + y^7)\}$$

be the $F_9^{3,7}$ contact simple complete intersection curve singularity. It is easy to see that the moduli algebra of $F_9^{3,7}$ is as follows:

$$\mathbb{C}\{x, y\}/(f, g, M_1) = \langle 1, x, x^2, y, y^2, y^3, y^4, y^5, y^6 \rangle.$$

The Lie algebra $\mathcal{NL}(V)$ has the following basis:

$$\begin{aligned} &e_1 = x\partial_1, \quad e_2 = y\partial_2, \quad e_3 = y^2\partial_2, \quad e_4 = y^3\partial_2, \ldots, \\ &e_6 = y^5\partial_2, \quad e_7 = x^2\partial_2, \quad e_8 = y^6\partial_2, \quad e_9 = x^2\partial_1, \quad e_{10} = y^6\partial_1. \end{aligned}$$

The nilradical $(\mathcal{NL}(V))^*$ of $\mathcal{NL}(V)$ is given by

$$(\mathcal{NL}(V))^* = \langle e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\rangle.$$

The nilradical $(\mathcal{NL}(V))^*$ has the following multiplication table:

$$[e_3, e_4] = e_5, \quad [e_3, e_5] = 2e_6, \quad [e_3, e_6] = 3e_8, \quad [e_4, e_5] = e_8.$$

The lower central series dimensions of $(\mathcal{NL}(V))^*$ are $[8, 3, 2, 1, 0]$.

Let

$$V = \{x, y \in \mathbb{C}^2 : (xy, x^4 + y^6)\}$$

be the $F_9^{4,6}$ contact simple complete intersection curve singularity. It is easy to see that the moduli algebra of $F_9^{4,6}$ is given by

$$\mathbb{C}\{x, y\}/(f, g, M_1) = \langle 1, x, x^2, y, y^2, y^3, y^4, y^5, x^3\rangle.$$

The Lie algebra $\mathcal{NL}(V)$ has the following basis:

$$\begin{aligned} &e_1 = x\partial_1, \quad e_2 = x^2\partial_1, \quad e_3 = y\partial_2, \quad e_4 = y^2\partial_2, \quad e_5 = y^3\partial_2, \\ &e_6 = y^4\partial_2, \quad e_7 = x^3\partial_2, \quad e_8 = y^5\partial_2, \quad e_9 = x^3\partial_1, \quad e_{10} = y^5\partial_1. \end{aligned}$$

The nilradical $(\mathcal{NL}(V))^*$ of $\mathcal{NL}(V)$ is given by

$$(\mathcal{NL}(V))^* = \langle e_2, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\rangle.$$

The nilradical $(\mathcal{NL}(V))^*$ has the following multiplication table:

$$[e_4, e_5] = e_6, \quad [e_4, e_6] = 2e_8.$$

The lower central series dimensions of $(\mathcal{NL}(V))^*$ are $[8, 2, 1, 0]$.

Let

$$V = \{x, y \in \mathbb{C}^2 : (xy, x^5 + y^5)\}$$

be the $F_9^{5,5}$ contact simple complete intersection curve singularity. It is easy to see that the moduli algebra of $F_9^{5,5}$ is as follows:

$$\mathbb{C}\{x, y\}/(f, g, M_1) = \langle 1, x, x^2, x^3, x^4, y, y^2, y^3, y^4\rangle.$$

The Lie algebra $\mathcal{NL}(V)$ has the following basis:

$$\begin{aligned} &e_1 = x\partial_1, \quad e_2 = x^2\partial_1, \quad e_3 = x^3\partial_1, \quad e_4 = y\partial_2, \quad e_5 = y^2\partial_2, \\ &e_6 = y^3\partial_2, \quad e_7 = x^4\partial_2, \quad e_8 = y^4\partial_2, \quad e_9 = x^4\partial_1, \quad e_{10} = y^4\partial_1. \end{aligned}$$

The nilradical $(\mathcal{NL}(V))^*$ of $\mathcal{NL}(V)$ is given by

$$(\mathcal{NL}(V))^* = \langle e_2, e_3, e_5, e_6, e_7, e_8, e_9, e_{10}\rangle.$$

The nilradical $(\mathcal{NL}(V))^*$ has the following multiplication table:

$$[e_2, e_3] = e_9, \quad [e_5, e_6] = e_8.$$

The lower central series dimensions of $(\mathcal{NL}(V))^*$ are $[8, 2, 0]$.

It follows from Propositions 2.3 and 2.5 that the nilradicals of $\mathcal{NL}(H_8)$ and $\mathcal{NL}(I_8)$ have lower central series dimensions $[9, 4, 1, 0]$ and $[9, 6, 4, 2, 1, 0]$, respectively. So, from all simple calculations above, it is easy see that the Lie algebra pairs in parts (a) and (d) have different lower central series dimensions of $(\mathcal{NL}(V))^*$.

Case (2): After simple calculations, similarly to above we can prove that the five-dimensional nilradicals $(\mathcal{NL}(V))^*$ of $\mathcal{NL}(H_6)$, $\mathcal{NL}(F_6^{2,5})$ and $\mathcal{NL}(F_6^{3,4})$ have different lower central series dimensions.

Cases (3)–(6): After simple calculations and from Propositions 2.3–2.5, it is easy to see that each pair of $\mathcal{NL}(V)$ in cases (3)–(6) has different dimensions of $(\mathcal{NL}(V))^*$.

Case (7): It follows from Proposition 2.3, when μ is odd and $\mu \geq 7$, that the minimal number of generators of the nilradical of $\mathcal{NL}(H_\mu)$ is

$$(\mathcal{NL}(V))^*/[(\mathcal{NL}(V))^*, (\mathcal{NL}(V))^*] = \langle e_2, e_3, e_4, e_{\mu+1}\rangle.$$

It is easy to see from Proposition 2.4, when $q \geq 7$, that the minimal number of generators of the nilradical of $\mathcal{NL}(I_{2q-1})$ is

$$(\mathcal{NL}(V))^*/[(\mathcal{NL}(V))^*, (\mathcal{NL}(V))^*] = \langle e_2, e_3\rangle.$$

Also note that, in case of $q = 4, 5, 6$, the minimal number of generators of the nilradical of $\mathcal{NL}(I_{2q-1})$ is

$$(\mathcal{NL}(V))^*/[(\mathcal{NL}(V))^*, (\mathcal{NL}(V))^*] = \langle e_2, e_3, e_{2q-3}\rangle.$$

Therefore,

$$\{\mathcal{NL}(I_{2q-1}) : q \geq 4\} \quad \text{and} \quad \{\mathcal{NL}(H_\mu) : \mu \text{ is odd and } \mu \geq 7\}$$

have different minimal numbers of generators of nilradicals of Lie algebras.

Case (8): It follows from Proposition 2.3, when μ is even and $\mu \geq 8$, that the minimal number of generators of the nilradical of $\mathcal{NL}(H_\mu)$ is

$$(\mathcal{NL}(V))^*/[(\mathcal{NL}(V))^*, (\mathcal{NL}(V))^*] = \langle e_2, e_3, e_4, e_{\mu-3}, e_{\mu+1}\rangle.$$

It is easy to see from Proposition 2.5, when $r \geq 4$, that the minimal number of generators of the nilradical of $\mathcal{NL}(I_{2r+2})$ is

$$(\mathcal{NL}(V))^*/[(\mathcal{NL}(V))^*, (\mathcal{NL}(V))^*] = \langle e_2, e_3\rangle.$$

Also note that, in case of $r = 3$, the minimal number of generators of the nilradical of $\mathcal{NL}(I_8)$ is

$$(\mathcal{NL}(V))^*/[(\mathcal{NL}(V))^*, (\mathcal{NL}(V))^*] = \langle e_2, e_3, e_5\rangle.$$

Therefore,

$$\{\mathcal{NL}(I_{2r+2}) : r \geq 3\} \quad \text{and} \quad \{\mathcal{NL}(H_\mu) : \mu \text{ is even and } \mu \geq 8\}$$

have different minimal numbers of generators of nilradicals of Lie algebras. □

It follows from Proposition 2.7 these eight cases are non-isomorphic. Therefore, we completely characterized the zero-dimensional simple complete intersection singularities by using Lie algebra $\mathcal{NL}(V)$. □

A Appendix

Maple code description. Query(Alg1, Alg2, parm, "Homomorphism") returns a 4-tuple TF, Eq, Soln, B. Here TF is true if Maple finds a set of values for the parameters for which the Matrix A is a homomorphism; Eq is the defining set of equations for the parameters parm in order that the matrix A be a homomorphism; Soln is a list of solutions to the equations Eq; and B is the list of Matrices obtained by evaluating A on the solutions in the list Soln.

A.1 Maple program of Proposition 2.1

```
> with(DifferentialGeometry):with(LieAlgebras):
> L1 := _DG([["LieAlgebra", Alg1, [5]], [[[1, 3, 2], 2], [[1, 4, 2], −1], [[1, 5, 3], −1], [[1, 5, 4], 2]
>DGsetup(L1):
> L2 := _DG([["LieAlgebra", Alg2, [5]], [[[1, 2, 3], 1], [[1, 3, 4], 1], [[2, 5, 4], 1]
```

> DGsetup(L2):
> A:= Matrix([[a11, a12, a13, a14, a15], [a21, a22, a23, a24, a25], [a31, a32, a33, a34, a35], [a41, a42, a43, a44, a45], [a51, a52, a53, a54, a55]])
> TF, EQ, SOLN, B:= Query(Alg1, Alg2, A, {a11, a12, a13, a14, a15, a21, a22, a23, a24, a25, a31, a32, a33, a34, a35, a41, a42, a43, a44, a45, a51, a52, a53, a54, a55}, "Homomorphism")

A.2 Maple program of Proposition 2.1

>with(DifferentialGeometry):with(LieAlgebras):
> $L1$:= _DG([["LieAlgebra", Alg3, [5]], [[[1, 3, 2], 2], [[1, 4, 2], −1], [[1, 5, 3], −1], [[1, 5, 4], 2]
> DGsetup(L1):
> $L2$:= _DG([["LieAlgebra", Alg4, [5]], [[[1, 2, 3], 1], [[1, 3, 4], 1], [[2, 5, 4], 1]
> DGsetup(L2, [f], [θ])
>

$$A := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 4 \\ 1 & 0 & -2 & 1 & 1 \\ 1 & -1 & 4 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

> ϕ:= Transformation(Alg3, Alg4, A)
> Query(Alg3, Alg4, A, "Homomorphism")
> true

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