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Classification of the nilradical of k-th Yau algebras arising from singularities

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Abstract: Every Lie algebra is a semi-direct product of semi-simple Lie algebra and a solvable Lie algebra. Brieskorn gave the connection between simple Lie algebras and simple singularities. Simple Lie algebras have been well understood, but not the solvable (nilpotent) Lie algebras. Classification of nilpotent Lie algebras with dimension up to 7 is known, but not for dimension greater than 7. Therefore it is important to establish connections between theory of singularities and theory of nilpotent Lie algebras. Let $(V, 0)$ be an isolated hypersurface singularity defined by the holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. The k-th Yau algebras $L^k(V), k \geq 0$ were introduced by the authors. It was defined to be the Lie algebra of derivations of the k-th moduli algebra $A^k(V)$. These Lie algebras are solvable in general and play an important role in the study of singularities. In this paper, we investigate the new connection between the nilpotent Lie algebras of dimension less than or equal to 7 and the nilradical of k-th Yau algebras.

Keywords: Derivation, nilpotent Lie algebra, isolated singularity, k-th Yau algebras.

1. Introduction

On the one hand, the classification theory of semi-simple Lie algebras over complex numbers included the killing form, root space decomposition, Dynkin diagrams, the Serre presentation, the theory of highest weight representation, and the Weyl character formula for finite-dimensional representations etc

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[Hm, Ja]. On the other hand, results of Seeley [Se] gave evidence that graphs and diagrams of nilpotent Lie algebras seem not good enough to classify the nilpotent Lie algebras. Historically, classification theories of semi-simple Lie algebras and solvable or nilpotent Lie algebras have a remarkable difference.

In the past, the progress towards a complete classification of nilpotent Lie algebras was quite slow. In [Um], nilpotent Lie algebras of dimension six were classified by Umlauf, a student of Engel. But this classification has several Lie algebras counted more than once because he was unaware of isomorphism among them. After that, several attempts have been made to develop some methods that reformulate the classification problem. Morozov [Mo] introduced inductive approach by considering the lower bound for the dimension of a maximal abelian ideal I of a nilpotent Lie algebra L. To study the inductive approach, one must know all smaller nilpotent Lie algebra and their finite-dimensional representations. Morozov's method is to consider L as a noncentral extension of L/I, where the abelian ideal I is a nontrivial L/I-module. Safiullina [Sa], compiled a list of seven-dimensional algebras by using Morozov's approach and knowledge of low-dimensional representations of two- and three-dimensional algebras. It is interesting to note that Morozov's method is useful if one is able to classify the irreducible finite-dimensional representation of all known algebras and also familiar with isomorphisms among the resulting new algebras.

Gauger studied metabelian Lie algebras [Ga], and developed a method to classify them. But this method is not applicable to study the general nilpotent Lie algebras. However, he attempted to classify the seven-dimensional metabelian Lie algebras and got partial information in dimensions eight and nine. Umlauf [Um] also classified a small subset of the algebras in dimensions seven, eight, and nine. He found that in these dimensions, infinite families of non-isomorphic nilpotent algebras occur. This phenomenon has also appeared in [Ch, Ga, San], during classification of Lie algebras.

Recently there has been a good progress in the study of nilpotent Lie algebras. Saullina made a first attempt to classify all seven-dimensional nilpotent Lie algebras. Magnin [Ma] introduced an inductive approach to classify the nilpotent Lie algebras. He classified algebras up to dimension 6 over the real field, and obtained partial information in dimension 7. Seeley [Se], classified the seven-dimensional nilpotent Lie algebras by using a different inductive approach and his classification consists of 161 tables. These 161 tables were divided into two main parts: first part has 130 indecomposable algebras and second part consists of 31 decomposable algebras with six infinite families parametrized by a single complex variable (see Theorem 1.2).

The lower central series of Lie algebras L is a sequence of ideal $L_{(i)}$ defined inductively by: $L_{(0)} = L$ and $L_{(i)} = [L, L_{(i-1)}]$, $i = 1, 2, 3, \cdots$. Recall that a Lie algebra L is called nilpotent if the lower central series of ideals: $L_{(0)}, L_{(1)}, L_{(2)}, \cdots$, terminates. The nilradical of a finite-dimensional Lie algebra L is its maximal nilpotent ideal. If $h \subset L$ is an ideal, then generalized center is $GC(h) = \{x \in L \mid [x, y] \in h, \quad \forall y \in L\}$. The upper central series of Lie algebra L is a sequence of ideal $C^i(L)$ defined inductively by $C^0(L) = GC(0)$ and $C^{i+1}(L) = GC(C^i(L))$. Note that $C^i(L) \subset C^{i+1}(L)$.

With the same notation as in [Ma, Se], we shall use the list of central series dimensions (lower central series is used in Theorem 1.1, and upper central series is used in Theorem 1.2) to denote nilpotent Lie algebras. For example, the algebras having a upper central series dimensions (resp. lower central series) $\{2, 4, 7\}$ (resp. $\{6, 3, 1\}$) are listed as $2, 4, 7_A, 2, 4, 7_B$ (resp. $6, 3, 1_A, 6, 3, 1_B$) and so forth. It is noted the subcripts A and B which are used to differentiate the two non-isomorphic nilpotent Lie algebras.

In the following theorem, the nilpotent Lie algebras of dimension 6 were classified using lower central series dimensions.

Theorem 1.1 ([Ma], pages 122–124)**.** *The classification of nilpotent Lie algebras of dimension less than or equal to* 6 *have the following list (here we use lower central series dimensions):*

Dimension 1 *(*1 *nilpotent Lie algebras)*

$\{g_1; \textit{abelian algebra}, 1_A\}$

Dimension 2 *(*1 *nilpotent Lie algebras)*

$\{(g_1)^2 = g_1 \times g_1; 2_A\}$

Dimension 3 *(*2 *nilpotent Lie algebras)*

$\{(g_1)^3; 3_A\}, \quad \{n : [X_1, X_2] = X_3; 3, 1_A\}$

Dimension 4 *(*3 *nilpotent Lie algebras)*

$\{(g_1)^4; 4_A\}, \quad \{n \times g_1; 4, 1_A\}, \quad \{g_4 : [X_1, X_2] = X_3, [X_1, X_3] = X_4; 4, 2, 1_A\}$

Dimension 5 *(*9 *nilpotent Lie algebras)*

$\{(g_1)^5; 5_A\}, \quad \{n \times g_1^2; 5, 1_A\}, \quad \{g_{5,1} : [X_1, X_2] = X_5, [X_3, X_4] = X_5; 5, 1_B\}$,
$\{g_4 \times g_1; 5, 2, 1_A\}, \quad \{g_{5,2} : [X_1, X_2] = X_4, [X_1, X_3] = X_5; 5, 2_A\}$,

$\{g_{5,3} : [X_1, X_2] = X_3, [X_1, X_3]{=}X_4, [X_2, X_5]{=}X_4; 5, 2, 1_B\}$, $\{g_{5,4} : [X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_2, X_3] = X_5; 5, 3, 2_A\}$, $\{g_{5,5} : [X_1, X_2] = X_3, [X_1, X_3]{=}X_4, [X_1, X_4]{=}X_5; 5, 3, 2, 1_A\}$, $\{g_{5,6} : [X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_1, X_4] = X_5, [X_2, X_3] = X_5; 5, 3, 2, 1_B\}$.

Dimension 6
Direct Product. (10 classes):

$$\{(g_1^6); 6_A\}, n \times n, \{n \times (g_1^3); 6, 1_A\}, g_4 \times (g_1^2), g_1 \times g_{5,i} (1 \le i \le 6),$$

$$(\textit{we use } 6, 1_B \textit{ to denote } g_1 \times g_{5,1}).$$

Other 22 classes are as follows:

$$\{[X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_1, X_5] = X_6; 6, 3, 1_A\},$$
$$\{[X_1, X_2] = X_6, [X_1, X_3] = X_4, [X_1, X_4] = X_5, [X_2, X_3] = X_5; 6, 3, 1_B\},$$
$$\vdots$$
$$[X_1, X_3] = X_4, [X_1, X_4] = X_6, [X_2, X_5] = X_6; 6, 3, 2, 1_C\}.$$

In the following theorem, the nilpotent Lie algebras of dimension 7 were classified using upper central dimensions. The classification of nilpotent Lie algebras consists of indecomposable and decomposable algebras. With similar notation as in [Se], $2 \oplus 1, 3, 5$ denotes the direct sum of 2-dimensional abelian Lie algebra with the unique algebra $\{1, 3, 5\}$ whose upper central series dimensions are $1, 3, 5$.

Theorem 1.2 ([Se], pages 482–494)**.** *The classification of nilpotent Lie algebras of dimension* 7 *have the following list (here we use upper central series dimensions):*

Decomposable algebras with large centers

$\{0(\textit{7-dimensional abelian}); 7\}$, $\{[a, b] = c; 4 \oplus 1, 3\}$, $\{[a, b] = d, [a, c] = e; 2 \oplus 2, 5\}$, $\{[a, b] = d, [a, c] = e, [b, c] = f; 1 \oplus 3, 6\}$, $\{[a, b] = c, [a, c] = d; 3 \oplus 1, 2, 4\}$, $\{[a, b] = c, [a, c] = d, [b, c] = e; 2 \oplus 2, 3, 5\}$.

Central series dimensions 3,7
Decomposables

$\{[a, b] = e, [c, d] = e; 2 \oplus 1, 5\}$, $\{[a, b] = e, [a, c] = f, [c, d] = e; 1 \oplus 2, 6\}$

$\{[a,b]=e,[c,d]=f;1\oplus 1,3\oplus 1,3\}$.

Indecomposables

$\{[a,b]=e,[b,c]=f,[b,d]=g;3,7_A\}$, $\{[a,b]=e,[b,c]=f,[c,d]=g;3,7_B\}$
$\{[a,b]=e,[b,c]=f,[c,d]=e,[b,d]=g;3,7_C\}$, $\{[a,b]=e,[a,c]=f,$
$[b,d]=g,[c,d]=e;3,7_D\}$.

Central series dimensions 3,5,7

$\vdots$

Central series dimensions 1,2,3,4,5,7

$\{[a,b]=c,[a,c]=d,[a,d]=e,[a,e]=f,[a,f]=g;1,2,3,4,5,7_A\}$,
$\{[a,b]=c,[a,c]=d,[a,d]=e,[a,e]=f,[a,f]=g,[b,c]=g;1,2,3,4,5,7_B\}$,

$\vdots$

$\{[a,b]=c,[a,c]=d,[a,d]=e,[a,e]=f,[a,f]=g,[b,c]=e,[b,d]=f,$
$[b,e]=\xi g,[c,d]=(1-\xi)g;1,2,3,4,5,7_I:\xi,(\xi\neq 1)\}$.

Let $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be the germ of a complex analytic function with an isolated critical point at the origin. Let $V=\{\mathbf{x}\in\mathbb{C}^n : f(\mathbf{x})=0\}$. Recall that the multiplicity of the singularity $(V,0)$ is defined to be the order of the lowest nonvanishing term in the power series Taylor expansion of f at 0. Let $\mathcal{O}_n$ denote the $\mathbb{C}$-algebra of germs of analytic functions defined at the origin of $\mathbb{C}^n$. The Milnor number μ of the singularity $(V,0)$ is defined by

$$\mu=\dim\mathcal{O}_n/(\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n}).$$

Recall that Siersma classified the isolated singularities with $\mu\leq 10$.

Theorem 1.3 ([Si])**.** *For $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ with isolated singularity at origin we have, either: $f\sim g+Q$ where g is a germ of one of the polynomials in the following table and $Q=x_{r+1}^2+\cdots+x_n^2$ (i.e., f is called stable equivalent to g).*

Name of singularity	Equation	Restriction	Milnor Number
A_k	x_1^{k+1}	$1\leq k\leq 10$	k

D_k	$x_1^2x_2+x_2^{k-1}$	$4\leq k\leq 10$	k
E_6	$x_1^3+x_2^4$		6
E_7	$x_1^3+x_1x_2^3$		7
E_8	$x_1^3+x_2^5$		8
J_{10}	$x_1^3+Ax_1x_2^4+Bx_2^6$	$4A^3+27B^2\neq 0$	10
X_9	$x_1^4+tx_1^2x_2^2+x_2^4$	$t^2\neq 4$	9
X_{10}	$x_1^4+x_1^2x_2^2+Ax_2^5$	$A\neq 0$	10
P_8	$x_1^3+x_2^2x_3+Ax_1x_3^2+Bx_3^3$	$4A^3+27B^2\neq 0$	8
P_9	$x_1x_2x_3+x_1^3+x_2^3+Ax_3^4$	$A\neq 0$	9
P_{10}	$x_1x_2x_3+x_1^3+x_2^3+Ax_3^5$	$A\neq 0$	10
Q_{10}	$x_1^3+x_2^2x_3+Ax_1x_3^3+x_3^4$		10
R_{10}	$x_1^3+x_1x_2x_3+x_2^4+Ax_3^4$	$A\neq 0$	10

For any isolated hypersurface singularity $(V,0)\subset(\mathbb{C}^n,0)$ where $V=V(f)=\{f=0\}$, one has the factor-algebra $A(V)=\mathcal{O}_n/(f,\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n})$ is finite-dimensional. This factor-algebra is called the moduli algebra of V.

Theorem 1.4 (Mather-Yau theorem [MY]). *Let V_1 and V_2 be two isolated hypersurface singularities and, $A(V_1)$ and $A(V_2)$ be the moduli algebras, then $(V_1,0)\cong(V_2,0)$ if and only if $A(V_1)\cong A(V_2)$.*

Yau [Ya1] introduced a Lie algebra of derivations of moduli algebra $A(V):=\mathcal{O}_n/(f,\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n})$, i.e., $L(V)=\mathrm{Der}(A(V),A(V))$. In [EK], [Yu], this finite-dimensional Lie algebra $L(V)$ was called Yau algebra and its dimension $\lambda(V)$ was called Yau number. The Yau algerba plays an important role in the classification of isolated hypersurface singularities ([SY], [Hu]). Yau and his collaborators have been systematically studying the Yau algebras of isolated hypersurface singularities begin from eighties ([Ya1]–[Ya3], [BY1, BY2], [SY], [YZ1, YZ2], [CYZ], [XY], [CHYZ], [CXY], [HYZ1]–[HYZ8]). Recently in [HYZ2], we extend the Yau algebra to the new k-th Yau algebra for an isolated hypersurface singularity. We first recall the definition of this new k-th Yau algebra.

The following theorem which is a slightly generalization of the well-known Mather-Yau theorem (cf. [GLS], Theorem 2.26).

Theorem 1.5 (generalized Mather-Yau theorem). *Let $f,g\in m\subset\mathcal{O}_n$. The following are equivalent:*

1) $(V(f),0)\cong(V(g),0)$;

2) For all $k\geq 0$, $\mathcal{O}_n/(f,m^kJ(f))\cong\mathcal{O}_n/(g,m^kJ(g))$ as $\mathbb{C}$-algebra;

3) There is some $k \geq 0$ such that $\mathcal{O}_n/(f, m^k J(f)) \cong \mathcal{O}_n/(g, m^k J(g))$ as $\mathbb{C}$-algebra, where $J(f) = (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$.

Therefore based on Theorem 1.5, it is natural to introduce the following Lie algebras to the singularity $(V(f), 0)$. The new series of k-th Yau algebras $L^k(V)$ (see Definition 2.4) are defined to be the Lie algebra of derivations of the k-th moduli algebra $A^k(V) = \mathcal{O}_n/(f, m^k J(f)), k \geq 0$, i.e., $L^k(V) = \mathrm{Der}(A^k(V), A^k(V))$. The dimension of $L^k(V)$ is denoted by $\lambda^k(V)$. These numbers $\lambda^k(V)(k \geq 0)$ are numerical analytic invariants of a singularity. We call it k-th Yau number. In particular, $L^0(V)$ is exactly the Yau algebra, thus $L^0(V) = L(V), \lambda^0(V) = \lambda(V)$. The nilradical of k-th Yau algebras is denoted by $(L^k(V))^*, k \geq 0$. We call the nilpotent Lie subalgebras of the k-th Yau algebras associated to an isolated hypersurface singularity geometric nilpotent Lie algebras.

Since a complete classification of nilpotent Lie algebras up to dimension seven was given in [Ma, Se]. A natural question is: whether these nilpotent Lie algebras in the above list in Theorem 1.1 and Theorem 1.2 (i.e., the classification nilpotent Lie algebras up to dimension seven) are geometric nilpotent Lie algebras? In general, this question is very hard. In this paper, we answer this question in some sense for the nilradical of k-th Yau algebras that arising from fewnomial singularity (see Definition 2.6). We shall prove the following result.

Main Theorem. *Let $(V(f), 0)$ be a fewnomial singularity defined by a weighted homogeneous polynomial $f(x_1, x_2, \cdots, x_n)$ and $(L^k(V(f)))^*, k \geq 0$, be the nilradical of k-th Yau algebras $(L^k(V(f)))$. Assume that dim $(L^k(V(f)))^* \leq 7$ and it is isomorphic to one of the nilpotent Lie algebras in the given list of classification of nilpotent Lie algebras of dimension ≤ 7 (cf. Theorem 1.1 and Theorem 1.2), then f's and its corresponding nilradicals $(L^k(V(f)))^*$ are as follows (we use lower (resp. upper) central series dimensions to denote the nilpotent Lie algebras of dimension ≤ 6 (resp.7)):*

Case 1. $k = 0$,

(1) $x_1^4 + x_2^2; 1_A$ (2) $x_1^5 + x_2^2; 2_A$ (3) $x_1^6 + x_2^2; 3, 1_A$ (4) $x_1^7 + x_2^2; 4, 2, 1_A$
(5) $x_1^8 + x_2^2; 5, 3, 2, 1_B$ (6) $x_1^9 + x_2^2; 6, 4, 3, 2, 1_C$ (7) $x_1^2 x_2 + x_2^4; 4, 2, 1_A$
(8) $x_1^2 x_2 + x_1 x_2^2; 2_A$, (9) $x_1^2 x_2 + x_2^5; 5, 2, 1_B$ (10) $x_1^2 x_2 + x_2^6; 6, 3, 1_C$
(11) $x_1^3 + x_2^4; 5, 2_A$ (12) $x_1^2 x_2 + x_1 x_2^3; 5, 2, 1_B$, (13) $x_1^2 x_2 + x_2^7; 1, 3, 5, 7_F$
(14) $x_1^3 x_2 + x_2^3; 1, 2, 4, 5, 7_E$ (15) $x_1^2 x_2 + x_1 x_2^4; 1, 3, 5, 7_F$
(16) $x_1^{10} + x_2^2; 1, 2, 3, 4, 5, 7_I(\xi = \dfrac{9}{10})$.

Case 2. $k = 1$,

(17) $x_1^3 + x_2^2; 3, 1_A$ (18) $x_1^5 + x_2^2; 5, 1_B$ (19)$x_1^6 + x_2^2; 6, 2, 1_A$ (20)$x_1^4 + x_2^2; 4, 1_A$ (21) $x_1^2 x_2 + x_2^2; 4, 1_A$ (22) $x_1^3 + x_2^2 + x_3^2; 5, 1_B$ (23) $x_1^3 x_2 + x_2^2; 6, 2, 1_A$ (24) $x_1^7 + x_2^7; 1, 4, 5, 7_B$ (25) $x_1^4 + x_2^2 + x_3^2; 6, 1_B$ (26) $x_1^5 + x_2^2 + x_3^2; 1, 7$.

Case 3. $k = 2$,

(27) $x_1^2 + x_2^2; 4_A$ (28) $x_1 x_2 + x_2^i (i \geq 2); 4_A$ (29) $x_1^2 + x_2^3; 1, 3, 5, 7_O$.

Case 4. $k = 3$,

(30) $x_1^2 + x_2^2; 6_A$ (31) $x_1 x_2 + x_2^i (i \geq 2); 6_A$.

Remark 1.1. *It follows from the classification result in Theorem 1.3, these isolated hypersurface singularities of Milnor number less than or equal to* 10 *which are not fewnomial singularities are listed as follows. We calculate their Yau numbers. It turns out that they are greater than 8. Thus, their nilradicals are not in the classification of nilpotent Lie algebras up to dimension seven. In general, a singularity with bigger Milnor number will have bigger Yau number. Based on the Remark 1.2 below, we conjecture that, in our main theorem, we have listed all geometric nilpotent Lie algebras with dimensions up to seven and their corresponding singularities up to stable equivalence.*

Name of singularity	Equation	Restriction	Milnor Number	Yau number
J_{10}	$x_1^3 + Ax_1 x_2^4 + Bx_2^6$	$4A^3 + 27B^2 \neq 0$	10	12
X_9	$x_1^4 + tx_1^2 x_2^2 + x_2^4$	$t^2 \neq 4$	9	11
X_{10}	$x_1^4 + x_1^2 x_2^2 + Ax_2^5$	$A \neq 0$	10	11
P_8	$x_1^3 + x_2^2 x_3 + Ax_1 x_3^2 + Bx_3^3$	$4A^3 + 27B^2 \neq 0$	8	10
P_9	$x_1 x_2 x_3 + x_1^3 + x_2^3 + Ax_3^4$	$A \neq 0$	9	10

P_{10}	$x_1x_2x_3 + x_1^3 + x_2^3 + Ax_3^5$	$A \neq 0$	10	11
Q_{10}	$x_1^3 + x_2^2x_3 + Ax_1x_3^3 + x_3^4$		10	13
R_{10}	$x_1^3 + x_1x_2x_3 + x_2^4 + Ax_3^4$	$A \neq 0$	11	11

Remark 1.2. *It is easy to check that the dimensions of nilradical of k-th Yau algebras $(L^k(V))^*, k \geq 4$ that arising from binomial singularity are greater than 7. It is also noted that in case of trinomial singularities (see Definition 2.6) with multiplicity ≥ 3 (see Propositions 2.3, 2.9, 2.10, 2.11, and 2.12), the dimensions of nilradical of Yau algebras $(L^0(V))^*$ are greater than 7. It is also easy to check through calculation, the dimensions of $(L^k(V))^*, k \geq 1$ that arising from trinomial singularities with multiplicity ≥ 3 are greater than 7. Furthermore, similarly we can check the dimensions of $(L^k(V))^*, k \geq 0$ that arising from n-nomial $(n > 3)$ singularities with multiplicity ≥ 3 are greater than 7.*

2. Basic results

Definition 2.1. *A Lie algebra is a vector space L over some field k (in this paper $k = \mathbb{C}$) together with a*

$$[\cdot,\ \cdot] : L \times L \to L$$

called the Lie bracket that satisfies the following axioms:

(1) Bilinear operator

$$[ax + by, z] = a[x, z] + b[y, z], \quad [z, ax + by] = a[z, x] + b[z, y]$$

for all scalars a, b in k and all elements x, y, z in L.

(2) Alternativity,

$$[x, x] = 0$$

for all x in L.

(3) The Jacobi identity,

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$$

for all x, y, z in L.

If $x \in L$, $y \to [x,y]$ is an endomorphism of L, which we denote by ad x.

Let L be a Lie algebra. For two subspaces A, B of L the symbol $[A,B]$ denotes the linear span of the set of all $[x,y]$ with x in A and y in B. A sub Lie algebra of L is a subspace, say J, of L that is closed under the bracket operation (i.e., $[J,J] \subset L$); J becomes then a Lie algebra with the linear and bracket operations inherited from L. A sub Lie algebra J is called an ideal of L if $[L,J] \subset J$ (if $x \in L$ and $y \in J$ implies $[x,y] \in J$). The centralizer C_S of a subset S of L is the set of those x in L that commute with all y in S (i.e., $[x,y]=0$). We say that two Lie algebras L, L' over k are isomorphic if there exists a vector space isomorphism $\phi : L \to L'$ satisfying $\phi([x,y]) = [\phi(x), \phi(y)]$.

We will basically deal with solvable and nilpotent Lie algebras so for completeness we recall the corresponding definitions.

Definition 2.2. *Given a Lie algebra L, there are two series of ideals: $L_{(*)} = \{L_{(i)}\}, L^{(*)} = \{L^{(i)}\}, L_{(0)} = L^{(0)} = L, L_{(1)} = L^{(1)} = [L,L], L_{(i)} = [L, L_{(i-1)}]$, $L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$, $i = 2,3,\ldots$. L is called nilpotent if the lower central series $L_{(*)}$ terminates. L is called solvable if the derived series $L^{(*)}$ terminates.*

The following basic concepts and results will be used to compute the derivation Lie algebras of isolated hypersurface singularities.

Let A, B be associative algebras over $\mathbb{C}$. The subalgebra of endomorphisms of A generated by the identity element and left and right multiplications by elements of A is called multiplication algebra $M(A)$ of A. The centroid $C(A)$ is defined as the set of endomorphisms of A which commute with all elements of $M(A)$. Obviously, $C(A)$ is a unital subalgebra of $\mathrm{End}(A)$. The following statement is a particular case of a general result from Proposition 1.2 of [Bl]. Let $S = A \otimes B$ be a tensor product of finite dimensional associative algebras with units. Then

$$\mathrm{Der} S \cong (\mathrm{Der} A) \otimes C(B) + C(A) \otimes (\mathrm{Der} B).$$

We will only use this result for commutative associative algebras with unit, in which case the centroid coincides with the algebra itself and one has following result for commutative associative algebras A, B:

Theorem 2.6. ([Bl])*For commutative associative algebras A, B,*

$$DerS \cong (DerA) \otimes B + A \otimes (DerB). \tag{2.1}$$

We shall use this formula in the sequel.

Definition 2.3. *Let J be an ideal in an analytic algebra S. Then $Der_J S \subseteq Der_{\mathbb{C}} S$ is Lie subalgebra of all $\sigma \in Der_{\mathbb{C}} S$ for which $\sigma(J) \subset J$.*

We shall use the following well-known result to compute the derivations.

Theorem 2.7. ([YZ2])*Let J be an ideal in $R = \mathbb{C}\{x_1, \cdots, x_n\}$. Then there is a natural isomorphism of Lie algebras*

$$(Der_J R)/(J \cdot Der_{\mathbb{C}} R) \cong Der_{\mathbb{C}}(R/J).$$

Recall that a derivation of commutative associative algebra A is defined as a linear endomorphism D of A satisfying the Leibniz rule: $D(ab) = D(a)b + aD(b)$. Thus for such an algebra A one can consider the Lie algebra of its derivations $Der(A, A)$ with the bracket defined by the commutator of linear endomorphisms.

Definition 2.4. *Let $f(x_1, \cdots, x_n)$ be a complex polynomial and $V = \{f = 0\}$ be a germ of an isolated hypersurface singularity at the origin in $\mathbb{C}^n$. Let $A^k(V) = \mathcal{O}_n/(f, m^k J(f)), k \geq 0$ be the k-th moduli algebra. Then $L^k(V) := Der(A^k(V), A^k(V))$ is the k-th Yau algebra of $(V, 0)$. The $\lambda^k(V)$ is the dimension of derivation Lie algebra $L^k(V)$, it is called k-th Yau number.*

It is noted that when $k = 0$, then derivation Lie algebra is called Yau algebra.

Definition 2.5. *A polynomial $f \in \mathbb{C}[x_1, x_2, \cdots, x_n]$ is called weighted homogeneous if there exist positive rational numbers $w_1, \ldots, w_n$ (called weights of indeterminates x_j) and d such that, for each monomial $\prod x_j^{k_j}$ appearing in f with non-zero coefficient, one has $\sum w_j k_j = d$. The number d is called the weighted degree of f with respect to weights w_j and is denoted by* $\deg f$. *The collection $(w; d) = (w_1, \cdots, w_n; d)$ is called the weight type of f.*

Definition 2.6. *An isolated hypersurface singularity in $\mathbb{C}^n$ is fewnomial if it can be defined by a n-nomial in n variables and it is a weighted homogeneous fewnomial isolated singularity if it can be defined by a weighted homogeneous fewnomial.* 3*-nomial isolated hypersurface singularity is also called trinomial singularity.*

Proposition 2.1. *Let f be a weighted homogeneous fewnomial isolated singularity with mult$(f) \geq 3$. Then f analytically equivalent to a linear combination of the following three series:*

Type A. $x_1^{a_1} + x_2^{a_2} + \cdots + x_{n-1}^{a_{n-1}} + x_n^{a_n}$, $n \geq 1$,
Type B. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n}$, $n \geq 2$,
Type C. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_1$, $n \geq 2$.

Proposition 2.1 has an immediate corollary.

Corollary 2.1. *Each binomial isolated singularity is analytically equivalent to one from the three series: A) $x_1^{a_1}+x_2^{a_2}$, B) $x_1^{a_1}x_2+x_2^{a_2}$, C) $x_1^{a_1}x_2+x_2^{a_2}x_1$.*

Wolfgang and Atsushi [ET] give the following classification of weighted homogeneous fewnomial singularities in case of three variables.

Proposition 2.2. ([ET]) *Let $f(x_1,x_2,x_3)$ be a weighted homogeneous fewnomial isolated singularity with mult$(f)\geq 3$. Then f is analytically equivalent to following five types:*

Type 1. $x_1^{a_1}+x_2^{a_2}+x_3^{a_3}$,
Type 2. $x_1^{a_1}x_2+x_2^{a_2}x_3+x_3^{a_3}$,
Type 3. $x_1^{a_1}x_2+x_2^{a_2}x_3+x_3^{a_3}x_1$,
Type 4. $x_1^{a_1}+x_2^{a_2}+x_3^{a_3}x_2$,
Type 5. $x_1^{a_1}x_2+x_2^{a_2}x_1+x_3^{a_3}$.

In order to prove the main theorem, we need to use following main results from [YZ2], [HYZ1], [HYZ2] and [HYZ3].

Proposition 2.3 ([YZ2])**.** *Let $(V,0)$ be a weighted homogeneous fewnomial isolated singularity of type A which is defined by $f=x_1^{a_1}+x_2^{a_2}+\cdots+x_n^{a_n}$ $(a_i\geq 3,\ 1\leq i\leq n)$ with weight type $(\frac{1}{a_1},\frac{1}{a_2},\cdots,\frac{1}{a_n};1)$. Then the Yau number*

$$\lambda(V)=n\prod_{i=1}^{n}(a_i-1)-\sum_{i}^{n}(a_1-1)(a_2-1)\cdots\widehat{(a_i-1)}\cdots(a_n-1),$$

where $\widehat{(a_i-1)}$ means that a_i-1 is omitted.

Proposition 2.4 ([YZ2])**.** *Let $(V,0)$ be a binomial isolated singularity of type B which is defined by $f=x_1^{a_1}x_2+x_2^{a_2}$ with weight type $(\frac{a_2-1}{a_1a_2},\frac{1}{a_2};1)$. Then the Yau number*

$$\lambda(V)=2a_1a_2-2a_1-3a_2+5.$$

Proposition 2.5 ([YZ2])**.** *Let $(V,0)$ be a binomial isolated singularity of type C which is defined by $f=x_1^{a_1}x_2+x_2^{a_2}x_1$ with weight type $(\frac{a_2-1}{a_1a_2-1},\frac{a_1-1}{a_1a_2-1};1)$. If mult$(f)\geq 4$, i.e., $a_1,a_2\geq 3$, then the Yau number*

$$\lambda(V)=2a_1a_2-2a_1-2a_2+6.$$

If mult$(f)=3$, i.e., $f=x_1^2x_2+x_2^{a_2}x_1$, then the Yau number is $\lambda(V)=2a_2$.

Proposition 2.6 ([HYZ2]). *Let $(V,0)$ be a weighted homogeneous fewnomial isolated singularity of type A which is defined by $f = x_1^{a_1} + x_2^{a_2}$ $(a_1 \geq 2,\ a_2 \geq 2)$ with weight type $(\frac{1}{a_1}, \frac{1}{a_2}; 1)$. Then*

$$\lambda^1(V) = \begin{cases} 2a_1a_2 - 3(a_1+a_2) + 10; & a_1 \geq 3, a_2 \geq 3 \\ a_1 + 2; & a_1 \geq 2, a_2 = 2. \end{cases}$$

Proposition 2.7 ([HYZ2]). *Let $(V,0)$ be a binomial isolated singularity of type B which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}$ $(a_1 \geq 1,\ a_2 \geq 2)$ with weight type $(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}; 1)$. Then*

$$\lambda^1(V) = \begin{cases} 2a_1a_2 - 2a_1 - 3a_2 + 11; & a_1 \geq 2, a_2 \geq 3 \\ 2a_1 + 2; & a_1 \geq 2, a_2 = 2 \\ 4; & a_1 = 1, a_2 \geq 2. \end{cases}$$

Proposition 2.8 ([HYZ2]). *Let $(V,0)$ be a binomial isolated singularity of type C which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_1$ $(a_1 \geq 1, a_2 \geq 1)$ with weight type $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}; 1)$. Then*

$$\lambda^1(V) = \begin{cases} 2a_1a_2 - 2a_1 - 2a_2 + 12; & a_1 \geq 3, a_2 \geq 3 \\ 2a_1 + 6; & a_1 \geq 2, a_2 = 2 \\ 4; & a_1 \geq 1, a_2 = 1 \\ 4; & a_1 = 1, a_2 \geq 2. \end{cases}$$

Proposition 2.9 ([HYZ1]). *Let $(V,0)$ be a fewnomial isolated singularity of type 2 which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}$ $(a_1 \geq 2, a_2 \geq 2, a_3 \geq 3)$ with weight type $(\frac{1-a_3+a_2a_3}{a_1a_2a_3}, \frac{a_3-1}{a_2a_3}, \frac{1}{a_3}; 1)$. Then the Yau number*

$$\lambda(V) = \begin{cases} 3a_1a_2a_3 - 2a_1a_3 - 4a_2a_3 + 6a_3 + 2a_1 - 2a_1a_2 + 2a_2 & \\ -7; & a_1 \geq 2, a_2 \geq 3, a_3 \geq 3 \\ 4a_1a_3 - 3a_3 - 2a_1 - 1; & a_1 \geq 2, a_2 = 2, a_3 \geq 3. \end{cases}$$

Proposition 2.10 ([HYZ1]). *Let $(V,0)$ be a fewnomial isolated singularity of type 3 which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_1$ $(a_1 \geq 2, a_2 \geq 2, a_3 \geq 2)$ with weight type*

$$\left(\frac{1-a_3+a_2a_3}{1+a_1a_2a_3}, \frac{1-a_1+a_1a_3}{1+a_1a_2a_3}, \frac{1-a_2+a_1a_2}{1+a_1a_2a_3}; 1\right).$$

Then the Yau number

$$\lambda(V) = \begin{cases} 12; & a_1 = 2, a_2 = 2, a_3 = 2 \\ 3a_1a_2a_3 - 2(a_1a_2 + a_1a_3 + a_2a_3) + 2(a_1 + a_2 + a_3) \\ -1; & Otherwise. \end{cases}$$

Proposition 2.11 ([HYZ3]). *Let $(V,0)$ be a fewnomial surface isolated singularity of type 4 which is defined by $f = x_1^{a_1} + x_2^{a_2} + x_3^{a_3}x_2$ $(a_1 \geq 3, a_2 \geq 3, a_3 \geq 2)$ with weight type $(\frac{1}{a_1}, \frac{1}{a_2}, \frac{a_2-1}{a_2a_3}; 1)$. Then*

$$\lambda(V) = \ 3a_1a_2a_3 - 4a_1a_2 - 4a_2a_3 - 2a_1a_3 + 6a_1 + 5a_2 + 2a_3 - 7.$$

Proposition 2.12 ([HYZ3]). *Let $(V,0)$ be a fewnomial surface isolated singularity of type 5 which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_1 + x_3^{a_3}$ $(a_1 \geq 2, a_2 \geq 2, a_3 \geq 3)$ with weight type $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}, \frac{1}{a_3}; 1)$. Then*

$$\lambda(V) = \begin{cases} 3a_1a_2a_3 - 4a_1a_2 - 2(a_2a_3 + a_1a_3) + 2(a_1 + a_2) + 6a_3 \\ -6; & a_1 \geq 3, a_2 \geq 3, a_3 \geq 3 \\ 4a_2a_3 - 6a_2; & a_1 = 2, a_2 \geq 2, a_3 \geq 3. \end{cases}$$

Proposition 2.13 ([HYZ3]). *Let $(V,0)$ be a weighted homogeneous fewnomial isolated singularity of type A which is defined by $f = x_1^{a_1} + x_2^{a_2}$ $(a_1 \geq 1,\ a_2 \geq 1)$ with weight type $(\frac{1}{a_1}, \frac{1}{a_2}; 1)$. Then*

$$\lambda^2(V) = \begin{cases} 2a_1a_2 - 3(a_1 + a_2) + 17; & a_1 \geq 5, a_2 \geq 5 \\ 3a_2 + 5; & a_1 = 3, a_2 \geq 4 \\ 13; & a_1 = 3, a_2 = 3 \\ 5a_2 + 4; & a_1 = 4, a_2 \geq 5 \\ 23; & a_1 = 4, a_2 = 4 \\ a_2 + 5; & a_1 = 2, a_2 \geq 3 \\ 6; & a_1 = 2, a_2 = 2 \\ 1; & a_1 = 1, a_2 \geq 1. \end{cases}$$

Proposition 2.14 ([HYZ3]). *Let $(V,0)$ be a binomial isolated singularity of type B which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}$ $(a_1 \geq 1,\ a_2 \geq 1)$ with weight type*

$(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}; 1)$. *Then*

$$\lambda^2(V) = \begin{cases} 2a_1a_2 - 2a_1 - 3a_2 + 20; & a_1 \geq 5, a_2 \geq 5 \\ 5a_2 + 12; & a_1 = 4, a_2 \geq 5 \\ 31; & a_1 = 4, a_2 = 4 \\ 4a_1 + 7; & a_1 \geq 3, a_2 = 3 \\ 2a_1 + 5; & a_1 \geq 2, a_2 = 2 \\ a_2 + 11; & a_1 = 2, a_2 \geq 3 \\ 6; & a_1 = 1, a_2 \geq 2 \\ 1; & a_1 \geq 1, a_2 = 1. \end{cases}$$

Proposition 2.15 ([HYZ3]). *Let $(V, 0)$ be a binomial isolated singularity of type C which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_1$ ($a_1 \geq 1,\ a_2 \geq 1$) with weight type $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}; 1)$. Then*

$$\lambda^2(V) = \begin{cases} 2a_1a_2 - 2(a_1 + a_2) + 21; & a_1 \geq 4, a_2 \geq 4 \\ 4a_2 + 13; & a_1 = 3, a_2 \geq 3 \\ 2a_2 + 10; & a_1 = 2, a_2 \geq 3 \\ 13; & a_1 = 2, a_2 = 2 \\ 6; & a_1 = 1, a_2 \geq 1. \end{cases}$$

3. Proof of main theorem

In order to prove the main theorem, we need to prove following propositions.

Proposition 3.16. *Let $(V, 0)$ be a fewnomial singularity defined by the weighted homogeneous polynomial $f(x_1, x_2, \cdots, x_n)$ and $(L^k(V))^*, k \geq 0$, be a nilradical of k-th Yau algebras with dimension ≤ 7. Then following nilpotent Lie algebras with notation [Ma] and [Se] are pairwise isomorphic:*

$(1) : (L^0(x_1^4 + x_2^2))^* \cong 1_A$ $\quad(2) : (L^0(x_1^5 + x_2^2))^* \cong 2_A$ $\quad(3) : (L^0(x_1^6 + x_2^2))^* \cong 3, 1_A$, $\quad(4) : (L^0(x_1^7 + x_2^2))^* \cong 4, 2, 1_A$ $\quad(5) : (L^0(x_1^8 + x_2^2))^* \cong 5, 3, 2, 1_B$ $\quad(6) : (L^0(x_1^9 + x_2^2))^* \cong 6, 4, 3, 2, 1_C$ $\quad(7) : (L^0(x_1^2x_2 + x_2^4))^* \cong 4, 2, 1_A$ $\quad(8) : (L^0(x_1^2x_2 + x_2^2x_1))^* \cong 2_A$ $\quad(9) : (L^0(x_1^2x_2 + x_2^5))^* \cong 5, 2, 1_B$ $\quad(10) : (L^0(x_1^2x_2 + x_2^6))^* \cong 6, 3, 1_C$ $\quad(11) : (L^0(x_1^3 + x_2^4))^* \cong 5, 2_A$ $\quad(12) : (L^0(x_1^2x_2 + x_2^3x_1))^* \cong 5, 2, 1_B$ $\quad(13) : (L^0(x_1^2x_2 + x_2^7))^* \cong 1, 3, 5, 7_F$ $\quad(14) : (L^0(x_1^3x_2 + x_2^3))^* \cong 1, 2, 4, 5, 7_E$ $\quad(15) : (L^0(x_1^2x_2 + x_2^4x_1))^* \cong 1, 3, 5, 7_F$ $\quad(16) : (L^0(x_1^{10} + x_2^2))^* \cong 1, 2, 3, 4, 5, 7_I$ $\quad(17) : (L^1(x_1^3 + x_2^2))^* \cong 3, 1_A$

$(18) : (L^1(x_1^5 + x_2^2))^* \cong 5, 1_B$ $\quad (19) : (L^1(x_1^6 + x_2^2))^* \cong 6, 2, 1_A$
$(20) : (L^1(x_1^4 + x_2^2))^* \cong 4, 1_A$ $\quad (21) : (L^1(x_1^2x_2 + x_2^2))^* \cong 4, 1_A$
$(22) : (L^1(x_1^3 + x_2^2 + x_3^2))^* \cong 5, 1_B$ $\quad (23) : (L^1(x_1^3x_2 + x_2^2))^* \cong 6, 2, 1_A$
$(24) : (L^1(x_1^7 + x_2^7))^* \cong 1, 4, 5, 7_B$ $\quad (25) : (L^1(x_1^4 + x_2^2 + x_3^2))^* \cong 6, 1_B$
$(26) : (L^1(x_1^5 + x_2^2 + x_3^2))^* \cong 1, 7$ $\quad (27) : (L^2(x_1^2 + x_2^2))^* \cong 4_A$
$(28) : (L^2(x_1x_2 + x_2^i), i \geq 2)^* \cong 4_A$ $\quad (29) : (L^2(x_1^2 + x_2^3))^* \cong 1, 3, 5, 7_O$
$(30) : (L^3(x_1^2 + x_2^2))^* \cong 6_A$ $\quad (31) : (L^3(x_1x_2 + x_2^i), i \geq 2)^* \cong 6_A$.

Proof. After simple calculation, we have the following table:

Table 3.3:

Nilradical of k-th Yau algebras	Multiplication table	Lower and upper central series dimensions
$(L^0(x_1^4 + x_2^2))^* = <x_1^2\partial_1>$	0	1
$(L^0(x_1^5 + x_2^2))^* = <x_1^2\partial_1, x_1^3\partial_1>$	0	2
$(L^0(x_1^6 + x_2^2))^* = <x_1^2\partial_1, x_1^3\partial_1, x_1^4\partial_1>$	$[e_1, e_2] = e_3$	$3, 1$
$(L^0(x_1^7 + x_2^2))^* = <x_1^2\partial_1, x_1^3\partial_1, x_1^4\partial_1, x_1^5\partial_1>$	$[e_1, e_2] = e_3, [e_1, e_3] = 2e_4$	$4, 2, 1$
$(L^0(x_1^8+x_2^2))^* = <x_1^2\partial_1, x_1^3\partial_1, x_1^4\partial_1, x_1^5\partial_1, x_1^6\partial_1>$	$[e_1, e_2] = e_3, [e_1, e_3] = 2e_4, [e_1, e_4] = 3e_5, [e_2, e_3] = e_5$	$5, 3, 2, 1$
$(L^0(x_1^9 + x_2^2))^* = <x_1^2\partial_1, x_1^3\partial_1, x_1^4\partial_1, x_1^5\partial_1, x_1^6\partial_1, x_1^7\partial_1>$	$[e_1, e_2] = e_3, [e_1, e_3] = 2e_4, [e_1, e_4] = 3e_5, [e_1, e_5] = 4e_6, [e_2, e_3] = e_5, [e_2, e_4] = 2e_6$	$6, 4, 3, 2, 1$

$(L^0(x_1^2x_2+x_2^4))^* = <x_2^3\partial_1, x_2^2\partial_2, 4x_2^2\partial_1+x_1\partial_2, x_2^3\partial_2>$	$[e_1,e_3]=e_4, [e_2,e_3]=8e_1$	$4,2,1$
$(L^0(x_1^2x_2+x_2^2x_1))^* = <x_1x_2\partial_1, x_1x_2\partial_2>$	0	2
$(L^0(x_1^2x_2+x_2^5))^* = <5x_2^3\partial_1+x_1\partial_2, x_2^4\partial_1, x_2^2\partial_2, x_2^3\partial_2, x_2^4\partial_2>$	$[e_1,e_2]=-e_5, [e_1,e_3]=-15e_2, [e_3,e_4]=e_5$	$5,2,1$
$(L^0(x_1^2x_2+x_2^6))^* = <6x_2^4\partial_1+x_1\partial_2, x_2^5\partial_1, x_2^2\partial_2, x_2^3\partial_2, x_2^4\partial_2, x_2^5\partial_2>$	$[e_1,e_2]=-e_6, [e_1,e_3]=-24e_2, [e_3,e_4]=e_5, [e_3,e_5]=2e_6$	$6,3,1$
$(L^0(x_1^3+x_2^4))^* = <x_1x_2\partial_1, x_1x_2^2\partial_1, x_1x_2\partial_2, x_1x_2^2\partial_2>$	$[e_1,e_3]=e_4, [e_1,e_5]=-e_2, [e_3,e_5]=e_4$	$5,2$
$(L^0(x_1^2x_2+x_2^3x_1))^* = <x_1x_2^2\partial_1, x_1x_2\partial_2, x_1x_2^2\partial_2, 2x_1x_2\partial_1+x_2^2\partial_2, 3x_1x_2\partial_1+2x_1\partial_2>$	$[e_1,e_5]=2e_3, [e_2,e_4]=-e_3, [e_2,e_5]=3e_3, [e_4,e_5]=15e_1$	$5,2,1$
$(L^0(x_1^2x_2+x_2^7))^* = <-x_2^2\partial_2, -x_2^3\partial_2, -x_2^4\partial_2, -x_2^5\partial_2, -x_2^6\partial_2, -7x_2^5\partial_1-x_1\partial_2, -x_2^6\partial_1>$	$[e_1,e_2]=-e_3, [e_1,e_3]=-2e_4, [e_1,e_4]=-3e_5, [e_1,e_6]=-35e_7, [e_2,e_3]=-e_5, [e_6,e_7]=e_5$	$1,3,5,7$
$(L^0(x_1^2x_2+x_2^4x_1))^* = <\frac{-3x_2^4\partial_1}{10}+\frac{x_2^4\partial_2}{5}, \frac{x_2^5\partial_1}{2}-\frac{x_1\partial_2}{2}, \frac{x_2^4\partial_2}{3}, \frac{x_2^5\partial_2}{2}, x_2^6\partial_2, \frac{-7x_2^5\partial_1}{2}+2x_1\partial_2+x_2^3\partial_2, x_2^6\partial_1>$	$[e_1,e_2]=\frac{3e_3}{20}+\frac{4e_7}{5}, [e_1,e_3]=\frac{4e_4}{15}, [e_1,e_4]=\frac{3e_5}{10}, [e_1,e_6]=\frac{-7e_7}{2}, [e_2,e_3]=\frac{e_5}{3}, [e_2,e_7]=\frac{e_5}{2}, [e_3,e_6]=e_5, [e_6,e_7]=-2e_5$	$1,3,5,7$

$(L^0(x_1^3x_2+x_2^3))^* = <-x_1x_2\partial_1, 3x_2\partial_1+2x_1^2\partial_2, -2x_1^2\partial_1 - 3x_1x_2\partial_2, -x_2^2\partial_1, -x_1x_2^2\partial_2, x_1x_2\partial_1 - x_2^2\partial_2, -x_1x_2^2\partial_1>$	$[e_1,e_2] = 3e_4, [e_1,e_3] = -3e_5, [e_1,e_6] = e_7, [e_2,e_3] = 18e_1 + 15e_6, [e_2,e_4] = -4e_5, [e_2,e_5] = -3e_7, [e_3,e_4] = -2e_7, [e_3,e_6] = -6e_5$	$1,2,4,5,7$
$(L^0(x_1^{10}+x_2^2))^* = <x_1^2\partial_1, x_1^3\partial_1, x_1^4\partial_1, x_1^5\partial_1, x_1^6\partial_1, x_1^7\partial_1, x_1^8\partial_1>$	$[e_1,e_2] = e_3, [e_1,e_3] = 2e_4, [e_1,e_4] = 3e_5, [e_1,e_5] = 4e_6, [e_1,e_6] = 5e_7, [e_2,e_3] = e_5, [e_2,e_4] = 2e_6, [e_2,e_5] = 3e_7, [e_3,e_4] = e_7$	$1,2,3,4,5,7$
$(L^1(x_1^3+x_2^2))^* = <x_1^2\partial_2, x_2\partial_1, x_1^2\partial_1>$	$[e_1,e_2] = e_3$	$3,1$
$(L^1(x_1^5+x_2^2))^* = <x_1^2\partial_1, x_1^3\partial_1, x_1^4\partial_2, x_2\partial_1, x_1^4\partial_1>$	$[e_1,e_2] = e_5, [e_3,e_4] = e_5$	$5,1$
$(L^1(x_1^6+x_2^2))^* = <x_1^2\partial_1, x_1^3\partial_1, x_1^4\partial_1, x_1^5\partial_1, x_2\partial_1, x_1^5\partial_2>$	$[e_1,e_2] = e_3, [e_1,e_3] = 2e_4, [e_5,e_6] = -e_4$	$6,2,1$
$(L^1(x_1^4+x_2^2))^* = <x_2\partial_1, x_1^2\partial_1, x_1^3\partial_1, x_1^3\partial_2,>$	$[e_1,e_2] = e_3$	$4,1$
$(L^1(x_1^3+x_2^2+x_3^2))^* = <x_1^2\partial_3, x_3\partial_1, x_1^2\partial_2, x_2\partial_1, x_1^2\partial_1>$	$[e_1,e_2] = e_5, \quad [e_3,e_4] = e_5$	$5,1$
$(L^1(x_1^4+x_2^2+x_3^2))^* = <-x_1^2\partial_1, -x_3\partial_1, -x_1^3\partial_3, x_2\partial_1, -x_1^3\partial_2, -x_1^3\partial_1>$	$[e_2,e_3] = e_6, \quad [e_4,e_5] = e_6$	$6,1$
$(L^1(x_1^5+x_2^2+x_3^2))^* = <x_1^2\partial_1, x_1^3\partial_1, x_1^4\partial_3, x_3\partial_1, x_2\partial_1, x_1^4\partial_2, x_1^4\partial_1>$	$[e_1,e_2] = e_7, \quad [e_3,e_4] = e_7, \quad [e_5,e_6] = e_7$	$1,7$

$(L^1(x_1^2x_2+x_2^2))^* = <x_1^3\partial_2, x_1^3\partial_1, x_1^2\partial_1, x_2\partial_1>$	$[e_1,e_2]=e_3$	$4,1$
$(L^1(x_1^3x_2+x_2^2))^* = <x_2\partial_1, x_1^3\partial_1, x_1^4\partial_1, x_1^5\partial_1, 2x_1^2\partial_1-3x_1^4\partial_2, x_1^5\partial_2>$	$[e_1,e_2]=-\frac{3}{2}e_4, [e_1,e_5]=e_3, [e_1,e_6]=-e_4, [e_2,e_5]=-2e_3, [e_3,e_5]=-4e_4$	$6,2,1$
$(L^1(x_1^7+x_2^2))^* = <x_1^2\partial_1, x_1^3\partial_1, x_1^4\partial_1, x_1^5\partial_1, x_1^6\partial_1, x_2\partial_1, x_1^6\partial_2>$	$[e_1,e_2]=e_3, [e_1,e_3]=2e_4, [e_1,e_4]=3e_5, [e_2,e_3]=e_5, [e_6,e_7]=-e_5$	$1,4,5,7$
$(L^2(x_1^2+x_2^2))^* = <x_2^2\partial_1, x_1x_2\partial_1, x_2^2\partial_2, x_1x_2\partial_2>$	0	4
$(L^2(x_1x_2+x_2^i), i\geq 2)^* = <x_2^2\partial_1, x_1^2\partial_1, x_2^2\partial_2, x_1^2\partial_2>$	0	4
$(L^2(x_1^2+x_2^3))^* = <-x_2^2\partial_2, -x_2^2\partial_1-x_1\partial_2, -x_2^2\partial_1, -x_2^3\partial_2, -x_1x_2\partial_2, -x_2^3\partial_1, -x_1x_2\partial_1>$	$[e_1,e_2]=2e_5-2e_6, [e_1,e_3]=-2e_6, [e_2,e_3]=e_1-2e_7, [e_2,e_6]=e_4, [e_2,e_7]=e_5, [e_3,e_5]=-e_4, [e_3,e_7]=-e_6$	$1,3,5,7$
$(L^3(x_1^2+x_2^2))^* = <x_2\partial_1-x_1\partial_2, x_2^2\partial_1-x_1x_2\partial_2, x_2^3\partial_1, x_1x_2\partial_1+x_2^2\partial_2, x_1x_2^2\partial_1, x_2^3\partial_2, x_1x_2^2\partial_2>$	0	6
$(L^3(x_1x_2+x_2^i), i\geq 2)^* = <x_2^3\partial_1, x_1^2\partial_1, x_1^3\partial_1, x_2^2\partial_2, x_2^3\partial_2, x_1^3\partial_2>$	0	6

Magnin [Ma] gave the complete classification of nilpotent Lie algebras of dimension ≤ 6. Since we deal with nilradical $((L^k(V))^*, k\geq 0)$ of k-th Yau algebras with dimension ≤ 7. So it is easy to see from [Ma] and [Se] we have the following possible classification of nilpotent Lie algebras which are equivalent to table 3.3.

Table 3.4: Classification of nilpotent Lie algebras([Ma],[Se])

Dimension of nilpotent Lie algebras	Multiplication table	Lower and upper central series dimensions
1	0	1_A
2	0	2_A
3	$[X_1, X_2] = X_3$	$3, 1_A$
4	$[X_1, X_2] = X_3, [X_1, X_3] = X_4$	$4, 2, 1_A$
4	0	4_A
4	$[X_1, X_2] = X_3$	$4, 1_A$
5	$[X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_1, X_4] = X_5, [X_2, X_3] = X_5$	$5, 3, 2, 1_B$
5	$[X_1, X_2] = X_5, [X_3, X_4] = X_5$	$5, 1_B$
5	$[X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_2, X_5] = X_4$	$5, 2, 1_B$
5	$[X_1, X_2] = X_4, [X_1, X_3] = X_5$	$5, 2_A$
6	$[X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_1, X_4] = X_5, [X_1, X_5] = X_6, [X_2, X_3] = X_5, [X_2, X_4] = X_6$	$6, 4, 3, 2, 1_C$
6	$[X_1, X_2] = X_5, [X_1, X_3] = X_4, [X_1, X_4] = X_6, [X_2, X_5] = X_6$	$6, 3, 1_C$
6	$[X_1, X_3] = X_4, [X_1, X_4] = X_6, [X_2, X_5] = X_6$	$6, 2, 1_A$
6	$[X_2, X_3] = X_6, [X_4, X_5] = X_6$	$6, 1_B$
7	$[a, b] = c, [a, c] = d, [a, d] = g, [b, c] = g, [e, f] = g$	$1, 4, 5, 7_B$
7	$[a, b] = c, [a, c] = e, [a, d] = f, [a, e] = g, [b, c] = g, [d, f] = g$	$1, 3, 5, 7_F$
7	$[a, b] = c, [a, c] = e, [a, d] = f, [a, f] = g, [b, c] = f, [b, e] = g$	$1, 3, 5, 7_O$
7	$[a, b] = c, [a, c] = d, [a, d] = e, [a, e] = f, [a, f] = g, [b, c] = e, [b, d] = f, [b, e] = \frac{9g}{10}, [c, d] = \frac{g}{10}$	$1, 2, 3, 4, 5, 7_I$
7	$[a, b] = c, [a, c] = d, [a, d] = f, [a, f] = g, [b, c] = f, [b, d] = g, [b, e] = f, [c, e] = g$	$1, 2, 4, 5, 7_E$
7	$[a, b] = g, [c, d] = g, [e, f] = g$	$1, 7$
6	0	6_A

It easily follows from tables 3.3 and 3.4, we only need to deal with cases (4), (5), (6), (7), (9), (10), (11), (12), (13), (14), (15), (16), (19), (23), (24), (29).

Case $(4)(L^0(x_1^7+x_2^2))^* \cong 4,2,1_A$.

It is easy to check that (4) is true under the following map:

$$(4): e_1 \longmapsto X_1+X_2+X_3+X_4, \quad e_2 \longmapsto X_2+X_3+X_4, \quad e_3 \longmapsto X_3+X_4, \quad e_4 \longmapsto \frac{1}{2}X_4.$$

Similarly, other cases are true under the following mapping:

$$\begin{aligned}
(5): & e_1 \longmapsto X_1+X_2+X_3+X_4+X_5, \quad e_2 \longmapsto \frac{X_2}{6}+X_3+X_4+X_5,\\
& e_3 \longmapsto \frac{X_3}{6}+X_4+\frac{11}{6}X_5, \quad e_4 \longmapsto \frac{X_4}{12}+\frac{7}{12}X_5, \quad e_5 \longmapsto \frac{X_5}{36}.\\
(6): & e_1 \longmapsto X_1+X_3+X_4+X_5+X_6, \quad e_2 \longmapsto \frac{X_2}{6}+X_3+X_4+X_5+X_6,\\
& e_3 \longmapsto \frac{X_3}{6}+X_4+\frac{5}{6}X_5+\frac{5}{6}X_6, \quad e_4 \longmapsto \frac{X_4}{12}+\frac{X_5}{2}+\frac{5}{12}X_6,\\
& e_5 \longmapsto \frac{X_5}{36}+\frac{X_6}{6}, \quad e_6 \longmapsto \frac{X_6}{144}.\\
(7): & e_1 \longmapsto -\frac{X_3}{8}-\frac{X_4}{8}, \quad e_2 \longmapsto X_2+X_3+X_4, \quad e_3 \longmapsto X_1+X_2+X_3,\\
& +X_4, \quad e_4 \longmapsto \frac{X_4}{8}.\\
(9): & e_1 \longmapsto X_1-15X_2+X_3+X_4+X_5, \quad e_2 \longmapsto -\frac{X_3}{15}+X_4,\\
& e_3 \longmapsto X_2+X_3+X_4+X_5, \quad e_4 \longmapsto X_3+X_4+\frac{X_5}{15}, \quad e_5 \longmapsto \frac{X_4}{15}.\\
(10): & e_1 \longmapsto X_2+X_3+X_4+X_5+X_6, \quad e_2 \longmapsto \frac{1}{24}(X_4+X_5-X_6),\\
& e_3 \longmapsto X_1-X_2+X_3+X_4+X_5+X_6, \quad e_4 \longmapsto -\frac{X_3}{12}+X_4+X_6,\\
& e_5 \longmapsto -\frac{X_4}{12}+X_6, \quad e_6 \longmapsto -\frac{X_6}{24}.\\
(11): & e_1 \longmapsto -X_1+X_2+X_3+X_4+X_5, \quad e_2 \longmapsto +2X_4+2X_5,\\
& e_3 \longmapsto X_2+X_3+X_4+X_5, \quad e_4 \longmapsto -X_4-X_5, \quad e_5 \longmapsto X_1+X_2\\
& +X_3+X_4+X_5.
\end{aligned}$$

$$(12):e_1 \longmapsto \frac{31}{45}X_3 - \frac{37}{45}X_4, \quad e_2 \longmapsto X_3 + X_4 + \frac{X_5}{30},$$
$$e_3 \longmapsto -\frac{31X_4}{90}, \quad e_4 \longmapsto -\frac{31}{3}X_2 + X_3 + X_4 + X_5,$$
$$e_5 \longmapsto X_1 + X_2 + X_3 + X_4 + X_5.$$
$$(13):e_1 \longmapsto 6a, \quad e_2 \longmapsto -6b, \quad e_3 \longmapsto 36c, \quad e_4 \longmapsto -108e,$$
$$e_5 \longmapsto 216g, \quad e_6 \longmapsto 6\sqrt{35}\sqrt{-1}d, \quad e_7 \longmapsto \frac{-36}{35}\sqrt{35}\sqrt{-1}f.$$
$$(14):e_1 \longmapsto \frac{-c}{108} + \frac{5e}{216}, \quad e_2 \longmapsto a, \quad e_3 \longmapsto \frac{-b}{6}, \quad e_4 \longmapsto \frac{d}{324},$$
$$e_5 \longmapsto \frac{-f}{1296}, \quad e_6 \longmapsto \frac{-e}{36}, \quad e_7 \longmapsto \frac{g}{3888}.$$
$$(15):e_1 \longmapsto \frac{6a}{5}, \quad e_2 \longmapsto \frac{3b}{2} - \frac{3\sqrt{35}d}{2}, \quad e_3 \longmapsto \frac{-36\sqrt{35}f}{35} + 12c,$$
$$e_4 \longmapsto 54e, \quad e_5 \longmapsto 216g, \quad e_6 \longmapsto 6\sqrt{35}d, \quad e_7 \longmapsto \frac{-72\sqrt{35}f}{35}.$$
$$(16):e_1 \longmapsto a, \quad e_2 \longmapsto \frac{b}{6}, \quad e_3 \longmapsto \frac{c}{6}, \quad e_4 \longmapsto \frac{d}{12}$$
$$e_5 \longmapsto \frac{e}{36}, \quad e_6 \longmapsto \frac{f}{144} \quad e_7 \longmapsto \frac{g}{720}.$$
$$(19):e_1 \longmapsto X_1 + X_2 + X_3 + X_4 + X_5 + X_6, \quad e_2 \longmapsto -2X_3 + X_4 + X_6,$$
$$e_3 \longmapsto -2X_4 + X_6, \quad e_4 \longmapsto -X_6, \quad e_5 \longmapsto X_2 - X_4 + 2X_5 + X_6,$$
$$e_6 \longmapsto -X_4 + X_5 + X_6.$$
$$(23):e_1 \longmapsto \frac{X_5 - X_3}{4}, \quad e_2 \longmapsto 6(X_2 + X_4 - X_5) + \frac{X_3}{2} \quad e_3 \longmapsto X_4,$$
$$e_4 \longmapsto X_6 \quad e_5 \longmapsto 4(X_1 - X_3 + X_5), \quad e_6 \longmapsto 4(X_2 + X_4 - X_5).$$
$$(24):e_1 \longmapsto a, \quad e_2 \longmapsto \frac{b}{6} \quad e_3 \longmapsto \frac{c}{6}, \quad e_4 \longmapsto \frac{d}{12}, \quad e_5 \longmapsto \frac{g}{36},$$
$$e_6 \longmapsto f \quad e_7 \longmapsto \frac{e}{36}.$$
$$(29):e_1 \longmapsto \frac{c}{2} + \frac{d}{4} - 2e - \frac{3f}{2} + \frac{9g}{4}, \quad e_2 \longmapsto a + \frac{b}{4} + c + d + e + f + g,$$
$$e_3 \longmapsto b - 4c + d + e + f + g, \quad e_4 \longmapsto \frac{g}{4}, \quad e_5 \longmapsto -\frac{e}{4} + \frac{f}{16},$$
$$e_6 \longmapsto \frac{f}{4} - g, \quad e_7 \longmapsto -\frac{c}{4} + \frac{d}{8} + e - \frac{f}{4} + g. \qquad \square$$

Remark 3.3. *These isomorphisms above can be obtained by using the Maple program which we list in the appendix.*

Proof of Main Theorem. It is an immediate corollary of Proposition 3.16 and Remark 1.2. □

Appendix

Maple code description:

Query(Alg1, Alg2, parm, "Homomorphism") returns a 4-tuple TF, Eq, Soln, B. Here TF is true if Maple finds a set of values for the parameters for which the Matrix A is a homomorphism; Eq is the defining set of equations for the parameters parm in order that the matrix A be a homomorphism; Soln is a list of solutions to the equations Eq; and B is the list of Matrices obtained by evaluating A on the solutions in the list Soln.

Maple program of propositions 3.16

> with(DifferentialGeometry):with(LieAlgebras):
> $L1 := \_DG([[\text{“}LieAlgebra\text{”}, Alg1, [7]], [[[1, 2, 4], 3], [[1, 3, 5], -3], [[1, 6, 7], 1], [[2, 3, 1], 18], [[2, 3, 6], 15], [[2, 4, 5], -4], [[2, 5, 7], -3], [[3, 4, 7], -2], [[3, 6, 5], -6]]])$
>DGsetup(L1):
> $L2 := \_DG([[\text{“}LieAlgebra\text{”}, Alg2, [7]], [[[1, 2, 3], 1], [[1, 3, 4], 1], [[1, 4, 6], 1], [[1, 6, 7], 1], [[2, 3, 6], 1], [[2, 4, 7], 1], [[2, 5, 6], 1], [[3, 5, 7], 1]]])$
> DGsetup(L2):
> A := Matrix([[a11, a12, a13, a14, a15, a16, a17], [a21, a22, a23, a24, a25, a26, a27], [a31, a32, a33, a34, a35, a36, a37], [a41, a42, a43, a44, a45, a46, a47], [a51, a52, a53, a54, a55, a56, a57], [a61, a62, a63, a64, a65, a66, a67], [a71, a72, a73, a74, a75, a76, a77]])
> TF, EQ, SOLN, B := Query(Alg1, Alg2, A, {a11, a12, a13, a14, a15, a16, a17, a21, a22, a23, a24, a25, a26, a27, a31, a32, a33, a34, a35, a36, a37, a41, a42, a43, a44, a45, a46, a47, a51, a52, a53, a54, a55, a56, a57, a61, a62, a63, a64, a65, a66, a67, a71, a72, a73, a74, a75, a76, a77}, "Homomorphism")

Maple program of Proposition 3.16

>with(DifferentialGeometry):with(LieAlgebras):
> $L1 := \_DG([[\text{“}LieAlgebr\text{”}, Alg3, [7]], [[[1, 2, 4], 3], [[1, 3, 5], -3], [[1, 6, 7], 1], [[2, 3, 1], 18], [[2, 3, 6], 15], [[2, 4, 5], -4], [[2, 5, 7], -3], [[3, 4, 7], -2], [[3, 6, 5], -6]]])$
> DGsetup(L1):
> $L2 := \_DG([[\text{“}LieAlgebra\text{”}, Alg4, [7]], [[[1, 2, 3], 1], [[1, 3, 4], 1], [[1, 4, 6], 1],$

$[[1, 6, 7], 1], [[2, 3, 6], 1], [[2, 4, 7], 1], [[2, 5, 6], 1], [[3, 5, 7], 1]]])$
$>$ DGsetup(L2, [f], [θ])
$>$A := Matrix(7, 7, [[0, 1, 0, 0, 0, 0, 0], [0, 0, $-1/6$, 0, 0, 0, 0], [$-1/108$, 0, 0, 0, 0, 0, 0], [0, 0, 0, 1/324, 0, 0, 0], [5/216, 0, 0, 0, 0, $-1/36$, 0], [0, 0, 0, 0, $-1/1296$, 0, 0], [0, 0, 0, 0, 0, 0, 1/3888]])
$>$ ϕ := Transformation(Alg3, Alg4, A)
$>$ Query(Alg3, Alg4, A, "Homomorphism")
$>$ true

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