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New k-th Yau algebras of isolated hypersurface singularities and weak Torelli-type theorem

Naveed Hussain, Stephen S.-T. Yau, and Huaiqing Zuo

Let V be a hypersurface with an isolated singularity at the origin defined by the holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. The Yau algebra $L(V)$ is defined to be the Lie algebra of derivations of the moduli algebra $A(V) := \mathcal{O}_n/(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$, i.e., $L(V) = \mathrm{Der}(A(V), A(V))$. It is known that $L(V)$ is a finite dimensional Lie algebra and its dimension $\lambda(V)$ is called Yau number. In this paper, we introduce a new series of Lie algebras, i.e., k-th Yau algebras $L^k(V)$, $k \geq 0$, which are a generalization of Yau algebra. $L^k(V)$ is defined to be the Lie algebra of derivations of the k-th moduli algebra $A^k(V) := \mathcal{O}_n/(f, m^k J(f)), k \geq 0$, i.e., $L^k(V) = \mathrm{Der}(A^k(V), A^k(V))$, where m is the maximal ideal of $\mathcal{O}_n$. The k-th Yau number is the dimension of $L^k(V)$ which we denote as $\lambda^k(V)$. In particular, $L^0(V)$ is exactly the Yau algebra, i.e., $L^0(V) = L(V), \lambda^0(V) = \lambda(V)$. These numbers $\lambda^k(V)$ are new numerical analytic invariants of singularities. In this paper we obtain the weak Torelli-type theorems of simple elliptic singularities using Lie algebras $L^1(V)$ and $L^2(V)$. We shall also characterize the simple singularities completely using $L^1(V)$.

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1. Introduction

Recall that the simple (Kleinian, rational double point) singularities which play significant role in singularity theory [AVZ], consist of two series A_k : $\{x_1^{k+1}+x_2^2=0\}\subset\mathbb{C}^2, k\geq 1$, $D_k : \{x_1^2x_2+x_2^{k-1}=0\}\subset\mathbb{C}^2, k\geq 4$, and three exceptional singularities E_6, E_7, E_8 defined in $\mathbb{C}^2$ by polynomials $x_1^3+x_2^4$, ${x_1}^3+x_1{x_2}^3, x_1^3+x_2^5$, respectively. It should be noted that each simple singularity belongs to one of these three series: A) $x_1^a+x_2^b$, B) $x_1^ax_2+x_2^b$, C) $x_1^ax_2+x_2^bx_1$. These are called binomial singularities, a special kind of fewnomial singularities (see Definition 2.3).

For any isolated hypersurface singularity $(V,0)\subset(\mathbb{C}^n,0)$ where $V=V(f)=\{f=0\}$, one has the factor-algebra $A(V)=\mathcal{O}_n/(f,\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n})$ is finite dimensional. This factor-algebra is called the moduli algebra of V and its dimension $\tau(V)$ is called Tyurina number. The order of the lowest nonvanishing term in the power series expansion of f at 0 is called the multiplicity (denoted by $mult(f)$) of the singularity $(V,0)$. It is well known that a polynomial $f\in\mathbb{C}[x_1,\cdots,x_n]$ is said to be weighted homogeneous if there exist positive rational numbers $w_1,\cdots,w_n$ (weights of $x_1,\cdots,x_n$) and d such that, $\sum a_iw_i=d$ for each monomial $\prod x_i^{a_i}$ appearing in f with nonzero coefficient. The number d is called weighted homogeneous degree (w-deg) of f with respect to weights w_j. The weight type of f is denoted as $(w_1,\cdots,w_n;d)$. Without loss of generality, we can assume that w-deg$f=1$. The Milnor number of the isolated hypersurface singularity is defined by $\mu=\dim\mathcal{O}_n/(\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n})$.

The well known Mather-Yau theorem [MY] stated that: Let V_1 and V_2 be two isolated hypersurface singularities and, $A(V_1)$ and $A(V_2)$ be the moduli algebras, then $(V_1,0)\cong(V_2,0)\iff A(V_1)\cong A(V_2)$. Motivated from the Mather-Yau theorem, Yau considered the Lie algebra of derivations of moduli algebra $A(V):=\mathcal{O}_n/(f,\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n})$, i.e., $L(V)=\mathrm{Der}(A(V),A(V))$. The finite dimensional Lie algebra $L(V)$ was called Yau algebra and its dimension $\lambda(V)$ was called Yau number (see [Yu], [EK]). The Yau algebra plays an important role in singularity theory [SY]. Yau and his collaborators have been systematically studying the Yau algebras of isolated hypersurface singularities begin from eighties (see [Ya1, Ya2, Ya3], [BY1], [BY2], [SY], [YZ1, YZ2], [CYZ], [CXY], [CCYZ], [Yu], [HYZ1, HYZ2, HYZ3, HYZ4], [Hu]). In particular, Yau algebras of simple singularities and simple elliptic singularities were computed and a number of elaborate applications to deformation theory were presented in [BY1] and [SY]. However, the Yau algebra cannot characterize the simple singularties completely. In [EK], it was shown that if X and Y are two simple singularities except the pair A_6 and

D_5, then $L(X) \cong L(Y)$ as Lie algebras if and only if X and Y are analytically isomorphic. Therefore, a natural question is to find new Lie algebras which can be used to distinguish singularities (at least for the simple singularities) completely. The aim of this paper is to introduce the series of new k-th Yau algebra, which will be used to characterize singularities, and we also compare it with Yau algebra that arising from isolated hypersurface singularities.

Recall that we have the following theorem.

Theorem 1.1. *([GLS], Theorem 2.26) Let $f, g \in m \subset \mathcal{O}_n$. The following are equivalent:*

1) $(V(f), 0) \cong (V(g), 0)$;

2) For all $k \geq 0$, $\mathcal{O}_n/(f, m^k J(f)) \cong \mathcal{O}_n/(g, m^k J(g))$ as $\mathbb{C}$-algebra;

3) There is some $k \geq 0$ such that $\mathcal{O}_n/(f, m^k J(f)) \cong \mathcal{O}_n/(g, m^k J(g))$ as $\mathbb{C}$-algebra, where $J(f) = (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$.

In particular, if $k = 0$ and $k = 1$ above, then the claim of the equivalence of 1) and 3) is exactly same as the Mather-Yau theorem.

Based on Theorem 1.1, it is natural for us to introduce the new series of k-th Yau algebras $L^k(V)$ which are defined to be the Lie algebras of derivations of the k-th moduli algebra $A^k(V) = \mathcal{O}_n/(f, m^k J(f)), k \geq 0$, i.e., $L^k(V) = \mathrm{Der}(A^k(V), A^k(V))$. Its dimension is denoted as $\lambda^k(V)$. This number $\lambda^k(V)$ is a new numerical analytic invariant of a singularity. We call it k-th Yau number. In particular, $L^0(V)$ is exactly the Yau algebra, thus $L^0(V) = L(V), \lambda^0(V) = \lambda(V)$. Therefore, we have reasons to believe that these new Lie algebras $L^k(V)$ and numerical invariants $\lambda^k(V)$ will also play an important role in the study of singularities.

On the one hand, since $L(V)$ can not characterize the simple singularities completely, so there is a natural question: whether these simple singularities (or which classes of more general singularities) can be characterized completely by the Lie algebra $L^k(V)$, $k \geq 1$? In this paper, we shall answer this question positively. We prove that the simple singularities V can be characterized completely using the $L^1(V)$. Therefore the k-th Yau algebra $L^k(V), k \geq 1$, is more subtle comparing to the Yau algebra $L(V)$ in some sense.

Furthermore, since derivations of moduli algebras are analogs of vector fields on smooth manifolds, such direction of research is in the spirit of the classical theorem of Pursell and Shanks stating that the Lie algebra of smooth vectors fields on a smooth manifold determines the diffeomorphism type of the manifold [PS]. In fact, Theorem B below yields a similar result for the simple singularities.

One of our main goals is to investigate whether the k-th Yau algebras determine the analytic structures of singularities, similar as the mentioned result of Pursell and Shanks. In this paper, we have computed $L^1(V)$ for some natural classes of singularities beyond simple singularities, in order to reveal the specific properties for simple singularities. In particular, we obtain more detailed information about $L^1(V)$ for three series of fewnomial isolated hypersurface singularities chosen in such a way that each simple singularity belongs to one of three series. Thus our results about computation of $L^1(V)$ that arising from simple singularities can be considered as extension of those presented in [EK].

On the other hand, Griffiths has studied the Torelli problem when a family of complex projective hypersurfaces in $\mathbb{C}P^n$ is given and his school asks whether the period map is injective on that family, i.e., whether the family of complex hypersurfaces can be distinguished by means of their Hodge structures. A complex projective hypersurface in $\mathbb{C}P^n$ can be viewed as a complex hypersurface with isolated singularity in $\mathbb{C}^{n+1}$. Let $V = \{z \in \mathbb{C}^{n+1} : f(z) = 0\}$ be a complex hypersurface with isolated singularity at the origin. Seeley and Yau investigated the family of isolated complex hypersurface singularities using Yau algebras and obtained two strong Torelli-type theorems for simple elliptic singularities $\tilde{E}_7$ and $\tilde{E}_8$ (cf. [SY]). There is a natural question arises: whether the family of isolated complex hypersurface singularities can be distinguished by means of their k-th Yau algebras $L^k(V), k \geq 1$. The family of hypersurface singularities here is not arbitrary. First of all, as in projective case, we are actually studying the complex structures of an isolated hypersurface singularity. In view of the theorem of Lê and Ramanujan [LR], we require that the Milnor number μ is constant along this family. Recall that the Tyurina number τ is a complex analytic invariant. So it suffices to consider only a (μ, τ)-constant family of isolated complex hypersurface singularities [SY]. The simple elliptic singularities are such families. Unfortunately, the methods in [SY] can not be generalized to $L^k, k \geq 1$ directly. We shall use completely new method to prove weak Torelli-type theorems of L^1 and L^2 for simple elliptic singularities $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$. However, to prove the weak Torelli-type theorem for $\tilde{E}_6$, we have to consider the 2-th Yau algebra $L^2(\tilde{E}_6)$. In fact, the family $L^1(\tilde{E}_6)$ is constant.

In this paper, we will prove the following main results.

Theorem A. *$L^2(\tilde{E}_6)$, $L^1(\tilde{E}_7)$, $L^2(\tilde{E}_7)$, $L^1(\tilde{E}_8)$ and $L^2(\tilde{E}_8)$ are non-trivial one-parameter families. Thus the weak Torelli-type theorems hold for simple elliptic singularities $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$.*

Theorem B. *If X and Y are two simple hypersurface singularities, then $L^1(X) \cong L^1(Y)$ as Lie algebras, if and only if X and Y are analytically isomorphic.*

Remark 1.1. *Using the same method as in our proof, though the calculations are extremely complicated, we conjecture that Theorem A and Theorem B are still true for $k > 2$ and $k > 1$ respectively.*

2. Preliminaries

2.1. Isolated hypersurface singularities

Let $\mathbb{C}[x_1, x_2, \cdots, x_n]$ be the algebra of complex polynomials in n indeterminates. We use $\mathcal{O}_n$ to denote the algebra of germs of holomorphic functions at the origin of $\mathbb{C}^n$. Obviously, $\mathcal{O}_n$ can be naturally identified with the algebra of convergent power series in n indeterminates with complex coefficients. For a polynomial $f \in \mathbb{C}[x_1, x_2, \cdots, x_n]$, denotes by $V = V(f)$ the germ at the origin of $\mathbb{C}^n$ of hypersurface $\{f = 0\} \subset \mathbb{C}^n$. We say that V is a germ of isolated hypersurface singularity if the origin is an isolated zero of the gradient of f. The local (function) algebra of V is defined as the (commutative associative) algebra $F(V) \cong \mathcal{O}_n/(f)$, where (f) is the principal ideal generated by the germ of f at the origin. According to Hilbert's Nullstellensatz for an isolated singularity $V = V(f) = \{f = 0\}$ the factor-algebra $A(V) = \mathcal{O}_n/(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$ is finite dimensional. This factor-algebra is called the moduli algebra of V and its dimension $\tau(V)$ is called Tyurina number.

Definition 2.1. A polynomial $f \in \mathbb{C}[x_1, x_2, \cdots, x_n]$ is called weighted homogeneous if there exist positive rational numbers $w_1, \cdots, w_n$ (called weights of indeterminates x_j) and d such that, for each monomial $\prod x_j^{k_j}$ appearing in f with non-zero coefficient, one has $\sum w_j k_j = d$. The number d is called the weighted homogeneous degree (w-deg) of f with respect to weights w_j and is denoted $\deg f$. The collection $(w; d) = (w_1, \cdots, w_n; d)$ is called the weight type of f.

It is well known that a polynomial f is a weighted homogeneous after a biholomorphic change of coordinates if and only if $\mu = \tau$ [Sa]. In the weighted homogeneous case, the Milnor number can be computed from the following simple formula.

Proposition 2.1. *[MO] For an isolated hypersurface singularity V defined by a weighted homogeneous polynomial of $(w_1, \cdots, w_n; d)$ type, one has*

$$\tau(V) = \mu(V) = \prod_{i=1}^{n} \frac{d - w_i}{w_i}.$$

2.2. Yau algebra

Recall that a derivation of commutative associative algebra A is defined as a linear endomorphism D of A satisfying the Leibniz rule: $D(ab) = D(a)b + aD(b)$. Thus for such an algebra A one can consider the Lie algebra of its derivations $\mathrm{Der}(A, A)$ with the bracket defined by the commutator of linear endomorphisms.

Definition 2.2. Let $f(x_1, \cdots, x_n)$ be a complex polynomial and $V = \{f = 0\}$ be a germ of an isolated hypersurface singularity at the origin in $\mathbb{C}^n$. Let $A(V)$ be the moduli algebra and $L(V) := \mathrm{Der}_{\mathbb{C}}(A(V), A(V))$. Yu [Yu] call $L(V)$ the Yau algebra of V. Its dimension is called the Yau number by Elashvili and Khimshiashvili [EK] and will be denoted $\lambda(V)$.

2.3. Fewnomial singularities

Since we shall also deal with new Lie algebras that arising from fewnomial isolated singularities, so we recall the definition of fewnomial isolated singularities. The concept of fewnomial is introduced in [Kh].

Definition 2.3. We say that a polynomial $f \in \mathbb{C}[x_1, x_2, \cdots, x_n]$ is fewnomial if the number of monomials appearing in f does not exceed n.

Obviously, the number of monomials in f may depend on the system of coordinates. In order to obtain a rigorous concept we shall only allow linear changes of coordinates and say that f (or rather its germ at the origin) is a k-nomial if k is the smallest natural number such that f becomes a k-nomial after (possibly) a linear change of coordinates. An isolated hypersurface singularity V is called k-nomial if there exists an isolated hypersurface singularity Y analytically isomorphic to V which can be defined by a k-nomial and k is the smallest such number. It was shown that a singularity defined by a fewnomial f can be isolated only if f is a n-nomial in n variables when its multiplicity at least 3 [CYZ].

Definition 2.4. We say that an isolated hypersurface singularity V is fewnomial if it can be defined by a fewnomial polynomial f. V is called weighted homogenous fewnomial isolated singularity if it can be defined by a weighted homogenous fewnomial polynomial f. 2-nomial (resp. 3-nomial) isolated hypersurface singularity is also called binomial (resp. trinomial) singularity.

The following proposition and corollary tell us that each simple singularity belongs to one of the following three types of series.

Proposition 2.2. *[YZ2] Let f be a weighted homogeneous fewnomial isolated hypersurface singularity with multiplicity at least 3. Then f analytically equivalent to a linear combination of the following three series:*
Type A. $x_1^{a_1} + x_2^{a_2} + \cdots + x_{n-1}^{a_{n-1}} + x_n^{a_n}$, $n \geq 1$,
Type B. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}$, $n \geq 2$,
Type C. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1$, $n \geq 2$.

Corollary 2.1. *[YZ2] Each binomial isolated singularity is analytically equivalent to one from the three series: A) $x_1^{a_1} + x_2^{a_2}$, B) $x_1^{a_1}x_2 + x_2^{a_2}$, C) $x_1^{a_1}x_2 + x_2^{a_2}x_1$.*

In many situations it is necessary to have an explicit basis of $A(V)$. It is well known that there always exist bases consisting of monomials. Such bases are called monomial bases and will often be used in the sequel. Recall that the monomial bases in moduli algebras of simple singularities $(A_k, D_k, E_6, E_7, E_8)$ are given in [AVZ].

2.4. Deformation and cohomology of Lie algebras

The main tool in our proof of Theorem A is the theory of deformation and cohomology of Lie algebra [NR].

Definition 2.5. A Lie algebra is a vector space L over some field k (in this paper $k = \mathbb{C}$) together with a

$$[\cdot,\ \cdot] : L \times L \to L$$

called the Lie bracket that satisfies the following axioms:

(1) Bilinear operator

$$[ax + by, z] = a[x, z] + b[y, z], \quad [z, ax + by] = a[z, x] + b[z, y]$$

for all scalars a, b in k and all elements x, y, z in L.

(2) Alternativity,

$$[x, x] = 0$$

for all x in L.

(3) The Jacobi identity,

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$$

for all x, y, z in L.

If $x \in L$, $y \to [x, y]$ is an endomorphism of L, which we denote by ad x.

Let L be a Lie algebra. For two subspaces A, B of L the symbol $[A, B]$ denotes the linear span of the set of all $[x, y]$ with x in A and y in B. A sub Lie algebra of L is a subspace, say J, of L that is closed under the bracket operation (i.e., $[J, J] \subset L$); J becomes then a Lie algebra with the linear and bracket operations inherited from L. A sub Lie algebra J is called an ideal of L if $[L, J] \subset J$ (if $x \in L$ and $y \in J$ implies $[x, y] \in J$). The centralizer C_S of a subset S of L is the set of those x in L that commute with all y in S (i.e., $[x, y] = 0$). We say that two Lie algebras L, L' over k are isomorphic if there exists a vector space isomorphism $\phi : L \to L'$ satisfying $\phi([x, y]) = [\phi(x), \phi(y)]$. We will basically deal with solvable and nilpotent Lie algebras so for completeness we recall the corresponding definitions.

Definition 2.6. Given a Lie algebra L, introduce two series of ideals: $L_{(*)} = \{L_{(i)}\}, L^{(*)} = \{L^{(i)}\}, L_{(0)} = L^{(0)} = L, L_{(1)} = L^{(1)} = [L, L], L_{(i)} = [L, L_{(i-1)}]$, $L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$, $i = 2, 3, \cdots, L$ is called nilpotent if the lower centeral series $L_{(*)}$ terminates. L is called solvable if the derived series $L^{(*)}$ terminates.

Definition 2.7. Let V be a finite dimensional vector space over some field k. Call $x \in \text{End}\, V$ semi-simple if the roots of its minimal polynomial over k are all distinct. Equivalently (k being algebraically closed), x is semi-simple if and only if x is diagonalizable.

Definition 2.8. A Cartan subalgebra C in Lie algebra L is a nilpotent subalgebra that is self-normalising (i.e., if $[x, y] \in C$ for all $x \in C$, then $y \in$

C). Equivalently, Cartan subalgebra is a maximal commutative subalgebra C such that, for each $h \in C$, ad h is semi-simple.

Cartan subalgebras exist for finite-dimensional Lie algebras L whenever the base field k is infinite. If the field k is algebraically closed of characteristic 0 and the algebra is finite-dimensional then all Cartan subalgebras are conjugate under automorphisms of the Lie algebra, and in particular are all isomorphic. Hence of the same dimension r which is called the rank rk$L$ of Lie algebra L [Bo]. According to Engel's theorem, a Lie algebra L is nilpotent if and only if all operators $\mathrm{ad}\, a : L \to L$ are nilpotent for $a \in L$ [Bo]. Another general result states that a solvable algebraic Lie algebra can be decomposed into a semidirect sum of a Cartan subalgebra and maximal nilpotent ideal $N(L)$ (the latter is called the nilpotent radical of L).

Let $L = (V, \eta)$ be a finite dimension Lie algebra where η is a Lie algebra multiplication and V is the based vector space. Define $C^n(L, L)$ to be the vector space of all alternating n-linear maps of V into itself. The coboundary operator $\delta : C^n(L, L) \to C^{n+1}(L, L)$ is given as follows: for any $\alpha \in C^n(L, L)$,

$$\begin{aligned}\delta\alpha(x_0, \cdots, x_n) = &\sum_{i=0}^{n}(-1)^i \eta(x_i, \alpha(x_0, \cdots, \hat{x_i}, \cdots, x_n)) \\ &+ \sum_{i<j}(-1)^{i+j}\alpha(\eta(x_i, x_j), x_0, \cdots, \hat{x_i}, \cdots, \hat{x_j}, \cdots, x_n),\end{aligned}$$

here $\hat{x_i}$ means that the variable should be omitted. Recall that $Z^n(L, L) = \mathrm{Kernel}(\delta : C^n(L, L) \to C^{n+1}(L, L))$, $B^n(L, L) = \mathrm{Image}(\delta : C^{n-1}(L, L) \to C^n(L, L))$, $B^n(L, L) \subseteq Z^n(L, L)$ and $H^n(L, L) = Z^n(L, L)/B^n(L, L)$.

Let $L = (V, \eta)$ be a Lie algebra and $\varphi \in C^2(L, L)$ be a alternating bilinear map of V into itself. Then $\eta' = \eta + \varphi$ is also a Lie algebra multiplication if and only if it satisfies the Jacobian identity

$$\eta'(x, \eta'(y, z)) + \eta'(y, \eta'(z, x)) + \eta'(z, \eta'(x, y)) = 0 \tag{1}$$

for any $x, y, z \in V$. It can be shown that (1) holds if and only if

$$\delta\varphi - [\varphi, \varphi]/2 = 0 \tag{2}$$

where $[\varphi, \varphi]$ is defined as follow:

$$[\varphi, \varphi](x, y, z) = 2\varphi(\varphi(x, y), z) + 2\varphi(\varphi(y, z), x) + 2\varphi(\varphi(z, x), y).$$

(2) is called the deformation equation. Let $\eta_t = \eta + t\varphi_1 + t^2\varphi_2 + \cdots$ be an one-parameter family of Lie algebra multiplications on V, where $\varphi_i \in C^2(L, L)$. Then $t\varphi_1 + t^2\varphi_2 + \cdots$ satisfies the deformation equation, which implies that $\delta\varphi_1 = 0$. Hence $\varphi_1 \in Z^2(L, L)$, and we call φ_1 an infinitesimal deformation of η.

A one parameter family of Lie algebra multiplications $\eta_t = \eta + t\varphi_1 + t^2\varphi_2 + \cdots$ is said to be trivial if $(V, \eta_t) \simeq (V, \eta_s)$ for any s, t. Then there exists an one-parameter family of invertible linear maps $I_t = I + t\alpha_1 + t^2\alpha_2 + \cdots$, where $\alpha_i \in C^1(L, L)$ is linear from V into itself and I is the identity map, such that

$$\eta_t(x, y) = I_t\eta((I_t)^{-1}x, (I_t)^{-1}y) \tag{3}$$

for any $x, y \in V$. It's easy to verify that (3) implies that $\varphi_1 = -\delta\alpha_1$. Hence $\varphi_1 \in B^2(L, L)$, and we call φ_1 a trivial infinitesimal deformation.

3. Proof of Theorem A

3.1. Simple elliptic singularity $\widetilde{E_7}$

The simple elliptic singularity $\widetilde{E_7}$ is defined by $\{(x, y, z) \in \mathbb{C}^3 \mid x^4 + y^4 + z^2 = 0\}$. In [SY], the authors showed that its (μ, τ)-constant family is given by

$$V_t = \{(x, y, z) \in \mathbb{C}^3 \mid f_t(x, y, z) = x^4 + y^4 + tx^2y^2 + z^2 = 0\}$$

with $t^2 \neq 4$, and they constructed a family of Lie algebra by associating to the singularities $(V_t, 0)$ a Lie algebra which is defined to be the algebra of derivations of the moduli algebra $A(V_t)$ (i.e. the Yau algebra of V_t). For $k = 1$, the moduli algebra of V_t is given by

$$\begin{aligned} A^1(V_t) = \mathbb{C}\{x, y, z\}/(f_t, x\frac{\partial f_t}{\partial x}, y\frac{\partial f_t}{\partial x}, z\frac{\partial f_t}{\partial x}, x\frac{\partial f_t}{\partial y}, y\frac{\partial f_t}{\partial y}, \\ z\frac{\partial f_t}{\partial y}, x\frac{\partial f_t}{\partial z}, y\frac{\partial f_t}{\partial z}, z\frac{\partial f_t}{\partial z}) \\ =< 1, x, y, z, xy, x^2, y^2, x^3, x^2y, xy^2, y^3, y^4 >, \end{aligned}$$

with multiplication rules

$$x^4 = y^5 = z^2 = x^2y^2 = xz = yz = 0.$$

By calculation, a basis for Lie algebra $L_t^1 := \mathrm{Der}(A^1(V_t), A^1(V_t))$ is the following (for $t^2 \neq 0, 4, 36$):

$$e_1 = \frac{x\partial_x}{4} + \frac{y\partial_y}{4}, \quad e_2 = \frac{-x^2t\partial_y}{6} - \frac{xyt^2\partial_x}{24}, \quad e_3 = \frac{2x^2t\partial_x}{-4+t^2} + \frac{y^2t^2\partial_x}{-4+t^2},$$
$$e_4 = \frac{-xyt\partial_x}{4}, \quad e_5 = \frac{2y^2t\partial_x}{-12+3t^2} + \frac{x^2t^2\partial_x}{-12+3t^2},$$
$$e_6 = \frac{-x^2yt\partial_y}{4}, \quad e_7 = \frac{y^2\partial_y}{2} + \frac{x^2t\partial_y}{4} + \frac{xy(-4+t^2)\partial_x}{16},$$
$$e_8 = \frac{xy\partial_x}{2} + \frac{4x^2\partial_x}{-4+t^2} + \frac{2y^2t\partial_x}{-4+t^2}, \quad e_9 = \frac{x^3\partial_x}{2}, \quad e_{10} = \frac{-x^2yt\partial_x}{2},$$
$$e_{11} = y^4\partial_z, \quad e_{12} = y^3\partial_z + \frac{x^2yt\partial_z}{2}, \quad e_{13} = xy^2\partial_z + \frac{2x^3\partial_z}{t}, \quad e_{14} = z\partial_z,$$
$$e_{15} = y^4\partial_y, \quad e_{16} = y^3\partial_y + \frac{x^2yt\partial_y}{2}, \quad e_{17} = -x^2y\partial_x + xy^2\partial_y,$$
$$e_{18} = x^3\partial_y + \frac{x^2yt\partial_x}{2}, \quad e_{19} = z\partial_y, \quad e_{20} = y^4\partial_x,$$
$$e_{21} = y^3\partial_x + \frac{x^2yt\partial_x}{2}, \quad e_{22} = xy^2\partial_x + \frac{2x^3\partial_x}{t}, \quad e_{23} = z\partial_x.$$

Now we need to prove L_t^1 ($t^2 \neq 4$) is not a trivial family. By calculation, the multiplication table of L_t^1 is given as:

$$[e_1, e_2] = -\frac{e_2}{4} \quad [e_1, e_3] = -\frac{e_3}{4}, \quad [e_1, e_4] = -\frac{e_4}{4}, \quad [e_1, e_5] = -\frac{e_5}{4},$$
$$[e_1, e_{10}] = -\frac{e_{10}}{2}, \quad [e_1, e_{11}] = -e_{11}, \quad [e_1, e_7] = -\frac{e_7}{4}, \quad [e_1, e_8] = -\frac{e_8}{4},$$
$$[e_1, e_9] = -\frac{e_9}{2}, \quad [e_1, e_6] = -\frac{e_6}{2}, \quad [e_1, e_{12}] = -\frac{3e_{12}}{4}, \quad [e_1, e_{13}] = -\frac{3e_{13}}{4},$$
$$[e_1, e_{15}] = -\frac{3e_{15}}{4}, \quad [e_1, e_{16}] = -\frac{e_{16}}{2}, \quad [e_1, e_{17}] = -\frac{e_{17}}{2}, \quad [e_1, e_{18}] = -\frac{e_{18}}{2},$$
$$[e_1, e_{19}] = \frac{e_{19}}{4}, \quad [e_1, e_{20}] = -\frac{3e_{20}}{4}, \quad [e_1, e_{21}] = -\frac{e_{21}}{2}, \quad [e_2, e_{16}] = -e_{15},$$
$$[e_1, e_{22}] = -\frac{e_{22}}{2}, \quad [e_2, e_4] = -\frac{e_6t}{3} - \frac{t^2e_9}{12},$$
$$[e_2, e_3] = -\frac{2t^2e_{18}}{3(-4+t^2)} - \frac{e_{17}t^3}{3(-4+t^2)} - \frac{t^4e_{21}}{24(-4+t^2)} + \frac{e_{10}(-20t^2 - t^4)}{24(-4+t^2)},$$
$$[e_2, e_5] = -\frac{2t^2e_{17}}{9(-4+t^2)} - \frac{t^3e_{10}}{6(-4+t^2)} - \frac{t^3e_{18}}{9(-4+t^2)} - \frac{t^3e_{21}}{36(-4+t^2)},$$
$$[e_2, e_7] = -te_6 - \frac{t^2e_{22}}{48},$$

$$[e_2, e_8] = -\frac{2te_{10}}{-4+t^2} - \frac{t^3e_{21}}{12(-4+t^2)} + \frac{e_{18}(-20t+t^3)}{12(-4+t^2)} + \frac{e_{17}(-36t^2+t^4)}{48(-4+t^2)},$$

$$[e_2, e_9] = -\frac{te_{15}}{6}, \quad [e_2, e_{10}] = -\frac{t^2e_{20}}{24}, \quad [e_2, e_{12}] = -e_{11},$$

$$[e_2, e_{21}] = -e_{20}, \quad [e_3, e_6] = \frac{t^2e_{20}}{-4+t^2}, \quad [e_4, e_6] = \frac{te_{15}}{4},$$

$$[e_3, e_4] = \frac{t^3e_{21}}{4(-4+t^2)} + \frac{e_{10}(4t+t^3)}{4(-4+t^2)}, \quad [e_3, e_5] = \frac{8te_9}{3(-4+t^2)} - \frac{2t^2e_{22}}{3(-4+t^2)},$$

$$[e_3, e_7] = \frac{e_{10}(-4-t^2)}{16} - \frac{t^2e_{18}}{-4+t^2} - \frac{t^3e_{17}}{2(-4+t^2)} + \frac{e_{21}(20t^2-t^4)}{16(-4+t^2)},$$

$$[e_3, e_8] = -e_6 - \frac{4te_9}{-4+t^2} - \frac{t^2e_{16}}{2(-4+t^2)} + \frac{t^2e_{22}}{-4+t^2},$$

$$[e_3, e_9] = \frac{2te_{20}}{-4+t^2}, \quad [e_3, e_{13}] = -e_{11}, \quad [e_3, e_{17}] = -e_{15}, \quad [e_3, e_{22}] = -e_{20},$$

$$[e_4, e_5] = -\frac{t^2e_{10}}{3(-4+t^2)} - \frac{t^2e_{21}}{6(-4+t^2)}, \quad [e_4, e_7] = -\frac{te_{22}}{8} - \frac{te_6}{2} + \frac{e_9(4-t^2)}{8},$$

$$[e_4, e_8] = \frac{te_{17}}{8} - \frac{t^2e_{21}}{2(-4+t^2)} + \frac{e_{10}(-4-t^2)}{2(-4+t^2)}, \quad [e_4, e_{10}] = \frac{te_{20}}{4},$$

$$[e_4, e_{12}] = -\frac{te_{11}}{2}, \quad [e_4, e_{16}] = -\frac{te_{15}}{2}, \quad [e_4, e_{17}] = e_{20}, \quad [e_4, e_{21}] = -\frac{te_{20}}{2},$$

$$[e_5, e_7] = -\frac{te_{10}}{4} - \frac{t^2e_{17}}{3(-4+t^2)} - \frac{t^3e_{18}}{6(-4+t^2)} + \frac{e_{21}(20t-t^3)}{24(-4+t^2)},$$

$$[e_5, e_8] = -\frac{8e_9}{-4+t^2} - \frac{te_{16}}{3(-4+t^2)} + \frac{2te_{22}}{-4+t^2}, \quad [e_5, e_9] = \frac{e_{20}(12-t^2)}{6(-4+t^2)},$$

$$[e_5, e_{13}] = -\frac{2e_{11}}{t}, \quad [e_5, e_{18}] = -e_{15}, \quad [e_5, e_{22}] = -\frac{2e_{20}}{t}, \quad [e_6, e_7] = -\frac{e_{15}}{2},$$

$$[e_7, e_8] = \frac{e_{18}(20t-t^3)}{8(-4+t^2)} + \frac{e_{21}(-20t+t^3)}{8(-4+t^2)} + \frac{e_{17}(-48+48t^2-t^4)}{32(-4+t^2)},$$

$$[e_7, e_9] = \frac{te_{15}}{4}, \quad [e_7, e_{10}] = \frac{e_{20}(-4+t^2)}{16}, \quad [e_8, e_9] = \frac{4e_{20}}{-4+t^2},$$

$$[e_8, e_{10}] = \frac{e_{15}}{2}, \quad [e_{11}, e_{14}] = -e_{11}, \quad [e_{11}, e_{19}] = -e_{15}, \quad [e_{11}, e_{23}] = -e_{20},$$

$$[e_{12}, e_{14}] = -e_{12}, \quad [e_{12}, \quad e_{23}] = -e_{21}, \quad [e_{13}, e_{14}] = -e_{13},$$

$$[e_{13}, e_{19}] = -e_{17} - \frac{2e_{18}}{t}, \quad [e_{13}, e_{23}] = -e_{22}, \quad [e_5, e_6] = \frac{2te_{20}}{3(-4+t^2)},$$

$$[e_4, e_{18}] = -\frac{te_{20}}{2}, \quad [e_6, e_8] = -\frac{2te_{20}}{-4+t^2}, \quad [e_{12}, e_{19}] = -e_{16},$$

$$[e_{14}, e_{19}] = -e_{19}, \quad [e_{14}, e_{23}] = -e_{23}.$$

Other Lie brackets $[e_i, e_j]$ are 0. Let V be the based vector space of L_t^1 and η_t be the Lie algebra multiplication of L_t^1. Then we can write

$$\eta_t = \eta_0 + t\varphi_1,$$

where η_0 is matrix which have following 23 rows and columns:

$\{\{0, -\frac{e_2}{4}, -\frac{e_3}{4}, -\frac{e_4}{4}, -\frac{e_5}{4}, -\frac{e_6}{2}, -\frac{e_7}{4}, -\frac{e_8}{4}, -\frac{e_9}{2}, -\frac{e_{10}}{2}, -e_{11}, -\frac{3e_{12}}{4}, -\frac{3e_{12}}{4}, 0, -\frac{3e_{15}}{4},$

$-\frac{e_{16}}{2}, -\frac{e_{17}}{2} - \frac{e_{18}}{2}, \frac{e_{19}}{4}, -\frac{3e_{20}}{4}, -\frac{e_{21}}{2}, -\frac{e_{22}}{2}, \frac{e_{23}}{4}\},$

$\{\frac{e_2}{4}, 0, \frac{7e_{10}}{24} + \frac{e_2}{9} + \frac{2e_{18}}{9} + \frac{e_{21}}{72}, -\frac{e_6}{3} - \frac{e_9}{12}, \frac{e_{10}}{18} + \frac{2e_{17}}{27} + \frac{e_{18}}{27} + \frac{e_{21}}{108}, 0, -\frac{e_{22}}{48} - e_6, \frac{2e_{10}}{3},$

$+\frac{35e_{17}}{144} + \frac{19e_{18}}{36} + \frac{e_{21}}{36}, -\frac{e_{15}}{6}, -\frac{e_{20}}{24}, 0, -e_{11}, 0, 0, 0, -e_{15}, 0, 0, 0, 0, -e_{20}, 0, 0\},$

$\{\frac{e_3}{4}, -\frac{7e_{10}}{24} - \frac{e_{17}}{9} - \frac{2e_{18}}{9} - \frac{e_{21}}{72}, 0, -\frac{5e_{10}}{12} - \frac{e_{21}}{12}, \frac{2e_{22}}{9} - \frac{8e_9}{9}, -\frac{e_{20}}{3}, -\frac{5e_{10}}{16} + \frac{e_{17}}{6} + \frac{e_{18}}{3},$

$-\frac{19e_{21}}{48} + \frac{e_{16}}{6} - \frac{e_{22}}{3} - e_6 + \frac{4e_9}{3}, -\frac{2e_{20}}{3}, 0, 0, 0, -e_{11}, 0, 0, 0, -e_{15}, 0, 0, 0, 0, -e_{20}, 0\},$

$\{\frac{e_4}{4}, \frac{e_6}{3} + \frac{e_9}{12}, \frac{5e_{10}}{12} + \frac{e_{21}}{12}, 0, \frac{e_{10}}{9} + \frac{e_{21}}{18}, \frac{e_{15}}{4}, -\frac{e_{22}}{8} - \frac{e_6}{2} + \frac{3e_9}{8}, \frac{5e_{10}}{6} + \frac{e_{17}}{8} + \frac{e_{21}}{6}, 0,$

$\frac{e_{20}}{4}, 0, -\frac{e_{11}}{2}, 0, 0, 0, -\frac{e_{15}}{2}, e_{20}, -\frac{e_{20}}{2}, 0, 0, -\frac{e_{20}}{2}, 0, 0\},$

$\{\frac{e_5}{4}, -\frac{e_{10}}{18} - \frac{2e_{17}}{27} - \frac{e_{18}}{27} - \frac{e_{21}}{108}, -\frac{2e_{22}}{9} + \frac{8e_9}{9}, -\frac{e_{10}}{9} - \frac{e_{21}}{18}, 0, -\frac{2e_{20}}{9}, -\frac{e_{10}}{4} + \frac{e_{17}}{9} + \frac{e_{18}}{18}$

$-\frac{19e_{21}}{72}, +\frac{e_{16}}{9} - \frac{2e_{22}}{3} + \frac{8e_9}{3}, -\frac{11e_{20}}{18}, 0, 0, 0, -2e_{11}, 0, 0, 0, 0, -e_{15}, 0, 0, 0, -2e_{20}, 0\},$

$\{\frac{e_6}{2}, 0, \frac{e_{20}}{3}, -\frac{e_{15}}{4}, \frac{2e_{20}}{9}, 0, -\frac{e_{15}}{2}, \frac{2e_{20}}{3}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\},$

$\{\frac{e_7}{4}, \frac{e_{22}}{48} + e_6, \frac{5e_{10}}{16} - \frac{e_{17}}{6} - \frac{e_{18}}{3} + \frac{19e_{21}}{48}, \frac{e_{22}}{8} + \frac{e_6}{2} - \frac{3e_9}{8}, \frac{e_{10}}{4} - \frac{e_{17}}{9} - \frac{e_{18}}{18} + \frac{19e_{21}}{72}, \frac{e_{15}}{2}, 0,$

$\frac{e_{17}}{96} - \frac{19e_{18}}{24} + \frac{19e_{21}}{24}, \frac{e_{15}}{4}, -\frac{3e_{20}}{16}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\},$

$\{\frac{e_8}{4}, -\frac{2e_{10}}{3} - \frac{35e_{17}}{144} - \frac{19e_{18}}{36} - \frac{e_{21}}{36}, -\frac{e_{16}}{6} + \frac{e_{22}}{3} + e_6 - \frac{4e_9}{3}, -\frac{5e_{10}}{6} - \frac{e_{17}}{8} - \frac{e_{21}}{6},$

$-\frac{e_{16}}{9} + \frac{2e_{22}}{3} - \frac{8e_9}{3}, -\frac{2e_{20}}{3}, -\frac{e_{17}}{96} + \frac{19e_{18}}{24} - \frac{19e_{21}}{24}, 0, -\frac{4e_{20}}{3}, \frac{e_{15}}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0,$

$0, 0, 0, 0\}, \{\frac{e_9}{2}, \frac{e_{15}}{6}, \frac{2e_{20}}{3}, 0, \frac{11e_{20}}{18}, 0, -\frac{e_{15}}{4}, \frac{4e_{20}}{3}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$

$\{\frac{e_{10}}{2}, \frac{e_{20}}{24}, 0, -\frac{e_{20}}{4}, 0, 0, \frac{3e_{20}}{16}, -\frac{e_{15}}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\},$

$$\{e_{11},0,0,0,0,0,0,0,0,0,0,0,0,-e_{11},0,0,0,0,-e_{15},0,0,0,-e_{20}\},$$

$$\left\{\tfrac{3e_{12}}{4},e_{11},0,\tfrac{e_{11}}{2},0,0,0,0,0,0,0,0,0,-e_{12},0,0,0,0,-e_{16},0,0,0,-e_{21}\right\},$$

$$\left\{\tfrac{3e_{13}}{4},0,e_{11},0,2e_{11},0,0,0,0,0,0,0,0,-e_{13},0,0,0,0,-e_{17}-2e_{18},0,0,0,-e_{22}\right\},$$

$$\{0,0,0,0,0,0,0,0,0,0,e_{11},e_{12},e_{13},0,0,0,0,0,-e_{19},0,0,0,-e_{23}\},$$

$$\left\{\tfrac{3e_{15}}{4},0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0\right\},$$

$$\left\{\tfrac{e_{16}}{2},e_{15},0,\tfrac{e_{15}}{2},0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0\right\},$$

$$\left\{\tfrac{e_{17}}{2},0,e_{15},-e_{20},0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0\right\},$$

$$\left\{\tfrac{e_{18}}{2},0,0,\tfrac{e_{20}}{2},e_{15},0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0\right\},$$

$$\left\{-\tfrac{e_{19}}{4},0,0,0,0,0,0,0,0,0,e_{15},e_{16},e_{17}+2e_{18},e_{19},0,0,0,0,0,0,0,0,0\right\},$$

$$\left\{\tfrac{3e_{20}}{4},0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0\right\},$$

$$\left\{\tfrac{e_{21}}{2},e_{20},0,\tfrac{e_{20}}{2},0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0\right\},$$

$$\left\{\tfrac{e_{22}}{2},0,e_{20},0,2e_{20},0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0\right\},$$

$$\left\{-\tfrac{e_{23}}{4},0,0,0,0,0,0,0,0,0,e_{20},e_{21},e_{22},e_{23},0,0,0,0,0,0,0,0,0\right\}\}.$$

And $\varphi_1 : W \times W \to W$ is given by

$$\{\{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0\},$$

$$\left\{0,0,\tfrac{29e_{10}}{36}+\tfrac{11e_{17}}{27}+\tfrac{16e_{18}}{27}+\tfrac{7e_{21}}{108},-\tfrac{e_6}{3}-\tfrac{e_9}{6},\tfrac{11e_{10}}{54}+\tfrac{16e_{17}}{81}+\tfrac{11e_{18}}{81}+\tfrac{11e_{21}}{324},0,\right.$$

$$\left.-\tfrac{e_{22}}{24},\tfrac{10e_{10}}{9}+\tfrac{137e_{17}}{216}+\tfrac{89e_{18}}{108}+\tfrac{11e_{21}}{108},-\tfrac{e_{15}}{6},-\tfrac{e_{20}}{12},0,0,0,0,0,0,0,0,0,0,0,0,0\right\},$$

$$\left\{0,-\tfrac{29e_{10}}{36}-\tfrac{11e_{17}}{27}-\tfrac{16e_{18}}{27}-\tfrac{7e_{21}}{108},0,-\tfrac{31e_{10}}{36}-\tfrac{11e_{21}}{36},\tfrac{16e_{22}}{27}-\tfrac{40e_9}{27},-\tfrac{8e_{20}}{9},\right.$$

$$\tfrac{e_{10}}{8}+\tfrac{11e_{17}}{18}+\tfrac{8e_{18}}{9}-\tfrac{73e_{21}}{72},\tfrac{4e_{16}}{9}-\tfrac{8e_{22}}{9}+\tfrac{20e_9}{9},-\tfrac{10e_{20}}{9},0,0,0,0,0,0,0,0,0,0,0,$$

$$\left.0,0,0\right\},\left\{0,\tfrac{e_6}{3}+\tfrac{e_9}{6},\tfrac{31e_{10}}{36}+\tfrac{11e_{21}}{36},0,\tfrac{8e_{10}}{27}+\tfrac{4e_{21}}{27},\tfrac{e_{15}}{4},-\tfrac{e_{22}}{8}-\tfrac{e_6}{2}-\tfrac{e_9}{4},\tfrac{8e_{10}}{9}+\tfrac{e_{17}}{8},\right.$$

$+\frac{4e_{21}}{9}, 0, \frac{e_{20}}{4}, 0, -\frac{e_{11}}{2}, , 0, 0, 0, -\frac{e_{15}}{2}, 0, -\frac{e_{20}}{2}, 0, 0, -\frac{e_{20}}{2}, 0, 0\Big\}$,

$\Big\{0, -\frac{11e_{10}}{54} - \frac{16e_{17}}{81} - \frac{11e_{18}}{81} - \frac{11e_{21}}{324}, -\frac{16e_{22}}{27} + \frac{40e_9}{27}, -\frac{8e_{10}}{27} - \frac{4e_{21}}{27}, 0, -\frac{10e_{20}}{27},$

$-\frac{e_{10}}{4} + \frac{8e_{17}}{27} + \frac{11e_{18}}{54} - \frac{89e_{21}}{216}, \frac{5e_{16}}{27} - \frac{10e_{22}}{9} + \frac{16e_9}{9}, -\frac{8e_{20}}{27},$

$0, 0, 0, 2e_{11}, 0, 0, 0, 0, 2e_{20}, 0\Big\}$,

$\Big\{0, 0, \frac{8e_{20}}{9}, -\frac{e_{15}}{4}, \frac{10e_{20}}{27}, 0, 0, \frac{10e_{20}}{9}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\Big\}$,

$\Big\{0, \frac{e_{22}}{24}, \frac{e_{10}}{8} - \frac{11e_{17}}{18} - \frac{8e_{18}}{9} + \frac{73e_{21}}{72}, \frac{e_{22}}{8} + \frac{e_6}{2} + \frac{e_9}{4}, \frac{e_{10}}{4} - \frac{8e_{17}}{27} - \frac{11e_{18}}{54} + \frac{89e_{21}}{216},$

$0, 0, -\frac{137e_{17}}{144} - \frac{89e_{18}}{72} + \frac{89e_{21}}{72}, \frac{e_{15}}{4}, \frac{e_{20}}{8}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\Big\}$,

$\Big\{0, -\frac{10e_{10}}{9} - \frac{137e_{17}}{216} - \frac{89e_{18}}{108} - \frac{11e_{21}}{108}, -\frac{4e_{16}}{9} + \frac{8e_{22}}{9} - \frac{20e_9}{9}, -\frac{8e_{10}}{9} - \frac{e_{17}}{8} - \frac{4e_{21}}{9},$

$-\frac{5e_{16}}{27}, -\frac{16e_9}{9}, -\frac{10e_{20}}{9}, \frac{137e_{17}}{144} + \frac{89e_{18}}{72} - \frac{89e_{21}}{72}, 0, -\frac{8e_{20}}{9},$

$0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\Big\}$,

$\Big\{0, \frac{e_{15}}{6}, \frac{10e_{20}}{9}, 0, \frac{8e_{20}}{27}, 0, -\frac{e_{15}}{4}, \frac{8e_{20}}{9}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\Big\}$,

$\Big\{0, \frac{e_{20}}{12}, 0, -\frac{e_{20}}{4}, 0, 0, -\frac{e_{20}}{8}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\Big\}$,

$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$,

$\Big\{0, 0, 0, \frac{e_{11}}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\Big\}$,

$\{0, 0, 0, 0, -2e_{11}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2e_{18}, 0, 0, 0, 0\}$,

$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$,

$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$,

$\Big\{0, 0, 0, \frac{e_{15}}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\Big\}$,

$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$,

$\Big\{0, 0, 0, \frac{e_{20}}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\Big\}$,

$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -2e_{18}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$,

$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$,

$\left\{0, 0, 0, \frac{e_{20}}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right\}$,

$\{0, 0, 0, 0, -2e_{20}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$,

$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}\}$.

Now we only need to check that φ_1 is not a trivial infinitesimal deformation, (i.e. $\varphi_1 \notin B^2(L, L)$ where $L = (V, \eta_0)$), then we can say L_t^1 is not a trivial family. Write $\eta_0(e_i, e_j) = \sum_{s=1}^{23} u_{ij}^s e_s$ and $\varphi_1(e_i, e_j) = \sum_{s=1}^{23} v_{ij}^s e_s$ for $i, j = 1, \ldots, 23$. If there exists a linear map $\alpha : V \to V$ such that $\delta\alpha = \varphi_1$, write $\alpha(e_i) = \sum_{j=1}^{23} a_{ij} e_j$, then we have

$$\begin{aligned}
\varphi_1(e_i, e_j) &= \delta\alpha(e_i, e_j) \\
&= \eta_0(e_i, \alpha(e_j)) - \eta_0(e_j, \alpha(e_i)) - \alpha(\eta_0(e_i, e_j)) \\
&= \eta_0(e_i, \sum_{k=1}^{23} a_{jk} e_k) - \eta_0(e_j, \sum_{k=1}^{23} a_{ik} e_k) - \alpha(\sum_{k=1}^{23} u_{ij}^k e_k) \\
&= \sum_{s=1}^{23} \sum_{k=1}^{23} (a_{jk} u_{ik}^s - a_{ik} u_{jk}^s - u_{ij}^k a_{ks}) e_s.
\end{aligned}$$

Hence $\sum_{k=1}^{n} (a_{jk} u_{ik}^s - a_{ik} u_{jk}^s - u_{ij}^k a_{ks}) = v_{ij}^s$ for $i, j, s = 1, \ldots, 23$. Then we get 12167 linear equations about 529 variables a_{ij}. We solve this system of linear equations with the help of computer and find that they have no solution. Hence $\varphi_1 \notin B^2(L, L)$ and the family is not trivial. In case of $k = 2$, the moduli algebra is defined by

$$A^2(V_t) = < 1, x, y, z, x^2, x^3, x^4, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, y^3, xy^3, y^4, xz, yz > .$$

By calculation, a basis for Lie algebra $L_t^2 := \mathrm{Der}(A^2(V_t), A^2(V_t))$ is the following (for $t^2 \neq 4$):

$$e_1 = \frac{x\partial_x}{3} + \frac{y\partial_y}{3} + \frac{2z\partial_z}{3}, \quad e_2 = y^3\partial_z, \quad e_3 = xy^2\partial_z, \quad e_4 = x^2y\partial_z,$$

$$e_5 = x^3\partial_z, \quad e_6 = -z\partial_y - y^3\partial_z - x^2yt\partial_z, \quad e_7 = -z\partial_x,$$
$$e_8 = -\frac{2xy\partial_x}{3} + \frac{y^2\partial_y}{3}, \quad e_9 = -\frac{2x^2\partial_x}{3} + \frac{xy\partial_y}{3},$$
$$e_{10} = \frac{x^2\partial_y}{3}, \quad e_{11} = y^2\partial_x, \quad e_{12} = xy\partial_x, \quad e_{13} = x^2\partial_x,$$
$$e_{14} = -\frac{xy^2\partial_x}{2} + \frac{y^3\partial_y}{2}, \quad e_{15} = -\frac{x^2y\partial_x}{2} + \frac{xy^2\partial_y}{2},$$
$$e_{16} = -\frac{x^3\partial_x}{2} + \frac{x^2y\partial_y}{2}, \quad e_{17} = \frac{x^3\partial_y}{2}, \quad e_{18} = y^3\partial_x, \quad e_{19} = xy^2\partial_x,$$
$$e_{20} = x^2y\partial_x, \quad e_{21} = x^3\partial_x, \quad e_{22} = yz\partial_z, \quad e_{23} = xz\partial_z, \quad e_{24} = y^4\partial_z,$$
$$e_{25} = xy^3\partial_z, \quad e_{26} = x^2y^2\partial_z, \quad e_{27} = x^3y\partial_z,$$
$$e_{28} = x^4\partial_z, \quad e_{29} = yz\partial_y, \quad e_{30} = xz\partial_y, \quad e_{31} = y^4\partial_y, \quad e_{32} = xy^3\partial_y,$$
$$e_{33} = x^2y^2\partial_y, \quad e_{34} = x^3y\partial_y, \quad e_{35} = x^4\partial_y, \quad e_{36} = yz\partial_x, \quad e_{37} = xz\partial_x,$$
$$e_{38} = y^4\partial_x, \quad e_{39} = xy^3\partial_x, \quad e_{40} = x^2y^2\partial_x, \quad e_{41} = x^3y\partial_x, \quad e_{42} = x^4\partial_x.$$

Next by using same steps as we used in case of $k = 1$, we get a system of linear equations which have no solution. So finally the family $L_t^2(\widetilde{E_7})$ is nontrivial.

3.2. Simple elliptic singularity $\widetilde{E_6}$

The simple elliptic singularity $\widetilde{E_6}$ is defined by $\{(x, y, z) \in \mathbb{C}^3 \mid x^3 + y^3 + z^3 = 0\}$. Its (μ, τ)-constant family is given by

$$V_t = \{(x, y, z) \in \mathbb{C}^3 \mid f_t(x, y, z) = x^3 + y^3 + z^3 + txyz = 0\}$$

with $t^3 + 27 \neq 0$ (cf. [Ya1]). In case of $k = 2$, the moduli algebra of V_t is given by

$$A^2(V_t) = <1, x, y, z, x^2, xy, x^2y, y^2, xy^2, y^3, xz, x^2z, yz, xyz, y^2z, z^2, xz^2, yz^2, z^3 > .$$

By calculation, a basis for the Lie algebra $L^2(V_t) = \mathrm{Der}(A^2(V_t), A^2(V_t))$, is the following (for $t^3 + 27 \neq 0$):

$$
\begin{aligned}
&e_1 = \frac{x\partial_x}{3} + \frac{y\partial_y}{3} + \frac{z\partial_z}{3}, \quad e_2 = -\frac{xz\partial_x}{2} - \frac{yz\partial_y}{2} + \frac{z^2\partial_z}{2},\\
&e_3 = -\frac{xy\partial_x}{2} - \frac{y^2\partial_y}{2} + \frac{yz\partial_z}{2}, \quad e_4 = -\frac{x^2\partial_x}{2} - \frac{xy\partial_y}{2} + \frac{xz\partial_z}{2}, \quad e_5 = \frac{y^2\partial_z}{2},\\
&e_6 = \frac{xy\partial_z}{2}, \quad e_7 = \frac{z^2\partial_x}{2} + \frac{x^2\partial_z}{2} + \frac{txy\partial_x}{2}, \quad e_8 = z^2\partial_y, \quad e_9 = yz\partial_y,\\
&e_{10} = xz\partial_y, \quad e_{11} = y^2\partial_y, \quad e_{12} = xy\partial_y, \quad e_{13} = x^2\partial_y, \quad e_{14} = z^2\partial_x\\
&e_{15} = yz\partial_x, \quad e_{16} = xz\partial_x, \quad e_{17} = y^2\partial_x, \quad e_{18} = xy\partial_x, \quad e_{19} = x^2\partial_x,\\
&e_{20} = z^3\partial_z, \quad e_{21} = yz^2\partial_z, \quad e_{22} = xz^2\partial_z, \quad e_{23} = y^2z\partial_z, \quad e_{24} = xyz\partial_z,\\
&e_{25} = x^2z\partial_z, \quad e_{26} = y^3\partial_z, \quad e_{27} = xy^2\partial_z, \quad e_{28} = x^2y\partial_z, \quad e_{29} = z^3\partial_y,\\
&e_{30} = yz^2\partial_y, \quad e_{31} = xz^2\partial_y, \quad e_{32} = y^2z\partial_y, \quad e_{33} = xyz\partial_y, \quad e_{34} = x^2z\partial_y,\\
&e_{35} = y^3\partial_y, \quad e_{36} = xy^2\partial_y, \quad e_{37} = x^2y\partial_y, \quad e_{38} = z^3\partial_x, \quad e_{39} = yz^2\partial_x,\\
&e_{40} = xz^2\partial_x, \quad e_{41} = y^2z\partial_x, \quad e_{42} = xyz\partial_x, \quad e_{43} = x^2z\partial_x, \quad e_{44} = y^3\partial_x,\\
&e_{45} = xy^2\partial_x, \quad e_{46} = x^2y\partial_x.
\end{aligned}
$$

We have following multiplication table:

$$
\begin{aligned}
&[e_1, e_i] = -\frac{e_i}{3} \quad, 2 \le i \le 19, \quad [e_1, e_j] = -\frac{2e_j}{3} \quad, 20 \le i \le 46,\\
&[e_2, e_3] = \frac{e_{21}}{2} - \frac{e_{32}}{2} - \frac{e_{42}}{2}, \quad [e_2, e_{13}] = \frac{e_{34}}{2},\\
&[e_2, e_4] = \frac{e_{22}}{2} - \frac{e_{33}}{2} - \frac{e_{43}}{2}, \quad [e_2, e_5] = e_{23} - \frac{e_{35}}{4} - \frac{e_{45}}{4},\\
&[e_2, e_6] = e_{24} - \frac{e_{36}}{4} - \frac{e_{46}}{4}, \quad [e_2, e_7] = e_{25} - \frac{e_{37}}{4} - \frac{e_{38}}{2} + \frac{e_{44}}{4} + \frac{te_{42}}{2},\\
&[e_2, e_9] = -\frac{e_{30}}{2}, \quad [e_2, e_{10}] = -\frac{e_{31}}{2}, \quad [e_2, e_{11}] = \frac{e_{32}}{2}, \quad [e_2, e_{12}] = \frac{e_{33}}{2},\\
&[e_2, e_{14}] = -\frac{3e_{38}}{2}, \quad [e_2, e_{15}] = -\frac{e_{39}}{2}, \quad [e_2, e_{16}] = -\frac{e_{40}}{2}, \quad [e_2, e_{17}] = \frac{e_{41}}{2},\\
&[e_2, e_{18}] = \frac{e_{42}}{2}, \quad [e_2, e_{19}] = \frac{e_{43}}{2}, \quad [e_3, e_5] = \frac{3e_{26}}{4}, \quad [e_3, e_6] = \frac{3e_{27}}{4},\\
&[e_3, e_7] = \frac{3e_{28}}{4} - \frac{3e_{39}}{4} + \frac{te_{45}}{4}, \quad [e_3, e_8] = \frac{e_{20}}{2} - 2e_{30} - \frac{e_{40}}{2},\\
&[e_3, e_{10}] = \frac{e_{22}}{2} - e_{33} - \frac{e_{43}}{2}, \quad [e_3, e_{11}] = \frac{e_{23}}{2} - \frac{e_{45}}{2}, \quad [e_3, e_{12}] = \frac{e_{24}}{2} - \frac{e_{46}}{2},
\end{aligned}
$$

$$[e_3,e_{13}]=\frac{e_{25}}{2}+\frac{e_{38}}{2}+\frac{e_{44}}{2}+\frac{te_{42}}{2},\quad [e_3,e_{14}]=-\frac{3e_{39}}{2},\quad [e_3,e_{15}]=-\frac{e_{41}}{2},$$
$$[e_3,e_9]=\frac{e_{21}}{2}-e_{32}-\frac{e_{42}}{2},\quad [e_2,e_8]=-\frac{3e_{29}}{2},\quad [e_4,e_5]=\frac{3e_{27}}{4},$$
$$[e_3,e_{16}]=-\frac{e_{42}}{2},\quad [e_3,e_{17}]=\frac{e_{44}}{2},\quad [e_3,e_{18}]=\frac{e_{45}}{2},\quad [e_3,e_{19}]=\frac{e_{46}}{2},$$
$$[e_4,e_6]=\frac{3e_{28}}{4},\quad [e_4,e_7]=-\frac{e_{20}}{2}-\frac{3e_{26}}{4}-\frac{e_{30}}{4}-e_{40}-\frac{te_{24}}{2}-\frac{te_{36}}{4},$$
$$[e_4,e_8]=-\frac{3e_{31}}{2},\quad [e_4,e_9]=-\frac{e_{33}}{2},\quad [e_4,e_{10}]=-\frac{e_{34}}{2},\quad [e_4,e_{11}]=\frac{e_{36}}{2},$$
$$[e_4,e_{12}]=\frac{e_{37}}{2},\quad [e_4,e_{13}]=-\frac{e_{29}}{2}-\frac{e_{35}}{2}-\frac{te_{33}}{2},$$
$$[e_4,e_{14}]=\frac{e_{20}}{2}-\frac{e_{30}}{2}-2e_{40},\quad [e_4,e_{15}]=\frac{e_{21}}{2}-\frac{e_{32}}{2}-e_{42},$$
$$[e_4,e_{16}]=\frac{e_{22}}{2}-\frac{e_{33}}{2}-e_{43},\quad [e_4,e_{17}]=\frac{e_{23}}{2}-\frac{e_{35}}{2},$$
$$[e_4,e_{18}]=\frac{e_{24}}{2}-\frac{e_{36}}{2},\quad [e_4,e_{19}]=\frac{e_{25}}{2}-\frac{e_{37}}{2},\quad [e_5,e_7]=-\frac{e_{41}}{2},$$
$$[e_5,e_8]=e_{21}-e_{32},\quad [e_5,e_9]=e_{23}-\frac{e_{35}}{2},\quad [e_5,e_{10}]=e_{24}-\frac{e_{36}}{2},$$
$$[e_5,e_{13}]=e_{28},\quad [e_5,e_{14}]=-e_{41},\quad [e_5,e_{15}]=-\frac{e_{44}}{2},\quad [e_5,e_{16}]=-\frac{e_{45}}{2},$$
$$[e_6,e_7]=\frac{e_{21}}{4}-\frac{e_{42}}{2}+\frac{te_{27}}{4},\quad [e_6,e_8]=\frac{e_{22}}{2}-e_{33},\quad [e_6,e_9]=\frac{e_{24}}{2}-\frac{e_{36}}{2},$$
$$[e_6,e_{10}]=\frac{e_{25}}{2}-\frac{e_{37}}{2},\quad [e_6,e_{11}]=\frac{e_{27}}{2},\quad [e_6,e_{12}]=\frac{e_{28}}{2},\quad [e_5,e_{12}]=e_{27},$$
$$[e_6,e_{13}]=-\frac{e_{20}}{2}-\frac{e_{26}}{2}-\frac{te_{24}}{2},\quad [e_6,e_{14}]=\frac{e_{21}}{2}-e_{42},\quad [e_6,e_{15}]=\frac{e_{23}}{2}-\frac{e_{45}}{2},$$
$$[e_6,e_{16}]=\frac{e_{24}}{2}-\frac{e_{46}}{2},\quad [e_6,e_{17}]=\frac{e_{26}}{2},\quad [e_6,e_{18}]=\frac{e_{27}}{2},\quad [e_6,e_{19}]=\frac{e_{28}}{2},$$
$$[e_7,e_8]=-e_{34}+\frac{te_{40}}{2},\quad [e_7,e_9]=-\frac{e_{37}}{2}+\frac{te_{42}}{2},\quad [e_7,e_{10}]=\frac{e_{35}}{2}+\frac{te_{43}}{2},$$
$$[e_7,e_{11}]=\frac{te_{45}}{2},\quad [e_7,e_{12}]=-\frac{e_{30}}{2}-\frac{te_{36}}{2}+\frac{te_{46}}{2},\quad [e_{13},e_{16}]=2e_{34},$$
$$[e_7,e_{13}]=-e_{31}-te_{37}-\frac{te_{38}}{2}-\frac{te_{44}}{2}-\frac{t^2e_{42}}{2},\quad [e_7,e_{14}]=e_{22}-e_{43}+\frac{te_{39}}{2},$$
$$[e_7,e_{15}]=e_{24}-\frac{e_{46}}{2}+\frac{te_{41}}{2},\quad [e_7,e_{16}]=e_{25}+\frac{e_{44}}{2}+\frac{te_{42}}{2},\quad [e_5,e_{11}]=e_{26},$$
$$[e_7,e_{17}]=e_{27}+\frac{te_{44}}{2},\quad [e_8,e_9]=-e_{29},\quad [e_8,e_{18}]=-e_{40},$$
$$[e_7,e_{18}]=e_{28}-\frac{e_{39}}{2},\quad [e_7,e_{19}]=-e_{20}-e_{26}-e_{40}-te_{24}-\frac{te_{46}}{2},$$
$$[e_8,e_{11}]=-2e_{30},\quad [e_8,e_{12}]=-e_{31},\quad [e_8,e_{15}]=-e_{38},\quad [e_8,e_{17}]=-2e_{39},$$
$$[e_9,e_{10}]=e_{31},\quad [e_9,e_{11}]=-e_{32},\quad [e_9,e_{13}]=e_{34},\quad [e_9,e_{15}]=-e_{39},$$
$$[e_9,e_{18}]=-e_{42},\quad [e_{10},e_{11}]=-2e_{33},\quad [e_{10},e_{12}]=-e_{34},\quad [e_{10},e_{14}]=e_{29},$$

$[e_{10}, e_{15}] = e_{30} - e_{40}$, $[e_{11}, e_{17}] = -2e_{44}$, $[e_9, e_{17}] = -2e_{41}$,
$[e_{10}, e_{16}] = e_{31}$, $[e_{10}, e_{17}] = e_{32} - 2e_{42}$, $[e_{10}, e_{18}] = e_{33} - e_{43}$,
$[e_{10}, e_{19}] = e_{34}$, $[e_{11}, e_{12}] = e_{36}$, $[e_{11}, e_{13}] = 2e_{37}$, $[e_{11}, e_{15}] = -e_{41}$,
$[e_{11}, e_{18}] = -e_{45}$, $[e_{12}, e_{13}] = -e_{29} - e_{35} - te_{33}$, $[e_{12}, e_{14}] = e_{30}$,
$[e_{12}, e_{15}] = e_{32} - e_{42}$, $[e_{12}, e_{16}] = e_{33}$, $[e_{12}, e_{17}] = e_{35} - 2e_{45}$,
$[e_{12}, e_{18}] = e_{36} - e_{46}$, $[e_{13}, e_{17}] = 2e_{36} - 2e_{46}$,
$[e_{12}, e_{19}] = e_{37}$, $[e_{13}, e_{14}] = 2e_{31}$, $[e_{13}, e_{15}] = 2e_{33} - e_{43}$,
$[e_{13}, e_{18}] = 2e_{37} + e_{38} + e_{44} + te_{42}$, $[e_{13}, e_{19}] = -2e_{29} - 2e_{35} - 2te_{33}$,
$[e_{14}, e_{16}] = -e_{38}$, $[e_{14}, e_{18}] = -e_{39}$, $[e_{14}, e_{19}] = -2e_{40}$, $[e_{15}, e_{16}] = -e_{39}$,
$[e_{15}, e_{18}] = -e_{41}$, $[e_{15}, e_{19}] = -2e_{42}$, $[e_{16}, e_{17}] = e_{41}$, $[e_{16}, e_{19}] = -e_{43}$,
$[e_{17}, e_{18}] = -e_{44}$, $[e_{17}, e_{19}] = -2e_{45}$, $[e_{18}, e_{19}] = -e_{46}$.

Other Lie brackets $[e_i, e_j]$ are 0. Let V be the based vector space of L_t^2 and η_t be the Lie algebra multiplication of L_t^2. Then we can write

$$\eta_t = \eta_0 + t\varphi_1$$

Now we only need to check that φ_1 is not a trivial infinitesimal deformation, (i.e. $\varphi_1 \notin B^2(L, L)$ where $L = (V, \eta_0)$), then we can say L_t^2 is not a trivial family. Write $\eta_0(e_i, e_j) = \sum_{s=1}^{46} u_{ij}^s e_s$ and $\varphi_1(e_i, e_j) = \sum_{s=1}^{46} v_{ij}^s e_s$ for $i, j = 1, \ldots, 46$. If there exists a linear map $\alpha : V \to V$ such that $\delta\alpha = \varphi_1$, write $\alpha(e_i) = \sum_{j=1}^{46} a_{ij} e_j$, then we have

$$\begin{aligned}
\varphi_1(e_i, e_j) &= \delta\alpha(e_i, e_j) \\
&= \eta_0(e_i, \alpha(e_j)) - \eta_0(e_j, \alpha(e_i)) - \alpha(\eta_0(e_i, e_j)) \\
&= \eta_0(e_i, \sum_{k=1}^{46} a_{jk} e_k) - \eta_0(e_j, \sum_{k=1}^{46} a_{ik} e_k) - \alpha(\sum_{k=1}^{46} u_{ij}^k e_k) \\
&= \sum_{s=1}^{46} \sum_{k=1}^{46} (a_{jk} u_{ik}^s - a_{ik} u_{jk}^s - u_{ij}^k a_{ks}) e_s.
\end{aligned}$$

Hence $\sum_{k=1}^{n} (a_{jk} u_{ik}^s - a_{ik} u_{jk}^s - u_{ij}^k a_{ks}) = v_{ij}^s$ for $i, j, s = 1, \ldots, 46$. Then we get 97336 linear equations about 2116 variables a_{ij}. We solve this system of linear equations with the help of computer and find that they have no solution. Hence $\varphi_1 \notin B^2(L, L)$ and the family is not trivial.

3.3. Simple elliptic singularity $\widetilde{E}_8$

The simple elliptic singularity $\widetilde{E}_8$ is defined by $\{(x,y,z)\in\mathbb{C}^3 \mid x^6+y^3+z^2=0\}$. In [SY], the authors had studied the $(\mu,\tau)-$ constant family of $\widetilde{E}_8$, which is given by

$$V_t=\{(x,y,z)\in\mathbb{C}^3 \mid f_t=x^6+y^3+z^2+tx^4y=0\},$$

with $4t^3+27\neq 0$. For $k=1$, the moduli algebra of V_t is given by

$$\begin{aligned}
&A^1(V_t)\\
&\quad=\mathbb{C}\{x,y,z\}/(f_t,x\frac{\partial f_t}{\partial x},y\frac{\partial f_t}{\partial x},z\frac{\partial f_t}{\partial x},x\frac{\partial f_t}{\partial y},y\frac{\partial f_t}{\partial y},z\frac{\partial f_t}{\partial y},x\frac{\partial f_t}{\partial z},y\frac{\partial f_t}{\partial z},z\frac{\partial f_t}{\partial z})\\
&\quad=<1,x,y,z,xy,x^2,y^2,x^3,x^2y,x^3y,x^4,x^5,x^6>,
\end{aligned}$$

with multiplication rules $x^7=y^4=z^2=x^3y^2=xz=yz=0$. By calculation, a basis for $L^1_t:=L^1(V_t)$ (for $4t^3+27\neq 0$ and $t\neq 0$), is the following:

$$\begin{aligned}
&e_1=\frac{x\partial_x}{6}+\frac{y\partial_y}{3},\quad e_2=-\frac{2yt\partial_x}{15}+\frac{4xyt^3\partial_y}{27+4t^3}-\frac{4x^3(-9t^2+2t^5)\partial_y}{15(27+4t^3)},\\
&e_3=-\frac{xyt\partial_x}{6}+\frac{2x^2yt^3\partial_y}{27+4t^3}+\frac{4y^2t^4\partial_y}{3(27+4t^3)},\\
&e_4=\frac{x^2\partial_x}{4}-\frac{3xy\partial_y}{4}+\frac{yt\partial_x}{6}+\frac{x^3t^2\partial_y}{6},\\
&e_5=-\frac{18xyt\partial_y}{27+4t^3}+\frac{4x^3t^3\partial_y}{27+4t^3},\quad e_6=-\frac{18x^2yt\partial_y}{27+4t^3}-\frac{12y^2t^2\partial_y}{27+4t^3},\\
&e_7=-\frac{2x^2yt\partial_x}{9}+\frac{4x^3yt^3\partial_y}{81},\quad e_8=\frac{x^3\partial_x}{3}-\frac{2x^2y\partial_y}{3}+\frac{2xyt\partial_x}{9}-\frac{4y^2t\partial_y}{9},\\
&e_9=\frac{27y^2\partial_y}{27+4t^3}-\frac{6x^2yt^2\partial_y}{27+4t^3},\quad e_{10}=-\frac{x^3yt\partial_x}{3},\\
&e_{11}=-\frac{3y^2\partial_x}{2t}+\frac{x^2yt\partial_x}{3}+\frac{x^3y(-27-4t^3)\partial_y}{54},\quad e_{12}=-\frac{2x^3yt\partial_y}{3},\\
&e_{13}=x^6\partial_z,\quad e_{14}=x^5\partial_z+\frac{2tx^3y\partial_z}{3},\quad e_{15}=x^4\partial_z+\frac{3y^2\partial_z}{t},\\
&e_{16}=z\partial_z,\quad e_{17}=x^6\partial_y,\quad e_{18}=x^5\partial_y+\frac{2x^3yt\partial_y}{3},\\
&e_{19}=x^4\partial_y+\frac{3y^2\partial_y}{t},\quad e_{20}=z\partial_y,\quad e_{21}=x^6\partial_x,\\
&e_{22}=x^5\partial_x+\frac{2x^3yt\partial_x}{3},\quad e_{23}=x^4\partial_x+\frac{3y^2\partial_x}{t},\quad e_{24}=z\partial_x.
\end{aligned}$$

For $k = 2$, the moduli algebra of V_t is defined by

$$A^2(V_t) = <1, x, y, z, x^2, x^3, x^4, x^5, x^6, x^7, xy, x^2y, x^3y, x^4y, y^2, xy^2, y^3, xz, yz > .$$

By calculation, a basis for Lie algebra $L^2(V_t) = \mathrm{Der}(A^2(V_t), A^2(V_t))$ $(4t^3 + 27 \neq 0)$, is given as:

$$e_1 = \frac{x\partial_x}{5} + \frac{2y\partial_y}{5} + \frac{3z\partial_z}{5}, \quad e_2 = \frac{z\partial_y}{2} + \frac{3y^2\partial_z}{2} + \frac{tx^4\partial_z}{2},$$
$$e_3 = -\frac{x^4\partial_z}{2} - \frac{3z\partial_y}{2t} - \frac{3y^2\partial_z}{2t}, \quad e_4 = -\frac{18tx^5\partial_z}{27+4t^3} - \frac{12t^2x^3y\partial_z}{27+4t^3},$$
$$e_5 = -z\partial_x \quad e_6 = x^6\partial_z, \quad e_7 = \frac{x^2\partial_x}{3} + \frac{2xy\partial_y}{3},$$
$$e_8 = \frac{x^3\partial_x}{2} + \frac{txy\partial_y}{3} - \frac{x^2(-27-4t^3)\partial_x}{24t^2},$$
$$e_9 = \frac{54xy\partial_x}{27+4t^3} - \frac{12t^2x^3\partial_x}{27+4t^3}, \quad e_{10} = \frac{y^2\partial_y}{2} + \frac{5t^2x^3\partial_x}{27+4t^3} - \frac{xy(81+2t^3)\partial_x}{3(27+4t^3)},$$
$$e_{11} = \frac{x^4\partial_y}{2} - \frac{15tx^3\partial_x}{27+4t^3} - \frac{xy(-81-2t^3)\partial_x}{t(27+4t^3)},$$
$$e_{12} = -\frac{27x^2y\partial_x}{27+4t^3} - \frac{9tx^5\partial_y}{27+4t^3} + \frac{6t^2x^4\partial_x}{27+4t^3} - \frac{6t^2x^3y\partial_y}{27+4t^3},$$
$$e_{13} = -\frac{3x^2y\partial_y}{2t} - \frac{45xy\partial_x}{27+4t^3} + \frac{3x^3(9+8t^3)\partial_x}{2t(27+4t^3)},$$
$$e_{14} = -\frac{6tx^3\partial_x}{27+4t^3} - \frac{4t^2xy\partial_x}{27+4t^3},$$
$$e_{15} = -\frac{18tx^4\partial_x}{27+4t^3} + \frac{18tx^3y\partial_y}{27+4t^3} - \frac{12t^2x^2y\partial_x}{27+4t^3} - \frac{4t^3x^5\partial_y}{27+4t^3},$$
$$e_{16} = y^2\partial_x + \frac{18tx^4\partial_x}{27+4t^3} - \frac{18tx^3y\partial_y}{27+4t^3} + \frac{12t^2x^2y\partial_x}{27+4t^3} + \frac{4t^3x^5\partial_y}{27+4t^3},$$
$$e_{17} = \frac{54x^3y\partial_y}{27+4t^3} - \frac{36tx^2y\partial_x}{27+4t^3} - \frac{12t^2x^5\partial_y}{27+4t^3} + x^4(1 - \frac{54}{27+4t^3})\partial_x,$$
$$e_{19} = \frac{36tx^4\partial_x}{27+4t^3} - \frac{36tx^3y\partial_y}{27+4t^3} + \frac{8t^3x^5\partial_y}{27+4t^3} + \frac{3x^2y(-27+4t^3)}{t(27+4t^3)}\partial_x,$$
$$e_{20} = \frac{x^5\partial_x}{3} + \frac{x^6\partial_y}{2t}, \quad e_{18} = x^6\partial_y,$$

$$e_{21} = \frac{x^3y\partial_x}{2} + \frac{x^5\partial_x}{4t} - \frac{x^6(-27-4t^3)}{24t^2}\partial_y, \quad e_{22} = \frac{x^6\partial_x}{2}, \quad e_{23} = yz\partial_z,$$
$$e_{24} = xz\partial_z, \quad e_{25} = y^3\partial_z - \frac{x^6\partial_z}{2}, \quad e_{26} = xy^2\partial_z + \frac{tx^5\partial_z}{3},$$
$$e_{27} = x^4y\partial_z + \frac{3x^6\partial_z}{2t}, \quad e_{28} = x^7\partial_z, \quad e_{29} = yz\partial_y \quad e_{30} = xz\partial_y,$$
$$e_{31} = y^3\partial_y - \frac{x^6\partial_y}{2}, \quad e_{32} = xy^2\partial_y + \frac{tx^5\partial_y}{3}, \quad e_{33} = x^4y\partial_y + \frac{3x^6\partial_y}{2t},$$
$$e_{34} = x^7\partial_y, \quad e_{35} = yz\partial_x \quad e_{36} = xz\partial_x \quad e_{37} = y^3\partial_x - \frac{x^6\partial_x}{2},$$
$$e_{38} = xy^2\partial_x + \frac{x^5t\partial_x}{3}, \quad e_{39} = x^4y\partial_x + \frac{3x^6\partial_x}{2t}, \quad e_{40} = x^7\partial_x.$$

using same steps as we used in case of $L^1(\tilde{E}_7)$, we can check that the families $L^1(\tilde{E}_8)$ and $L^2(\tilde{E}_8)$ is non-trivial.

4. Computing the Lie algebras

In this section we shall use the computation of Lie algebra $L^1(V)$ ([HYZ2]), for binomial singularities, which include the simple hypersurface singularities. As an application, we prove that the simple hypersurface singularities can be characterized completely by using Lie algebras $L^1(V)$.

Proposition 4.1. *([HYZ2]) Let $(V,0)$ be a weighted homogeneous fewnomial isolated singularity of type A which is defined by $f = x_1^{a_1} + x_2^{a_2}$ $(a_1 \geq 2,\ a_2 \geq 2)$ with weight type $(\frac{1}{a_1}, \frac{1}{a_2}; 1)$. Then*

$$\lambda^1(V) = \begin{cases} 2a_1a_2 - 3(a_1+a_2) + 10; & a_1 \geq 3, a_2 \geq 3 \\ a_1 + 2; & a_1 \geq 2, a_2 = 2. \end{cases}$$

Proposition 4.2. *([HYZ2]) Let $(V,0)$ be a binomial isolated singularity of type B which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}$ $(a_1 \geq 1,\ a_2 \geq 2)$ with weight type $(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}; 1)$. Then*

$$\lambda^1(V) = \begin{cases} 2a_1a_2 - 2a_1 - 3a_2 + 11; & a_1 \geq 2, a_2 \geq 3 \\ 2a_1 + 2; & a_1 \geq 1, a_2 = 2 \\ 4; & a_1 = 1, a_2 \geq 3. \end{cases}$$

Proposition 4.3. *([HYZ2]) Let $(V,0)$ be a binomial isolated singularity of type C which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_1$ $(a_1 \geq 1, a_2 \geq 1)$ with weight* *type*

$$(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}; 1).$$

Then

$$\lambda^1(V) = \begin{cases} 2a_1a_2 - 2a_1 - 2a_2 + 12; & a_1 \geq 3, a_2 \geq 3 \\ 2a_1 + 6; & a_1 \geq 2, a_2 = 2 \\ 4; & a_1 \geq 1, a_2 = 1 \\ 4; & a_1 = 1, a_2 \geq 2. \end{cases}$$

In order to prove Theorem B, we need the following proposition.

Proposition 4.4. *The following four cases of Lie algebras $L^1(V)$ that arising from simple hypersurface singularities are not isomorphic:*

(1) $L^1(D_6) \ncong L^1(A_9)$,

(2) $L^1(E_6) \ncong L^1(D_7), L^1(E_6) \ncong L^1(A_{10}), L^1(D_7) \ncong L^1(A_{10})$,

(3) $L^1(E_7) \ncong L^1(D_8), L^1(E_7) \ncong L^1(A_{11}), L^1(D_8) \ncong L^1(A_{11})$,

(4) $L^1(E_8) \ncong L^1(D_{10}), L^1(E_8) \ncong L^1(A_{13}), L^1(D_{10}) \ncong L^1(A_{13})$.

Proof. (1) It is easy to see from Proposition 4.2, $L^1(D_6)$ is a 12-dimensional complex Lie algebra spanned by following basis:

$$< x_1\partial_1, x_2^2\partial_1, x_1x_2\partial_1, x_2^3\partial_1, x_2^4\partial_1, x_1\partial_2, x_1^2\partial_2, x_1x_2\partial_2, x_2\partial_2, x_2^2\partial_2, x_2^3\partial_2, x_2^4\partial_2 > .$$

Set $e_1 = x_1\partial_1, \cdots, e_{12} = x_2^4\partial_2$. We obtain following multiplication table.

$$\begin{aligned}
&[e_1, e_2] = e_2, [e_1, e_4] = -e_4, [e_1, e_5] = -e_5, [e_1, e_6] = e_6, [e_1, e_7] = 2e_7, \\
&[e_1, e_8] = e_8, [e_2, e_6] = e_7, [e_3, e_4] = -e_5, [e_3, e_6] = -e_2 + e_8, [e_3, e_9] = -e_3, \\
&[e_4, e_6] = e_{11}, [e_4, e_8] = e_{12}, [e_4, e_9] = -3e_4, [e_4, e_{10}] = -3e_5, [e_5, e_6] = e_{12}, \\
&[e_5, e_9] = -4e_5, [e_6, e_8] = e_7, [e_6, e_9] = e_6, [e_6, e_{10}] = 2e_8, [e_7, e_9] = e_7, \\
&[e_9, e_{10}] = e_{10}, [e_9, e_{11}] = 2e_{11}, [e_9, e_{12}] = 3e_{12}, [e_{10}, e_{11}] = e_{12}.
\end{aligned}$$

The Cartan subalgebra is generated by $< e_1, e_9 >$. Therefore the rank of $L^1(D_6)$ is 2. It follows from multiplication table the sequence of dimensions of derived series are $\{12, 10, 6, 0\}$. It is easy to see from Proposition 4.1,

$L^1(A_9)$ is a 12-dimensional complex Lie algebra spanned by following basis:

$$< x_1\partial_1, x_2^9\partial_1, x_1\partial_2, x_2\partial_2, x_2^2\partial_2, x_2^3\partial_2, x_2^4\partial_2, x_2^5\partial_2, x_2^6\partial_2, x_2^7\partial_2, x_2^8\partial_2, x_2^9\partial_2 > .$$

Set $d_1 = x_1\partial_1, \cdots, d_{12} = x_2^9\partial_2$. We obtain following multiplication table.

$$\begin{aligned}
&[d_1, d_2] = -d_2, [d_1, d_3] = d_3, [d_2, d_3] = d_{12}, [d_2, d_4] = -9d_2, [d_3, d_4] = d_3,\\
&[d_4, d_5] = d_5, [d_4, d_6] = 2d_6, [d_4, d_7] = 3d_7, [d_4, d_8] = 4d_8, [d_4, d_9] = 5d_9,\\
&[d_4, d_{10}] = 6d_{10}, [d_4, d_{11}] = 7d_{11}, [d_4, d_{12}] = 8d_{12}, [d_5, d_6] = d_7, [d_5, d_7] = 2d_8,\\
&[d_5, d_8] = 3d_9, [d_5, d_9] = 4d_{10}, [d_5, d_{10}] = 5d_{11}, [d_5, d_{11}] = 6d_{12}, [d_6, d_7] = d_9,\\
&[d_6, d_8] = 2d_{10}, [d_6, d_9] = 3d_{11}, [d_6, d_{10}] = 4d_{12}, [d_7, d_8] = d_{11}, [d_7, d_9] = 2d_{12}.
\end{aligned}$$

The Cartan subalgebra is generated by $< d_1, d_4 >$. Therefore the rank of $L^1(A_9)$ is 2. It follows from multiplication table the sequence of dimensions of derived series are $\{12, 10, 6, 2, 0\}$. It should be noted that both Lie algebras $L^1(D_6)$ and $L^1(A_9)$ have different sequences of dimensions of derived series. Therefore we conclude that $L^1(D_6)$ and $L^1(A_9)$ are non-isomorphic.

Case (2) It is easy to see from Proposition 4.1, the $L^1(E_6)$ is a 13-dimensional complex Lie algebra spanned by the following basis:

$$\begin{aligned}
<&x_1\partial_1, x_1^2\partial_1, x_1x_2\partial_1, x_1x_2^2\partial_1, x_2^2\partial_1,\\
&x_2^3\partial_1, x_1\partial_2, x_1^2\partial_2, x_2\partial_2, x_2^2\partial_2, x_1x_2\partial_2, x_1x_2^2\partial_2, x_2^3\partial_2 > .
\end{aligned}$$

Set $e_1 = x_1\partial_1, \cdots, e_{13} = x_2^3\partial_2$. We obtain following multiplication table.

$$\begin{aligned}
&[e_1, e_2] = e_2, [e_1, e_5] = -e_5, [e_1, e_6] = -e_6, [e_1, e_7] = e_7, [e_1, e_8] = 2e_8,\\
&[e_1, e_{11}] = e_{11}, [e_1, e_{12}] = e_{12}, [e_2, e_5] = -2e_4, [e_2, e_7] = e_8, [e_3, e_5] = -e_6,\\
&[e_3, e_7] = -e_2 + e_{11}, [e_3, e_9] = -e_3[e_3, e_{10}] = -e_4, [e_3, e_{11}] = e_{12},\\
&[e_4, e_7] = e_{12}, [e_4, e_9] = -2e_4, [e_5, e_7] = -2e_3 + e_{11}, [e_5, e_8] = 2e_{12},\\
&[e_5, e_9] = -2e_5, [e_5, e_{10}] = -2e_6, [e_5, e_{11}] = -2e_4 + e_{13}, [e_6, e_7] = -3e_4 + e_{13},\\
&[e_6, e_9] = -3e_6, [e_7, e_9] = e_7, [e_7, e_{10}] = 2e_{11}, [e_7, e_{12}] = e_8, [e_7, e_{13}] = 3e_{12},\\
&[e_8, e_9] = e_8, [e_9, e_{10}] = e_{10}, [e_9, e_{12}] = e_{12}, [e_9, e_{13}] = 2e_{13}, [e_{10}, e_{11}] = -e_{12}.
\end{aligned}$$

The Cartan subalgebra is generated by $< e_1, e_9 >$. Therefore the rank of $L^1(E_6)$ is 2. It follows from multiplication table the sequence of dimensions of derived series are $\{13, 11, 8, 1, 0\}$.

It is easy to see from Proposition 4.2, $L^1(D_7)$ is a 13-dimensional complex Lie algebra spanned by following basis:

$$<x_1\partial_1, x_1^2\partial_1, x_1x_2\partial_1, x_2^4\partial_1, x_2^5\partial_1, x_1\partial_2, x_1^2\partial_2,\\ x_1x_2\partial_2, x_2\partial_2, x_2^2\partial_2, x_2^3\partial_2, x_2^4\partial_2, x_2^5\partial_2 > .$$

Set $d_1 = x_1\partial_1, \cdots, d_{13} = x_2^5\partial_2$. We obtain following multiplication table.

$$[d_1, d_2] = d_2, [d_1, d_4] = -d_4, [d_1, d_5] = -d_5, [d_1, d_6] = d_6, [d_1, d_7] = 2d_7,\\ [d_1, d_8] = d_8, [d_2, d_6] = d_7, [d_3, d_4] = -d_5, [d_3, d_6] = d_8 - d_2, [d_3, d_9] = -d_3,\\ [d_4, d_6] = d_{12}, [d_4, d_8] = d_{13}, [d_4, d_9] = -4d_4, [d_4, d_{10}] = -4d_5, [d_5, d_6] = d_{13},\\ [d_5, d_9] = -5d_5, [d_6, d_8] = d_7, [d_6, d_9] = d_6, [d_6, d_{10}] = 2d_8, [d_7, d_9] = d_7,\\ [d_9, d_{10}] = d_{10}, [d_9, d_{11}] = 2d_{11}, [d_9, d_{12}] = 3d_{12}, [d_9, d_{13}] = 4d_{13},\\ [d_{10}, d_{11}] = d_{12}, [d_{10}, d_{12}] = 2d_{13}.$$

The Cartan subalgebra is generated by $< d_1, d_9 >$. Therefore the rank of $L^1(D_7)$ is 2. It follows from multiplication table, the sequence of dimensions of derived series are $\{13, 11, 6, 0\}$. It is easy to see from Proposition 4.1, $L^1(A_{10})$ is a 13-dimensional complex Lie algebra spanned by following basis:

$$<x_1\partial_1, x_2^{10}\partial_1, x_1\partial_2, x_2\partial_2, x_2^2\partial_2, x_2^3\partial_2, x_2^4\partial_2,\\ x_2^5\partial_2, x_2^6\partial_2, x_2^7\partial_2, x_2^8\partial_2, x_2^9\partial_2, x_2^{10}\partial_2 > .$$

Set $a_1 = x_1\partial_1, \cdots, a_{13} = x_2^{10}\partial_2$. We obtain following multiplication table.

$$[a_1, a_2] = -a_2, \quad [a_1, a_3] = a_3, \quad [a_2, a_3] = a_{13}, \quad [a_2, a_4] = -10a_2,\\ [a_3, a_4] = a_3, [a_4, a_5] = a_5, \quad [a_4, a_6] = 2a_6, \quad [a_4, a_7] = 3a_7,\\ [a_4, a_8] = 4a_8, \quad [a_4, a_9] = 5a_9, \quad [a_4, a_{10}] = 6a_{10}, \quad [a_4, a_{11}] = 7a_{11},\\ [a_4, a_{12}] = 8a_{12}, \quad [a_4, a_{13}] = 9a_{13}, \quad [a_5, a_6] = a_7, \quad [a_5, a_7] = 2a_8,\\ [a_5, a_8] = 3a_9, [a_5, a_9] = 4a_{10}, \quad [a_5, a_{10}] = 5a_{11}, \quad [a_5, a_{11}] = 6a_{12},\\ [a_5, a_{12}] = 7a_{13}, [a_6, a_7] = a_9, \quad [a_6, a_8] = 2a_{10}, \quad [a_6, a_9] = 3a_{11},\\ [a_6, a_{10}] = 4a_{12}, [a_6, a_{11}] = 5a_{13}, \quad [a_7, a_8] = a_{11}, \quad [a_7, a_9] = 2a_{12},\\ [a_7, a_{10}] = 3a_{13}, \quad [a_8, a_9] = a_{13}.$$

The Cartan subalgebra is generated by $< a_1, a_4 >$. Therefore the rank of $L^1(A_{10})$ is 2. It follows from multiplication table the sequence of dimensions of derived series are $\{13, 11, 7, 3, 0\}$. it follows that the Lie algebra $L^1(D_7)$, $L^1(E_6)$ and $L^1(A_{10})$ have different sequence of dimensions of derived series.

Therefore these three Lie algebras are pairwise non-isomorphic. Similarly we can prove cases (3) and (4). □

Proof of Theorem B.

Proof. It follows from Proposition 4.1, the series $A_k : \{x_1^{k+1} + x_2^2 = 0\} \subset \mathbb{C}, k \geq 1$, have following basis of Lie algebra:
$L^1(A_k) =< x_1\partial_1; x_1\partial_2; x_2^k\partial_1; x_2^{i_2}\partial_2, 1 \leq i_2 \leq k >$. Therefore we get $\lambda^1(A_k) = k+3$ and Cartan subalgebra that arising from $L^1(A_k)$ generated by $< x_1\partial_1, x_2\partial_2 >$. Let $g(V)$ be the nilradical of Lie algebra $L^1(V)$. For A_k singularity $k \geq 6$, we have

$$g(V) =< x_2^k\partial_1; x_1\partial_2; x_2^{i_2}\partial_2, 2 \leq i_2 \leq k > .$$

By setting $e_1 = x_2^k\partial_1, e_2 = x_1\partial_2, \cdots, e_{k+1} = x_2^k\partial_2$, we have the following multiplication table:

Case 1. k is even and $k = 2l+4 \geq 6, l \geq 1$, then

$$\begin{aligned}
&[e_1, e_2] = e_{k+1},\\
&[e_3, e_4] = e_5,\ [e_3, e_5] = 2e_6,\ [e_3, e_6] = 3e_7,\ \cdots, [e_3, e_k] = (k-3)e_{k+1},\\
&[e_4, e_5] = e_7,\ [e_4, e_6] = 2e_8,\ [e_4, e_7] = 3e_9,\ \cdots, [e_4, e_{k-1}] = (k-5)e_{k+1},\\
&[e_5, e_6] = e_9,\ [e_5, e_7] = 2e_{10},\ [e_5, e_8] = 3e_{11},\ \cdots, [e_5, e_{k-2}] = (k-7)e_{k+1},\\
&[e_6, e_7] = e_{11},\ [e_6, e_8] = 2e_{12},\ [e_6, e_9] = 3e_{13},\ \cdots, [e_6, e_{k-3}] = (k-9)e_{k+1},\\
&\quad\vdots\\
&[e_{l+3}, e_{l+4}] = e_{2l+5}.
\end{aligned}$$

Case 2. k is odd and $k = 2l+5 \geq 7, l \geq 1$, then

$$\begin{aligned}
&[e_1, e_2] = e_{k+1},\\
&[e_3, e_4] = e_5,\ [e_3, e_5] = 2e_6,\ [e_3, e_6] = 3e_7,\ \cdots, [e_3, e_k] = (k-3)e_{k+1},\\
&[e_4, e_5] = e_7,\ [e_4, e_6] = 2e_8,\ [e_4, e_7] = 3e_9,\ \cdots, [e_4, e_{k-1}] = (k-5)e_{k+1},\\
&[e_5, e_6] = e_9,\ [e_5, e_7] = 2e_{10},\ [e_5, e_8] = 3e_{11},\ \cdots, [e_5, e_{k-2}] = (k-7)e_{k+1},\\
&[e_6, e_7] = e_{11},\ [e_6, e_8] = 2e_{12},\ [e_6, e_9] = 3e_{13},\ \cdots, [e_6, e_{k-3}] = (k-9)e_{k+1},\\
&\quad\vdots\\
&[e_{l+3}, e_{l+4}] = e_{2l+5},\ [e_{l+3}, e_{l+5}] = 2e_{2l+6}.
\end{aligned}$$

The type of $A_k (k \geq 6)$ singularity =dim $g(V)/[g(V), g(V)] = 4$.

It follows from Proposition 4.2, the series $D_k : \{x_1^2 x_2 + x_2^{k-1} = 0\} \subset \mathbb{C}^2$, $k \geq 4$, have following basis of Lie algebra $L^1(V)$:

$$< x_1\partial_1; x_1^2\partial_1; x_1x_2\partial_1; x_2^{k-3}\partial_1; \\ x_2^{k-2}\partial_1; x_1\partial_2; x_1^2\partial_2; x_1x_2\partial_2; x_2^{i_2}\partial_2, 1 \leq i_2 \leq k-2 > .$$

Therefore we get $\lambda^1(D_k) = k + 6$ and Cartan subalgebra that arising from $L^1(D_k)$ generated by $< x_1\partial_1, x_2\partial_2 >$. The nilradical of Lie algebra of D_k is defined by:

$$g(V) = < x_1^2\partial_1; x_1x_2\partial_1; x_2^{k-3}\partial_1; x_2^{k-2}\partial_1; \\ x_1\partial_2; x_1^2\partial_2; x_1x_2\partial_2; x_2^{i_2}\partial_2, 2 \leq i_2 \leq k-2 > .$$

By setting $e_1 = x_2^k\partial_1, e_2 = x_1x_2\partial_2, \cdots, e_{k+4} = x_2^{k-2}\partial_2$. For D_k singularity $k \geq 8$, we have the following multiplication table:
Case 1. when k is odd and $k = 2l + 7$, $l \geq 1$, then

$$\begin{aligned}
&[e_1, e_5] = e_6,\ [e_2, e_3] = -e_4,\ [e_2, e_5] = -e_1 + e_7,\ [e_3, e_5] = e_{k+3},\\
&[e_3, e_7] = e_{k+4},\ [e_3, e_8] = -(k-3)e_4,\ [e_4, e_5] = e_{k+4},\ [e_5, e_7] = e_6,\\
&[e_5, e_8] = 2e_7,\ [e_8, e_9] = e_{10},\ [e_8, e_{10}] = 2e_{11},\ [e_8, e_{11}] = 3e_{12}, \cdots,\\
&[e_8, e_{k+3}] = (k-5)e_{k+4},\ [e_9, e_{10}] = e_{12},\ [e_9, e_{11}] = 2e_{13},\\
&[e_9, e_{12}] = 3e_{14}, \cdots, [e_9, e_{k+2}] = (k-7)e_{k+4},\\
&\quad \vdots\\
&[e_{2l+7}, e_{2l+8}] = e_{2l+10},\ [e_{2l+7}, e_{2l+9}] = e_{2l+11}.
\end{aligned}$$

Case 2. when k is even and $k = 2l + 6$, $l \geq 1$, then

$$\begin{aligned}
&[e_1, e_5] = e_6,\ [e_2, e_3] = -e_4,\ [e_2, e_5] = -e_1 + e_7,\ [e_3, e_5] = e_{k+3},\\
&[e_3, e_7] = e_{k+4},\\
&[e_3, e_8] = -(k-3)e_4,\ [e_4, e_5] = e_{k+4},\ [e_5, e_7] = e_6,\ [e_5, e_8] = 2e_7,\\
&[e_8, e_9] = e_{10},\ [e_8, e_{10}] = 2e_{11},\ [e_8, e_{11}] = 3e_{12}, \cdots, [e_8, e_{k+3}] = (k-5)e_{k+4},\\
&[e_9, e_{10}] = e_{12},\ [e_9, e_{11}] = 2e_{13},\ [e_9, e_{12}] = 3e_{14}, \cdots, [e_9, e_{k+2}] = (k-7)e_{k+4},\\
&\quad \vdots\\
&[e_{2l+7}, e_{2l+8}] = e_{2l+10}.
\end{aligned}$$

The type of D_k singularity (for $k \geq 8$) =dim $g(V)/[g(V), g(V)] = 5$. Therefore when $k \geq 8$, then the series A_k and D_k have different type of singularity.

The type of singularity for Lie algebras $L^1(D_4)$, $L^1(D_5)$, $L^1(D_6)$ and $L^1(D_7)$ are 3, 3, 4 and 5 respectively. The type of singularity for Lie algebras $L^1(A_7)$, $L^1(A_8)$ and $L^1(A_9)$ are 4. It is easy to see from Propositions 4.1 and 4.2 the $\lambda^1(E_6) = 13, \lambda^1(E_7) = 14, \lambda^1(E_8) = 16$ and Cartan subalgebra that arising from $L^1(E_6)$, $L^1(E_7)$ and $L^1(E_8)$ generated by $< x_1\partial_1, x_2\partial_2 >$. Therefore in case of Lie algebra $L^1(V)$, the simple hypersurface singularities (ADE singularities) have rank 2. Next we need to distinguish the remaining pairs which have same rank, same dimensions of Lie algebras and same type of singularity. It is noted that we only need to treat four cases:

1. $L^1(D_6) \ncong L^1(A_9)$,
2. $L^1(E_6) \ncong L^1(D_7)$, $\quad L^1(E_6) \ncong L^1(A_{10})$, $\quad L^1(D_7) \ncong L^1(A_{10})$,
3. $L^1(E_7) \ncong L^1(D_8)$, $\quad L^1(E_7) \ncong L^1(A_{11})$, $\quad L^1(D_8) \ncong L^1(A_{11})$,
4. $L^1(E_8) \ncong L^1(D_{10})$, $\quad L^1(E_8) \ncong L^1(A_{13})$, $\quad L^1(D_{10}) \ncong L^1(A_{13})$.

It follows from Proposition 4.4 these four cases are non-isomorphic. Therefore we completely characterized the simple hypersurface singularities by using Lie algebra $L^1(V)$. □

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Department of Mathematical Sciences, Tsinghua University
Beijing, 100084, P. R. China
Current address: Department of Mathematics and Statistics
University of Agriculture, Faisalabad 38000, Pakistan
E-mail address: dr.nhussain@uaf.edu.pk

Department of Mathematical Sciences, Tsinghua University
Beijing, 100084, P. R. China
Yanqi Lake Beijing Institute of Mathematical Sciences and Applications, Beijing, 101400, P. R. China
E-mail address: yau@uic.edu

Department of Mathematical Sciences, Tsinghua University
Beijing, 100084, P. R. China
E-mail address: hqzuo@mail.tsinghua.edu.cn

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