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An inequality conjecture and a weak Torelli-type theorem for isolated complete intersection singularities ☆

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ABSTRACT

In this paper, we propose an inequality conjecture for the dimensions of derivation Lie algebras associated to isolated complete intersection singularities. We verify this conjecture for simple and unimodal isolated complete intersection singularities. We also construct several new one-parameter families of solvable Lie algebras from T_{10}, R_9, U_{11}, V_{10}, Y_{11}, and M_{11} singularities and show that the weak Torelli-type theorem holds.

1. Introduction

We will consider isolated complete intersection singularities $(V^n, 0)$ in $(\mathbb{C}^m, 0)$ which are defined by weighted homogeneous polynomials. Recall that a polynomial $f(x_1, \dots, x_m)$ is said to be weighted homogeneous with weights $(w_1, \dots, w_m)$, $w_j \in \mathbb{Q}$ and $w_j > 1$, if for every monomial $\alpha x_1^{a_1} \dots x_m^{a_m}$ one has

$$a_1 w_1 + \cdots + a_m w_m = 1.$$

We say that f has weight type $(w_1, \cdots, w_m; 1)$. Sometimes we also use integer weight type in the following manner. Write $w_j = u_j/v_j$, where u_j and v_j are positive integers without common factor. Let $d = \operatorname{lcm}(v_1, \dots, v_m)$, and define $w'_j = d w_j$. Then f has integer weight type $(w'_1, \cdots, w'_m; d)$.

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In this paper we consider weighted homogenous complete intersection variety with isolated singularity $V = V(f_1, \ldots, f_k) = V(f_1) \cap \ldots \cap V(f_k) \subset \mathbb{C}^m$, where (i) each f_i is a weighted homogeneous polynomial with weights $(w_1, \ldots, w_m)$ independent of i, and (ii) for all $V(f_1, \ldots, f_k)$ is a complete intersection with an isolated singularity at the origin in $\mathbb{C}^m$. The weight type of $V = V(f_1, \ldots, f_k)$ is denoted as $(w_1, \cdots, w_m; d_1, \cdots, d_m)$ where d_i is weight degree of f_i with respect to $(w_1, \cdots, w_m)$.

Observe that if we let $f : \mathbb{C}^m \to \mathbb{C}^k$ be the holomorphic function with coordinates $f_1, \ldots, f_k$, then (ii) implies that the $k \times m$ Jacobian matrix $\left(\partial f_i / \partial z_j\right)$ has rank k everywhere in some neighborhood of the origin in V except possibly at the origin itself.

Let $\mathcal{O}_m$ denote the $\mathbb{C}$-algebra of germs of analytic functions defined at the origin of $\mathbb{C}^m$ and $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_m$. Let V be an analytic space at the origin of $\mathbb{C}^m$ defined by an ideal $I_V = (f_1, \cdots, f_k) \subset \mathfrak{m}^2$ as the fiber of the corresponding map germ $f : (\mathbb{C}^m, 0) \to (\mathbb{C}^k, 0)$. It is well-known [24] that, in the case $m \geq k$, the map germ f is finitely contact determined if and only if $(V, 0)$ is an isolated complete intersection singularity (This will be abbreviated in the sequel to ICIS). Thus the ICIS $(V, 0)$ is determined by the Artinian $\mathbb{C}$-algebra $\mathcal{O}_n/(I_V + \mathfrak{m}^{d+1})$ where d is the order of contact-determinacy of the map germ f. In this paper, we consider a different Artinian $\mathbb{C}$-algebra, more geometrically associated to V, which can play a similar role. More precisely, if V is an ICIS defined by an ideal I_V as above, then one can consider the singular subspace of V, which is the analytic space germ SV defined by the ideal $SI_V \subset \mathfrak{m}$ generated by the f_i and all the $k \times k$ minors in the Jacobian matrix $(\frac{\partial f_i}{\partial x_j})$, $i = 1, \cdots, k$; $j = 1, \cdots, m$. Since SV depends only on the isomorphism class of V, it follows that $\mathcal{O}_m/SI_V$, which is the coordinate ring of SV, is an invariant of $(V, 0)$. In one-dimensional ICIS case, the $\dim \mathcal{O}_m/SI_V$ is exactly the Tjurina number of $(V, 0)$. That is to say that if $f, g \in \mathbb{C}\{x, y, z\}$ are analytic functions defining an isolated curve singularity $(V, 0)$, then the Tjurina number of $(V, 0)$ (i.e., the dimension of the tangent space of the base space of the semiuniversal deformation of $(V, 0)$), $\tau(V, 0) = \dim_{\mathbb{C}} \mathbb{C}\{x, y, z\}/(f, g, M_1, M_2, M_3)$, where M_1, M_2, M_3 are the 2-minors of the Jacobian matrix of f, g, i.e.,

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{pmatrix},$$

and

$$M_1 = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix}, M_2 = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial z} \end{vmatrix}, M_3 = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{vmatrix}.$$

Recall that for any isolated hypersurface singularity $(V, 0) \subset (\mathbb{C}^m, 0)$ defined by f, Yau [27] first studied the Lie algebra of derivations of moduli algebra $A(V) = \mathcal{O}_m/(f, J(f))$ ($J(f)$ is the Jacobian ideal), i.e., $L(V) = \mathrm{Der}(A(V), A(V))$. It is known that $L(V)$ is a finite dimensional solvable Lie algebra. The $L(V)$ is called the Yau algebra and its dimension $\lambda(V)$ is called Yau number in [30] and [18] in order to distinguish from Lie algebras of other types appearing in singularity theory [7]. The Yau algebra plays an important role in singularity theory [23]. Yau and his collaborators have been systematically studying the Lie algebras of isolated hypersurface singularities beginning from eighties (see, e.g., [3], [4], [5], [9–16], [27–29], [26], [31,32]).

Recently, in [16], the authors generalize the construction of Yau algebra. For any isolated hypersurface singularity $(V, 0) \subset (\mathbb{C}^m, 0)$ defined by the holomorphic function $f(x_1, \cdots, x_m)$, let $Hess(f)$ be the Hessian matrix (f_{ij}) of the second-order partial derivatives of f and $h(f)$, the Hessian of f, be the determinant of the matrix $Hess(f)$. More generally, for each k satisfying $0 \leq k \leq m$ we denote by $h_k(f)$ the ideal in $\mathcal{O}_m$ generated by all $k \times k$-minors in the matrix $Hess(f)$. In particular, the ideal $h_m(f) = (h(f))$ is a principal ideal. For each k as above, consider the graded k-th Hessian algebra of the polynomial f defined by

$$H_k(f) = \mathcal{O}_m/(f + J(f) + h_k(f)).$$

In particular, $H_0(f)$ is exactly the well-known moduli algebra $A(V)$.

It is easy to check that the isomorphism class of the local k-th Hessian algebra $H_k(f)$ is contact invariant of f, i.e. depends only on the isomorphism class of the germ $(V, 0)$ ([6], Lemma 2.1). In [16], the authors introduced a new series Lie algebra $L_k(V)$ which is the Lie algebra of derivations of k-th Hessian algebra $H_k(f)$. The dimension of $L_k(V)$, denoted by $\lambda_k(V)$, is a new numerical analytic invariant of an isolated hypersurface singularity.

In the theory of isolated singularities, one always wants to find invariants associated to the isolated singularities. Hopefully with enough invariants found, one can distinguish between isolated singularities. However, not many invariants are known for ICIS. In this paper, we generalize the construction of Yau algebra $L(V)$ to the case of ICIS $(V, 0)$. We introduce a new derivation Lie algebra as follows.

Definition 1.1. For each ICIS $(V, 0)$ in $(\mathbb{C}^m, 0)$, the new derivation Lie algebra $\mathcal{NL}(V)$ is defined to be the Lie algebra of derivations of the local Artinian algebra $\mathcal{O}_m/SI_V$, i.e., $\mathcal{NL}(V) = Der(\mathcal{O}_m/SI_V, \mathcal{O}_m/SI_V)$. Its dimension is denoted as $\upsilon(V)$.

This number $\upsilon(V)$ is also a new numerical analytic invariant. The new Lie algebra is a generalization of the Yau algebra $L(V)$.

It is interesting to bound the Yau number with a number which depends on weight type. In [32], Yau and Zuo firstly proposed the sharp upper estimate conjecture that bound the Yau number $\lambda(V)$. They also proved that this conjecture holds in case of binomial isolated hypersurface singularities. Furthermore, in [9], this conjecture was verified for trinomial singularities (the definitions of fewnomial, binomial, and trinomial singularities can be found in [32]).

In [16], we proposed the following conjecture:

Conjecture 1.1. *For each k, let $h_k(a_1, \cdots, a_m)$ denote $\lambda_k(\{x_1^{a_1} + \cdots + x_m^{a_m} = 0\})$. Let $(V, 0) = \{(x_1, x_2, \cdots, x_m) \in \mathbb{C}^m : f(x_1, x_2, \cdots, x_m) = 0\}$, $(m \geq 2)$ be an isolated singularity defined by the weighted homogeneous polynomial $f(x_1, x_2, \cdots, x_m)$ of weight type $(w_1, w_2, \cdots, w_m; 1)$. Then $\lambda_k(V) \leq h_k(1/w_1, \cdots, 1/w_m)$.*

The Conjecture 1.1 tells us that the Brieskorn singularity has the maximal dimension of derivation Lie algebra of L_k when fixing the weight type of the singularity. This gives a sharp upper bound for $\lambda_k(V)$. The conjecture was only proven for $k = 0$, and 1 for binomial and trinomial singularities in [32], [9], and [16] respectively.

It is well-known that the ICIS of Brieskorn type is a generalization of Brieskorn hypersurface singularity. It is a natural question to consider the similar properties of Brieskorn ICIS.

Recall that a germ $(W, 0) \subset \left(\mathbb{C}^m, 0\right)$ of an n-dimensional Brieskorn ICIS is defined by

$$W = \left\{(x_i) \in \mathbb{C}^m \mid q_{j1}x_1^{a_1} + \cdots + q_{jm}x_m^{a_m} = 0, \, j = 1, \ldots, k\right\}$$

where $a_i \geq 2$ are integers and $n = m - k$. $(W, 0)$ has an isolated singularity at 0 if and only if every maximal minor of the matrix $\left(q_{ji}\right)$ does not vanish (cf. [[17], Section 7]).

A natural interesting question is: whether one can give a sharp bound for the new introduced $\upsilon(V)$ of an ICIS $(V, 0)$. We proposed the following sharp upper estimate conjecture which is a natural generalization of Conjecture 1.1 to ICIS.

Conjecture 1.2. *Let $(V, 0) = \{(x_1, x_2, \cdots, x_m) \in \mathbb{C}^m : f_i(x_1, x_2, \cdots, x_m) = 0, 1 \leq i \leq k\}$, $(m \geq 2)$ be an n-dimensional $(n = m - k)$ ICIS defined by the weighted homogeneous polynomials $f_1, \cdots, f_k$ of weight type $(w_1, \cdots, w_m; d_1, \cdots, d_k)$. Let $(W, 0)$ be an n-dimensional Brieskorn ICIS defined by polynomials*

$$W = \left\{(x_i) \in \mathbb{C}^m \mid q_{j1}x_1^{a_1} + \cdots + q_{jm}x_m^{a_m} = 0, \, j = 1, \ldots, k\right\},$$

where

$$d = max\{d_1, \cdots, d_k\}, a_l = \lceil d/w_l \rceil, 1 \leq l \leq m.$$

Then

$$\upsilon(V) \leq \upsilon(W).$$

Recall that the classifications of contact simple and unimodal complete intersection singularities were done by Giusti [8] and Aleksandrov [1]. The classification of the contact simple complete intersection (SCI) which is not hypersurface singularities (i.e., with modality 0) is as follows [8].

(1) Zero-dimensional simple complete intersection singularities.

$$\text{Type } F_{q+r-1}^{q,r} \quad (xy, \ x^q + y^r); \ q, r \geq 2,$$

$$\text{Types } \begin{cases} G_5 & \left(x^2, \ y^3\right), \\ G_7 & \left(x^2, \ y^4\right), \end{cases}$$

$$\text{Type } H_\mu \quad (x^2 + y^{\mu-3}, \ xy^2), \ \mu \geq 6,$$

$$\text{Types } \begin{cases} I_{2q-1} & \left(x^2 + y^3, \ y^q\right), \ q \geq 4, \\ I_{2r+2} & \left(x^2 + y^3, \ xy^r\right), \ r \geq 3, \end{cases}$$

(2) Simple complete intersection curve singularities.

$$\text{Type } S_\mu \quad (x^2 + y^2 + z^{\mu-3}, \ yz), \ \mu \geq 5,$$

$$\text{Types } \begin{cases} T_7 & \left(x^2 + y^3 + z^3, \ yz\right), \\ T_8 & \left(x^2 + y^3 + z^4, \ yz\right), \\ T_9 & \left(x^2 + y^3 + z^5, \ yz\right), \end{cases}$$

$$\text{Types } \begin{cases} U_7 & \left(x^2 + yz, \ xy + z^3\right), \\ U_8 & \left(x^2 + yz + z^3, \ xy\right), \\ U_9 & \left(x^2 + yz, \ xy + z^4\right), \end{cases}$$

$$\text{Types}\begin{cases} W_8 & \left(x^2+z^3,\ y^2+xz\right),\\ W_9 & \left(x^2+yz^2,\ y^2+xz\right),\end{cases}$$
$$\text{Types}\begin{cases} Z_9 & \left(x^2+z^3,\ y^2+z^3\right),\\ Z_{10} & \left(x^2+yz^2,\ y^2+z^3\right).\end{cases}$$

Note that all SCI singularities are weighted homogeneous singularities.

Aleksandrov [[1], page 21] and Wall [25] have obtained following classification for weighted homogeneous unimodal complete intersection singularities (here we use the notations in Aleksandrov's article).

$Type(\mathrm{V})$	Equations
T_{10}	$\{(x^2+y^3+z^6,\ ax+yz),\ 27a^6+4\neq 0\}$
T_k	$\{(x^2+y^3+z^{k-4},\ yz),\ k\geq 11\}$
R_9	$\{(x^2+y^4+z^4,\ ax+yz),\ 4a^4-1\neq 0\}$
R_k	$\{(x^2+y^4+z^{k-5},\ yz),\ k\geq 10\}$
$L_{2,q,r}$	$\{(x^2+y^q+z^r,\ yz),\ q,r\geq 5\}$
U_{11}	$\{(x^2+yz+z^4+axz^2,\ xy),\ a^2-4\neq 0\}$
U_{13}	$(x^2+yz,\ xy+z^6)$
U_{14}	$(x^2+z^5+yz,\ xy)$
U_{15}	$(x^2+yz,\ xy+z^7)$
V_{10}	$\{(x^3+yz+z^3+ax^2z,\ xy),\ 4a^3+27\neq 0\}$
V_{12}	$(x^4+yz+z^3,\ xy)$
V_{13}	$(yz+z^3,\ xy+x^4)$
V_{11}	$(x^5+yz+z^3,\ xy)$
Q_{13}	$(x^3+yz,\ xy+z^4)$
Q_{11}	$(x^3+yz+z^4,\ xy)$
$L_{3,2,4}$	$(x^4+y^2+z^3,\ xy)$
$L_{3,2,5}$	$(x^5+y^2+z^3,\ xy)$
Y_{11}	$\{(x^2+y^3+z^3+ay^2z,\ xy),\ 4a^3+27\neq 0\}$
G_{14}	$(x^2+y^3z+z^3,\ xy)$
H_{13}	$(x^2+y^2z,\ xy+z^3)$
H_{14}	$(x^2+y^3,\ xy+z^3)$
M_{11}	$\{(x^2+z^4,\ y^2+z^3+axz),\ a^2+1\neq 0\}$
M_{12}	$(x^2+yz^3,\ y^2+z^3)$
M_{13}	$(x^2+z^5,\ y^2+z^3)$
M_{14}	$(x^2+yz^4,\ y^2+z^3)$
N_{13}	$(x^2+yz^3,\ y^2+xz)$
N_{14}	$(x^2+z^5,\ y^2+xz)$

In this paper, we obtain the following results.

Theorem A. *The Conjecture 1.2 is true for the following classes of singularities:*
1) Contact simple complete intersection zero-dimensional singularities,
2) Contact simple complete intersection curve singularities,
3) Weighted homogeneous contact unimodal complete intersection curve singularities.

On the one hand, since derivations of moduli algebras are analogs of vector fields on smooth manifolds, such direction of research is in the spirit of the classical theorem of Pursell and Shanks stating that the Lie algebra of smooth vectors fields on a smooth manifold determines the diffeomorphism type of the manifold [22].

The result of Pursell and Shanks motivates us to investigate whether the new derivation Lie algebras $\mathcal{NL}(V)$ determines the analytic structure of the singularity $(V,0)$. In this paper, we have computed $\mathcal{NL}(V)$ for some natural classes of ICIS, in order to reveal the specific properties for simple and unimodal ICIS. In particular, we obtain more detailed information about $\mathcal{NL}(V)$ of unimodal ICIS. Thus our results about the computation of $\mathcal{NL}(V)$ that arising from unimodal ICIS can be considered as an extension of those presented in [7].

On the other hand, Griffiths has studied the Torelli problem when a family of complex projective hypersurfaces in $\mathbb{CP}^n$ is given and his school asks whether the period map is injective on that family, i.e., whether the family of complex hypersurfaces can be distinguished by means of their Hodge structures. A complex projective hypersurface in $\mathbb{CP}^n$ can be viewed as a complex hypersurface with an isolated singularity in $\mathbb{C}^{n+1}$. Let $V=\{z\in\mathbb{C}^{n+1}: f(z)=0\}$ be a complex hypersurface with an isolated singularity at the origin. Seeley and Yau investigated the family of isolated complex hypersurface singularities using Yau algebras and obtained two strong Torelli-type theorems for simple elliptic singularities $\tilde{E}_7$ and $\tilde{E}_8$ (cf. [23]). A question arises naturally whether the complex structures of a family of ICIS V_t can be distinguished by means of the corresponding new Lie algebras $\mathcal{NL}(V_t)$. Unfortunately, the methods in [23] cannot be generalized to $\mathcal{NL}(V)$ directly. We shall use a completely new method to prove the weak Torelli-type theorem of $\mathcal{NL}(V)$ for unimodal ICIS T_{10}, R_9, U_{11}, V_{10}, Y_{11} and M_{11}.

The study of Torelli-type theorems of singularities has an important application in classification theory of Lie algebras [4]. Recall that finite dimensional Lie algebras are semi-direct product of the semi-simple Lie algebras and solvable Lie algebras. Brieskorn [2] gave the connection between simple Lie algebras and simple singularities. Simple Lie algebras have been well understood, but not the solvable (nilpotent) Lie algebras. It is extremely important to establish connections between singularities and solvable (nilpotent) Lie algebras. In this article, a new natural connection between the set of complex analytic ICIS and the set of finite dimensional solvable (nilpotent) Lie algebras have been constructed. We construct finite dimensional solvable (nilpotent) Lie algebras naturally from ICIS. These constructions help us to understand the solvable (nilpotent) Lie algebras from the geometric point of view. Moreover, it is known that the classification of nilpotent Lie algebras in higher dimensions (> 7) remains to be an open area. There are one-parameter families of non-isomorphic nilpotent Lie algebras (but no two-parameter families) in dimension seven. Dimension seven is the watershed of the existence of such families. It is well-known that no such family exists in dimension less than seven, while it is hard to construct one-parameter family in dimension greater than seven. In this article, we are able to construct six one-parameter families of solvable Lie algebras of dimensions 11, 12, 15, 16, 14, and 17 from T_{10}, R_9, U_{11}, V_{10}, Y_{11} and M_{11}, and show that the weak Torelli-type theorem holds. This shed a light on the construction and classification of families of higher dimensional solvable (nilpotent) Lie algebras.

Theorem B. *The new derivation Lie algebras $\mathcal{NL}(V)$ of T_{10}, R_9, U_{11}, V_{10}, Y_{11}, and M_{11} are non-trivial one-parameter families. Thus the weak Torelli-type theorems hold for these one-parameter families of singularities T_{10}, R_9, U_{11}, V_{10}, Y_{11}, and M_{11}.*

2. Preliminaries

2.1. Isolated hypersurface singularities

Let $\mathcal{O}_n$ be the algebra of germs of holomorphic functions at the origin of $\mathbb{C}^n$. Obviously, $\mathcal{O}_n$ can be naturally identified with the algebra of convergent power series in n indeterminates with complex coefficients. For $f \in \mathcal{O}_n$, we denote by $V = V(f)$ (or $(V, 0)$) the germ at the origin of $\mathbb{C}^n$ of hypersurface $\{f = 0\} \subset \mathbb{C}^n$. We say that V is a germ of isolated hypersurface singularity if the origin is an isolated zero of the gradient of f. The local (function) algebra of V is defined as the (commutative associative) algebra $F(V) \cong \mathcal{O}_n/(f)$, where (f) is the principal ideal generated by the germ of f at the origin. According to Hilbert's Nullstellensatz for an isolated singularity $V = V(f) = \{f = 0\}$ the factor-algebra $A(V) = \mathcal{O}_n/(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$ is finite dimensional. This factor-algebra is called the moduli algebra of V and its dimension $\tau(V)$ is called Tjurina number. The well-known Mather-Yau theorem states that

Theorem 2.1. *[20] The analytic isomorphism type of an isolated hypersurface singularity is determined by the isomorphism class of its moduli algebras i.e.,*

$$(V_1, 0) \cong (V_2, 0) \Longleftrightarrow A(V_1) \cong A(V_2).$$

Definition 2.2. The new derivation Lie algebras $\mathcal{NL}(X)$ which is defined to be the Lie algebra of derivations of the local Artinian algebra $\mathcal{O}_n/SI_X$, i.e., $\mathcal{NL}(X) = Der(\mathcal{O}_n/SI_X)$. Its dimension is denoted by $\upsilon(V)$. This number $\upsilon(V)$ is also a new numerical analytic invariant.

2.2. Cohomology of Lie algebras

[19] There exists a general study of the cohomology of Lie algebra g by considering the cohomology with values on a g-module.

Let L be a Lie algebra. A p-dimensional cochain of L (with values in L) is a p-linear alternating mapping of L^p in L ($p \in \mathbb{N}^*$). A 0-cochain is a constant function from L to L.

We denote by $C^p(L, L)$ as the space of the p-cochains and

$$C^*(L, L) = \oplus_{p \geq 0} C^p(L, L).$$

We can provide $C^p(L, L)$ of a L-module structure by putting

$$(x\Phi)(x_1, \cdots, x_p) = [x, \Phi(x_1, \cdots, x_p)] - \sum_{1 \leq i \leq p} \Phi(x_1, \cdots, [x, x_i], \cdots, x_p)$$

for all $x_1, \cdots, x_p \in L$.

On the space $C^*(L, L)$ we define the endomorphism

$$\delta : C^*(L, L) \to C^*(L, L)$$
$$\Phi \to \delta\Phi$$

by putting

$$\begin{aligned}\delta\Phi(x) &= x\Phi \text{ if } \Phi \in C^0(L,L)\\ \delta\Phi(x_1,\cdots,x_{p+1}) &= \sum_{1\le s\le p+1}(-1)^{s+1}(x_s\Phi)(x_1,\cdots\hat{x}_s,\cdots,x_{p+1})+\\ &\quad+\sum_{1\le s\le t\le p+1}(-1)^{s+t}\Phi([x_s,x_t],x_1,\cdots,.\hat{x}_s,\cdots,\hat{x}_s,\cdots,x_{p+1})\\ &\quad \text{if } \Phi\in C^p(L,L),\, p\ge 1.\end{aligned}$$

By this definition, $\delta(C^p(L,L)) \subset C^{p+1}(L,L)$ and we can verify that

$$\delta\circ\delta = 0.$$

We denote by

$$\begin{cases} Z^p(L,L) = \mathrm{Ker}\delta|_{C^p(L,L)} & p\ge 1\\ B^p(L,L) = \mathrm{Im}\delta|_{C^p(L,L)} & p \ge 1\end{cases}$$

and $H^p(L,L) = Z^p(L,L)/B^p(L,L)$, $p \ge 1$.

This quotient space is called the cohomology space of L of degree p (with values in L). For $p = 0$, then we put $B^0(L,L) = \{0\}$ and $H^0(L,L) = Z^0(L,L)$. This last space can be identified to the space of all L-invariant elements that is

$$\{x \in L \text{ such that } \mathrm{ad}\,y(x) = 0,\ \forall\ y \in L\}.$$

Then $Z^0(L,L) = C_L$ (the center of L).

Definition 2.3. A derivation f of a Lie algebra L is a linear mapping

$$f : L \to L$$

satisfying

$$[f(x),y] + [x,f(y)] - f[x,y] = 0, \quad \forall (x,y) \in L^2.$$

We denote by Der$L$ the set of derivations of L.

For all x in L, the endomorphism ad x is a derivation of L. The derivations f of L which are of type $f = \mathrm{ad}x$ for $x \in L$ are called inner derivations.

We have

$$Z^1(L,L) = \{f : L \to L \mid \delta f = 0\}.$$

But $\delta f(x,y) = [f(x),y] + [x,f(y)] - f[x,y]$. Then $Z^1(L,L)$ is nothing but the algebra of derivation of L:

$$Z^1(L,L) = \mathrm{Der}L.$$

It is the same for:

$$B^1(L,L) = \{adx, x \in L\}.$$

Thus the space $H^1(L,L)$ can be interpreted as the set of the outer derivations of the Lie algebra L.
The main tool of our proof in theorem B is the theory of deformation and cohomology of Lie algebra.

2.3. Deformation of Lie algebras

[21] Let $L = (V,\mu)$ be a finite dimension Lie algebra where μ is a Lie algebra multiplication and V is the based vector space. Let $C^n(L,L)$ be the vector space of all alternating n-linear maps of V into itself and the coboundary operator is $\delta : C^n(L,L) \to C^{n+1}(L,L)$. Note that $Z^n(L,L) = \mathrm{Kernel}(\delta : C^n(L,L) \to C^{n+1}(L,L))$, $B^n(L,L) = \mathrm{Image}(\delta : C^{n-1}(L,L) \to C^n(L,L))$, $B^n(L,L) \subseteq Z^n(L,L)$ and $H^n(L,L) = Z^n(L,L)/B^n(L,L)$.

Let $L = (V,\mu)$ be a Lie algebra and $\varphi \in C^2(L,L)$ be an alternating bilinear map of V into itself. Then $\mu' = \mu + \varphi$ is also a Lie algebra multiplication if and only if it satisfies the Jacobian identity

$$\mu'(x,\mu'(y,z)) + \mu'(y,\mu'(z,x)) + \mu'(z,\mu'(x,y)) = 0 \tag{2.1}$$

for any $x,y,z \in V$. It can be shown that (2.1) holds if and only if

$$\delta\varphi - [\varphi, \varphi]/2 = 0 \tag{2.2}$$

where $[\varphi, \varphi]$ is defined as follows:

$$[\varphi, \varphi](x, y, z) = 2\varphi(\varphi(x, y), z) + 2\varphi(\varphi(y, z), x) + 2\varphi(\varphi(z, x), y).$$

(2.2) is called the deformation equation. Let $\mu_t = \mu + t\varphi_1 + t^2\varphi_2 + \cdots$ be an one-parameter family of Lie algebra multiplications on V, where $\varphi_i \in C^2(L, L)$. Then $t\varphi_1 + t^2\varphi_2 + \cdots$ satisfies the deformation equation, which implies that $\delta\varphi_1 = 0$. Hence $\varphi_1 \in Z^2(L, L)$, and we call φ_1 an infinitesimal deformation of μ.

A one-parameter family of Lie algebra multiplications $\mu_t = \mu + t\varphi_1 + t^2\varphi_2 + \cdots$ is said to be trivial if $(V, \mu_t) \simeq (V, \mu_s)$ for any s, t. Then there exists an one-parameter family of invertible linear maps $I_t = I + t\alpha_1 + t^2\alpha_2 + \cdots$, where $\alpha_i \in C^1(L, L)$ is linear from V into itself and I is the identity map, such that

$$\mu_t(x, y) = I_t\mu((I_t)^{-1}x, (I_t)^{-1}y) \tag{2.3}$$

for any $x, y \in V$. It's easy to verify that (2.3) implies that $\varphi_1 = -\delta\alpha_1$. Hence $\varphi_1 \in B^2(L, L)$, and we call φ_1 a trivial infinitesimal deformation.

3. Proof of theorems

To prove the main theorems we need to prove following propositions.

Proposition 3.1. *Let $V = \{x, y \in \mathbb{C}^2 : (x^a + y^b, x^a + 2y^b), a, b \geq 2\}$ be the complete intersection curve singularity. Then*

$$\upsilon(V) = 2ab - (a + b).$$

Proof. It is easy to see that moduli algebra $\mathbb{C}\{x, y\}/(f, g, M_1)$ has dimension $ab - 1$ and has a monomial basis of the form

$$\{x^i y^j; 0 \leq i \leq a-2, 0 \leq j \leq b-1; x^{a-1}y^j; 0 \leq j \leq b-2\}.$$

After simple calculation the Lie algebra $\mathcal{NL}(V)$ have the following bases:

$$x^i y^j \partial_1,\ 1 \leq i \leq a-2, 0 \leq j \leq b-1; x^{a-1}y^j\partial_1,\ 0 \leq j \leq b-2; y^{b-1}\partial_1;$$
$$x^i y^j \partial_2,\ 0 \leq i \leq a-2, 1 \leq j \leq b-1; x^{a-1}y^j\partial_2,\ 0 \leq j \leq b-2.$$

Therefore we have the following formula

$$\upsilon(V) = 2ab - (a + b). \quad \square$$

Proposition 3.2. *Let $V = \{x, y, z \in \mathbb{C}^3 : (x^a + y^b + z^c, x^a + 2y^b + 3z^c), a, b, c \geq 2\}$ be the complete intersection curve singularity. Then*

$$\upsilon(V) = 4abc - 2(ab + ac + bc) - (a + b + c) + 5.$$

Proof. It is easy to see that moduli algebra $\mathbb{C}\{x, y, z\}/(f, g, M_1, M_2, M_3)$ has dimension $2abc - (ab + ac + bc) + 1$ and has a monomial basis of the form

$$\{x^i y^j z^k; 0 \leq i \leq a-2, 0 \leq j \leq b-2, 0 \leq k \leq 2c-2; x^i y^{b-1} z^k; 0 \leq i \leq a-2, 0 \leq k \leq c-2;$$
$$x^{a-1} y^j z^k; 0 \leq j \leq b-2, 0 \leq k \leq c-2\}.$$

After simple calculation the Lie algebra $\mathcal{NL}(V)$ have the following bases:

$$x^i y^j z^k \partial_1 + x^{i-1} y^{j+1} z^k \partial_2 + x^{i-1} y^j z^{k+1} \partial_3,\ 1 \leq i \leq a-1, 0 \leq j \leq b-2, 0 \leq k \leq c-2;$$
$$x^i y^j z^{c-1} \partial_1,\ 0 \leq i \leq a-2, 0 \leq j \leq b-2; x^i y^j z^k \partial_1,\ 0 \leq i \leq a-2, 0 \leq j \leq b-2,$$
$$c \leq k \leq 2c-2; x^i y^{b-1} z^k \partial_1,\ 0 \leq i \leq a-2, 0 \leq k \leq c-2; x^i y^j z^k \partial_2,\ 0 \leq i \leq a-2,$$
$$0 \leq j \leq b-2, c-1 \leq k \leq 2c-1; x^{a-1} y^j z^k \partial_2,\ 1 \leq j \leq b-2, 0 \leq k \leq c-2; x^i y^j z^k \partial_3,$$
$$0 \leq i \leq a-2, 0 \leq j \leq b-2, c \leq k \leq 2c-2; x^i y^{b-1} z^k \partial_3,\ 0 \leq i \leq a-2, 1 \leq k \leq c-2;$$
$$x^{a-1} y^j z^k \partial_3,\ 0 \leq j \leq b-2, 1 \leq k \leq c-2.$$

Therefore we have the following formula

$$\upsilon(V) = 4abc - 2(ab + ac + bc) - (a + b + c) + 5. \quad \square$$

Proof of Theorem A. After simple calculation and Proposition 3.1 and Proposition 3.2, we have the following table:

$Type$(V)	Equations	$\upsilon(V)$	Weights; Degress	Brieskorn (W)	$\upsilon(W)$
$F^{q,r}_{q+r-1}$	$\{(xy,\ x^q+y^r),\ q,r\geq 2\}$	$q+r$	$r,q;q+r,qr$	(x^q+y^r,x^q+2y^r)	$2qr-(q+r)$
G_5	$(x^2,\ y^3)$	7	1, 1; 2, 3	(x^2+y^3,x^2+2y^3)	7
G_7	$(x^2,\ y^4)$	10	1, 1; 2, 4	(x^2+y^4,x^2+2y^4)	10
H_6	$(x^2+y^3,\ xy^2)$	8	3, 2; 6, 7	(x^3+y^4,x^3+2y^4)	17
H_μ	$\{(x^2+y^{\mu-3},\ xy^2),\ \mu\geq 7\}$	$\mu+2$	$\mu-3,2;2\mu-6,\mu+1$	$(x^2+y^{\mu-3},x^2+2y^{\mu-3})$	$3\mu-11$
I_{2q-1}	$\{(x^2+y^3,\ y^q),\ q\geq 4\}$	$2q+1$	$3,2;6,2q$	(x^q+y^q,x^q+2y^q)	$2q(q-1)$
I_{2r+2}	$\{(x^2+y^3,\ xy^r),\ r\geq 3\}$	$2r+4$	$3,2;6,3+2r$	$(x^r+y^{r+2},x^r+2y^{r+2})$	$2r(r+2)-(2r+2)$
S_μ	$\{(x^2+y^2+z^{\mu-3},\ yz),\ \mu\geq 5\}$	$\mu+2$	$\mu-3,\mu-3,2;2\mu-6,\mu-1$	$(x^2+y^2+z^{\mu-3},x^2+2y^2+3z^{\mu-3})$	$7(\mu-4)$
T_7	$(x^2+y^3+z^3,\ yz)$	9	3, 2, 2; 6, 4	$(x^2+y^3+z^3,x^2+2y^3+3z^3)$	27
T_8	$(x^2+y^3+z^4,\ yz)$	10	6, 4, 3; 12, 7	$(x^2+y^3+z^4,x^2+2y^3+3z^4)$	40
T_9	$(x^2+y^3+z^5,\ yz)$	11	15, 10, 6; 30, 16	$(x^2+y^3+z^5,x^2+2y^3+3z^5)$	53
U_7	$(x^2+yz,\ xy+z^3)$	10	4, 5, 3; 8, 9	$(x^3+y^2+z^3,x^3+2y^2+3z^3)$	27
U_8	$(x^2+yz+z^3,\ xy)$	11	3, 4, 2; 6, 7	$(x^3+y^2+z^4,x^3+2y^2+3z^4)$	40
U_9	$(x^2+yz,\ xy+z^4)$	13	5, 7, 3; 10, 12	$(x^3+y^2+z^4,x^3+2y^2+3z^4)$	40
W_8	$(x^2+z^3,\ y^2+xz)$	12	6, 5, 4; 12, 10	$(x^2+y^3+z^3,x^2+2y^3+3z^3)$	27
W_9	$(x^2+yz^2,\ y^2+xz)$	13	5, 4, 3; 10, 8	$(x^2+y^3+z^4,x^2+2y^3+3z^4)$	40
Z_9	$(x^2+z^3,\ y^2+z^3)$	14	3, 3, 2; 6, 6	$(x^2+y^2+z^3,x^2+2y^2+3z^3)$	14
Z_{10}	$(x^2+yz^2,\ y^2+z^3)$	15	7, 6, 4; 14, 12	$(x^2+y^3+z^4,x^2+2y^3+3z^4)$	40

After simple calculation and Proposition 3.2 we have the following table:

$Type$(V)	Equations	$\upsilon(V)$	Weights; Degress	Brieskorn (W)	$\upsilon(W)$
T_{10}	$\{(x^2+y^3+z^6,\ ax+yz),\ 27a^6+4\neq 0\}$	12	3, 2, 1; 6, 3	$(x^2+y^3+z^6,x^2+2y^3+3z^6)$	66
T_k	$\{(x^2+y^3+z^{k-4},\ yz),\ k\geq 11\}$	$k+2$	$3k-12,2k-8,6;6k-24,2k-2$	$(x^2+y^3+z^{k-4},x^2+2y^3+3z^{k-4})$	$13k-64$
R_9	$\{(x^2+y^4+z^4,\ ax+yz),\ 4a^4-1\neq 0\}$	11	2, 1, 1; 4, 2	$(x^2+y^4+z^4,x^2+2y^4+3z^4)$	59
R_k	$\{(x^2+y^4+z^{k-5},\ yz),\ k\geq 10\}$	$k+2$	$2k-10,k-5,4;4k-20,k-1$	$(x^2+y^4+z^{k-5},x^2+2y^4+3z^{k-5})$	$19k-112$
$L_{2,q,r}$	$\{(x^2+y^q+z^r,\ yz),\ q,r\geq 5\}$	$q+r+3$	$qr,2r,2q;2qr,2(q+r)$	$(x^2+y^q+z^r,x^2+2y^q+3z^r)$	$6qr-5(q+r)+3$
U_{11}	$\{(x^2+yz+z^4+axz^2,\ xy),\ a^2-4\neq 0\}$	15	2, 3, 1; 4, 5	$(x^3+y^2+z^5,x^3+2y^2+3z^5)$	53
U_{13}	$(x^2+yz,\ xy+z^6)$	19	7, 11, 3; 14, 18	$(x^3+y^2+z^6,x^3+2y^2+3z^6)$	66
U_{14}	$(x^2+z^5+yz,\ xy)$	19	5, 8, 2; 10, 13	$(x^3+y^2+z^7,x^3+2y^2+3z^7)$	79
U_{15}	$(x^2+yz,\ xy+z^7)$	21	8, 13, 3; 16, 21	$(x^3+y^2+z^7,x^3+2y^2+3z^7)$	79
V_{10}	$\{(x^3+yz+z^3+ax^2z,\ xy),\ 4a^3+27\neq 0\}$	14	1, 2, 1; 3, 3	$(x^3+y^2+z^3,x^3+2y^2+3z^3)$	27
V_{12}	$(x^4+yz+z^3,\ xy)$	17	3, 8, 4; 12, 11	$(x^4+y^2+z^3,x^4+2y^2+3z^3)$	40
V_{13}	$(yz+z^3,\ xy+x^4)$	19	2, 6, 3; 9, 8	$(x^5+y^2+z^3,x^5+2y^2+3z^3)$	53
V_{11}	$(x^5+yz+z^3,\ xy)$	20	3, 10, 5; 15, 13	$(x^5+y^2+z^3,x^5+2y^2+3z^3)$	53
Q_{13}	$(x^3+yz,\ xy+z^4)$	20	5, 11, 4; 15, 16	$(x^4+y^2+z^4,x^4+2y^2+3z^4)$	59
Q_{11}	$(x^3+yz+z^4,\ xy)$	20	4, 9, 3; 12, 13	$(x^4+y^2+z^5,x^4+2y^2+3z^5)$	78
$L_{3,2,4}$	$(x^4+y^2+z^3,\ xy)$	20	3, 6, 4; 12, 9	$(x^4+y^2+z^3,x^4+2y^2+3z^3)$	40
$L_{3,2,5}$	$(x^5+y^2+z^3,\ xy)$	23	6, 15, 10; 30, 21	$(x^5+y^2+z^3,x^5+2y^2+3z^3)$	53
Y_{11}	$\{(x^2+y^3+z^3+ay^2z,\ xy),\ 4a^3+27\neq 0\}$	17	3, 2, 2; 6, 5	$(x^2+y^3+z^3,x^2+2y^3+3z^3)$	27
G_{14}	$(x^2+y^3z+z^3,\ xy)$	21	9, 4, 6; 18, 13	$(x^2+y^5+z^3,x^2+2y^5+3z^3)$	53
H_{13}	$(x^2+y^2z,\ xy+z^3)$	20	7, 5, 4; 14, 12	$(x^2+y^3+z^4,x^2+2y^3+3z^4)$	40
H_{14}	$(x^2+y^3,\ xy+z^3)$	23	9, 6, 5; 18, 15	$(x^2+y^3+z^4,x^2+2y^3+3z^4)$	40
M_{11}	$\{(x^2+z^4,\ y^2+z^3+axz),\ a^2+1\neq 0\}$	16	4, 3, 2; 8, 6	$(x^2+y^3+z^4,x^2+2y^3+3z^4)$	40
M_{12}	$(x^2+yz^3,\ y^2+z^3)$	17	9, 6, 4; 18, 12	$(x^2+y^3+z^5,x^2+2y^3+3z^5)$	53
M_{13}	$(x^2+z^5,\ y^2+z^3)$	18	5, 3, 2; 10, 6	$(x^2+y^4+z^5,x^2+2y^4+3z^5)$	78
M_{14}	$(x^2+yz^4,\ y^2+z^3)$	19	11, 6, 4; 22, 12	$(x^2+y^4+z^6,x^2+2y^4+3z^6)$	97
N_{13}	$(x^2+yz^3,\ y^2+xz)$	19	7, 5, 3; 14, 10	$(x^2+y^3+z^5,x^2+2y^3+3z^5)$	53
N_{14}	$(x^2+z^5,\ y^2+xz)$	22	10, 7, 4; 20, 14	$(x^2+y^3+z^5,x^2+2y^3+3z^5)$	53

From above tables it is easy to see that 1)- 3) satisfy the conjecture $\upsilon(V)\leq\upsilon(W)$.

Proof of Theorem B. The unimodal complete intersection singularity R_9 is defined by $\{(x^2+y^4+z^4,\ tx+yz),\ 4t^4-1\neq 0\}$. The moduli algebra have the following bases

$$\mathbb{C}\{x,y,z\}/(f,g,M_1,M_2,M_3)=<1,x,y,z,z^2,z^3,z^4,y^2,y^3>.$$

By calculation, a basis for Lie algebra $\mathcal{NL}(R_9)$ is the following (for $4t^4-1\neq 0$):

$$e_1=\frac{2x\partial_1}{3}+\frac{y\partial_2}{3}+\frac{z\partial_3}{3},\quad e_2=2tz^3\partial_1+\frac{4z^2\partial_2t^2}{3\left(-1+4t^4\right)}-\frac{y^2\partial_2\left(1-12t^4\right)}{3\left(-1+4t^4\right)},\quad e_3=\frac{y^3\partial_2}{2}$$
$$e_4=\frac{y^3\partial_1}{2t}+\frac{x\partial_2}{4t}-\frac{y^2\partial_3}{1-4t^4}+\frac{2z^2\partial_3t^2}{-1+4t^4},\quad e_5=2y^3\partial_1t+\frac{4y^2\partial_3t^2}{3\left(-1+4t^4\right)}-\frac{z^2\partial_3\left(1-12t^4\right)}{3\left(-1+4t^4\right)},$$

$$e_6 = \frac{z^3\partial_1}{2t} + \frac{x\partial_3}{4t} - \frac{z^2\partial_2}{1-4t^4} + \frac{2y^2\partial_2 t^2}{-1+4t^4}, \quad e_7 = \frac{z^3\partial_3}{2}, \quad e_8 = y^3\partial_3 - \frac{z^4\partial_1}{t}, \quad e_9 = z^4\partial_3,$$
$$e_{10} = z^4\partial_2, \quad e_{11} = z^3\partial_2 - \frac{z^4\partial_1}{t}.$$

Now we need to prove $\mathcal{NL}(R_9)_t$ $(4t^4-1 \neq 0)$ is not a trivial family. By calculation, the multiplication table of $\mathcal{NL}(R_9)_t$ is given by:

$$[e_1,e_2] = \frac{e_2}{3}, \quad [e_1,e_3] = \frac{2e_3}{3}, \quad [e_1,e_4] = \frac{e_4}{3}, \quad [e_1,e_5] = \frac{e_5}{3}, \quad [e_1,e_6] = \frac{e_6}{3},$$
$$[e_1,e_7] = \frac{2e_7}{3} \quad [e_1,e_8] = \frac{2e_8}{3}, \quad [e_1,e_9] = e_9, \quad [e_1,e_{10}] = e_{10} \quad [e_1,e_{11}] = \frac{2e_{11}}{3}$$
$$[e_2,e_3] = -\frac{e_{10}\left(1+12t^4\right)}{6\left(-1+4t^4\right)}, \quad [e_2,e_4] = \frac{2e_8}{3\left(-1+4t^4\right)} - \frac{e_{11}\left(1+12t^4\right)}{6\left(-1+4t^4\right)},$$
$$[e_2,e_5] = -\frac{8e_{11}t^2}{9\left(-1+4t^4\right)} + \frac{8e_8t^2}{9\left(-1+4t^4\right)}, \quad [e_2,e_6] = e_7, \quad [e_2,e_7] = -\frac{4e_{10}t^2}{3\left(-1+4t^4\right)},$$
$$[e_2,e_8] = e_9, \quad [e_3,e_4] = -\frac{e_9}{1-4t^4}, \quad [e_3,e_5] = \frac{4e_9t^2}{3\left(-1+4t^4\right)}, \quad [e_3,e_6] = \frac{2e_{10}t^2}{-1+4t^4},$$
$$[e_4,e_5] = -e_3, \quad [e_4,e_6] = -\frac{e_{11}\left(-1-12t^4\right)}{8t^2\left(-1+4t^4\right)} - \frac{e_8\left(1+12t^4\right)}{8t^2\left(-1+4t^4\right)}, \quad [e_4,e_7] = -\frac{2e_9t^2}{-1+4t^4}$$
$$[e_4,e_8] = \frac{e_{10}}{4t^2}, \quad [e_4,e_{11}] = \frac{e_{10}}{4t^2}, \quad [e_5,e_6] = \frac{2e_{11}}{3\left(-1+4t^4\right)} - \frac{e_8\left(1+12t^4\right)}{6\left(-1+4t^4\right)},$$
$$[e_5,e_7] = -\frac{e_9\left(1+12t^4\right)}{6\left(-1+4t^4\right)}, \quad [e_5,e_{11}] = e_{10}, \quad [e_6,e_7] = -\frac{e_{10}}{-1+4t^4}, \quad [e_6,e_8] = \frac{e_9}{4t^2}$$
$$[e_6,e_{11}] = \frac{e_9}{4t^2}.$$

Other Lie brackets $[e_i,e_j]$ are 0. Let $V = < e_1, \cdots, e_{11} >$ be the based vector space of $\mathcal{NL}(R_9)_t$ and μ_t be the Lie algebra multiplication of $\mathcal{NL}(R_9)_t$. Then we can write

$$\mu_t = \mu_0 + (t-2)\varphi_1 + (t-2)^2\varphi_2 + \cdots$$

where μ_0 is matrix which have following 11 rows and columns:

$$\left\{\left\{0, \frac{e_2}{3}, \frac{2e_3}{3}, \frac{e_4}{3}, \frac{e_5}{3}, \frac{e_6}{3}, \frac{2e_7}{3}, \frac{2e_8}{3}, e_9, e_{10}, \frac{2e_{11}}{3}\right\},\right.$$
$$\left\{-\frac{e_2}{3}, 0, -\frac{193e_{10}}{378}, -\frac{193e_{11}}{378} + \frac{2e_8}{189}, -\frac{32e_{11}}{567} + \frac{32e_8}{567}, e_7, -\frac{16e_{10}}{189}, +e_9, 0, 0, 0\right\},$$
$$\left\{-\frac{2e_3}{3}, \frac{193e_{10}}{378}, 0, \frac{e_9}{63}, \frac{16e_9}{189}, \frac{8e_{10}}{63}, 0, 0, 0, 0, 0\right\},$$
$$\left\{-\frac{e_4}{3}, \frac{193e_{11}}{378} - \frac{2e_8}{189}, -\frac{e_9}{63}, 0, -e_3, \frac{193e_{11}}{2016} - \frac{193e_8}{2016}, -\frac{8e_9}{63}, \frac{e_{10}}{16}, 0, 0, \frac{e_{10}}{16}\right\},$$
$$\left\{-\frac{e_5}{3}, \frac{32e_{11}}{567} - \frac{32e_8}{567}, -\frac{16e_9}{189}, e_3, 0, \frac{2e_{11}}{189} - \frac{193e_8}{378}, -\frac{193e_9}{378}, 0, 0, 0, e_{10}\right\},$$
$$\left\{-\frac{e_6}{3}, -e_7, -\frac{8e_{10}}{63}, -\frac{193e_{11}}{2016} + \frac{193e_8}{2016}, -\frac{2e_{11}}{189} + \frac{193e_8}{378}, 0, -\frac{e_{10}}{63}, \frac{e_9}{16}, 0, 0, \frac{e_9}{16}\right\},$$
$$\left\{-\frac{2e_7}{3}, \frac{16e_{10}}{189}, 0, \frac{8e_9}{63}, \frac{193e_9}{378}, \frac{e_{10}}{63}, 0, 0, 0, 0, 0\right\},$$
$$\left\{-\frac{2e_8}{3}, -e_9, 0, -\frac{e_{10}}{16}, 0, -\frac{e_9}{16}, 0, 0, 0, 0, 0\right\},$$
$$\{-e_9, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\},$$
$$\{-e_{10}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$$

$$\left\{-\frac{2e_{11}}{3}, 0, 0, -\frac{e_{10}}{16}, -e_{10}, -\frac{e_9}{16}, 0, 0, 0, 0, 0\right\}\Bigg\}.$$

And $\varphi_1 : V \times V \to V$ is given by

$$\{\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\},$$
$$\left\{0, 0, \frac{256e_{10}}{11907}, \frac{256e_{11}}{11907} - \frac{256e_8}{11907}, \frac{2080e_{11}}{35721} - \frac{2080e_8}{35721}, 0, \frac{1040e_{10}}{11907}, 0, 0, 0, 0\right\},$$
$$\left\{0, -\frac{256e_{10}}{11907}, 0, -\frac{128e_9}{3969}, -\frac{1040e_9}{11907}, -\frac{520e_{10}}{3969}, 0, 0, 0, 0, 0\right\},$$
$$\left\{0, -\frac{256e_{11}}{11907} + \frac{256e_8}{11907}, \frac{128e_9}{3969}, 0, 0, -\frac{12671e_{11}}{127008} + \frac{12671e_8}{127008}, \frac{520e_9}{3969}, -\frac{e_{10}}{16}, 0, 0, -\frac{e_{10}}{16}\right\},$$
$$\left\{0, -\frac{2080e_{11}}{35721} + \frac{2080e_8}{35721}, \frac{1040e_9}{11907}, 0, 0, -\frac{256e_{11}}{11907} + \frac{256e_8}{11907}, \frac{256e_9}{11907}, 0, 0, 0, 0\right\},$$
$$\left\{0, 0, \frac{520e_{10}}{3969}, \frac{12671e_{11}}{127008} - \frac{12671e_8}{127008}, \frac{256e_{11}}{11907} - \frac{256e_8}{11907}, 0, \frac{128e_{10}}{3969}, -\frac{e_9}{16}, 0, 0, -\frac{e_9}{16}\right\},$$
$$\left\{0, -\frac{1040e_{10}}{11907}, 0, -\frac{520e_9}{3969}, -\frac{256e_9}{11907}, -\frac{128e_{10}}{3969}, 0, 0, 0, 0, 0\right\},$$
$$\left\{0, 0, 0, \frac{e_{10}}{16}, 0, \frac{e_9}{16}, 0, 0, 0, 0, 0\right\},$$
$$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\},$$
$$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\},$$
$$\left\{0, 0, 0, \frac{e_{10}}{16}, 0, \frac{e_9}{16}, 0, 0, 0, 0, 0\right\}\Bigg\}.$$

Now we only need to check that φ_1 is not a trivial infinitesimal deformation, (i.e. $\varphi_1 \notin B^2(\mathcal{NL}(R_9)_2, \mathcal{NL}(R_9)_2)$ where $\mathcal{NL}(R_9)_2 = (V, \mu_0)$), then we can say $\mathcal{NL}(R_9)_t$ is not a trivial family. Write $\mu_0(e_i, e_j) = \sum_{s=1}^{11} u_{ij}^s e_s$ and $\varphi_1(e_i, e_j) = \sum_{s=1}^{11} v_{ij}^s e_s$ for $i, j = 1, \ldots, 11$. If there exists a linear map $\alpha : V \to V$ such that $\delta\alpha = \varphi_1$, write $\alpha(e_i) = \sum_{j=1}^{11} a_{ij} e_j$, then we have

$$\begin{aligned}
\varphi_1(e_i, e_j) &= \delta\alpha(e_i, e_j) \\
&= \mu_0(e_i, \alpha(e_j)) - \mu_0(e_j, \alpha(e_i)) - \alpha(\mu_0(e_i, e_j)) \\
&= \mu_0(e_i, \sum_{k=1}^{11} a_{jk} e_k) - \mu_0(e_j, \sum_{k=1}^{11} a_{ik} e_k) - \alpha(\sum_{k=1}^{11} u_{ij}^k e_k) \\
&= \sum_{s=1}^{11} \sum_{k=1}^{11} (a_{jk} u_{ik}^s - a_{ik} u_{jk}^s - u_{ij}^k a_{ks}) e_s.
\end{aligned}$$

Hence

$$\sum_{k=1}^{n} (a_{jk} u_{ik}^s - a_{ik} u_{jk}^s - u_{ij}^k a_{ks}) = v_{ij}^s$$

for $i, j, s = 1, \ldots, 11$. Then we get 1331 linear equations about 121 variables a_{ij}. We solve this system of linear equations with the help of computer and find that they have no solution. Hence $\varphi_1 \notin B^2(\mathcal{NL}(R_9)_2, \mathcal{NL}(R_9)_2)$ and the family is not trivial.

Similarly we have checked that the unimodal complete intersection singularities T_{10}, U_{11}, V_{10}, Y_{11}, and M_{11} also satisfy the weak Torelli-type theorems. Due to the large computation, we just give only bases and multiplication tables of Lie algebra $\mathcal{NL}$ for the remaining five cases as follows.

(1). The unimodal complete intersection singularity T_{10} is defined by $\{(x^2 + y^3 + z^6,\ tx + yz),\ 27t^6 + 4 \neq 0\}$. The moduli algebra have the following basis

$$\mathbb{C}\{x, y, z\}/(f, g, M_1, M_2, M_3) = < 1, x, y, z, z^2, z^3, z^4, z^5, z^6, y^2 > .$$

By calculation, a basis for Lie algebra $\mathcal{NL}(T_{10})$ is the following (for $27t^6+4\neq 0$):

$$e_1=\frac{3x\partial_1}{4}+\frac{y\partial_2}{2}+\frac{z\partial_3}{4},\quad e_2=\frac{3z^5\partial_1 t}{2}-\frac{3z^4\partial_2 t^2}{4+27t^6}-\frac{y^2\partial_2\left(-2-27t^6\right)}{2\left(4+27t^6\right)},$$
$$e_3=-\frac{2y\partial_3}{4+27t^6}+\frac{15z^4\partial_1 t}{4+27t^6}-\frac{15z^3\partial_2 t^2}{4+27t^6}+\frac{9z^2\partial_3 t^4}{4+27t^6}-\frac{x\partial_2\left(-2-81t^6\right)}{3t\left(4+27t^6\right)}-\frac{y^2\partial_1\left(-1+27t^6\right)}{t\left(4+27t^6\right)},$$
$$e_4=-\frac{24y\partial_3 t^2}{5\left(4+27t^6\right)}+\frac{36z^4\partial_1 t^3}{4+27t^6}-\frac{36z^3\partial_2 t^4}{4+27t^6}+\frac{54x\partial_2 t^7}{4+27t^6}-\frac{z^2\partial_3\left(-4-135t^6\right)}{5\left(4+27t^6\right)},$$
$$-\frac{3y^2\partial_1\left(-4t+27t^7\right)}{2\left(4+27t^6\right)},\quad e_5=\frac{z^3\partial_3}{4}+\frac{9}{8}z^5\partial_1 t^3-\frac{9z^4\partial_2 t^4}{4+27t^6}+\frac{81y^2\partial_2 t^8}{4\left(4+27t^6\right)},$$
$$e_6=\frac{z^4\partial_3}{3}-\frac{3}{2}z^6\partial_1 t^3,\quad e_7=-2z^4\partial_2+\frac{x\partial_3}{3t}+\frac{3}{2}z^3\partial_3 t^2+\frac{9}{2}y^2\partial_2 t^4-\frac{z^5\partial_1\left(-4-27t^6\right)}{4t},$$
$$e_8=\frac{z^5\partial_3}{2},\quad e_9=y^2\partial_3+3z^4\partial_3 t^2-\frac{z^6\partial_1\left(4+27t^6\right)}{2t},\quad e_{10}=z^6\partial_3,\quad e_{11}=z^6\partial_2,$$
$$e_{12}=z^5\partial_2-\frac{z^6\partial_1}{t}.$$

By calculation, the multiplication table of $\mathcal{NL}(T_{10})_t$ is given by:

$$[e_1,e_2]=\frac{e_2}{2},\quad [e_1,e_3]=\frac{e_3}{4},\quad [e_1,e_4]=\frac{e_4}{4},\quad [e_1,e_5]=\frac{e_5}{2},$$
$$[e_1,e_6]=\frac{3e_6}{4},\quad [e_1,e_7]=\frac{e_7}{2},\quad [e_1,e_8]=e_8,\quad [e_1,e_9]=\frac{3e_9}{4},$$
$$[e_1,e_{10}]=\frac{5e_{10}}{4},\quad [e_1,e_{11}]=e_{11},\quad [e_1,e_{12}]=\frac{3e_{12}}{4},$$
$$[e_2,e_3]=-\frac{e_9\left(2+27t^6\right)}{\left(4+27t^6\right)^2}+\frac{9e_6t^2}{4+27t^6}+\frac{27e_{12}t^6}{4+27t^6},\quad [e_2,e_4]=\frac{108e_6t^4}{5\left(4+27t^6\right)}-\frac{12e_9\left(2t^2+27t^8\right)}{5\left(4+27t^6\right)^2}$$
$$+\frac{3e_{12}\left(4t^2+135t^8\right)}{5\left(4+27t^6\right)},\quad [e_2,e_5]=\frac{3e_{11}t^2}{4},\quad [e_2,e_7]=e_8+\frac{9e_{11}t^4}{2},\quad [e_2,e_9]=e_{10},$$
$$[e_3,e_4]=-e_2,\quad [e_3,e_5]=-\frac{27e_6t^4}{4+27t^6}-\frac{3e_{12}\left(-28t^2+27t^8\right)}{8\left(4+27t^6\right)}+\frac{9e_9\left(4t^2+45t^8\right)}{4\left(4+27t^6\right)^2},$$
$$[e_3,e_6]=-\frac{12e_8t^4}{4+27t^6}-\frac{e_{11}\left(-32t^2-81t^8\right)}{2\left(4+27t^6\right)},\quad [e_3,e_7]=-\frac{e_{12}\left(2+27t^6\right)}{12t^2}-\frac{e_9\left(-2-27t^6\right)}{6t^2\left(4+27t^6\right)},$$
$$[e_3,e_8]=-\frac{9e_{10}t^4}{4+27t^6},\quad [e_3,e_9]=2e_8-\frac{e_{11}\left(-2-81t^6\right)}{6t^2},\quad [e_3,e_{11}]=\frac{2e_{10}}{4+27t^6},$$
$$[e_3,e_{12}]=\frac{4e_8}{4+27t^6}-\frac{e_{11}\left(-2-81t^6\right)}{3t^2\left(4+27t^6\right)},\quad [e_4,e_5]=-\frac{3e_6\left(-4+405t^6\right)}{20\left(4+27t^6\right)}+\frac{27e_9\left(4t^4+45t^{10}\right)}{5\left(4+27t^6\right)^2}$$
$$-\frac{9e_{12}\left(-44t^4+135t^{10}\right)}{20\left(4+27t^6\right)},\quad [e_4,e_6]=-\frac{4e_8\left(-4+81t^6\right)}{15\left(4+27t^6\right)}+\frac{9e_{11}\left(4t^4+9t^{10}\right)}{4+27t^6},\quad [e_4,e_7]=\frac{3e_9\left(4+45t^6\right)}{10\left(4+27t^6\right)}$$
$$-\frac{1}{10}e_{12}\left(16+135t^6\right),\quad [e_4,e_8]=-\frac{3e_{10}\left(-4+45t^6\right)}{10\left(4+27t^6\right)},\quad [e_4,e_9]=\frac{24e_8t^2}{5}+27e_{11}t^6,$$
$$[e_4,e_{11}]=\frac{24e_{10}t^2}{5\left(4+27t^6\right)},\quad [e_4,e_{12}]=\frac{48e_8t^2}{5\left(4+27t^6\right)}-\frac{e_{11}\left(-4-81t^6\right)}{4+27t^6},\quad [e_5,e_6]=\frac{e_{10}}{12},$$
$$[e_5,e_7]=-2e_{11}+3e_8t^2,\quad [e_5,e_9]=3e_{10}t^2,\quad [e_6,e_7]=e_{10}t^2,$$
$$[e_7,e_9]=-\frac{e_{10}\left(-4-27t^6\right)}{6t^2},\quad [e_7,e_{12}]=\frac{e_{10}}{3t^2}.$$

(2). The unimodal complete intersection singularity U_{11} is defined by $\{(x^2+yz+z^4+txz^2\,,xy),\ t^2-4\neq 0\}$. The moduli algebra have the following basis

$$\mathbb{C}\{x,y,z\}/(f,g,M_1,M_2,M_3) = <1,x,y,z,z^2,z^3,z^4,z^5,z^6,xz,xz^2>.$$

By calculation, a basis for Lie algebra $\mathcal{NL}(U_{11})_t$ is the following (for $t^2-4\neq 0$):

$$e_1=\frac{x\partial_1}{2}+\frac{3y\partial_2}{4}+\frac{z\partial_3}{4},\quad e_2=\frac{20z^5\partial_2\left(-3+t^2\right)}{3t}-\frac{2z^3\partial_3\left(-3+t^2\right)}{t}-\frac{4y\partial_3\left(9-6t^2+t^4\right)}{t\left(-4+t^2\right)}$$
$$-\frac{2z^4\partial_1\left(-18+3t^2+t^4\right)}{9\left(-4+t^2\right)}-\frac{2xz^2\partial_1\left(9t-9t^3+2t^5\right)}{9\left(-4+t^2\right)},\quad e_3=\frac{162y\partial_1\left(-3+t^2\right)}{5\left(-108+27t^2+8t^4\right)}-\frac{36x\partial_3\left(-3+t^2\right)^2}{-108+27t^2+8t^4}$$
$$-\frac{2z^2\partial_3\left(-3+t^2\right)\left(9t+2t^3\right)}{-108+27t^2+8t^4}-\frac{6z^3\partial_1\left(324-123t^2+5t^4\right)}{5\left(-108+27t^2+8t^4\right)}+\frac{6z^4\partial_2\left(12t-7t^3+t^5\right)}{-108+27t^2+8t^4}$$
$$-\frac{4xz\partial_1\left(378t-201t^3+25t^5\right)}{5\left(-108+27t^2+8t^4\right)},\quad e_4=-\frac{24x\partial_3 t\left(-3+t^2\right)}{-108+27t^2+8t^4}-\frac{3z^2\partial_3\left(12+t^2\right)}{-108+27t^2+8t^4}$$
$$-\frac{3y\partial_1 t\left(324-117t^2+8t^4\right)}{20\left(-3+t^2\right)\left(-108+27t^2+8t^4\right)}-\frac{3z^3\partial_1 t\left(828-319t^2+16t^4\right)}{10\left(-3+t^2\right)\left(-108+27t^2+8t^4\right)}$$
$$-\frac{3z^4\partial_2\left(144-104t^2+17t^4\right)}{2\left(-3+t^2\right)\left(-108+27t^2+8t^4\right)}-\frac{3xz\partial_1\left(-1440+2148t^2-719t^4+56t^6\right)}{20\left(-3+t^2\right)\left(-108+27t^2+8t^4\right)},\quad e_5=z^5\partial_1,$$
$$e_6=\frac{2z^3\partial_3\left(-3+t^2\right)}{t}+\frac{8z^4\partial_1\left(-3+t^2\right)}{3\left(-4+t^2\right)}+\frac{4xz^2\partial_1 t\left(-3+t^2\right)}{3\left(-4+t^2\right)}-\frac{z^5\partial_2\left(-20+7t^2\right)}{t}+\frac{4y\partial_3\left(9-6t^2+t^4\right)}{t\left(-4+t^2\right)},$$
$$e_7=\frac{z^5\partial_2}{3}-\frac{y\partial_3\left(-3+t^2\right)}{2\left(-4+t^2\right)}-\frac{z^4\partial_1 t\left(-3+t^2\right)}{9\left(-4+t^2\right)}-\frac{2xz^2\partial_1\left(9-6t^2+t^4\right)}{9\left(-4+t^2\right)},\quad e_8=\frac{z^4\partial_3}{3}+\frac{z^5\partial_1 t}{6\left(-3+t^2\right)},$$
$$e_9=\frac{z^5\partial_3}{2},\quad e_{10}=xz^2\partial_3-\frac{3z^5\partial_1\left(-4+t^2\right)}{4\left(-3+t^2\right)^2}+\frac{z^4\partial_3 t}{2\left(-3+t^2\right)},\quad e_{11}=zx\partial_3+\frac{3y\partial_3}{t}+\frac{2z^3\partial_3}{t}$$
$$-\frac{5z^5\partial_2\left(-4+t^2\right)}{t\left(-3+t^2\right)},\quad e_{12}=z^6\partial_3,\quad e_{13}=xz^2\partial_2-\frac{9y\partial_1}{20\left(-3+t^2\right)}-\frac{3z^3\partial_1}{10\left(-3+t^2\right)}-\frac{3xz\partial_1 t}{20\left(-3+t^2\right)}$$
$$+\frac{z^4\partial_2 t}{2\left(-3+t^2\right)},\quad e_{14}=z^6\partial_2,\quad e_{15}=z^6\partial_1.$$

By calculation, the multiplication table of $\mathcal{NL}(U_{11})_t$ is given by:

$$[e_1,e_2]=\frac{e_2}{2},\quad [e_1,e_3]=\frac{e_3}{4},\quad [e_1,e_4]=\frac{e_4}{4},\quad [e_1,e_5]=\frac{3e_5}{4},$$
$$[e_1,e_6]=\frac{e_6}{2},\quad [e_1,e_7]=\frac{e_7}{2},\quad [e_1,e_8]=\frac{3e_8}{4},\quad [e_1,e_9]=e_9,$$
$$[e_1,e_{10}]=\frac{3e_{10}}{4},\quad [e_1,e_{11}]=\frac{e_{11}}{2},\quad [e_1,e_{12}]=\frac{5e_{12}}{4},\quad [e_1,e_{13}]=\frac{e_{13}}{4},$$
$$[e_1,e_{14}]=\frac{3e_{14}}{4},\quad [e_1,e_{15}]=e_{15},\quad [e_2,e_3]=\frac{400e_{14}t^2\left(-3+t^2\right)^2}{3\left(-108+27t^2+8t^4\right)}-\frac{108e_8\left(9-6t^2+t^4\right)}{-108+27t^2+8t^4}$$
$$+\frac{16e_{10}\left(-4374+5832t^2-2943t^4+675t^6-63t^8+t^{10}\right)}{3t\left(432-216t^2-5t^4+8t^6\right)},\quad [e_2,e_4]=e_5-\frac{300e_{14}\left(-4+t^2\right)\left(-3+t^2\right)}{t\left(-108+27t^2+8t^4\right)}$$
$$-\frac{72e_8\left(-3t+t^3\right)}{-108+27t^2+8t^4}+\frac{8e_{10}\left(1620-1647t^2+594t^4-87t^6+4t^8\right)}{3\left(432-216t^2-5t^4+8t^6\right)},\quad [e_2,e_6]=\frac{4e_{15}t\left(-3+t^2\right)}{3\left(-4+t^2\right)}$$
$$-\frac{8e_9\left(9-6t^2+t^4\right)}{3\left(-4+t^2\right)},\quad [e_2,e_7]=\frac{2e_{15}\left(-3+t^2\right)}{-4+t^2}-\frac{e_9\left(-3t+t^3\right)}{-4+t^2},\quad [e_2,e_8]=\frac{2e_{12}\left(-3+t^2\right)}{t},$$
$$[e_2,e_{10}]=e_{12},\quad [e_2,e_{13}]=e_{14}+\frac{4e_{10}\left(9-6t^2+t^4\right)}{t\left(-4+t^2\right)},\quad [e_2,e_{14}]=\frac{4e_{12}\left(9-6t^2+t^4\right)}{t\left(-4+t^2\right)},$$
$$[e_3,e_4]=-\frac{45e_2\left(-4+t^2\right)}{-108+27t^2+8t^4}-\frac{45e_6\left(-4+t^2\right)}{-108+27t^2+8t^4}-\frac{9e_{11}\left(-3t^2+t^4\right)}{5\left(-108+27t^2+8t^4\right)},$$

$$[e_3,e_5]=-\frac{504e_{15}\left(-3t+t^3\right)}{5\left(-108+27t^2+8t^4\right)},\quad [e_3,e_6]=\frac{20e_{14}t^2\left(-3+t^2\right)\left(-20+7t^2\right)}{-108+27t^2+8t^4}-\frac{20e_5\left(-3t^3+t^5\right)}{-108+27t^2+8t^4}$$
$$-\frac{8e_{10}\left(8748-10935t^2+4995t^4-945t^6+45t^8+4t^{10}\right)}{3t\left(432-216t^2-5t^4+8t^6\right)},\quad [e_3,e_7]=-\frac{6e_5\left(-36+9t^2+t^4\right)}{-108+27t^2+8t^4}$$
$$-\frac{3e_8\left(54t-21t^3+t^5\right)}{-108+27t^2+8t^4}-\frac{2e_{14}\left(108t-93t^3+19t^5\right)}{3\left(-108+27t^2+8t^4\right)}-\frac{2e_{10}\left(972-1377t^2+684t^4-141t^6+10t^8\right)}{3\left(432-216t^2-5t^4+8t^6\right)},$$
$$[e_3,e_8]=\frac{2e_{15}\left(2916-1107t^2-20t^4\right)}{15\left(-108+27t^2+8t^4\right)}+\frac{4e_9\left(81t-39t^3+4t^5\right)}{3\left(-108+27t^2+8t^4\right)},\quad [e_3,e_9]=-\frac{6e_{12}\left(9t-6t^3+t^5\right)}{-108+27t^2+8t^4},$$
$$[e_3,e_{10}]=\frac{15e_{15}t^3\left(-4+t^2\right)}{\left(-3+t^2\right)\left(-108+27t^2+8t^4\right)},\quad [e_3,e_{11}]=\frac{100e_{14}t^2\left(-4+t^2\right)}{-108+27t^2+8t^4}-\frac{216e_{10}\left(9-6t^2+t^4\right)}{t\left(-108+27t^2+8t^4\right)},$$
$$[e_3,e_{13}]=-\frac{27e_{11}\left(-3t+t^3\right)}{5\left(-108+27t^2+8t^4\right)},\quad [e_3,e_{14}]=-\frac{162e_{15}\left(-3+t^2\right)}{5\left(-108+27t^2+8t^4\right)},\quad [e_3,e_{15}]=\frac{36e_{12}\left(-3+t^2\right)^2}{-108+27t^2+8t^4},$$
$$[e_4,e_5]=\frac{48e_9\left(-3t+t^3\right)}{-108+27t^2+8t^4}+\frac{3e_{15}\left(2160+48t^2-419t^4+56t^6\right)}{20\left(-3+t^2\right)\left(-108+27t^2+8t^4\right)},\quad [e_4,e_6]=\frac{45e_5\left(-4+t^2\right)}{-108+27t^2+8t^4}$$
$$-\frac{45e_{14}\left(-4+t^2\right)\left(-20+7t^2\right)}{t\left(-108+27t^2+8t^4\right)}-\frac{120e_{10}\left(9-6t^2+t^4\right)}{-108+27t^2+8t^4},\quad [e_4,e_7]=\frac{9e_8\left(-12+19t^2\right)}{4\left(-108+27t^2+8t^4\right)}-\frac{4e_5\left(12t+t^3\right)}{-108+27t^2+8t^4}$$
$$+\frac{9e_{14}\left(88-58t^2+9t^4\right)}{2\left(-3+t^2\right)\left(-108+27t^2+8t^4\right)}-\frac{e_{10}\left(-324t+405t^3-147t^5+16t^7\right)}{3\left(432-216t^2-5t^4+8t^6\right)},\quad [e_4,e_8]=\frac{12e_9\left(-4+3t^2\right)}{-108+27t^2+8t^4},$$
$$+\frac{3e_{15}t\left(364-147t^2+8t^4\right)}{5\left(-3+t^2\right)\left(-108+27t^2+8t^4\right)},\quad [e_4,e_9]=\frac{3e_{12}\left(-36+17t^2\right)}{2\left(-108+27t^2+8t^4\right)},$$
$$[e_4,e_{10}]=-\frac{135e_{15}\left(-4+t^2\right)^2}{4\left(-3+t^2\right)^2\left(-108+27t^2+8t^4\right)},$$
$$[e_4,e_{11}]=-\frac{225e_{14}\left(-4+t^2\right)^2}{t\left(-3+t^2\right)\left(-108+27t^2+8t^4\right)}-\frac{e_{10}\left(-324+117t^2-8t^4\right)}{-108+27t^2+8t^4},\quad [e_4,e_{13}]=-\frac{18e_{11}t^2}{5\left(-108+27t^2+8t^4\right)},$$
$$[e_4,e_{14}]=\frac{3e_{15}t\left(324-117t^2+8t^4\right)}{20\left(-3+t^2\right)\left(-108+27t^2+8t^4\right)},\quad [e_4,e_{15}]=\frac{24e_{12}t\left(-3+t^2\right)}{-108+27t^2+8t^4},$$
$$[e_5,e_{11}]=e_{12},\quad [e_5,e_{13}]=-\frac{3e_{15}t}{20\left(-3+t^2\right)},\quad [e_6,e_7]=-\frac{8e_{15}\left(-3+t^2\right)}{3\left(-4+t^2\right)}+\frac{4e_9\left(-3t+t^3\right)}{3\left(-4+t^2\right)},$$
$$[e_6,e_8]=-\frac{2e_{12}\left(-3+t^2\right)}{t},\quad [e_6,e_{13}]=-\frac{4e_{10}\left(9-6t^2+t^4\right)}{t\left(-4+t^2\right)},\quad [e_6,e_{14}]=-\frac{4e_{12}\left(9-6t^2+t^4\right)}{t\left(-4+t^2\right)},$$
$$[e_7,e_8]=\frac{e_{12}}{3},\quad [e_7,e_{10}]=\frac{e_{12}t}{2\left(-3+t^2\right)},\quad [e_7,e_{13}]=-\frac{e_{10}\left(3-t^2\right)}{2\left(-4+t^2\right)}+\frac{e_{14}t}{2\left(-3+t^2\right)},$$
$$[e_7,e_{14}]=-\frac{e_{12}\left(3-t^2\right)}{2\left(-4+t^2\right)},\quad [e_8,e_{11}]=\frac{2e_{12}}{t},\quad [e_8,e_{13}]=-\frac{3e_{15}}{10\left(-3+t^2\right)},$$
$$[e_{11},e_{13}]=-\frac{3e_{10}}{t},\quad [e_{11},e_{14}]=-\frac{3e_{12}}{t},\quad [e_{13},e_{14}]=\frac{9e_{15}}{20\left(-3+t^2\right)}.$$

(3). The unimodal complete intersection singularity M_{11} is defined by $\{(x^2+z^4\,,\,y^2+z^3+txz),\ t^2+2\neq 0\}$. The moduli algebra have the following basis

$$\mathbb{C}\{x,y,z\}/(f,g,M_1,M_2,M_3)=<1,x,y,z,z^2,z^3,z^4,z^5,yz,yz^2,xz>.$$

By calculation, a basis for Lie algebra $\mathcal{NL}(M_{11})_t$ is the following (for $t^2+2\neq 0$):

$$e_1=\frac{2x\partial_1}{3}+\frac{y\partial_2}{2}+\frac{z\partial_3}{3},\quad e_2=yz\partial_1-\frac{y\partial_3 t}{3}+x\partial_2\left(-1-t^2\right),\quad e_3=z^4\partial_1+\frac{yz^2\partial_2 t}{2\left(1+t^2\right)}$$

$$e_4=\frac{10z^3\partial_1}{5+8t^2}+\frac{5x\partial_3}{5+8t^2}+\frac{12xz\partial_1 t}{5+8t^2}+\frac{4yz\partial_2 t}{5+8t^2}-\frac{4z^2\partial_3 t}{5+8t^2},\quad e_5=-z^3\partial_2-\frac{yz\partial_3}{1+t^2}-\frac{yz^2\partial_1 t}{1+t^2}$$

$$e_6=\frac{3yz\partial_2}{5+8t^2}-\frac{12z^3\partial_1 t}{5+8t^2}-\frac{6x\partial_3 t}{5+8t^2}-\frac{4xz\partial_1\left(-1+2t^2\right)}{5+8t^2}+\frac{2z^2\partial_3\left(1+4t^2\right)}{5+8t^2},\quad e_7=z^4\partial_2$$

$$e_8=2z^3\partial_2-\frac{yz\partial_3}{-1-t^2}+\frac{2yz^2\partial_1 t}{1+t^2},\quad e_9=\frac{z^3\partial_3}{3}-\frac{z^4\partial_1 t}{3}+\frac{yz^2\partial_2}{2\left(1+t^2\right)},\quad e_{10}=\frac{z^4\partial_3}{2}$$

$$e_{11}=xz\partial_2-z^3\partial_3 t+z^4\partial_1\left(1+t^2\right),\quad e_{12}=yz^2\partial_3+z^4\partial_2\left(1+t^2\right),\quad e_{13}=z^5\partial_3$$

$$e_{14}=-yz^2\partial_1+xz\partial_2-z^3\partial_2 t,\quad e_{15}=z^5\partial_2,\quad e_{16}=z^5\partial_1.$$

By calculation, the multiplication table of $\mathcal{NL}(M_{11})_t$ is given by:

$$[e_1,e_2]=\frac{e_2}{6},\quad [e_1,e_3]=\frac{2e_3}{3},\quad [e_1,e_4]=\frac{e_4}{3},\quad [e_1,e_5]=\frac{e_5}{2},$$

$$[e_1,e_6]=\frac{e_6}{3},\quad [e_1,e_7]=\frac{5e_7}{6},\quad [e_1,e_8]=\frac{e_8}{2},\quad [e_1,e_9]=\frac{2e_9}{3}$$

$$[e_1,e_{10}]=e_{10},\quad [e_1,e_{11}]=\frac{2e_{11}}{3},\quad [e_1,e_{12}]=\frac{5e_{12}}{6},\quad [e_1,e_{13}]=\frac{4e_{13}}{3},$$

$$[e_1,e_{14}]=\frac{e_{14}}{2},\quad [e_1,e_{15}]=\frac{7e_{15}}{6},\quad [e_1,e_{16}]=e_{16},\quad [e_2,e_3]=\frac{1}{3}e_7\left(3+2t^2\right)$$

$$+\frac{e_{12}t^2}{6\left(1+t^2\right)},\quad [e_2,e_4]=\frac{4e_{14}\left(6t+7t^3\right)}{3\left(5+8t^2\right)}-\frac{e_8\left(-15-31t^2-16t^4\right)}{3\left(5+8t^2\right)}+\frac{4e_5\left(t^2+t^4\right)}{3\left(5+8t^2\right)},$$

$$[e_2,e_5]=-e_3+e_9t-\frac{e_{11}\left(-3-2t^2\right)}{3\left(1+t^2\right)},\quad [e_2,e_6]=-\frac{e_{14}\left(-1+6t^2+8t^4\right)}{5+8t^2}+\frac{8e_5\left(t+2t^3+t^5\right)}{3\left(5+8t^2\right)}$$

$$-\frac{e_8\left(11t+19t^3+8t^5\right)}{3\left(5+8t^2\right)},\quad [e_2,e_7]=-e_{16}+\frac{2e_{10}t}{3},\quad [e_2,e_8]=-\frac{e_{11}\left(3+2t^2\right)}{3\left(1+t^2\right)},$$

$$[e_2,e_9]=-\frac{e_7t}{3}-\frac{e_{12}\left(t+2t^3\right)}{6\left(1+t^2\right)},\quad [e_2,e_{11}]=e_{12}\left(1+t^2\right),\quad [e_2,e_{14}]=\frac{e_{11}t}{3},$$

$$[e_2,e_{15}]=\frac{e_{13}t}{3},\quad [e_2,e_{16}]=e_{15}\left(1+t^2\right),\quad [e_3,e_4]=\frac{10e_{10}}{5+8t^2}+\frac{8e_{16}t}{5+8t^2},$$

$$[e_3,e_55]=-\frac{e_{15}t}{2\left(1+t^2\right)},\quad [e_3,e_6]=-\frac{12e_{10}t}{5+8t^2}-\frac{4e_{16}\left(1+4t^2\right)}{5+8t^2},\quad [e_3,e_{11}]=e_{13},$$

$$[e_3,e_{14}]=e_{15},\quad [e_4,e_5]=-\frac{3e_{12}t}{5+13t^2+8t^4},\quad [e_4,e_6]=-\frac{8e_3\left(1+t^2\right)}{5+8t^2},\quad [e_4,e_8]=\frac{4e_7t}{5+8t^2}$$

$$-\frac{2e_{12}t}{5+13t^2+8t^4},\quad [e_4,e_9]=\frac{32e_{10}t}{3\left(5+8t^2\right)}-\frac{2e_{16}\left(15+2t^2\right)}{3\left(5+8t^2\right)},\quad [e_4,e_{10}]=\frac{64e_{13}t}{5+8t^2},$$

$$[e_4,e_{11}]=\frac{4e_{16}t\left(1+t^2\right)}{5+8t^2},\quad [e_4,e_{12}]=\frac{4e_{15}t\left(1+t^2\right)}{5+8t^2},\quad [e_4,e_{14}]=\frac{5e_{12}}{5+8t^2},$$

$$[e_4,e_{16}]=-\frac{5e_{13}}{5+8t^2},\quad [e_5,e_6]=e_7-\frac{e_{12}\left(-1+2t^2\right)}{5+13t^2+8t^4},\quad [e_5,e_7]=\frac{e_{13}}{1+t^2},$$

$$[e_5,e_8]=\frac{2e_{10}}{1+t^2}+\frac{2e_{16}t}{1+t^2},\quad [e_5,e_9]=e_{15}\left(1+\frac{1}{2\left(1+t^2\right)}\right),\quad [e_5,e_{12}]=e_{13},$$

$$[e_5,e_{14}]=-e_{16},\quad [e_6,e_7]=e_{15},\quad [e_6,e_8]=\frac{8e_7\left(1+t^2\right)}{5+8t^2}-\frac{e_{12}\left(-1-4t^2\right)}{5+13t^2+8t^4},$$

$$[e_6,e_9]=-\frac{4e_{10}\left(-1+8t^2\right)}{3\left(5+8t^2\right)}-\frac{4e_{16}\left(-7t+2t^3\right)}{3\left(5+8t^2\right)},\quad [e_6,e_{10}]=-\frac{2e_{13}\left(-1+2t^2\right)}{5+8t^2},$$

$$[e_6,e_{11}]=\frac{8e_{16}\left(1+t^2\right)^2}{5+8t^2},\quad [e_6,e_{12}]=\frac{8e_{15}\left(1+t^2\right)^2}{5+8t^2},\quad [e_6,e_{14}]=-\frac{6e_{12}t}{5+8t^2},$$

$$[e_6,e_{16}]=\frac{6e_{13}t}{5+8t^2},\quad [e_7,e_8]=\frac{e_{13}}{1+t^2},\quad [e_8,e_9]=-2e_{15},\quad [e_9,e_{11}]=-e_{13}t,$$
$$[e_9,e_{14}]=-e_{15}t.$$

(4). The unimodal complete intersection singularity V_{10} is defined by $\{(x^3+yz+z^3+tx^2z\,,xy),\ 4t^3+27\neq 0\}$. The moduli algebra have the following basis

$$\mathbb{C}\{x,y,z\}/(f,g,M_1,M_2,M_3)=<1,x,y,z,z^2,z^3,z^4,x^2,xz,x^2z>.$$

By calculation, a basis for Lie algebra $\mathcal{NL}(V_{10})_t$ is the following (for $4t^3+27\neq 0$):

$$e_1=\frac{x\partial_1}{3}+\frac{2y\partial_2}{3}+\frac{z\partial_3}{3},\quad e_2=-3xz\partial_1 t-\frac{4z^3\partial_2 t\left(9+2t^3\right)}{6+t^3}+\frac{12xz\partial_3\left(9+2t^3\right)}{27+4t^3}+\frac{24y\partial_3\left(5t+t^4\right)}{27+4t^3}$$
$$+\frac{3z^2\partial_3\left(15t+4t^4\right)}{27+4t^3}+\frac{12z^2\partial_1\left(5t^2+t^5\right)}{\left(6+t^3\right)\left(27+4t^3\right)}+\frac{4y\partial_1\left(14t^2+3t^5\right)}{\left(6+t^3\right)\left(27+4t^3\right)}-\frac{4x^2\partial_1\left(162+46t^3+3t^6\right)}{\left(6+t^3\right)\left(27+4t^3\right)},$$
$$e_3=-\frac{9xz\partial_1 t}{2\left(6+t^3\right)}-\frac{6z^3\partial_2 t\left(27+5t^3\right)}{\left(6+t^3\right)^2}+\frac{27z^2\partial_3\left(21t+4t^4\right)}{2\left(6+t^3\right)\left(27+4t^3\right)}+\frac{6y\partial_3\left(72t+13t^4\right)}{\left(6+t^3\right)\left(27+4t^3\right)}-\frac{xz\partial_3\left(-324-39t^3+4t^6\right)}{\left(6+t^3\right)\left(27+4t^3\right)}$$
$$-\frac{2y\partial_1\left(-270t^2-39t^5+2t^8\right)}{3\left(6+t^3\right)^2\left(27+4t^3\right)}-\frac{2z^2\partial_1\left(-27t^2+6t^5+2t^8\right)}{\left(6+t^3\right)^2\left(27+4t^3\right)}-\frac{4x^2\partial_1\left(729+216t^3+21t^6+t^9\right)}{3\left(6+t^3\right)^2\left(27+4t^3\right)}$$
$$e_4=\frac{9z^3\partial_1\left(6+t^3\right)}{2\left(27+4t^3\right)}+\frac{3x^2z\partial_1 t\left(6+t^3\right)}{2\left(27+4t^3\right)},\quad e_5=\frac{3xz\partial_1 t^2}{2\left(6+t^3\right)}-\frac{6y\partial_3\left(432+160t^3+15t^6\right)}{t\left(6+t^3\right)\left(27+4t^3\right)}$$
$$+\frac{2z^3\partial_2\left(486+180t^3+17t^6\right)}{t\left(6+t^3\right)^2}-\frac{3z^2\partial_3\left(1296+471t^3+44t^6\right)}{2t\left(6+t^3\right)\left(27+4t^3\right)}+\frac{3xz\partial_3\left(108t+41t^4+4t^7\right)}{\left(6+t^3\right)\left(27+4t^3\right)}$$
$$+\frac{12z^2\partial_1\left(135+77t^3+15t^6+t^9\right)}{\left(6+t^3\right)^2\left(27+4t^3\right)}+\frac{2y\partial_1\left(-216-26t^3+13t^6+2t^9\right)}{\left(6+t^3\right)^2\left(27+4t^3\right)}+\frac{2x^2\partial_1 t\left(432+205t^3+34t^6+2t^9\right)}{\left(6+t^3\right)^2\left(27+4t^3\right)},$$
$$e_6=\frac{3xz\partial_1}{6+t^3}+\frac{8y\partial_3}{6+t^3}+\frac{9z^2\partial_3}{6+t^3}-\frac{2xz\partial_3 t^2}{6+t^3}-\frac{4z^3\partial_2\left(27+4t^3\right)}{\left(6+t^3\right)^2}$$
$$-\frac{4y\partial_1\left(-6t+t^4\right)}{9\left(6+t^3\right)^2}-\frac{4z^2\partial_1\left(3t+t^4\right)}{3\left(6+t^3\right)^2}-\frac{2x^2\partial_1\left(24t^2+5t^5\right)}{9\left(6+t^3\right)^2},\quad e_7=-\frac{z^3\partial_1 t^2\left(6+t^3\right)}{2\left(27+4t^3\right)}-\frac{3x^2z\partial_1\left(36+12t^3+t^6\right)}{2\left(27+4t^3\right)},$$
$$e_8=\frac{z^3\partial_3}{2}-\frac{9z^3\partial_1 t}{8\left(27+4t^3\right)}-\frac{3x^2z\partial_1 t^2}{8\left(27+4t^3\right)},\quad e_9=x^2z\partial_3+\frac{3z^3\partial_1}{2\left(6+t^3\right)}+\frac{x^2z\partial_1 t}{2\left(6+t^3\right)}+\frac{z^3\partial_3 t^2}{3\left(6+t^3\right)}$$
$$e_{10}=x^2\partial_3+\frac{4y\partial_3}{t}+\frac{3z^2\partial_3}{t}+\frac{2y\partial_1}{6+t^3}+\frac{3z^2\partial_1}{2\left(6+t^3\right)}+\frac{x^2\partial_1 t}{2\left(6+t^3\right)}-\frac{3z^3\partial_2\left(27+4t^3\right)}{2t\left(6+t^3\right)},\quad e_{11}=z^4\partial_3$$
$$e_{12}=x^2z\partial_2+\frac{8y\partial_1}{9\left(6+t^3\right)}+\frac{2z^2\partial_1}{3\left(6+t^3\right)}+\frac{2x^2\partial_1 t}{9\left(6+t^3\right)}+\frac{z^3\partial_2 t^2}{3\left(6+t^3\right)},\quad e_{13}=z^4\partial_2,\quad e_{14}=z^4\partial_1.$$

By calculation, the multiplication table of $\mathcal{NL}(V_{10})_t$ is given by:

$$[e_1,e_2]=\frac{e_2}{3},\quad [e_1,e_3]=\frac{e_3}{3},\quad [e_1,e_4]=\frac{2e_4}{3},\quad [e_1,e_5]=\frac{e_5}{3},$$
$$[e_1,e_6]=\frac{e_6}{3},\quad [e_1,e_7]=\frac{2e_7}{3},\quad [e_1,e_8]=\frac{2e_8}{3},\quad [e_1,e_9]=\frac{2e_9}{3},$$
$$[e_1,e_{10}]=\frac{e_{10}}{3},\quad [e_1,e_{11}]=e_{11},\quad [e_1,e_{12}]=\frac{e_{12}}{3},\quad [e_1,e_{13}]=\frac{2e_{13}}{3},$$
$$[e_1,e_{14}]=e_{14},\quad [e_2,e_3]=e_4-\frac{9e_7 t}{6+t^3}-\frac{2e_8 t^2}{6+t^3}+\frac{18e_{13}\left(33t^2+5t^5\right)}{\left(6+t^3\right)^2}+\frac{3e_9\left(-72-13t^3+4t^6\right)}{2\left(27+4t^3\right)},$$
$$[e_2,e_4]=-\frac{18e_{14}t}{27+4t^3}-\frac{12e_{11}\left(9+2t^3\right)}{27+4t^3},\quad [e_2,e_5]=\frac{3e_7t^2}{6+t^3}+\frac{6e_8\left(8+t^3\right)}{6+t^3}-\frac{6e_{13}\left(108+17t^3\right)}{6+t^3}$$
$$-\frac{3e_4\left(8t+t^4\right)}{6+t^3}-\frac{9e_9\left(64t+35t^4+4t^7\right)}{2\left(27+4t^3\right)},\quad [e_2,e_6]=-\frac{12e_7}{6+t^3}-\frac{4e_8 t}{6+t^3}+\frac{28e_{13}\left(27t+4t^4\right)}{\left(6+t^3\right)^2}+\frac{3e_9\left(7t^2+4t^5\right)}{27+4t^3},$$

$$[e_2,e_7]=\frac{2e_{14}\left(162+47t^3+3t^6\right)}{\left(6+t^3\right)\left(27+4t^3\right)},\quad [e_2,e_8]=-\frac{3e_{14}\left(47t^2+12t^5\right)}{4\left(6+t^3\right)\left(27+4t^3\right)}-\frac{3e_{11}\left(396t+137t^4+12t^7\right)}{2\left(6+t^3\right)\left(27+4t^3\right)},$$
$$[e_2,e_9]=\frac{4e_{11}\left(9+t^3\right)}{\left(6+t^3\right)^2}-\frac{e_{14}t\left(33+4t^3\right)}{\left(6+t^3\right)^2},\quad [e_2,e_{10}]=-\frac{18e_9t}{27+4t^3}+\frac{9e_{13}\left(8+t^3\right)\left(27+4t^3\right)}{2\left(6+t^3\right)^2},$$
$$[e_2,e_{12}]=\frac{4e_{13}\left(9+t^3\right)}{\left(6+t^3\right)^2}-\frac{24e_9\left(5t+t^4\right)}{27+4t^3},\quad [e_2,e_{13}]=-\frac{24e_{11}\left(5t+t^4\right)}{27+4t^3}-\frac{4e_{14}\left(14t^2+3t^5\right)}{\left(6+t^3\right)\left(27+4t^3\right)},$$
$$[e_3,e_4]=\frac{9e_{14}t\left(9+2t^3\right)}{\left(6+t^3\right)\left(27+4t^3\right)}-\frac{e_{11}\left(324+39t^3-4t^6\right)}{\left(6+t^3\right)\left(27+4t^3\right)},\quad [e_3,e_5]=-\frac{9e_4t}{\left(6+t^3\right)^2}-\frac{4e_8t^3}{\left(6+t^3\right)^2}$$
$$-\frac{54e_{13}\left(27+4t^3\right)}{\left(6+t^3\right)^3}-\frac{3e_9\left(432t+151t^4+12t^7\right)}{162+51t^3+4t^6},\quad [e_3,e_6]=-\frac{27e_7}{\left(6+t^3\right)^2}+\frac{4e_8t^4}{3\left(6+t^3\right)^2}+\frac{60e_{13}\left(27t+4t^4\right)}{\left(6+t^3\right)^3}$$
$$+\frac{2e_9\left(27t^2+14t^5\right)}{162+51t^3+4t^6},\ [e_3,e_7]=-\frac{e_{14}\left(-5832-1890t^3-195t^6-8t^9\right)}{6\left(6+t^3\right)^2\left(27+4t^3\right)},\ [e_3,e_8]=-\frac{9e_{11}\left(405t+139t^4+12t^7\right)}{2\left(6+t^3\right)^2\left(27+4t^3\right)}$$
$$-\frac{e_{14}\left(351t^2-96t^5-32t^8\right)}{8\left(6+t^3\right)^2\left(27+4t^3\right)},\quad [e_3,e_9]=-\frac{27e_{14}t}{2\left(6+t^3\right)^3}-\frac{3e_{11}t^3}{\left(6+t^3\right)^3},\quad [e_3,e_{10}]=\frac{27e_{13}\left(27+4t^3\right)}{2\left(6+t^3\right)^3}$$
$$-\frac{3e_9\left(108t+39t^4+4t^7\right)}{2\left(162+51t^3+4t^6\right)},\quad [e_3,e_{12}]=-\frac{3e_{13}t^3}{\left(6+t^3\right)^3}-\frac{6e_9\left(72t+13t^4\right)}{162+51t^3+4t^6},\quad [e_3,e_{13}]=-\frac{6e_{11}\left(72t+13t^4\right)}{\left(6+t^3\right)\left(27+4t^3\right)}$$
$$+\frac{2e_{14}\left(-270t^2-39t^5+2t^8\right)}{3\left(6+t^3\right)^2\left(27+4t^3\right)},\quad [e_4,e_5]=\frac{3e_{14}t^2\left(9+2t^3\right)}{\left(6+t^3\right)\left(27+4t^3\right)}+\frac{3e_{11}\left(108t+41t^4+4t^7\right)}{\left(6+t^3\right)\left(27+4t^3\right)},$$
$$[e_4,e_6]=-\frac{6e_{14}}{6+t^3}-\frac{2e_{11}t^2}{6+t^3},\quad [e_5,e_6]=\frac{9e_7t}{\left(6+t^3\right)^2}-\frac{4e_8t^2}{6+t^3}-\frac{2e_4\left(27+4t^3\right)}{\left(6+t^3\right)^2}-\frac{12e_9\left(-54-5t^3+t^6\right)}{162+51t^3+4t^6}$$
$$-\frac{4e_{13}\left(2187+783t^3+68t^6\right)}{t\left(6+t^3\right)^3},\quad [e_5,e_7]=-\frac{e_{14}\left(1728t+838t^4+139t^7+8t^{10}\right)}{2\left(6+t^3\right)^2\left(27+4t^3\right)},$$
$$[e_5,e_8]=-\frac{3e_{14}\left(4320+2473t^3+480t^6+32t^9\right)}{8\left(6+t^3\right)^2\left(27+4t^3\right)}+\frac{3e_{11}\left(7776+4149t^3+739t^6+44t^9\right)}{2t\left(6+t^3\right)^2\left(27+4t^3\right)},$$
$$[e_5,e_9]=\frac{9e_{14}t^2}{2\left(6+t^3\right)^3}+\frac{e_{11}t^4}{\left(6+t^3\right)^3},\quad [e_5,e_{10}]=-\frac{9e_{13}t\left(27+4t^3\right)}{2\left(6+t^3\right)^3}$$
$$+\frac{9e_9\left(864+428t^3+71t^6+4t^9\right)}{2t\left(162+51t^3+4t^6\right)},\quad [e_5,e_{12}]=\frac{e_{13}t^4}{\left(6+t^3\right)^3}+\frac{6e_9\left(432+160t^3+15t^6\right)}{t\left(162+51t^3+4t^6\right)},$$
$$[e_5,e_{13}]=\frac{6e_{11}\left(432+160t^3+15t^6\right)}{t\left(6+t^3\right)\left(27+4t^3\right)}-\frac{2e_{14}\left(-216-26t^3+13t^6+2t^9\right)}{\left(6+t^3\right)^2\left(27+4t^3\right)},\quad [e_6,e_7]=-\frac{e_{14}\left(6t^2-t^5\right)}{9\left(6+t^3\right)^2},$$
$$[e_6,e_8]=-\frac{e_{11}\left(27+5t^3\right)}{\left(6+t^3\right)^2}-\frac{e_{14}\left(-21t-16t^4\right)}{12\left(6+t^3\right)^2},\quad [e_6,e_9]=\frac{3e_{14}\left(27+4t^3\right)}{\left(6+t^3\right)^3}+\frac{2e_{11}t^2\left(27+4t^3\right)}{3\left(6+t^3\right)^3},$$
$$[e_6,e_{10}]=-\frac{2e_9\left(3+t^3\right)}{6+t^3}-\frac{3e_{13}\left(27+4t^3\right)^2}{t\left(6+t^3\right)^3},\quad [e_6,e_{12}]=-\frac{8e_9}{6+t^3}+\frac{2e_{13}t^2\left(27+4t^3\right)}{3\left(6+t^3\right)^3},$$
$$[e_6,e_{13}]=-\frac{8e_{11}}{6+t^3}+\frac{4e_{14}\left(-6t+t^4\right)}{9\left(6+t^3\right)^2},\quad [e_7,e_{10}]=e_{11}+\frac{e_{14}t}{2\left(6+t^3\right)},\quad [e_7,e_{12}]=\frac{2e_{14}t}{9\left(6+t^3\right)},$$
$$[e_8,e_{10}]=\frac{3e_{11}}{t}+\frac{3e_{14}}{2\left(6+t^3\right)},\quad [e_8,e_{12}]=\frac{2e_{14}}{3\left(6+t^3\right)},\quad [e_{10},e_{12}]=-\frac{4e_9}{t},$$
$$[e_{10},e_{13}]=-\frac{4e_{11}}{t}-\frac{2e_{14}}{6+t^3},\quad [e_{12},e_{13}]=-\frac{8e_{14}}{9\left(6+t^3\right)}.$$

(5). The unimodal complete intersection singularity Y_{11} is defined by $\{(x^2+y^3+z^3+ty^2z\ ,xy),\ 4t^3+27\neq 0\}$. The moduli algebra have the following basis

$$\mathbb{C}\{x,y,z\}/(f,g,M_1,M_2,M_3) = <1,x,y,z,z^2,z^3,z^4,yz,y^2,y^2z,xz>.$$

By calculation, a basis for Lie algebra $\mathcal{NL}(Y_{11})_t$ is the following (for $4t^3+27\neq 0$):

$$e_1=\frac{3x\partial_1}{5}+\frac{2y\partial_2}{5}+\frac{2z\partial_3}{5},\quad e_2=-\frac{18x\partial_2}{27+4t^3}+\frac{18yz\partial_1 t}{27+4t^3}-\frac{24z^2\partial_1 t^2}{5\left(27+4t^3\right)}-\frac{4x\partial_3 t^2}{27+4t^3}$$
$$-\frac{y^2\partial_1\left(-135+8t^3\right)}{5\left(27+4t^3\right)},\quad e_3=\frac{3z^3\partial_1\left(45+8t^3\right)}{5\left(27+4t^3\right)}-\frac{y^2z\partial_1\left(-45t-8t^4\right)}{5\left(27+4t^3\right)}$$
$$e_4=\frac{6x\partial_2 t}{27+4t^3}-\frac{6yz\partial_1 t^2}{27+4t^3}-\frac{x\partial_3\left(-45-8t^3\right)}{27+4t^3}+\frac{6z^2\partial_1\left(45+8t^3\right)}{5\left(27+4t^3\right)}-\frac{y^2\partial_1\left(-45t-16t^4\right)}{5\left(27+4t^3\right)}$$
$$e_5=\frac{2yz\partial_3\left(9+2t^3\right)\left(45+8t^3\right)}{3\left(27+4t^3\right)}+\frac{y^2\partial_2\left(45+8t^3\right)^2}{15\left(27+4t^3\right)}+\frac{z^2\partial_2\left(45+8t^3\right)^2}{5t\left(27+4t^3\right)}-\frac{xz\partial_1\left(45t+8t^4\right)}{2\left(27+4t^3\right)}-\frac{z^2\partial_3\left(45t+8t^4\right)}{5\left(27+4t^3\right)}$$
$$-\frac{y^2\partial_3\left(-30375-11520t^3-1088t^6\right)}{60t\left(27+4t^3\right)},\quad e_6=\frac{yz\partial_2}{2}+\frac{81xz\partial_1}{4\left(27+4t^3\right)}+\frac{8yz\partial_3 t^5}{3\left(27+4t^3\right)}-\frac{z^2\partial_3\left(-135-8t^3\right)}{10\left(27+4t^3\right)}$$
$$+\frac{2z^2\partial_2 t\left(27+8t^3\right)}{5\left(27+4t^3\right)}+\frac{2y^2\partial_2 t^2\left(27+8t^3\right)}{15\left(27+4t^3\right)}-\frac{y^2\partial_3\left(-135t-56t^4\right)}{10\left(27+4t^3\right)},\quad e_7=-\frac{z^3\partial_2 t^2\left(45+8t^3\right)}{15\left(27+4t^3\right)}$$
$$-\frac{y^2z\partial_2\left(2025+720t^3+64t^6\right)}{60\left(27+4t^3\right)},\ e_8=-yz\partial_2 t-\frac{27xz\partial_1 t}{4\left(27+4t^3\right)}-\frac{z^2\partial_3\left(135t+16t^4\right)}{10\left(27+4t^3\right)}-\frac{yz\partial_3\left(1215+540t^3+64t^6\right)}{12\left(27+4t^3\right)}$$
$$-\frac{z^2\partial_2\left(2025+696t^3+64t^6\right)}{t\left(27+4t^3\right)}-\frac{y^2\partial_2\left(6075+1692t^3+128t^6\right)}{60\left(27+4t^3\right)}-\frac{y^2\partial_3\left(10125+4200t^3+448t^6\right)}{40t\left(27+4t^3\right)},$$
$$e_9=2y^2\partial_2+5yz\partial_3+\frac{6z^2\partial_2}{t}+\frac{15y^2\partial_3}{2t},\quad e_{10}=\frac{3z^3\partial_2\left(45+8t^3\right)}{5\left(27+4t^3\right)}-\frac{y^2z\partial_2\left(-45t-8t^4\right)}{5\left(27+4t^3\right)},$$
$$e_{11}=\frac{z^3\partial_3}{2}-\frac{9z^3\partial_2 t}{5\left(27+4t^3\right)}-\frac{3y^2z\partial_2 t^2}{5\left(27+4t^3\right)},\quad e_{12}=\frac{3z^3\partial_1}{5}+xz\partial_3+\frac{y^2z\partial_1 t}{5},$$
$$e_{13}=y^2z\partial_3+\frac{18z^3\partial_2}{45+8t^3}+\frac{6y^2z\partial_2 t}{45+8t^3}+\frac{4z^3\partial_3 t^2}{45+8t^3},\quad e_{14}=z^4\partial_3,\quad e_{15}=xz\partial_2-\frac{2}{15}z^3\partial_1 t^2$$
$$+\frac{1}{30}y^2z\partial_1\left(-45-8t^3\right),\quad e_{16}=z^4\partial_2,\quad e_{17}=z^4\partial_1.$$

By calculation, the multiplication table of $\mathcal{NL}(Y_{11})_t$ is given by:

$$[e_1,e_2]=\frac{e_2}{5},\quad [e_1,e_3]=\frac{3e_3}{5},\quad [e_1,e_4]=\frac{e_4}{5},\quad [e_1,e_5]=\frac{2e_5}{5},$$
$$[e_1,e_6]=\frac{2e_6}{5},\quad [e_1,e_7]=\frac{4e_7}{5},\quad [e_1,e_8]=\frac{2e_8}{5},\quad [e_1,e_9]=\frac{2e_9}{5},$$
$$[e_1,e_{10}]=\frac{4e_{10}}{5},\quad [e_1,e_{11}]=\frac{4e_{11}}{5},\quad [e_1,e_{12}]=\frac{3e_{12}}{5},\quad [e_1,e_{13}]=\frac{4e_{13}}{5},$$
$$[e_1,e_{14}]=\frac{6e_{14}}{5},\quad [e_1,e_{15}]=\frac{3e_{15}}{5},\quad [e_1,e_{16}]=\frac{6e_{16}}{5},\quad [e_1,e_{17}]=e_{17},$$
$$[e_2,e_3]=\frac{18e_{10}}{27+4t^3}+\frac{8e_{11}t^2}{27+4t^3}+\frac{4e_{13}\left(45t^3+8t^6\right)}{5\left(27+4t^3\right)^2},\quad [e_2,e_4]=\frac{6e_9 t}{27+4t^3},$$
$$[e_2,e_5]=-e_3-\frac{2e_{12}\left(12150+4905t^3+488t^6\right)}{5\left(27+4t^3\right)^2}-\frac{e_{15}\left(18225t+6120t^4+512t^7\right)}{5\left(27+4t^3\right)^2},$$
$$[e_2,e_6]=-\frac{e_{12}\left(135t^2+272t^5\right)}{5\left(27+4t^3\right)^2}-\frac{e_{15}\left(-1215+1224t^3+256t^6\right)}{10\left(27+4t^3\right)^2},\quad [e_2,e_7]=-\frac{e_{17}\left(135-8t^3\right)}{5\left(27+4t^3\right)},$$
$$[e_2,e_8]=-\frac{e_{12}\left(-18225-8910t^3-1088t^6\right)}{10\left(27+4t^3\right)^2}-\frac{e_{15}\left(-19845t-6288t^4-512t^7\right)}{10\left(27+4t^3\right)^2},\quad [e_2,e_9]=-\frac{90e_{12}}{27+4t^3}$$
$$-\frac{48e_{15}t}{27+4t^3},\quad [e_2,e_{10}]=-\frac{18e_{17}t}{27+4t^3},\quad [e_2,e_{11}]=\frac{24e_{17}t^2}{5\left(27+4t^3\right)},\quad [e_2,e_{12}]=\frac{3e_{13}\left(45+8t^3\right)}{5\left(27+4t^3\right)},$$

$$[e_2,e_{15}]=-\frac{2e_{13}\left(45t^2+8t^5\right)}{15\left(27+4t^3\right)},\quad [e_2,e_{17}]=\frac{18e_{16}}{27+4t^3}+\frac{4e_{14}t^2}{27+4t^3},\quad [e_3,e_4]=\frac{6e_{10}t}{27+4t^3}$$

$$+\frac{2e_{11}\left(45+8t^3\right)}{27+4t^3}-\frac{e_{13}\left(-2025t-720t^4-64t^7\right)}{5\left(27+4t^3\right)^2},\quad [e_3,e_5]=\frac{e_{17}\left(45t+8t^4\right)}{2\left(27+4t^3\right)},\quad [e_3,e_6]=-\frac{81e_{17}}{4\left(27+4t^3\right)},$$

$$[e_3,e_8]=\frac{27e_{17}t}{4\left(27+4t^3\right)},\quad [e_3,e_{12}]=e_{14},\quad [e_3,e_{15}]=e_{16},\quad [e_4,e_5]=2e_3t$$

$$-\frac{e_{12}\left(-18225t-7200t^4-704t^7\right)}{10\left(27+4t^3\right)^2}-\frac{e_{15}\left(-182250-97875t^3-17400t^6-1024t^9\right)}{5t\left(27+4t^3\right)^2},\quad [e_4,e_6]=-\frac{135e_3}{2\left(45+8t^3\right)}$$

$$+\frac{9e_{12}\left(675+280t^3+64t^6\right)}{20\left(27+4t^3\right)^2}-\frac{e_{15}\left(-9315t-4728t^4-512t^7\right)}{10\left(27+4t^3\right)^2},\quad [e_4,e_7]=-\frac{e_{17}\left(45t+16t^4\right)}{5\left(27+4t^3\right)},$$

$$[e_4,e_8]=\frac{45e_3t}{2\left(45+8t^3\right)}-\frac{9e_{12}\left(3375t+1280t^4+128t^7\right)}{20\left(27+4t^3\right)^2}-\frac{e_{15}\left(182250+96255t^3+17136t^6+1024t^9\right)}{10t\left(27+4t^3\right)^2},$$

$$[e_4,e_9]=\frac{30e_{12}t}{27+4t^3}+\frac{12e_{15}\left(45+8t^3\right)}{t\left(27+4t^3\right)},\quad [e_4,e_{10}]=\frac{6e_{17}t^2}{27+4t^3},\quad [e_4,e_{11}]=-\frac{6e_{17}\left(45+8t^3\right)}{5\left(27+4t^3\right)},$$

$$[e_4,e_{12}]=-\frac{e_{13}\left(45t+8t^4\right)}{5\left(27+4t^3\right)},\quad [e_4,e_{15}]=\frac{e_{13}\left(2025+720t^3+64t^6\right)}{30\left(27+4t^3\right)},\quad [e_4,e_{17}]=-\frac{6e_{16}t}{27+4t^3}$$

$$-\frac{e_{14}\left(45+8t^3\right)}{27+4t^3},\quad [e_5,e_6]=e_7-4e_{11}t-\frac{e_{10}\left(135+8t^3\right)}{12t}+\frac{4e_{13}\left(675t^2+300t^5+32t^8\right)}{75\left(27+4t^3\right)},$$

$$[e_5,e_7]=-\frac{e_{16}\left(10125+3240t^3+256t^6\right)}{60\left(27+4t^3\right)}-\frac{e_{14}\left(30375+11520t^3+1088t^6\right)}{60t\left(27+4t^3\right)},\quad [e_5,e_8]=-2e_7t$$

$$+\frac{1}{12}e_{10}\left(135+16t^3\right)-\frac{e_{11}\left(75+8t^3\right)}{4t}-\frac{e_{13}\left(50625+25650t^3+4400t^6+256t^9\right)}{75\left(27+4t^3\right)},$$

$$[e_5,e_9]=-5e_{10}+\frac{15e_{11}}{t}-\frac{e_{13}\left(675+300t^3+32t^6\right)}{5\left(27+4t^3\right)},\quad [e_5,e_{10}]=-\frac{2e_{14}\left(9+2t^3\right)\left(45+8t^3\right)}{3\left(27+4t^3\right)}$$

$$-\frac{e_{16}\left(45t+8t^4\right)}{27+4t^3},\quad [e_5,e_{11}]=-\frac{6e_{14}\left(15t+2t^4\right)}{5\left(27+4t^3\right)}-\frac{e_{16}\left(2025+705t^3+64t^6\right)}{5t\left(27+4t^3\right)},\quad [e_5,e_{12}]=-e_{17}t,$$

$$[e_5,e_{13}]=\frac{45e_{14}}{45+8t^3}-\frac{30e_{16}t}{45+8t^3},\quad [e_5,e_{15}]=-\frac{3e_{17}}{2},\quad [e_6,e_7]=-\frac{e_{14}\left(135t+56t^4\right)}{10\left(27+4t^3\right)}$$

$$-\frac{e_{16}\left(243t^2+32t^5\right)}{30\left(27+4t^3\right)},\quad [e_6,e_8]=-\frac{5e_{11}t}{2}+\frac{135e_7}{2\left(45+8t^3\right)}-\frac{75e_{10}\left(27+4t^3\right)}{8t\left(45+8t^3\right)}-\frac{e_{13}\left(45t^2+8t^5\right)}{5\left(27+4t^3\right)},$$

$$[e_6,e_9]=\frac{405e_{10}}{2t\left(45+8t^3\right)}+\frac{90e_{11}t}{45+8t^3}-\frac{6e_{13}\left(15t^2+4t^5\right)}{5\left(27+4t^3\right)},\quad [e_6,e_{10}]=-\frac{8e_{14}t^5}{3\left(27+4t^3\right)}-\frac{e_{16}\left(-27+2t^3\right)}{27+4t^3},$$

$$[e_6,e_{11}]=-\frac{e_{14}\left(-6075+180t^3+128t^6\right)}{20\left(27+4t^3\right)\left(45+8t^3\right)}-\frac{e_{16}t\left(6075+2304t^3+256t^6\right)}{10\left(27+4t^3\right)\left(45+8t^3\right)},\quad [e_6,e_{12}]=\frac{81e_{17}}{2\left(45+8t^3\right)},$$

$$[e_6,e_{13}]=\frac{1215e_{16}}{\left(45+8t^3\right)^2}+\frac{270e_{14}t^2}{\left(45+8t^3\right)^2},\quad [e_6,e_{15}]=-\frac{9e_{17}t^2}{45+8t^3},\quad [e_7,e_8]=-\frac{e_{16}\left(6075+1782t^3+128t^6\right)}{60\left(27+4t^3\right)}$$

$$-\frac{e_{14}\left(10125+4200t^3+448t^6\right)}{40t\left(27+4t^3\right)},\quad [e_7,e_9]=2e_{16}+\frac{15e_{14}}{2t},\quad [e_8,e_9]=-\frac{135e_{10}}{2\left(45+8t^3\right)}-\frac{30e_{11}t^2}{45+8t^3}$$

$$+\frac{3e_{13}\left(2025+700t^3+64t^6\right)}{20\left(27+4t^3\right)},\quad [e_8,e_{10}]=\frac{e_{16}t\left(27+8t^3\right)}{2\left(27+4t^3\right)}-\frac{e_{14}\left(-1215-540t^3-64t^6\right)}{12\left(27+4t^3\right)},$$

$$[e_8,e_{11}]=-\frac{e_{14}\left(-6075t-2700t^4-256t^7\right)}{20\left(27+4t^3\right)\left(45+8t^3\right)}-\frac{e_{16}\left(-91125-47925t^3-8448t^6-512t^9\right)}{10t\left(27+4t^3\right)\left(45+8t^3\right)},$$

$$[e_8, e_{12}] = -\frac{27e_{17}t}{2\left(45+8t^3\right)}, \quad [e_8, e_{13}] = -\frac{405e_{16}t}{\left(45+8t^3\right)^2} - \frac{90e_{14}t^3}{\left(45+8t^3\right)^2}, \quad [e_8, e_{15}] = \frac{3e_{17}t^3}{45+8t^3},$$
$$[e_9, e_{10}] = -5e_{14}, \quad [e_9, e_{11}] = -\frac{6e_{16}}{t}.$$

It follows from the multiplication tables above, it is easy to check that the new Lie algebras $\mathcal{NL}(V)$ of T_{10}, R_9, U_{11}, V_{10}, Y_{11}, and M_{11} are solvable. It is natural for us to propose the following conjecture.

Conjecture 3.3. *Let $(V, 0) \subset (\mathbb{C}^m, 0)$ be a ICIS of dimension n ($n < m$), then the Lie algebra $\mathcal{NL}(V)$ is solvable.*

Remark 3.1. The conjecture was proved by the second author when $n = m - 1$ [29].

References

[1] A.G. Aleksandrov, Normal forms of one dimensional quasihomogeneous complete intersection singularities, Sb. Math. 117 (1) (1982) 3–31, 159.
[2] E. Brieskorn, Singular elements of semi-simple algebraic groups, Actes Congress Intern. Math. 2 (1970) 279–284.
[3] B. Chen, H. Chen, S.S.-T. Yau, H. Zuo, The non-existence of negative weight derivations on positive dimensional isolated singularities: generalized Wahl conjecture, J. Differ. Geom. 115 (2) (2020) 195–224.
[4] B. Chen, N. Hussain, S.S.-T. Yau, H. Zuo, Variation of complex structures and variation of Lie algebras II: new Lie algebras arising from singularities, J. Differ. Geom. 115 (3) (2020) 437–473.
[5] H. Chen, S.S.-T. Yau, H. Zuo, Non-existence of negative weight derivations on positively graded Artinian algebras, Trans. Am. Math. Soc. 372 (4) (2019) 2493–2535.
[6] A. Dimca, G. Sticlaru, Hessian ideals of a homogeneous polynomial and generalized Tjurina algebras, Doc. Math. 20 (2015) 689–705.
[7] A. Elashvili, G. Khimshiashvili, Lie algebras of simple hypersurface singularities, J. Lie Theory 16 (2006) 621–649.
[8] M. Giusti, Classification des Singularitiés isolées d'intersectios compleètes simples, C. R. Acad. Sci. Paris Sér. A–B 284 (3) (1977) A167–A170.
[9] N. Hussain, S.S.-T. Yau, H. Zuo, On the derivation Lie algebras of fewnomial singularities, Bull. Aust. Math. Soc. 98 (1) (2018) 77–88.
[10] N. Hussain, S.S.-T. Yau, H. Zuo, On the new k-th Yau algebras of isolated hypersurface singularities, Math. Z. 294 (1–2) (2020) 331–358.
[11] N. Hussain, S.S.-T. Yau, H. Zuo, Generalized Cartan matrices arising from new derivation Lie algebras of isolated hypersurface singularities, Pac. J. Math. 305 (1) (2020) 189–217.
[12] N. Hussain, S.S.-T. Yau, H. Zuo, k-th Yau number of isolated hypersurface singularities and an inequality conjecture, J. Aust. Math. Soc. 110 (1) (2021) 94–118.
[13] N. Hussain, S.S.-T. Yau, H. Zuo, On two inequality conjectures for the k-th Yau numbers of isolated hypersurface singularities, Geom. Dedic. 212 (2021) 57–71.
[14] N. Hussain, S.S.-T. Yau, H. Zuo, On the generalized Cartan matrices arising from k-th Yau algebras of isolated hypersurface singularities, Algebr. Represent. Theory 24 (4) (2021) 1101–1124.
[15] N. Hussain, S.S.-T. Yau, H. Zuo, Inequality conjectures on derivations of local k-th Hessian algebras associated to isolated hypersurface singularities, Math. Z. 298 (2021) 1813–1829.
[16] N. Hussain, S.S.-T. Yau, H. Zuo, New k-th Yau algebras of isolated hypersurface singularities and weak Torelli-type theorem, Math. Res. Lett. (2022), 26pp. In ms, to appear.
[17] M. Jankins, W.D. Neumann, Lectures on Seifert Manifolds, Brandeis Lecture Notes, vol. 2, Brandeis University, Waltham, MA, 1983.
[18] G. Khimshiashvili, Yau Algebras of Fewnomial Singularities, Universiteit Utrecht Preprints, vol. 1352, 2006, http://www.math.uu.nl/publications/preprints/1352.pdf.
[19] J.L. Koszul, Homologie et cohomologie des algèbres de Lie, Bull. Soc. Math. Fr. 78 (1950) 41–64.
[20] J. Mather, S.S.-T. Yau, Classification of isolated hypersurface singularities by their moduli algebras, Invent. Math. 69 (1982) 243–251.
[21] A. Nijenhuis, R.W. Richardson, Deformations of Lie algebra structures, J. Math. Mech. 17 (1) (1967) 89–105.
[22] L. Pursell, M. Shanks, The Lie algebra of a smooth manifold, Trans. Am. Math. Soc. 5 (1954) 468–472.
[23] C. Seeley, S.S.-T. Yau, Variation of complex structure and variation of Lie algebras, Invent. Math. 99 (1990) 545–565.
[24] C.T.C. Wall, Finite determinacy of smooth map-germs, Bull. Lond. Math. Soc. 13 (1981) 481–539.
[25] C.T.C. Wall, Classification of unimodal isolated singularities of complete intersections, Proc. Symp. Pure Math. 40 (2) (1983) 625–640.
[26] Y.-J. Xu, S.S.-T. Yau, Micro-local characterization quasi-homogeneous singularities, Am. J. Math. 118 (2) (1996) 389–399.
[27] S.S.-T. Yau, Continuous family of finite-dimensional representations of a solvable Lie algebra arising from singularities, Proc. Natl. Acad. Sci. USA 80 (1983) 7694–7696.
[28] S.S.-T. Yau, Solvable Lie algebras and generalized Cartan matrices arising from isolated singularities, Math. Z. 191 (1986) 489–506.
[29] S.S.-T. Yau, Solvability of Lie algebras arising from isolated singularities and nonisolatedness of singularities defined by $sl(2, \mathbb{C})$ invariant polynomials, Am. J. Math. 113 (1991) 773–778.
[30] Y. Yu, On Jacobian ideals invariant by reducible sl(2; C) action, Trans. Am. Math. Soc. 348 (1996) 2759–2791.
[31] S.S.-T. Yau, H. Zuo, Derivations of the moduli algebras of weighted homogeneous hypersurface singularities, J. Algebra 457 (2016) 18–25.
[32] S.S.-T. Yau, H. Zuo, A sharp upper estimate conjecture for the Yau number of weighted homogeneous isolated hypersurface singularity, Pure Appl. Math. Q. 12 (1) (2016) 165–181.