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Variation of complex structures and variation of Lie algebras *

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§ 0. Introduction

Given a family of complex projective hypersurfaces in $\mathbf{CP}^n$, the Torelli problem studied by P. Griffiths and his school asks whether the period map is injective on that family, i.e., whether the family of complex hypersurfaces can be distinguished by means of their Hodge structures. A complex projective hypersurface in $\mathbf{CP}^n$ can be viewed as a complex hypersurface with isolated singularity in $\mathbf{C}^{n+1}$. Let $V=\{z\in\mathbf{C}^{n+1}: f(z)=0\}$ be a complex hypersurface with isolated singularity at the origin. The moduli algebra of $(V,0)$ is $A(V)$ $:=\mathbf{C}\{z_0 m \ldots, z_n\}\Big/\!\!\Big(f, \dfrac{\partial f}{\partial z_0}, \ldots, \dfrac{\partial f}{\partial z_n}\Big)$. It is a finite dimensional commutative local algebra. In [2], Mather and the second author proved that the complex structures of $(V,0)$ determines and is determined by its moduli algebra. Subsequently the second author [6] introduced the Lie algebra $L(V)$ to $(V,0)$, which is the Lie algebra of derivations of $A(V)$. He proved that $L(V)$ is solvable if $n\leqq 5$ (cf. [7]). The natural question arises: whether the family of isolated complex hypersurface singularities can be distinguished by means of their Lie algebras. The family of hypersurface singularities here is not arbitrary. First of all, as in projective case, we are really studying the complex structures of an isolated hypersurface singularity. In view of the theorem of *Lê* and Ramanujan [1], we require that the Milnor number μ is constant along this family. Recall that the dimension of the moduli algebra (denoted by τ) is a complex analytic invariant. So it suffices to consider only a (μ, τ)-constant family of isolated complex hypersurface singularities.

Let $(V,0)$ be an isolated hypersurface singularity with $\mathbf{C}^*$-action. Let S_E be the (μ,τ)-constant strata in the semi-universal deformation of $(V,0)$. In § 1, we shall show that $(S_E, 0)$ is isomorphic to $(\mathbf{C}^m, 0)$. In § 2, we construct a family of Lie algebras $\hat{L}(V_t)$ over S_E. In § 3, we shall prove a Torelli type theorem for simple elliptic singularities $\tilde{E}_7$ and $\tilde{E}_8$. There are several advantages of our approach. First of all, it works for general complex hypersurface singularities

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which are not necessarily homogeneous. Second, it allows us to construct a continuous invariant explicitly. Third, it gives a general method for producing a continuous family of nilpotent Lie algebras.

§ 1. (μ, τ) Constant deformation for hypersurface singularities with $\mathbf{C}^*$-action

Let $(V, 0) \subseteq (\mathbf{C}^n, 0)$ be an isolated hypersurface singularity with the local defining equation $f(x_1, \ldots, x_n) = 0$. Then the semi-universal deformation of $(V, 0)$ is given by

$$\begin{array}{ccccc} (\mathscr{V}, 0) & \longrightarrow & (\mathbf{C}^n \times \mathbf{C}^k, 0) & (x_1, \ldots, x_n, t_1, \ldots, t_k) & \\ \downarrow & & \downarrow & \downarrow & \\ (S, 0) & = & (\mathbf{C}^k, 0) & (t_1, \ldots, t_k). & \end{array}$$

Here $\mathscr{V} = \left\{(x_1, \ldots, x_n, t_1, \ldots, t_k) : f(x) + \sum_{i=1}^{k} t_i g_i(x) = 0\right\}$, where $g_1(x), \ldots, g_k(x)$ are monomials in $x_1, \ldots, x_n$ which represent a linear basis of the complex vector space $A(V) = \mathbf{C}[\![x_1, \ldots, x_n]\!] \Big/ \left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$. We are particularly interested in the case when f is weighted-homogeneous, i.e. when there exist positive integers $q_1, \ldots, q_n$ and d such that for any $t \in \mathbf{C}^* = \mathbf{C} - \{0\}$

$$f(t^{q_1} x_1, t^{q_2} x_2, \ldots, t^{q_n} x_n) = t^d f(x_1, \ldots, x_n). \tag{1.1}$$

In the rest of this paper, we shall always assume that f is weighted homogeneous.

Let us give the variable x_i the weight q_i. Then each monomial $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ which appears in f has total weight $d = \alpha_1 q_1 + \alpha_2 q_2 + \ldots + \alpha_n q_n$.

Theorem 1.1. *Let f be a weighted homogeneous polynomial with isolated singularity at the origin as above. Then*

$$\{(x_1, \ldots, x_n, t_1, \ldots, t_m) : f(x) + t_1 g_1(x) + \ldots + t_m g_m(x) = 0\},$$

where $g_1, \ldots, g_m$ are monomials in a basis for the moduli algebra

$$A(V) = \mathbf{C}[\![x_1, \ldots, x_n]\!] \Big/ \left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right),$$

is a (μ, τ)-constant deformation of $V = \{x : f(x) = 0\}$ if and only if weight (g_i) $=$ weight(f) for all $1 \leq i \leq m$.

Proof. "$\Leftarrow$" For each fixed $t = (t_1, t_2, \ldots, t_m)$, $f_t(x) = f(x) + t_1 g_1(x) + \ldots + t_m g_m(x)$ is a weighted homogeneous polynomial total weight equal to that of f. Since by [3], the Milnor number of a weighted homogeneous polynomial is determined by $q_1, \ldots, q_n$ and d, we conclude that the Milnor number μ is independent of $t=(t_1, \ldots, t_m)$. Since $f_t(x)$ is weighted homogeneous for all t, $\tau=\mu$ is independent of $t=(t_1, \ldots, t_m)$ also.

"$\Rightarrow$" If $\{(x,t)\}: f(x)+t_1 g_1(x)+\ldots+t_m g_m(x)=0\}$ is a (μ, τ)-constant deformation, then in particular $\tau(V_t)$ is independent of $t=(t_1, \ldots, t_m)$ where $V_t=\{x\in\mathbf{C}^n: f(x)+t_1 g(x)+\ldots+t_m g(x)=0\}$.

This implies that $\mathbf{C}\{x_1, \ldots, x_n, t_i\}\Big/\left(F, \dfrac{\partial F}{\partial x_1}, \ldots, \dfrac{\partial F}{\partial x_n}\right)$ is flat over $\mathbf{C}\{t_i\}$ where $F(x, t_i)=f(x)+t_i g_i(x)$. Let $D=q_1 x_i \dfrac{\partial}{\partial x_1}+\ldots+q_n x_n \dfrac{\partial}{\partial x_n}$ be the Euler derivation. Since f is weighted homogeneous with total weight d, $Df=df$. Hence $DF-dF =Df-df+t_i(Dg_i-dg_i)=t(Dg_i-dg_i)$ is in (t_i), the ideal generated by t_i in $\mathbf{C}\{x_1, \ldots, x_n, t_i\}$. Clearly $DF-dF$ is in $\left(F, \dfrac{\partial F}{\partial x_1}, \ldots, \dfrac{\partial F}{\partial x_n}\right)$. So $DF-dF$ is in $(t_i)\cap\left(F, \dfrac{\partial F}{\partial x_1}, \ldots, \dfrac{\partial F}{\partial x_n}\right)$. As

$$0\rightarrow \mathbf{C}\{t_i\}\xrightarrow{t_i}\mathbf{C}\{t_i\}$$

is exact, so is

$$0\rightarrow \mathbf{C}\{x, t_i\}\Big/\left(f, \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}\right)\xrightarrow{t_i}\mathbf{C}\{x, t_i\}\Big/\left(F, \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}\right) \tag{1.2}$$

by the flatness of $\mathbf{C}\{x, t_i\}\Big/\left(f, \dfrac{\partial F}{\partial x_1}, \ldots, \dfrac{\partial F}{\partial x_n}\right)$ over $C\{t_i\}$.

It follows that

$$(t_i)\cap\left(F, \frac{\partial F}{\partial x_i}, \ldots, \frac{\partial F}{\partial x_m}\right)\subseteq(t_i)\left(F, \frac{\partial F}{\partial x_i}, \ldots, \frac{\partial F}{\partial x_n}\right).$$

Therefore we have

$$t_i(Dg_i-dg_i)=DF-dF\in(t_i)\left(F, \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}\right).$$

There exist $b_{ij}(x)\in\mathbf{C}\{x\}$ such that

$$\begin{aligned} t_i(Dg_i-dg_i)=&(b_{01}(x)t_i+b_{02}(x)t_i^2+b_{03}(x)t_i^3+\ldots)(f(x)+t_i g_i(x))\\ &+(b_{11}(x)t_i+b_{12}(x)t_i+b_{13}(x)t_i^3+\ldots)\left(\frac{\partial F}{\partial x_1}+t_i^2\frac{\partial g_i}{\partial x_1}\right)\\ &+\ldots\\ &+(b_{n1}(x)t_i+b_{n2}(x)t_i^2+b_{n3}(x)t_i^3+\ldots)\left(\frac{\partial f}{\partial x_n}+t_i\frac{\partial g_i}{\partial x_n}\right)\\ \Rightarrow Dg_i-dg_i=&\,b_{01}(x)f(x)+b_{11}(x)\frac{\partial f}{\partial x_1}+\ldots+b_{n1}(x)\frac{\partial f}{\partial x_n}\\ \equiv&\,0 \quad \text{in } \mathbf{C}\{x_1, \ldots, x_n\}\Big/\left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right). \end{aligned}$$

On the other hand

$$Dg_i - dg_i = \left(q_1 x_i \frac{\partial}{\partial x_1} + \ldots + q_n x_n \frac{\partial}{\partial x_n}\right)(g_i) - dg_i = (wt(g_i) - d)g_i.$$

Since g_i is a basis element in $\mathbf{C}\{x_1, \ldots, x_n\} \Big/ \left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$, we conclude that $wt(g_i) = d$. This is true for all $1 \leqq i \leqq m$. Q.E.D.

Remark. Since the initial fiber is weighted homogeneous, we know that $\mu = \tau$. Hence we see immediately that $f_t(x) = f(x) + t_1 g_1(x) + \ldots + t_m g_m(x)$ is quasi-homogeneous for each t. By a theorem of Saito [4], $f_t(x)$ is weighted homogeneous with respect to a certain coordinate system. The point of the above theorem is that $f_t(x)$ is weighted homogeneous with respect to $x_1, \ldots, x_n$ for all t.

It is well known that the (μ, τ)-constant strata $S_E = \{t \in \mathbf{C}^k : (\mu(V_t), \tau(V_t)) = (\mu, \tau)\}$ forms a subvariety in the parameter space of the semi-universal deformation of $(V, 0)$. In case $(V, 0)$ has a $\mathbf{C}^*$-action, we shall show that $(S_E, 0)$ is isomorphic to $(\mathbf{C}^m, 0)$, where m is the dimension of A_d (elements in the moduli algebra $A(V)$ of weight d).

Theorem 1.2. *Let f be a weighted homogeneous polynomial with isolated singularity at the origin as in (1.1). Let $g_1, \ldots, g_m$ be elements in a monomial basis of the moduli algebra $A(V) = \mathbf{C}[\![x_i, \ldots, x_n]\!] \Big/ \left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$ such that weight (g_i) = weight(f) for all $1 \leqq i \leqq m$. Then the (μ, τ)-constant strata S_E is $\mathbf{C}^m$ and the equitopological deformation $(\mathscr{W}, 0) \to (S_E, 0)$ of $(V, 0)$ is given by*

$$\begin{tikzcd}
(\mathscr{W}, 0) \arrow[r] \arrow[d] & (\mathscr{V}, 0) \arrow[r, equal] \arrow[d] & (\mathbf{C}^n \times \mathbf{C}^k, 0) \arrow[d, "Pr_2"] \\
(S_E, 0) \arrow[r] & (S, 0) \arrow[r, equal] & (\mathbf{C}^k, 0)
\end{tikzcd}$$

where $\mathscr{W} = \{(x_1, \ldots, x_n, t_1, \ldots, t_m) : f(x) + t_1 g_1(x) + \ldots + t_m g_m(x) = 0\}$

(Pr_2 = projection onto the second factor)

and

$$(S_{E,0}) = (\mathbf{C}^m, 0).$$

Proof. By Theorem 1.1, we know that $(\mathbf{C}^m, 0) \subseteq (S_E, 0)$. Suppose that $(\mathbf{C}^m, 0) \neq (S_E, 0)$. Then by the theorem on resolution of singularities, we can find a curve α: $(\mathbf{C}, 0) \to (S_E, 0)$ such that $\alpha(t) = (\alpha_1(t), \alpha_2(t), \ldots, \alpha_k(t))$ lies in $S_E - \mathbf{C}^m$ for any t in $\mathbf{C} - \{0\}$ and $\alpha(0) = (0, \ldots, 0)$. (i.e. there exists $i \geqq m+1$ such that $\alpha_i(t)$ is not identically zero.) The proof of Theorem 1.1 shows that for any non-negative integer p

$$(t^p) \cap \left(F, \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}\right) = (t^p)\left(F, \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}\right) \quad \forall p \geqq 0 \qquad (1.3)$$

where (t^p) is the ideal generated by t^p in $\mathbf{C}\{x, t\}$. Let r be

$$\min\{o(\alpha_{m+1}(t)), o(\alpha_{m+2}(t)), \ldots, o(\alpha_k(t))\}$$

where we denote $o(\alpha_i(t))$ to be the vanishing order of $\alpha_i(t)$ at origin. Observe that $1 \leqq r < \infty$ because $\alpha_i(t)$ is not identically zero for some i between $m+1$ and k. Let $D = q_1 x_1 \frac{\partial}{\partial x_1} + \ldots + q_n x_n \frac{\partial}{\partial x_n}$ be the Euler derivation. Then

$$\begin{aligned} DF(x) - dF(x) &= Df(x) - df(x) + \alpha_1(t)(Dg_1(x) - dg_1(x)) + \ldots \\ &\quad + \alpha_k(t)(Dg_k(x) - dg_k(x)) \\ &= c_{m+1}\,\alpha_{m+1}(t)\,g_{m+1}(x) + c_{m+2}\,\alpha_{m+2}(t)\,g_{m+2}(x) + \ldots \\ &\quad + c_k\,\alpha_k(t)\,g_k(x) \end{aligned}$$

where c_i, for $m+1 \leqq i \leqq k$, are nonzero constants. Write $\alpha_{m+1}(t) = t^r \beta_{m+1}(t)$, $\alpha_{m+2}(t) = t^r \beta_{m+2}(t), \ldots, \alpha_k(t) = t^r \beta_k(t)$. Then $\beta_j(0) \neq 0$ for some $m+1 \leqq j \leqq k$ by our choice of r.

$$\begin{aligned} DF(x) - dF(x) &= t^r [c_{m+1}\,\beta_{m+1}(t)\,g_{m+1}(x) + \ldots + c_k\,\beta_k(t)\,g_k(x)] \\ &\in (t^r) \cap \left(F, \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}\right) \\ &= (t^r)\left(F, \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}\right) \end{aligned}$$

by (1.3). We can write

$$\begin{aligned} &t^r [c_{m+1}\,\beta_{m+1}(t)\,g_{m+1}(x) + \ldots + c_k\,\beta_k(t)\,g_k(x)] \\ &\quad = \left(t^t b_0 F + t^r b_1 \frac{\partial F}{\partial x_1} + \ldots + t^r b_n \frac{\partial F}{\partial x_n}\right) \end{aligned}$$

where b_i is in $\mathbf{C}\{x_1, \ldots, x_n, t\}$ for all $0 \leqq i \leqq n$. Expanding by powers of t, comparing the coefficient of t^r on both sides and then setting $t = 0$, we get

$$\begin{aligned} &c_{m+1}\,\beta_{m+1}(0)\,g_{m+1}(x) + \ldots + c_k\,\beta_k(0)\,g_k(x) \\ &\quad = b_0(0)\,f + b_1(0)\,\frac{\partial f}{\partial x_1} + \ldots + b_n(0)\,\frac{\partial f}{\partial x_n} \\ &\quad \equiv 0 \quad \text{in } \mathbf{C}\{x_1, \ldots, x_n\} \Big/ \left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right). \end{aligned}$$

Since $\{g_{m+1}(x), \ldots, g_k(x)\}$ is a linearly independent set in $\mathbf{C}\{x_1, \ldots, x_n\} \Big/ \left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$, we have $c_i \beta_i(0) = 0$ for all $m+1 \leqq i \leqq n$. This contradicts to our previous assertions that $c_i \neq 0$ for all $m+1 \leqq i \leqq k$ and $\beta_j(0) \neq 0$ for some $m+1 \leqq j \leqq k$. Q.E.D.

§ 2. Construction of a family of solvable Lie algebras over the (μ, τ)-constant strata S_E

Let $(V, 0)$ be a hypersurface singularity defined by a weighted homogenous polynomial $f(x_1, \dots, x_n)$. In § 1 we have shown that $S_E = \mathbf{C}^m$ and the equitopological deformation is given by

$$\begin{array}{ccc} \{(x_1, \dots, x_n, t_1, \dots, t_m): f(x) + t_1 g_1(x) + \dots + t_m g_m(x) = 0\} & \hookrightarrow & \mathbf{C}^n \times \mathbf{C}^m \\ \downarrow & & \downarrow{\scriptstyle pr_2} \\ S_E & = & \mathbf{C}^m \end{array}$$

where g_i are those monomials in a monomial basis of $\mathbf{C}\{x_1, \dots, x_n\} \Big/ \left(f, \dfrac{\partial f}{\partial x_1}, \dots, \dfrac{\partial f}{\partial x_n}\right)$ such that $wt(g_i) = wt(f)$. In this section we shall construct a family of solvable Lie algebras over S_E. Recall that in [6], we have associated to an isolated singularity $(V, 0)$ a finite dimensional Lie algebra $L(V)$, which is defined to be the algebra of derivations of the moduli algebra $A(V)$. $L(V)$ is solvable if $n \leqq 5$ [7]. We shall define a Lie subalgebra $\tilde{L}(V)$ of $L(V)$. This Lie subalgebra $\tilde{L}(V)$ admits a natural deformation over the parameter space S_E. Recall that by Theorem 1.2, S_E is isomorphic to $\mathbf{C}^m$ with coordinates $t_1, \dots, t_m$.

Definition. A derivation $D_0 \in L(V)$ is liftable to S_E if there exist differential operators D_{τ_i} such that

$$D = D_0 + \sum_{|\tau_1| = 1} t^{\tau_1} D_{\tau_1} + \sum_{|t_2| = 2} t^{\tau_2} D_{\tau_2} + \dots$$

leaves the ideal $(f_{x_1} + t_1 g_{1x_1} + \dots + t_m g_{mx_1}, f_{x_2} + t_1 g_{1x_2} + \dots + t_m g_{mx_2}, \dots, f_{x_n} + t_1 g_{1x_n} + \dots + t_m x_n)$ in $\mathbf{C}\{x_1, \dots, x_n, t_1, \dots, t_m\}$ invariant. (By differential operator, we mean operator of the form $d_1(x) \dfrac{\partial}{\partial x_1} + \dots + d_n(x) \dfrac{\partial}{\partial x_n}$ with $d_j(x)$ a linear combination of monomial basis elements of the moduli algebra $A(V)$.)

Here we use the standard notation for multi-indices. For example if $\alpha = (\alpha_1, \dots, \alpha_n)$, then $|\alpha| = \alpha_1 + \dots \alpha_n$ and

$$t^\alpha D_\alpha = t_1^{\alpha_1} \dots t_n^{\alpha_n} \left(d_1^\alpha(x) \frac{\partial}{\partial x_1} + \dots + d_n^\alpha(x) \frac{\partial}{\partial x_n} \right).$$

Definition 2.2. The Liftable Lie algebra $\tilde{L}(V)$ is defined to be the set of those $D_0 \in L(V)$ such that D_0 is liftable to S_E.

Clearly $\tilde{L}(V)$ is a Lie subalgebra of $L(V)$ and has a natural deformation over the parameter space S_E. Let D be an operator as above. To say that

D is a lifting of D_0, for all $1 \leqq i \leqq n$ there must exist ${}_i a_v^j \in \mathbf{C}\{x_1, \ldots, x_n\}$ for $1 \leqq i, \leqq n$ and $|v| \geqq 1$ such that

$$\begin{aligned} &D(f_{x_i} + t_1 g_{1x_i} + t_2 g_{2x_i} + \ldots + t_m g_{mx_i}) \\ &= ({}_i a_0^1 + \sum_{|v_1|=1} {}_i a_{v_1}^1 t^{v_1} + \sum_{|v_2|=1} {}_i a_{v_2}^1 t^{v_2} + \ldots)(f_{x_1} + \sum_{|\mu_1|=1} (g_{x_1} t)^{\mu_1}) \\ &\quad + ({}_i a_0^2 + \sum_{|v_1|=1} {}_i a_{v_1}^2 t^{v_1} + \sum_{|v_2|=2} {}_i a_{v_2}^n t^{v_2} + \ldots)(f_{x_n} + \sum_{|\mu_2|=1} (g_{x_n} t)^{\mu_2}) \\ &\quad + \ldots \\ &\quad + ({}_i a_0^n + \sum_{|v_1|=1} {}_i a_{v_1}^n t^{v_1} + \sum_{|v_2|=2} {}_i a_{v_2}^n t^{v_2} + \ldots)(f_{x_n} + \sum_{|\mu_n|=1} (g_{x_n} t)^{\mu_n}) \end{aligned} \tag{2.1}$$

where μ_i is a multi-index with m entries. We use the notations $(g_{x_1} t)^{(0,0,1,0,\ldots,0)}$ to denote $g_{3x_1} t_3$ and $g_{x_1}^{(0,0,1,0,\ldots,0)}$ to denote g_{3x_1}, for example.

The left hand side of (2.1) is given by

$$\begin{aligned} D_0 f_{x_i} &+ \sum_{|\tau_1|=1} (D_0 g_{x_i}^{\tau_1} + D_{\tau_1} f_{x_i}) t^{\tau_1} \\ &+ \sum_{|\tau_2|=2} \Big[\sum_{\substack{|\tau_1|=1 \\ \tau_2 \geqq t_1}} (D_{\tau_1} g_{x_i}^{\tau_2-\tau_1}) + D_{\tau_2} f x_i \Big] t^{t_2} \end{aligned}$$

while the right hand side of (2.1) is given by

$$\begin{aligned} &{}_i a_0^1 f_{x_1} + {}_i a_0^2 f_{x_2} + \ldots + {}_i a_0^n f_{x_n} \\ &\quad + \sum_{|v_1|=1} \big[{}_i a_0^1 g_{x_1}^{v_1} + {}_i a_0^2 g_{x_2}^{v_1} + \ldots + {}_i a_0^n g_{x_n}^{v_1} + {}_i a_{v_1}^1 f_{x_1} + {}_i a_{v_1}^2 f_{x_2} + \ldots + {}_i a_{v_1}^n f_{x_n} \big] t^{v_1} \\ &\quad + \sum_{|v_2|=2} \Big[\sum_{\substack{|v_1|=1 \\ v_2 \geqq v_1}} {}_i a_{v_1}^1 (g_{x_1})^{v_2-v_1} + {}_i a_{v_1}^2 (g_{x_2})^{v_2-v_1} + \ldots + {}_i a_{v_1}^n (g_{x_n})^{v_2-v_1} \\ &\quad + ({}_i a_{v_2}^1 f_{x_1} + {}_i a_{v_2}^2 f_{x_2} + \ldots + {}_i a_{v_2}^n f_{x_n}) \Big] t^{v_2} + \ldots . \end{aligned}$$

By comparing the coefficients of t, we conclude that

$$D_0 f_{x_i} = {}_i a_0^1 f_{x_1} + {}_i a_0^2 f_{x_2} + \ldots + {}_i a_0^n f_{x_n} \tag{0}$$

$$\begin{aligned} &D_{v_1} f_{x_i} - ({}_i a_{v_1}^1 f_{x_1} + {}_i a_{v_1}^2 f_{x_2} + \ldots + {}_i a_{v_1}^n f_{x_n}) \\ &\quad = -D_o g_{x_i}^{v_1} + ({}_i a_0^1 g_{x_1}^{v_1} + {}_i a_0^2 g_{x_2}^{v_1} + \ldots + {}_i a_0^1 g_{x_n}^{v_1}) \qquad \forall |v_1| = 1 \end{aligned} \tag{1}$$

$$\begin{aligned} &D_{v_2} f_{x_i} - ({}_i a_{v_2}^1 f_{x_1} + {}_i a_{v_2}^2 f_{x_2} + \ldots + {}_i a_{v_2}^n f_{x_n}) \\ &\quad = -\sum_{\substack{|v_1|=1 \\ v_2 \geqq v_1}} D_{v_1} g_{x_i}^{v_2-v_1} + \sum_{\substack{|v_1|=1 \\ v_2 \geqq v_1}} ({}_i a_{v_1}^1 g_{x_1}^{v_2-v_1} + {}_i a_{v_1}^2 g_{x_2}^{v_2-v_1} \\ &\qquad + \ldots + {}_i a_{v_1}^n g_{x_n}^{v_2 v_1}) \qquad \forall |v_2| = 2 \end{aligned} \tag{2}$$

etc. If D_0 is in $L(V)$, then there exist ${}_i a_0^j$ such that Eq. (0) is satisfied. In order to show that D_0 is liftable, the first step is to find D_{v_1} and ${}_i a_{v_1}^j$ such that Eqs. (1) are satisfied for all $|v_1| = 1$. The second step is to find D_{v_2} and ${}_i a_{v_2}^j$ such that

Eqs. (2) are satisfied for all $|v_2|=2$, etc. (Actually we can use this the definition of liftability for D_0.) Restricting ourselves to the three variable case with $m=1$ (i.e. 1-parameter family of deformations), the above equations read as follows:

$$\begin{aligned} D_0(f_x) &= a_0^1 f_x + a_0^2 f_y + a_0^3 f_z \\ D_0(f_y) &= b_0^1 f_x + b_0^2 f_y + b_0^3 f_z \\ D_0(f_z) &= c_0^1 f_x + c_0^2 f_y + c_0^3 f_z \end{aligned} \tag{0'}$$

$$\begin{aligned} D_1(f_x)-(a_1^1 f_x + a_z^2 f_y + a_1^3 f_z) &= -D_0 g_x + (a_0^1 g_x + a_0^2 g_y + a_0^3 g_z) \\ D_1(f_y)-(b_1^1 f_x + b_1^2 f_y + b_1^3 f_z) &= -D_0 g_y + (b_0^1 g_x + b_0^2 g_y + b_0^3 g_z) \\ D_1(f_z)-(c_1^1 f_x + c_1^2 f_y + c_1^3 f_z) &= -D_0 g_z + (c_0^1 g_x + c_0^2 g_y + c_0^3 g_z) \end{aligned} \tag{1'}$$

$$\begin{aligned} D_2(f_x)-(a_2^1 f_x + a_2^2 f_y + a_2^3 f_z) &= -D_1 g_x + (a_1^1 g_x + a_1^2 g_4 + a_1^3 g_z) \\ D_2(f_y)-(b_2^1 f_x + b_2^2 f_y + b_2^3 f_z) &= -D_1 g_y + (b_1^1 g_x + b_1^2 g_y + b_1^3 g_z) \\ D_2(f_z)-(c_2^1 f_x + c_2^2 f_y + c_2^3 f_z) &= -D_1 g_z + (c_1^1 g_x + c_1^2 g_4 + c_1^3 g_z) \end{aligned} \tag{2'}$$

etc.

Example. Let $V=\{(x,y,z): f(x,y,z)=x^3+y^3+z^3=0\}$. Then the moduli algebra is given by the vector space spanned by $1, x, y, z, xy, yz, zx$ and xyz.

The weight of $g(x,g,z)=xyz$ is three, which is exactly the weight of f. In view of Theorem 2, the equitopological deformation of V is given by

$$V_i=\{(x,y,z): x^3+y^3+z^3+txyz=0\} \qquad t^3+27\neq 0.$$

It is easy to see that the Lie algebra $L(V_0)$ associated to $V=V_0$ is given by

$$L(V_0)=\left\langle x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, z\frac{\partial}{\partial z}, xy\frac{\partial}{\partial x}, zx\frac{\partial}{\partial x}, xy\frac{\partial}{\partial y}, yz\frac{\partial}{\partial y}, \right.$$
$$\left. yz\frac{\partial}{\partial z}, zx\frac{\partial}{\partial z}, xyz\frac{\partial}{\partial x}, xyz\frac{\partial}{\partial y}, xyz\frac{\partial}{\partial z}\right\rangle.$$

We claim that $x\dfrac{\partial}{\partial x}$ is not liftable. To see this, we observe that

$$x\frac{\partial}{\partial x}(f_x)=2f_x+0f_y+0\cdot f_z$$
$$\Rightarrow a_0^1=2, \qquad a_0^2=0, \qquad a_0^3=0.$$

Suppose that there exist a_1^1, a_1^2, a_1^3 and

$$D_1=(\alpha_1 x+\alpha_2 y+\alpha_3 z)\frac{\partial}{\partial x}+(\beta_1 x+\beta_2 y+\beta_3 z)\frac{\partial}{\partial y}+(\gamma_1 x+\gamma_2 y+\gamma_3 z)\frac{\partial}{\partial z}$$

such that (1′) is satisfied. Then

$$\begin{aligned}a_1^1 f_x + a_1^2 f_y + a_1^3 f_z &= D_1 f_x + D_0 g_x - (a_0^1 g_x + a_0^2 g_y + a_0^3 g_z)\\ &= 6\alpha_1 x^2 + 6\alpha_2 xy + 6\alpha_3 xz - 2yz.\end{aligned}$$

Because of the appearance of $-2yz$ on the right hand side, there is no choice of a_1^1, a_1^2, a_1^3, α_1, α_2 and α_3 which makes above equations true.

On the other hand, we claim that $D_0 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ is liftable. In fact, D_0 itself preserves the ideal generated by $3x^2 + tyz$, $3y^2 + txz$, and $3z^2 + txy$. We can see that

$$\begin{aligned}\tilde{L}(V_0) = \Big\langle & x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},\ xy\frac{\partial}{\partial x},\ zx\frac{\partial}{\partial x},\ xy\frac{\partial}{\partial y},\ yz\frac{\partial}{\partial y},\\ & yz\frac{\partial}{\partial z},\ zx\frac{\partial}{\partial z},\ xyz\frac{\partial}{\partial x},\ xyz\frac{\partial}{\partial y},\ xyz\frac{\partial}{\partial z}\Big\rangle.\end{aligned}$$

Indeed, the equitopological deformation $\{V_t\}$ gives the following deformation of $\tilde{L}(V_0)$.

$$\begin{aligned}\tilde{L}(V_t) = \Big\langle & x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},\ xy\frac{\partial}{\partial x} - \frac{t}{6} zx\frac{\partial}{\partial y},\ zx\frac{\partial}{\partial x} - \frac{t}{6} xy\frac{\partial}{\partial z},\\ & xy\frac{\partial}{\partial y} - \frac{t}{6} yz\frac{\partial}{\partial x},\ yz\frac{\partial}{\partial y} - \frac{t}{6} xy\frac{\partial}{\partial z},\ yz\frac{\partial}{\partial z} - \frac{t}{6} zx\frac{\partial}{\partial y},\\ & zx\frac{\partial}{\partial z} - \frac{t}{6} yz\frac{\partial}{\partial x},\ xyz\frac{\partial}{\partial x},\ xyz\frac{\partial}{\partial y},\ xyz\frac{\partial}{\partial z}\Big\rangle.\end{aligned}$$

This deformation is actually a trivial family (as a family of Lie algebras).

§ 3. Torelli type problems

In § 2, we have constructed a family of Lie algebras $\tilde{L}(V_t)$ over S_E. It is natural to study the following Torelli type problem: If $\tilde{L}(V_{t_1}) \simeq \tilde{L}(V_{t_2})$ as Lie algebra t_1, t_2 in S_E, is V_{t_1} biholomorphically equivalent to V_{t_2}. In what follows, we shall study this problem for simple elliptic singularities $\tilde{E}_7$ and $\tilde{E}_8$.

Definition. Let $\{(x_1, y_1), (x_2 : y_2), (x_3 : y_3), (x_4 : y_4)\}$ be an ordered set of four distinct points in $\mathbf{CP}^1$. The cross-ratio associated to this ordered set is

$$r = \frac{(x_1 y_3 - x_3 y_1)(x_2 y_4 - x_4 y_2)}{(x_1 y_4 - x_4 y_1)(x_2 y_3 - x_3 y_2)}.$$

Remark. Consider a point in $\mathbf{CP}^1$ as a line in $\mathbf{C}^2$. Given 4 lines, choose a basis vector for each. Write $(x_3, y_3)=a(x_1, y_1)+b(x_2, y_2)$, $(x_4, y_4)=c(x_1, y_1)+d(x_2, y_2)$. Then the cross-ratio is in fact equal to $\frac{bc}{ad}$.

Remark. Under all possible orderings of four distinct points $\{(x_1:y_1), (x_2:y_2), (x_3:y_3), (x_4:y_4)\}$ in $\mathbf{CP}^1$, six cross-ratios occur, four times apiece. The set of six cross ratios is $\left\{r, r^{-1}, \frac{r}{r-1}, \frac{r-1}{r}, \frac{1}{1-r}, 1-r\right\}$. The ratios occur in pairs (inverses of each other) and the Klein-4 subgroup of S_4, acting as permutations of the four points, leaves the cross-ratio unchanged.

The following proposition is easy to prove and is well-known.

Proposition 3. *Let $A=\{(x_1:y_1), (x_2:y_2), (x_3:y_3), (x_4:y_4)\}$ and $B=\{(z_1:w_1), (z_2:w_2), (z_3:w_3), (z_4:w_4)\}$ be two sets of four points in $\mathbf{CP}^1$. Then there exists a linear automorphism of $\mathbf{CP}^1$ which carries the set A to the set B if and only if the sets of six cross-ratios associated to A and B respectively are the same: i.e.*

$$\left\{r_A, r_A^{-1}, \frac{r_A}{r_A-1}, \frac{r_A-1}{r_A}, \frac{1}{1-r_A}, 1-r_A\right\}$$
$$=\left\{r_B, r_b^{-1}, \frac{r_B}{r_B-1}, \frac{r_B-1}{r_B}, \frac{1}{1-r_B}, 1-r_B\right\}.$$

Let $\tilde{E}_7$ be a simple elliptic singularity defined by $\{(x, y, z)\in\mathbf{C}^3: x^4+y^4+z^2=0\}$. It is clear from Theorem 1.2 that the (μ, τ)-constant family is given by

$$V_t=\{(x, y, z): f_t(x, y, z)=x^4+y^4+tx^2y^2+z^2=0\} \quad \textit{with} \quad t^2 \neq 4 \tag{3.1}$$

Hence $S_E=\mathbf{C}-\{\pm 2\}$.

Theorem 3.1. *A Torelli type theorem holds for simple elliptic singularities $\tilde{E}_7$. I.e., $\tilde{L}(V_{t_1})\cong\tilde{L}(V_{t_2})$ as Lie algebras for $t_1 \neq t_2$ in S_E if and only if V_{t_1} is biholomorphically equivalent to V_{t_2}.*

Proof. Recall that by Mather-Yau [2] there is a one to one correspondence between the complex structure of the singularity V_t and its moduli algebra

$$A_t=\mathbf{C}\{x, y, z\}\Big/\left(\frac{\partial f_t}{\partial x}, \frac{\partial f_t}{\partial y}, \frac{\partial f_t}{\partial z}\right)$$
$$=\langle 1, x, y, x^2, xy, y^2, x^2y, xy^2, x^2y^2\rangle$$

with multiplication rules

$$\begin{aligned} x^3 &= -\frac{t}{2}xy^2 \\ y^3 &= -\frac{t}{2}x^2y. \\ x^3y &= xy^3=0 \end{aligned} \tag{3.2}$$

We now compute a basis of $L(V_t) := \mathrm{Der}_{\mathbf{C}}(A_t)$. Observe that $A_t = \mathbf{C}\{x, y\}/I_t$ where $I_t = \left(\frac{\partial f_t}{\partial x}, \frac{\partial f_t}{\partial y}\right) = (4x^3 + 2txy^2, 4y^3 + 2tx^2y)$. Any element $D \in \mathrm{Der}_{\mathbf{C}}(A_t)$ can be written as

$$\begin{aligned} D = (&a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{21}x^2y \\ &+ a_{12}xy^2 + C_{22}x^2y^2)\frac{\partial}{\partial x} + (b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy \\ &+ b_{02}y^2 + b_{21}x^2y + b_{12}xy^2 + b_{22}x^2y^2)\frac{\partial}{\partial y} \end{aligned}$$

which leaves $\left(\frac{\partial f_t}{\partial x}, \frac{\partial f_t}{\partial y}\right)$ invariant.

$$\begin{aligned} D\left(\frac{\partial f_t}{\partial x}\right) &= D(4x^3 + 2txy^2) \\ &\equiv [12a_{00}]x^2 + [2ta_{00}]y^2 + [4tb_{00}]xy \\ &\quad + [(12 - t^2)a_{01} + 4tb_{10}]x^2y + [-4ta_{10} + 4tb_{01}]xy^2 \\ &\quad + [-4ta_{20} + (12 - t^2)a_{02} + 4tb_{11}]x^2y^2 \\ &\equiv 0 \pmod{I_t}. \end{aligned} \tag{3.3}$$

Interchanging x and y yields similar facts. Setting the coefficients of each basis vector in A_t equal to 0, using $t^2 \neq 0, 4, 36$, restricts the values of the a_{ij}'s and b_{ij}'s. We find that a basis for $L(V_t)$ is then the following:

$$e_0 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

$$e_1 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \qquad e_2 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}$$

$$e_3 = (t^2 - 12)xy\frac{\partial}{\partial x} + 4tx^2\frac{\partial}{\partial y}, \quad e_4 = 4ty^2\frac{\partial}{\partial x} + (t^2 - 12)xy\frac{\partial}{\partial y}$$

$$e_5 = x^2y\frac{\partial}{\partial x}, \quad e_6 = xy^2\frac{\partial}{\partial y}, \quad e_7 = xy^2\frac{\partial}{\partial x}, \quad e_8 = x^2y\frac{\partial}{\partial y}$$

$$e_9 = x^2y^2\frac{\partial}{\partial x}, \quad e_{10} = x^2y^2\frac{\partial}{\partial y}.$$

For $t = 0$, $\{e_0\}$ is replaced by $\left\{x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\right\}$.

For $t = 6$, $\{e_0\}$ is replaced by $\left\{e_0, y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right\}$, $t = -6$, $\{e_0\}$ is replaced by $\left\{e_0, y\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right\}$.

The Liftable Lie algebra $\tilde{L}(V_t)$ defined in the previous section is spanned by $\langle e_0, e_1, \ldots, e_{10}\rangle$. The nilradical N_t of $\tilde{L}(V_t)$ is of dimension 10 spanned by $\langle e_1, \ldots, e_{10}\rangle$. We shall show that the mapping $\{V_t\} \to \{\tilde{L}(V_t)\}$ gives a one-to-one correspondence between the complex structures of V_t and isomorphism classes of the solvable Lie algebras $\tilde{L}(V_t)$. For this purpose it suffices to show that the natural mapping $\{V_t\} \to \{N_t\}$ gives a one-to-one correspondence between the complex structures of V_t and isomorphism classes of the nilpotent Lie algebras N_t. Using equations (3.2), we have the following multiplication table

$$
\begin{aligned}
&[e_1, e_2]=0 && [e_3, e_4]=(t^2+36)(t^2-4)(e_5-e_6)\\
&[e_1, e_3]=-3(t^2-4)e_6 && [e_3, e_5]=-(t^2+12)e_9\\
&[e_1, e_4]=3(t^2-4)e_8 && [e_3, e_6]=(-t^2+12)e_9\\
&[e_1, e_5]=-e_{10} && [e_3, e_7]=-8\,t\,e_{10}\\
&[e_1, e_6]=2\,e_{10} && [e_3, e_8]=-24\,e_{10}.\\
&[e_1, e_7]=e_9 && \\
&[e_1, e_8]=0 &&
\end{aligned}
$$

All brackets among e_5, e_6, e_7, e_8, e_9, e_{10} are zero because A_t has no nonzero monomials of degree bigger than 4. Brackets incolving e_2 and e_4 can be copied from the above using $x \leftrightarrow y$ symmetry, which induces $e_1 \leftrightarrow e_2$, $e_3 \leftrightarrow \varepsilon_4, \ldots, e_9 \leftrightarrow e_{10}$.

We now simplify the multiplication table by introducting a more convenient basis. The factor (t^2-4) which appears in all brackets among e_1, e_2, e_3, e_4 can be eliminated, as can a few other unpleasantries. Replace e_5 by (t^2-4) $(e_5+2\,e_6)$ and e_6 by (t^2-4) $(e_6+2\,e_5)$. (These are linearly independent.) Multiply each of e_7, e_8, e_9, and e_{10} by t^2-4. Replace e_3 by $e_3+\frac{t^2+36}{3}\,e_2$ and e_4 by $e_4+\frac{t^2+36}{3}\,e_1$.

The multiplication table with respect to this new basis is:

$$
\begin{aligned}
&[e_1, e_3]=e_6-2\,e_5 && [e_2, e_3]=3\,e_7\\
&[e_1, e_4]=3\,e_8 && [e_2, e_4]=e_5-2\,e_6\\
&[e_1, e_5]=3\,e_{10} && [e_2, e_6]=3\,e_9\\
&[e_1, e_7]=e_9 && [e_2, e_8]=e_{10}\\
&[e_3, e_5]=-3(t^2-4)e_9 && [e_4, e_5]=-2(t^2-12)e_{10}\\
&[e_3, e_6]=-2(t^2-12)e_9 && [e_4, e_6]=-3(t^2-4)e_{10}\\
&[e_3, e_7]=-8\,t\,e_{10} && [e_4, e_7]=\tfrac{1}{3}(t^2-36)e_9\\
&[e_3, e_8]=\tfrac{1}{3}(t^2-36)e_{10} && [e_4, e_8]=-8\,t\,e_9
\end{aligned}
$$

other brackets are zero.

Let

$$\begin{aligned} Z &:= \text{center}(N_t) = \langle e_9, e_{10} \rangle \\ Z^2 &:= \{n \in N_t\colon \text{Image}(ad_n) \subseteq Z\} \\ &= \langle e_5, e_6, \ldots, e_{10} \rangle \\ Z^3 &:= \{n \in N_t\colon \text{Image}(ad_n) \subseteq Z^z\} \\ &= N_t. \end{aligned}$$

Although $\langle e_5, e_6, e_7, e_8 \rangle$ is not an invariant subspace under Lie algebra automorphisms, the quotient space Z^2/Z (on which Lie algebra automorphisms of N_t act) has a basis represented by e_5, e_6, e_7 and e_8.

For $t=0$, $\text{Image}(ad_{e_7}) = \mathbf{C} e_9$ and $\text{Image}(ad_{e_8}) = \mathbf{C} e_{10}$. Generically,

$$\text{Image}(ad_{a_5 e_5 + a_6 e_6 + a_7 e_7 + a_8 e_8}) = \mathbf{C} e_9 + \mathbf{C} e_{10}.$$

We shall see that for most values of t, there are exactly four vectors, unique up to scalar multiple modulo Z, which lie in Z^2 and have one-dimensional adjoint image. These vectors are linearly independent and form a basis for Z^2/Z. Each of the four one-dimensional images is of course in Z. These four lines in $\mathbf{C}^2$ have a cross-ratio, relative to a choice of order. The 4! possible orderings yield (in general) six different cross rations. These six numbers are continuous invariants of N_t. It will thus be seen that seen that $N_s = N_t$ if and only if V_s is biholomorphically equivalent to V_t.

We will show that $A_s \simeq A_t \to N_s \simeq N_t \to s \in \left\{ \pm t, \pm\left(\frac{12-2t}{2+t}\right), \pm\left(\frac{12+2t}{2-t}\right) \right\}$ $\to A_s \simeq A_t$.

It is clear that N_t depends on the analytic type of the singularity (i.e. N_t is an analytic invariant), hence first implication. The last is easy to check. (E.g. $f_t(x', y', z) = f_{\frac{12+2t}{2-t}}(x, y, z)$ where $x' = kx + ky$, $y' = ikx - iky$, $i^2 = -1$, and $k^4 = \frac{1}{2-t}$. Also $f_t(x, iy, z) = f_{-t}(x, y, z)$. These two substitutions of s for t generate all six such substitutions.)

We shall now see that the second implication holds. Therefore for $\tilde{E}_7$ singularities the isomorphism classes of N_t differ for singularities which are not of the same analytic type. Consider $ad_{ae_5 + be_6 + ce_7 + de_8}$ acting on the left. Its kernel includes the span of $e_5, e_6, e_7, e_8, e_9, e_{10}$; it sends

$$\begin{aligned} -e_1 &\mapsto c e_9 + 3 a e_{10} \\ -e_2 &\mapsto 3 b e_9 + d e_{10} \\ -e_3 &\mapsto (-2(t^2-12)b - 3(t^2-4)a)e_9 + (\tfrac{1}{3}(t^2-36)d - 8tc)e_{10} \\ -e_4 &\mapsto (\tfrac{1}{3}(t^2-36)c - 8td)e_9 + (-2(t^2-12)a - 3(t^2-4)b)e_{10}. \end{aligned}$$

We shall find the 4-tuples (a, b, c, d) for which the image of $ad_{ae_5+be_6+ce_7+de_8}$ is one-dimensional. The following fractions must be equal, or of the form $\frac{0}{0}$ (corresponding to $0e_9+0e_{10}$):

$$\frac{c}{3a}=\frac{3b}{d}=\frac{-2(t^2-12)b-3(t^2-4)a}{\frac{1}{3}(t^2-36)d-8tc}=\frac{\frac{1}{3}(t^2-36)c-8td}{-2(t^2-12)a-3(t^2-4)b}.$$

A laborious calculation leads to the following (easily verifiable) result for the case a, b, c and $d \neq 0$ and $t^2 \neq 0, 4, 20, 36$. Choose a branch for the square root function. Let

$$B := \frac{-(t^2-52)+8\sqrt{36-t^2}}{t^2-20}$$

$$C := \pm 3\sqrt{\frac{6+\sqrt{36-t^2}}{t}}.$$

Two choices of coefficients are

$$a=1, \quad b=B, \quad c=C, \quad d=9BC^{-1}$$

and

$$a=1, \quad b=B, \quad c=-C, \quad d=-9BC^{-1}.$$

The other two, corresponding to $a=1$ and $b=B^{-1}$, can be renormalized so that

$$a=B, \quad b=1, \quad c=9BC^{-1}, \quad d=C$$

and

$$a=B, \quad b=1, \quad c=-9BC^{-1}, \quad d=-C.$$

The x, y symmetry, which interchanges e_{2i-1} and e_{2i}; $i=1, 2, 3, 4, 5$, is apparent.

If one of the coefficients a, b, c, or d is zero t^2 must equal 0 or 20. For $t^2=0$, 36 there are only two special vectors.

In summary

$$j(t)=1\begin{cases} t=0 & e_7, e_8 & \text{two distinct special vectors} \\ t=6 & e_5+e_6+3e_7+3e_8 & \text{two distinct special vectors} \\ & e_5+e_6-3e_7-3e_8 & \\ t=-6 & e_5+e_6+3ie_7+3ie_8 & \text{two distinct special vectors} \\ & e_5+e_6-3ie_7+3ie_8 & \end{cases}$$

$$
j(t)=\frac{32}{27}\begin{cases}
t=2\sqrt{5} & e_5+\frac{3}{\sqrt[4]{5}}\,e_7 \\
 & e_5-\frac{3}{\sqrt[4]{5}}\,e_7 \quad \text{four distinct special vectors} \\
 & e_6+\frac{3}{\sqrt[4]{5}}\,e_8 \\
 & e_6-\frac{3}{\sqrt[4]{5}}\,e_8 \\
t=-2\sqrt{5} & e_5+\frac{3}{\sqrt[4]{5}}\,i\,e_7 \quad \text{four distinct special vectors} \\
 & e_5-\frac{3}{\sqrt[4]{5}}\,i\,e_7 \\
 & e_6+\frac{3}{\sqrt[4]{5}}\,i\,e_8 \\
 & e_6-\frac{3}{\sqrt[4]{5}}\,i\,e_8 .
\end{cases}
$$

The first two special vectors for $t=\pm 2\sqrt{5}$ are the limiting cases for $b=0$, while the last two special vectors for $t=\pm 2\sqrt{5}$ are the limiting cases for $c=0$. Other t ($t^2 \neq 0, 4, 20, 36$) yield

$$f_+=e_5+Be_6+C\,e_7+\frac{9B}{C}\,e_8 \quad \text{four distinct special vectors}$$

$$f_-=e_5+Be_6-Ce_7-\frac{9B}{C}\,e_8$$

$$g_+=Be_5+e_6+\frac{9B}{C}\,e_7+Ce_8$$

$$g_-=Be_5+e_6-\frac{9B}{C}\,e_7-Ce_8.$$

The image of the adjoint action of a special vector is spanned by its value on e_1 or e_2

$$-[f_+,e_1]=Ce_9+3e_{10}=X_+ \qquad -[g_+,e_2]=3e_9+Ce_{10}=Y_+$$

$$-[f_-,e_1]=-Ce_9+3e_{10}=X_- \qquad -[g_-,e_2]=3e_9-Ce_{10}=-Y_-.$$

These four vectors X_+, X_-, Y_+, and Y_-, are unique up to scalar multiples, and are pairwise linearly independent for $t^2 \neq 0, 4, 36$ (i.e. for $C^2 \neq 0, \pm 9$). Any linear combination of two or more of the four special vectors has a two-dimensional adjoint image. This shows that there are no more than four special vectors modulo scalar multiplication.

Generically, the six cross-ratios associated to the four vectors are $\left\{\delta, \frac{1}{\delta}, \frac{1}{1-\delta}, 1-\delta, \frac{\delta}{\delta-1}, \frac{\delta-1}{\delta}\right\}$ where

$$\delta = \left(\frac{9+c^2}{6c}\right)^2 = \frac{\left(9+\frac{9}{t}(6t+\sqrt{36-t^2})\right)^2}{36\left(\frac{9}{t}(6+\sqrt{36-t^2})\right)}$$

$$= \frac{(t+(6+\sqrt{36-t^2}))^2}{4t(6+\sqrt{36-t^2})} = \frac{72+12t+(12+2t)\sqrt{36-t^2}}{4t(6+\sqrt{36-t^2})}$$

$$= \frac{6+t}{2t}.$$

By Proposition 3, $N_t \simeq N_s$ implies

$$\left\{\frac{6+t}{2t}, \frac{2t}{6+t}, \frac{6+t}{6-1}, \frac{2t}{t-6}, \frac{t-6}{2t}\right\}$$

$$= \left\{\frac{6+s}{2s}, \frac{2s}{6+s}, \frac{6+s}{6-s}, \frac{6-s}{6+s}, \frac{2s}{s-6}, \frac{2s}{s-6}, \frac{s-6}{2s}\right\}$$

which is equivalent to $s = \pm t$, $\pm\left(\frac{12-2t}{2+t}\right)$ or $\pm\left(\frac{12+2t}{2-t}\right)$. But as shown before, the latter condition implies that V_t is biholomorphically equivalent to V_s. Q.E.D.

Remark. Let $J(\delta) = \frac{4(\delta^2-\delta+1)^3}{(\delta-2)^2(\delta+1)^2(2\delta-1)^2}$. It is easy to check that $J(\delta) = J(\delta^{-1}) = J\left(\frac{\delta-1}{\delta}\right) = J\left(\frac{\delta}{\delta-1}\right) = J(1-\delta) = J\left(\frac{1}{1-\delta}\right)$. Recall that $N_s \cong N_t$ if $s = \pm t$, $\pm\left(\frac{12-2t}{2+t}\right)$ or $\pm\left(\frac{12+2t}{2-t}\right)$. Then J is a function of the cross ratios and hence is well-defined on isomorphism classes among the N_t's. It is clear that $J(\delta(t)) = \frac{(t^2+12)^3}{108(t^2-4)^2}$ is a modulus for the family $\{N_t\}$ and hence $\{V_t\}$. This agrees with the work of Saito [5]. He found that the j-invariant of the elliptic curve which is the exceptional divisor in the resolution of singularity for each value of t is $j(t) = \frac{(t^2+12)^3}{108(t^2-4)^2}$.

Further investigations into the structure of N_t show how to continue to find a more canonical basis unique up to scalar multiplies modulo the invariant subspaces $Z(N_t)$ and $Z^2(N_t)$. Each of f_+, f_-, g_+, g_- has a 3-dimensional adjoint kernel in N_t/Z^2. These kernels are pairwise disjoint. Taking intersections, three at a time, gives a set of four invariant vectors (up to scalar multiplies) expressed as linear combinations of e_1, e_2, e_3, e_4. The multiplication table can then be completed (if need be) by computing Lie brackets among these four new vectors. This approach yields the best basis for studying the structure of N_t.

Let $\tilde{E}_8$ be a simple elliptic singularity defined by $\{(x, y, z) \in \mathbf{C}^3\colon x^6 + y^3 + z^2 = 0\}$. It is clear from Theorem 1.2 that the (μ, τ) constant family is given by

$$V_1 = \{(x, y, z) \in \mathbf{C}^3\colon f_t(x, y, z) = x^6 + y^3 + z^2 + t x^4 y = 0\}$$

with $4t^3 + 27 \neq 0$. Hence $S_E = \mathbf{C} - \{t \in \mathbf{C}\colon 4t^3 + 27 = 0\}$.

Theorem 3.2. *A Torelli type theorem holds for simple elliptic singularities $\tilde{E}_8$. I.e., $\tilde{L}(V_{t_1}) \cong \tilde{L}(V_{T_2})$ as Lie algebra for $t_1 \neq t_2$ in S_E if and only if V_{t_1} is biholomorphically equivalent to V_{t_2}.*

Proof. By the theorem of Mather-Yau [2] there is a one to one correspondence between the complex structure of the singularity V_t and its moduli algebra

$$\begin{aligned} A_t &= \mathbf{C}\{x, y, z\} \Big/ \left(\frac{\partial f_t}{\partial x}, \frac{\partial f_t}{\partial y}, \frac{\partial f_t}{\partial z}\right) \\ &= \langle 1, x, x^2, y, x^3, xy, x^4, x^2 y, x^3 y, x^4 y \rangle \end{aligned}$$

with multiplication rules

$$y^2 = -\frac{t}{3} x^4$$

$$x^5 = -\frac{2t}{3} x^3 y$$

A_t is a graded algebra with $\deg x = 1$ and $\deg y = 2$. Observe that $A_t = \mathbf{C}\{x, y\}/I_t$ where $I_t = \left(\frac{\partial f_t}{\partial x}, \frac{\partial f_t}{\partial y}\right) = (3x^5 + 2t x^3 y,\ 3y^2 + t x^4)$. Any element $D \in \mathrm{Der}_{\mathbf{C}}(A_t)$ can be written as

$$\begin{aligned} D = {} & (a_0 + a_1 x + a_2 x^2 + a_2^1 y + a_3 x_3 + a_3^1 x y + a_4 x^4 \\ & + a_4^1 x^2 y + a_5^1 x^3 y + a_6^1 x^4 y) \frac{\partial}{\partial x} + (b_0 + b_1 x + b_2 x^2 + b_2^1 y + b_3 x^3 \\ & + b_3^1 x y + b_4 x^4 + b_4^1 x^2 y + b_5^1 x^3 y + b_6^1 x^4 y) \frac{\partial}{\partial y}. \end{aligned}$$

The subscripts refer to degrees of monomials, with a_i^1, b_i^1 the coefficients of a monomial containing y.

(Computations, similar to those in the $\tilde{E}_7$ case, have been omitted.) A basis for L_t is, for $t \neq 0$, the following:

deg 0 $\quad e_0 = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$

deg 1 $\quad e_1 = x^2\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y},\ e_2 = 2ty\frac{\partial}{\partial x} + (2t^2x^3 - 15xy)\frac{\partial}{\partial y}$

deg 2 $\quad e_3 = (2t^2x^4 - 9x^2y)\frac{\partial}{\partial y},\ e_4 = 9x^3\frac{\partial}{\partial x} + 4t^2x^4\frac{\partial}{\partial y}$

$\quad e_5 = -3xy\frac{\partial}{\partial x} + 2tx^4\frac{\partial}{\partial y}$

deg 3 $\quad e_6 = x^4\frac{\partial}{\partial x},\ e_7 = x^2y\frac{\partial}{\partial z},\ e_8 = x^3y\frac{\partial}{\partial y}$

deg 4 $\quad e_9 = x^3y\frac{\partial}{\partial x},\ e_{10} = x^4y\frac{\partial}{\partial y}$

deg 5 $\quad e_{11} = x^4y\frac{\partial}{\partial x}$.

For $t=0$, $\{e_0\}$ is replaced by $\left\{x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\right\}$. The Lie algebra $\tilde{L}_t$ defined in the previous section is spanned by $\langle e_0, e_1, \ldots, e_{11}\rangle$. The nilradical N_t of $\tilde{L}_t$ is of dimension 11, spanned by $\langle e_1, e_2, \ldots, e_{11}\rangle$. We shall show that the mapping $\{V_t\} \to \{\tilde{L}_t\}$ gives a one-to-one correspondence between the complex structures of V_t an the isomorphism classes of the solvable Lie algebras $\tilde{L}_t$. Again, we will do this by studying the nilradical N_t.

$[e_1, e_2] = \frac{5}{3}e_3 \qquad [e_2, e_3] = -4t^3e_6 + 18te_7$

$[e_1, e_3] = -\frac{(8t+54)}{3}e_8 \qquad [e_2, e_4] = -8t^2e_6 + 54te_7 + (135 + 28t^3)e_8$

$[e_1, e_4] = 9e_6 - \frac{54 + 16t^3}{3}e_8 \qquad [e_2, e_5] = -8t^2e_6 + 45e_7 + 4t^2e_8$

$[e_1, e_5] = -3e_7 - \frac{4t^2}{3}e_8 \qquad [e_2, e_6] = 8te_9 + (15 + 4t^3)e_{10}$

$[e_1, e_6] = -\frac{4t}{3}e_9 - 2e_{10} \qquad [e_2, e_7] = -\frac{135 + 4t^3}{9}e_9 - \frac{8t^2}{3}e_{10}$

$[e_1, e_7] = 2e_9 - \frac{4t^2}{9}e_{10} \qquad [e_2, e_8] = -2te_9$

$[e_1, e_8] = 3e_{10} \qquad [e_2, e_9] = -15e_{11}$

$[e_1, e_9] = 3e_{11} \qquad [e_2, e_{10}] = -2te_{11}$

$[e_3, e_4] = (24t^3 + 162)e_{10} \qquad [e_4, e_5] = (8t^3 + 54)e_9 \qquad [e_5, e_6] = -9e_{11}$

$[e_3, e_5] = (4t^3 + 27)e_9 \qquad [e_4, e_6] = -6te_{11} \qquad [e_5, e_7] = -2t^2e_{11}$

$[e_3, e_7] = -\frac{4t^3 + 27}{3}e_{11} \qquad [e_4, e_7] = -\frac{27 + 8t^3}{3}e_{11} \qquad [e_5, e_8] = 3e_{11}$

Other brackets $[e_i, e_j]$, $i<j$ are zero. There are some invariant subspaces which show explicitly the structure of N_t. Let

$$\begin{aligned}
Z &:= \mathrm{center}(N_t) = \langle e_{11}\rangle \\
Z^2 &:= \{x\in N_t\colon \mathrm{Image}(ad_x)\subseteq Z\} = \langle e_9, e_{10}, e_{11}\rangle \\
Z^3 &:= \{x\in N_t\colon \mathrm{Image}(ad_x)\subseteq Z^2\} = \langle e_6, e_7, e_8, e_9, e_{10}, e_{11}\rangle \\
Z^4 &:= \{x\in N_t\colon \mathrm{Image}(ad_x)\subseteq Z^3\} = \langle e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}\rangle \\
Z^5 &:= \langle x\in N_t\colon \mathrm{Image}(ad_x)\subseteq Z^4\} = N_t \\
N^{(1)} &= [N, N] = \langle e_3, e_6, e_7, e_8, e_9, e_{10}, e_{11}\rangle \\
N^{(2)} &= [N, N^{(1)}] = \left\langle e_7 - \frac{2t^2}{9} e_6, e_8, e_9, e_{10}, e_{11}\right\rangle \\
N^{(3)} &= [N, N^{(2)}] = \langle e_9, e_{10}, e_{11}\rangle \\
N^{(4)} &= [N, N^{(3)}] = \langle e_{11}\rangle.
\end{aligned}$$

The quotient space $Z^2/Z = N^{(3)}/N^{(4)}$ is two-dimensional. spanned by the images of e_9 and e_{10}. There are four invariant lines in this space (i.e. each is preserved under all automorphism of N). Their ordered cross-ratio is a complex number which is also invariant under all automorphisms, and will therefore distinguish N_t from N_s unless $N_t \cong N_s$. Let

$$\begin{aligned}
&l_1 = Z^4/Z^3 \cap N^{(1)}/Z^3 = \mathbf{C}\bar{e}_3 \subseteq Z^4/Z^3 \\
&P_2 = \ker(ad_{l_1}) = \mathbf{C}\bar{e}_6 \oplus \mathbf{C}\bar{e}_8 \subseteq Z^3/Z^2 \\
&\qquad \text{where } ad_{l_1}\colon Z^3/Z^2 \to Z \\
&P_3 = \mathrm{Image}(ad_{l_1}) = \mathbf{C}\bar{e}_8 \oplus \mathbf{C}\overline{(9e_7 - 2t^2 e_6)} \subseteq Z^3/Z^2 \\
&\qquad \text{where } ad_{l_1}\colon Z^5/Z^4 \to Z^3/Z^2 \\
&l_4 = P_2 \cap P_3 = \mathbf{C}\bar{e}_8 \subseteq Z^3/Z^2 \\
&l_5 = \{\bar{x}\in Z^5/Z^4\colon ad_{l_1}(\bar{x}) \subseteq l_4\} = \mathbf{C}\bar{e}_1 \subseteq Z^5/Z^4 \\
&\qquad \text{where } ad_{l_1}\colon Z^5/Z^4 \to Z^3/Z^2 \\
&l_6 = [l_4, l_5] = \mathbf{C}\bar{e}_{10} \subseteq Z^2/Z \\
&l_7 = \ker(ad_{l_5}) = \mathbf{C}\overline{\left(e_6 + \frac{2t}{3} e_7 + \frac{8t^3+54}{81} e_8\right)} \subseteq Z^3/Z^2 \\
&\qquad \text{where } ad_{l_5}\colon Z^3/Z^2 \to Z^2/Z \\
&l_8 = \{\bar{x}\in Z^4/Z^3\colon ad_{l_5}(\bar{x}) \subseteq l_7\} = \mathbf{C}\overline{(4e_3 - 3e_4 + 6t e_5)} \\
&\qquad \text{where } ad_{l_5}\colon Z^4/Z^3 \to Z^3/Z^2 \\
&l_9 = [l_1, l_8] = \mathbf{C}\overline{(t e_9 - 3e_{10})} \subseteq Z^2/Z \\
&l_{10} = \ker(ad_{l_9}) = \mathbf{C}\overline{(3e_1 + e_2)} \subseteq Z^5/Z^4 \\
&\qquad \text{where } ad_{l_9}\colon Z^5/Z^4 \to Z \\
&l_{11} = [l_{10}, l_4] = \mathbf{C}\overline{(-2t e_9 + 9e_{10})} \subseteq Z^2/Z \\
&l_{12} = [l_1, l_{10}] = \mathbf{C}\overline{(2t^3 e_6 - 9t e_7 + (4t^3+27) e_8)} \subseteq Z^3/Z^2 \\
&l_{13} = [l_{10}, l_{12}] = \mathbf{C}\overline{(t e_9 + (2t^3+9) e_{10})} \subseteq Z^2/Z.
\end{aligned}$$

It is clear that any Lie algebra isomorphism from N_t to N_s induces an isomorphism from Z_t^2/Z_t to Z_s^2/Z_s which sends the ordered set $\{l_6(t), l_9(t), l_{11}(t), l_{13}(t)\}$ to the ordered set $\{l_6(s), l_9(s), l_{11}(s), l_{13}(s)\}$. The cross ratios of these two ordered sets are $\frac{2}{3}(2t^3+12)$ and $\frac{2}{3}(2s^3+12)$. Consequently $N_t \cong N_s$ implies that $s^3=t^3$. Conversely if $s^3=t^3$, then $s=\rho t$ for some ρ with $\rho^3=1$ and V_t is biholomorphically equivalent to V_s. The biholomorphism is given by $f_t(x, \rho y, z)=f_s(x, y, z)$. In particular N_s is isomorphic to N_t as a Lie algebra if $s^3=t^3$. Thus t^3 can be considered as the modulus of the analytic type of the $\tilde{E}_8$ singularities. Q.E.D.

Again this agrees with the work of Saito [5]. He found that $j(t)=\dfrac{4t^3}{4t^3+27}$, which is a one-to-one function of our modulus t^3.

Remark. A complete basis for N_t of $\tilde{E}_8$ can be obtained (for $t^3 \neq 0, -\frac{27}{4}$) by defining two more lines l_{14}, and l_{15} below, and choosing representative vectors for $l_5, l_{10}, l_1, l_8, l_{14}, l_4, l_7, l_{12}, l_6, l_9, l_{15}$. These vectors will be unique, up to scalar multiples, modulo higher centers Z^i.

$$\begin{aligned}
l_{14} &= \{\bar{x} \in Z^4/Z^3 : ad_{l_5}(\bar{x}) \subseteq l_{12}\} \\
&= \mathbf{C}\left(\left(\frac{32t^6}{9}+36t^3+81\right)e_3-\frac{16t^6+108t^3}{9}e_4-(24t^4+162t)e_5\right) \\
&\subseteq Z^4/Z^3 \\
&\quad \text{where } ad_{l_5}: Z^4/Z^3 \to Z^3/Z^2 \\
l_{15} &= \mathbf{C}e_{11} = Z.
\end{aligned}$$

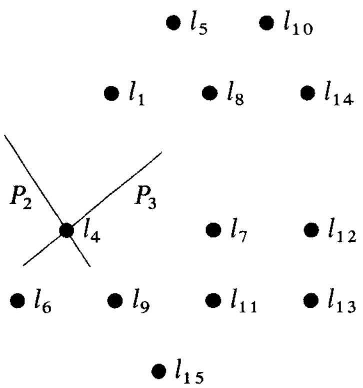
Two sets of Lie algebra generators are easily seen to be $\{e_1, e_2, e_4, e_5\}$ and

$$\Big\{e_1, 3e_1+e_2, 4e_3-3e_4+6te_5,$$
$$\left(\frac{32t^6}{9}+36t^3+81\right)e_3-\left(\frac{16t^6}{9}+12t^3\right)e_4-(24t^4+162t)e_5\Big\}.$$

The second set represents $\{l_5, l_{10}, l_8, l_{14}\}$.

Notice that $\tilde{L}_t$ is a graded Lie algebra. In fact each e_i is of pure degree acting on A_t. For example $e_{11}=x^4y\dfrac{\partial}{\partial x}$ raises degree by $5=5\deg x+\deg y-\deg x$.

deg 0	e_0
deg 1	e_1
	e_2
deg 2	e_3
	e_4
	e_5
deg 3	e_6
	e_7
	e_8
deg 4	e_9
	e_{10}
deg 5	e_{11}



A dot ● represents a complex line and a segment —— represents a complex plane. Notice that, for instance, Z^2 is the span of the degree 4 and degree 5 derivations, although the degree 4 subspace is not invariant under all automorphisms of N_t.

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