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Generalized Cartan Matrices Associated to *k*-th Yau Algebras of Singularities and Characterization Theorem

Naveed Hussain[1] · **Stephen S.-T. Yau**[2,3] · **Huaiqing Zuo**[4]

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Abstract

Let $(V, 0)$ be an isolated hypersurface singularity defined by the holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. The k-th Yau algebra $L^k(V)$ is defined to be the Lie algebra of derivations of the k-th moduli algebra $A^k(V) := \mathcal{O}_n/(f, m^k J(f))$, where $k \geq 0$, m is the maximal ideal of $\mathcal{O}_n$. The Generalized Cartan matrix $C^k(V)$ is an object associated to $L^k(V)$. We previously proposed a conjecture that ADE singularities can be completely characterized by $C^k(V)$, and verified it for $k = 1$ in our previous work. In this paper, we continue this work and verify this conjecture for $k = 2$.

Keywords Isolated singularity · Lie algebra · Generalized Cartan matrix

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✉ Stephen S.-T. Yau
yau@uic.edu

✉ Huaiqing Zuo
hqzuo@mail.tsinghua.edu.cn

Naveed Hussain
naveed1573@gmail.com

1 School of Data Sciences, Guangzhou Huashang College, Guangzhou, 511300, People's Republic of China

2 Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, People's Republic of China

3 Yanqi Lake Beijing Institute of Mathematical Sciences and Applications, Huairou, 101400, People's Republic of China

4 Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, People's Republic of China

1 Introduction

A Lie algebras L is called solvable (resp. nilpotent) if the derived series: $L^{(0)} = L, L^{(1)} = [L, L], L^{(i)} = [L^{(i-1)}, L^{(i-1)}], i = 2, 3, \ldots$ (resp. the lower central series: $L_0 = L, L_1 = [L, L], L_i = [L, L_{(i-1)}], i = 2, 3, \ldots$) terminates. Every nilpotent Lie algebra is solvable. It is well-known that finite dimensional Lie algebras are semi-direct product of the semi-simple Lie algebras and solvable Lie algebras. Simple Lie algebras have been well understood, but not the solvable (nilpotent) Lie algebras. The problem of classifying nilpotent Lie algebra was studied first time in 1891 by a student of Engle, Umlauf, who gave the complete list over $\mathbb{C}$ up to the dimension 6 and a certain continuous families at the dimensions 7, 8 and 9 [29]. Moreover, it is known that the classification of nilpotent Lie algebras in higher dimensions (> 7) remains to be a vast open area. There are one-parameter families of non-isomorphic nilpotent Lie algebras (but no two-parameter families) in dimension seven. Dimension seven is the watershed of the existence of such families. It is well-known that no such family exists in dimension less than seven, while it is very hard to construct one-parameter family in dimension greater than seven. Bratzlavsky [4] and Gabriel [12, 13] introduced the root systems to study nilpotent Lie algebras. The concept of root system constitutes an important step in the classification of nilpotent Lie algebras. By using these root systems, Santharoubane [26] established a link between the nilpotent Lie algebras and the Kac-Mody Lie algebras (which are infinite dimensional version generalization of the semi-simple Lie algebra).

The class of simple (ADE) singularities, consist of two series $A_k : \{x_1^{k+1} + x_2^2 = 0\} \subset \mathbb{C}^2, k \geq 1$, $D_k : \{x_1^2 x_2 + x_2^{k-1} = 0\} \subset \mathbb{C}^2, k \geq 4$, and three exceptional singularities E_6, E_7, E_8 defined in $\mathbb{C}^2$ by polynomials $x_1^3 + x_2^4, x_1^3 + x_1 x_2^3, x_1^3 + x_2^5$, respectively. ADE singularities have been studied since the nineteenth century, and there are many number of ways of characterising them (see Durfee [10]). They appear in many areas of geometry, algebraic geometry, singularity theory, group theory, etc.

Brieskorn [5] gave a beautiful connection between simple Lie algebras and simple singularities. Thus it is extremely important to establish connection between singularities and solvable (nilpotent) Lie algebras. Recently, in [7, 17, 18, 23], Yau, Zuo and Hussain gave many new natural connections between the set of complex analytic isolated hypersurface singularities and the set of finite dimensional solvable (nilpotent) Lie algebras. They introduced two different ways to associate Lie algebras to isolated hypersurface singularities. These constructions are helpful to understand the solvable (nilpotent) Lie algebras from the geometric point of view [7].

On the one hand, in [7], the authors firstly introduced a new finite dimensional solvable (nilpotent) Lie algebras to isolated hypersurface singularities as follows. For an isolated hypersurface singularity $(V, 0)$ defined by the holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, the new Lie algebra $L^*(V) := \mathrm{Der}(A^*(V), A^*(V))$, was defined to be the Lie algebra of derivations of the Artinian algebra

$$A^*(V) = \mathcal{O}_n/(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}, \mathrm{Det}(\frac{\partial^2 f}{\partial x_i \partial x_j})_{i,j=1,\cdots,n}),$$

and $\lambda^*(V)$ is the dimension of $L^*(V)$. In [7], it was shown that $L^*(V)$ completely distinguish ADE singularities. Furthermore, the authors have proven Torelli-type theorems for some simple elliptic singularities. Therefore, this new Lie algebra $L^*(V)$ is a subtle invariant of isolated hypersurface singularities. It is a natural question whether we can distinguish singularities by only using part of information of $L^*(V)$. In [19], we studied generalized Cartan matrices of the new Lie algebra $L^*(V)$ for simple hypersurface singularities and simple elliptic singularities. We introduced many other numerical invariants, namely, dimension of the maximal nilpotent subalgebras (i.e., nilradical of nilpotent Lie algebra) $g(V)$ of $L^*(V)$; dimension of maximal torus of $g(V)$, etc. We have proven that the generalized Cartan matrix of $L^*(V)$ can be used to characterize the ADE singularities except the pair of A_6 and D_5 singularities [19].

In the theory of isolated singularities, one always wants to find invariants associated to the isolated singularities. Hopefully with enough invariants found, one can distinguish between isolated singularities. However, not many invariants are known. Later, more general derivation Lie algebras $L_k(V), 0 \le k \le n+1$ associated to the isolated hypersurface singularity $(V, 0)$ defined by the holomorphic function $f(x_1, \cdots, x_n)$ are introduced. Let $Hess(f)$ be the Hessian matrix (f_{ij}) of the second order partial derivatives of f and $h(f)$, the Hessian of f, be the determinant of the matrix $Hess(f)$. More generally, for each k satisfying $0 \le k \le n$ we denote by $h_k(f)$ the ideal in $\mathcal{O}_n$ generated by all $k \times k$-minors in the matrix $Hess(f)$. In particular, the ideal $h_n(f) = (h(f))$ is a principal ideal. For each k as above, consider the graded k-th Hessian algebra of the polynomial f defined by

$$H_k(f) = \mathcal{O}_n/(f + J(f) + h_k(f)).$$

It is easy to check that the isomorphism class of the local k-th Hessian algebra $H_k(f)$ is contact invariant of f, i.e. depends only on the isomorphism class of the germ $(V, 0)$ ([9], Lemma 2.1). In [23], we investigated the new Lie algebra $L_k(V)$ which is the Lie algebra of derivations of k-th Hessian algebra $H_k(f)$. The dimension of $L_k(V)$, denoted by $\lambda_k(V)$, is a new numerical analytic invariant of an isolated hypersurface singularity. In particular $L_n(V) = L^*(V)$, so $L_k(V)$ is a generalization of $L^*(V)$.

On the other hand, for any isolated hypersurface singularity $(V, 0) \subset (\mathbb{C}^n, 0)$ where $V = V(f) = \{f = 0\}$, in the early 80s, Yau introduced the Lie algebra of derivations of moduli algebra $A(V) := \mathcal{O}_n/(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$, i.e., $L(V) := \mathrm{Der}(A(V), A(V))$. It is known that $L(V)$ is a finite dimensional solvable Lie algebra [30, 32]. $L(V)$ is called the Yau algebra of V (its dimension $\lambda(V)$ is called Yau number) in [33] and [11] in order to distinguish from Lie algebras of other types appearing in singularity theory [1, 3]. The Yau algebras play an improtant role in singularities. Yau and his collaborators have been systematically studying the Lie algebras of isolated hypersurface singularities since early eighties (see, e.g., [2, 6, 8, 16–24, 27, 28, 30–32, 34, 35]).

The Mather-Yau theorem was slightly generalized in ([14], Theorem 2.26) (without assuming isolated singularity):

Theorem 1.1 *Let $f, g \in m \subset \mathcal{O}_n$. The following are equivalent:*
1) $(V(f), 0) \cong (V(g), 0)$;
2) For all $k \ge 0$, $\mathcal{O}_n/(f, m^k J(f)) \cong \mathcal{O}_n/(g, m^k J(g))$ as $\mathbb{C}$-algebra;

3) There is some $k \geq 0$ such that $\mathcal{O}_n/(f, m^k J(f)) \cong \mathcal{O}_n/(g, m^k J(g))$ as $\mathbb{C}$-algebra, where $J(f) = (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$.

In particular, if $k = 0$ and $k = 1$ above, then the claim of the equivalence of 1) and 3) is exactly the Mather-Yau theorem [25].

Motivated from Theorem 1.1, in [17, 18], we introduced the new series of k-th Yau algebras $L^k(V)$ (or $L^k((V, 0))$, see Definition 2.2) which are defined to be the Lie algebra of derivations of the moduli algebra $A^k(V) = \mathcal{O}_n/(f, m^k J(f)), k \geq 0$, where m is the maximal ideal. I.e., $L^k(V) := \mathrm{Der}(A^k(V), A^k(V))$. Its dimension is denoted as $\lambda^k(V)$ (or $\lambda^k((V, 0))$). This series of integers $\lambda^k(V)$ are new numerical analytic invariants of singularities. It is natural to call it k-th Yau number. In particular, when $k = 0$, those are exactly the previous Yau algebra and Yau number, i.e., $L(V) = L^0(V), \lambda^0(V) = \lambda(V)$. In [30], Yau observed that the Yau algebra for the one-parameter family of simple elliptic singularities $\tilde{E}_6$ is constant. It turns out that the 1-st Yau algebra $L^1(V)$ is also constant for the family of simple elliptic singularities $\tilde{E}_6$. However, Torelli-type theorem for $L^k(V)$ for all $k > 1$ do hold on $\tilde{E}_6$ [15]. In general, the invariant $L^k(V), k \geq 1$ are more subtle than Yau algebra (i.e., $L^0(V)$). We have many reasons to believe that these new Lie algebras $L^k(V)$ will play an important role in the study of singularities.

Since $L^k(V)$ potentially contains all information of singularities, it is natural to ask what kind of partial information of these k-th Yau algebras can be used to distinguish singularities. In this paper we shall answer this question partially. We introduce many other numerical invariants to k-th Yau algebra, namely, dimension of the maximal nilpotent subalgebras (i.e., nilradical of nilpotent Lie algebra) $g(V)$ of $L(V)$; dimension of maximal torus of $g(V)$; generalized Cartan matrix $C^k(V), k \geq 0$ (see Definition 2.5); type and nilpotency of singularity. Notice that the generalized Cartan matrixes of $L^*(V)$ and $L(V)$ can not completely characterize the ADE singularities. There are an exceptinal case, i.e., the pair of A_6 and D_5 (cf. [19, 20]). However, we proposed the following conjecture.

Conjecture 1.1 [20] For every $k \geq 1$, the generalized Cartan matrices $C^k(V)$ arising from k-th Yau algebra $L^k(V)$ can characterize the ADE singularities completely. Equivalently, if X and Y are two ADE singularities, then $C^k(X) = C^k(Y)$ if and only if X and Y are analytically isomorphic.

The Conjecture 1.1 was verified when $k = 1$ in [20]. In this paper we shall investigate the 2-nd Yau algebra $L^2(V)$ of ADE singularities and verify the Conjecture 1.1 when $k = 2$. Moreover, we compute different numerical invariants such as dimension of maximal torus of $g(V)$; type and nilpotency of a singularity and generalized Cartan matrix $C^2(V)$ and so on. Our main result below provides evidence for Conjecture 1.1 and gives a new characterization theorem for ADE singularities which extends the results in [10, 11].

Main Theorem *The generalized Cartan matrix $C^2(V)$ arising from 2-nd Yau algebra $L^2(V)$ characterizes the ADE singularities. Equivalently, if X and Y are two ADE singularities, then $C^2(X) = C^2(Y)$ if and only if X and Y are analytically isomorphic.*

Remark 1.1 The ADE singularities in the above main theorem include: $A_k : \{x_1^{k+1} + x_2^2 = 0\} \subset \mathbb{C}^2, k \geq 1$, $D_k : \{x_1^2 x_2 + x_2^{k-1} = 0\} \subset \mathbb{C}^2, k \geq 4$, and three exceptional singularities E_6, E_7, E_8 defined in $\mathbb{C}^2$ by polynomials $x_1^3 + x_2^4, x_1^3 + x_1 x_2^3, x_1^3 + x_2^5$ respectively.

2 Preliminaries

2.1 Isolated Hypersurface Singularities

Let $\mathcal{O}_n$ be the algebra of germs of holomorphic functions at the origin of $\mathbb{C}^n$. Obviously, $\mathcal{O}_n$ can be naturally identified with the algebra of convergent power series in n indeterminates with complex coefficients. For $f \in \mathcal{O}_n$, we denote by $V = V(f)$ (or $(V, 0)$) the germ at the origin of $\mathbb{C}^n$ of hypersurface $\{f = 0\} \subset \mathbb{C}^n$. We say that V is a germ of isolated hypersurface singularity if the origin is an isolated zero of the gradient of f. The local (function) algebra of V is defined as the (commutative associative) algebra $F(V) \cong \mathcal{O}_n/(f)$, where (f) is the principal ideal generated by the germ of f at the origin. According to Hilbert's Nullstellensatz for an isolated singularity $V = V(f) = \{f = 0\}$ the factor-algebra $A(V) = \mathcal{O}_n/(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$ is finite dimensional. This factor-algebra is called the moduli algebra of V and its dimension $\tau(V)$ is called Tjurina number. The well-known Mather-Yau theorem states that

Theorem 2.1 [25] *The analytic isomorphism type of an isolated hypersurface singularity is determined by the isomorphism class of its moduli algebras i.e.,*

$$(V_1, 0) \cong (V_2, 0) \Longleftrightarrow A(V_1) \cong A(V_2).$$

2.2 k-th Yau Algebra

Recall that a derivation of commutative associative algebra A is defined as a linear endomorphism D of A satisfying the Leibniz rule: $D(ab) = D(a)b + aD(b)$. Thus for such an algebra A one can consider the Lie algebra of its derivations $\mathrm{Der}(A, A)$ with the bracket defined by the commutator of linear endomorphisms.

Definition 2.1 Let $V = V(f) = \{f = 0\}$ be a germ of an isolated hypersurface singularity at the origin in $\mathbb{C}^n$. Let $A(V)$ be the moduli algebra and $L(V) := \mathrm{Der}(A(V), A(V))$. Yu [33] call $L(V)$ the Yau algebra of V. The dimension of $L(V)$ is called the Yau number by Elashvili and Khimshiashvili [11] and is denoted by $\lambda(V)$.

The Definition 2.1 was slightly generalized as follows in [17].

Definition 2.2 Let $V = V(f) = \{f = 0\}$ be a germ of an isolated hypersurface singularity at the origin in $\mathbb{C}^n$. Let $A^k(V) = \mathcal{O}_n/(f, m^k J(f)), k \geq 0$, be a k-th moduli algebra. Let $L^k(V) := Der(A^k(V), A^k(V))$, which is called the k-th Yau algebra. The k-th Yau number $\lambda^k(V)$ is the dimension of derivation Lie algebra $L^k(V)$.

It is noted that 0-th Yau algebra is precisely the Yau algebra.

2.3 Kac-Moody Lie Algebras of Isolated Hypersurface Singularities

Let $(V, 0)$ be an isolated hypersurface singularity. Let $g(V)$ be the maximal ideal of $L(V)$ consisting of nilpotent elements. It follows from [26] a generalized Cartan matrix $C(V)$, constructed from $g(V)$, is an invaraint of $(V, 0)$ (cf. [31]).

Definition 2.3 An $l \times l$ matrix with entries in $\mathbb{Z}$, $C = (c_{ij})$ is a generalized Cartan matrix if
a) $c_{ii} = 2 \quad \forall i = 1, \cdots, l$,
b) $c_{ij} \leq 0 \quad \forall i, j = 1, \cdots, l, i \neq j$,
c) $c_{ij} = 0$ if and only if $c_{ji} = 0 \; \forall i, j = 1, \cdots, l, i \neq j$.

To each generalized Cartan matrix $C(V)$, one can associate a Lie algebra $KM(C)$ (called a Kac-Moody Lie algebra) defined by generators:

$$\{f_1, \cdots, f_l, h_1, \cdots, h_l, e_1, \cdots, e_l\}$$

and relations:

$$\begin{aligned}
&[h_i, e_j] = c_{ij}e_j, \quad [h_i, f_j] = -c_{ij}f_j, \quad (\forall i, j = 1, \cdots, l),\\
&[h_i, h_j] = 0, \quad (\forall i, j = 1, \cdots, l), \quad [e_i, f_i] = h_i,\\
&[e_i, f_j] = 0, \quad (ade_i)^{-c_{ij}+1}e_j = 0 = (adf_i)^{-c_{ij}+1}f_j, \quad (\forall i \neq j).
\end{aligned}$$

Let $H = \mathbb{C}h_i + \cdots + \mathbb{C}h_l$; denote $\xi_+(C)$ (resp. $\xi_-(C)$) the subalgebra of $KM(C)$ generated by $\{e_1, \cdots, e_l\}$ (resp. $(f_1, \cdots, f_l)$) one shows that:

$$KM(C) = \xi_+(C) \oplus H \oplus \xi_-(C).$$

One can also define $\xi_+(C)$ by generators:$\{e_1, \cdots, e_l\}$ and relations:

$$(\mathrm{ad}e_i)^{-c_{ij}+1}e_j = 0 \quad \forall i, j = 1, \cdots, l, i \neq j.$$

We shall construct the generalized Cartan matrix from an isolated hypersurface singularity $(V, 0)$. Let $g(V)$ be the set of all nilpotent elements in $L(V)$, then $g(V)$ is the maximal nilpotent Lie subalgebra of $L(V)$ and $Der(g(V))$ be its derivation algebra.

Definition 2.4 A torus on $g(V)$ is a commutative subalgebra of $Der(g(V))$ whose elements are semisimple endomorphism. A maximal torus is a torus not contain in any other torus. The dimension of maximal torus is called generalized Mostow number (GMN). GMN is an invariant of the isolated singularity $(V, 0)$.

Theorem 2.2 (Mostow's theorem, [26]) *If T_1 and T_2 are maximal tori of $g(V)$, then there exist $\varphi \in Autg(V)$ (automorphism group of $g(V)$) such that $\varphi T_1 \varphi^{-1} = T_2$.*

Let T be a maximal torus and consider the root space decomposition of $g(V)$ relatively to T [26]:

$$g(V) = \sum_{\beta \in R(T)} g(V)^\beta,$$

$$g(V)^\beta = \{x \in g(V) : tx = \beta(t)x, \forall t \in T\},$$

and

$$R(T) = \{\beta \in T^* : g(V)^\beta \neq (0)\}(\text{root system}),$$

$$R^1(T) = \{\beta \in R(T) : g(V)^\beta \nsubseteq [g(V), g(V)]\},$$

$$l_\beta = dim\left(\frac{g(V)^\beta}{[g(V), g(V)] \cap g(V)^\beta}\right), \forall \beta \in R^1(T),$$

$$d_\beta = dim(g(V)^\beta), \beta \in R^1(T).$$

The map: $\beta \mapsto d_\beta \quad R^1(T) \to \mathbb{N}^*$ gives the partition:

$$R^1(T) = R^1(T)_{p_1} \cup \cdots \cup R^1(T)_{p_q}, \quad p_1 < \cdots < p_q, \quad R^1(T)_{p_i} \neq \emptyset,$$

$$R^1(T)_p = \{\beta \in R^1(T); d_\beta = p\}.$$

Set $s_i = \sharp R^1(T)_{p_i}$ and $s = s_1 + \cdots + s_q$. We let $d_{\beta_i} = d_i$ and $l_{\beta_i} = l_i$.

Let $f : \{1, \cdots, l\} \longrightarrow \{1, \cdots, s\}$ be defined by:

$$f_i = \begin{cases} 1; \ 1 \leqslant i \leqslant l_1, \\ 2; \ l_1 \leqslant i \leqslant l_1 + l_2, \\ \vdots \\ s; \ l_1 + l_2 + \cdots + l_{s-1} \leqslant i \leqslant l. \end{cases}$$

Theorem 2.3 [26] *For $i, j \in \{1, \cdots, l\}$, $i \neq j$, let*

$$-c_{ij}(T) = min\{-n \in N; (\mathrm{ad}v)^{-n+1}w = 0, \quad \forall v \in g^{\beta_{f(i)}}, \quad \forall w \in g^{\beta_{f(j)}}\},$$

with $(\mathrm{ad}0)^0 = 0$ *and let* $c_{ii}(T) = 2$ *for* $i = 1, \cdots, l$. *Then*

$$C(T) = (c_{ij}(T))_{1 \leq i,j \leq l}$$

is a generalized Cartan matrix.

Thus, we define $C(V) := C(T)$, is called the generalized Cartan matrix of the isolated hypersurface singularity $(V, 0)$.

Definition 2.5 The generalized Cartan matrix of the k-th Yau algebra $L^k(V)$ is denoted as $C^k(V)$.

3 Proof of the Main Theorem

Now we apply the above theory to study the 2-nd Yau algebra $L^2(V)$ of simple hpersurface singularities. We use the following convention: $g^1 = [g, g], \cdots, g^{p+1} = [g, g^p]$. We use N to denote the set of positive integers.

Proposition 3.1 *Let* $V = \{(x_1, x_2) \in \mathbb{C}^2 : x_1^2 + x_2^{k+1} = 0\}$ *be the* A_k *singularity,* $k \geq 1$ *and* $L^2(V)$ *be a derivation Lie algebra. Then*

$$C^2(A_k) = \begin{cases} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}; & k=1, \\ \begin{pmatrix} 2 & -1 & -2 \\ -1 & 2 & -1 \\ -3 & -2 & 2 \end{pmatrix}; & k=2, \\ \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}; & k=3, \\ \begin{pmatrix} 2 & -1 & -3 & -3 & 0 \\ -1 & 2 & -3 & -3 & 0 \\ -1 & -1 & 2 & 0 & -2 \\ -1 & -1 & 0 & 2 & -2 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}; & k=4, \\ \begin{pmatrix} 2 & -1 & -1 & 0 & -1 \\ -1 & 2 & 0 & -3 & -1 \\ -1 & 0 & 2 & -3 & -1 \\ 0 & -2 & -2 & 2 & 0 \\ -1 & -2 & -2 & 0 & 2 \end{pmatrix}; & k=5, \\ \begin{pmatrix} 2 & 0 & 0 & -3 & -3 \\ 0 & 2 & -1 & -2 & -2 \\ 0 & -1 & 2 & -2 & -2 \\ -(k-2) & -1 & -1 & 2 & 0 \\ -(k-2) & -1 & -1 & 0 & 2 \end{pmatrix}; & k \text{ is even and } k \geq 6, \\ \begin{pmatrix} 2 & -1 & -2 & -2 & -1 \\ -1 & 2 & -2 & -2 & -1 \\ -\frac{k+1}{2} & -\frac{k+1}{2} & 2 & 0 & -(k-2) \\ -\frac{k+1}{2} & -\frac{k+1}{2} & 0 & 2 & -(k-2) \\ -2 & -2 & -\frac{k-1}{2} & -\frac{k-1}{2} & 2 \end{pmatrix}; & k \text{ is odd and } k \geq 7. \end{cases}$$

Proof It is easy to see that the Lie algebra $L^2(V)$ that arising from series

$$A_k : \{x_1^2 + x_2^{k+1} = 0\} \subset \mathbb{C}^2, k \geq 1,$$

have following dimension :

$$\lambda^2(V) = \begin{cases} k+6; & k \geq 2 \\ 6; & k = 1. \end{cases}$$

In case of $k \geq 2$, the Lie algebra $L^2(V)$ has following basis:

$$e_1 = (k+1)x_1\partial_1 + 2x_2\partial_2, \quad e_2 = x_2^k\partial_1, \quad e_3 = x_2^2\partial_2, e_4 = x_2^3\partial_2, \cdots, e_k = x_2^{k-1}\partial_2,$$
$$e_{k+1} = x_2^k\partial_1 + x_1\partial_2, \quad e_{k+2} = x_2^k\partial_2, \quad e_{k+3} = x_1x_2\partial_2, \quad e_{k+4} = x_2^{k+1}\partial_2, \quad e_{k+5} = x_1x_2\partial_1,$$
$$e_{k+6} = x_2^{k+1}\partial_1.$$

In case of $k = 1$, the Lie algebra $L^2(V)$ has following basis:

$$e_1 = x_1\partial_1 + x_2\partial_2, \quad e_2 = x_2\partial_1 - x_1\partial_2, \quad e_3 = x_1x_2\partial_2, \quad e_4 = x_2^2\partial_2, \quad e_5 = x_1x_2\partial_1,$$
$$e_6 = x_2^2\partial_1.$$

For A_1 singularity,

$$g(V) = < e_3, e_4, e_5, e_6 > .$$

It is easy to see that $[g(V), g(V)] = 0$. The type of A_1 singularity:=dim $g(V)/[g(V), g(V)] = 4$. The nilpotency of A_1 singularity=min$\{p \in N \cup \{0\} : g(V)^{p+1} = 0\} = 0$. It is easy to see from [2] that the torus T of $g(V)$ is spanned by

$$\begin{array}{llll} t_1 : g(V) \longrightarrow g(V) & t_2 : g(V) \longrightarrow g(V) & t_3 : g(V) \longrightarrow g(V) & t_4 : g(V) \longrightarrow g(V) \\ e_3 \longrightarrow e_3 & e_3 \longrightarrow 0 & e_3 \longrightarrow 0 & e_3 \longrightarrow 0 \\ e_4 \longrightarrow 0 & e_4 \longrightarrow e_4 & e_4 \longrightarrow 0 & e_4 \longrightarrow 0 \\ e_5 \longrightarrow 0 & e_5 \longrightarrow 0 & e_5 \longrightarrow e_5 & e_5 \longrightarrow 0 \\ e_6 \longrightarrow 0, & e_6 \longrightarrow 0, & e_6 \longrightarrow 0, & e_6 \longrightarrow e_6. \end{array}$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2 + \mathbb{C}t_3 + \mathbb{C}t_4$. Since dim $T = 4 =$ the type of A_1, therefore T is maximal torus of $g(V)$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2, 3, 4$.

$$\begin{aligned} g(V) &= g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_3} \oplus g^{\beta_4} \\ &= \mathbb{C}e_3 \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_5 \oplus \mathbb{C}e_6. \end{aligned}$$

(e_3, e_4, e_5, e_6) is a T-minimal system of generators. The generalized Cartan matrix is

$$C^2(A_1) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

For A_2 singularity,

$$g(V) = < e_2, e_3, e_4, e_5, e_6, e_7, e_8 > .$$

$$[e_2, e_3] = e_4 - 2e_7, \quad [e_2, e_4] = -2e_8, \quad [e_2, e_5] = e_6, \quad [e_2, e_7] = e_8, \quad [e_3, e_4] = 2e_5 - 2e_8,$$
$$[e_3, e_7] = -e_5, \quad [e_3, e_8] = -e_6.$$

It is easy to see that $[g(V), g(V)] = 4$. The type of A_2 singularity:=dim $g(V)/[g(V), g(V)] = 3$. The nilpotency of A_2 singularity=min$\{p \in N \cup \{0\} : g(V)^{p+1} =$

$0\} = 3$. It is easy to see from [2] that the torus T of $g(V)$ is spanned by

$$\begin{aligned}
&t_1 : g(V) \longrightarrow g(V) && t_2 : g(V) \longrightarrow g(V)\\
&e_2 \longrightarrow e_2 && e_2 \longrightarrow 0\\
&e_3 \longrightarrow e_3 && e_3 \longrightarrow e_2 - 4e_3\\
&e_4 \longrightarrow 2e_4 && e_4 \longrightarrow -2e_4 + 4e_7\\
&e_5 \longrightarrow 3e_5 && e_5 \longrightarrow -8e_5 + e_8\\
&e_6 \longrightarrow 4e_6 && e_6 \longrightarrow -8e_6\\
&e_7 \longrightarrow 2e_7 && e_7 \longrightarrow e_4 - 2e_7\\
&e_8 \longrightarrow 3e_8 && e_8 \longrightarrow -4e_8.
\end{aligned}$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2$.

$$\begin{aligned}
g(V) &= g^{\beta_1} \oplus g^{4\beta_1-8\beta_2} \oplus g^{3\beta_1+4\beta_2} \oplus g^{2\beta_1-4\beta_2} \oplus g^{\beta_1-4\beta_2} \oplus g^{2\beta_1} \oplus g^{3\beta_1-8\beta_2}\\
&= \mathbb{C}e_2 \oplus \mathbb{C}e_6 \oplus \mathbb{C}e_8 \oplus \mathbb{C}(\frac{-e_4}{2} + e_7) \oplus \mathbb{C}(\frac{-e_2}{4} + e_3) \oplus \mathbb{C}(\frac{e_4}{2} + e_7) \oplus \mathbb{C}(-12e_5 + e_8).
\end{aligned}$$

$(e_2, \frac{-e_4}{2} + e_7, \frac{-e_2}{4} + e_3)$ is a T-minimal system of generators. The generalized Cartan matrix is

$$C^2(A_2) = \begin{pmatrix} 2 & -1 & -2\\ -1 & 2 & -1\\ -3 & -2 & 2 \end{pmatrix}.$$

For A_3 singularity,

$$g(V) =< e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9 > .$$

$$[e_2, e_3] = -3e_9, \quad [e_2, e_4] = e_5, \quad [e_2, e_6] = e_7, \quad [e_2, e_8] = e_9, \quad [e_3, e_4] = -2e_6 + 3e_9,$$
$$[e_3, e_5] = e_7, \quad [e_4, e_8] = -e_6, \quad [e_4, e_9] = -e_7.$$

It is easy to see that $[g(V), g(V)] = 4$. The type of A_3 singularity :=dim $g(V)/[g(V), g(V)] = 4$. The nilpotency of A_3 singularity=min$\{p \in N \cup \{0\} : g(V)^{p+1} = 0\} = 2$. It is easy to see from [2] that the torus T of $g(V)$ is spanned by

$$\begin{aligned}
&t_1 : g(V) \longrightarrow g(V) && t_2 : g(V) \longrightarrow g(V)\\
&e_2 \longrightarrow e_2 && e_2 \longrightarrow 0\\
&e_3 \longrightarrow 0 && e_3 \longrightarrow e_3\\
&e_4 \longrightarrow e_4 && e_4 \longrightarrow 0\\
&e_5 \longrightarrow 2e_5 && e_5 \longrightarrow 0\\
&e_6 \longrightarrow e_6 && e_6 \longrightarrow e_6\\
&e_7 \longrightarrow 2e_7 && e_7 \longrightarrow e_7\\
&e_8 \longrightarrow 0 && e_8 \longrightarrow e_8\\
&e_9 \longrightarrow e_9 && e_9 \longrightarrow e_9.
\end{aligned}$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2$.

$$\begin{aligned}
g(V) &= g^{\beta_1} \oplus g^{\beta_2} \oplus g^{2\beta_1} \oplus g^{\beta_1+\beta_2} \oplus g^{2\beta_1+\beta_2}\\
&= \mathbb{C}e_2 \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_8 \oplus \mathbb{C}e_5 \oplus \mathbb{C}e_6 \oplus \mathbb{C}e_9 \oplus \mathbb{C}e_7.
\end{aligned}$$

(e_2, e_4, e_3, e_8) is a T-minimal system of generators. The generalized Cartan matrix is

$$C^2(A_3) = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}.$$

For A_4 singularity,

$$g(V) =< e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10} > .$$

$$[e_2, e_3] = -4e_{10}, \quad [e_2, e_5] = e_6, \quad [e_2, e_7] = e_8, \quad [e_2, e_9] = e_{10}, \quad [e_3, e_4] = e_6,$$
$$[e_3, e_5] = 4e_{10} - 2e_7, \quad [e_3, e_6] = 2e_8, \quad [e_5, e_9] = -e_7, \quad [e_5, e_{10}] = -e_8.$$

It is easy to see that $[g(V), g(V)] = 4$. The type of A_4 singularity :=dim $g(V)/[g(V), g(V)] = 5$. The nilpotency of A_4 singularity=min$\{p \in N \cup \{0\} : g(V)^{p+1} = 0\} = 2$. It is easy to see from [2] that the torus T of $g(V)$ is spanned by

$$\begin{aligned} t_1 : g(V) &\longrightarrow g(V) & t_2 : g(V) &\longrightarrow g(V) \\ e_2 &\longrightarrow e_2 & e_2 &\longrightarrow 0 \\ e_3 &\longrightarrow 0 & e_3 &\longrightarrow e_3 \\ e_4 &\longrightarrow 2e_4 & e_4 &\longrightarrow -e_4 \\ e_5 &\longrightarrow e_5 & e_5 &\longrightarrow 0 \\ e_6 &\longrightarrow 2e_6 & e_6 &\longrightarrow 0 \\ e_7 &\longrightarrow e_7 & e_7 &\longrightarrow e_7 \\ e_8 &\longrightarrow 2e_8 & e_8 &\longrightarrow e_8 \\ e_9 &\longrightarrow 0 & e_9 &\longrightarrow e_9 \\ e_{10} &\longrightarrow e_{10} & e_9 &\longrightarrow e_{10}. \end{aligned}$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2$.

$$\begin{aligned} g(V) &= g^{\beta_1} \oplus g^{\beta_2} \oplus g^{2\beta_1} \oplus g^{\beta_1+\beta_2} \oplus g^{2\beta_1+\beta_2} \oplus g^{2\beta_1-\beta_2} \\ &= \mathbb{C}e_2 \oplus \mathbb{C}e_5 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_9 \oplus \mathbb{C}e_6 \oplus \mathbb{C}e_7 \oplus \mathbb{C}e_{10} \oplus \mathbb{C}e_8 \oplus \mathbb{C}e_4. \end{aligned}$$

$(e_2, e_5, e_3, e_9, e_4)$ is a T-minimal system of generators. The generalized Cartan matrix is

$$C^2(A_4) = \begin{pmatrix} 2 & -1 & -3 & -3 & 0 \\ -1 & 2 & -3 & -3 & 0 \\ -1 & -1 & 2 & 0 & -2 \\ -1 & -1 & 0 & 2 & -2 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}.$$

For A_5 singularity,

$$g(V) =< e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11} > .$$

$$[e_2, e_3] = -5e_{11}, \quad [e_2, e_6] = e_7, \quad [e_2, e_8] = e_9, \quad [e_2, e_{10}] = e_{11}, \quad [e_3, e_4] = e_5, \quad [e_3, e_5] = 2e_7,$$
$$[e_3, e_6] = 5e_{11} - 2e_8, \quad [e_3, e_7] = 3e_9, \quad [e_4, e_5] = e_9, \quad [e_6, e_{10}] = -e_8, \quad [e_6, e_{11}] = -e_9.$$

It is easy to see that $[g(V), g(V)] = 5$. The type of A_5 singularity :=dim $g(V)/[g(V), g(V)] = 5$. The nilpotency of A_5 singularity=min$\{p \in N \cup \{0\} : g(V)^{p+1} =$

$0\} = 3$. It is easy to see from [2] that the torus T of $g(V)$ is spanned by

$$
\begin{array}{ll}
t_1 : g(V) \longrightarrow g(V) & t_2 : g(V) \longrightarrow g(V) \\
e_2 \longrightarrow e_2 & e_2 \longrightarrow 0 \\
e_3 \longrightarrow 0 & e_3 \longrightarrow e_3 \\
e_4 \longrightarrow 0 & e_4 \longrightarrow 2e_4 \\
e_5 \longrightarrow 0 & e_5 \longrightarrow 3e_5 \\
e_6 \longrightarrow -e_6 & e_6 \longrightarrow 4e_6 \\
e_7 \longrightarrow 0 & e_7 \longrightarrow 4e_7 \\
e_8 \longrightarrow -e_8 + 5e_{11} & e_8 \longrightarrow 5e_8 - 10e_{11} \\
e_9 \longrightarrow 0 & e_9 \longrightarrow 5e_9 \\
e_{10} \longrightarrow 0 & e_{10} \longrightarrow e_{10} \\
e_{11} \longrightarrow e_{11} & e_{11} \longrightarrow e_{11}.
\end{array}
$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2$.

$$
\begin{aligned}
g(V) &= g^{\beta_1} \oplus g^{\beta_2} \oplus g^{2\beta_2} \oplus g^{3\beta_2} \oplus g^{-\beta_1+4\beta_2} \oplus g^{4\beta_2} \oplus g^{-\beta_1+5\beta_2} \oplus g^{5\beta_2} \oplus g^{\beta_1+\beta_2} \\
&= \mathbb{C}e_2 \oplus (\mathbb{C}e_3 \oplus \mathbb{C}e_{10}) \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_5 \oplus \mathbb{C}e_6 \oplus \mathbb{C}e_7 \oplus \mathbb{C}(\frac{-2e_8}{5} + e_{11}) \oplus \mathbb{C}e_9 \oplus \mathbb{C}e_{11}.
\end{aligned}
$$

$(e_2, e_3, e_{10}, e_4, e_6)$ is a T-minimal system of generators. The generalized Cartan matrix is

$$
C^2(A_5) = \begin{pmatrix} 2 & -1 & -1 & 0 & -1 \\ -1 & 2 & 0 & -3 & -1 \\ -1 & 0 & 2 & -3 & -1 \\ 0 & -2 & -2 & 2 & 0 \\ -1 & -2 & -2 & 0 & 2 \end{pmatrix}.
$$

For A_k singularity, $k \geq 6$,

$$
g(V) = < e_2, e_3, e_4, \cdots, e_{k+6} > .
$$

Case 1. k is even and $k = 2l + 4, \quad l \geq 3$.
The nilradical $g(V)$ has the following multiplication table:

$$
\begin{aligned}
&[e_2, e_3] = -ke_{k+6}, \quad [e_2, e_{k+1}] = e_{k+2}, \quad [e_2, e_{k+3}] = e_{k+4}, \quad [e_2, e_{k+5}] = e_{k+6}, \\
&[e_3, e_4] = e_5, \quad [e_3, e_5] = 2e_6, \quad [e_3, e_6] = 3e_7, \quad \cdots, \quad [e_3, e_{k-1}] = (k-4)e_k, \\
&[e_3, e_k] = (k-3)e_{k+2}, \quad [e_3, e_{k+1}] = -2e_{k+3} + ke_{k+6}, \quad [e_3, e_{k+2}] = (k-2)e_{k+4}, \\
&[e_4, e_5] = e_7, \quad [e_4, e_6] = 2e_8, \quad [e_4, e_7] = 3e_9, \quad \cdots, \quad [e_4, e_{k-2}] = (k-6)e_k,
\end{aligned}
$$

$$[e_4, e_{k-1}] = (k-5)e_{k+2}, \quad [e_4, e_k] = (k-4)e_{k+4},$$
$$[e_5, e_6] = e_9, \quad [e_5, e_7] = 2e_{10}, \quad [e_5, e_8] = 3e_{11}, \quad \cdots, \quad [e_5, e_{k-3}] = (k-8)e_k,$$
$$[e_5, e_{k-2}] = (k-7)e_{k+2}, \quad [e_5, e_{k-1}] = (k-6)e_{k+4},$$
$$\vdots$$
$$[e_{l+3}, e_{l+4}] = e_{2l+6}, \quad [e_{l+3}, e_{l+5}] = 2e_{2l+8}, \quad [e_{2l+5}, e_{2l+9}] = -e_{2l+7}, \quad [e_{2l+5}, e_{2l+10}] = -e_{2l+8}.$$

Remark 3.1 In fact, the above table is true for $l \geq 3$. Only few equations need to be adjusted when $l = 1, 2$, eg. $[e_4, e_7] = 0$ when $l = 1$ and $[e_4, e_7] = 3e_{10}$ when $l = 2$ (see the following). So does the tables appear later (on page 13, page 20, and page 22, in order to save space, we omit the case $l = 1, 2$).

For $l = 1$, we have following multiplication table:

$$[e_2, e_3] = -6e_{12}, \quad [e_2, e_7] = e_8, \quad [e_2, e_9] = e_{10}, \quad [e_2, e_{11}] = e_{12}, \quad [e_3, e_4] = e_5,$$
$$[e_3, e_5] = 2e_6, \quad [e_3, e_6] = 3e_8, \quad [e_3, e_7] = 6e_{12} - 2e_9, \quad [e_3, e_8] = 4e_{10},$$
$$[e_4, e_5] = e_8, \quad [e_4, e_6] = 2e_{10}, \quad [e_7, e_{11}] = -e_9, \quad [e_7, e_{12}] = -e_{10}.$$

For $l = 2$, we have following multiplication table:

$$[e_2, e_3] = -8e_{14}, \quad [e_2, e_9] = e_{10}, \quad [e_2, e_{11}] = e_{12}, \quad [e_2, e_{13}] = e_{14}, \quad [e_3, e_4] = e_5,$$
$$[e_3, e_5] = 2e_6, \quad [e_3, e_6] = 3e_7, \quad [e_3, e_7] = 4e_8, \quad [e_3, e_8] = 5e_{10}, \quad [e_3, e_9] = -2e_{11} + 8e_{14},$$
$$[e_3, e_{10}] = 6e_{12}, \quad [e_4, e_5] = e_7, \quad [e_4, e_6] = 2e_8, \quad [e_4, e_7] = 3e_{10}, \quad [e_4, e_8] = 4e_{12},$$
$$[e_5, e_6] = e_{10}, \quad [e_5, e_7] = 2e_{12}, \quad [e_9, e_{13}] = -e_{11}, \quad [e_9, e_{14}] = -e_{12}.$$

The type of $A_k(k \geq 6)$ singularity =dim $g(V)/[g(V), g(V)] = 5$. The nilpotency of $A_k(k \geq 6) = min\{p \in N \cup \{0\} : g(V)^{p+1} = 0\} = k - 3$. It is easy to see from [2] that the torus T of $g(V)$ is spanned by

$$t : g(V) \longrightarrow g(V)$$
$$e_2 \longrightarrow e_2$$
$$e_3 \longrightarrow \frac{2e_3}{k-1}$$
$$e_4 \longrightarrow \frac{4e_4}{k-1}$$
$$e_5 \longrightarrow \frac{6e_5}{k-1}$$
$$\vdots$$
$$e_k \longrightarrow \frac{(2k-4)e_k}{k-1}$$

$$\begin{aligned}
e_{k+1} &\longrightarrow e_{k+1}\\
e_{k+2} &\longrightarrow 2e_{k+2}\\
e_{k+3} &\longrightarrow \frac{(k+1)e_{k+3}}{k-1}\\
e_{k+4} &\longrightarrow \frac{(2k)e_{k+4}}{k-1}\\
e_{k+5} &\longrightarrow \frac{2e_{k+5}}{k-1}\\
e_{k+6} &\longrightarrow \frac{(k+1)e_{k+6}}{k-1}.
\end{aligned}$$

Thus $T = \mathbb{C}t$ is a unique maximal torus of $g(V)$. Let $\beta : T \longrightarrow \mathbb{C}$ be a linear map with $\beta(t) = 1$.

$$\begin{aligned}
g(V) &= g^{\beta} \oplus g^{\frac{2\beta}{k-1}} \oplus g^{\frac{4\beta}{k-1}} \oplus g^{\frac{6\beta}{k-1}} \oplus \cdots \oplus g^{\frac{2(k-2)\beta}{k-1}} \oplus g^{2\beta} \oplus g^{\frac{(k+1)\beta}{k-1}} \oplus g^{\frac{2k\beta}{k-1}}\\
&= \mathbb{C}e_2 \oplus \mathbb{C}e_{k+1} \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_{k+5} \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_5 \oplus \mathbb{C}e_6 \oplus \cdots \oplus \mathbb{C}e_k \oplus \mathbb{C}e_{k+2}\\
&\quad \oplus \mathbb{C}e_{k+3} \oplus \mathbb{C}e_{k+6} \oplus \mathbb{C}e_{k+4}.
\end{aligned}$$

$(e_4, e_2, e_{k+1}, e_3, e_{k+5})$ is a T-minimal system of generators. The generalized Cartan matrix is

$$C^2(A_k) = \begin{pmatrix} 2 & 0 & 0 & -3 & -3\\ 0 & 2 & -1 & -2 & -2\\ 0 & -1 & 2 & -2 & -2\\ -(k-2) & -1 & -1 & 2 & 0\\ -(k-2) & -1 & -1 & 0 & 2 \end{pmatrix}.$$

Case 2. k is odd and $k = 2l + 5, \quad l \geq 1$, then

$$\begin{aligned}
&[e_2, e_3] = -ke_{k+6}, \quad [e_2, e_{k+1}] = e_{k+2}, \quad [e_2, e_{k+3}] = e_{k+4}, \quad [e_2, e_{k+5}] = e_{k+6},\\
&[e_3, e_4] = e_5, \quad [e_3, e_5] = 2e_6, \quad [e_3, e_6] = 3e_7, \quad \cdots, \quad [e_3, e_{k-1}] = (k-4)e_k,\\
&[e_3, e_k] = (k-3)e_{k+2}, \quad [e_3, e_{k+1}] = -2e_{k+3} + ke_{k+6}, \quad [e_3, e_{k+2}] = (k-2)e_{k+4},\\
&[e_4, e_5] = e_7, \quad [e_4, e_6] = 2e_8, \quad [e_4, e_7] = 3e_9, \quad \cdots, \quad [e_4, e_{k-2}] = (k-6)e_k,\\
&[e_4, e_{k-1}] = (k-5)e_{k+2}, \quad [e_4, e_k] = (k-4)e_{k+4},\\
&[e_5, e_6] = e_9, \quad [e_5, e_7] = 2e_{10}, \quad [e_5, e_8] = 3e_{11}, \quad \cdots, \quad [e_5, e_{k-3}] = (k-8)e_k,\\
&[e_5, e_{k-2}] = (k-7)e_{k+2}, \quad [e_5, e_{k-1}] = (k-6)e_{k+4},\\
&\quad \vdots\\
&[e_{l+4}, e_{l+5}] = e_{2l+9}, \quad [e_{2l+6}, e_{2l+10}] = -e_{2l+8}, \quad [e_{2l+6}, e_{2l+11}] = -e_{2l+9}.
\end{aligned}$$

It is easy to see from [2] that the torus T of $g(V)$ is spanned by

$$
\begin{aligned}
t : g(V) &\longrightarrow g(V) \\
e_2 &\longrightarrow e_2 \\
e_3 &\longrightarrow \frac{e_3}{l+2} \\
e_4 &\longrightarrow \frac{2e_4}{l+2} \\
e_5 &\longrightarrow \frac{3e_5}{l+2} \\
&\vdots \\
e_k &\longrightarrow \frac{(2l+3)e_k}{l+2} \\
e_{k+1} &\longrightarrow e_{k+1} \\
e_{k+2} &\longrightarrow 2e_{k+2} \\
e_{k+3} &\longrightarrow \frac{(l+3)e_{k+3}}{l+2} \\
e_{k+4} &\longrightarrow \frac{(2l+5)e_{k+4}}{l+2} \\
e_{k+5} &\longrightarrow \frac{e_{k+5}}{l+2} \\
e_{k+6} &\longrightarrow \frac{(l+3)e_{k+6}}{l+2}
\end{aligned}
$$

Thus $T = \mathbb{C}t$ is a unique maximal torus of $g(V)$. Let $\beta : T \longrightarrow \mathbb{C}$ be a linear map with $\beta(t) = 1$.

$$
\begin{aligned}
g(V) &= g^{\beta} \oplus g^{\frac{\beta}{l+2}} \oplus g^{\frac{2\beta}{l+2}} \oplus g^{\frac{3\beta}{l+2}} \oplus g^{\frac{4\beta}{l+2}} \oplus \cdots \oplus g^{\frac{(l+1)\beta}{l+2}} \oplus g^{\frac{(l+3)\beta}{l+2}} \oplus g^{2\beta} \oplus g^{\frac{(2l+5)\beta}{l+2}} \\
&\quad \oplus g^{\frac{(l+4)\beta}{l+2}} \oplus g^{\frac{(l+5)\beta}{l+2}} \oplus g^{\frac{(l+6)\beta}{l+2}} \oplus \cdots \oplus g^{\frac{(k-2)\beta}{l+2}} \\
&= \mathbb{C}e_2 \oplus \mathbb{C}e_{l+4} \oplus \mathbb{C}e_{k+1} \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_{k+5} \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_5 \oplus \mathbb{C}e_6 \oplus \cdots \oplus \mathbb{C}e_{l+3} \\
&\quad \oplus \mathbb{C}e_{k+3} \oplus \mathbb{C}e_{k+6} \oplus \mathbb{C}e_{l+5} \oplus \mathbb{C}e_{k+2} \oplus \mathbb{C}e_{k+4} \oplus \mathbb{C}e_{l+6} \oplus \mathbb{C}e_{l+7} \oplus \mathbb{C}e_{l+8} \oplus \cdots \oplus \mathbb{C}e_k.
\end{aligned}
$$

$(e_2, e_{k+1}, e_3, e_{k+5}, e_4)$ is a T-minimal system of generators. The generalized Cartan matrix is

$$
C^2(A_k) = \begin{pmatrix}
2 & -1 & -2 & -2 & -1 \\
-1 & 2 & -2 & -2 & -1 \\
-\frac{k+1}{2} & -\frac{k+1}{2} & 2 & 0 & -(k-2) \\
-\frac{k+1}{2} & -\frac{k+1}{2} & 0 & 2 & -(k-2) \\
-2 & -2 & -\frac{k-1}{2} & -\frac{k-1}{2} & 2
\end{pmatrix}.
$$

□

Proposition 3.2 *Let* $V = \{(x_1, x_2) \in \mathbb{C}^2 : x_1^2 x_2 + x_2^{k-1} = 0\}$ *be the* D_k *singularity,* $k \geq 4$ *and* $L^2(V)$ *be a derivation Lie algebra. Then*

$$C^2(D_k) = \begin{cases} \begin{pmatrix} 2 & -1 & -1 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 & -1 & -1 \\ -1 & -1 & 2 & -1 & -1 & -1 \\ -1 & -1 & -1 & 2 & -1 & -1 \\ -1 & -1 & -1 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & -1 & 2 \end{pmatrix}; & k\text{=4}, \\ \begin{pmatrix} 2 & -1 & -1 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 & -1 & -1 \\ -2 & -2 & 2 & -1 & -1 & -2 \\ -1 & -1 & -2 & 2 & -1 & -1 \\ -1 & -1 & -2 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & -1 & 2 \end{pmatrix}; & k\text{=5}, \\ \begin{pmatrix} 2 & -1 & -1 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & -1 & 0 & 0 & -1 \\ -1 & -1 & 2 & -1 & -2 & -2 & -1 \\ -1 & -1 & -1 & 2 & -2 & -2 & -1 \\ 0 & 0 & -1 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 & 2 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 2 \end{pmatrix}; & k\text{=6}, \\ \begin{pmatrix} 2 & -1 & -1 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & -1 & 0 & -1 & 0 \\ -1 & -1 & 2 & -1 & -(k-4) & -1 & -2 \\ -1 & -1 & -1 & 2 & -(k-4) & -1 & -2 \\ 0 & 0 & -(k-4) & -(k-4) & 2 & 0 & -1 \\ 0 & -1 & -1 & -1 & 0 & 2 & 0 \\ -1 & 0 & -1 & -1 & -1 & 0 & 2 \end{pmatrix}; & k \text{ is even and } k \geq 8, \\ \begin{pmatrix} 2 & -1 & -1 & 0 & -1 & 0 & -1 \\ -1 & 2 & -1 & -2 & -1 & -(k-4) & -1 \\ -1 & -1 & 2 & -2 & -1 & -(k-4) & -1 \\ 0 & -1 & -1 & 2 & 0 & -1 & -1 \\ -1 & -1 & -1 & 0 & 2 & 0 & 0 \\ 0 & -(k-4) & -(k-4) & -1 & 0 & 2 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 2 \end{pmatrix}; & k \text{ is odd and } k \geq 7, \end{cases}$$

Proof It is easy to see that the Lie algebra $L^2(V)$ that arising from series

$$D_k : \{x_1^2 x_2 + x_2^{k-1} = 0\} \subset \mathbb{C}^2, k \geq 4,$$

have following dimension :

$$\lambda^2(V) = \begin{cases} k + 10; & k \geq 5 \\ 13; & k = 4. \end{cases}$$

In case of $k \geq 5$, the Lie algebra $L^2(V)$ has following basis:

$$e_1 = (k-2)x_1\partial_1 + 2x_2\partial_2, \quad e_2 = (x_2^{k-2} + x_1^2)\partial_1, \quad e_3 = -x_1x_2\partial_1 + x_2^2\partial_2,$$
$$e_4 = x_2^{k-3}\partial_1, \quad e_5 = (x_2^{k-2} + x_1^2)\partial_2, \quad e_6 = x_2^2\partial_2, \quad e_7 = x_2^{k-2}\partial_1,$$
$$e_8 = x_2^3\partial_2, \quad e_9 = x_2^4\partial_2, \quad e_{10} = x_2^5\partial_2, \quad \cdots \quad , e_{k+2} = x_2^{k-3}\partial_2,$$
$$e_{k+3} = x_2^{k-2}\partial_1 + x_1x_2\partial_2, \quad e_{k+4} = x_2^{k-2}\partial_2, \quad e_{k+5} = x_1^3\partial_2, \quad e_{k+6} = x_1x_2^2\partial_2,$$
$$e_{k+7} = x_2^{k-1}\partial_2, \quad e_{k+8} = x_1^3\partial_1, \quad e_{k+9} = x_1x_2^2\partial_1, \quad e_{k+10} = x_2^{k-1}\partial_1.$$

In case of $k = 4$, the Lie algebra $L^2(V)$ has following basis:

$$e_1 = x_1\partial_1 + x_2\partial_2, \quad e_2 = x_1^2\partial_1 - x_1x_2\partial_2, \quad e_3 = x_2^2\partial_1 + x_1x_2\partial_2, \quad e_4 = -x_1x_2\partial_1 + x_2^2\partial_2,$$
$$e_5 = x_1^2\partial_2, \quad e_6 = x_2^2\partial_2, \quad e_7 = -x_1x_2\partial_2, \quad e_8 = x_1^3\partial_2, \quad e_9 = x_1x_2^2\partial_2, \quad e_{10} = x_2^3\partial_2,$$
$$e_{11} = x_1^3\partial_1, \quad e_{12} = x_1x_2^2\partial_1, \quad e_{13} = x_2^3\partial_1.$$

For D_4 singularity,

$$g(V) =< e_2, e_3, e_4, \cdots, e_{13} > .$$

$$[e_2, e_3] = -4e_{12}, \quad [e_2, e_4] = -2e_{13} - 2e_9, \quad [e_2, e_5] = 3e_8, \quad [e_2, e_6] = -e_9, \quad [e_2, e_7] = e_{10},$$
$$[e_3, e_4] = -2e_{13} + 2e_9, \quad [e_3, e_5] = 2e_{13} - e_8 + 2e_9, \quad [e_3, e_6] = -2e_{13} + e_9,$$
$$[e_3, e_7] = -e_{10} + 2e_{12}, \quad [e_4, e_5] = 4e_{10} + e_{11}, \quad [e_4, e_6] = e_{12}, \quad [e_4, e_7] = e_{13} + 2e_9,$$
$$[e_5, e_6] = -2e_{10}, \quad [e_5, e_7] = -e_8, \quad [e_6, e_7] = e_9.$$

It is easy to see that $[g(V), g(V)] = 6$. The type of D_4 singularity:=dim $g(V)/[g(V), g(V)] = 6$. The nilpotency of D_4 singularity=min$\{p \in N \cup \{0\} : g(V)^{p+1} = 0\} = 1$. It is easy to see from [2] that the torus T of $g(V)$ is spanned by

$t_1 : g(V) \longrightarrow g(V)$	$t_2 : g(V) \longrightarrow g(V)$
$e_2 \longrightarrow e_2$	$e_2 \longrightarrow e_3 + 4e_7$
$e_3 \longrightarrow e_3$	$e_3 \longrightarrow e_2 - 4e_7$
$e_4 \longrightarrow e_4$	$e_4 \longrightarrow e_4$
$e_5 \longrightarrow e_5$	$e_5 \longrightarrow 4e_4 + 3e_5 - 6e_6$
$e_6 \longrightarrow e_6$	$e_6 \longrightarrow e_6$
$e_7 \longrightarrow e_7$	$e_7 \longrightarrow -e_2 + e_3 + 5e_7$
$e_8 \longrightarrow 2e_8$	$e_8 \longrightarrow 4e_8 - 2e_{13}$
$e_9 \longrightarrow 2e_9$	$e_9 \longrightarrow 4e_9 + 2e_{13}$
$e_{10} \longrightarrow 2e_{10}$	$e_{10} \longrightarrow 4e_{10} - 2e_{12}$
$e_{11} \longrightarrow 2e_{11}$	$e_{11} \longrightarrow 4e_{11} + 2e_{12}$
$e_{12} \longrightarrow 2e_{12}$	$e_{12} \longrightarrow 2e_{12}$
$e_{13} \longrightarrow 2e_{13}$	$e_{13} \longrightarrow 2e_{13}$

It is easy to check that $T = \mathbb{C}t_1 + \mathbb{C}t_2$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2$.

$$\begin{aligned} g(V) &= g^{2(\beta_1+\beta_2)} \oplus g^{\beta_1+\beta_2} \oplus g^{\beta_1+3\beta_2} \oplus g^{2(\beta_1+2\beta_2)} \\ &= \mathbb{C}e_{12} \oplus \mathbb{C}e_{13} \oplus \mathbb{C}(-e_2 + e_7) \oplus \mathbb{C}(e_2 + e_3) \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_6 \oplus \mathbb{C}(\frac{-e_2}{4} + \frac{e_3}{4} + e_7) \\ &\oplus \mathbb{C}(\frac{-2e_4}{3} - \frac{e_5}{3} + e_6) \oplus \mathbb{C}(-e_8 + e_{13}) \oplus \mathbb{C}(-e_{10} + e_{12}) \oplus \mathbb{C}(e_{10} + e_{11}) \oplus \mathbb{C}(e_8 + e_9). \end{aligned}$$

$(-e_2+e_7, e_2+e_3, e_4, e_6, \frac{-e_2}{4}+\frac{e_3}{4}+e_7, \frac{-2e_4}{3}-\frac{e_5}{3}+e_6)$ is a T-minimal system of generators. The generalized Cartan matrix is

$$C^2(D_4) = \begin{pmatrix} 2 & -1 & -1 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 & -1 & -1 \\ -1 & -1 & 2 & -1 & -1 & -1 \\ -1 & -1 & -1 & 2 & -1 & -1 \\ -1 & -1 & -1 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & -1 & 2 \end{pmatrix}.$$

For D_5 singularity,

$$\begin{aligned} &[e_2, e_3] = 2e_{14}, \quad [e_2, e_4] = 5e_{15}, \quad [e_2, e_5] = -2e_{10}, \quad [e_2, e_6] = 3e_{15}, \quad [e_3, e_4] = 3e_7, \\ &[e_3, e_5] = -2e_{11}, \quad [e_3, e_6] = 2e_7 - e_8, \quad [e_3, e_8] = -2e_{14} - e_9, \quad [e_3, e_9] = 2e_{15}, \\ &[e_3, e_{11}] = -e_{12}, \quad [e_3, e_{14}] = -e_{15}, \quad [e_4, e_5] = -5e_{12} - e_{13}, \quad [e_4, e_6] = -e_{14}, \\ &[e_4, e_7] = -4e_{15}, \quad [e_4, e_8] = 2e_{11} - 3e_{15}, \quad [e_4, e_9] = -e_{12}, \quad [e_5, e_6] = 3e_{12}, \\ &[e_5, e_8] = -e_{10}, \quad [e_6, e_7] = -3e_{15}, \quad [e_6, e_8] = e_{11} - 3e_{15}, \quad [e_6, e_9] = -e_{12}, \quad [e_7, e_8] = -e_{12}. \end{aligned}$$

It is easy to see that $[g(V), g(V)] = 8$. The type of D_5 singularity:=dim $g(V)/[g(V), g(V)] = 6$. The nilpotency of D_5 singularity=min$\{p \in N \cup \{0\} : g(V)^{p+1} = 0\} = 2$. It is easy to see from [2] that the torus T of $g(V)$ is spanned by

$$t : g(V) \longrightarrow g(V)$$

$$\begin{aligned} &e_2 \longrightarrow e_2 && e_3 \longrightarrow \frac{e_3}{3} \\ &e_4 \longrightarrow \frac{2e_4}{3} && e_5 \longrightarrow \frac{4e_5}{3} \\ &e_6 \longrightarrow \frac{2e_6}{3} && e_7 \longrightarrow e_7 \\ &e_8 \longrightarrow e_8 && e_9 \longrightarrow \frac{4e_9}{3} \\ &e_{10} \longrightarrow \frac{7e_{10}}{3} && e_{11} \longrightarrow \frac{5e_{11}}{3} \\ &e_{12} \longrightarrow 2e_{12} && e_{13} \longrightarrow 2e_{13} \\ &e_{14} \longrightarrow \frac{4e_{14}}{3} && e_{15} \longrightarrow \frac{5e_{15}}{3}. \end{aligned}$$

Let $\beta : T \longrightarrow \mathbb{C}$ be a linear map with $\beta(t) = 1$.

$$\begin{aligned} g(V) &= g^{\beta} \oplus g^{\frac{\beta}{3}} \oplus g^{\frac{2\beta}{3}} \oplus g^{\frac{4\beta}{3}} \oplus g^{\frac{7\beta}{3}} \oplus g^{\frac{5\beta}{3}} \\ &= \mathbb{C}e_2 \oplus \mathbb{C}e_7 \oplus \mathbb{C}e_8 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_6 \oplus \mathbb{C}e_5 \oplus \mathbb{C}e_9 \oplus \mathbb{C}e_{14} \oplus \mathbb{C}e_{10} \oplus \mathbb{C}e_{11} \oplus \mathbb{C}e_{15} \\ &\oplus \mathbb{C}e_{12} \oplus \mathbb{C}e_{13}. \end{aligned}$$

$(e_2, e_3, e_4, e_5, e_6, e_8)$ is a T-minimal system of generators. The generalized Cartan matrix is

$$C^2(D_5) = \begin{pmatrix} 2 & -1 & -1 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 & -1 & -1 \\ -2 & -2 & 2 & -1 & -1 & -2 \\ -1 & -1 & -2 & 2 & -1 & -1 \\ -1 & -1 & -2 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & -1 & 2 \end{pmatrix}.$$

For D_6 singularity,

$$\begin{aligned}
&[e_2, e_3] = -6e_{16}, \quad [e_2, e_5] = 2e_{11}, \quad [e_2, e_6] = -4e_{16}, \quad [e_3, e_4] = 4e_7, \quad [e_3, e_5] = 6e_{13} + e_{14},\\
&[e_3, e_6] = e_{15}, \quad [e_3, e_7] = 5e_{16}, \quad [e_3, e_8] = e_{10}, \quad [e_3, e_9] = -2e_{12} + 4e_{16},\\
&[e_3, e_{10}] = 2e_{13}, \quad [e_4, e_6] = -3e_7, \quad [e_4, e_8] = -3e_{16}, \quad [e_4, e_9] = e_{10}, \quad [e_4, e_{12}] = e_{13},\\
&[e_4, e_{15}] = e_{16}, \quad [e_5, e_6] = -4e_{13}, \quad [e_5, e_9] = e_{11}, \quad [e_6, e_7] = 4e_{16}, \quad [e_6, e_8] = e_{10},\\
&[e_6, e_9] = -e_{12} + 4e_{16}, \quad [e_6, e_{10}] = 2e_{13}, \quad [e_7, e_9] = e_{13}.
\end{aligned}$$

It is easy to see that $[g(V), g(V)] = 8$. The type of D_6 singularity:=dim $g(V)/[g(V), g(V)] = 7$. The nilpotency of D_6 singularity=min$\{p \in N \cup \{0\} : g(V)^{p+1} = 0\} = 2$. It is easy to see from [2] that the torus T of $g(V)$ is spanned by

$$\begin{array}{llll}
t_1 : g(V) \longrightarrow g(V) & t_2 : g(V) \longrightarrow g(V) & t_3 : g(V) \longrightarrow g(V) & t_4 : g(V) \longrightarrow g(V) \\
e_2 \longrightarrow e_2, & e_2 \longrightarrow 0, & e_2 \longrightarrow 0, & e_2 \longrightarrow 0 \\
e_3 \longrightarrow 0, & e_3 \longrightarrow 0, & e_3 \longrightarrow e_3, & e_3 \longrightarrow 0 \\
e_4 \longrightarrow e_4, & e_4 \longrightarrow 0, & e_4 \longrightarrow -e_4, & e_4 \longrightarrow 0 \\
e_5 \longrightarrow 4e_{10}, & e_5 \longrightarrow 4e_{10}, & e_5 \longrightarrow -2e_{10}, & e_5 \longrightarrow e_5 - 2e_{10} \\
e_6 \longrightarrow 0, & e_6 \longrightarrow 0, & e_6 \longrightarrow e_6, & e_6 \longrightarrow 0 \\
e_7 \longrightarrow e_7, & e_7 \longrightarrow 0, & e_7 \longrightarrow 0, & e_7 \longrightarrow 0
\end{array}$$

$$\begin{array}{llll}
e_8 \longrightarrow 2e_8 + 6e_{15} & e_8, \longrightarrow 2e_8 + 6e_{15}, & e_8 \longrightarrow -2e_8 - 6e_{15}, & e_8 \longrightarrow 0 \\
e_9 \longrightarrow e_9, & e_9 \longrightarrow e_2 - \dfrac{10e_7}{3} + e_9, & e_9 \longrightarrow 0, & e_9 \longrightarrow 0 \\
e_{10} \longrightarrow 2e_{10}, & e_{10} \longrightarrow 2e_{10}, & e_{10} \longrightarrow -e_{10}, & e_{10} \longrightarrow 0 \\
e_{11} \longrightarrow e_{11}, & e_{11} \longrightarrow 0, & e_{11} \longrightarrow 0, & e_{11} \longrightarrow e_{11} \\
e_{12} \longrightarrow e_{12}, & e_{12} \longrightarrow 2e_{12} + \dfrac{4e_{16}}{3}, & e_{12} \longrightarrow e_{12}, & e_{12} \longrightarrow 0 \\
e_{13} \longrightarrow 2e_{13}, & e_{13} \longrightarrow 2e_{13}, & e_{13} \longrightarrow 0, & e_{13} \longrightarrow 0 \\
e_{14} \longrightarrow -4e_{14}, & e_{14} \longrightarrow -4e_{13}, & e_{14} \longrightarrow 2e_{13} + e_{14}, & e_{14} \longrightarrow 2e_{13} + e_{14} \\
e_{15} \longrightarrow 0, & e_{15} \longrightarrow 0, & e_{15} \longrightarrow 2e_{15}, & e_{15} \longrightarrow 0 \\
e_{16} \longrightarrow e_{16}, & e_{16} \longrightarrow 0, & e_{16} \longrightarrow e_{16}, & e_{16} \longrightarrow 0.
\end{array}$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2 + \mathbb{C}t_3 + \mathbb{C}t_4$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2, 3, 4$.

$$\begin{aligned} g(V) &= g^{\beta_1+\beta_3} \oplus g^{2\beta_3} \oplus g^{\beta_2+\beta_3} \oplus g^{\beta_1+\beta_4} \oplus g^{\beta_4} \oplus g^{\beta_1} \oplus g^{\beta_3} \oplus g^{\beta_1-\beta_3} \oplus g^{\beta_1+2\beta_2+\beta_3} \oplus g^{2\beta_1+2\beta_2} \\ &\quad \oplus g^{2\beta_1+2\beta_2-\beta_3} \oplus g^{2(\beta_1-\beta_3)+\beta_2} \oplus g^{\beta_1+\beta_2} \\ &= \mathbb{C}e_{16} \oplus \mathbb{C}e_{15} \oplus \mathbb{C}(2e_{13} + e_{14}) \oplus \mathbb{C}e_{11} \oplus \mathbb{C}(\frac{-e_5}{2} + e_{10}) \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_7 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_6 \oplus \mathbb{C}e_4 \\ &\quad \oplus \mathbb{C}(\frac{3e_{12}}{2} + e_{16}) \oplus \mathbb{C}e_{13} \oplus \mathbb{C}e_{10} \oplus \mathbb{C}(\frac{-e_8}{3} + e_{15}) \oplus \mathbb{C}(e_2 - \frac{10e_7}{3} + e_9). \end{aligned}$$

$(\frac{-e_5}{2} + e_{10}, e_2, e_7, e_3, e_6, e_4, \frac{-e_8}{3} + e_{15}, e_2 - \frac{10e_7}{3} + e_9)$ is a T-minimal system of generators. The generalized Cartan matrix is

$$C^2(D_6) = \begin{pmatrix} 2 & -1 & -1 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & -1 & 0 & 0 & -1 \\ -1 & -1 & 2 & -1 & -2 & -2 & -1 \\ -1 & -1 & -1 & 2 & -2 & -2 & -1 \\ 0 & 0 & -1 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 & 2 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 2 \end{pmatrix}.$$

Let $g(V)$ be the nilradical of Lie algebra $L^2(V)$ that arising from D_k series spanned by:

$$g(V) =< e_2, e_3, e_4, \cdots, e_{k+10} > .$$

The nilradical $g(V)$ has the following multiplication table:
Case 1. k is even and $k = 2l + 6, \quad l \geq 1$, then

$$\begin{aligned} &[e_2, e_3] = -ke_{k+10}, \quad [e_2, e_5] = 2e_{k+5}, \quad [e_2, e_6] = -(k-2)e_{k+10}, \\ &[e_3, e_4] = (k-2)e_7, \quad [e_3, e_5] = ke_{k+7} + e_{k+8}, \quad [e_3, e_6] = e_{k+9}, \quad [e_3, e_7] = (k-1)e_{k+10}, \\ &[e_3, e_8] = e_9, \quad [e_3, e_9] = 2e_{10}, \quad [e_3, e_{10}] = 3e_{11}, \quad \cdots, \quad [e_3, e_{k+2}] = (k-5)e_{k+4}, \\ &[e_3, e_{k+3}] = -2e_{k+6} + (k-2)e_{k+10}, \quad [e_3, e_{k+4}] = (k-4)e_{k+7}, \\ &[e_4, e_6] = -(k-3)e_7, \quad [e_4, e_8] = -(k-3)e_{k+10}, \quad [e_4, e_{k+3}] = e_{k+4}, \quad [e_4, e_{k+6}] = e_{k+7}, \\ &[e_4, e_{k+9}] = e_{k+10}, \quad [e_5, e_6] = -(k-2)e_{k+7}, \quad [e_5, e_{k+3}] = e_{k+5}, \quad [e_6, e_7] = (k-2)e_{k+10}, \\ &[e_6, e_8] = e_9, \quad [e_6, e_9] = 2e_{10}, \quad [e_6, e_{10}] = 3e_{11}, \quad \cdots, \quad [e_6, e_{k+2}] = (k-5)e_{k+4}, \\ &[e_6, e_{k+3}] = -e_{k+6} + (k-2)e_{k+10}, \quad [e_6, e_{k+4}] = (k-4)e_{k+7}, \quad [e_7, e_{k+3}] = e_{k+7}, \\ &[e_8, e_9] = e_{11}, \quad [e_8, e_{10}] = 2e_{12}, \quad [e_8, e_{11}] = 3e_{13}, \quad \cdots, \quad [e_8, e_k] = (k-8)e_{k+2}, \\ &[e_8, e_{k+1}] = (k-7)e_{k+4}, \quad [e_8, e_{k+2}] = (k-6)e_{k+7}, \\ &[e_9, e_{10}] = e_{13}, \quad [e_9, e_{11}] = 2e_{14}, \quad [e_9, e_{12}] = 3e_{15}, \quad \cdots, \quad [e_9, e_{k-1}] = (k-10)e_{k+2}, \\ &[e_9, e_k] = (k-9)e_{k+4}, \quad [e_9, e_{k+1}] = (k-8)e_{k+7}, \\ &[e_{10}, e_{11}] = e_{15}, \quad [e_{10}, e_{12}] = 2e_{16}, \quad [e_{10}, e_{13}] = 3e_{17}, \quad \cdots, \quad [e_{10}, e_{k-2}] = (k-12)e_{k+2}, \\ &[e_{10}, e_{k-1}] = (k-11)e_{k+4}, \quad [e_{10}, e_k] = (k-10)e_{k+7}, \\ &\quad \vdots \\ &[e_{l+7}, e_{l+8}] = e_{2l+10}, \quad [e_{l+7}, e_{l+9}] = 2e_{2l+13}. \end{aligned}$$

The type of D_k ($k \geq 7$) singularity =dim $g(V)/[g(V), g(V)] = 7$. The nilpotency of D_k singularity $= min\{p \in N \cup \{0\} : g(V)^{p+1} = 0\} = k - 4$. It is easy to see from [2] that the torus T of $g(V)$ is spanned by

$$
\begin{aligned}
t_1 &: g(V) \longrightarrow g(V) & t_2 &: g(V) \longrightarrow g(V)\\
e_2 &\longrightarrow e_2, & e_2 &\longrightarrow 0,\\
e_3 &\longrightarrow \frac{e_3}{l+2}, & e_3 &\longrightarrow \frac{e_3}{l+2},\\
e_4 &\longrightarrow \frac{(l+1)e_4}{l+2}, & e_4 &\longrightarrow \frac{-e_4}{l+2},\\
e_5 &\longrightarrow \frac{(k-3)e_{k+4}}{l+1}, & e_5 &\longrightarrow \frac{(k-3)e_{k+4}}{l+1},\\
e_6 &\longrightarrow \frac{e_6}{l+2}, & e_6 &\longrightarrow \frac{e_6}{l+2},\\
e_7 &\longrightarrow e_7, & e_7 &\longrightarrow 0,\\
e_8 &\longrightarrow \frac{2e_8}{l+} & e_8 &\longrightarrow \frac{2e_8}{l+2},
\end{aligned}
$$

$$
\begin{aligned}
e_9 &\longrightarrow \frac{3e_9}{l+2}, & e_9 &\longrightarrow \frac{3e_9}{l+2},\\
e_{10} &\longrightarrow \frac{4e_{10}}{l+2}, & e_{10} &\longrightarrow \frac{4e_{10}}{l+2},\\
&\vdots \qquad\qquad \vdots & &\\
e_{k+2} &\longrightarrow \frac{(k-4)e_{k+2}}{l+2}, & e_{k+2} &\longrightarrow \frac{(k-4)e_{k+2}}{l+2}.\\
e_{k+3} &\longrightarrow e_{k+3}, & e_{k+3} &\longrightarrow \frac{-(2(k-1)+2l)e_7}{k-3} + 2e_{k+3} + e_2,\\
e_{k+4} &\longrightarrow \frac{(k-3)e_{k+4}}{l+2}, & e_{k+4} &\longrightarrow \frac{(k-3)e_{k+4}}{l+2},\\
e_{k+5} &\longrightarrow e_{k+5}, & e_{k+5} &\longrightarrow 0,\\
e_{k+6} &\longrightarrow \frac{(l+3)e_{k+6}}{l+2}, & e_{k+6} &\longrightarrow \frac{(k-1)e_{k+6}}{l+2} + \frac{(k-2)e_{k+10}}{k-3},\\
e_{k+7} &\longrightarrow 2e_{k+7}, & e_{k+7} &\longrightarrow 2e_{k+7},\\
e_{k+8} &\longrightarrow \frac{-(2k-6)e_{k+7}}{l+2} + \frac{e_{k+8}}{l+2}, & e_{k+8} &\longrightarrow \frac{-(2k-6)e_{k+7}}{l+2} + \frac{e_{k+8}}{l+2},\\
e_{k+9} &\longrightarrow \frac{2e_{k+9}}{l+2}, & e_{k+9} &\longrightarrow \frac{2e_{k+9}}{l+2},\\
e_{k+10} &\longrightarrow \frac{(l+3)e_{k+10}}{l+2}, & e_{k+10} &\longrightarrow \frac{e_{k+10}}{l+2},
\end{aligned}
$$

$$
\begin{aligned}
t_3 : g(V) &\longrightarrow g(V) \\
e_2 &\longrightarrow 0, \\
e_3 &\longrightarrow 0, \\
e_4 &\longrightarrow 0, \\
e_5 &\longrightarrow e_5 - \frac{(l+2)e_{k+4}}{l+1}, \\
e_6 &\longrightarrow 0, \\
e_7 &\longrightarrow 0, \\
e_8 &\longrightarrow 0, \\
e_9 &\longrightarrow 0, \\
e_{10} &\longrightarrow 0, \\
&\vdots \\
e_{k+7} &\longrightarrow 0, \\
e_{k+8} &\longrightarrow 2e_{k+7} + e_{k+8},
\end{aligned}
$$

$$
\begin{aligned}
e_{k+9} &\longrightarrow 0, \\
e_{k+10} &\longrightarrow 0.
\end{aligned}
$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2 + \mathbb{C}t_3$ is a unique maximal torus of $g(V)$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2, 3$.

$$
\begin{aligned}
g(V) &= g^{\beta_1+2\beta_2} \oplus g^{\frac{(l+3)\beta_1}{l+2}+\frac{(k-1)\beta_2}{l+2}} \oplus g^{\frac{\beta_1+\beta_2}{l+2}+\beta_3} \oplus g^{\beta_1} \oplus g^{\frac{\beta_1+\beta_2}{l+2}} \oplus g^{\frac{2(\beta_1+\beta_2)}{l+2}} \oplus g^{\beta_3} \oplus g^{\frac{(l+1)\beta_1}{l+2}-\frac{\beta_2}{l+2}} \\
&\quad \oplus g^{\frac{3(\beta_1+\beta_2)}{l+2}} \oplus g^{\frac{4(\beta_1+\beta_2)}{l+2}} \oplus g^{\frac{5(\beta_1+\beta_2)}{l+2}} \oplus \cdots \oplus g^{\frac{(k-3)(\beta_1+\beta_2)}{l+2}} \oplus g^{\beta_1+\beta_3} \oplus g^{2(\beta_1+\beta_2)} \oplus g^{\frac{(l+3)\beta_1}{l+2}+\frac{\beta_2}{l+2}} \\
&= \mathbb{C}\left(\frac{e_2}{2} - \frac{(k+l-1)e_7}{7} + e_{k+3}\right) \oplus \mathbb{C}\left(\frac{(k-3)e_{k+6}}{l+2} + e_{10}\right) \oplus \mathbb{C}(2e_{k+7} + e_{k+8}) \oplus \mathbb{C}(e_2 \oplus e_7) \\
&\quad \oplus \mathbb{C}(e_3 \oplus e_6) \oplus \mathbb{C}(e_{k+9} \oplus e_8) \oplus \mathbb{C}\left(-\frac{(l+1)e_5}{l+2} + e_{k+4}\right) \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_9 \oplus \mathbb{C}e_{10} \oplus \mathbb{C}e_{11} \\
&\quad \oplus \cdots \oplus \mathbb{C}e_{k+2} \oplus \mathbb{C}e_{k+4} \oplus \mathbb{C}e_{k+5} \oplus \mathbb{C}e_{k+7} \oplus \mathbb{C}e_{k+10}.
\end{aligned}
$$

$(\frac{e_2}{2} - \frac{(k+l-1)e_7}{7} + e_{k+3}, e_2, e_3, e_6, e_8, -\frac{(l+1)e_5}{l+2} + e_{k+4}, e_4)$ is a T-minimal system of generators. The generalized Cartan matrix is

$$
C^2(D_k) = \begin{pmatrix}
2 & -1 & -1 & -1 & 0 & 0 & -1 \\
-1 & 2 & -1 & -1 & 0 & -1 & 0 \\
-1 & -1 & 2 & -1 & -(k-4) & -1 & -2 \\
-1 & -1 & -1 & 2 & -(k-4) & -1 & -2 \\
0 & 0 & -(k-4) & -(k-4) & 2 & 0 & -1 \\
0 & -1 & -1 & -1 & 0 & 2 & 0 \\
-1 & 0 & -1 & -1 & -1 & 0 & 2
\end{pmatrix}.
$$

Case 2. k is odd and $k = 2l + 5, \quad l \geq 1$, then

$$
\begin{aligned}
&[e_2, e_3] = -ke_{k+10}, \quad [e_2, e_5] = 2e_{k+5}, \quad [e_2, e_6] = -(k-2)e_{k+10},\\
&[e_3, e_4] = (k-2)e_{k-4}, \quad [e_3, e_5] = ke_{k+7} + e_{k+8}, \quad [e_3, e_6] = e_{k+9}, \quad [e_3, e_7] = (k-1)e_{k+10},\\
&[e_3, e_8] = e_9, \quad [e_3, e_9] = 2e_{10}, \quad [e_3, e_{10}] = 3e_{11}, \quad \cdots, \quad [e_3, e_{k+1}] = (k-6)e_{k+2},\\
&[e_3, e_{k+2}] = (k-5)e_{k+4}, \quad [e_3, e_{k+3}] = -2e_{k+6} + (k-2)e_{k+10}, \quad [e_3, e_{k+4}] = (k-4)e_{k+7},\\
&[e_4, e_6] = -(k-3)e_7, \quad [e_4, e_8] = -(k-3)e_{k+10}, \quad [e_4, e_{k+3}] = e_{k+4}, \quad [e_4, e_{k+6}] = e_{k+7},\\
&[e_4, e_{k+9}] = e_{k+10}, \quad [e_5, e_6] = -(k-2)e_{k+7}, \quad [e_5, e_{k+3}] = e_{k+5}, \quad [e_6, e_7] = (k-2)e_{k+10},\\
&[e_6, e_8] = e_9, \quad [e_6, e_9] = 2e_{10}, \quad [e_6, e_{10}] = 3e_{11}, \quad \cdots, \quad [e_6, e_{k+1}] = (k-6)e_{k+2},\\
&[e_6, e_{k+2}] = (k-5)e_{k+3}, \quad [e_6, e_{k+3}] = -e_{k+6} + (k-2)e_{k+10}, \quad [e_6, e_{k+4}] = (k-4)e_{k+7},\\
&[e_7, e_{k+3}] = e_{k+7}, \quad [e_8, e_9] = e_{11}, \quad [e_8, e_{10}] = 2e_{12}, \quad \cdots, \quad [e_8, e_k] = (k-8)e_{k+2},
\end{aligned}
$$

$$
\begin{aligned}
&[e_8, e_{k+1}] = (k-7)e_{k+4}, \quad [e_8, e_{k+2}] = (k-6)e_{k+7},\\
&[e_9, e_{10}] = e_{13}, \quad [e_9, e_{11}] = 2e_{14}, \quad [e_9, e_{12}] = 3e_{15}, \quad \cdots, \quad [e_9, e_{k-1}] = (k-10)e_{k+2},\\
&[e_9, e_k] = (k-9)e_{k+4}, \quad [e_9, e_{k+1}] = (k-8)e_{k+7},\\
&[e_{10}, e_{11}] = e_{15}, \quad [e_{10}, e_{12}] = 2e_{16}, \quad [e_{10}, e_{13}] = 3e_{17}, \quad \cdots, \quad [e_{10}, e_{k-2}] = (k-12)e_{k+2},\\
&[e_{10}, e_{k-1}] = (k-11)e_{k+4}, \quad [e_{10}, e_k] = (k-10)e_{k+7},\\
&\quad \vdots\\
&[e_{l+7}, e_{l+8}] = e_{2l+12}.
\end{aligned}
$$

It is easy to see from ([2]) that the torus T of $g(V)$ is spanned by

$$
\begin{aligned}
t_1 &: g(V) \longrightarrow g(V) & t_2 &: g(V) \longrightarrow g(V)\\
e_2 &\longrightarrow e_2, & e_2 &\longrightarrow 0,\\
e_3 &\longrightarrow \frac{2e_3}{k-2}, & e_3 &\longrightarrow \frac{2e_3}{k-2},\\
e_4 &\longrightarrow \frac{(k-4)e_4}{k-2}, & e_4 &\longrightarrow \frac{-2e_4}{k-2},\\
e_5 &\longrightarrow \frac{(k-1+2l)e_{k+4}}{k-4}, & e_5 &\longrightarrow \frac{(k+2l-1)e_{k+4}}{k-4},\\
e_6 &\longrightarrow \frac{2e_6}{k-2}, & e_6 &\longrightarrow \frac{2e_6}{k-4},\\
e_7 &\longrightarrow e_7, & e_7 &\longrightarrow 0,\\
e_8 &\longrightarrow \frac{4e_8}{k-2}, & e_8 &\longrightarrow \frac{4e_8}{k-2},\\
e_9 &\longrightarrow \frac{6e_9}{k-2}, & e_9 &\longrightarrow \frac{6e_9}{k-2},
\end{aligned}
$$

$$\begin{aligned}
e_{10} &\longrightarrow \frac{8e_{10}}{k-2}, & e_{10} &\longrightarrow \frac{8e_{10}}{k-2},\\
&\vdots & &\vdots\\
e_{k+2} &\longrightarrow \frac{(k+l)e_{k+2}}{k-2}, & e_{k+2} &\longrightarrow \frac{(k+l)e_{k+2}}{k-2},\\
e_{k+3} &\longrightarrow e_{k+3}, & e_{k+3} &\longrightarrow \frac{-5(k+4)e_7}{2(k+l-2)} + 2e_{k+3} + e_2,\\
e_{k+4} &\longrightarrow \frac{(k-1+l)e_{k+4}}{k-4}, & e_{k+4} &\longrightarrow \frac{(k+2l-1)e_{k+4}}{k-2},\\
e_{k+5} &\longrightarrow e_{k+5}, & e_{k+5} &\longrightarrow 0,\\
e_{k+6} &\longrightarrow \frac{ke_{k+6}}{k-2}, & e_{k+6} &\longrightarrow \frac{(2k-2)e_{k+6}}{k-2} + \frac{(k-2)e_{k+10}}{k-3},\\
e_{k+7} &\longrightarrow 2e_{k+7}, & e_{k+7} &\longrightarrow 2e_{k+7},
\end{aligned}$$

$$\begin{aligned}
e_{k+8} &\longrightarrow \frac{-2(k+2l-1)e_{k+7}}{k-2} + \frac{2e_{k+8}}{k-2}, & e_{k+8} &\longrightarrow \frac{-2(k+2l-1)e_{k+7}}{k-2} + \frac{2e_{k+8}}{k-2},\\
e_{k+9} &\longrightarrow \frac{4e_{k+9}}{k-2}, & e_{k+9} &\longrightarrow \frac{4e_{k+9}}{k-2},\\
e_{k+10} &\longrightarrow \frac{ke_{k+10}}{k-2}, & e_{k+10} &\longrightarrow \frac{2e_{k+10}}{k-2},
\end{aligned}$$

$$\begin{aligned}
t_3 &: g(V) \longrightarrow g(V)\\
e_2 &\longrightarrow 0\\
e_3 &\longrightarrow 0\\
e_4 &\longrightarrow 0\\
e_5 &\longrightarrow e_5 - \frac{(k-2)e_{k+4}}{k-4}\\
e_6 &\longrightarrow 0\\
e_7 &\longrightarrow 0\\
e_8 &\longrightarrow 0\\
e_9 &\longrightarrow 0\\
e_{10} &\longrightarrow 0\\
&\vdots\\
e_{k+4} &\longrightarrow 0\\
e_{k+5} &\longrightarrow e_{k+5}\\
e_{k+6} &\longrightarrow 0\\
e_{k+7} &\longrightarrow 2e_{k+7} + e_{k+8}\\
e_{k+8} &\longrightarrow 0\\
e_{k+9} &\longrightarrow 0\\
e_{k+10} &\longrightarrow 0.
\end{aligned}$$

Thus $T = \mathbb{C}t_1 + \mathbb{C}t_2 + \mathbb{C}t_3$. is a unique maximal torus of $g(V)$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2, 3$.

$$g(V) = g^{\beta_1} \oplus g^{\frac{2(\beta_1+\beta_2)}{k-2}} \oplus g^{\frac{(k-4)\beta_1}{k-2} - \frac{2\beta_2}{k-2}} \oplus g^{\beta_3} \oplus g^{\frac{4(\beta_1+\beta_2)}{k-2}} \oplus g^{\frac{6(\beta_1+\beta_2)}{k-2}} \oplus \cdots \oplus g^{\frac{2(k-4)(\beta_1+\beta_2)}{k-2}}$$
$$\oplus g^{\beta_1+2\beta_2} \oplus g^{\frac{2(k-3)(\beta_1+\beta_2)}{k-2}} \oplus g^{\beta_1+\beta_3} \oplus g^{\frac{k\beta_1}{k-2}+\frac{(k+2(l+1)+1)\beta_2}{k-2}} \oplus g^{2(\beta_1+\beta_2+\beta_3)} \oplus g^{\frac{2(\beta_1+\beta_2)}{k-2}+2\beta_3}$$
$$\oplus g^{\frac{4(\beta_1+\beta_2)}{k-2}} \oplus g^{\frac{k\beta_1}{k-2}+\frac{2\beta_2}{k-2}}$$
$$= \mathbb{C}(e_2 \oplus e_7) \oplus \mathbb{C}(e_3 \oplus e_6) \oplus \mathbb{C}e_4 \oplus \mathbb{C}(-\frac{(k-4)e_5}{k-2} + e_{k+4}) \oplus \mathbb{C}e_8 \oplus \mathbb{C}e_9 \oplus \cdots \oplus \mathbb{C}e_{k+2}$$

$$\oplus \mathbb{C}(\frac{e_2}{2} - \frac{5(k+4)e_7}{2(k+l-2)} + e_{k+3}) \oplus \mathbb{C}e_{k+4} \oplus \mathbb{C}e_{k+5} \oplus \mathbb{C}(\frac{(k+l+2)e_{k+6}}{k-2} + e_{k+10}) \oplus \mathbb{C}e_{k+7}$$
$$\oplus \mathbb{C}(2e_{k+7} + e_{k+8}) \oplus \mathbb{C}e_{k+9} \oplus \mathbb{C}e_{k+10}.$$

$(e_2, e_3, e_6, e_4, -\frac{(k-4)e_5}{k-2} + e_{k+4}, e_8, \frac{e_2}{2} - \frac{5(k+4)e_7}{2(k+l-2)} + e_{k+3})$ is a T-minimal system of generators. The generalized Cartan matrix is

$$C^2(D_k) = \begin{pmatrix} 2 & -1 & -1 & 0 & -1 & 0 & -1 \\ -1 & 2 & -1 & -2 & -1 & -(k-4) & -1 \\ -1 & -1 & 2 & -2 & -1 & -(k-4) & -1 \\ 0 & -1 & -1 & 2 & 0 & -1 & -1 \\ -1 & -1 & -1 & 0 & 2 & 0 & 0 \\ 0 & -(k-4) & -(k-4) & -1 & 0 & 2 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 2 \end{pmatrix}.$$

□

Proposition 3.3 *Let* $V = \{(x_1, x_2) \in \mathbb{C}^2 : x_1^3 + x_2^4 = 0\}$ *be the* E_6 *singularity and* $L^2(V)$ *be a derivation Lie algebra. Then*

$$C^2(E_6) = \begin{pmatrix} 2 & -1 & -1 & -2 & -2 \\ -2 & 2 & -2 & -2 & -2 \\ -2 & -2 & 2 & -2 & -2 \\ -2 & -1 & -1 & 2 & -1 \\ -2 & -1 & -1 & -1 & 2 \end{pmatrix}.$$

Proof It is easy to see that $A^2(V) =< 1, x_2, x_2^2, x_2^3, x_2^4, x_1, x_1x_2, x_1x_2^2, x_1x_2^3, x_1^2, x_1^2x_2 >$. It is easy to see that $L^2(E_6)$ is 17-dimensional complex Lie algebra spanned by following basis:

$$e_1 = 4x_1\partial_1 + 3x_2\partial_2, \quad e_2 = x_2^2\partial_1, \quad e_3 = 2x_1x_2\partial_1 - x_2^2\partial_2, \quad e_4 = x_2^3\partial_1 + x_1^2\partial_2,$$
$$e_5 = x_1^2\partial_1 - x_1x_2\partial_2, \quad e_6 = x_2^2\partial_2, \quad e_7 = x_1x_2\partial_2, \quad e_8 = x_1x_2^2\partial_1, \quad e_9 = x_1^2\partial_2,$$
$$e_{10} = x_1x_2^2\partial_2, \quad e_{11} = x_1x_2^2\partial_1 - x_2^3\partial_2, \quad e_{12} = x_1^2x_2\partial_2, \quad e_{13} = x_1x_2^3\partial_2, \quad e_{14} = x_2^4\partial_2,$$
$$e_{15} = x_1^2x_2\partial_1, \quad e_{16} = x_1x_2^3\partial_1, \quad e_{17} = x_2^4\partial_1.$$

Let $g(V)$ be the nilradical of Lie algebra $L^2(V)$. The nilradical of Lie algebra $L^2(E_6)$ is spanned by:

$$g(V) =< e_2, e_3, e_4, \cdots, e_{17} > .$$

The nilradical of Lie algebra $L^2(E_6)$ has following multiplication table:

$$[e_2,e_3]=4e_4+4e_9,[e_2,e_4]=2e_{10}-2e_{15},[e_2,e_5]=e_{11}-3e_8,[e_2,e_6]=-2e_4-2e_9,$$
$$[e_2,e_7]=e_{11}-e_8,[e_2,e_8]=e_{17},[e_2,e_9]=-2e_{10}+2e_{15},[e_2,e_{10}]=e_{14}-2e_{16},$$
$$[e_2,e_{11}]=-3e_{17},[e_2,e_{12}]=2e_{13},[e_2,e_{15}]=2e_{16},[e_3,e_4]=6e_{12}-3e_{17},$$
$$[e_3,e_5]=3e_{10}-4e_{15},[e_3,e_6]=-2e_8,[e_3,e_7]=3e_{10}-2e_{15},[e_3,e_8]=-2e_{16},$$
$$[e_3,e_9]=-6e_{12}-2e_{17},[e_3,e_{10}]=2e_{13},[e_3,e_{11}]=-e_{14},[e_4,e_5]=-2e_{14}-5e_{16},$$
$$[e_4,e_6]=2e_{12}-3e_{17},[e_4,e_7]=-3e_{16},[e_4,e_9]=-2e_{13},[e_5,e_6]=e_{10},[e_5,e_7]=-e_{12},$$
$$[e_5,e_8]=-e_{13},[e_5,e_9]=-3e_{14},[e_5,e_{11}]=3e_{13},[e_6,e_7]=-e_{10},[e_6,e_8]=2e_{16},$$
$$[e_6,e_9]=2e_{12},[e_6,e_{11}]=e_{14}-2e_{16},[e_7,e_8]=-e_{13},[e_7,e_9]=-e_{14},[e_7,e_{11}]=3e_{13}.$$

The type of E_6 singularity=dim $g(V)/[g(V),g(V)]=5$. The nilpotency of E_6 singularity $=\min\{p\in N\cup\{0\}:g(V)^{p+1}=0\}=3$. It is easy to see from [2] that the torus T of $g(V)$ is spanned by

$$t:g(V)\longrightarrow g(V)$$

$$\begin{array}{ll} e_2\longrightarrow e_2 & e_3\longrightarrow \frac{3e_3}{2}\\ e_4\longrightarrow \frac{5e_4}{2} & e_5\longrightarrow 2e_5\\ e_6\longrightarrow \frac{3e_6}{2} & e_7\longrightarrow 2e_7\\ e_8\longrightarrow 3e_8 & e_9\longrightarrow \frac{5e_9}{2}\\ e_{10}\longrightarrow \frac{7e_{10}}{2} & e_{11}\longrightarrow 3e_{11}\\ e_{12}\longrightarrow 4e_{12} & e_{13}\longrightarrow 5e_{13}\\ e_{14}\longrightarrow \frac{9e_{14}}{2} & e_{15}\longrightarrow \frac{7e_{15}}{2}\\ e_{16}\longrightarrow \frac{9e_{16}}{2} & e_{17}\longrightarrow 4e_{17}. \end{array}$$

Thus $T=\mathbb{C}t$ is a unique maximal torus of $g(V)$. Let $\beta:T\longrightarrow\mathbb{C}$ be a linear map with $\beta(t)=1$.

$$\begin{aligned} g(V)&=g^{\beta}\oplus g^{\frac{3\beta}{2}}\oplus g^{\frac{5\beta}{2}}\oplus g^{2\beta}\oplus g^{3\beta}\oplus g^{\frac{7\beta}{2}}\oplus g^{4\beta}\oplus g^{5\beta}\oplus g^{\frac{9\beta}{2}}\\ &=\mathbb{C}e_2\oplus\mathbb{C}e_3\oplus\mathbb{C}e_6\oplus\mathbb{C}e_4\oplus\mathbb{C}e_9\oplus\mathbb{C}e_5\oplus\mathbb{C}e_7\oplus\mathbb{C}e_8\oplus\mathbb{C}e_{11}\oplus\mathbb{C}e_{10}\oplus\mathbb{C}e_{15}\oplus\mathbb{C}e_{12}\\ &\oplus\mathbb{C}e_{17}\oplus\mathbb{C}e_{13}\oplus\mathbb{C}e_{14}\oplus\mathbb{C}e_{16}. \end{aligned}$$

(e_2,e_3,e_5,e_6,e_7) is a T-minimal system of generators. The generalized Cartan matrix is

$$C^2(E_6)=\begin{pmatrix} 2&-1&-1&-2&-2\\ -2&2&-2&-2&-2\\ -2&-2&2&-2&-2\\ -2&-1&-1&2&-1\\ -2&-1&-1&-1&2 \end{pmatrix}.$$

□

Proposition 3.4 *Let* $V = \{(x_1, x_2) \in \mathbb{C}^2 : x_1^3 x_2 + x_2^3 = 0\}$ *be the* E_7 *singularity and* $L^2(V)$ *be a derivation Lie algebra. Then*

$$C^2(E_7) = \begin{pmatrix} 2 & -1 & -3 & -3 & -2 \\ -1 & 2 & -3 & -3 & -2 \\ -5 & -5 & 2 & -4 & -5 \\ -2 & -2 & -1 & 2 & -1 \\ -2 & -2 & -2 & -1 & 2 \end{pmatrix}.$$

Proof It is easy to see that $A^2(V) =< 1, x_1, x_2, x_2^2, x_2^3, x_1x_2, x_1x_2^2, x_1^2, x_1^2x_2, x_1^2x_2^2, x_1^3, x_1^4 >$. We have the following basis of Lie algebra $L^2(V)$ of E_7 singularity,

$$e_1 = 2x_1\partial_1 + 3x_2\partial_2, \quad e_2 = -x_1^2\partial_1 + 2x_1x_2\partial_2, \quad e_3 = x_2^2\partial_1 + 2x_1x_2\partial_2,$$
$$e_4 = -x_1x_2\partial_1 + 2x_2^2\partial_2, \quad e_5 = x_1^2\partial_2, \quad e_6 = x_2^2\partial_2, \quad e_7 = -x_1x_2\partial_2,$$
$$e_8 = x_1^3\partial_1 + 3x_2^2\partial_1 + 2x_1^2x_2\partial_2, \quad e_9 = x_1x_2^2\partial_1, \quad e_{10} = -x_1^2x_2\partial_1 + x_1x_2^2\partial_2,$$
$$e_{11} = x_1^3\partial_2 + 3x_2^2\partial_2, \quad e_{12} = x_1x_2^2\partial_2, \quad e_{13} = -x_1^2x_2\partial_2, \quad e_{14} = x_1^4\partial_2 + 3x_1x_2^2\partial_2,$$
$$e_{15} = x_1^2x_2^2\partial_2, \quad e_{16} = x_2^3\partial_2, \quad e_{17} = x_1^4\partial_1 + 3x_1x_2^2\partial_1, \quad e_{18} = x_1^2x_2^2\partial_1, \quad e_{19} = x_2^3\partial_1.$$

Let $g(V)$ be the nilradical of Lie algebra $L^2(V)$. The nilradical of Lie algebra $L^2(E_7)$ Spanned by:

$$g(V) =< e_2, e_3, e_4, \cdots, e_{19} > .$$

The nilradical of Lie algebra $L^2(E_7)$ has following multiplication table:

$$[e_2, e_3] = -6e_9, [e_2, e_4] = -3e_{10} - 3e_{12}, [e_2, e_5] = 4e_{11} - 12e_6, [e_2, e_6] = -2e_{12}, [e_2, e_7] = e_{13},$$
$$[e_2, e_8] = e_{17} - 21e_9, [e_2, e_9] = -5e_{18}, [e_2, e_{10}] = -3e_{15} - 2e_{19}, [e_2, e_{11}] = -21e_{12} + 5e_{14},$$
$$[e_2, e_{12}] = -e_{15}, [e_2, e_{13}] = 2e_{16}, [e_2, e_{14}] = -21e_{15}, [e_2, e_{17}] = -21e_{18},$$
$$[e_3, e_4] = -4e_{15} + 4e_{19}, [e_3, e_5] = -2e_{10} - 3e_{12} + e_{14}, [e_3, e_6] = -e_{15} + 2e_{19},$$
$$[e_3, e_7] = e_{16} - 2e_9, [e_3, e_8] = -5e_{18}, [e_3, e_{11}] = -9e_{15} + 4e_{19}, [e_3, e_{13}] = -2e_{18},$$
$$[e_4, e_5] = -5e_{13} + 3e_3 - e_8, [e_4, e_6] = -e_9, [e_4, e_7] = -e_{10} - 2e_{12}, [e_4, e_8] = 8e_{15} - 15e_{19},$$

$$[e_4, e_{11}] = -7e_{16} - e_{17}, [e_4, e_{12}] = -e_{18}, [e_4, e_{13}] = -4e_{15} - e_{19}, [e_5, e_6] = 2e_{13},$$
$$[e_5, e_7] = e_{11} - 3e_6, [e_5, e_8] = 6e_{10}, [e_5, e_9] = 2e_{15} + 2e_{19}, [e_5, e_{10}] = 4e_{16} + e_{17} - 3e_9,$$
$$[e_5, e_{11}] = 6e_{13}, [e_5, e_{12}] = 2e_{16}, [e_5, e_{13}] = -3e_{12} + e_{14}, [e_5, e_{14}] = 6e_{16},$$
$$[e_5, e_{16}] = -3e_{15}, [e_5, e_{17}] = 6e_{19}, [e_5, e_{19}] = -3e_{18}, [e_6, e_7] = -e_{12}, [e_6, e_8] = 2e_{15} - 6e_{19},$$
$$[e_6, e_{10}] = e_{18}, [e_6, e_{11}] = -2e_{16}, [e_6, e_{13}] = -e_{15}, [e_7, e_8] = -2e_{16} + 6e_9, [e_7, e_9] = 2e_{18},$$
$$[e_7, e_{10}] = 2e_{15} + e_{19}, [e_7, e_{11}] = 6e_{12} - e_{14}, [e_7, e_{12}] = e_{15}, [e_7, e_{14}] = 6e_{15},$$
$$[e_7, e_{17}] = 6e_{18}, [e_8, e_{11}] = -12e_{15} + 12e_{19}, [e_8, e_{13}] = -6e_{18}, [e_{11}, e_{13}] = -6e_{15}.$$

The type of E_7 singularity=dim $g(V)/[g(V), g(V)] = 5$. The nilpotency of E_7 singularity =min$\{p \in N \cup \{0\} : g(V)^{p+1} = 0\} = 5$. It is easy to see from [2] that the torus T of $g(V)$ is spanned by

$$t : g(V) \longrightarrow g(V)$$

$$\begin{array}{ll}
e_2 \longrightarrow e_2 - \frac{4e_{13}}{3} & e_3 \longrightarrow 2e_3 + \dfrac{2e_9}{3} - \dfrac{4e_{16}}{3} \\
e_4 \longrightarrow \frac{3e_4}{2} + \frac{e_{10}}{3} + \frac{5e_{12}}{3} & e_5 \longrightarrow \dfrac{e_5}{2} \\
e_6 \longrightarrow \frac{3e_6}{2} + \frac{2e_{12}}{3} & e_7 \longrightarrow e_7 + \dfrac{e_{13}}{3} \\
e_8 \longrightarrow 2e_8 + e_9 - \frac{8e_{16}}{3} + \frac{e_{17}}{3} & e_9 \longrightarrow 3e_9 + e_{18} \\
e_{10} \longrightarrow \frac{5e_{10}}{2} + \frac{5e_{15}}{3} + \frac{2e_{18}}{3} & e_{11} \longrightarrow \dfrac{3e_{11}}{2} + e_{12} + \dfrac{e_{14}}{3} \\
e_{12} \longrightarrow \frac{5e_{12}}{2} + e_{15} & e_{13} \longrightarrow 2e_{13} + \dfrac{2e_{16}}{3} \\
e_{14} \longrightarrow \frac{5e_{14}}{2} + e_{15} & e_{15} \longrightarrow \dfrac{7e_{15}}{2} \\
e_{16} \longrightarrow 3e_{16} & e_{17} \longrightarrow 3e_{17} + e_{18} \\
e_{18} \longrightarrow 4e_{18} & e_{19} \longrightarrow \dfrac{7e_{19}}{2}.
\end{array}$$

Thus $T = \mathbb{C}t$ is a unique maximal torus of $g(V)$. Let $\beta : T \longrightarrow \mathbb{C}$ be a linear map with $\beta(t) = 1$.

$$\begin{aligned}
g(V) &= g^{\beta} \oplus g^{\frac{\beta}{2}} \oplus g^{2\beta} \oplus g^{\frac{5\beta}{2}} \oplus g^{4\beta} \oplus g^{\frac{7\beta}{2}} \oplus g^{\frac{3\beta}{2}} \oplus g^{3\beta} \\
&= \mathbb{C}(\frac{-9e_2}{4} - 3e_{13} + e_{16}) \oplus \mathbb{C}(\frac{e_2}{4} + e_7) \oplus \mathbb{C}e_5 \oplus \mathbb{C}(3e_3 - 2e_9 + 6e_{13} + e_{18}) \\
&\oplus \mathbb{C}(6e_3 - 3e_8 - e_9 + e_{17}) \oplus \mathbb{C}(\frac{-3e_{13}}{2} + e_{16}) \oplus \mathbb{C}(-e_{12} + e_{15}) \oplus \mathbb{C}(-e_{12} + e_{14})
\end{aligned}$$

$$\begin{aligned}
&\oplus \mathbb{C}(\frac{-9e_{10}}{4} + \frac{15e_{12}}{4} + e_{18}) \oplus \mathbb{C}e_{18} \oplus \mathbb{C}e_{19} \oplus \mathbb{C}e_{15} \oplus \mathbb{C}(\frac{45e_4}{4} - \frac{75e_6}{2} - \frac{15e_{10}}{4} \\
&+ \frac{25e_{12}}{4} + e_{18}) \oplus \mathbb{C}(3e_6 - 2e_{12} + e_{15}) \oplus \mathbb{C}(6e_6 - 3e_{11} - e_{12} + e_{14}) \oplus \mathbb{C}(-e_9 + e_{18}) \\
&\oplus \mathbb{C}(-e_9 + e_{17}) \oplus \mathbb{C}e_{16}.
\end{aligned}$$

$(\frac{-9e_2}{4} - 3e_{13} + e_{16}, \frac{e_2}{4} + e_7, e_5, 6e_3 - 3e_8 - e_9 + e_{17}, \frac{45e_4}{4} - \frac{75e_6}{2} - \frac{15e_{10}}{4} + \frac{25e_{12}}{4} + e_{18})$ is a T-minimal system of generators. The generalized Cartan matrix is

$$C^2(E_7) = \begin{pmatrix} 2 & -1 & -3 & -3 & -2 \\ -1 & 2 & -3 & -3 & -2 \\ -5 & -5 & 2 & -4 & -5 \\ -2 & -2 & -1 & 2 & -1 \\ -2 & -2 & -2 & -1 & 2 \end{pmatrix}.$$

□

Proposition 3.5 *Let $V = \{(x_1, x_2) \in \mathbb{C}^2 : x_1^3 + x_2^5 = 0\}$ be the E_8 singularity. Then*

$$C^2(E_8) = \begin{pmatrix} 2 & -1 & -1 & -1 & -1 & -1 \\ -2 & 2 & -3 & -3 & -3 & -2 \\ -2 & -3 & 2 & -3 & -3 & -2 \\ -2 & -1 & -1 & 2 & -1 & -1 \\ -2 & -1 & -1 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & -1 & 2 \end{pmatrix}.$$

Proof It is noted that $A^2(V) =< 1, x_2, x_2^2, x_2^3, x_2^4, x_2^5, x_1, x_1x_2, x_1x_2^2, x_1x_2^3, x_1x_2^4, x_1^2, x_1^2x_2 >$. We have the following basis of the Lie algebra $L^2(V)$ of E_8 singularity,

$$e_1 = 5x_1\partial_1 + 3x_2\partial_2, \quad e_2 = x_2^3\partial_1, \quad e_3 = 3x_1x_2\partial_1 - x_2^2\partial_2, \quad e_4 = x_2^4\partial_1 + x_1^2\partial_2,$$
$$e_5 = -x_1^2\partial_1 + x_1x_2\partial_2, \quad e_6 = x_2^2\partial_2, \quad e_7 = -x_1x_2^2\partial_1 + x_2^3\partial_2, \quad e_8 = x_1x_2\partial_2,$$
$$e_9 = x_1x_2^3\partial_1 + 3x_2^2\partial_1, \quad e_{10} = 2x_1x_2^2\partial_1 - x_2^3\partial_2, \quad e_{11} = -x_1^2\partial_2, \quad e_{12} = x_1x_2^2\partial_2,$$
$$e_{13} = x_1x_2^3\partial_2, \quad e_{14} = -x_1x_2^3\partial_1 + x_2^4\partial_2, \quad e_{15} = x_1^2x_2\partial_2, \quad e_{16} = x_1x_2^4\partial_2, \quad e_{17} = x_2^5\partial_2,$$
$$e_{18} = x_1^2x_2\partial_1, \quad e_{19} = x_1x_2^4\partial_1, \quad e_{20} = x_2^5\partial_1$$

The nilradical of Lie algebra $L^2(V)$ of E_8 singularity is spanned by:

$$g(V) =< e_2, e_3, e_4, \cdots, e_{20} > .$$

The multiplication table of nilradical $g(V)$ is given as:

$$[e_2, e_3] = -6e_{11} - 6e_4, [e_2, e_4] = -2e_{13}, [e_2, e_5] = -e_{14} + 4e_9, [e_2, e_6] = 3e_{11} + 3e_4,$$
$$[e_2, e_7] = 4e_{20}, [e_2, e_8] = -e_{14} + 2e_9, [e_2, e_{10}] = -5e_{20}, [e_2, e_{11}] = 2e_{13}$$
$$[e_2, e_{12}] = -e_{17} + 3e_{19}, [e_2, e_{15}] = -2e_{16}, [e_2, e_{18}] = -2e_{19}, [e_3, e_4] = -8e_{15} + 4e_{20},$$
$$[e_3, e_5] = -4e_{12} + 6e_{18}, [e_3, e_6] = 3e_{10} + 3e_7, [e_3, e_7] = e_{14} + 2e_9, [e_3, e_8] = -4e_{12} + 3e_{18},$$
$$[e_3, e_9] = 3e_{19}, [e_3, e_{10}] = -e_{14}, [e_3, e_{11}] = 8e_{15} + 3e_{20}, [e_3, e_{12}] = -3e_{13}, [e_3, e_{13}] = -2e_{16},$$
$$[e_3, e_{14}] = 2e_{17}, [e_4, e_5] = 2e_{17} + 6e_{19}, [e_4, e_6] = -2e_{15} + 4e_{20}, [e_4, e_8] = 4e_{19}, [e_4, e_{11}] = 2e_{16},$$

$$[e_5, e_6] = -e_{12}, [e_5, e_7] = -3e_{13}, [e_5, e_8] = e_{15}, [e_5, e_9] = e_{16}, [e_5, e_{10}] = 4e_{13}, [e_5, e_{11}] = 3e_{17},$$
$$[e_5, e_{14}] = -4e_{16}, [e_6, e_7] = -e_{14} + e_9, [e_6, e_8] = e_{12}, [e_6, e_9] = -3e_{19}, [e_6, e_{10}] = e_{14} - 3e_9,$$
$$[e_6, e_{11}] = -2e_{15}, [e_6, e_{13}] = -e_{16}, [e_6, e_{14}] = -2e_{17} + 3e_{19}, [e_7, e_8] = 3e_{13}, [e_7, e_{10}] = -2e_{19},$$
$$[e_7, e_{12}] = 2e_{16}, [e_8, e_9] = e_{16}[e_8, e_{10}] = 4e_{13}, [e_8, e_{11}] = e_{17}, [e_8, e_{14}] = -4e_{16}, [e_{10}, e_{12}] = -3e_{16}.$$

The type of E_8 singularity=dim $g(V)/[g(V), g(V)] = 6$. The nilpotency of E_8 singularity $= min\{p \in N \cup \{0\} : g(V)^{p+1} = 0\} = 3$. It is easy to see from [2] that the torus T of $g(V)$ is spanned by

$$t : g(V) \longrightarrow g(V)$$

$$\begin{array}{ll} e_2 \longrightarrow e_2 & e_3 \longrightarrow \dfrac{3e_3}{4} \\ e_4 \longrightarrow \frac{7e_4}{4} & e_5 \longrightarrow \dfrac{5e_5}{4} \\ e_6 \longrightarrow \frac{3e_6}{4} & e_7 \longrightarrow \dfrac{3e_7}{2} \\ e_8 \longrightarrow \frac{5e_8}{4} & e_9 \longrightarrow \dfrac{9e_9}{4} \\ e_{10} \longrightarrow \frac{3e_{10}}{2} & e_{11} \longrightarrow \dfrac{7e_{11}}{4} \\ e_{12} \longrightarrow 2e_{12} & e_{13} \longrightarrow \dfrac{11e_{13}}{4} \\ e_{14} \longrightarrow \frac{9e_{14}}{4} & e_{15} \longrightarrow \dfrac{5e_{15}}{2} \\ e_{16} \longrightarrow \frac{7e_{16}}{2} & e_{17} \longrightarrow 3e_{17} \\ e_{18} \longrightarrow 2e_{18} & e_{19} \longrightarrow 3e_{19} \\ e_{20} \longrightarrow \frac{5e_{20}}{2}. & \end{array}$$

Thus $T = \mathbb{C}t$ is a unique maximal torus of $g(V)$. Let $\beta : T \longrightarrow \mathbb{C}$ be a linear map with $\beta(t) = 1$.

$$\begin{aligned} g(V) &= g^{\beta} \oplus g^{\frac{3\beta}{4}} \oplus g^{\frac{7\beta}{4}} \oplus g^{\frac{5\beta}{4}} \oplus g^{\frac{3\beta}{2}} \oplus g^{\frac{9\beta}{4}} \oplus g^{2\beta} \oplus g^{\frac{11\beta}{4}} \oplus g^{\frac{5\beta}{2}} \oplus g^{\frac{7\beta}{2}} \oplus g^{3\beta} \\ &= \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_6 \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_{11} \oplus \mathbb{C}e_5 \oplus \mathbb{C}e_8 \oplus \mathbb{C}e_7 \oplus \mathbb{C}e_{10} \oplus \mathbb{C}e_9 \oplus \mathbb{C}e_{14} \oplus \mathbb{C}e_{12} \\ &\quad \oplus \mathbb{C}e_{18} \oplus \mathbb{C}e_{13} \oplus \mathbb{C}e_{15} \oplus \mathbb{C}e_{20} \oplus \mathbb{C}e_{16} \oplus \mathbb{C}e_{17} \oplus \mathbb{C}e_{19}. \end{aligned}$$

$(e_2, e_3, e_5, e_6, e_7, e_8)$ is a T-minimal system of generators. The generalized Cartan matrix is

$$C^2(E_8) = \begin{pmatrix} 2 & -1 & -1 & -1 & -1 & -1 \\ -2 & 2 & -3 & -3 & -3 & -2 \\ -2 & -3 & 2 & -3 & -3 & -2 \\ -2 & -1 & -1 & 2 & -1 & -1 \\ -2 & -1 & -1 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & -1 & 2 \end{pmatrix}.$$

□

Proof of the Main Theorem The Main Theorem is an immediate corollary of Propositions 3.1, 3.2, 3.3, 3.4, and 3.5. □

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