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NON-EXISTENCE OF NEGATIVE WEIGHT DERIVATIONS OF THE LOCAL 1-ST HESSIAN ALGEBRAS OF SINGULARITIES*

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Abstract. In our previous work, we proposed a conjecture about the non-existence of negative weight derivations of the k-th Tjurina algebras of weighted homogeneous hypersurface singularities. In this paper, we verify this conjecture for three dimensional fewnomial singularities.

Key words. derivation Lie algebra, isolated singularity, weighted homogeneous.

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1. Introduction. Let $(V,0) \subset (\mathbb{C}^{n+1},0)$ be an isolated hypersurface singularity defined by a holomorphic function $f : (\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$. A holomorphic function f is called to be quasi-homogeneous if $f \in J(f)$, where $J(f) := (\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n})$ is the Jacobian ideal. A polynomial $f(z_0,\dots,z_n)$ is called to be weighted homogeneous of type $(\alpha_0,\dots,\alpha_n;d)$, where $\alpha_0,\dots,\alpha_n$ and d are fixed positive integers, if it can be expressed as a linear combination of momomials $z_0^{i_0} z_1^{i_1} \dots z_n^{i_n}$ for which $\alpha_0 i_0 + \cdots + \alpha_n i_n = d$. According to a beautiful theorem of Saito [27], if f defines an isolated singularity, then f is a weighted homogeneous polynomial after a biholomorphic change of coordinates if and only if f is quasi-homogeneous. Recall that the order of the lowest nonvanishing term in the power series expansion of f at 0 is called the multiplicity (denoted by $mult(f)$) of the singularity $(V,0)$.

For any isolated hypersurface singularity $(V,0) \subset (\mathbb{C}^{n+1},0)$ defined by the holomorphic function $f : (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$, one has the Tjurina algebra $A(V) := \mathcal{O}_{n+1}/(f, \frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n})$ which is finite-dimensional. The well-known Mather-Yau theorem [22] states that: If $(V_1,0)$ and $(V_2,0)$ are two isolated hypersurface singularities with the same dimension, then $(V_1,0)$ is biholomorphic to $(V_2,0)$ if and only if $A(V_1)$ is isomorphic to $A(V_2)$. In 1983, motivated from the Mather-Yau theorem, the second author introduced the Lie algebra of derivations of the Tjurina algebra $A(V)$, i.e., $L(V) = \mathrm{Der}(A(V), A(V))$. The finite-dimensional Lie algebra $L(V)$ was called Yau algebra and its dimension $\lambda(V)$ was called Yau number ([11], [20], [38]).

The Yau algebra plays an important role in singularity theory and is used to distinguish complex analytic structure of isolated hypersurface singularities [28]. Yau and his collaborators have been systematically studying the Yau algebras of isolated hypersurface singularities and its generalizations beginning from eighties (cf. [1, 2], [3], [4], [6], [7], [12]-[19], [28], [32], [33]-[35], [36, 37]). In [19], Hussain-Yau-Zuo introduced a new derivation Lie algebra arising from isolated hypersurface singularities. This Lie algebra is a more subtle invariant of singularities compared with previous Lie algebras.

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It was defined as follows.

For any isolated hypersurface singularity $(V,0) \subset (\mathbb{C}^{n+1},0)$ defined by the holomorphic function $f(z_0,\cdots,z_n)$, let $\mathrm{Hess}(f)$ be the Hessian matrix (f_{ij}) of the second order partial derivatives of f and $h(f)$ be the Hessian of f, i.e. the determinant of this matrix $\mathrm{Hess}(f)$. More generally, for each k satisfying $0 \leq k \leq n+1$ we denote by $h_k(f)$ the ideal in $\mathcal{O}_{n+1}$ generated by all $k \times k$-minors in the matrix $\mathrm{Hess}(f)$. In particular, the ideal $h_{n+1}(f) = (h(f))$ is a principal ideal. For each k as above, consider the k-th Hessian algebra of $(V,0)$ defined by

$$H_k(V) = \mathcal{O}_{n+1}/(f + J(f) + h_k(f)).$$

In particular, $H_0(V)$ is exactly the well-known Tjurina algebra $A(V)$. The isomorphism class of the local k-th Hessian algebra $H_k(V)$ is a contact invariant of $(V,0)$, i.e. depends only on the isomorphism class of the germ $(V,0)$ [9].

In particular, $H_{n+1}(f)$ has geometric meaning due to the following beautiful theorem.

Theorem 1.1 (Dimca [8]). *Two zero-dimensional isolated complete intersection singularities X and Y are isomorphic if and only if their singular subspaces Sing(X) and Sing(Y) are isomorphic.*

Remark 1.2. Let $V = V(f)$ be an isolated quasi-homogeneous hypersurface singularity. Assume that X defined by $(\frac{\partial f}{\partial z_0},\ldots,\frac{\partial f}{\partial z_n})$ is a zero-dimensional isolated complete intersection singularities. Then $\mathrm{Sing}(X)$ is defined by

$$(f, \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}, h(f)).$$

Theorem 1.1 implies that in order to study analytic isomorphism type of a zero-dimensional isolated complete intersection singularity X, we only need to consider the Artinian local algebra $H_{n+1}(f)$ which is the coordinate ring of $\mathrm{Sing}(X)$.

Combining Theorem 1.1 with Mather-Yau theorem, we know that $H_{n+1}(f)$ is a complete invariant of quasi-homogeneous isolated hypersurface singularities (i.e., $H_{n+1}(f)$ determines and is determined by the analytic isomorphism type of the singularity). In [4], the $H_{n+1}(f)$ is called the generalized Tjurina algebra of V. In [19], the authors introduced the following new invariants for isolated hypersurface singularities.

Definition 1.3. Let $(V,0) = \{(z_0,\ldots,z_n) \in \mathbb{C}^{n+1} \colon f(z_0,\ldots,z_n) = 0\}$ be a germ of isolated hypersurface singularity at the origin of $\mathbb{C}^{n+1}$ defined by $f(z_0,\ldots,z_n)$ ($n \geq 1$). The series of new derivation Lie algebras arising from the isolated hypersurface singularity $(V,0)$ are defined as $L_k(V) := \mathrm{Der}(H_k(V), H_k(V))$ or $\mathrm{Der}(H_k(V))$ for short, where $H_k(V) = \mathcal{O}_{n+1}/(f + J(f) + h_k(f))$ $(0 \leq k \leq n+1)$. Its dimension is denoted by $\lambda_k(V)$.

It is known that the Yau algebra can not characterize the ADE singularities completely. In fact, Elashvili and Khimshiashvili proved a beautiful result in [11]: if X and Y are two simple singularities except the pair A_6 and D_5, then $L(X) \cong L(Y)$ as Lie algebras if and only if X and Y are analytically isomorphic. However, in [4], the authors have proven that the ADE singularities can be characterized completely by the new Lie algebra $L_{n+1}(V)$. We have reasons to believe that this new Lie algebra

$L_k(V)$ and numerical invariant $\lambda_k(V)$ where $1 \le k \le n+1$ will also play an important role in the study of singularities.

THEOREM 1.4 ([4]). *If X and Y are two n-dimensional ADE singularities, then $L_{n+1}(X) \cong L_{n+1}(Y)$ as Lie algebras if and only if X and Y are analytically isomorphic.*

The derivation Lie algebra is also important in rational homotopy theory. Let A be a weighted homogeneous zero-dimensional complete intersection, i.e., a commutative algebra of the form

$$A = \mathbb{C}[z_0, z_1, \dots, z_n]/I$$

where the ideal $I = (f_0, f_1, \dots, f_n)$ is generated by a regular sequence of length $n+1$. Here all f_i are assumed to be weighted homogeneous with respect to strictly positive integral weights denoted by $wt(z_i) = \alpha_i (0 \le i \le n)$. Consequently, A is graded and one may speak about its homogeneous degree k derivations where k is an integer. Recall that a linear map $D : A \to A$ is a derivation if $D(ab) = D(a)b + aD(b)$ for any $a, b \in A$. A derivation D belongs to $\mathrm{Der}^k(A)$ if $D : A^* \to A^{*+k}$. That is to say, D has degree k.

On the one hand, one of the most prominent open problems in rational homotopy theory is related to the vanishing of the above derivations in strictly negative degrees:

HALPERIN CONJECTURE ([21]). *If A is as above, then* $\mathrm{Der}^{<0}(A) = 0$.

The Halperin Conjecture has been verified in several particular cases (see [5], [6], [25], [31], [36]). For recent progress, please see [7].

Let $(V, 0) = \{(z_0, z_1, \dots, z_n) \in \mathbb{C}^{n+1} \colon f(z_0, z_1, \dots, z_n) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial $f(z_0, z_1, \dots, z_n)$ of weighted type $(\alpha_0, \alpha_1, \dots, \alpha_n; d)$. Then by a well-known result of Saito [27], we can always assume without loss of generality that $d \ge 2\alpha_i > 0$ for all $0 \le i \le n$. We give the variable z_i weight α_i for $0 \le i \le n$, thus the Tjurina algebra $A(V)$ is a graded algebra, i.e., $A(V) = \bigoplus_{i=0}^{\infty} A_i(V)$, and the Lie algebra of derivations $\mathrm{Der}(A(V))$ is also graded. Thus $L(V)$ is graded. Similarly, $H_k(V)$ and $L_k(V)$ are also graded.

On the other hand, the second author discovered independently the following conjecture on the non-existence of negative weight derivations which is a special case of Halperin Conjecture.

YAU CONJECTURE (cf. [5], [6]). *Let $(V, 0) = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \colon f(z_0, \dots, z_n) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial $f(z_0, \dots, z_n)$ of weight type $(\alpha_0, \dots, \alpha_n; d)$. Assume that $d \ge 2\alpha_0 \ge 2\alpha_1 \ge \cdots \ge 2\alpha_n > 0$ without loss of generality. Then there is no non-zero negative weight derivation on the Tjurina algebra (= Milnor algebra) $A(V) = \mathcal{O}_{n+1}/(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n})$, i.e., $L(V)$ is non-negatively graded.*

This conjecture is still open and has only been proved in the low-dimensional case $n \le 3$ ([5], [6]) by explicit calculations. It has also been proved for the high-dimensional singularities under certain condition [36] and homogeneous singularities in [32].

It is a very interesting question to know whether a positvely graded algebra has negative weight derivations due to many applications in algebraic geometry, singularity theory and rational homotopy theory ([21], [26], [29, 30]). Assume that f is a weighted homogeneous polynomial, since the k-th Hessian algebra $H_k(V)$ and $L_k(V)$ are also naturally graded, it is natural to propose the following new conjecture:

Conjecture 1.5. *Let* $(V,0)=\{(z_0,\ldots,z_n)\in\mathbb{C}^{n+1}\colon f(z_0,\ldots,z_n)=0\}$ *be an isolated singularity defined by the weighted homogeneous polynomial* $f(z_0,\ldots,z_n)$ *of weight type* $(\alpha_0,\ldots,\alpha_n;d)$*. Assume that* $d\geq 2\alpha_0\geq 2\alpha_1\geq\cdots\geq 2\alpha_n>0$ *without loss of generality. Let* $H_k(V)$ *be the* k*-th Hessian algebra. Furthermore, in the case of* $1\leq k\leq n$*, we need to assume that* $mult(f)\geq 5$*. Then for any* $0\leq k\leq n+1$*, there does not exist negative weight derivations of* $H_k(V)$*, i.e.,* $L_k(V)$ *is non-negatively graded.*

When $k=0$, it is exactly the long-standing Yau Conjecture which was verified for $n\leq 3$ ([5]).

When $k=n+1$, it was verified in [23] for $n\leq 3$.

When $1<k\leq n$, it was verified in [24] for $n\leq 2$.

When $k=2$ or $k=3$, it was also verified in [24] for $n=3$.

The case when $n=1$ is trivial. However, the proof of the Conjecture 1.5 for the case of $k=1$ is completely different from other cases and seems very hard in general. In this paper, we shall verify Conjecture 1.5 for the case $n=3$ and $k=1$ (see Theorem A). We first recall some definitions.

Definition 1.6. An isolated hypersurface singularity in $\mathbb{C}^n$ is fewnomial if it can be defined by an n-nomial in n variables and it is a weighted homogeneous fewnomial isolated singularity if it can be defined by a weighted homogeneous fewnomial. 2 (resp. 3)-nomial isolated hypersurface singularity is also called binomial (resp. trinomial) singularity.

Proposition 1.7. *[37] Let* f *be a weighted homogeneous fewnomial isolated singularity with* $mult(f)\geq 3$*. Then* f *is analytically equivalent to a linear combination of the following three types:*
Type (I). $z_0^{n_0}+z_1^{n_1}+\cdots+z_{r-1}^{n_{r-1}}+z_r^{n_r}$, $r\geq 0$,
Type (II). $z_0^{n_0}z_1+z_1^{n_1}z_2+\cdots+z_{r-1}^{n_{r-1}}z_r+z_r^{n_r}$, $r\geq 1$,
Type (III). $z_0^{n_0}z_1+z_1^{n_1}z_2+\cdots+z_{r-1}^{n_{r-1}}z_r+z_r^{n_r}z_0$, $r\geq 1$.

The above three types are also called "the Brieskorn type", "the chain type", and "the loop type" respectively. According to Ebeling and Takahashi [10], the fewnomial singularity is also called invertible singularity which plays an important role in mirror symmetry.

We introduce the following definition.

Definition 1.8. Let $f(z_0,z_1,z_2,z_3)$ be a weighted homogeneous fewnomial.

f is called Type A fewnomial if f is one of Type (I), Type (II) or Type (III) above.

f is called Type B fewnomial if f can be written as the sum of two weighted homogeneous polynomial $f_1(z_0,z_1,z_2)$ and $f_2(z_3)=z_3^{n_3}$ (after a biholomorphic transformation if necessary) where f_1 is Type (II) or Type (III) above.

f is called Type C fewnomial if f can be written as the sum of two weighted homogeneous polynomial $f_1(z_0,z_1)$ and $f_2(z_2,z_3)$ where both of f_1 and f_2 are Type (I), Type (II) or Type (III) above but they are not Type (I) at the same time.

In this paper, we prove the following main results: Theorem A and Theorem B. The Theorem A verifies Conjecture 1.5 partially, and Theorem B gives a complete classification of the singularities which have negative weight derivations.

Theorem A. *Let* $(V,0)=\{(z_0,z_1,z_2,z_3)\in\mathbb{C}^4\colon f(z_0,z_1,z_2,z_3)=0\}$ *be an isolated singularity defined by the weighted homogeneous fewnomial* $f(z_0,z_1,z_2,z_3)$ *of* *weight type $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ where mult $(f) \geq 3$. Let $H_1(V)$ be the 1-st Hessian algebra. If f is Type A fewnomial with mult$(f) \geq 5$, Type B fewnomial with mult$(f) \geq 4$ or Type C fewnomial, there does not exist negative weight derivation of $H_1(V)$.*

Proof. The Theorem A follows from the Theorem B. ∎

The condition $mult(f) \geq 5$ for Type A and $mult(f) \geq 4$ for Type B in Theorem A cannot be omitted. In Theorem B below, we list all the possibilities of $(V, 0)$ when there exists negative weight derivation of $H_1(V)$.

Theorem B. *Let $(V, 0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 : f(z_0, z_1, z_2, z_3) = 0\}$ be an isolated singularity defined by the weighted homogeneous fewnomial $f(z_0, z_1, z_2, z_3)$ of weight type $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ where mult $(f) \geq 3$. Let $H_1(V)$ be the 1-st Hessian algebra. There exists a negative weight derivation of $H_1(V)$ if and only if f is in one of the following forms after renumbering the variables z_0, z_1, z_2 and z_3 so that $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$:*

(1) *when f is a Type A fewnomial:*

(*i*) $f = z_0^3 z_3 + z_1^3 z_2 + z_2^3 z_0 + z_3^{n_3} (n_3 \geq 5)$. *In this case, the set of negative weight derivations of $H_1(V)$ is* $\left\{D | D = c z_2 z_3^k \frac{\partial}{\partial z_1}, c \neq 0, 0 \leq k < \frac{n_3-4}{27}, k \in \mathbb{Z}\right\}$;

(*ii*) $f = z_0^3 + z_1^3 z_2 + z_2^3 z_0 + z_3^{n_3} z_1 \, (n_3 \geq 21)$. *In this case, the set of negative weight derivations of $H_1(V)$ is* $\left\{D | D = c z_2 z_3^k \frac{\partial}{\partial z_1}, c \neq 0, 1 \leq k < \frac{n_3}{20}, k \in \mathbb{Z}\right\}$;

(*iii*) $f = z_0^3 + z_1^2 z_0 + z_2^3 z_3 + z_3^3 z_1$. *In this case, the set of negative weight derivations of $H_1(V)$ is* $\left\{D | D = c z_3 \frac{\partial}{\partial z_2}, c \neq 0\right\}$;

(*iv*) $f = z_0^2 z_3 + z_1^3 z_2 + z_2^3 z_0 + z_3^{n_3} (n_3 \geq 5)$. *In this case, the set of negative weight derivations of $H_1(V)$ is* $\left\{D | D = c z_2 \frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(*v*) $f = z_0^2 z_3 + z_1^3 z_2 + z_2^4 z_0 + z_3^{n_3} (n_3 \geq 8)$. *In this case, the set of negative weight derivations of $H_1(V)$ is* $\left\{D | D = c z_2^2 \frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(*vi*) $f = z_0^2 z_3 + z_1^4 z_2 + z_2^3 z_0 + z_3^{n_3} (n_3 \geq 6)$. *In this case, the set of negative weight derivations of $H_1(V)$ is* $\left\{D | D = c z_2 \frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(*vii*) $f = z_0^2 z_2 + z_1^3 z_3 + z_2^4 + z_3^3 z_0$. *In this case, the set of negative weight derivations of $H_1(V)$ is* $\left\{D | D = c z_3 \frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(*viii*) $f = z_0^2 z_2 + z_1^3 z_3 + z_2^3 z_1 + z_3^3 z_0$. *In this case, the set of negative derivations of $H_1(V)$ is* $\left\{D | D = c z_3 \frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(*ix*) $f = z_0^2 z_3 + z_1^3 z_2 + z_2^3 z_0 + z_3^{n_3} z_1 \, (n_3 \geq 4)$. *In this case, the set of negative derivations of $H_1(V)$ is* $\left\{D | D = c z_2 \frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(*x*) $f = z_0^2 z_3 + z_1^4 z_2 + z_2^3 z_0 + z_3^{n_3} z_1 \, (n_3 \geq 5)$. *In this case, the set of negative derivations of $H_1(V)$ is* $\left\{D | D = c z_2 \frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(*xi*) $f = z_0^2 z_3 + z_1^3 z_2 + z_2^4 z_0 + z_3^{n_3} z_1 \, (n_3 \geq 6)$. *In this case, the set of negative derivations of $H_1(V)$ is* $\left\{D | D = c z_2^2 \frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(*xii*) $f = z_0^3 z_3 + z_1^3 z_2 + z_2^3 z_0 + z_3^{n_3} z_1 \, (n_3 \geq 24)$. *In this case, the set of negative derivations of $H_1(V)$ is* $\left\{D | D = c z_2 z_3^k \frac{\partial}{\partial z_1}, c \neq 0, 1 \leq k < \frac{n_3-3}{20}, k \in \mathbb{Z}\right\}$.

(2) *when f is a Type B fewnomial:*

(*i*) $f = z_0^3 + z_1^3 + z_2^3 z_3 + z_3^3 z_0$. *In this case, the set of negative weight derivations of $H_1(V)$ is* $\left\{D | D = c z_3 \frac{\partial}{\partial z_2}, c \neq 0\right\}$;

(ii) $f = z_0^3 + z_1^3 z_3 + z_2^4 + z_3^3 z_0$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D|D = cz_3\frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(iii) $f = z_0^3 + z_1^3 z_2 + z_2^3 z_0 + z_3^{n_3}\,(n_3 \geq 5)$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D|D = cz_2\frac{\partial}{\partial z_1}, c \neq 0\right\}$.

When f *is Type C fewnomial, there does not exist any negative weight derivation* D.

Proof. By Proposition 2.1, Proposition 3.1, and Proposition 4.1, we complete the proof of Theorem B. □

Remark 1.9. Actually, in Theorem A and Theorem B we just need to consider the weighted homogeneous polynomials with $mult(f) \geq 3$ due to the cases $mult(f) = 1, 2$ are trivial.

2. Type A Fewnomial Case. In this section, we will discuss the Type A fewnomial case where $mult\,(f) \geq 3$. There are three types to discuss:

Type (I): $f(z_0, z_1, z_2, z_3) = z_0^{n_0} + z_1^{n_1} + z_2^{n_2} + z_3^{n_3}$.

Type (II): $f(z_0, z_1, z_2, z_3) = z_0^{n_0} z_1 + z_1^{n_1} z_2 + z_2^{n_2} z_3 + z_3^{n_3}$.

Type (III): $f(z_0, z_1, z_2, z_3) = z_0^{n_0} z_1 + z_1^{n_1} z_2 + z_2^{n_2} z_3 + z_3^{n_3} z_0$.

In the above forms, the weights orders of α_0, α_1, α_2 and α_3 are not determined. The overall conclusion is written in Proposition 2.1.

Proposition 2.1 (Type A fewnomial case of Theorem B). *Let* $(V, 0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 : f(z_0, z_1, z_2, z_3) = 0\}$ *be an isolated singularity defined by the Type A fewnomial* $f(z_0, z_1, z_2, z_3)$ *of weight type* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ *where* $mult\,(f) \geq 3$. *Let* $H_1(V)$ *be the 1-st Hessian algebra. There exists negative weight derivation of* $H_1(V)$ *if and only if* f *is in one of the following forms after renumbering the variables* z_0, z_1, z_2 *and* z_3 *so that* $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$ *(we combine the cases that can be transformed into each other by simply renumbering the variables, which is caused by the equal weights of asymmetrical variables):*

(i) $f = z_0^3 z_3 + z_1^3 z_2 + z_2^3 z_0 + z_3^{n_3}(n_3 \geq 5)$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D|D = cz_2 z_3^k\frac{\partial}{\partial z_1}, c \neq 0, 0 \leq k < \frac{n_3-4}{27}, k \in \mathbb{Z}\right\}$;

(ii) $f = z_0^3 + z_1^3 z_2 + z_2^3 z_0 + z_3^{n_3} z_1\,(n_3 \geq 21)$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D|D = cz_2 z_3^k\frac{\partial}{\partial z_1}, c \neq 0, 1 \leq k < \frac{n_3}{20}, k \in \mathbb{Z}\right\}$;

(iii) $f = z_0^3 + z_1^2 z_0 + z_2^3 z_3 + z_3^3 z_1$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D|D = cz_3\frac{\partial}{\partial z_2}, c \neq 0\right\}$;

(iv) $f = z_0^2 z_3 + z_1^3 z_2 + z_2^3 z_0 + z_3^{n_3}(n_3 \geq 5)$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D|D = cz_2\frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(v) $f = z_0^2 z_3 + z_1^3 z_2 + z_2^4 z_0 + z_3^{n_3}(n_3 \geq 8)$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D|D = cz_2^2\frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(vi) $f = z_0^2 z_3 + z_1^4 z_2 + z_2^3 z_0 + z_3^{n_3}(n_3 \geq 6)$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D|D = cz_2\frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(vii) $f = z_0^2 z_2 + z_1^3 z_3 + z_2^4 + z_3^3 z_0$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D|D = cz_3\frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(viii) $f = z_0^2 z_2 + z_1^3 z_3 + z_2^3 z_1 + z_3^3 z_0$. *In this case, the set of negative derivations of* $H_1(V)$ *is* $\left\{D|D = cz_3\frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(ix) $f = z_0^2 z_3 + z_1^3 z_2 + z_2^3 z_0 + z_3^{n_3} z_1\,(n_3 \geq 4)$. *In this case, the set of negative derivations of* $H_1(V)$ *is* $\left\{D|D = cz_2\frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(x) $f = z_0^2 z_3 + z_1^4 z_2 + z_2^3 z_0 + z_3^{n_3} z_1\,(n_3 \geq 5)$. *In this case, the set of negative derivations of* $H_1(V)$ *is* $\left\{D|D = cz_2\frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(xi) $f = z_0^2 z_3 + z_1^3 z_2 + z_2^4 z_0 + z_3^{n_3} z_1\,(n_3 \geq 6)$. *In this case, the set of negative derivations of* $H_1(V)$ *is* $\left\{D|D = cz_2^2\frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(xii) $f = z_0^3 z_3 + z_1^3 z_2 + z_2^3 z_0 + z_3^{n_3} z_1\,(n_3 \geq 24)$. *In this case, the set of negative derivations of* $H_1(V)$ *is* $\left\{D|D = cz_2 z_3^k\frac{\partial}{\partial z_1}, c \neq 0, 1 \leq k < \frac{n_3-3}{20}, k \in \mathbb{Z}\right\}$.

Therefore, if $mult(f) \geq 5$*, there does not exist any negative weight derivation of* $H_1(V)$.

Proof. This proof is tedious but simple calculations. We omit the details and readers can find those in the rest of this section. By Proposition 2.2, Proposition 2.3, and Proposition 2.57, the proof is clear. □

2.1. Type (I). Next we will discuss the case

$$f = z_0^{n_0} + z_1^{n_1} + z_2^{n_2} + z_3^{n_3}$$

where $mult(f) \geq 3$. The weights orders of $\alpha_0, \alpha_1, \alpha_2$ and α_3 are not determined. All results of this subsection are summarized in Proposition 2.2.

Proposition 2.2 (Type (I) of Proposition 2.1). *Let* $(V, 0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 \colon f(z_0, z_1, z_2, z_3) = 0\}$ *be an isolated singularity defined by* $f(z_0, z_1, z_2, z_3) = z_0^{n_0} + z_1^{n_1} + z_2^{n_2} + z_3^{n_3}$ *of weight type* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ *where* $mult(f) \geq 3$. *Let* $H_1(V)$ *be the* 1*-st Hessian algebra. There does not exist any negative weight derivation of* $H_1(V)$.

Proof. Without loss of generality, set $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$. After renumbering, if there exists some negative weight derivation D, D must be in the form of $D = p_0(z_1, z_2, z_3)\frac{\partial}{\partial z_0} + p_1(z_2, z_3)\frac{\partial}{\partial z_1} + cz_3^k\frac{\partial}{\partial z_2}$.

We can get $n_0 \geq 3$, $n_1 \geq 3$, $n_2 \geq 3$ and $n_3 \geq 3$ from $mult(f) \geq 3$. Regardless of difference of constants, we obtain

$$\mathrm{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0^{n_0-2} & 0 & 0 & 0 \\ * & z_1^{n_1-2} & 0 & 0 \\ * & * & z_2^{n_2-2} & 0 \\ * & * & * & z_3^{n_3-2} \end{bmatrix}.$$

From $D\left(z_0^{n_0-2}\right) = p_0(z_1, z_2, z_3)(n_0 - 2)z_0^{n_0-3} = 0$, we have $p_0(z_1, z_2, z_3) = 0$. From $D\left(z_1^{n_1-2}\right) = p_1(z_2, z_3)(n_1 - 2)z_1^{n_1-3} \in \left(z_0^{n_0-2}\right)$, we have $p_1(z_2, z_3) = 0$. From $D\left(z_2^{n_2-2}\right) = cz_3^k(n_2 - 2)z_2^{n_2-3} \in \left(z_0^{n_0-2}, z_1^{n_1-2}\right)$, we have $c = 0$.

So $D = 0$, which contradicts to the assumption that D is negatively weighted. There does not exist any negative weight derivation. □

2.2. Type (II). Next we will discuss the case

$$f = z_0^{n_0} z_1 + z_1^{n_1} z_2 + z_2^{n_2} z_3 + z_3^{n_3}$$

where $mult(f) \geq 3$. The weight orders of $\alpha_0, \alpha_1, \alpha_2$ and α_3 are not determined. All results of this subsection are summarized in Proposition 2.3.

Proposition 2.3 (Type (II) of Proposition 2.1). *Let* $(V,0)=\{(z_0,z_1,z_2,z_3)\in \mathbb{C}^4\colon f(z_0,z_1,z_2,z_3)=0\}$ *be an isolated singularity defined by* $f(z_0,z_1,z_2,z_3)=z_0^{n_0}z_1+z_1^{n_1}z_2+z_2^{n_2}z_3+z_3^{n_3}$ *of weight type* $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ *where* $mult(f)\geq 3$. *Let* $H_1(V)$ *be the* 1*-st Hessian algebra. There exists negative weight derivation of* $H_1(V)$ *if and only if* f *is in one of the following forms after renumbering the variables* z_0, z_1, z_2 *and* z_3 *so that* $\alpha_0\geq\alpha_1\geq\alpha_2\geq\alpha_3$ *(we combine the cases that can be transformed into each other by simply renumbering the variables, which is caused by the equal weights of asymmetrical variables):*

(i) $f=z_0^3z_3+z_1^3z_2+z_2^3z_0+z_3^{n_3}(n_3\geq 5)$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D|D=cz_2z_3^k\frac{\partial}{\partial z_1},c\neq 0,0\leq k<\frac{n_3-4}{27},k\in\mathbb{Z}\right\}$;

(ii) $f=z_0^3+z_1^3z_2+z_2^3z_0+z_3^{n_3}z_1\,(n_3\geq 21)$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D|D=cz_2z_3^k\frac{\partial}{\partial z_1},c\neq 0,1\leq k<\frac{n_3}{20},k\in\mathbb{Z}\right\}$;

(iii) $f=z_0^3+z_1^2z_0+z_2^3z_3+z_3^3z_1$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D|D=cz_3\frac{\partial}{\partial z_2},c\neq 0\right\}$;

(iv) $f=z_0^2z_3+z_1^3z_2+z_2^3z_0+z_3^{n_3}(n_3\geq 5)$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D|D=cz_2\frac{\partial}{\partial z_1},c\neq 0\right\}$;

(v) $f=z_0^2z_3+z_1^3z_2+z_2^4z_0+z_3^{n_3}(n_3\geq 8)$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D|D=cz_2^2\frac{\partial}{\partial z_1},c\neq 0\right\}$;

(vi) $f=z_0^2z_3+z_1^4z_2+z_2^3z_0+z_3^{n_3}(n_3\geq 6)$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D|D=cz_2\frac{\partial}{\partial z_1},c\neq 0\right\}$;

(vii) $f=z_0^2z_2+z_1^3z_3+z_2^4+z_3^3z_0$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D|D=cz_3\frac{\partial}{\partial z_1},c\neq 0\right\}$.

Proof. By $mult(f)\geq 3$, we get $wt(f)>2max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\}$. There are two cases to discuss:

(i) $wt(f)>3max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\}$;

(ii) $2max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\}<wt(f)\leq 3max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\}$.

They correspond to Proposition 2.4 and Proposition 2.29 respectively.

By the two propositions, we complete the proof. □

For $f(z_0,z_1,z_2,z_3)=z_0^{n_0}z_1+z_1^{n_1}z_2+z_2^{n_2}z_3+z_3^{n_3}$, discussions when $wt(f)>3max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\}$ are summarized in Proposition 2.4.

Proposition 2.4 (Case (i) of Proposition 2.3). *Let* $(V,0)=\{(z_0,z_1,z_2,z_3)\in\mathbb{C}^4\colon f(z_0,z_1,z_2,z_3)=0\}$ *be an isolated singularity defined by* $f(z_0,z_1,z_2,z_3)=z_0^{n_0}z_1+z_1^{n_1}z_2+z_2^{n_2}z_3+z_3^{n_3}$ *of weight type* $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ *where* $wt(f)>3max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\}$. *Let* $H_1(V)$ *be the* 1*-st Hessian algebra. Then there exists negative weight derivation of* $H_1(V)$ *if and only if* f *is in the form of* $f=z_0^3z_3+z_1^3z_2+z_2^3z_0+z_3^{n_3}(n_3\geq 5)$ *after renumbering the variables* z_0, z_1, z_2 *and* z_3 *so that* $\alpha_0\geq\alpha_1\geq\alpha_2\geq\alpha_3$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D|D=cz_2z_3^k\frac{\partial}{\partial z_1},c\neq 0,0\leq k<\frac{n_3-4}{27},k\in\mathbb{Z}\right\}$ *after renumbering.*

Proof. The calculation process is lengthy. There are 24 cases with respect to the weight order of z_0, z_1, z_2 and z_3.

(i) $\alpha_0\geq\alpha_1\geq\alpha_2\geq\alpha_3$;

(ii) $\alpha_0\geq\alpha_1\geq\alpha_3\geq\alpha_2$;

(iii) $\alpha_0\geq\alpha_2\geq\alpha_1\geq\alpha_3$;

(iv) $\alpha_0 \geq \alpha_2 \geq \alpha_3 \geq \alpha_1$;
(v) $\alpha_0 \geq \alpha_3 \geq \alpha_1 \geq \alpha_2$;
(vi) $\alpha_0 \geq \alpha_3 \geq \alpha_2 \geq \alpha_1$;
(vii) $\alpha_1 \geq \alpha_0 \geq \alpha_2 \geq \alpha_3$;
(viii) $\alpha_1 \geq \alpha_0 \geq \alpha_3 \geq \alpha_2$;
(ix) $\alpha_1 \geq \alpha_2 \geq \alpha_0 \geq \alpha_3$;
(x) $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_0$;
(xi) $\alpha_1 \geq \alpha_3 \geq \alpha_0 \geq \alpha_2$;
(xii) $\alpha_1 \geq \alpha_3 \geq \alpha_2 \geq \alpha_0$;
(xiii) $\alpha_2 \geq \alpha_0 \geq \alpha_1 \geq \alpha_3$;
(xiv) $\alpha_2 \geq \alpha_0 \geq \alpha_3 \geq \alpha_1$;
(xv) $\alpha_2 \geq \alpha_1 \geq \alpha_0 \geq \alpha_3$;
(xvi) $\alpha_2 \geq \alpha_1 \geq \alpha_3 \geq \alpha_0$;
(xvii) $\alpha_2 \geq \alpha_3 \geq \alpha_0 \geq \alpha_1$;
(xviii) $\alpha_2 \geq \alpha_3 \geq \alpha_1 \geq \alpha_0$;
(xix) $\alpha_3 \geq \alpha_0 \geq \alpha_1 \geq \alpha_2$;
(xx) $\alpha_3 \geq \alpha_0 \geq \alpha_2 \geq \alpha_1$;
(xxi) $\alpha_3 \geq \alpha_1 \geq \alpha_0 \geq \alpha_2$;
(xxii) $\alpha_3 \geq \alpha_1 \geq \alpha_2 \geq \alpha_0$;
(xxiii) $\alpha_3 \geq \alpha_2 \geq \alpha_0 \geq \alpha_1$;
(xxiv) $\alpha_3 \geq \alpha_2 \geq \alpha_1 \geq \alpha_0$.

One can look up more details in the following lemmas respectively (from Lemma 2.5 to Lemma 2.28). □

LEMMA 2.5 (Case (i) of Proposition 2.4). *Let* $(V,0) = \{(z_0,z_1,z_2,z_3) \in \mathbb{C}^4 \colon f(z_0,z_1,z_2,z_3) = 0\}$ *be an isolated singularity defined by* $f(z_0,z_1,z_2,z_3) = z_0^{n_0}z_1 + z_1^{n_1}z_2 + z_2^{n_2}z_3 + z_3^{n_3}$ *of weight type* $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ *where* $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$ *and* $wt(f) > 3max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\} = 3\alpha_0$. *Let* $H_1(V)$ *be the* 1*-st Hessian algebra. Then there does not exist negative weight derivation of* $H_1(V)$.

Proof. The form of f does not change after renumbering to make $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$. After renumbering, if there exists some negative weight derivation D, D must be in the form of $D = p_0(z_1,z_2,z_3)\frac{\partial}{\partial z_0} + p_1(z_2,z_3)\frac{\partial}{\partial z_1} + cz_3^k\frac{\partial}{\partial z_2}$.

After renumbering, it is clear that $n_0 \geq 3$. From $2\alpha_i + \alpha_j \leq 3\alpha_0 < wt(f) = n_i\alpha_i + \alpha_j$, we get $n_i > 2$ for $(i,j) = (1,2)$ and $(2,3)$. So we have $n_1 \geq 3$ and $n_2 \geq 3$. From $3\alpha_0 < wt(f) = n_3\alpha_3 \leq n_3\alpha_0$, we have $n_3 > 3$, which is equivalent to $n_3 \geq 4$. Regardless of difference of constants, we get the equations below.

$$\text{Hess }(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0^{n_0-2}z_1 & z_0^{n_0-1} & 0 & 0 \\ * & z_1^{n_1-2}z_2 & z_1^{n_1-1} & 0 \\ * & * & z_2^{n_2-2}z_3 & z_2^{n_2-1} \\ * & * & * & z_3^{n_3-2} \end{bmatrix}.$$

From $D\left(z_0^{n_0-2}z_1\right) = p_0\left(z_1,z_2,z_3\right)(n_0-2)z_0^{n_0-3}z_1 + p_1\left(z_2,z_3\right)z_0^{n_0-2} = 0$, we get $p_0\left(z_1,z_2,z_3\right)(n_0-2)z_1 + p_1\left(z_2,z_3\right)z_0 = 0$. Therefore, we have $p_0\left(z_1,z_2,z_3\right) = 0, p_1\left(z_2,z_3\right) = 0$ and $D = cz_3^k\frac{\partial}{\partial z_2}$.

Since $D\left(z_1^{n_1-2}z_2\right) = cz_3^k z_1^{n_1-2} \in \left(z_0^{n_0-2}z_1, z_0^{n_0-1}\right)$, it is easy to see $c = 0$. Therefore, $D = 0$.

In conclusion, there does not exist negative weight derivation of $H_1(V)$ and we complete the proof. □

Lemma 2.6 (Case (ii) of Proposition 2.4). *Let $(V,0)=\{(z_0,z_1,z_2,z_3)\in\mathbb{C}^4\colon f(z_0,z_1,z_2,z_3)=0\}$ be an isolated singularity defined by $f(z_0,z_1,z_2,z_3)=z_0^{n_0}z_1+z_1^{n_1}z_2+z_2^{n_2}z_3+z_3^{n_3}$ of weight type $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ where $\alpha_0\geq\alpha_1\geq\alpha_3\geq\alpha_2$ and $wt(f)>3max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\}=3\alpha_0$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there does not exist negative weight derivation of $H_1(V)$.*

Proof. After renumbering to make $\alpha_0\geq\alpha_1\geq\alpha_2\geq\alpha_3$, the form of f changes to $f=z_0^{n_0}z_1+z_1^{n_1}z_3+z_2^{n_2}+z_3^{n_3}z_2$. After renumbering, if there exists some negative weight derivation D, D must be in the form of $D=p_0(z_1,z_2,z_3)\frac{\partial}{\partial z_0}+p_1(z_2,z_3)\frac{\partial}{\partial z_1}+cz_3^k\frac{\partial}{\partial z_2}$.

After renumbering, it is clear that $n_0\geq 3$. From $2\alpha_i+\alpha_j\leq 3\alpha_0<wt(f)=n_i\alpha_i+\alpha_j$, we get $n_i>2$ for $(i,j)=(1,3)$ and $(3,2)$. So we have $n_1\geq 3$ and $n_3\geq 3$. From $3\alpha_0<wt(f)=n_2\alpha_2\leq n_2\alpha_0$, we have $n_2>3$, which is equivalent to $n_2\geq 4$. Regardless of difference of constants, we get the equations below.

$$\text{Hess }(f)=\begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03}\\ f_{10} & f_{11} & f_{12} & f_{13}\\ f_{20} & f_{21} & f_{22} & f_{23}\\ f_{30} & f_{31} & f_{32} & f_{33}\end{bmatrix}=\begin{bmatrix} z_0^{n_0-2}z_1 & z_0^{n_0-1} & 0 & 0\\ * & z_1^{n_1-2}z_3 & 0 & z_1^{n_1-1}\\ * & * & z_2^{n_2-2} & z_3^{n_3-1}\\ * & * & * & z_3^{n_3-2}z_2\end{bmatrix}.$$

From $D\left(z_0^{n_0-2}z_1\right)=p_0\left(z_1,z_2,z_3\right)(n_0-2)z_0^{n_0-3}z_1+p_1\left(z_2,z_3\right)z_0^{n_0-2}=0$, we get $p_0\left(z_1,z_2,z_3\right)(n_0-2)z_1+p_1\left(z_2,z_3\right)z_0=0$. Therefore, $p_0\left(z_1,z_2,z_3\right)=0, p_1\left(z_2,z_3\right)=0$ and $D=cz_3^k\frac{\partial}{\partial z_2}$.

From $D\left(z_2^{n_2-2}\right)=c\left(n_2-2\right)z_2^{n_2-3}z_3^k\in\left(z_0^{n_0-2}z_1,z_0^{n_0-1},z_1^{n_1-2}z_3,z_1^{n_1-1}\right)$, we have $c=0$. Therefore, $D=0$.

In conclusion, there does not exist a negative weight derivation of $H_1(V)$ and we complete the proof. □

Lemma 2.7 (Case (iii) of Proposition 2.4). *Let $(V,0)=\{(z_0,z_1,z_2,z_3)\in\mathbb{C}^4\colon f(z_0,z_1,z_2,z_3)=0\}$ be an isolated singularity defined by $f(z_0,z_1,z_2,z_3)=z_0^{n_0}z_1+z_1^{n_1}z_2+z_2^{n_2}z_3+z_3^{n_3}$ of weight type $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ where $\alpha_0\geq\alpha_2\geq\alpha_1\geq\alpha_3$ and $wt(f)>3max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\}=3\alpha_0$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there does not exist a negative weight derivation of $H_1(V)$.*

Proof. After renumbering to make $\alpha_0\geq\alpha_1\geq\alpha_2\geq\alpha_3$, the form of f changes to $f=z_0^{n_0}z_2+z_1^{n_1}z_3+z_2^{n_2}z_1+z_3^{n_3}$. After renumbering, if there exists some negative weight derivation D, D must be in the form of $D=p_0(z_1,z_2,z_3)\frac{\partial}{\partial z_0}+p_1(z_2,z_3)\frac{\partial}{\partial z_1}+cz_3^k\frac{\partial}{\partial z_2}$.

After renumbering, it is clear that $n_0\geq 3$. From $2\alpha_i+\alpha_j\leq 3\alpha_0<wt(f)=n_i\alpha_i+\alpha_j$, we get $n_i>2$ for $(i,j)=(1,3)$ and $(2,1)$. So we have $n_1\geq 3$ and $n_2\geq 3$. From $3\alpha_0<wt(f)=n_3\alpha_3\leq n_3\alpha_0$, we have $n_3>3$, which is equivalent to $n_3\geq 4$. Regardless of difference of constants, we get the equations below.

$$\text{Hess }(f)=\begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03}\\ f_{10} & f_{11} & f_{12} & f_{13}\\ f_{20} & f_{21} & f_{22} & f_{23}\\ f_{30} & f_{31} & f_{32} & f_{33}\end{bmatrix}=\begin{bmatrix} z_0^{n_0-2}z_2 & 0 & z_0^{n_0-1} & 0\\ * & z_1^{n_1-2}z_3 & z_2^{n_2-1} & z_1^{n_1-1}\\ * & * & z_2^{n_2-2}z_1 & 0\\ * & * & * & z_3^{n_3-2}\end{bmatrix}.$$

From $D\left(z_0^{n_0-2}z_2\right)=p_0\left(z_1,z_2,z_3\right)(n_0-2)z_0^{n_0-3}z_2+cz_3^kz_0^{n_0-2}=0$, we get $p_0\left(z_1,z_2,z_3\right)(n_0-2)z_2+cz_3^kz_0=0$. Therefore, we can obtain $c=0$, $p_0\left(z_1,z_2,z_3\right)=0$ and $D=p_1\left(z_2,z_3\right)\frac{\partial}{\partial z_1}$.

Since $D\left(z_1^{n_1-2}z_3\right)=p_1\left(z_2,z_3\right)\left(n_1-2\right)z_1^{n_1-3}z_3\in\left(z_0^{n_0-2}z_2,z_0^{n_0-1}\right)$, it is easy to see $p_1\left(z_2,z_3\right)=0$. Therefore, $D=0$.

In conclusion, there does not exist negative weight derivation of $H_1(V)$ and we complete the proof. □

LEMMA 2.8 (Case (iv) of Proposition 2.4). *Let* $(V,0) = \{(z_0,z_1,z_2,z_3) \in \mathbb{C}^4\colon f(z_0,z_1,z_2,z_3) = 0\}$ *be an isolated singularity defined by* $f(z_0,z_1,z_2,z_3) = z_0^{n_0}z_1+z_1^{n_1}z_2+z_2^{n_2}z_3+z_3^{n_3}$ *of weight type* $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ *where* $\alpha_0 \geq \alpha_2 \geq \alpha_3 \geq \alpha_1$ *and* $wt(f) > 3max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\} = 3\alpha_0$. *Let* $H_1(V)$ *be the* 1*-st Hessian algebra. Then there does not exist negative weight derivation of* $H_1(V)$.

Proof. After renumbering to make $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, the form of f changes to $f = z_0^{n_0}z_3+z_1^{n_1}z_2+z_2^{n_2}+z_3^{n_3}z_1$. After renumbering, if there exists some negative weight derivation D, D must be in the form of $D = p_0(z_1,z_2,z_3)\frac{\partial}{\partial z_0}+p_1(z_2,z_3)\frac{\partial}{\partial z_1}+cz_3^k\frac{\partial}{\partial z_2}$.

After renumbering, it is clear that $n_0 \geq 3$. From $2\alpha_i+\alpha_j \leq 3\alpha_0 < wt(f) = n_i\alpha_i+\alpha_j$, we get $n_i > 2$ for $(i,j) = (1,2)$ and $(3,1)$. So we have $n_1 \geq 3$ and $n_3 \geq 3$. From $3\alpha_0 < wt(f) = n_2\alpha_2 \leq n_2\alpha_0$, we have $n_2 > 3$, which is equivalent to $n_2 \geq 4$. Regardless of difference of constants, we get the equations below.

$$\text{Hess } (f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0^{n_0-2}z_3 & 0 & 0 & z_0^{n_0-1} \\ * & z_1^{n_1-2}z_2 & z_1^{n_1-1} & z_3^{n_1-1} \\ * & * & z_2^{n_2-2} & 0 \\ * & * & * & z_3^{n_3-2}z_1 \end{bmatrix}.$$

From $D\left(z_0^{n_0-2}z_3\right) = p_0\left(z_1,z_2,z_3\right)(n_0-2)z_0^{n_0-3}z_3 = 0$, we get $p_0\left(z_1,z_2,z_3\right) = 0$.

Since $D\left(z_1^{n_1-2}z_2\right) = p_1\left(z_2,z_3\right)(n_1-2)\,z_1^{n_1-3}z_2+cz_3^kz_1^{n_1-2} \in \left(z_0^{n_0-2}z_3, z_0^{n_0-1}\right)$, it is easy to see $p_1\left(z_2,z_3\right)(n_1-2)\,z_1^{n_1-3}z_2 + cz_3^kz_1^{n_1-2} = 0$. Therefore, $p_1\left(z_2,z_3\right)(n_1-2)\,z_2+cz_3^kz_1 = 0$, and it follows that $c = 0$ and $p_1\left(z_2,z_3\right) = 0$.

Therefore, $D = 0$.

In conclusion, there does not exist negative weight derivation of $H_1(V)$ and we complete the proof. □

LEMMA 2.9 (Case (v) of Proposition 2.4). *Let* $(V,0) = \{(z_0,z_1,z_2,z_3) \in \mathbb{C}^4\colon f(z_0,z_1,z_2,z_3) = 0\}$ *be an isolated singularity defined by* $f(z_0,z_1,z_2,z_3) = z_0^{n_0}z_1+z_1^{n_1}z_2+z_2^{n_2}z_3+z_3^{n_3}$ *of weight type* $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ *where* $\alpha_0 \geq \alpha_3 \geq \alpha_1 \geq \alpha_2$ *and* $wt(f) > 3max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\} = 3\alpha_0$. *Let* $H_1(V)$ *be the* 1*-st Hessian algebra. Then there does not exist negative weight derivation of* $H_1(V)$.

Proof. After renumbering to make $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, the form of f changes to $f = z_0^{n_0}z_2+z_1^{n_1}+z_2^{n_2}z_3+z_3^{n_3}z_1$. After renumbering, if there exists some negative weight derivation D, D must be in the form of $D = p_0(z_1,z_2,z_3)\frac{\partial}{\partial z_0}+p_1(z_2,z_3)\frac{\partial}{\partial z_1}+cz_3^k\frac{\partial}{\partial z_2}$.

After renumbering, it is clear that $n_0 \geq 3$. From $2\alpha_i+\alpha_j \leq 3\alpha_0 < wt(f) = n_i\alpha_i+\alpha_j$, we get $n_i > 2$ for $(i,j) = (2,3)$ and $(3,1)$. So we have $n_2 \geq 3$ and $n_3 \geq 3$. From $3\alpha_0 < wt(f) = n_1\alpha_1 \leq n_1\alpha_0$, we have $n_1 > 3$, which is equivalent to $n_1 \geq 4$. Regardless of difference of constants, we get the equations below.

$$\text{Hess } (f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0^{n_0-2}z_2 & 0 & z_0^{n_0-1} & 0 \\ * & z_1^{n_1-2} & 0 & z_3^{n_3-1} \\ * & * & z_2^{n_2-2}z_3 & z_2^{n_2-1} \\ * & * & * & z_3^{n_3-2}z_1 \end{bmatrix}.$$

From $D\left(z_0^{n_0-2}z_2\right) = p_0\left(z_1,z_2,z_3\right)(n_0-2)z_0^{n_0-3}z_2 + cz_3^kz_0^{n_0-2} = 0$, we get $p_0\left(z_1,z_2,z_3\right)(n_0-2)z_2+cz_3^kz_0 = 0$. Therefore, $c = 0$ and $p_0\left(z_1,z_2,z_3\right) = 0$.

Since $D\left(z_1^{n_1-2}\right)=p_1\left(z_2,z_3\right)\left(n_1-2\right)z_1^{n_1-3}\in\left(z_0^{n_0-2}z_2,z_0^{n_0-1}\right)$, it is easy to see $p_1\left(z_2,z_3\right)=0$.

Therefore, $D=0$.

In conclusion, there does not exist negative weight derivation of $H_1(V)$ and we complete the proof. □

LEMMA 2.10 (Case (vi) of Proposition 2.4). *Let $(V,0)=\{(z_0,z_1,z_2,z_3)\in\mathbb{C}^4\colon f(z_0,z_1,z_2,z_3)=0\}$ be an isolated singularity defined by $f(z_0,z_1,z_2,z_3)=z_0^{n_0}z_1+z_1^{n_1}z_2+z_2^{n_2}z_3+z_3^{n_3}$ of weight type $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ where $\alpha_0\geq\alpha_3\geq\alpha_2\geq\alpha_1$ and $wt(f)>3max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\}=3\alpha_0$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there does not exist negative weight derivation of $H_1(V)$.*

Proof. After renumbering to make $\alpha_0\geq\alpha_1\geq\alpha_2\geq\alpha_3$, the form of f changes to $f=z_0^{n_0}z_3+z_1^{n_1}+z_2^{n_2}z_1+z_3^{n_3}z_2$. After renumbering, if there exists some negative weight derivation D, D must be in the form of $D=p_0(z_1,z_2,z_3)\frac{\partial}{\partial z_0}+p_1(z_2,z_3)\frac{\partial}{\partial z_1}+cz_3^k\frac{\partial}{\partial z_2}$.

After renumbering, it is clear that $n_0\geq 3$. From $2\alpha_i+\alpha_j\leq 3\alpha_0<wt(f)=n_i\alpha_i+\alpha_j$, we get $n_i>2$ for $(i,j)=(2,1)$ and $(3,2)$. So we have $n_2\geq 3$ and $n_3\geq 3$. From $3\alpha_0<wt(f)=n_1\alpha_1\leq n_1\alpha_0$, we have $n_1>3$, which is equivalent to $n_1\geq 4$. Regardless of difference of constants, we get the equations below.

$$\text{Hess }(f)=\begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03}\\ f_{10} & f_{11} & f_{12} & f_{13}\\ f_{20} & f_{21} & f_{22} & f_{23}\\ f_{30} & f_{31} & f_{32} & f_{33}\end{bmatrix}=\begin{bmatrix} z_0^{n_0-2}z_3 & 0 & 0 & z_0^{n_0-1}\\ * & z_1^{n_1-2} & z_2^{n_2-1} & 0\\ * & * & z_2^{n_2-2}z_1 & z_3^{n_3-1}\\ * & * & * & z_3^{n_3-2}z_2\end{bmatrix}.$$

From $D\left(z_0^{n_0-2}z_3\right)=p_0\left(z_1,z_2,z_3\right)(n_0-2)z_0^{n_0-3}z_3=0$, we get $p_0\left(z_1,z_2,z_3\right)=0$.

Since $D\left(z_1^{n_1-2}\right)=p_1\left(z_2,z_3\right)\left(n_1-2\right)z_1^{n_1-3}\in\left(z_0^{n_0-2}z_3,z_0^{n_0-1}\right)$, it is easy to see $p_1\left(z_2,z_3\right)=0$. So D is in the form of $D=cz_3^k\frac{\partial}{\partial z_2}$. By the relation $D\left(z_2^{n_2-1}\right)=cz_3^k\left(n_2-1\right)z_2^{n_2-2}\in\left(z_0^{n_0-2}z_3,z_0^{n_0-1},z_1^{n_1-2}\right)$ we get $c=0$. Therefore, $D=0$.

In conclusion, there does not exist negative weight derivation of $H_1(V)$ and we complete the proof. □

LEMMA 2.11 (Case (vii) of Proposition 2.4). *Let $(V,0)=\{(z_0,z_1,z_2,z_3)\in\mathbb{C}^4\colon f(z_0,z_1,z_2,z_3)=0\}$ be an isolated singularity defined by $f(z_0,z_1,z_2,z_3)=z_0^{n_0}z_1+z_1^{n_1}z_2+z_2^{n_2}z_3+z_3^{n_3}$ of weight type $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ where $\alpha_1\geq\alpha_0\geq\alpha_2\geq\alpha_3$ and $wt(f)>3max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\}=3\alpha_1$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there does not exist negative weight derivation of $H_1(V)$.*

Proof. After renumbering to make $\alpha_0\geq\alpha_1\geq\alpha_2\geq\alpha_3$, the form of f changes to $f=z_0^{n_0}z_2+z_1^{n_1}z_0+z_2^{n_2}z_3+z_3^{n_3}$. After renumbering, if there exists some negative weight derivation D, D must be in the form of $D=p_0(z_1,z_2,z_3)\frac{\partial}{\partial z_0}+p_1(z_2,z_3)\frac{\partial}{\partial z_1}+cz_3^k\frac{\partial}{\partial z_2}$.

After renumbering, it is clear that $n_0\geq 3$. From $2\alpha_i+\alpha_j\leq 3\alpha_0<wt(f)=n_i\alpha_i+\alpha_j$, we get $n_i>2$ for $(i,j)=(1,0)$ and $(2,3)$. So we have $n_1\geq 3$ and $n_2\geq 3$. From $3\alpha_0<wt(f)=n_3\alpha_3\leq n_3\alpha_0$, we have $n_3>3$, which is equivalent to $n_3\geq 4$. Regardless of difference of constants, we get the equations below.

$$\text{Hess}(f)=\begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03}\\ f_{10} & f_{11} & f_{12} & f_{13}\\ f_{20} & f_{21} & f_{22} & f_{23}\\ f_{30} & f_{31} & f_{32} & f_{33}\end{bmatrix}=\begin{bmatrix} z_0^{n_0-2}z_2 & z_1^{n_1-1} & z_0^{n_0-1} & 0\\ * & z_1^{n_1-2}z_0 & 0 & 0\\ * & * & z_2^{n_2-2}z_3 & z_2^{n_2-1}\\ * & * & * & z_3^{n_3-2}\end{bmatrix}.$$

From $D\left(z_0^{n_0-2}z_2\right)=p_0\left(z_1,z_2,z_3\right)(n_0-2)z_0^{n_0-3}z_2+cz_3^kz_0^{n_0-2}=0$, we get $p_0\left(z_1,z_2,z_3\right)(n_0-2)z_2+cz_3^kz_0=0$. Therefore, $c=0$ and $p_0\left(z_1,z_2,z_3\right)=0$.

Since $D\left(z_1^{n_1-1}\right) = p_1(z_2,z_3)(n_1-1)\,z_1^{n_1-2} \in \left(z_0^{n-2}z_2\right)$, it is easy to see $p_1(z_2,z_3)=0$.

Therefore, $D=0$.

In conclusion, there does not exist negative weight derivation of $H_1(V)$ and we complete the proof. □

Lemma 2.12 (Case (viii) of Proposition 2.4). *Let $(V,0)=\{(z_0,z_1,z_2,z_3)\in\mathbb{C}^4\colon f(z_0,z_1,z_2,z_3)=0\}$ be an isolated singularity defined by $f(z_0,z_1,z_2,z_3)=z_0^{n_0}z_1+z_1^{n_1}z_2+z_2^{n_2}z_3+z_3^{n_3}$ of weight type $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ where $\alpha_1\geq\alpha_0\geq\alpha_3\geq\alpha_2$ and $wt(f)>3max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\}=3\alpha_1$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there does not exist negative weight derivation of $H_1(V)$.*

Proof. After renumbering to make $\alpha_0\geq\alpha_1\geq\alpha_2\geq\alpha_3$, the form of f changes to $f=z_0^{n_0}z_3+z_1^{n_1}z_0+z_2^{n_2}+z_3^{n_3}z_2$. After renumbering, if there exists some negative weight derivation D, D must be in the form of $D=p_0(z_1,z_2,z_3)\frac{\partial}{\partial z_0}+p_1(z_2,z_3)\frac{\partial}{\partial z_1}+cz_3^k\frac{\partial}{\partial z_2}$.

After renumbering, it is clear that $n_0\geq 3$. From $2\alpha_i+\alpha_j\leq 3\alpha_0<wt(f)=n_i\alpha_i+\alpha_j$, we get $n_i>2$ for $(i,j)=(1,0),(3,2)$. So we have $n_1\geq 3$ and $n_3\geq 3$. From $3\alpha_0<wt(f)=n_2\alpha_2\leq n_2\alpha_0$, we have $n_2>3$, which is equivalent to $n_2\geq 4$. Regardless of difference of constants, we get the equations below.

$$\text{Hess}\ (f)=\begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03}\\ f_{10} & f_{11} & f_{12} & f_{13}\\ f_{20} & f_{21} & f_{22} & f_{23}\\ f_{30} & f_{31} & f_{32} & f_{33}\end{bmatrix}=\begin{bmatrix} z_0^{n_0-2}z_3 & z_1^{n_1-1} & 0 & z_0^{n_0-1}\\ * & z_1^{n_1-2}z_0 & 0 & 0\\ * & * & z_2^{n_2-2} & z_3^{n_3-1}\\ * & * & * & z_3^{n_3-2}z_2\end{bmatrix}.$$

From $D\left(z_0^{n_0-2}z_3\right)=p_0(z_1,z_2,z_3)(n_0-2)z_0^{n_0-3}z_3=0$, we get $p_0(z_1,z_2,z_3)=0$.

From $D\left(z_1^{n_1-1}\right)=p_1(z_2,z_3)(n_1-1)\,z_1^{n_1-2}\in\left(z_0^{n_0-2}z_3\right)$, we get $p_1(z_2,z_3)=0$.

From $D\left(z_2^{n_2-2}\right)=cz_3^k(n_2-2)\,z_2^{n_2-3}\in\left(z_0^{n_0-2}z_3,z_1^{n_1-1},z_0^{n_0-1},z_1^{n_1-2}z_0\right)$, we get $c=0$.

Therefore, $D=0$.

In conclusion, there does not exist negative weight derivation of $H_1(V)$ and we complete the proof. □

Lemma 2.13 (Case (ix) of Proposition 2.4). *Let $(V,0)=\{(z_0,z_1,z_2,z_3)\in\mathbb{C}^4\colon f(z_0,z_1,z_2,z_3)=0\}$ be an isolated singularity defined by $f(z_0,z_1,z_2,z_3)=z_0^{n_0}z_1+z_1^{n_1}z_2+z_2^{n_2}z_3+z_3^{n_3}$ of weight type $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ where $\alpha_1\geq\alpha_2\geq\alpha_0\geq\alpha_3$ and $wt(f)>3max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\}=3\alpha_1$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there does not exist negative weight derivation of $H_1(V)$.*

Proof. After renumbering to make $\alpha_0\geq\alpha_1\geq\alpha_2\geq\alpha_3$, the form of f changes to $f=z_0^{n_0}z_1+z_1^{n_1}z_3+z_2^{n_2}z_0+z_3^{n_3}$. After renumbering, if there exists some negative weight derivation D, D must be in the form of $D=p_0(z_1,z_2,z_3)\frac{\partial}{\partial z_0}+p_1(z_2,z_3)\frac{\partial}{\partial z_1}+cz_3^k\frac{\partial}{\partial z_2}$.

After renumbering, it is clear that $n_0\geq 3$. From $2\alpha_i+\alpha_j\leq 3\alpha_0<wt(f)=n_i\alpha_i+\alpha_j$, we get $n_i>2$ for $(i,j)=(1,3)$ and $(2,0)$. So $n_1\geq 3$ and $n_2\geq 3$. From $3\alpha_0<wt(f)=n_3\alpha_3\leq n_3\alpha_0$, we have $n_3>3$, which is equivalent to $n_3\geq 4$. Regardless of difference of constants, we get the equations below.

$$\text{Hess}(f)=\begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03}\\ f_{10} & f_{11} & f_{12} & f_{13}\\ f_{20} & f_{21} & f_{22} & f_{23}\\ f_{30} & f_{31} & f_{32} & f_{33}\end{bmatrix}=\begin{bmatrix} z_0^{n_0-2}z_1 & z_0^{n_0-1} & z_2^{n_2-1} & 0\\ * & z_1^{n_1-2}z_3 & 0 & z_1^{n_1-1}\\ * & * & z_2^{n_2-2}z_0 & 0\\ * & * & * & z_3^{n_3-2}\end{bmatrix}.$$

From $D\left(z_0^{n_0-2}z_1\right) = p_0\left(z_1,z_2,z_3\right)(n_0-2)z_0^{n_0-3}z_1 + p_1\left(z_2,z_3\right)z_0^{n_0-2} = 0$, we get $p_0\left(z_1,z_2,z_3\right)(n_0-2)z_1 + p_1\left(z_2,z_3\right)z_0 = 0$. Therefore, $p_1\left(z_2,z_3\right) = 0$ and $p_0\left(z_1,z_2,z_3\right) = 0$.

Since $D\left(z_2^{n_2-1}\right) = cz_3^k\left(n_2-1\right)z_2^{n_2-2} \in \left(z_0^{n_0-2}z_1, z_0^{n_0-1}, z_1^{n_1-2}z_3\right)$, it is easy to see $c=0$.

Therefore, $D=0$.

In conclusion, there does not exist negative weight derivation of $H_1(V)$ and we complete the proof. □

Lemma 2.14 (Case (x) of Proposition 2.4). *Let $(V,0) = \{(z_0,z_1,z_2,z_3)\in \mathbb{C}^4\colon f(z_0,z_1,z_2,z_3) = 0\}$ be an isolated singularity defined by $f(z_0,z_1,z_2,z_3) = z_0^{n_0}z_1+z_1^{n_1}z_2+z_2^{n_2}z_3+z_3^{n_3}$ of weight type $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ where $\alpha_1\geq\alpha_2\geq\alpha_3\geq\alpha_0$ and $wt(f) > 3max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\} = 3\alpha_1$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there does not exist negative weight derivation of $H_1(V)$.*

Proof. After renumbering to make $\alpha_0\geq\alpha_1\geq\alpha_2\geq\alpha_3$, the form of f changes to $f = z_0^{n_0}z_1+z_1^{n_1}z_2+z_2^{n_2}+z_3^{n_3}z_0$. After renumbering, if there exists some negative weight derivation D, D must be in the form of $D = p_0(z_1,z_2,z_3)\frac{\partial}{\partial z_0}+p_1(z_2,z_3)\frac{\partial}{\partial z_1}+cz_3^k\frac{\partial}{\partial z_2}$.

After renumbering, it is clear that $n_0\geq 3$. From $2\alpha_i+\alpha_j\leq 3\alpha_0 < wt(f) = n_i\alpha_i+\alpha_j$, we get $n_i>2$ for $(i,j)=(1,2)$ and $(3,0)$. So we have $n_1\geq 3$ and $n_3\geq 3$. From $3\alpha_0 < wt(f) = n_2\alpha_2 \leq n_2\alpha_0$, we have $n_2>3$, which is equivalent to $n_2\geq 4$. Regardless of difference of constants, we get the equations below.

$$\text{Hess }(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03}\\ f_{10} & f_{11} & f_{12} & f_{13}\\ f_{20} & f_{21} & f_{22} & f_{23}\\ f_{30} & f_{31} & f_{32} & f_{33}\end{bmatrix} = \begin{bmatrix} z_0^{n_0-2}z_1 & z_0^{n_0-1} & 0 & z_3^{n_3-1}\\ * & z_1^{n_1-2}z_2 & z_1^{n_1-1} & 0\\ * & * & z_2^{n_2-2} & 0\\ * & * & * & z_3^{n_3-2}z_0\end{bmatrix}.$$

From $D\left(z_0^{n_0-2}z_1\right) = p_0\left(z_1,z_2,z_3\right)(n_0-2)z_0^{n_0-3}z_1 + p_1\left(z_2,z_3\right)z_0^{n_0-2} = 0$, we get $p_0\left(z_1,z_2,z_3\right)(n_0-2)z_1 + p_1\left(z_2,z_3\right)z_0 = 0$. Therefore, $p_1\left(z_2,z_3\right) = 0$ and $p_0\left(z_1,z_2,z_3\right) = 0$.

Since $D\left(z_1^{n_1-2}z_2\right) = cz_3^kz_1^{n_1-2} \in \left(z_0^{n_0-2}z_1, z_0^{n_0-1}, z_3^{n_3-1}\right)$, it is easy to see $cz_3^kz_1^{n_1-2}\in\left(z_3^{n_3-1}\right)$.

If $c\neq 0, cz_3^kz_2^{n_1-2}$ can be divided by $z_3^{n_3-1}$ and $wt\left(z_3^{n_3-1}\right)\leq wt\left(z_3^k\right)$. From the weight relationship $wt\left(z_3^k\right) < \alpha_2\leq\alpha_0$ and $wt\left(z_3^{n_3-1}\right) = \left(n_3-1\right)\alpha_3 = wt(f)-\alpha_0-\alpha_3 = n_0\alpha_0+\alpha_1-\alpha_0-\alpha_3 \geq (n_0-1)\alpha_0 > \alpha_0$, we have $wt\left(z_3^{n_3-1}\right) > wt\left(z_3^k\right)$, which contradicts with the fact that $wt\left(z_3^{n_3-1}\right)\leq wt\left(z_3^k\right)$. So $c=0$ and $D=0$.

In conclusion, there does not exist negative weight derivation of $H_1(V)$ and we complete the proof. □

Lemma 2.15 (Case (xi) of Proposition 2.4). *Let $(V,0) = \{(z_0,z_1,z_2,z_3)\in \mathbb{C}^4\colon f(z_0,z_1,z_2,z_3) = 0\}$ be an isolated singularity defined by $f(z_0,z_1,z_2,z_3) = z_0^{n_0}z_1+z_1^{n_1}z_2+z_2^{n_2}z_3+z_3^{n_3}$ of weight type $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ where $\alpha_1\geq\alpha_3\geq\alpha_0\geq\alpha_2$ and $wt(f) > 3max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\} = 3\alpha_1$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there does not exist negative weight derivation of $H_1(V)$.*

Proof. After renumbering to make $\alpha_0\geq\alpha_1\geq\alpha_2\geq\alpha_3$, the form of f changes to $f = z_0^{n_0}z_3+z_1^{n_1}+z_2^{n_2}z_0+z_3^{n_3}z_1$. After renumbering, if there exists some negative weight derivation D, D must be in the form of $D = p_0(z_1,z_2,z_3)\frac{\partial}{\partial z_0}+p_1(z_2,z_3)\frac{\partial}{\partial z_1}+cz_3^k\frac{\partial}{\partial z_2}$.

After renumbering, it is clear that $n_0\geq 3$. From $2\alpha_i+\alpha_j\leq 3\alpha_0 < wt(f) = n_i\alpha_i+\alpha_j$, we get $n_i>2$ for $(i,j)=(2,0)$ and $(3,1)$. So we have $n_2\geq 3$ and $n_3\geq 3$.

From $3\alpha_0 < wt(f) = n_1\alpha_1 \le n_1\alpha_0$, we have $n_1 > 3$, which is equivalent to $n_1 \ge 4$. Regardless of difference of constants, we get the equations below.

$$\mathrm{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0^{n_0-2}z_3 & 0 & z_2^{n_2-1} & z_0^{n_0-1} \\ * & z_1^{n_1-2} & 0 & z_3^{n_1-1} \\ * & * & z_2^{n_2-2}z_0 & 0 \\ * & * & * & z_3^{n_3-2}z_1 \end{bmatrix}.$$

From $D\left(z_0^{n_0-2}z_3\right) = p_0\left(z_1,z_2,z_3\right)(n_0-2)z_0^{n_0-3}z_3 = 0$, we get $p_0\left(z_1,z_2,z_3\right) = 0$.

From $D\left(z_2^{n_2-1}\right) = cz_3^k\left(n_2-1\right)z_2^{n_2-2} \in \left(z_0^{n_0-2}z_3, z_1^{n_1-2}\right)$, we get $c = 0$.

Since $D\left(z_1^{n_1-2}\right) = p_1\left(z_2,z_3\right)\left(n_1-2\right)z_1^{n_1-3} \in \left(z_0^{n_0-2}z_3, z_2^{n_2-1}, z_0^{n_0-1}\right)$, it is easy to see $p_1\left(z_2,z_3\right)\left(n_1-2\right)z_1^{n_1-3} \in \left(z_2^{n_2-1}\right)$.

So $p_1\left(z_2,z_3\right)$ can be divided by $z_2^{n_2-1}$. If $p_1\left(z_2,z_3\right) \ne 0$, we have $wt\left(z_2^{n_2-1}\right) \le wt\left(p_1\left(z_2,z_3\right)\right)$. From the weight relationship $wt\left(p_1\left(z_2,z_3\right)\right) < \alpha_1 \le \alpha_0$ and $wt\left(z_2^{n_2-1}\right) = \left(n_2-1\right)\alpha_2 = wt(f)-\alpha_0-\alpha_2 = n_0\alpha_0+\alpha_3-\alpha_0-\alpha_2 > (n_0-2)\alpha_0 \ge \alpha_0$, we have $wt\left(z_2^{n_2-1}\right) > wt\left(p_1\left(z_2,z_3\right)\right)$, which contradicts with the fact that $wt\left(z_2^{n_2-1}\right) \le wt\left(p_1\left(z_2,z_3\right)\right)$. Thus $p_1\left(z_2,z_3\right) = 0$ and $D = 0$.

In conclusion, there does not exist negative weight derivation of $H_1(V)$ and we complete the proof. □

LEMMA 2.16 (Case (xii) of Proposition 2.4). *Let $(V,0) = \{(z_0,z_1,z_2,z_3) \in \mathbb{C}^4\colon f(z_0,z_1,z_2,z_3) = 0\}$ be an isolated singularity defined by $f(z_0,z_1,z_2,z_3) = z_0^{n_0}z_1 + z_1^{n_1}z_2 + z_2^{n_2}z_3 + z_3^{n_3}$ of weight type $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ where $\alpha_1 \ge \alpha_3 \ge \alpha_2 \ge \alpha_0$ and $wt(f) > 3max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\} = 3\alpha_1$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there does not exist negative weight derivation of $H_1(V)$.*

Proof. After renumbering to make $\alpha_0 \ge \alpha_1 \ge \alpha_2 \ge \alpha_3$, the form of f changes to $f = z_0^{n_0}z_2 + z_1^{n_1} + z_2^{n_2}z_1 + z_3^{n_3}z_0$. After renumbering, if there exists some negative weight derivation D, D must be in the form of $D = p_0(z_1,z_2,z_3)\frac{\partial}{\partial z_0} + p_1(z_2,z_3)\frac{\partial}{\partial z_1} + cz_3^k\frac{\partial}{\partial z_2}$.

After renumbering, it is clear that $n_0 \ge 3$. From $2\alpha_i + \alpha_j \le 3\alpha_0 < wt(f) = n_i\alpha_i + \alpha_j$, we get $n_i > 2$ for $(i,j) = (2,1)$ and $(3,0)$. So we have $n_2 \ge 3$ and $n_3 \ge 3$. From $3\alpha_0 < wt(f) = n_1\alpha_1 \le n_1\alpha_0$, we have $n_1 > 3$, which is equivalent to $n_1 \ge 4$. Regardless of difference of constants, we get the equations below.

$$\mathrm{Hess}\ (f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0^{n_0-2}z_2 & 0 & z_0^{n_0-1} & z_3^{n_3-1} \\ * & z_1^{n_1-2} & z_2^{n_2-1} & 0 \\ * & * & z_2^{n_2-2}z_1 & 0 \\ * & * & * & z_3^{n_3-2}z_0 \end{bmatrix}.$$

From $D\left(z_0^{n_0-2}z_2\right) = p_0\left(z_1,z_2,z_3\right)(n_0-2)z_0^{n_0-3}z_2 + cz_3^kz_0^{n_0-2} = 0$, we get $p_0\left(z_1,z_2,z_3\right)(n_0-2)z_2 + cz_3^kz_0 = 0$. Therefore, $c = 0$ and $p_0\left(z_1,z_2,z_3\right) = 0$.

Since $D\left(z_1^{n_1-2}\right) = p_1\left(z_2,z_3\right)\left(n_1-2\right)z_1^{n_1-3} \in \left(z_0^{n_0-2}z_2, z_0^{n_0-1}, z_3^{n_3-1}\right)$, it is easy to see $p_1\left(z_2,z_3\right) \in \left(z_3^{n_3-1}\right)$. So $p_1\left(z_2,z_3\right)$ can be divided by $z_3^{n_3-1}$. If $p_1\left(z_2,z_3\right) \ne 0$, we have $wt\left(z_3^{n_3-1}\right) \le wt\left(p_1\left(z_2,z_3\right)\right)$. From the weight relationship $wt\left(p_1\left(z_2,z_3\right)\right) < \alpha_1 \le \alpha_0$ and $wt\left(z_3^{n_3-1}\right) = \left(n_3-1\right)\alpha_3 = wt(f) - \alpha_0 - \alpha_3 = n_0\alpha_0 + \alpha_2 - \alpha_0 - \alpha_3 \ge (n_0-1)\alpha_0 > \alpha_0$, we have $wt\left(z_3^{n_3-1}\right) > wt\left(p_1\left(z_2,z_3\right)\right)$, which contradicts with the fact that $wt\left(z_3^{n_3-1}\right) \le wt\left(p_1\left(z_2,z_3\right)\right)$. Thus $p_1\left(z_2,z_3\right) = 0$ and $D = 0$.

In conclusion, there does not exist negative weight derivation of $H_1(V)$ and we complete the proof. □

LEMMA 2.17 (Case (xiii) of Proposition 2.4). *Let* $(V,0)=\{(z_0,z_1,z_2,z_3)\in\mathbb{C}^4\colon f(z_0,z_1,z_2,z_3)=0\}$ *be an isolated singularity defined by* $f(z_0,z_1,z_2,z_3)=z_0^{n_0}z_1+z_1^{n_1}z_2+z_2^{n_2}z_3+z_3^{n_3}$ *of weight type* $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ *where* $\alpha_2\ge\alpha_0\ge\alpha_1\ge\alpha_3$ *and* $wt(f)>3max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\}=3\alpha_2$. *Let* $H_1(V)$ *be the* 1*-st Hessian algebra. Then there exists negative weight derivation of* $H_1(V)$ *if and only if* f *is in the form of* $f=z_0^3z_3+z_1^3z_2+z_2^3z_0+z_3^{n_3}(n_3\ge 5)$ *after renumbering the variables* z_0, z_1, z_2 *and* z_3 *so that* $\alpha_0\ge\alpha_1\ge\alpha_2\ge\alpha_3$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D|D=cz_2z_3^k\frac{\partial}{\partial z_1},c\neq 0,0\le k<\frac{n_3-4}{27},k\in\mathbb{Z}\right\}$ *after renumbering.*

Proof. After renumbering to make $\alpha_0\ge\alpha_1\ge\alpha_2\ge\alpha_3$, the form of f changes to $f=z_0^{n_0}z_3+z_1^{n_1}z_2+z_2^{n_2}z_0+z_3^{n_3}$. After renumbering, if there exists some negative weight derivation D, D must be in the form of $D=p_0(z_1,z_2,z_3)\frac{\partial}{\partial z_0}+p_1(z_2,z_3)\frac{\partial}{\partial z_1}+cz_3^k\frac{\partial}{\partial z_2}$.

After renumbering, it is clear that $n_0\ge 3$. From $2\alpha_i+\alpha_j\le 3\alpha_0<wt(f)=n_i\alpha_i+\alpha_j$, we get $n_i>2$ for $(i,j)=(1,2)$ and $(2,0)$. So we have $n_1\ge 3$ and $n_2\ge 3$. From $3\alpha_0<wt(f)=n_3\alpha_3\le n_3\alpha_0$, we have $n_3>3$, which is equivalent to $n_3\ge 4$. Regardless of difference of constants, we get the equations below.

$$\text{Hess }(f)=\begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03}\\ f_{10} & f_{11} & f_{12} & f_{13}\\ f_{20} & f_{21} & f_{22} & f_{23}\\ f_{30} & f_{31} & f_{32} & f_{33}\end{bmatrix}=\begin{bmatrix} z_0^{n_0-2}z_3 & 0 & z_2^{n_2-1} & z_0^{n_0-1}\\ * & z_1^{n_1-2}z_2 & z_1^{n_1-1} & 0\\ * & * & z_2^{n_2-2}z_0 & 0\\ * & * & * & z_3^{n_3-2}\end{bmatrix}.$$

From $D\left(z_0^{n_0-2}z_3\right)=p_0\left(z_1,z_2,z_3\right)(n_0-2)z_0^{n_0-3}z_3=0$, we get $p_0\left(z_1,z_2,z_3\right)=0$. From $D\left(z_2^{n_2-1}\right)=cz_3^k\left(n_2-1\right)z_2^{n_2-2}\in\left(z_0^{n_0-2}z_3,z_1^{n_1-2}z_2\right)$, we get $c=0$. Therefore, $D=p_1\left(z_2,z_3\right)\frac{\partial}{\partial z_1}$.

From $D\left(z_1^{n_1-2}z_2\right)=p_1\left(z_2,z_3\right)(n_1-2)z_1^{n_1-3}z_2\in\left(z_0^{n_0-2}z_3,z_2^{n_2-1},z_0^{n_0-1}\right)$, we get $p_1\left(z_2,z_3\right)z_1^{n_1-3}\ z_2\in\left(z_2^{n_2-1}\right)$. So $p_1\left(z_2,z_3\right)$ can be divided by $z_2^{n_2-2}$. If $p_1\left(z_2,z_3\right)\neq 0$, we have $wt\left(z_2^{n_2-2}\right)\le wt\left(p_1\left(z_2,z_3\right)\right)<\alpha_1\le\alpha_0$. From the weight relationship $wt\left(z_2^{n_2-2}\right)=\frac{n_2-2}{n_2}\left(wt(f)-\alpha_0\right)=\frac{n_2-2}{n_2}\left(n_0\alpha_0+\alpha_3-\alpha_0\right)>\frac{n_2-2}{n_2}2\alpha_0$, we have $\frac{n_2-2}{n_2}2\alpha_0<\alpha_0$. Therefore, $n_2<4$. Note that $n_2\ge 3$, we get $n_2=3$.

From $wt\left(z_2^{n_2-2}\right)=wt(f)-\alpha_0-2\alpha_2<\alpha_1$, we get $wt(f)<\alpha_0+\alpha_1+2\alpha_2\le 4\alpha_0$. Therefore, $n_0\alpha_0\le n_0\alpha_0+\alpha_3=wt(f)<4\alpha_0$. We can get $n_0<4$. Since $n_0\ge 3$, it is clear that $n_0=3$.

Therefore, f is in the form of $f=z_0^3z_3+z_1^{n_1}z_2+z_2^3z_0+z_3^{n_3}$.

From $n_1\alpha_1<wt(f)=n_1\alpha_1+\alpha_2\le(n_1+1)\alpha_1$, we can see that $\alpha_1\in\left[\frac{wt(f)}{n_1+1},\frac{wt(f)}{n_1}\right)$.

From $3\alpha_0<wt(f)=3\alpha_0+\alpha_3<4\alpha_0$, we can see that $\alpha_0\in\left(\frac{wt(f)}{4},\frac{wt(f)}{3}\right)$. Therefore, $\alpha_2=\frac{wt(f)-\alpha_0}{3}\in\left(\frac{2wt(f)}{9},\frac{wt(f)}{4}\right)$.

By $\frac{2wt(f)}{9}<\alpha_2\le\alpha_1<\frac{wt(f)}{n_1}$, we obtain $n_1<\frac{9}{2}$. Since $n_1\ge 3$, it is clear that $n_1=3$ or $n_1=4$.

Regardless of difference of constants, we get the equations below.

$$\text{Hess }(f)=\begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03}\\ f_{10} & f_{11} & f_{12} & f_{13}\\ f_{20} & f_{21} & f_{22} & f_{23}\\ f_{30} & f_{31} & f_{32} & f_{33}\end{bmatrix}=\begin{bmatrix} z_0z_3 & 0 & z_2^2 & z_0^2\\ * & z_1^{n_1-2}z_2 & z_1^{n_1-1} & 0\\ * & * & z_0z_2 & 0\\ * & * & * & z_3^{n_3-2}\end{bmatrix}.$$

We check the conditions of $D\left(z_1^{n_1-2}z_2\right)\in\left(z_0z_3,z_2^2,z_0^2\right)$ and $D(z_1^{n_1-1})\in\left(z_0z_3,z_2^2,z_0^2,z_1^{n_1-2}z_2\right)$.

The restriction that $D\left(z_1^{n_1-2}z_2\right) = p_1\left(z_2, z_3\right)(n_1-2)z_1^{n_1-3}z_2 \in \left(z_0z_3, z_2^2, z_0^2\right)$ is equivalent to the restriction that $p_1(z_2, z_3)$ can be divided by z_2.

The restriction that $D\left(z_1^{n_1-1}\right) = p_1\left(z_2, z_3\right)(n_1-1)z_1^{n_1-2} \in \left(z_0z_3, z_2^2, z_0^2, z_1^{n_1-2}z_2\right)$ is equivalent to the restriction that $p_1(z_2, z_3)$ can be divided by z_2.

Note that z_2^2 and $z_3^{n_3-2}$ are in the ideal generated by elements of Hess (f). Therefore, if D is nonzero, $p_1(z_2, z_3)$ must be in the form of $p_1(z_2, z_3) = c_1z_2z_3^{k_1}(0 \le k_1 \le n_3-3)$. Accordingly, D is in the form of $D = c_1z_2z_3^{k_1}\frac{\partial}{\partial z_1}(0 \le k_1 \le n_3-3)$.

When $n_1 = 3$, f is in the form of $f = z_0^3z_3 + z_1^3z_2 + z_2^3z_0 + z_3^{n_3}$.

From the weight relationship

$$\begin{cases} 3\alpha_0 + \alpha_3 = wt(f) \\ 3\alpha_1 + \alpha_2 = wt(f) \\ 3\alpha_2 + \alpha_0 = wt(f) \\ \quad n_3\alpha_3 = wt(f) \end{cases},$$

we obtain

$$\begin{cases} \alpha_0 = \frac{1}{3}(1-\frac{1}{n_3})wt(f) \\ \alpha_1 = \frac{1}{27}(7-\frac{1}{n_3})wt(f) \\ \alpha_2 = \frac{1}{9}(2+\frac{1}{n_3})wt(f) \\ \alpha_3 = \frac{1}{n_3}wt\,(f) \end{cases}.$$

Consider the restriction $\alpha_0 \ge \alpha_1 \ge \alpha_2 \ge \alpha_3$, we obtain $n_3 \ge 4$. Since D is negatively weighted, we can see that $\alpha_2 + k_1\alpha_3 < \alpha_1$. The necessary and sufficient condition for such integer k_1 satisfying $0 \le k_1 \le n_3 - 3$ to exist is $\alpha_2 < \alpha_1$, from which we get $n_3 > 4$. Therefore, $n_3 \ge 5$.

Note that $\alpha_2 + k_1\alpha_3 < \alpha_1$ is equivalent to $k_1 < \frac{1}{27}(n_3-4)$. Since $\frac{1}{27}(n_3-4) < n_3 - 4 < n_3 - 3$, we can see k_1 is qualified if and only if $0 \le k_1 < \frac{1}{27}(n_3-4)$.

Therefore, when $n_1 = 3$, there exists negative weight derivation of $H_1(V)$ if and only if $n_3 \ge 5$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D|D = cz_2z_3^k\frac{\partial}{\partial z_1}, c \ne 0,\ \ 0 \le k < \frac{n_3-4}{27}, k \in \mathbb{Z}\right\}$.

When $n_1 = 4$, f is in the form of $f = z_0^3z_3 + z_1^4z_2 + z_2^3z_0 + z_3^{n_3}$.

From the weight relationship

$$\begin{cases} 3\alpha_0 + \alpha_3 = wt(f) \\ 4\alpha_1 + \alpha_2 = wt(f) \\ 3\alpha_2 + \alpha_0 = wt(f) \\ \quad n_3\alpha_3 = wt(f) \end{cases},$$

we obtain

$$\begin{cases} \alpha_0 = \frac{1}{3}(1-\frac{1}{n_3})wt(f) \\ \alpha_1 = \frac{1}{36}(7-\frac{1}{n_3})wt(f) \\ \alpha_2 = \frac{1}{9}(2+\frac{1}{n_3})wt(f) \\ \alpha_3 = \frac{1}{n_3}wt\,(f) \end{cases}.$$

Consider the restriction $\alpha_1 \ge \alpha_2$, we obtain $-\frac{5}{n_3} \ge 1$, which is absurd.

Therefore there does not exist negative weight derivation of $H_1(V)$ when $n_1 = 4$.

In conclusion, there exists negative weight derivation of $H_1(V)$ if and only if f is in the form of $f = z_0^3z_3 + z_1^3z_2 + z_2^3z_0 + z_3^{n_3}(n_3 \ge 5)$ after renumbering the variables z_0, z_1, z_2 and z_3 so that $\alpha_0 \ge \alpha_1 \ge \alpha_2 \ge \alpha_3$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D|D = cz_2 z_3^k \frac{\partial}{\partial z_1}, c \neq 0, 0 \leq k < \frac{n_3-4}{27}, k \in \mathbb{Z}\right\}$ after renumbering. □

Lemma 2.18 (Case (xiv) of Proposition 2.4). *Let* $(V,0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 : f(z_0, z_1, z_2, z_3) = 0\}$ *be an isolated singularity defined by* $f(z_0, z_1, z_2, z_3) = z_0^{n_0} z_1 + z_1^{n_1} z_2 + z_2^{n_2} z_3 + z_3^{n_3}$ *of weight type* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ *where* $\alpha_2 \geq \alpha_0 \geq \alpha_3 \geq \alpha_1$ *and* $wt(f) > 3max\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\} = 3\alpha_2$. *Let* $H_1(V)$ *be the* 1*-st Hessian algebra. Then there does not exist negative weight derivation of* $H_1(V)$.

Proof. After renumbering to make $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, the form of f changes to $f = z_0^{n_0} z_2 + z_1^{n_1} z_3 + z_2^{n_2} + z_3^{n_3} z_0$. After renumbering, if there exists some negative weight derivation D, D must be in the form of $D = p_0(z_1, z_2, z_3)\frac{\partial}{\partial z_0} + p_1(z_2, z_3)\frac{\partial}{\partial z_1} + cz_3^k \frac{\partial}{\partial z_2}$.

After renumbering, it is clear that $n_0 \geq 3$. From $2\alpha_i + \alpha_j \leq 3\alpha_0 < wt(f) = n_i\alpha_i + \alpha_j$, we get $n_i > 2$ for $(i,j) = (1,3)$ and $(3,0)$. So we have $n_1 \geq 3$ and $n_3 \geq 3$. From $3\alpha_0 < wt(f) = n_2\alpha_2 \leq n_2\alpha_0$, we have $n_2 > 3$, which is equivalent to $n_2 \geq 4$. Regardless of difference of constants, we get the equations below.

$$\text{Hess }(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0^{n_0-2} z_2 & 0 & z_0^{n_0-1} & z_3^{n_3-1} \\ * & z_1^{n_1-2} z_3 & 0 & z_1^{n_1-1} \\ * & * & z_2^{n_2-2} & 0 \\ * & * & * & z_3^{n_3-2} z_0 \end{bmatrix}.$$

From $D\left(z_0^{n_0-2} z_2\right) = p_0(z_1, z_2, z_3)(n_0 - 2)z_0^{n_0-3} z_2 + cz_3^k z_0^{n_0-2} = 0$, we get $p_0(z_1, z_2, z_3)(n_0 - 2)z_2 + cz_3^k z_0 = 0$. Therefore, $c = 0$ and $p_0(z_1, z_2, z_3) = 0$.

Since $D\left(z_1^{n_1-2} z_3\right) = p_1(z_2, z_3)(n_1 - 2)z_1^{n_1-3} z_3 \in \left(z_0^{n-2} z_2, z_0^{n_0-1}, z_3^{n_3-1}\right)$, it is easy to see $p_1(z_2, z_3) \in \left(z_3^{n_3-2}\right)$.

So $p_1(z_2, z_3)$ can be divided by $z_3^{n_3-2}$.

If $p_1(z_2, z_3) \neq 0$, we have $wt\left(z_3^{n_3-2}\right) \leq wt(p_1(z_2, z_3))$. From the weight relationship $wt(p_1(z_2, z_3)) < \alpha_1 \leq \alpha_0$ and $wt\left(z_3^{n_3-2}\right) = (n_3 - 2)\alpha_3 = \frac{n_3-2}{n_3}(wt(f) - \alpha_0) = \frac{n_3-2}{n_3}(n_0\alpha_0 + \alpha_2 - \alpha_0) > \frac{n_3-2}{n_3}(n_0 - 1)\alpha_0 \geq 2\frac{n_3-2}{n_3}\alpha_0$, we have $2\frac{n_3-2}{n_3}\alpha_0 < \alpha_0$. Therefore, $n_3 < 4$.

Since $n_3 \geq 3$, we can get $n_3 = 3$. In other words, $p_1(z_2, z_3)$ can be divided by z_3. Thus from $\frac{n_3-2}{n_3}(n_0 - 1)\alpha_0 = \frac{(n_0-1)\alpha_0}{3} < \alpha_0$, we have $n_0 < 4$. Note that $n_0 \geq 3$, so $n_0 = 3$ and f is in the form of $f = z_0^3 z_2 + z_1^{n_1} z_3 + z_2^{n_2} + z_3^3 z_0$.

From $3\alpha_0 + \alpha_2 = 3\alpha_3 + \alpha_0$, we get $2\alpha_0 + \alpha_2 = 3\alpha_3$. Considering $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, we know $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3$. Since $p_1(z_2, z_3)$ can be divided by z_3, it is clear that $\alpha_3 \leq wt(p_1(z_2, z_3)) < \alpha_1 = \alpha_3$. This leads to a contradiction.

Therefore, we get $p_1(z_2, z_3) = 0$ and $D = 0$.

In conclusion, there does not exist negative weight derivation of $H_1(V)$ and we complete the proof. □

Lemma 2.19 (Case (xv) of Proposition 2.4). *Let* $(V,0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 : f(z_0, z_1, z_2, z_3) = 0\}$ *be an isolated singularity defined by* $f(z_0, z_1, z_2, z_3) = z_0^{n_0} z_1 + z_1^{n_1} z_2 + z_2^{n_2} z_3 + z_3^{n_3}$ *of weight type* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ *where* $\alpha_2 \geq \alpha_1 \geq \alpha_0 \geq \alpha_3$ *and* $wt(f) > 3max\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\} = 3\alpha_2$. *Let* $H_1(V)$ *be the* 1*-st Hessian algebra. Then there does not exist negative weight derivation of* $H_1(V)$.

Proof. After renumbering to make $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, the form of f changes to $f = z_0^{n_0} z_3 + z_1^{n_1} z_0 + z_2^{n_2} z_1 + z_3^{n_3}$. After renumbering, if there exists some negative weight derivation D, D must be in the form of $D = p_0(z_1, z_2, z_3)\frac{\partial}{\partial z_0} + p_1(z_2, z_3)\frac{\partial}{\partial z_1} + cz_3^k \frac{\partial}{\partial z_2}$.

After renumbering, it is clear that $n_0 \geq 3$. From $2\alpha_i + \alpha_j \leq 3\alpha_0 < wt(f) = n_i\alpha_i + \alpha_j$, we get $n_i > 2$ for $(i,j) = (1,0)$ and $(2,1)$. So we have $n_1 \geq 3$ and $n_2 \geq 3$. From $3\alpha_0 < wt(f) = n_3\alpha_3 \leq n_3\alpha_0$, we have $n_3 > 3$, which is equivalent to $n_3 \geq 4$. Regardless of difference of constants, we get the equations below.

$$\mathrm{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0^{n_0-2}z_3 & z_1^{n_1-1} & 0 & z_0^{n_0-1} \\ * & z_1^{n_1-2}z_0 & z_2^{n_2-1} & 0 \\ * & * & z_2^{n_2-2}z_1 & 0 \\ * & * & * & z_3^{n_3-2} \end{bmatrix}.$$

From $D\left(z_0^{n_0-2}z_3\right) = p_0\left(z_1,z_2,z_3\right)(n_0-2)z_0^{n_0-3}z_3 = 0$, we get $p_0\left(z_1,z_2,z_3\right) = 0$.

From $D\left(z_1^{n_1-1}\right) = p_1\left(z_2,z_3\right)\left(n_1-1\right)z_1^{n_1-2} \in \left(z_0^{n_0-2}z_3\right)$, we get $p_1\left(z_2,z_3\right) = 0$.

From $D\left(z_2^{n_2-1}\right) = cz_3^k\left(n_2-1\right)z_2^{n_2-2} \in \left(z_0^{n_0-2}z_3, z_1^{n_1-1}, z_0^{n_0-1}, z_1^{n_1-2}z_0\right)$, we get $c = 0$.

So $D = 0$.

In conclusion, there does not exist negative weight derivation of $H_1(V)$ and we complete the proof. □

Lemma 2.20 (Case (xvi) of Proposition 2.4). *Let $(V,0) = \{(z_0,z_1,z_2,z_3) \in \mathbb{C}^4 : f(z_0,z_1,z_2,z_3) = 0\}$ be an isolated singularity defined by $f(z_0,z_1,z_2,z_3) = z_0^{n_0}z_1 + z_1^{n_1}z_2 + z_2^{n_2}z_3 + z_3^{n_3}$ of weight type $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ where $\alpha_2 \geq \alpha_1 \geq \alpha_3 \geq \alpha_0$ and $wt(f) > 3max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\} = 3\alpha_2$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there does not exist negative weight derivation of $H_1(V)$.*

Proof. After renumbering to make $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, the form of f changes to $f = z_0^{n_0}z_2 + z_1^{n_1}z_0 + z_2^{n_2} + z_3^{n_3}z_1$. After renumbering, if there exists some negative weight derivation D, D must be in the form of $D = p_0(z_1,z_2,z_3)\frac{\partial}{\partial z_0} + p_1(z_2,z_3)\frac{\partial}{\partial z_1} + cz_3^k\frac{\partial}{\partial z_2}$.

After renumbering, it is clear that $n_0 \geq 3$. From $2\alpha_i + \alpha_j \leq 3\alpha_0 < wt(f) = n_i\alpha_i + \alpha_j$, we get $n_i > 2$ for $(i,j) = (1,0)$ and $(3,1)$. So we have $n_1 \geq 3$ and $n_3 \geq 3$. From $3\alpha_0 < wt(f) = n_2\alpha_2 \leq n_2\alpha_0$, we have $n_2 > 3$, which is equivalent to $n_2 \geq 4$. Regardless of difference of constants, we get the equations below.

$$\mathrm{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0^{n_0-2}z_2 & z_1^{n_1-1} & z_0^{n_0-1} & 0 \\ * & z_1^{n_1-2}z_0 & 0 & z_3^{n_3-1} \\ * & * & z_2^{n_2-2} & 0 \\ * & * & * & z_3^{n_3-2}z_1 \end{bmatrix}.$$

From $D\left(z_0^{n_0-2}z_2\right) = p_0\left(z_1,z_2,z_3\right)(n_0-2)z_0^{n_0-3}z_2 + cz_3^kz_0^{n_0-2} = 0$, we get $p_0\left(z_1,z_2,z_3\right)(n_0-2)z_2 + cz_3^kz_0 = 0$. Therefore, we have $c = 0$ and $p_0\left(z_1,z_2,z_3\right) = 0$.

From $D\left(z_1^{n_1-1}\right) = p_1\left(z_2,z_3\right)\left(n_1-1\right)z_1^{n_1-2} \in \left(z_0^{n_0-2}z_2\right)$, we get $p_1\left(z_2,z_3\right) = 0$.

Therefore, $D = 0$.

In conclusion, there does not exist negative weight derivation of $H_1(V)$ and we complete the proof. □

Lemma 2.21 (Case (xvii) of Proposition 2.4). *Let $(V,0) = \{(z_0,z_1,z_2,z_3) \in \mathbb{C}^4 : f(z_0,z_1,z_2,z_3) = 0\}$ be an isolated singularity defined by $f(z_0,z_1,z_2,z_3) = z_0^{n_0}z_1 + z_1^{n_1}z_2 + z_2^{n_2}z_3 + z_3^{n_3}$ of weight type $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ where $\alpha_2 \geq \alpha_3 \geq \alpha_0 \geq \alpha_1$ and $wt(f) > 3max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\} = 3\alpha_2$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there does not exist negative weight derivation of $H_1(V)$.*

Proof. After renumbering to make $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, the form of f changes to $f = z_0^{n_0}z_1 + z_1^{n_1} + z_2^{n_2}z_3 + z_3^{n_3}z_0$. After renumbering, if there exists some negative weight derivation D, D must be in the form of $D = p_0(z_1,z_2,z_3)\frac{\partial}{\partial z_0} + p_1(z_2,z_3)\frac{\partial}{\partial z_1} + cz_3^k\frac{\partial}{\partial z_2}$.

After renumbering, it is clear that $n_0 \geq 3$. From $2\alpha_i + \alpha_j \leq 3\alpha_0 < wt(f) = n_i\alpha_i + \alpha_j$, we get $n_i > 2$ for $(i,j) = (2,3)$ and $(3,0)$. So we have $n_2 \geq 3$ and $n_3 \geq 3$. From $3\alpha_0 < wt(f) = n_1\alpha_1 \leq n_1\alpha_0$, we have $n_1 > 3$, which is equivalent to $n_1 \geq 4$. Regardless of difference of constants, we get the equations below.

$$\mathrm{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0^{n_0-2}z_1 & z_0^{n_0-1} & 0 & z_3^{n_3-1} \\ * & z_1^{n_1-2} & 0 & 0 \\ * & * & z_2^{n_2-2}z_3 & z_2^{n_2-1} \\ * & * & * & z_3^{n_3-2}z_0 \end{bmatrix}.$$

From $D\left(z_0^{n_0-2}z_1\right) = p_0(z_1,z_2,z_3)(n_0-2)z_0^{n_0-3}z_1 + p_1(z_2,z_3)z_0^{n_0-2} = 0$, we get $p_0(z_1,z_2,z_3)(n_0-2)z_1 + p_1(z_2,z_3)z_0 = 0$. Therefore, $p_1(z_2,z_3) = 0$ and $p_0(z_1,z_2,z_3) = 0$.

From $D\left(z_2^{n_2-2}z_3\right) = cz_3^{k+1}(n_2-2)z_2^{n_2-3} \in \left(z_0^{n_0-2}z_1, z_0^{n_0-1}, z_3^{n_3-1}, z_1^{n_1-2}\right)$, we get $cz_3^{k+1}(n_2-2)\, z_2^{n_2-3} \in \left(z_3^{n_3-1}\right)$.

If $c \neq 0$, we have $k \geq n_3 - 2$ and $wt\left(z_3^k\right) \geq wt\left(z_3^{n_3-2}\right)$. However, we can also see $wt\left(z_3^k\right) < \alpha_2 \leq \alpha_0$ and $wt\left(z_3^{n_3-2}\right) = wt(f) - \alpha_0 - 2\alpha_3 = n_0\alpha_0 + \alpha_1 - \alpha_0 - 2\alpha_3 \geq 2\alpha_0 - \alpha_3 \geq \alpha_0$. Contradiction.

So we get $c = 0$ and $D = 0$.

In conclusion, there does not exist negative weight derivation of $H_1(V)$ and we complete the proof. ∎

LEMMA 2.22 (Case (xviii) of Proposition 2.4). *Let $(V,0) = \{(z_0,z_1,z_2,z_3) \in \mathbb{C}^4 \colon f(z_0,z_1,z_2,z_3) = 0\}$ be an isolated singularity defined by $f(z_0,z_1,z_2,z_3) = z_0^{n_0}z_1 + z_1^{n_1}z_2 + z_2^{n_2}z_3 + z_3^{n_3}$ of weight type $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ where $\alpha_2 \geq \alpha_3 \geq \alpha_1 \geq \alpha_0$ and $wt(f) > 3max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\} = 3\alpha_2$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there does not exist negative weight derivation of $H_1(V)$.*

Proof. After renumbering to make $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, the form of f changes to $f = z_0^{n_0}z_1 + z_1^{n_1} + z_2^{n_2}z_0 + z_3^{n_3}z_2$. After renumbering, if there exists some negative weight derivation D, D must be in the form of $D = p_0(z_1,z_2,z_3)\frac{\partial}{\partial z_0} + p_1(z_2,z_3)\frac{\partial}{\partial z_1} + cz_3^k\frac{\partial}{\partial z_2}$.

After renumbering, it is clear that $n_0 \geq 3$. From $2\alpha_i + \alpha_j \leq 3\alpha_0 < wt(f) = n_i\alpha_i + \alpha_j$, we get $n_i > 2$ for $(i,j) = (2,0)$ and $(3,2)$. So we have $n_2 \geq 3$ and $n_3 \geq 3$. From $3\alpha_0 < wt(f) = n_1\alpha_1 \leq n_1\alpha_0$, we have $n_1 > 3$, which is equivalent to $n_1 \geq 4$. Regardless of difference of constants, we get the equations below.

$$\mathrm{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0^{n_0-2}z_1 & z_0^{n_0-1} & z_2^{n_2-1} & 0 \\ * & z_1^{n_1-2} & 0 & 0 \\ * & * & z_2^{n_2-2}z_0 & z_3^{n_3-1} \\ * & * & * & z_3^{n_3-2}z_2 \end{bmatrix}.$$

From $D\left(z_0^{n_0-2}z_1\right) = p_0(z_1,z_2,z_3)(n_0-2)z_0^{n_0-3}z_1 + p_1(z_2,z_3)z_0^{n_0-2} = 0$, we get $p_0(z_1,z_2,z_3)(n_0-2)z_1 + p_1(z_2,z_3)z_0 = 0$. Therefore, we obtain $p_1(z_2,z_3) = 0$ and $p_0(z_1,z_2,z_3) = 0$.

From $D\left(z_2^{n_2-1}\right) = cz_3^k(n_2-1)z_2^{n_2-2} \in \left(z_0^{n_0-2}z_1, z_0^{n_0-1}, z_1^{n_1-2}\right)$, we get $cz_3^k(n_2-1)z_2^{n_2-2} = 0$. Therefore, we obtain $c = 0$ and $D = 0$.

In conclusion, there does not exist negative weight derivation of $H_1(V)$ and we complete the proof. ∎

LEMMA 2.23 (Case (xix) of Proposition 2.4). *Let $(V,0) = \{(z_0,z_1,z_2,z_3) \in \mathbb{C}^4 \colon f(z_0,z_1,z_2,z_3) = 0\}$ be an isolated singularity defined by $f(z_0,z_1,z_2,z_3) = z_0^{n_0}z_1 +$*

$z_1^{n_1}z_2 + z_2^{n_2}z_3 + z_3^{n_3}$ *of weight type* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ *where* $\alpha_3 \geq \alpha_0 \geq \alpha_1 \geq \alpha_2$ *and* $wt(f) > 3max\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\} = 3\alpha_3$. *Let* $H_1(V)$ *be the* 1*-st Hessian algebra. Then there does not exist negative weight derivation of* $H_1(V)$.

Proof. After renumbering to make $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, the form of f changes to $f = z_0^{n_0}+z_1^{n_1}z_2+z_2^{n_2}z_3+z_3^{n_3}z_0$. After renumbering, if there exists some negative weight derivation D, D must be in the form of $D = p_0(z_1, z_2, z_3)\frac{\partial}{\partial z_0} + p_1(z_2, z_3)\frac{\partial}{\partial z_1} + cz_3^k\frac{\partial}{\partial z_2}$.

After renumbering, it is clear that $n_0 \geq 4$. Therefore, we have $wt\,(f) = n_0\alpha_0 \geq 4\alpha_0$. From $3\alpha_i + \alpha_j \leq 4\alpha_0 \leq wt(f) = n_i\alpha_i + \alpha_j$, we get $n_j \geq 3$ for $(i, j) = (1, 2), (2, 3)$ and $(3, 0)$. So we have $n_1 \geq 3$, $n_2 \geq 3$ and $n_3 \geq 3$. Regardless of difference of constants, we get the equations below.

$$\mathrm{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0^{n_0-2} & 0 & 0 & z_3^{n_3-1} \\ * & z_1^{n_1-2}z_2 & z_1^{n_1-1} & 0 \\ * & * & z_2^{n_2-2}z_3 & z_2^{n_2-1} \\ * & * & * & z_3^{n_3-2}z_0 \end{bmatrix}.$$

From $D\left(z_0^{n_0-2}\right) = p_0\,(z_1, z_2, z_3)\,(n_0 - 2)z_0^{n_0-3} = 0$, we get $p_0\,(z_1, z_2, z_3) = 0$.

From $D\left(z_1^{n_1-2}z_2\right) = p_1\,(z_2, z_3)\,(n_1 - 2)\,z_1^{n_1-3}z_2 + cz_3^kz_1^{n_1-2} \in \left(z_0^{n_0-2}, z_3^{n_3-1}\right)$, we get $p_1\,(z_2, z_3)\;(n_1 - 2)\,z_1^{n_1-3}z_2 + cz_3^kz_1^{n_1-2} \in \left(z_3^{n_3-1}\right)$.

From $wt\left(z_3^{n_3-1}\right) = wt(f) - \alpha_0 - \alpha_3 \geq 3\alpha_0 - \alpha_3 \geq 2\alpha_0 > \alpha_0$ and $wt\left(z_3^k\right) < \alpha_2 \leq \alpha_0$, we can know that $z_3^kz_1^{n_1-2}$ cannot be divided by $z_3^{n_3-1}$.

If $c \neq 0$, since $cz_3^kz_1^{n_1-2}$ cannot be divided by z_2, $cz_3^kz_1^{n_1-2}$ cannot be eliminated by $p_1\,(z_2, z_3)\;(n_1 - 2)\,z_1^{n_1-3}z_2$, which implies that $p_1\,(z_2, z_3)\;(n_1 - 2)\,z_1^{n_1-3}z_2 + cz_3^kz_1^{n_1-2}$ cannot be divided by $z_3^{n_3-1}$. Contradiction.

Therefore, we obtain $c = 0$. It follows that $p_1\,(z_2, z_3) \in \left(z_3^{n_3-1}\right)$. If $p_1\,(z_2, z_3) \neq 0$, we have $wt\,(p_1\,(z_2, z_3)) \geq wt\left(z_3^{n_3-1}\right)$. Note that $wt\,(p_1\,(z_2, z_3)) < \alpha_1 \leq \alpha_0 < wt\left(z_3^{n_3-1}\right)$, which leads to a contradiction. Therefore, it is clear that $p_1\,(z_2, z_3) = 0$ and $D = 0$.

In conclusion, there does not exist negative weight derivation of $H_1(V)$ and we complete the proof. □

Lemma 2.24 (Case (xx) of Proposition 2.4). *Let* $(V, 0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 \colon f(z_0, z_1, z_2, z_3) = 0\}$ *be an isolated singularity defined by* $f(z_0, z_1, z_2, z_3) = z_0^{n_0}z_1 + z_1^{n_1}z_2 + z_2^{n_2}z_3 + z_3^{n_3}$ *of weight type* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ *where* $\alpha_3 \geq \alpha_0 \geq \alpha_2 \geq \alpha_1$ *and* $wt(f) > 3max\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\} = 3\alpha_3$. *Let* $H_1(V)$ *be the* 1*-st Hessian algebra. Then there does not exist negative weight derivation of* $H_1(V)$.

Proof. After renumbering to make $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, the form of f changes to $f = z_0^{n_0}+z_1^{n_1}z_3+z_2^{n_2}z_0+z_3^{n_3}z_2$. After renumbering, if there exists some negative weight derivation D, D must be in the form of $D = p_0(z_1, z_2, z_3)\frac{\partial}{\partial z_0} + p_1(z_2, z_3)\frac{\partial}{\partial z_1} + cz_3^k\frac{\partial}{\partial z_2}$.

After renumbering, it is clear that $n_0 \geq 4$. Therefore, we have $wt\,(f) = n_0\alpha_0 \geq 4\alpha_0$. From $3\alpha_i + \alpha_j \leq 4\alpha_0 \leq wt(f) = n_i\alpha_i + \alpha_j$, we get $n_j \geq 3$ for $(i, j) = (1, 3), (2, 0)$ and $(3, 2)$. So we have $n_1 \geq 3, n_2 \geq 3$ and $n_3 \geq 3$. Regardless of difference of constants, we get the equations below.

$$\mathrm{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0^{n_0-2} & 0 & z_2^{n_2-1} & 0 \\ * & z_1^{n_1-2}z_3 & 0 & z_1^{n_1-1} \\ * & * & z_2^{n_2-2}z_0 & z_3^{n_3-1} \\ * & * & * & z_3^{n_3-2}z_2 \end{bmatrix}.$$

From $D\left(z_0^{n_0-2}\right) = p_0\left(z_1,z_2,z_3\right)(n_0-2)z_0^{n_0-3} = 0$, we get $p_0\left(z_1,z_2,z_3\right) = 0$.

From $D\left(z_1^{n_1-2}z_3\right) = p_1\left(z_2,z_3\right)\left(n_1-2\right)z_1^{n_1-3}z_3 \in \left(z_0^{n_0-2}, z_2^{n_2-1}\right)$, we get $p_1\left(z_2,z_3\right)\left(n_1-2\right) z_1^{n_1-3}z_3 \in \left(z_2^{n_2-1}\right)$.

So $p_1\left(z_2,z_3\right)$ can be divided by $z_2^{n_2-1}$. If $p_1\left(z_2,z_3\right) \neq 0$, we get $wt\left(p_1\left(z_2,z_3\right)\right) \geq wt\left(z_2^{n_2-1}\right)$ since $wt\left(z_2^{n_2-1}\right) = wt(f) - \alpha_0 - \alpha_2 \geq 3\alpha_0 - \alpha_2 \geq 2\alpha_0 > \alpha_0$ and $wt\left(p_1\left(z_2,z_3\right)\right) < \alpha_1 \leq \alpha_0$, we get $wt\left(p_1\left(z_2,z_3\right)\right) < wt\left(z_2^{n_2-1}\right)$. This leads to a contradiction. Thus we obtain $p_1\left(z_2,z_3\right) = 0$.

From $D\left(z_2^{n_2-1}\right) = cz_3^k\left(n_2-1\right)z_2^{n_2-2} \in \left(z_0^{n_0-2}, z_1^{n_1-2}z_3\right)$, we get $cz_3^k\left(n_2-1\right)z_2^{n_2-2} = 0$. Therefore, it is obvious that $c = 0$ and $D = 0$.

In conclusion, there does not exist negative weight derivation of $H_1(V)$ and we complete the proof. □

LEMMA 2.25 (Case (xxi) of Proposition 2.4). *Let $(V,0) = \{(z_0,z_1,z_2,z_3) \in \mathbb{C}^4 \colon f(z_0,z_1,z_2,z_3) = 0\}$ be an isolated singularity defined by $f(z_0,z_1,z_2,z_3) = z_0^{n_0}z_1 + z_1^{n_1}z_2 + z_2^{n_2}z_3 + z_3^{n_3}$ of weight type $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ where $\alpha_3 \geq \alpha_1 \geq \alpha_0 \geq \alpha_2$ and $wt(f) > 3max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\} = 3\alpha_3$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there does not exist negative weight derivation of $H_1(V)$.*

Proof. After renumbering to make $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, the form of f changes to $f = z_0^{n_0} + z_1^{n_1}z_3 + z_2^{n_2}z_1 + z_3^{n_3}z_0$. After renumbering, if there exists some negative weight derivation D, D must be in the form of $D = p_0(z_1,z_2,z_3)\frac{\partial}{\partial z_0} + p_1(z_2,z_3)\frac{\partial}{\partial z_1} + cz_3^k\frac{\partial}{\partial z_2}$.

After renumbering, it is clear that $n_0 \geq 4$. Therefore, we have $wt\left(f\right) = n_0\alpha_0 \geq 4\alpha_0$. From $3\alpha_i + \alpha_j \leq 4\alpha_0 \leq wt(f) = n_i\alpha_i + \alpha_j$, we get $n_j \geq 3$ for $(i,j) = (1,3), (2,1)$ and $(3,0)$. So $n_1 \geq 3, n_2 \geq 3$ and $n_3 \geq 3$. Regardless of difference of constants, we get the equations below.

$$\text{Hess }(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0^{n_0-2} & 0 & 0 & z_3^{n_3-1} \\ * & z_1^{n_1-2}z_3 & z_2^{n_2-1} & z_1^{n_1-1} \\ * & * & z_2^{n_2-2}z_1 & 0 \\ * & * & * & z_3^{n_3-2}z_0 \end{bmatrix}.$$

From $D\left(z_0^{n_0-2}\right) = p_0\left(z_1,z_2,z_3\right)(n_0-2)z_0^{n_0-3} = 0$, we get $p_0\left(z_1,z_2,z_3\right) = 0$.

From $D\left(z_1^{n_1-2}z_3\right) = p_1\left(z_2,z_3\right)\left(n_1-2\right)z_1^{n_1-3}z_3 \in \left(z_0^{n_0-2}, z_3^{n_3-1}\right)$, we get $p_1\left(z_2,z_3\right)\left(n_1-2\right) z_1^{n_1-3}z_3 \in \left(z_3^{n_3-1}\right)$.

So $p_1\left(z_2,z_3\right)$ can be divided by $z_3^{n_3-2}$. If $p_1\left(z_2,z_3\right) \neq 0$, we get $wt\left(p_1\left(z_2,z_3\right)\right) \geq wt\left(z_3^{n_3-2}\right)$. Since $wt\left(z_3^{n_3-2}\right) = wt(f) - \alpha_0 - 2\alpha_3 \geq 3\alpha_0 - 2\alpha_3 \geq \alpha_0$ and $wt\left(p_1\left(z_2,z_3\right)\right) < \alpha_1 \leq \alpha_0$, we get $wt\left(p_1\left(z_2,z_3\right)\right) < wt\left(z_3^{n_3-2}\right)$. This leads to a contradiction. Thus $p_1\left(z_2,z_3\right) = 0$.

Therefore, D is in the form of $D = cz_3^k\frac{\partial}{\partial z_2}$.

From $D\left(z_2^{n_2-1}\right) = c\left(n_2-1\right)z_3^kz_2^{n_2-2} \in \left(z_0^{n_0-2}, z_3^{n_3-1}, z_1^{n_1-2}z_3\right)$, we get $cz_3^kz_2^{n_2-2} \in \left(z_3^{n_3-1}\right)$. If $c \neq 0$, we can get $wt\left(z_3^k\right) \geq wt\left(z_3^{n_3-1}\right)$. From $wt\left(z_3^{n_3-1}\right) = wt\left(f\right) - \alpha_0 - \alpha_3$ and $wt\left(z_3^k\right) < \alpha_2$, we can get $\alpha_2 > wt\left(f\right) - \alpha_0 - \alpha_3$. Therefore, $wt\left(f\right) < \alpha_0 + \alpha_2 + \alpha_3 \leq 3\alpha_0$, which is in contradiction with $wt\left(f\right) \geq 4\alpha_0$. Therefore, $c = 0$ and $D = 0$.

In conclusion, there does not exist negative weight derivation of $H_1(V)$ and we complete the proof. □

LEMMA 2.26 (Case (xxii) of Proposition 2.4). *Let $(V,0) = \{(z_0,z_1,z_2,z_3) \in \mathbb{C}^4 \colon f(z_0,z_1,z_2,z_3) = 0\}$ be an isolated singularity defined by $f(z_0,z_1,z_2,z_3) = z_0^{n_0}z_1 + z_1^{n_1}z_2 + z_2^{n_2}z_3 + z_3^{n_3}$ of weight type $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ where $\alpha_3 \geq \alpha_1 \geq \alpha_2 \geq \alpha_0$ and*

$wt(f) > 3max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\} = 3\alpha_3$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there does not exist negative weight derivation of $H_1(V)$.

Proof. After renumbering to make $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, the form of f changes to $f = z_0^{n_0}+z_1^{n_1}z_2+z_2^{n_2}z_0+z_3^{n_3}z_1$. After renumbering, if there exists some negative weight derivation D, D must be in the form of $D = p_0(z_1,z_2,z_3)\frac{\partial}{\partial z_0}+p_1(z_2,z_3)\frac{\partial}{\partial z_1}+cz_3^k\frac{\partial}{\partial z_2}$.

After renumbering, it is clear that $n_0 \geq 4$. Therefore, we have $wt\,(f) = n_0\alpha_0 \geq 4\alpha_0$. From $3\alpha_i+\alpha_j \leq 4\alpha_0 \leq wt(f) = n_i\alpha_i+\alpha_j$, we get $n_j \geq 3$ for $(i,j) = (1,2),(2,0)$ and $(3,1)$. So we have $n_1 \geq 3, n_2 \geq 3$ and $n_3 \geq 3$. Regardless of difference of constants, we get the equations below.

$$\mathrm{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0^{n_0-2} & 0 & z_2^{n_2-1} & 0 \\ * & z_1^{n_1-2}z_2 & z_1^{n_1-1} & z_3^{n_3-1} \\ * & * & z_2^{n_2-2}z_0 & 0 \\ * & * & * & z_3^{n_3-2}z_1 \end{bmatrix}.$$

From $D\left(z_0^{n_0-2}\right) = p_0\left(z_1,z_2,z_3\right)(n_0-2)z_0^{n_0-3} = 0$, we get $p_0\left(z_1,z_2,z_3\right) = 0$.

From $D\left(z_2^{n_2-1}\right) = cz_3^k\left(n_2-1\right)z_2^{n_2-2} \in \left(z_0^{n_0-2}, z_1^{n_1-2}z_2\right)$, we get $c = 0$.

From $D\left(z_1^{n_1-2}z_2\right) = p_1\left(z_2,z_3\right)\left(n_1-2\right)z_1^{n_1-3}z_2 \in \left(z_0^{n_0-2}, z_2^{n_2-1}\right)$, we get $p_1\left(z_2,z_3\right)\left(n_1-2\right)\, z_1^{n_1-3}z_2 \in \left(z_2^{n_2-1}\right)$.

So $p_1\left(z_2,z_3\right)z_2$ can be divided by $z_2^{n_2-1}$. If $p_1\left(z_2,z_3\right) \neq 0$, we get $wt\left(p_1\left(z_2,z_3\right)z_2\right) \geq wt\left(z_2^{n_2-1}\right)$. Since $wt\left(z_2^{n_2-1}\right) = wt(f)-\alpha_0-\alpha_2 \geq 3\alpha_0-\alpha_2 \geq 2\alpha_0$ and $wt\left(p_1\left(z_2,z_3\right)z_2\right) < \alpha_1+\alpha_2 \leq 2\alpha_0$, we get $wt\left(p_1\left(z_2,z_3\right)z_2\right) < wt\left(z_2^{n_2-1}\right)$. Contradiction. Thus $p_1\left(z_2,z_3\right) = 0$ and $D = 0$.

In conclusion, there does not exist negative weight derivation of $H_1(V)$ and we complete the proof. □

Lemma 2.27 (Case (xxiii) of Proposition 2.4). *Let $(V,0) = \{(z_0,z_1,z_2,z_3) \in \mathbb{C}^4 \colon f(z_0,z_1,z_2,z_3) = 0\}$ be an isolated singularity defined by $f(z_0,z_1,z_2,z_3) = z_0^{n_0}z_1+z_1^{n_1}z_2+z_2^{n_2}z_3+z_3^{n_3}$ of weight type $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ where $\alpha_3 \geq \alpha_2 \geq \alpha_0 \geq \alpha_1$ and $wt(f) > 3max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\} = 3\alpha_3$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there does not exist negative weight derivation of $H_1(V)$.*

Proof. After renumbering to make $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, the form of f changes to $f = z_0^{n_0}+z_1^{n_1}z_0+z_2^{n_2}z_3+z_3^{n_3}z_1$. After renumbering, if there exists some negative weight derivation D, D must be in the form of $D = p_0(z_1,z_2,z_3)\frac{\partial}{\partial z_0}+p_1(z_2,z_3)\frac{\partial}{\partial z_1}+cz_3^k\frac{\partial}{\partial z_2}$.

After renumbering, it is clear that $n_0 \geq 4$. Therefore, we have $wt\,(f) = n_0\alpha_0 \geq 4\alpha_0$. From $3\alpha_i+\alpha_j \leq 4\alpha_0 \leq wt(f) = n_i\alpha_i+\alpha_j$, we get $n_j \geq 3$ for $(i,j) = (1,0),(2,3)$ and $(3,1)$. So we have $n_1 \geq 3, n_2 \geq 3$ and $n_3 \geq 3$. Regardless of difference of constants, we get the equations below.

$$\mathrm{Hess}\ (f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0^{n_0-2} & z_1^{n_1-1} & 0 & 0 \\ * & z_1^{n_1-2}z_0 & 0 & z_3^{n_3-1} \\ * & * & z_2^{n_2-2}z_3 & z_2^{n_2-1} \\ * & * & * & z_3^{n_3-2}z_1 \end{bmatrix}.$$

From $D\left(z_0^{n_0-2}\right) = p_0\left(z_1,z_2,z_3\right)(n_0-2)z_0^{n_0-3} = 0$, we get $p_0\left(z_1,z_2,z_3\right) = 0$.

From $D\left(z_1^{n_1-1}\right) = p_1\left(z_2,z_3\right)\left(n_1-1\right)z_1^{n_1-2} \in \left(z_0^{n_0-2}\right)$, we get $p_1\left(z_2,z_3\right) = 0$.

From $D\left(z_2^{n_2-2}z_3\right) = cz_3^k\left(n_2-2\right)z_2^{n_2-3}z_3 \in \left(z_0^{n_0-2}, z_1^{n_1-1}, z_1^{n_1-2}z_0, z_3^{n_3-1}\right)$, we get $cz_3^k\left(n_2-2\right)\, z_2^{n_2-3}z_3 \in \left(z_3^{n_3-1}\right)$.

If $c \neq 0$, we have $k \geq n_3 - 2$ and $wt\left(z_3^k\right) + \alpha_3 \geq wt\left(z_3^{n_3-1}\right)$. However, we can also get $wt\left(z_3^k\right) + \alpha_3 < \alpha_2 + \alpha_3 \leq 2\alpha_0$ and $wt\left(z_3^{n_3-1}\right) = wt(f) - \alpha_1 - \alpha_3 \geq 4\alpha_0 - 2\alpha_0 = 2\alpha_0$. This leads to a contradiction.

Thus we have $c = 0$ and $D = 0$.

In conclusion, there does not exist negative weight derivation of $H_1(V)$ and we complete the proof. □

Lemma 2.28 (Case (xxiv) of Proposition 2.4). *Let $(V, 0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 : f(z_0, z_1, z_2, z_3) = 0\}$ be an isolated singularity defined by $f(z_0, z_1, z_2, z_3) = z_0^{n_0} z_1 + z_1^{n_1} z_2 + z_2^{n_2} z_3 + z_3^{n_3}$ of weight type $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ where $\alpha_3 \geq \alpha_2 \geq \alpha_1 \geq \alpha_0$ and $wt(f) > 3max\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\} = 3\alpha_3$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there does not exist negative weight derivation of $H_1(V)$.*

Proof. After renumbering to make $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, the form of f changes to $f = z_0^{n_0} + z_1^{n_1} z_0 + z_2^{n_2} z_1 + z_3^{n_3} z_2$. After renumbering, if there exists some negative weight derivation D, D must be in the form of $D = p_0(z_1, z_2, z_3)\frac{\partial}{\partial z_0} + p_1(z_2, z_3)\frac{\partial}{\partial z_1} + cz_3^k \frac{\partial}{\partial z_2}$.

After renumbering, it is clear that $n_0 \geq 4$. Therefore, we have $wt(f) = n_0\alpha_0 \geq 4\alpha_0$. From $3\alpha_i + \alpha_j \leq 4\alpha_0 \leq wt(f) = n_i\alpha_i + \alpha_j$, we get $n_j \geq 3$ for $(i, j) = (1, 0), (2, 1)$ and $(3, 2)$. So we have $n_1 \geq 3, n_2 \geq 3$ and $n_3 \geq 3$. Regardless of difference of constants, we get the equations below.

$$\text{Hess }(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0^{n_0-2} & z_1^{n_1-1} & 0 & 0 \\ * & z_1^{n_1-2} z_0 & z_2^{n_2-1} & 0 \\ * & * & z_2^{n_2-2} z_1 & z_3^{n_3-1} \\ * & * & * & z_3^{n_3-2} z_2 \end{bmatrix}.$$

From $D\left(z_0^{n_0-2}\right) = p_0(z_1, z_2, z_3)(n_0 - 2)z_0^{n_0-3} = 0$, we get $p_0(z_1, z_2, z_3) = 0$.

From $D\left(z_1^{n_1-1}\right) = p_1(z_2, z_3)(n_1 - 1)z_1^{n_1-2} \in \left(z_0^{n_0-2}\right)$, we get $p_1(z_2, z_3) = 0$.

From $D\left(z_2^{n_2-1}\right) = cz_3^k(n_2 - 1)z_2^{n_2-2} \in \left(z_0^{n_0-2}, z_1^{n_1-1}, z_1^{n_1-2} z_0\right)$, we get $cz_3^k(n_2 - 1)z_2^{n_2-2} = 0$. Therefore, we have $c = 0$ and $D = 0$.

In conclusion, there does not exist negative weight derivation of $H_1(V)$ and we complete the proof. □

For $f(z_0, z_1, z_2, z_3) = z_0^{n_0} z_1 + z_1^{n_1} z_2 + z_2^{n_2} z_3 + z_3^{n_3}$, discussions when $2max\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\} < wt(f) \leq 3max\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$ are summarized in Proposition 2.29.

Proposition 2.29 (Case (ii) of Proposition 2.3). *Let $(V, 0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 : f(z_0, z_1, z_2, z_3) = 0\}$ be an isolated singularity defined by $f(z_0, z_1, z_2, z_3) = z_0^{n_0} z_1 + z_1^{n_1} z_2 + z_2^{n_2} z_3 + z_3^{n_3}$ of weight type $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ where $2max\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\} < wt(f) \leq 3max\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$. Let $H_1(V)$ be the 1-st Hessian algebra. There exists negative weight derivation of $H_1(V)$ if and only if f is in one of the following forms after renumbering the variables z_0, z_1, z_2 and z_3 so that $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$ (we combine the cases that can be transformed into each other by simply renumbering the variables, which is caused by the equal weights of asymmetrical variables):*

(i) $f = z_0^3 + z_1^3 z_2 + z_2^3 z_0 + z_3^{n_3} z_1\ (n_3 \geq 21)$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D | D = cz_2 z_3^k \frac{\partial}{\partial z_1}, c \neq 0, 1 \leq k < \frac{n_3}{20}, k \in \mathbb{Z}\right\}$;

(ii) $f = z_0^3 + z_1^2 z_0 + z_2^3 z_3 + z_3^3 z_1$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D | D = cz_3 \frac{\partial}{\partial z_2}, c \neq 0\right\}$;

(iii) $f = z_0^2 z_3 + z_1^3 z_2 + z_2^3 z_0 + z_3^{n_3}\ (n_3 \geq 5)$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D | D = cz_2 \frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(iv) $f = z_0^2 z_3 + z_1^3 z_2 + z_2^4 z_0 + z_3^{n_3} (n_3 \geq 8)$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{ D|D = cz_2^2 \frac{\partial}{\partial z_1}, c \neq 0 \right\}$;

(v) $f = z_0^2 z_3 + z_1^4 z_2 + z_2^3 z_0 + z_3^{n_3} (n_3 \geq 6)$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{ D|D = cz_2 \frac{\partial}{\partial z_1}, c \neq 0 \right\}$;

(vi) $f = z_0^2 z_2 + z_1^3 z_3 + z_2^4 + z_3^3 z_0$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{ D|D = cz_3 \frac{\partial}{\partial z_1}, c \neq 0 \right\}$.

Proof. We renumber the variables z_0, z_1, z_2 and z_3 to satisfy the weight relationship $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$. After renumbering, there are 2 cases:
(i) $f = z_0^3 + \ldots$;
(ii) $f = z_0^2 z_i + \ldots$.
They correspond to Proposition 2.31 and Proposition 2.38 respectively. □

LEMMA 2.30. *Let* $(V, 0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 \colon f(z_0, z_1, z_2, z_3) = 0\}$ *be an isolated singularity defined by* $f(z_0, z_1, z_2, z_3) = z_0^{n_0} z_1 + z_1^{n_1} z_2 + z_2^{n_2} z_3 + z_3^{n_3}$ *of weight type* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ *where* $2max\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\} < wt(f) \leq 3max\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$. *Let* $H_1(V)$ *be the 1-st Hessian algebra. We renumber the variables* z_0, z_1, z_2 *and* z_3 *to satisfy the weight relationship* $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$. *If we get* $f = z_0^3 + \ldots$ *after renumbering, whenever there exists any negative weight derivation* D *of* $H_1(V)$, D *must be in the form of* $D = p_1(z_2, z_3) \frac{\partial}{\partial z_1} + cz_3^k \frac{\partial}{\partial z_2}$.

Proof. Regardless of difference of constants, $f_{00} = z_0$. So $D = p_1(z_2, z_3) \frac{\partial}{\partial z_1} + cz_3^k \frac{\partial}{\partial z_2}$. □

In Proposition 2.31, we will discuss one case of Proposition 2.29. That is, for $f(z_0, z_1, z_2, z_3) = z_0^{n_0} z_1 + z_1^{n_1} z_2 + z_2^{n_2} z_3 + z_3^{n_3}$ satisfying $2\alpha_0 < wt(f) \leq 3\alpha_0$, f takes the form of $f = z_0^2 z_i + \cdots$ after we renumber the variables z_0, z_1, z_2 and z_3 to satisfy the weight relationship $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$.

PROPOSITION 2.31 (Case (i) of Proposition 2.29). *Let* $(V, 0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 \colon f(z_0, z_1, z_2, z_3) = 0\}$ *be an isolated singularity defined by* $f(z_0, z_1, z_2, z_3) = z_0^{n_0} z_1 + z_1^{n_1} z_2 + z_2^{n_2} z_3 + z_3^{n_3}$ *of weight type* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$. *Let* $H_1(V)$ *be the 1-st Hessian algebra. We renumber the variables* z_0, z_1, z_2 *and* z_3 *to satisfy the weight relationship* $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$. *If we get* $f = z_0^3 + \ldots$ *after renumbering, there exists negative weight derivation if and only if* f *is in one of the two forms after renumbering:*

(i) $f = z_0^3 + z_1^3 z_2 + z_2^3 z_0 + z_3^{n_3} z_1 (n_3 \geq 21)$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{ D|D = cz_2 z_3^k \frac{\partial}{\partial z_1}, c \neq 0, 1 \leq k < \frac{n_3}{20}, k \in \mathbb{Z} \right\}$;

(ii) $f = z_0^3 + z_1^2 z_0 + z_2^3 z_3 + z_3^3 z_1$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{ D|D = cz_3 \frac{\partial}{\partial z_2}, c \neq 0 \right\}$.

Proof. There are 6 cases of f after renumbering:
(i) $f = z_0^3 + z_1^{n_1} z_2 + z_2^{n_2} z_3 + z_3^{n_3} z_0$;
(ii) $f = z_0^3 + z_1^{n_1} z_3 + z_2^{n_2} z_0 + z_3^{n_3} z_2$;
(iii) $f = z_0^3 + z_1^{n_1} z_3 + z_2^{n_2} z_1 + z_3^{n_3} z_0$;
(iv) $f = z_0^3 + z_1^{n_1} z_2 + z_2^{n_2} z_0 + z_3^{n_3} z_1$;
(v) $f = z_0^3 + z_1^{n_1} z_0 + z_2^{n_2} z_3 + z_3^{n_3} z_1$;
(vi) $f = z_0^3 + z_1^{n_1} z_0 + z_2^{n_2} z_1 + z_3^{n_3} z_2$.
The calculation requires much effort. One can refer to the lemmas below (from Lemma 2.32 to Lemma 2.37) for further details. □

Lemma 2.32 (Case (i) of Proposition 2.31). *Let* $(V,0) = \{(z_0,z_1,z_2,z_3) \in \mathbb{C}^4 \colon f(z_0,z_1,z_2,z_3) = 0\}$ *be an isolated singularity defined by* $f(z_0,z_1,z_2,z_3) = z_0^3 + z_1^{n_1}z_2 + z_2^{n_2}z_3 + z_3^{n_3}z_0$ *of weight type* $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ *where* $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$ *. Let* $H_1(V)$ *be the* 1*-st Hessian algebra. Then there does not exist negative weight derivation of* $H_1(V)$*.*

Proof. By Lemma 2.30, whenever there exists any negative weight derivation D of $H_1(V)$, D must be in the form of $D = p_1(z_2,z_3)\frac{\partial}{\partial z_1} + cz_3^k\frac{\partial}{\partial z_2}$ after renumbering the variables.

From $2\alpha_i + \alpha_j \leq 3\alpha_0 = wt(f) = n_i\alpha_i + \alpha_j$ for $(i,j) = (1,2),(2,3)$ and $(3,0)$, we get $n_1 \geq 2, n_2 \geq 2$ and $n_3 \geq 2$. Regardless of difference of constants and monomials in the ideal generated by elements of Hess(f), we get the equations below.

$$\text{Hess }(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0 & 0 & 0 & z_3^{n_3-1} \\ * & z_1^{n_1-2}z_2 & z_1^{n_1-1} & 0 \\ * & * & z_2^{n_2-2}z_3 & z_2^{n_2-1} \\ * & * & * & 0 \end{bmatrix}.$$

If $n_1 = 2$, the equations become

$$\text{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0 & 0 & 0 & z_3^{n_3-1} \\ * & z_2 & z_1 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}.$$

Then the nonzero elements of $H_1(V)$ do not contain z_1 or z_2. Since $D = p_1(z_2,z_3)\frac{\partial}{\partial z_1} + cz_3^k\frac{\partial}{\partial z_2}$, we have $D = 0$.

If $n_1 > 2$, we have the following discussions.

It is obvious that $D(z_0) = 0$. From $D\left(z_1^{n_1-2}z_2\right) = (n_1-2)p_1(z_2,z_3)z_1^{n_1-3}z_2 + cz_3^kz_1^{n_1-2} \in \left(z_0, z_3^{n_3-1}\right)$, we get $(n_1-2)p_1(z_2,z_3)z_1^{n_1-3}z_2 + cz_3^kz_1^{n_1-2} \in \left(z_3^{n_3-1}\right)$.

We claim that both $p_1(z_2,z_3)$ and cz_3^k can be divided by $z_3^{n_3-1}$. In fact, if $p_1(z_2,z_3)$ cannot be divided by $z_3^{n_3-1}$, there exists some monomial with respect to z_0, z_1, z_2 and z_3 in $(n_1-2)p_1(z_2,z_3)z_1^{n_1-3}z_2$ that cannot be divided by $z_3^{n_3-1}$. It cannot be eliminated by $cz_3^kz_1^{n_1-2}$. If cz_3^k cannot be divided by $z_3^{n_3-1}$, $cz_3^kz_1^{n_1-2}$ cannot be eliminated by any monomial with respect to z_0, z_1, z_2 and z_3 in $(n_1-2)p_1(z_2,z_3)z_1^{n_1-3}z_2$. Both cases are in contradiction to $(n_1-2)p_1(z_2,z_3)z_1^{n_1-3}z_2 + cz_3^kz_1^{n_1-2} \in \left(z_3^{n_3-1}\right)$.

If $p_1(z_2,z_3) \neq 0$, we get $wt(p_1(z_2,z_3)) \geq wt\left(z_3^{n_3-1}\right)$. Since $wt\left(z_3^{n_3-1}\right) = wt(f) - \alpha_0 - \alpha_3 = 2\alpha_0 - \alpha_3 \geq \alpha_0$ and $wt(p_1(z_2,z_3)) < \alpha_1 \leq \alpha_0$, it is clear that $wt(p_1(z_2,z_3)) < wt\left(z_3^{n_3-1}\right)$. This leads to a contradiction. Thus $p_1(z_2,z_3) = 0$.

If $c \neq 0$, we get $wt\left(z_3^k\right) \geq wt\left(z_3^{n_3-1}\right)$. From $wt\left(z_3^{n_3-1}\right) = wt(f) - \alpha_0 - \alpha_3 = 2\alpha_0 - \alpha_3 \geq \alpha_0$ and $wt\left(z_3^k\right) < \alpha_2 \leq \alpha_0$, we get $wt\left(z_3^k\right) < wt\left(z_3^{n_3-1}\right)$. This leads to a contradiction. Thus we get $c = 0$. Therefore, $D = 0$ and we get a contradiction from our discussions.

Therefore, there does not exist negative weight derivation of $H_1(V)$. ☐

Lemma 2.33 (Case (ii) of Proposition 2.31). *Let* $(V,0) = \{(z_0,z_1,z_2,z_3) \in \mathbb{C}^4 \colon f(z_0,z_1,z_2,z_3) = 0\}$ *be an isolated singularity defined by* $f(z_0,z_1,z_2,z_3) = z_0^3 + z_1^{n_1}z_3 + z_2^{n_2}z_0 + z_3^{n_3}z_2$ *of weight type* $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ *where* $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$*. Let* $H_1(V)$ *be the* 1*-st Hessian algebra. Then there does not exist negative weight derivation of* $H_1(V)$*.*

Proof. By Lemma 2.30, whenever there exists any negative weight derivation D of $H_1(V)$, D must be in the form of $D = p_1(z_2, z_3)\frac{\partial}{\partial z_1} + cz_3^k \frac{\partial}{\partial z_2}$ after renumbering the variables.

From $2\alpha_i + \alpha_j \le 3\alpha_0 = wt(f) = n_i\alpha_i + \alpha_j$ for $(i,j) = (1,3), (2,0)$ and $(3,2)$, we get $n_1 \ge 2, n_2 \ge 2$ and $n_3 \ge 2$. Regardless of difference of constants and monomials in the ideal generated by elements of Hess(f), we get the equations below.

$$\text{Hess }(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0 & 0 & z_2^{n_2-1} & 0 \\ * & z_1^{n_1-2}z_3 & 0 & z_1^{n_1-1} \\ * & * & 0 & z_3^{n_3-1} \\ * & * & * & z_3^{n_3-2}z_2 \end{bmatrix}.$$

If $n_1 = 2$, we get the relation $2\alpha_1 + \alpha_3 = 3\alpha_0$. Considering $\alpha_0 \ge \alpha_1 \ge \alpha_2 \ge \alpha_3$, we have $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3$ and $n_1 = 2, n_2 = 2, n_3 = 2$. Thus the equations become

$$\text{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0 & 0 & z_2 & 0 \\ * & z_3 & 0 & z_1 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}.$$

Then the nonzero elements of Hessian algebra do not contain z_1, z_2 or z_3. From $D = p_1(z_2, z_3)\frac{\partial}{\partial z_1} + cz_3^k \frac{\partial}{\partial z_2}$, we get $D = 0$.

If $n_1 > 2$, we have the following discussions.

Firstly, $D(z_0) = 0$ is obvious. From $D\left(z_2^{n_2-1}\right) = cz_3^k(n_2-1)z_2^{n_2-2} \in \left(z_0, z_1^{n_1-2}z_3\right)$, we get $c = 0$. From $D\left(z_1^{n_1-2}z_3\right) = (n_1-2)p_1(z_2,z_3)z_1^{n_1-3}z_3 \in \left(z_0, z_2^{n_2-1}\right)$, we get $(n_1-2)p_1(z_2,z_3)\, z_1^{n_1-3}z_3 \in \left(z_2^{n_2-1}\right)$.

So $p_1(z_2, z_3)$ is divided by $z_2^{n_2-1}$.

If $p_1(z_2,z_3) \ne 0$, we get $wt(p_1(z_2,z_3)) \ge wt\left(z_2^{n_2-1}\right)$. Since $wt\left(z_2^{n_2-1}\right) = wt(f) - \alpha_0 - \alpha_2 = 2\alpha_0 - \alpha_2 \ge \alpha_0$ and $wt(p_1(z_2,z_3)) < \alpha_1 \le \alpha_0$, we get $wt(p_1(z_2,z_3)) < wt\left(z_2^{n_2-1}\right)$. This leads to a contradiction. Thus we get $p_1(z_2,z_3) = 0$ and $D = 0$.

Therefore, there does not exist negative weight derivation of $H_1(V)$. □

LEMMA 2.34 (Case (iii) of Proposition 2.31). *Let $(V,0) = \{(z_0,z_1,z_2,z_3) \in \mathbb{C}^4 \colon f(z_0,z_1,z_2,z_3) = 0\}$ be an isolated singularity defined by $f(z_0,z_1,z_2,z_3) = z_0^3 + z_1^{n_1}z_3 + z_2^{n_2}z_1 + z_3^{n_3}z_0$ of weight type $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ where $\alpha_0 \ge \alpha_1 \ge \alpha_2 \ge \alpha_3$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there does not exist negative weight derivations of $H_1(V)$.*

Proof. By Lemma 2.30, whenever there exists any negative weight derivation D of $H_1(V)$, D must be in the form of $D = p_1(z_2, z_3)\frac{\partial}{\partial z_1} + cz_3^k \frac{\partial}{\partial z_2}$ after renumbering the variables.

From $2\alpha_i + \alpha_j \le 3\alpha_0 = wt(f) = n_i\alpha_i + \alpha_j$ for $(i,j) = (1,3), (2,1)$ and $(3,0)$, we get $n_1 \ge 2, n_2 \ge 2$ and $n_3 \ge 2$. Regardless of difference of constants and monomials in the ideal generated by elements of Hess(f), we get the equations below.

$$\text{Hess }(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0 & 0 & 0 & z_3^{n_3-1} \\ * & z_1^{n_1-2}z_3 & z_2^{n_2-1} & z_1^{n_1-1} \\ * & * & z_2^{n_2-2}z_1 & 0 \\ * & * & * & 0 \end{bmatrix}.$$

If $n_1=2$, we get the relation $2\alpha_1+\alpha_3=3\alpha_0$. Since $\alpha_0\geq\alpha_1\geq\alpha_2\geq\alpha_3$, we have $\alpha_0=\alpha_1=\alpha_2=\alpha_3$, $n_1=2, n_2=2$ and $n_3=2$. The equations become

$$\mathrm{Hess}(f)=\begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03}\\ f_{10} & f_{11} & f_{12} & f_{13}\\ f_{20} & f_{21} & f_{22} & f_{23}\\ f_{30} & f_{31} & f_{32} & f_{33}\end{bmatrix}=\begin{bmatrix} z_0 & 0 & 0 & 0\\ * & z_3 & z_2 & z_1\\ * & * & 0 & 0\\ * & * & * & 0\end{bmatrix}.$$

Then the nonzero elements of $H_1(V)$ do not contain z_1, z_2 or z_3. Since $D=p_1(z_2,z_3)\frac{\partial}{\partial z_1}+cz_3^k\frac{\partial}{\partial z_2}$, we have $D=0$.

If $n_3=2$, we get $2\alpha_3+\alpha_0=3\alpha_0$. Since $\alpha_0\geq\alpha_1\geq\alpha_2\geq\alpha_3$, we have $\alpha_0=\alpha_1=\alpha_2=\alpha_3$, $n_1=2, n_2=2$ and $n_3=2$. The equations become

$$\mathrm{Hess}(f)=\begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03}\\ f_{10} & f_{11} & f_{12} & f_{13}\\ f_{20} & f_{21} & f_{22} & f_{23}\\ f_{30} & f_{31} & f_{32} & f_{33}\end{bmatrix}=\begin{bmatrix} z_0 & 0 & 0 & 0\\ * & z_3 & z_2 & z_1\\ * & * & 0 & 0\\ * & * & * & 0\end{bmatrix}.$$

Then the nonzero elements of Hessian algebra do not contain z_1, z_2 or z_3. Since $D=p_1(z_2,z_3)\frac{\partial}{\partial z_1}+cz_3^k\frac{\partial}{\partial z_2}$, so we have $D=0$.

If $n_1>2$ and $n_3>2$, we can deduct $\alpha_0>\alpha_3$. In fact, when $\alpha_0=\alpha_3$, we get $\alpha_0=\alpha_1=\alpha_2=\alpha_3$. We can get $n_1=2$ and $n_3=2$, which leads to a contradiction.

In this case, if $n_2=2$, we get $2\alpha_2+\alpha_1=3\alpha_0$. Considering $\alpha_0\geq\alpha_1\geq\alpha_2\geq\alpha_3$, we have $\alpha_0=\alpha_1=\alpha_2>\alpha_3$. The equations become

$$\mathrm{Hess}(f)=\begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03}\\ f_{10} & f_{11} & f_{12} & f_{13}\\ f_{20} & f_{21} & f_{22} & f_{23}\\ f_{30} & f_{31} & f_{32} & f_{33}\end{bmatrix}=\begin{bmatrix} z_0 & 0 & 0 & z_3^{n_3-1}\\ * & 0 & z_2 & 0\\ * & * & z_1 & 0\\ * & * & * & 0\end{bmatrix}.$$

Then the nonzero elements of $H_1(V)$ do not contain z_1 or z_2. Since $D=p_1(z_2,z_3)\frac{\partial}{\partial z_1}+cz_3^k\frac{\partial}{\partial z_2}$, we have $D=0$.

Thus we only need to consider the case when $n_1>2, n_2>2$ and $n_3>2$. It follows that $\alpha_0>\alpha_2$ since we can get $n_2=2$ when $\alpha_0=\alpha_2$. From $D\left(z_1^{n_1-2}z_3\right)=p_1(z_2,z_3)\,(n_1-2)\,z_1^{n_1-3}z_3\in\left(z_0,z_3^{n_3-1}\right)$, we get $p_1(z_2,z_3)\in\left(z_3^{n_3-2}\right)$. So $p_1(z_2,z_3)$ is divided by $z_3^{n_3-2}$. From $D\left(z_2^{n_2-1}\right)=cz_3^k(n_2-1)\,z_2^{n_2-2}\in\left(z_0,z_3^{n_3-1},z_1^{n_1-2}z_3\right)$, we get $cz_3^k(n_2-1)\,z_2^{n_2-2}\in\left(z_3^{n_3-1}\right)$. If $c\neq 0, z_3^k$ is divided by $z_3^{n_3-1}$ and we get $wt\left(z_3^k\right)\geq wt\left(z_3^{n_3-1}\right)$. Since $wt\left(z_3^{n_3-1}\right)=wt(f)-\alpha_0-\alpha_3=2\alpha_0-\alpha_3\geq 2\alpha_0-\alpha_2>\alpha_0$ and $wt\left(z_3^k\right)<\alpha_2<\alpha_0$, we get $wt\left(z_3^k\right)<wt\left(z_3^{n_3-1}\right)$. This leads to a contradiction. Thus $c=0$.

We consider $D\left(z_1^{n_1-1}\right)=p_1(z_2,z_3)\,(n_1-1)\,z_1^{n_1-2}\in\left(z_0,z_3^{n_3-1},z_1^{n_1-2}z_3,z_2^{n_2-1},z_2^{n_2-2}z_1\right)$. Since $p_1(z_2,z_3)$ is divided by $z_3^{n_3-2}$, we get that $p_1(z_2,z_3)$ is divided by z_3.

So it is obvious that $p_1(z_2,z_3)\,(n_1-1)\,z_1^{n_1-2}\in\left(z_1^{n_1-2}z_3\right)$.

From $D\left(z_2^{n_2-2}z_1\right)=p_1(z_2,z_3)\,z_2^{n_2-2}\in\left(z_0,z_3^{n_3-1},z_1^{n_1-2}z_3,z_2^{n_2-1},z_1^{n_1-1}\right)$, we get $p_1(z_2,z_3)\,z_2^{n_2-2}\in\left(z_3^{n_3-1},z_2^{n_2-1}\right)$.

If $p_1(z_2,z_3)\neq 0$, since $wt\left(z_3^{n_3-1}\right)=wt(f)-\alpha_0-\alpha_3=2\alpha_0-\alpha_3\geq 2\alpha_0-\alpha_2>\alpha_0\geq\alpha_1>wt\left(p_1(z_2,z_3)\right)$, we have $p_1(z_2,z_3)\,z_2^{n_2-2}$ cannot be divided by $z_3^{n_3-1}$. So $p_1(z_2,z_3)\,z_2^{n_2-2}$ is divided by $z_2^{n_2-1}$, $p_1(z_2,z_3)$ is divided by z_2 and $p_1(z_2,z_3)$ is divided by $z_2z_3^{n_3-2}$.

On the one hand, we have $wt\left(p_1\left(z_2,z_3\right)\right) \geq wt\left(z_2 z_3^{n_3-2}\right)$. On the other hand, we have $wt\left(z_2 z_3^{n_3-2}\right) = \alpha_2 + \left(n_3-2\right)\alpha_3 \geq \left(n_3-1\right)\alpha_3 = wt\left(z_3^{n_3-1}\right) > wt\left(p_1\left(z_2,z_3\right)\right)$. This leads to a contradiction. It follows that $p_1\left(z_2,z_3\right)=0$ and $D=0$.

Therefore, there does not exist negative weight derivations of $H_1\left(V\right)$. $\square$

Lemma 2.35 (Case (iv) of Proposition 2.31). *Let $(V,0) = \{(z_0,z_1,z_2,z_3) \in \mathbb{C}^4\colon f(z_0,z_1,z_2,z_3) = 0\}$ be an isolated singularity defined by $f(z_0,z_1,z_2,z_3) = z_0^3 + z_1^{n_1}z_2 + z_2^{n_2}z_0 + z_3^{n_3}z_1$ of weight type $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ where $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there exists negative weight derivation of $H_1\left(V\right)$ if and only if f is in the form of $f = z_0^3 + z_1^3 z_2 + z_2^3 z_0 + z_3^{n_3} z_1\left(n_3 \geq 21\right)$. In this case, the set of negative weight derivations of $H_1\left(V\right)$ is $\left\{D | D = c z_2 z_3^k \frac{\partial}{\partial z_1}, c \neq 0, 1 \leq k < \frac{n_3}{20}, k \in \mathbb{Z}\right\}$.*

Proof. By Lemma 2.30, whenever there exists any negative weight derivation D of $H_1\left(V\right)$, D must be in the form of $D = p_1\left(z_2,z_3\right)\frac{\partial}{\partial z_1} + c z_3^k \frac{\partial}{\partial z_2}$ after renumbering the variables.

From $2\alpha_i + \alpha_j \leq 3\alpha_0 = wt(f) = n_i\alpha_i + \alpha_j$, for $(i,j) = (1,2),(2,0)$ and $(3,1)$, we get $n_1 \geq 2, n_2 \geq 2$ and $n_3 \geq 2$. Regardless of difference of constants and monomials in the ideal generated by elements of Hess(f), we get the equations below.

$$\mathrm{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0 & 0 & z_2^{n_2-1} & 0 \\ * & z_1^{n_1-2}z_2 & z_1^{n_1-1} & z_3^{n_3-1} \\ * & * & 0 & 0 \\ * & * & * & z_3^{n_3-2}z_1 \end{bmatrix}.$$

If $n_1 = 2$, we get $2\alpha_1 + \alpha_2 = 3\alpha_0$. Since $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, we have $\alpha_0 = \alpha_1 = \alpha_2 \geq \alpha_3$ and $n_2 = 2$. Thus both z_1 and z_2 are in the ideal of ideal generated by elements of Hess (f). Then the nonzero elements of Hessian algebra do not contain z_1 or z_2. Since $D = p_1\left(z_2,z_3\right)\frac{\partial}{\partial z_1} + c z_3^k \frac{\partial}{\partial z_2}$, we have $D=0$.

If $n_1 > 2$, from $3\alpha_2 \leq 2\alpha_1 + \alpha_2 < n_1\alpha_1 + \alpha_2 = wt\left(f\right) = 3\alpha_0$, we have $\alpha_0 > \alpha_2 \geq \alpha_3$. Thus from $2\alpha_i + \alpha_j < 3\alpha_0 = wt(f) = n_i\alpha_i + \alpha_j$, we get $n_j > 2$ for $(i,j) = (1,2),(2,0)$ and $(3,1)$. So we have $n_1 > 2, n_2 > 2$ and $n_3 > 2$. From $D\left(z_2^{n_2-1}\right) = c z_3^k\left(n_2-1\right)z_2^{n_2-2} \in \left(z_0, z_1^{n_1-2}z_2\right)$, we get $c=0$. Thus $D = p_1\left(z_2,z_3\right)\frac{\partial}{\partial z_1}$. By the relation $D\left(z_1^{n_1-2}z_2\right) = \left(n_1-2\right)p_1\left(z_2,z_3\right)z_1^{n_1-3}z_2 \in \left(z_0, z_2^{n_2-1}\right)$, we obtain $\left(n_1-2\right)p_1\left(z_2,z_3\right)z_1^{n_1-3}z_2 \in \left(z_2^{n_2-1}\right)$. So $p_1\left(z_2,z_3\right)$ is divided by $z_2^{n_2-2}$.

$D\left(z_1^{n_1-1}\right)$ should satisfy $D\left(z_1^{n_1-1}\right) = p_1\left(z_2,z_3\right)\left(n_1-1\right)z_1^{n_1-2} \in \left(z_0, z_2^{n_2-1}, z_1^{n_1-2}z_2\right)$. Since $p_1\left(z_2,z_3\right)$ is divided by $z_2^{n_2-2}$, $p_1\left(z_2,z_3\right)$ is divided by z_2. By the relation $D\left(z_1^{n_1-1}\right) = p_1\left(z_2,z_3\right)\left(n_1-1\right)z_1^{n_1-2} \in \left(z_1^{n_1-2}z_2\right)$, it is obvious that $D\left(z_1^{n_1-1}\right) = p_1\left(z_2,z_3\right)\left(n_1-1\right)z_1^{n_1-2} \in \left(z_0, z_2^{n_2-1}, z_1^{n_1-2}z_2\right)$ holds.

$D\left(z_3^{n_3-2}z_1\right)$ should satisfy $D\left(z_3^{n_3-2}z_1\right) = p_1\left(z_2,z_3\right)z_3^{n_3-2} \in \left(z_0, z_2^{n_2-1}, z_1^{n_1-2}z_2, z_1^{n_1-1}, z_3^{n_3-1}\right)$, from which we obtain $p_1\left(z_2,z_3\right)z_3^{n_3-2} \in \left(z_2^{n_2-1}, z_3^{n_3-1}\right)$. If $p_1\left(z_2,z_3\right) \neq 0$, we have $wt\left(z_2^{n_2-1}\right) = wt(f) - \alpha_0 - \alpha_2 = 3\alpha_0 - \alpha_0 - \alpha_2 > \alpha_0 \geq \alpha_1 > wt\left(p_1\left(z_2,z_3\right)\right)$. However, if $z_2^{n_2-1}$ is a factor of $p_1\left(z_2,z_3\right)$, we have $wt\left(z_2^{n_2-1}\right) \leq wt\left(p_1\left(z_2,z_3\right)\right)$. This leads to a contradiction. Therefore, $z_2^{n_2-1}$ is not a factor of $p_1\left(z_2,z_3\right)$ and it follows that $p_1\left(z_2,z_3\right)z_3^{n_3-2} \in \left(z_3^{n_3-1}\right)$.

In summary, nonzero $p_1\left(z_2,z_3\right)$ exists only if $p_1\left(z_2,z_3\right)$ is divided by $z_2^{n_2-2}z_3$.

Solving the equations

$$\begin{cases} 3\alpha_0 = n_2\alpha_2 + \alpha_0 \\ 3\alpha_0 = n_1\alpha_1 + \alpha_2 \end{cases},$$

we get

$$\begin{cases} \alpha_1 = \frac{1}{n_1}\left(3 - \frac{2}{n_2}\right)\alpha_0 \\ \alpha_2 = \frac{2}{n_2}\alpha_0 \end{cases}.$$

By $(n_2-2)\alpha_2 < (n_2-2)\alpha_2 + \alpha_3 \le wt\left(p_1(z_2,z_3)\right) < \alpha_1$, we have $(n_2-2)\frac{2}{n_2}\alpha_0 < \frac{1}{n_1}\left(3-\frac{2}{n_2}\right)\alpha_0$. So $n_1 < \frac{3n_2-2}{2(n_2-2)}$. Note that $n_1 \ge 3$, we have $\frac{3n_2-2}{2(n_2-2)} > 3$, which means $n_2 < \frac{10}{3}$. Note that $n_2 > 2$ we have $n_2 = 3$. So $n_1 < \frac{3n_2-2}{2(n_2-2)} = \frac{7}{2}$. Note that $n_1 > 2$ we have $n_1 = 3$. So we get $f = z_0^3 + z_1^3 z_2 + z_2^3 z_0 + z_3^{n_3} z_1$, $\alpha_1 = \frac{7}{9}\alpha_0$ and $\alpha_2 = \frac{2}{3}\alpha_0$. The constraint $(n_2-2)\alpha_2 + \alpha_3 < \alpha_1$ is equal to $\alpha_3 < \frac{1}{9}\alpha_0$. It follows that $\alpha_3 < \alpha_2$. From $3\alpha_0 = n_3\alpha_3 + \alpha_1$, we have $\alpha_3 = \frac{20}{9n_3}\alpha_0$. So $n_3 > 20$, which is equivalent to $n_3 \ge 21$. The necessary and sufficient condition for $wt\left(z_2^{n_2-2}z_3\right) < \alpha_1$ and $\alpha_3 \le \alpha_2 \le \alpha_1 \le \alpha_0$ is that f is in the form of $f = z_0^3 + z_1^3 z_2 + z_2^3 z_0 + z_3^{n_3} z_1\,(n_3 \ge 21)$.

In this case, the equations become

$$\operatorname{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0 & 0 & z_2^2 & 0 \\ * & z_1 z_2 & z_1^2 & z_3^{n_3-1} \\ * & * & 0 & 0 \\ * & * & * & z_3^{n_3-2}z_1 \end{bmatrix}$$

regardless of constants and useless polynomials. Since z_2^2 and $z_3^{n_3-1}$ are in the ideal generated by elements of $\operatorname{Hess}(f)$ and $p_1(z_2,z_3)$ is divided by z_2z_3, it is clear that $p_1(z_2,z_3) = c_1 z_2 z_3^{k_1}\,(1 \le k_1 \le n_3 - 2, c_1 \ne 0)$. The derivation is negatively weighted if and only if $\alpha_2 + k_1\alpha_3 < \alpha_1$, which is equivalent to $\frac{2}{3}\alpha_0 + k_1\frac{20}{9n_3}\alpha_0 < \frac{7}{9}\alpha_0$. We get $1 \le k_1 < \frac{n_3}{20}$.

From the above discussions, when $n_3 \ge 21$, we have verified that such $D = c_1 z_2 z_3^{k_1}\frac{\partial}{\partial z_1}\left(c_1 \ne 0,\ \ 1 \le k_1 < \frac{n_3}{20}, k_1 \in \mathbb{Z}\right)$ does satisfy the restrictions of negative weight derivations. Therefore, the set of negative weight derivations of f is $\left\{D | D = c z_2 z_3^k \frac{\partial}{\partial z_1}, c \ne 0, 1 \le k < \frac{n_3}{20}, k \in \mathbb{Z}\right\}$.

Therefore, there exists negative weight derivation if and only if f is in the form of $f = z_0^3 + z_1^3 z_2 + z_2^3 z_0 + z_3^{n_3} z_1\,(n_3 \ge 21)$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D | D = c z_2 z_3^k \frac{\partial}{\partial z_1}, c \ne 0, 1 \le k < \frac{n_3}{20}, k \in \mathbb{Z}\right\}$. □

Lemma 2.36 (Case (v) of Proposition 2.31). *Let $(V,0) = \{(z_0,z_1,z_2,z_3) \in \mathbb{C}^4 \colon f(z_0,z_1,z_2,z_3) = 0\}$ be an isolated singularity defined by $f(z_0,z_1,z_2,z_3) = z_0^3 + z_1^{n_1}z_0 + z_2^{n_2}z_3 + z_3^{n_3}z_1$ of weight type $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ where $\alpha_0 \ge \alpha_1 \ge \alpha_2 \ge \alpha_3$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there exists negative weight derivation of $H_1(V)$ if and only if f is in the form of $f = z_0^3 + z_1^2 z_0 + z_2^3 z_3 + z_3^3 z_1$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D | D = c z_3 \frac{\partial}{\partial z_2}, c \ne 0\right\}$.*

Proof. By Lemma 2.30, whenever there exists any negative weight derivation D of $H_1(V)$, D must be in the form of $D = p_1(z_2,z_3)\frac{\partial}{\partial z_1} + c z_3^k \frac{\partial}{\partial z_2}$ after renumbering the variables.

From $2\alpha_i + \alpha_j \leq 3\alpha_0 = wt(f) = n_i\alpha_i + \alpha_j$ for $(i, j) = (1, 0), (2, 3)$ and $(3, 1)$, we get $n_1 \geq 2, n_2 \geq 2$ and $n_3 \geq 2$. Regardless of difference of constants and monomials in the ideal generated by elements of Hess(f), we get the equations below.

$$\mathrm{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0 & z_1^{n_1-1} & 0 & 0 \\ * & 0 & 0 & z_3^{n_3-1} \\ * & * & z_2^{n_2-2}z_3 & z_2^{n_2-1} \\ * & * & * & z_3^{n_3-2}z_1 \end{bmatrix}.$$

It is obvious that $D(z_0) = 0$. From $D\left(z_1^{n_1-1}\right) = p_1(z_2, z_3)(n_1 - 1)z_1^{n_1-2} \in (z_0)$, we obtain $p_1(z_2, z_3) = 0$ and $D = cz_3^k\frac{\partial}{\partial z_2}$.

If $n_2 = 2$, we have $2\alpha_2 + \alpha_3 = 3\alpha_0$. By $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, we have $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3$. Therefore, we get $n_1 = 2$ and $n_3 = 2$. The equations become

$$\mathrm{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0 & z_1 & 0 & 0 \\ * & 0 & 0 & z_3 \\ * & * & 0 & z_2 \\ * & * & * & 0 \end{bmatrix}.$$

Thus both z_1 and z_2 are in the ideal of ideal generated by elements of Hess(f). Then the nonzero elements of Hessian algebra do not contain z_1 or z_2. From $D = cz_3^k\frac{\partial}{\partial z_2}$, we can get $D = 0$.

If $n_2 > 2$, we can get $\alpha_0 > \alpha_3$. Otherwise, it is clear that $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3$, from which we get $n_2 = 2$. This leads to a contradiction. Therefore, from the relation $3\alpha_0 = wt(f) = n_3\alpha_3 + \alpha_1 < (n_3 + 1)\alpha_0$, we get $n_3 > 2$, which is equivalent to $n_3 \geq 3$. From $D\left(z_2^{n_2-2}z_3\right) = cz_3^k(n_2 - 2)z_2^{n_2-3}z_3 \in \left(z_0, z_1^{n_1-1}, z_3^{n_3-1}\right)$, we have $cz_3^k(n_2 - 2)$ $z_2^{n_2-3}z_3 \in \left(z_3^{n_3-1}\right)$.

If $c \neq 0$, on the one hand, it is clear that $k \geq n_3 - 2$ and $wt\left(z_3^k\right) \geq wt\left(z_3^{n_3-2}\right)$; on the other hand, we notice that $wt\left(z_3^k\right) < \alpha_2$ and $wt\left(z_3^{n_3-2}\right) = wt(f) - \alpha_1 - 2\alpha_3 \geq n_2\alpha_2 - \alpha_1 - \alpha_3$. Therefore, $n_2\alpha_2 < \alpha_1 + \alpha_2 + \alpha_3$. From $n_1\alpha_1 + \alpha_0 = wt(f) = n_2\alpha_2 + \alpha_3 < \alpha_1 + \alpha_2 + 2\alpha_3 \leq 3\alpha_1 + \alpha_0$, we get $n_1 < 3$. Note that $n_1 \geq 2$, it is easy to see that $n_1 = 2$. Therefore, from the weight relationship $3\alpha_0 = 2\alpha_1 + \alpha_0 = n_2\alpha_2 + \alpha_3 = n_3\alpha_3 + \alpha_1$, we get $\alpha_1 = \alpha_0$, $\alpha_2 = \frac{1}{n_2}\left(3 - \frac{2}{n_3}\right)\alpha_0$, $\alpha_3 = \frac{2}{n_3}\alpha_0$. Substituting them for $n_2\alpha_2 < \alpha_1 + \alpha_2 + \alpha_3$, we have $n_2 < \frac{3}{2} + \frac{2}{n_3-2} \leq \frac{7}{2}$. Note that $n_2 > 2$, we get $n_2 = 3$. From $3 = n_2 \leq \frac{3}{2} + \frac{2}{n_3-2}$, we get $n_3 \leq \frac{10}{3}$. Note that $n_3 \geq 3$, we have $n_3 = 3$.

Thus f is in the form of $f = z_0^3 + z_1^2z_0 + z_2^3z_3 + z_3^3z_1$. We have $\alpha_1 = \alpha_0$, $\alpha_2 = \frac{7}{9}\alpha_0$ and $\alpha_3 = \frac{2}{3}\alpha_0$.

Regardless of difference of constants and monomials in the ideal generated by elements of Hess(f), we get the equations below.

$$\mathrm{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0 & z_1 & 0 & 0 \\ * & 0 & 0 & z_3^2 \\ * & * & z_2z_3 & z_2^2 \\ * & * & * & 0 \end{bmatrix}.$$

Since z_3^2 is in the ideal generated by elements of Hess(f), we have $0 \leq k \leq 1$. Since $D\left(z_2^2\right) = 2cz_3^kz_2$ is in the ideal generated by elements of Hess(f), we have $k = 1$. Therefore, D is in the form of $D = cz_3\frac{\partial}{\partial z_2}$. It is easy to verify that this form of D is qualified.

Therefore, there exists negative weight derivation if and only if f is in the form of $f = z_0^3 + z_1^2 z_0 + z_2^3 z_3 + z_3^3 z_1$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D|D = cz_3 \frac{\partial}{\partial z_2}, c \neq 0\right\}$. □

LEMMA 2.37 (Case (vi) of Proposition 2.31). *Let* $(V, 0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 : f(z_0, z_1, z_2, z_3) = 0\}$ *be an isolated singularity defined by* $f(z_0, z_1, z_2, z_3) = z_0^3 + z_1^{n_1} z_0 + z_2^{n_2} z_1 + z_3^{n_3} z_2$ *of weight type* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ *where* $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$ *. Let* $H_1(V)$ *be the* 1*-st Hessian algebra. Then there does not exist any negative weight derivation of* $H_1(V)$*.*

Proof. By Lemma 2.30, whenever there exists any negative weight derivation D of $H_1(V)$, D must be in the form of $D = p_1(z_2, z_3) \frac{\partial}{\partial z_1} + cz_3^k \frac{\partial}{\partial z_2}$ after renumbering the variables.

From $2\alpha_i + \alpha_j \leq 3\alpha_0 = wt(f) = n_i\alpha_i + \alpha_j$ for $(i, j) = (1, 0), (2, 1)$ and $(3, 2)$, we get $n_1 \geq 2, n_2 \geq 2$ and $n_3 \geq 2$. Regardless of difference of constants and monomials in the ideal generated by elements of Hess(f), we get the equations below.

$$\mathrm{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0 & z_1^{n_1-1} & 0 & 0 \\ * & 0 & z_2^{n_2-1} & 0 \\ * & * & z_2^{n_2-2} z_1 & z_3^{n_3-1} \\ * & * & * & z_3^{n_3-2} z_2 \end{bmatrix}.$$

It is obvious that the condition $D(z_0) = 0$ is satisfied. From the condition $D\left(z_1^{n_1-1}\right) = p_1(z_2, z_3)(n_1 - 1) z_1^{n_1-2} \in (z_0)$, we can see $p_1(z_2, z_3) = 0$. From the condition $D\left(z_2^{n_2-1}\right) = cz_3^k(n_2 - 1) z_2^{n_2-2} \in \left(z_0, z_1^{n_1-1}\right)$, we can see $c = 0$. It is clear that $D = 0$.

Therefore, there does not exist negative weight derivation of $H_1(V)$. □

In Proposition 2.38, we will discuss the other case of Proposition 2.29. That is, for $f(z_0, z_1, z_2, z_3) = z_0^{n_0} z_1 + z_1^{n_1} z_2 + z_2^{n_2} z_3 + z_3^{n_3}$ satisfying $2\alpha_0 < wt(f) \leq 3\alpha_0$, f takes the form of $f = z_0^2 z_i + \cdots$ after we renumber the variables z_0, z_1, z_2 and z_3 to satisfy the weight relationship $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$.

PROPOSITION 2.38 (Case (ii) of Proposition 2.29). *Let* $(V, 0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 : f(z_0, z_1, z_2, z_3) = 0\}$ *be an isolated singularity defined by* $f(z_0, z_1, z_2, z_3) = z_0^{n_0} z_1 + z_1^{n_1} z_2 + z_2^{n_2} z_3 + z_3^{n_3}$ *of weight type* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$*. Let* $H_1(V)$ *be the* 1*-st Hessian algebra. We renumber the variables* z_0*,* z_1*,* z_2 *and* z_3 *to satisfy the weight relation* $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$*. If we get* $f = z_0^2 z_i + \cdots$ *after renumbering, there exists negative weight derivation of* $H_1(V)$ *if and only if* f *is in one of the following forms:*

(i) $f = z_0^2 z_3 + z_1^3 z_2 + z_2^3 z_0 + z_3^{n_3} (n_3 \geq 5)$*. In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D|D = cz_2 \frac{\partial}{\partial z_1}, c \neq 0\right\}$*;*

(ii) $f = z_0^2 z_3 + z_1^3 z_2 + z_2^4 z_0 + z_3^{n_3} (n_3 \geq 8)$*. In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D|D = cz_2^2 \frac{\partial}{\partial z_1}, c \neq 0\right\}$*;*

(iii) $f = z_0^2 z_3 + z_1^4 z_2 + z_2^3 z_0 + z_3^{n_3} (n_3 \geq 6)$*. In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D|D = cz_2 \frac{\partial}{\partial z_1}, c \neq 0\right\}$*;*

(iv) $f = z_0^2 z_2 + z_1^3 z_3 + z_2^4 + z_3^3 z_0$*. In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D|D = cz_3 \frac{\partial}{\partial z_1}, c \neq 0\right\}$*;*

(v) $f = z_0^2 z_1 + z_1^3 + z_2^3 z_3 + z_3^3 z_0$*. In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D|D = cz_3 \frac{\partial}{\partial z_2}, c \neq 0\right\}$*.*

Proof. After renumbering, z_0 and z_i are in the ideal generated by elements of Hess(f). Thus there does not exist any nonzero monomial or polynomial with respect to z_0, z_1, z_2 and z_3 that is divided by z_0 or z_i in $H_1(V)$. Since $wt(f)$ is more than $2\alpha_0$, the multiplicity of each monomial with respect to z_0, z_1, z_2 and z_3 is more than 2.

To simplify the problem, we renumber the variables again by letting the bigger weight variable left be z_{j_0} and the smaller weight variable left be z_{j_1}.

In this case, if there exists some negative weight derivation D, D must be in the form of $D = cz_{j_1}^k \frac{\partial}{\partial z_{j_0}} + c_{j_1} \frac{\partial}{\partial z_{j_1}}$.

If z_{j_1} is an element in the ideal generated by elements of Hess(f), there does not exist any nonzero element which is divided by z_{j_1} in $H_1(V)$. Thus $c_{j_1} = 0$.

If z_{j_1} is not an element in the ideal generated by elements of Hess(f), regardless of difference of constants, there exists an positive integer k_1 such that $z_{j_1}^{k_1+1}$ is in the ideal generated by elements of Hess(f) while $z_{j_1}^{k_1}$ is not. From the fact that $D\left(z_{j_1}^{k_1+1}\right) = c_{j_1}(k_1+1) z_{j_1}^{k_1}$ is in the ideal, we get $c_{j_1} = 0$.

In conclusion, $c_{j_1} = 0$ and $D = cz_{j_1}^k \frac{\partial}{\partial z_{j_0}}$.

There exists some positive integer p such that $z_{j_0}^p$ is in the ideal, while $z_{j_0}^{p-1}$ is not. Therefore, $D(z_{j_0}^p) = cpz_{j_1}^k z_{j_0}^{p-1}$ is in the ideal, from which we get $k \geq 1$. Therefore, $\alpha_{j_0} > k\alpha_{j_1} \geq \alpha_{j_1}$.

We will discuss what element the ideal contain when both z_{j_0} and z_{j_1} are not in the ideal generated by elements of Hess(f). Otherwise, it is clear that such negative weight derivation D does not exist.

Similar to the cases in the proof of Proposition 2.4, we only need to check 18 cases after renumbering to satisfy the weight relationship $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$:

(i)$f = z_0^2 z_1 + z_1^{n_1} z_2 + z_2^{n_2} z_3 + z_3^{n_3} (n_1 \geq 3, n_2 \geq 3, n_3 \geq 4)$ and $D = cz_3^k \frac{\partial}{\partial z_2} (k \geq 1, c \neq 0)$;

(ii) $f = z_0^2 z_1 + z_1^{n_1} z_3 + z_2^{n_2} + z_3^{n_3} z_2 (n_1 \geq 3, n_2 \geq 4, n_3 \geq 3)$ and $D = cz_3^k \frac{\partial}{\partial z_2} (k \geq 1, c \neq 0)$;

(iii) $f = z_0^2 z_2 + z_1^{n_1} z_3 + z_2^{n_2} z_1 + z_3^{n_3} (n_1 \geq 3, n_2 \geq 3, n_3 \geq 4)$ and $D = cz_3^k \frac{\partial}{\partial z_1} (k \geq 1, c \neq 0)$;

(iv) $f = z_0^2 z_3 + z_1^{n_1} z_2 + z_2^{n_2} + z_3^{n_3} z_1 (n_1 \geq 3, n_2 \geq 4, n_3 \geq 3)$ and $D = cz_2^k \frac{\partial}{\partial z_1} (k \geq 1, c \neq 0)$;

(v) $f = z_0^2 z_2 + z_1^{n_1} + z_2^{n_2} z_3 + z_3^{n_3} z_1 (n_1 \geq 4, n_2 \geq 3, n_3 \geq 3)$ and $D = cz_3^k \frac{\partial}{\partial z_1} (k \geq 1, c \neq 0)$;

(vi) $f = z_0^2 z_3 + z_1^{n_1} + z_2^{n_2} z_1 + z_3^{n_3} z_2 (n_1 \geq 4, n_2 \geq 3, n_3 \geq 3)$ and $D = cz_2^k \frac{\partial}{\partial z_1} (k \geq 1, c \neq 0)$;

(vii) $f = z_0^2 z_2 + z_1^{n_1} z_0 + z_2^{n_2} z_3 + z_3^{n_3} (n_1 \geq 3, n_2 \geq 3, n_3 \geq 4)$ and $D = cz_3^k \frac{\partial}{\partial z_1} (k \geq 1, c \neq 0)$;

(viii) $f = z_0^2 z_3 + z_1^{n_1} z_0 + z_2^{n_2} + z_3^{n_3} z_2 (n_1 \geq 3, n_2 \geq 4, n_3 \geq 3)$ and $D = cz_2^k \frac{\partial}{\partial z_1} (k \geq 1, c \neq 0)$;

(ix) $f = z_0^2 z_1 + z_1^{n_1} z_3 + z_2^{n_2} z_0 + z_3^{n_3} (n_1 \geq 3, n_2 \geq 3, n_3 \geq 4)$ and $D = cz_3^k \frac{\partial}{\partial z_2} (k \geq 1, c \neq 0)$;

(x) $f = z_0^2 z_1 + z_1^{n_1} z_2 + z_2^{n_2} + z_3^{n_3} z_0 (n_1 \geq 3, n_2 \geq 4, n_3 \geq 3)$ and $D = cz_3^k \frac{\partial}{\partial z_2} (k \geq 1, c \neq 0)$;

(xi) $f = z_0^2 z_3 + z_1^{n_1} + z_2^{n_2} z_0 + z_3^{n_3} z_1 (n_1 \geq 4, n_2 \geq 3, n_3 \geq 3)$ and $D = cz_2^k \frac{\partial}{\partial z_1} (k \geq 1, c \neq 0)$;

(xii) $f = z_0^2z_2 + z_1^{n_1} + z_2^{n_2}z_1 + z_3^{n_3}z_0 (n_1 \geq 4, n_2 \geq 3, n_3 \geq 3)$ and $D = cz_3^k\frac{\partial}{\partial z_1}\,(k \geq 1, c \neq 0)$;

(xiii) $f = z_0^2z_3+z_1^{n_1}z_2+z_2^{n_2}z_0+z_3^{n_3} (n_1 \geq 3, n_2 \geq 3)$ and $D = cz_2^k\frac{\partial}{\partial z_1}\,(k \geq 1, c \neq 0)$;

(xiv) $f = z_0^2z_2+z_1^{n_1}z_3+z_2^{n_2}+z_3^{n_3}z_0 (n_1 \geq 3, n_3 \geq 3)$ and $D = cz_3^k\frac{\partial}{\partial z_1}\,(k \geq 1, c \neq 0)$;

(xv) $f = z_0^2z_3+z_1^{n_1}z_0+z_2^{n_2}z_1+z_3^{n_3} (n_1 \geq 3, n_2 \geq 3)$ and $D = cz_2^k\frac{\partial}{\partial z_1}\,(k \geq 1, c \neq 0)$;

(xvi) $f = z_0^2z_2+z_1^{n_1}z_0+z_2^{n_2}+z_3^{n_3}z_1 (n_1 \geq 3, n_3 \geq 3)$ and $D = cz_3^k\frac{\partial}{\partial z_1}\,(k \geq 1, c \neq 0)$;

(xvii) $f = z_0^2z_1 + z_1^{n_1} + z_2^{n_2}z_3 + z_3^{n_3}z_0 (n_2 \geq 3, n_3 \geq 3)$ and $D = cz_3^k\frac{\partial}{\partial z_2}\,(k \geq 1, c \neq 0)$;

(xviii) $f = z_0^2z_1 + z_1^{n_1} + z_2^{n_2}z_0 + z_3^{n_3}z_2 (n_2 \geq 3, n_3 \geq 3)$ and $D = cz_3^k\frac{\partial}{\partial z_2}\,(k \geq 1, c \neq 0)$;

The calculation process is lengthy. One can look it up in the following lemmas (from Lemma 2.39 to Lemma 2.56). □

Lemma 2.39 (Case (i) of Proposition 2.38). *Let $(V, 0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 \colon f(z_0, z_1, z_2, z_3) = 0\}$ be an isolated singularity defined by $f(z_0, z_1, z_2, z_3) = z_0^2z_1 + z_1^{n_1}z_2 + z_2^{n_2}z_3 + z_3^{n_3} (n_1 \geq 3, n_2 \geq 3, n_3 \geq 4)$ of weight type $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ where $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there does not exist any negative weight derivation of $H_1(V)$ in the form of $D = cz_3^k\frac{\partial}{\partial z_2}\,(k \geq 1, c \neq 0)$.*

Proof. Assume that there exists some D in the form of $D = cz_3^k\frac{\partial}{\partial z_2}\,(k \geq 1, c \neq 0)$.

Regardless of difference of constants and monomials in the ideal generated by elements of Hess(f), we get the equations below.

$$\text{Hess }(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_1 & z_0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & z_2^{n_2-2}z_3 & z_2^{n_2-1} \\ * & * & * & z_3^{n_3-2} \end{bmatrix}.$$

By $D\left(z_2^{n_2-2}z_3\right) = c\,(n_2-2)\,z_3^{k+1}z_2^{n_2-3} \in (z_1, z_0)$, we obtain $c = 0$. There does not exist any negative weight derivation in the form of $D = cz_3^k\frac{\partial}{\partial z_2}\,(k \geq 1, c \neq 0)$.

Therefore, we complete the proof. □

Lemma 2.40 (Case (ii) of Proposition 2.38). *X Let $(V, 0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 \colon f(z_0, z_1, z_2, z_3) = 0\}$ be an isolated singularity defined by $f(z_0, z_1, z_2, z_3) = z_0^2z_1 + z_1^{n_1}z_3 + z_2^{n_2} + z_3^{n_3}z_2 (n_1 \geq 3, n_2 \geq 4, n_3 \geq 3)$ of weight type $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ where $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there does not exist any negative weight derivation of $H_1(V)$ in the form of $D = cz_3^k\frac{\partial}{\partial z_2}\,(k \geq 1, c \neq 0)$.*

Proof. Assume that there exists some D in the form of $D = cz_3^k\frac{\partial}{\partial z_2}\,(k \geq 1, c \neq 0)$.

Regardless of difference of constants and monomials in the ideal generated by elements of Hess(f), we get the equations below.

$$\text{Hess }(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_1 & z_0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & z_2^{n_2-2} & z_3^{n_3-1} \\ * & * & * & z_3^{n_3-2}z_2 \end{bmatrix}.$$

By $D\left(z_2^{n_2-2}\right) = c\,(n_2-2)\,z_3^kz_2^{n_2-3} \in (z_1, z_0)$, we obtain $c = 0$. There does not exist any negative weight derivation in the form of $D = cz_3^k\frac{\partial}{\partial z_2}\,(k \geq 1, c \neq 0)$.

Therefore, we complete the proof. □

Lemma 2.41 (Case (*iii*) of Proposition 2.38). *Let* $(V,0)=\{(z_0,z_1,z_2,z_3)\in \mathbb{C}^4\colon f(z_0,z_1,z_2,z_3)=0\}$ *be an isolated singularity defined by* $f(z_0,z_1,z_2,z_3)=z_0^2z_2+z_1^{n_1}z_3+z_2^{n_2}z_1+z_3^{n_3}(n_1\geq 3,n_2\geq 3,n_3\geq 4)$ *of weight type* $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ *where* $\alpha_0\geq\alpha_1\geq\alpha_2\geq\alpha_3$ *. Let* $H_1(V)$ *be the* 1*-st Hessian algebra. Then there does not exist any negative weight derivation of* $H_1(V)$ *in the form of* $D=cz_3^k\frac{\partial}{\partial z_1}\,(k\geq 1,c\neq 0)$.

Proof. Assume that there exists some D in the form of $D=cz_3^k\frac{\partial}{\partial z_1}\,(k\geq 1,c\neq 0)$.

Regardless of difference of constants and monomials in the ideal generated by elements of Hess(f), we get the equations below.

$$\text{Hess }(f)=\begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03}\\ f_{10} & f_{11} & f_{12} & f_{13}\\ f_{20} & f_{21} & f_{22} & f_{23}\\ f_{30} & f_{31} & f_{32} & f_{33}\end{bmatrix}=\begin{bmatrix} z_2 & 0 & z_0 & 0\\ * & z_1^{n_1-2}z_3 & 0 & z_1^{n_1-1}\\ * & * & 0 & 0\\ * & * & * & z_3^{n_3-2}\end{bmatrix}.$$

By $D\left(z_1^{n_1-2}z_3\right)=c\,(n_1-2)\,z_1^{n_1-3}z_3^{k+1}\in(z_2,z_0)$, we obtain $c=0$. There does not exist any negative weight derivation in the form of $D=cz_3^k\frac{\partial}{\partial z_1}\,(k\geq 1,c\neq 0)$.

Therefore, we complete the proof. □

Lemma 2.42 (Case (*iv*) of Proposition 2.38). *Let* $(V,0)=\{(z_0,z_1,z_2,z_3)\in \mathbb{C}^4\colon f(z_0,z_1,z_2,z_3)=0\}$ *be an isolated singularity defined by* $f(z_0,z_1,z_2,z_3)=z_0^2z_3+z_1^{n_1}z_2+z_2^{n_2}+z_3^{n_3}z_1(n_1\geq 3,n_2\geq 4,n_3\geq 3)$ *of weight type* $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ *where* $\alpha_0\geq\alpha_1\geq\alpha_2\geq\alpha_3$ *. Let* $H_1(V)$ *be the* 1*-st Hessian algebra. Then there does not exist any negative weight derivation of* $H_1(V)$ *in the form of* $D=cz_2^k\frac{\partial}{\partial z_1}\,(k\geq 1,c\neq 0)$.

Proof. After renumbering, assume that there exists some D in the form of $D=cz_2^k\frac{\partial}{\partial z_1}\,(k\geq 1,c\neq 0)$.

Regardless of difference of constants and monomials in the ideal generated by elements of Hess(f), we get the equations below.

$$\text{Hess }(f)=\begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03}\\ f_{10} & f_{11} & f_{12} & f_{13}\\ f_{20} & f_{21} & f_{22} & f_{23}\\ f_{30} & f_{31} & f_{32} & f_{33}\end{bmatrix}=\begin{bmatrix} z_3 & 0 & 0 & z_0\\ * & z_1^{n_1-2}z_2 & z_1^{n_1-1} & 0\\ * & * & z_2^{n_2-2} & 0\\ * & * & * & 0\end{bmatrix}.$$

By $D\left(z_1^{n_1-2}z_2\right)=c\,(n_1-2)\,z_1^{n_1-3}z_2^{k+1}\in(z_3,z_0)$, we obtain $c=0$. There does not exist any negative weight derivation in the form of $D=cz_2^k\frac{\partial}{\partial z_1}\,(k\geq 1,c\neq 0)$.

Therefore, we complete the proof. □

Lemma 2.43 (Case (*v*) of Proposition 2.38). *Let* $(V,0)=\{(z_0,z_1,z_2,z_3)\in \mathbb{C}^4\colon f(z_0,z_1,z_2,z_3)=0\}$ *be an isolated singularity defined by* $f(z_0,z_1,z_2,z_3)=z_0^2z_2+z_1^{n_1}+z_2^{n_2}z_3+z_3^{n_3}z_1(n_1\geq 4,n_2\geq 3,n_3\geq 3)$ *of weight type* $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ *where* $\alpha_0\geq\alpha_1\geq\alpha_2\geq\alpha_3$*. Let* $H_1(V)$ *be the* 1*-st Hessian algebra. Then there does not exist any negative weight derivation of* $H_1(V)$ *in the form of* $D=cz_3^k\frac{\partial}{\partial z_1}\,(k\geq 1,c\neq 0)$.

Proof. Assume that there exists some D in the form of $D=cz_3^k\frac{\partial}{\partial z_1}\,(k\geq 1,c\neq 0)$.

Regardless of difference of constants and monomials in the ideal generated by elements of Hess(f), we get the equations below.

$$\text{Hess }(f)=\begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03}\\ f_{10} & f_{11} & f_{12} & f_{13}\\ f_{20} & f_{21} & f_{22} & f_{23}\\ f_{30} & f_{31} & f_{32} & f_{33}\end{bmatrix}=\begin{bmatrix} z_2 & 0 & z_0 & 0\\ * & z_1^{n_1-2} & 0 & z_3^{n_3-1}\\ * & * & 0 & 0\\ * & * & * & z_3^{n_3-2}z_1\end{bmatrix}.$$

By $D\left(z_1^{n_1-2}\right) = c\,(n_1-2)\,z_1^{n_1-3}z_3^k \in (z_2, z_0)$, we obtain $c = 0$. There does not exist any negative weight derivation in the form of $D = cz_3^k\frac{\partial}{\partial z_1}\,(k \geq 1, c \neq 0)$.

Therefore, we complete the proof. □

LEMMA 2.44 (Case (*vi*) of Proposition 2.38). *Let* $(V,0) = \{(z_0,z_1,z_2,z_3) \in \mathbb{C}^4\colon f(z_0,z_1,z_2,z_3) = 0\}$ *be an isolated singularity defined by* $f(z_0,z_1,z_2,z_3) = z_0^2z_3 + z_1^{n_1} + z_2^{n_2}z_1 + z_3^{n_3}z_2(n_1 \geq 4, n_2 \geq 3, n_3 \geq 3)$ *of weight type* $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ *where* $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$. *Let* $H_1(V)$ *be the* 1*-st Hessian algebra. Then there does not exist any negative weight derivation of* $H_1(V)$ *in the form of* $D = cz_2^k\frac{\partial}{\partial z_1}\,(k \geq 1, c \neq 0)$.

Proof. Assume that there exists some D in the form of $D = cz_2^k\frac{\partial}{\partial z_1}\,(k \geq 1, c \neq 0)$.

Regardless of difference of constants and monomials in the ideal generated by elements of Hess(f), we get the equations below.

$$\mathrm{Hess}\ (f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_3 & 0 & 0 & z_0 \\ * & z_1^{n_1-2} & z_2^{n_2-1} & 0 \\ * & * & z_2^{n_2-2}z_1 & 0 \\ * & * & * & 0 \end{bmatrix}.$$

By $D\left(z_1^{n_1-2}\right) = c\,(n_1-2)\,z_1^{n_1-3}z_2^k \in (z_3, z_0)$, we obtain $c = 0$. There does not exist any negative weight derivation in the form of $D = cz_2^k\frac{\partial}{\partial z_1}\,(k \geq 1, c \neq 0)$.

Therefore, we complete the proof. □

LEMMA 2.45 (Case (*vii*) of Proposition 2.38). *Let* $(V,0) = \{(z_0,z_1,z_2,z_3) \in \mathbb{C}^4\colon f(z_0,z_1,z_2,z_3) = 0\}$ *be an isolated singularity defined by* $f(z_0,z_1,z_2,z_3) = z_0^2z_2 + z_1^{n_1}z_0 + z_2^{n_2}z_3 + z_3^{n_3}(n_1 \geq 3, n_2 \geq 3, n_3 \geq 4)$ *of weight type* $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ *where* $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$. *Let* $H_1(V)$ *be the* 1*-st Hessian algebra. Then there does not exist any negative weight derivation of* $H_1(V)$ *in the form of* $D = cz_3^k\frac{\partial}{\partial z_1}\,(k \geq 1, c \neq 0)$.

Proof. Assume that there exists some D in the form of $D = cz_3^k\frac{\partial}{\partial z_1}\,(k \geq 1, c \neq 0)$.

Regardless of difference of constants and monomials in the ideal generated by elements of Hess(f), we get the equations below.

$$\mathrm{Hess}\ (f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_2 & z_1^{n_1-1} & z_0 & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & z_3^{n_3-2} \end{bmatrix}.$$

By $D\left(z_1^{n_1-1}\right) = c\,(n_1-1)\,z_1^{n_1-2}z_3^k \in (z_2)$, we obtain $c = 0$. There does not exist any negative weight derivation in the form of $D = cz_3^k\frac{\partial}{\partial z_1}\,(k \geq 1, c \neq 0)$.

Therefore, we complete the proof. □

LEMMA 2.46 (Case (*viii*) of Proposition 2.38). *Let* $(V,0) = \{(z_0,z_1,z_2,z_3) \in \mathbb{C}^4\colon f(z_0,z_1,z_2,z_3) = 0\}$ *be an isolated singularity defined by* $f(z_0,z_1,z_2,z_3) = z_0^2z_3 + z_1^{n_1}z_0 + z_2^{n_2} + z_3^{n_3}z_2(n_1 \geq 3, n_2 \geq 4, n_3 \geq 3)$ *of weight type* $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ *where* $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$. *Let* $H_1(V)$ *be the* 1*-st Hessian algebra. Then there does not exist any negative weight derivation of* $H_1(V)$ *in the form of* $D = cz_2^k\frac{\partial}{\partial z_1}\,(k \geq 1, c \neq 0)$.

Proof. Assume that there exists some D in the form of $D = cz_2^k\frac{\partial}{\partial z_1}\,(k \geq 1, c \neq 0)$.

Regardless of difference of constants and monomials in the ideal generated by elements of Hess(f), we get the equations below.

$$\text{Hess }(f)=\begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix}=\begin{bmatrix} z_3 & z_1^{n_1-1} & 0 & z_0 \\ * & 0 & 0 & 0 \\ * & * & z_2^{n_2-2} & 0 \\ * & * & * & 0 \end{bmatrix}.$$

By $D\left(z_1^{n_1-1}\right)=c\,(n_1-1)\,z_1^{n_1-2}z_2^k\in(z_3)$, we obtain $c=0$. There does not exist any negative weight derivation in the form of $D=cz_2^k\frac{\partial}{\partial z_1}\,(k\geq 1, c\neq 0)$.

Therefore, we complete the proof. □

Lemma 2.47 (Case (ix) of Proposition 2.38). *Let $(V,0)=\{(z_0,z_1,z_2,z_3)\in\mathbb{C}^4\colon f(z_0,z_1,z_2,z_3)=0\}$ be an isolated singularity defined by $f(z_0,z_1,z_2,z_3)=z_0^2z_1+z_1^{n_1}z_3+z_2^{n_2}z_0+z_3^{n_3}(n_1\geq 3, n_2\geq 3, n_3\geq 4)$ of weight type $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ where $\alpha_0\geq\alpha_1\geq\alpha_2\geq\alpha_3$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there does not exist any negative weight derivation of $H_1(V)$ in the form of $D=cz_3^k\frac{\partial}{\partial z_2}\,(k\geq 1, c\neq 0)$.*

Proof. Assume that there exists some D in the form of $D=cz_3^k\frac{\partial}{\partial z_2}\,(k\geq 1, c\neq 0)$.

Regardless of difference of constants and monomials in the ideal generated by elements of Hess(f), we get the equations below.

$$\text{Hess }(f)=\begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix}=\begin{bmatrix} z_1 & z_0 & z_2^{n_2-1} & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & z_3^{n_3-2} \end{bmatrix}.$$

By $D\left(z_2^{n_2-1}\right)=c\,(n_2-1)\,z_2^{n_2-2}z_3^k\in(z_1,z_0)$, we obtain $c=0$. There does not exist any negative weight derivation in the form of $D=cz_3^k\frac{\partial}{\partial z_2}\,(k\geq 1, c\neq 0)$.

Therefore, we complete the proof. □

Lemma 2.48 (Case (x) of Proposition 2.38). *Let $(V,0)=\{(z_0,z_1,z_2,z_3)\in\mathbb{C}^4\colon f(z_0,z_1,z_2,z_3)=0\}$ be an isolated singularity defined by $f(z_0,z_1,z_2,z_3)=z_0^2z_1+z_1^{n_1}z_2+z_2^{n_2}+z_3^{n_3}z_0(n_1\geq 3, n_2\geq 4, n_3\geq 3)$ of weight type $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ where $\alpha_0\geq\alpha_1\geq\alpha_2\geq\alpha_3$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there does not exist any negative weight derivation of $H_1(V)$ in the form of $D=cz_3^k\frac{\partial}{\partial z_2}\,(k\geq 1, c\neq 0)$.*

Proof. Assume that there exists some D in the form of $D=cz_3^k\frac{\partial}{\partial z_2}\,(k\geq 1, c\neq 0)$.

Regardless of difference of constants and monomials in the ideal generated by elements of Hess(f), we get the equations below.

$$\text{Hess }(f)=\begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix}=\begin{bmatrix} z_1 & z_0 & 0 & z_3^{n_3-1} \\ * & 0 & 0 & 0 \\ * & * & z_2^{n_2-2} & 0 \\ * & * & * & 0 \end{bmatrix}.$$

From the weight relaionship

$$\begin{cases} 2\alpha_0+\alpha_1=wt(f) \\ n_1\alpha_1+\alpha_2=wt(f) \\ n_2\alpha_2=wt(f) \\ n_3\alpha_3+\alpha_0=wt(f) \end{cases},$$

we obtain

$$\begin{cases} \alpha_0 = \left(\frac{1}{2} - \frac{1}{2n_1} + \frac{1}{2n_1 n_2}\right) wt(f) \\ \alpha_1 = \left(\frac{1}{n_1} - \frac{1}{n_1 n_2}\right) wt(f) \\ \alpha_2 = \frac{1}{n_2} wt(f) \\ \alpha_3 = \frac{1}{n_3}\left(\frac{1}{2} + \frac{1}{2n_1} - \frac{1}{2n_1 n_2}\right) wt\left(f\right) \end{cases}.$$

The only restriction we need to consider is that $D\left(z_2^{n_2-2}\right) = c\left(n_2-2\right) z_3^k z_2^{n_2-3} \in \left(z_1, z_0, z_3^{n_3-1}\right)$. Therefore, we have $k \geq n_3 - 1$, or $wt\left(z_3^k\right) \geq wt\left(z_3^{n_3-1}\right)$. However, it is clear that $wt\left(z_3^k\right) < \alpha_2$. We can get $wt\left(z_3^{n_3-1}\right) < \alpha_2$. Therefore, we have

$$\left(1 - \frac{1}{n_3}\right)\left(\frac{1}{2} + \frac{1}{2n_1} - \frac{1}{2n_1 n_2}\right) < \frac{1}{n_2},$$

which is equivalent to

$$n_2\left(1 + \frac{1}{n_1}\right) - \frac{1}{n_1} < \frac{2}{1 - \frac{1}{n_3}}.$$

Therefore,

$$4\left(1 + \frac{1}{n_1}\right) - \frac{1}{n_1} < \frac{2}{1 - \frac{1}{n_3}},$$

which is equivalent to

$$4 + \frac{3}{n_1} < \frac{2}{1 - \frac{1}{n_3}}.$$

However,

$$\frac{2}{1 - \frac{1}{n_3}} \leq 3 < 4 + \frac{3}{n_1}.$$

This leads to a contradiction. There does not exist any negative weight derivation in the form of $D = cz_3^k \frac{\partial}{\partial z_2}\left(k \geq 1, c \neq 0\right)$.

Therefore, we complete the proof. □

Lemma 2.49 (Case (xi) of Proposition 2.38). *Let $(V, 0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 \colon f(z_0, z_1, z_2, z_3) = 0\}$ be an isolated singularity defined by $f(z_0, z_1, z_2, z_3) = z_0^2 z_3 + z_1^{n_1} + z_2^{n_2} z_0 + z_3^{n_3} z_1 (n_1 \geq 4, n_2 \geq 3, n_3 \geq 3)$ and $D = cz_2^k \frac{\partial}{\partial z_1}\left(k \geq 1, c \neq 0\right)$ of weight type $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ where $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there does not exist any negative weight derivation of $H_1\left(V\right)$ in the form of $D = cz_2^k \frac{\partial}{\partial z_1}\left(k \geq 1, c \neq 0\right)$.*

Proof. Assume that there exists some D in the form of $D = cz_2^k \frac{\partial}{\partial z_1}\left(k \geq 1, c \neq 0\right)$.

Regardless of difference of constants and monomials in the ideal generated by elements of Hess(f), we get the equations below.

$$\text{Hess}\left(f\right) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_3 & 0 & z_2^{n_2-1} & z_0 \\ * & z_1^{n_1-2} & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}.$$

From the weight relaionship

$$\begin{cases} 2\alpha_0+\alpha_3=wt(f) \\ n_1\alpha_1=wt(f) \\ n_2\alpha_2+\alpha_0=wt(f) \\ n_3\alpha_3+\alpha_1=wt(f) \end{cases},$$

we obtain

$$\begin{cases} \alpha_0=\left(\frac{1}{2}-\frac{1}{2n_3}+\frac{1}{2n_1n_3}\right)wt(f) \\ \alpha_1=\frac{1}{n_1}wt(f) \\ \alpha_2=\frac{1}{n_2}\left(\frac{1}{2}+\frac{1}{2n_3}-\frac{1}{2n_1n_3}\right)wt(f) \\ \alpha_3=\left(\frac{1}{n_3}-\frac{1}{n_1n_3}\right)wt\left(f\right) \end{cases}.$$

The only restriction we need to consider is that $D\left(z_1^{n_1-2}\right)=c\left(n_1-2\right)z_2^kz_1^{n_1-3}\in\left(z_3,z_2^{n_2-1},z_0\right)$. Therefore, we have $k\geq n_2-1$, or $wt\left(z_2^k\right)\geq wt\left(z_2^{n_2-1}\right)$. However, it is clear that $wt\left(z_2^k\right)<\alpha_1$. We can get $wt\left(z_2^{n_2-1}\right)<\alpha_1$. Therefore, we have

$$\left(1-\frac{1}{n_2}\right)\left(\frac{1}{2}+\frac{1}{2n_3}-\frac{1}{2n_1n_3}\right)<\frac{1}{n_1},$$

which is equivalent to

$$n_1\left(1+\frac{1}{n_3}\right)-\frac{1}{n_3}<\frac{2}{1-\frac{1}{n_2}}.$$

Therefore,

$$4\left(1+\frac{1}{n_3}\right)-\frac{1}{n_3}<\frac{2}{1-\frac{1}{n_2}},$$

which is equivalent to

$$4+\frac{3}{n_3}<\frac{2}{1-\frac{1}{n_2}}.$$

However,

$$\frac{2}{1-\frac{1}{n_2}}\leq 3<4+\frac{3}{n_3}.$$

This leads to a contradiction. There does not exist any negative weight derivation in the form of $D=cz_2^k\frac{\partial}{\partial z_1}\left(k\geq 1,c\neq 0\right)$.

Therefore, we complete the proof. ∎

Lemma 2.50 (Case (*xii*) of Proposition 2.38). *Let* $(V,0)=\{(z_0,z_1,z_2,z_3)\in\mathbb{C}^4\colon f(z_0,z_1,z_2,z_3)=0\}$ *be an isolated singularity defined by* $f(z_0,z_1,z_2,z_3)=z_0^2z_2+z_1^{n_1}+z_2^{n_2}z_1+z_3^{n_3}z_0(n_1\geq 4,n_2\geq 3,n_3\geq 3)$ *of weight type* $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ *where* $\alpha_0\geq\alpha_1\geq\alpha_2\geq\alpha_3$. *Let* $H_1(V)$ *be the* 1*-st Hessian algebra. Then there does not exist any negative weight derivation of* $H_1\left(V\right)$ *in the form of* $D=cz_3^k\frac{\partial}{\partial z_1}\left(k\geq 1,c\neq 0\right)$.

Proof. Assume that there exists some D in the form of $D=cz_3^k\frac{\partial}{\partial z_1}\left(k\geq 1,c\neq 0\right)$.

Regardless of difference of constants and monomials in the ideal generated by elements of Hess(f), we get the equations below.

$$\text{Hess }(f)=\begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix}=\begin{bmatrix} z_2 & 0 & z_0 & z_3^{n_3-1} \\ * & z_1^{n_1-2} & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}.$$

From the weight relaionship

$$\begin{cases} 2\alpha_0+\alpha_2=wt(f) \\ n_1\alpha_1=wt(f) \\ n_2\alpha_2+\alpha_1=wt(f) \\ n_3\alpha_3+\alpha_0=wt(f) \end{cases},$$

we obtain

$$\begin{cases} \alpha_0=\left(\frac{1}{2}-\frac{1}{2n_2}+\frac{1}{2n_1n_2}\right)wt(f) \\ \alpha_1=\frac{1}{n_1}wt(f) \\ \alpha_2=\frac{1}{n_2}\left(1-\frac{1}{n_1}\right)wt(f) \\ \alpha_3=\frac{1}{n_3}\left(\frac{1}{2}+\frac{1}{2n_2}-\frac{1}{2n_1n_2}\right)wt\left(f\right) \end{cases}.$$

The only restriction we need to consider is that $D\left(z_1^{n_1-2}\right)=c\left(n_1-2\right)z_3^kz_1^{n_1-3}\in\left(z_2,z_0,z_3^{n_3-1}\right)$. Therefore, we have $k\geq n_3-1$, or $wt\left(z_3^k\right)\geq wt\left(z_3^{n_3-1}\right)$. However, it is clear that $wt\left(z_3^k\right)<\alpha_1$. We can get $wt\left(z_3^{n_3-1}\right)<\alpha_1$. Therefore, we have

$$\left(1-\frac{1}{n_3}\right)\left(\frac{1}{2}+\frac{1}{2n_2}-\frac{1}{2n_1n_2}\right)<\frac{1}{n_1},$$

which is equivalent to

$$n_1\left(1+\frac{1}{n_2}\right)-\frac{1}{n_2}<\frac{2}{1-\frac{1}{n_3}}.$$

Therefore,

$$4\left(1+\frac{1}{n_2}\right)-\frac{1}{n_2}<\frac{2}{1-\frac{1}{n_3}},$$

which is equivalent to

$$4+\frac{3}{n_2}<\frac{2}{1-\frac{1}{n_3}}.$$

However,

$$\frac{2}{1-\frac{1}{n_3}}\leq 3<4+\frac{3}{n_3}.$$

This leads to a contradiction. There does not exist any negative weight derivation in the form of $D=cz_3^k\frac{\partial}{\partial z_1}\left(k\geq 1,c\neq 0\right)$.

Therefore, we complete the proof. □

LEMMA 2.51 (Case (*xiii*) of Proposition 2.38). *Let* $(V,0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 \colon f(z_0, z_1, z_2, z_3) = 0\}$ *be an isolated singularity defined by* $f(z_0, z_1, z_2, z_3) = z_0^2 z_3 + z_1^{n_1} z_2 + z_2^{n_2} z_0 + z_3^{n_3} (n_1 \geq 3, n_2 \geq 3)$ *of weight type* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ *where* $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$*. Let* $H_1(V)$ *be the 1-st Hessian algebra. Then there exists negative weight derivation of* $H_1(V)$ *in the form of* $D = cz_2^k \frac{\partial}{\partial z_1}$ $(k \geq 1, c \neq 0)$ *if and only if* f *is in one of the following forms:*

(*i*) $f = z_0^2 z_3 + z_1^3 z_2 + z_2^3 z_0 + z_3^{n_3} (n_3 \geq 5)$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D | D = cz_2 \frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(*ii*) $f = z_0^2 z_3 + z_1^3 z_2 + z_2^4 z_0 + z_3^{n_3} (n_3 \geq 8)$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D | D = cz_2^2 \frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(*iii*) $f = z_0^2 z_3 + z_1^4 z_2 + z_2^3 z_0 + z_3^{n_3} (n_3 \geq 6)$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D | D = cz_2 \frac{\partial}{\partial z_1}, c \neq 0\right\}$.

Proof. Assume that there exists some D in the form of $D = cz_2^k \frac{\partial}{\partial z_1}$ $(k \geq 1, c \neq 0)$. Therefore, we have $\alpha_3 \leq \alpha_2 < \alpha_1 \leq \alpha_0$.

By the weight relationship $3\alpha_3 < 2\alpha_0 + \alpha_3 = n_3\alpha_3$, we have $n_3 > 3$, which is equivalent to $n_3 \geq 4$.

Regardless of difference of constants and monomials in the ideal generated by elements of Hess(f), we get the equations below.

$$\text{Hess}\ (f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_3 & 0 & z_2^{n_2-1} & z_0 \\ * & z_1^{n_1-2} z_2 & z_1^{n_1-1} & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}.$$

From the weight relaionship

$$\begin{cases} 2\alpha_0 + \alpha_3 = wt(f) \\ n_1\alpha_1 + \alpha_2 = wt(f) \\ n_2\alpha_2 + \alpha_0 = wt(f) \\ n_3\alpha_3 = wt(f) \end{cases},$$

we obtain

$$\begin{cases} \alpha_0 = \left(\frac{1}{2} - \frac{1}{2n_3}\right) wt(f) \\ \alpha_1 = \frac{1}{n_1} \left(1 - \frac{1}{2n_2} - \frac{1}{2n_2 n_3}\right) wt(f) \\ \alpha_2 = \frac{1}{n_2} \left(\frac{1}{2} + \frac{1}{2n_3}\right) wt(f) \\ \alpha_3 = \frac{1}{n_3} wt\,(f) \end{cases}.$$

It is easy to verify that $D\left(z_1^{n_1-1}\right) = c\,(n_1 - 1)\, z_1^{n_1-2} z_2^k \in \left(z_3, z_2^{n_2-1}, z_0, z_1^{n_1-2} z_2\right)$. The only restriction of D we need to verify is that $D\left(z_1^{n_1-2} z_2\right) = c\,(n_1 - 2)\, z_1^{n_1-3} z_2^{k+1} \in \left(z_3, z_2^{n_2-1}, z_0\right)$. By our assumption that $c \neq 0$, we have $k + 1 \geq n_2 - 1$, which is equivalent to $wt(z_2^k) \geq wt(z_2^{n_2-2})$. Since D is negative weight, we have $wt(z_2^k) < \alpha_1$. Therefore, we have $wt(z_2^{n_2-2}) < \alpha_1$.

Substituting the weights of α_1 and α_2 for it, we get

$$n_1 < \frac{1}{1 - \frac{2}{n_2}} \left(\frac{1}{\frac{1}{2} + \frac{1}{2n_3}} - \frac{1}{n_2}\right) < \frac{2 - \frac{1}{n_2}}{1 - \frac{2}{n_2}} = 2 + \frac{3}{n_2 - 2}.$$

If $n_1 = 3$, we obtain $3 < 2 + \frac{3}{n_2-2}$, which is equivalent to $n_2 < 5$. Note that $n_2 \geq 3$, we get $n_2 = 3$ or $n_2 = 4$ when $n_1 = 3$.

If $n_1 = 4$, we obtain $4 < 2 + \frac{3}{n_2-2}$, which is equivalent to $n_2 < \frac{7}{2}$. Note that $n_2 \geq 3$, we get $n_2 = 3$ when $n_1 = 4$.

If $n_1 \geq 5$, we obtain $5 < 2 + \frac{3}{n_2-2}$, which is equivalent to $n_2 < 3$. Note that $n_2 \geq 3$, we get a contradiction when $n_1 \geq 5$.

There are 3 cases left:

Case 1: $n_1 = 3$, $n_2 = 3$;

Case 2: $n_1 = 3$, $n_2 = 4$;

Case 3: $n_1 = 4$, $n_2 = 3$.

In Case 1, the weights are

$$\begin{cases} \alpha_0 = \frac{1}{2}\left(1 - \frac{1}{n_3}\right) wt(f) \\ \alpha_1 = \frac{1}{18}\left(5 - \frac{1}{n_3}\right) wt(f) \\ \alpha_2 = \frac{1}{6}\left(1 + \frac{1}{n_3}\right) wt(f) \\ \alpha_3 = \frac{1}{n_3} wt\left(f\right) \end{cases}.$$

By $\alpha_0 \geq \alpha_1 > \alpha_2 \geq \alpha_3$, $wt(z_2^{n_2-2}) < \alpha_1$ and $n_3 \geq 4$, we get $n_3 \geq 5$. The restrictions of k are $k \geq n_2 - 2$ and $k\alpha_2 < \alpha_1$. Therefore, $1 \leq k < \frac{5n_3-1}{3(n_3+1)} < \frac{5}{3}$. Therefore, $k = 1$. Since $\alpha_1 > \alpha_2$ when $n_3 \geq 5$, we know $k = 1$ is valid when $n_3 \geq 5$.

Therefore, in Case 1, there exists negative weight derivation D in the form of $D = cz_2^k \frac{\partial}{\partial z_1} (k \geq 1, c \neq 0)$ if and only if f is in the form of $f = z_0^2 z_3 + z_1^3 z_2 + z_2^3 z_0 + z_3^{n_3} (n_3 \geq 5)$. Accordingly, the set of negative weight derivations of $H_1(V)$ is $\left\{D | D = cz_2 \frac{\partial}{\partial z_1}, c \neq 0\right\}$.

In Case 2, the weights are

$$\begin{cases} \alpha_0 = \frac{1}{2}\left(1 - \frac{1}{n_3}\right) wt(f) \\ \alpha_1 = \frac{1}{24}\left(7 - \frac{1}{n_3}\right) wt(f) \\ \alpha_2 = \frac{1}{8}\left(1 + \frac{1}{n_3}\right) wt(f) \\ \alpha_3 = \frac{1}{n_3} wt\left(f\right) \end{cases}.$$

By $\alpha_0 \geq \alpha_1 > \alpha_2 \geq \alpha_3$, $wt(z_2^{n_2-2}) < \alpha_1$ and $n_3 \geq 4$, we get $n_3 > 7$, which is equivalent to $n_3 \geq 8$. The restrictions of k are $k \geq n_2 - 2$ and $k\alpha_2 < \alpha_1$. Therefore, $2 \leq k < \frac{7n_3-1}{3(n_3+1)} < \frac{7}{3}$. Therefore, $k = 2$. Since $wt(z_2^2) < \alpha_1$ when $n_3 \geq 8$, we know $k = 2$ is valid when $n_3 \geq 8$.

Therefore, in Case 2, there exists negative weight derivation D in the form of $D = cz_2^k \frac{\partial}{\partial z_1} (k \geq 1, c \neq 0)$ if and only if f is in the form of $f = z_0^2 z_3 + z_1^3 z_2 + z_2^4 z_0 + z_3^{n_3} (n_3 \geq 8)$. Accordingly, the set of negative weight derivations of $H_1(V)$ is $\left\{D | D = cz_2^2 \frac{\partial}{\partial z_1}, c \neq 0\right\}$.

In Case 3, the weights are

$$\begin{cases} \alpha_0 = \frac{1}{2}\left(1 - \frac{1}{n_3}\right) wt(f) \\ \alpha_1 = \frac{1}{24}\left(5 - \frac{1}{n_3}\right) wt(f) \\ \alpha_2 = \frac{1}{6}\left(1 + \frac{1}{n_3}\right) wt(f) \\ \alpha_3 = \frac{1}{n_3} wt\left(f\right) \end{cases}.$$

By $\alpha_0 \geq \alpha_1 > \alpha_2 \geq \alpha_3$, $wt(z_2^{n_2-2}) < \alpha_1$ and $n_3 \geq 4$, we get $n_3 > 5$, which is equivalent to $n_3 \geq 6$. The restrictions of k are $k \geq n_2 - 2$ and $k\alpha_2 < \alpha_1$. Therefore, $1 \leq k < \frac{5n_3-1}{4(n_3+1)} < \frac{5}{4}$. Therefore, $k = 1$. Since $\alpha_1 > \alpha_2$ when $n_3 \geq 6$, we know $k = 1$ is valid when $n_3 \geq 6$.

Therefore, in Case 3, there exists negative weight derivation D in the form of $D = cz_2^k \frac{\partial}{\partial z_1} (k \geq 1, c \neq 0)$ if and only if f is in the form of $f = z_0^2 z_3 + z_1^4 z_2 + z_2^3 z_0 + z_3^{n_3} (n_3 \geq 6)$. Accordingly, the set of negative weight derivations of $H_1(V)$ is $\left\{D | D = cz_2 \frac{\partial}{\partial z_1}, c \neq 0\right\}$.

Therefore, we complete the proof. □

Lemma 2.52 (Case (*xiv*) of Proposition 2.38). *Let* $(V,0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 \colon f(z_0, z_1, z_2, z_3) = 0\}$ *be an isolated singularity defined by* $f(z_0, z_1, z_2, z_3) = z_0^2 z_2 + z_1^{n_1} z_3 + z_2^{n_2} + z_3^{n_3} z_0 (n_1 \geq 3, n_3 \geq 3)$ *of weight type* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ *where* $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$. *Let* $H_1(V)$ *be the 1-st Hessian algebra. Then there exists negative weight derivation of* $H_1(V)$ *in the form of* $D = cz_3^k \frac{\partial}{\partial z_1} (k \geq 1, c \neq 0)$ *if and only if* f *is in one of the following forms:*

(*i*) $f = z_0^2 z_2 + z_1^3 z_3 + z_2^4 + z_3^3 z_0$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D | D = cz_3 \frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(*ii*) $f = z_0^2 z_2 + z_1^3 z_3 + z_2^5 + z_3^3 z_0$. *In this case, the set of negative weight derivations of* $H_1(V)$ *is* $\left\{D | D = cz_3 \frac{\partial}{\partial z_1}, c \neq 0\right\}$.

Proof. Assume that there exists some D in the form of $D = cz_3^k \frac{\partial}{\partial z_1} (k \geq 1, c \neq 0)$. From the weight relationship $3\alpha_2 \leq 2\alpha_0 + \alpha_2 = n_2\alpha_2$, we have $n_2 \geq 3$.

Regardless of difference of constants and monomials in the ideal generated by elements of Hess(f), we get the equations below.

$$\text{Hess } (f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_2 & 0 & z_0 & z_3^{n_3-1} \\ * & z_1^{n_1-2} z_3 & 0 & z_1^{n_1-1} \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}.$$

From the weight relaionship

$$\begin{cases} 2\alpha_0 + \alpha_2 = wt(f) \\ n_1\alpha_1 + \alpha_3 = wt(f) \\ n_2\alpha_2 = wt(f) \\ n_3\alpha_3 + \alpha_0 = wt(f) \end{cases},$$

we obtain

$$\begin{cases} \alpha_0 = \frac{1}{2}\left(1 - \frac{1}{n_2}\right) wt(f) \\ \alpha_1 = \frac{1}{n_1}\left(1 - \frac{1}{2n_3} - \frac{1}{2n_2 n_3}\right) wt(f) \\ \alpha_2 = \frac{1}{n_2} wt(f) \\ \alpha_3 = \frac{1}{n_3}\left(\frac{1}{2} + \frac{1}{2n_2}\right) wt(f) \end{cases}.$$

It is easy to verify that $D\left(z_1^{n_1-1}\right) = c(n_1 - 1) z_1^{n_1-2} z_3^k \in \left(z_2, z_0, z_3^{n_3-1}, z_1^{n_1-2} z_3\right)$. The only restriction of D we need to verify is that $D\left(z_1^{n_1-2} z_3\right) = c(n_1 - 2) z_1^{n_1-3} z_3^{k+1} \in \left(z_2, z_0, z_3^{n_3-1}\right)$. By our assumption that $c \neq 0$, we have

$k \geq n_3 - 2$, which is equivalent to $wt(z_3^k) \geq wt(z_3^{n_3-2})$. Since D is negative weight, we have $wt(z_3^k) < \alpha_1$. Therefore, we have $wt(z_3^{n_3-2}) < \alpha_1$.

Substituting the weights of α_1 and α_3 for it, we get

$$n_1 < \frac{1}{1-\frac{2}{n_3}}\left(\frac{1}{\frac{1}{2}+\frac{1}{2n_2}} - \frac{1}{n_3}\right) < \frac{2-\frac{1}{n_3}}{1-\frac{2}{n_3}} = 2 + \frac{3}{n_3-2}.$$

If $n_1 = 3$, we obtain $3 < 2 + \frac{3}{n_3-2}$, which is equivalent to $n_3 < 5$. Note that $n_3 \geq 3$, we get $n_3 = 3$ or $n_3 = 4$ when $n_1 = 3$.

If $n_1 = 4$, we obtain $4 < 2 + \frac{3}{n_3-2}$, which is equivalent to $n_3 < \frac{7}{2}$. Note that $n_3 \geq 3$, we get $n_3 = 3$ when $n_1 = 4$.

If $n_1 \geq 5$, we obtain $5 < 2 + \frac{3}{n_3-2}$, which is equivalent to $n_3 < 3$. Note that $n_3 \geq 3$, we get a contradiction when $n_1 \geq 5$.

There are 3 cases left:

Case 1: $n_1 = 3$, $n_3 = 3$;

Case 2: $n_1 = 3$, $n_3 = 4$;

Case 3: $n_1 = 4$, $n_3 = 3$.

In Case 1, the weights are

$$\begin{cases} \alpha_0 = \frac{1}{2}\left(1-\frac{1}{n_2}\right)wt(f) \\ \alpha_1 = \frac{1}{18}\left(5-\frac{1}{n_2}\right)wt(f) \\ \alpha_2 = \frac{1}{n_2}wt(f) \\ \alpha_3 = \frac{1}{6}\left(1+\frac{1}{n_2}\right)wt\left(f\right) \end{cases}.$$

By $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, $wt(z_3^{n_3-2}) < \alpha_1$ and $n_2 \geq 3$, we get $\frac{19}{5} \leq n_2 \leq 5$. Therefore, we have $n_2 = 4$ or $n_2 = 5$. The restrictions of k are $k \geq n_3 - 2 = 1$ and $k\alpha_3 < \alpha_1$. Therefore, $1 \leq k < \frac{5n_2-1}{3(n_2+1)} < \frac{5}{3}$. Therefore, we have $k = 1$. Since $\alpha_1 > \alpha_3$ when $n_2 = 4$ or $n_2 = 5$, we know $k = 1$ is valid when $n_2 = 4$ or $n_2 = 5$.

Therefore, in Case 1, there exists negative weight derivation D in the form of $D = cz_3^k\frac{\partial}{\partial z_1}\,(k \geq 1, c \neq 0)$ if and only if f is in one of the following forms:

(i) $f = z_0^2z_2 + z_1^3z_3 + z_2^4 + z_3^3z_0$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D|D = cz_3\frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(ii) $f = z_0^2z_2 + z_1^3z_3 + z_2^5 + z_3^3z_0$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D|D = cz_3\frac{\partial}{\partial z_1}, c \neq 0\right\}$.

In Case 2, the weights are

$$\begin{cases} \alpha_0 = \frac{1}{2}\left(1-\frac{1}{n_2}\right)wt(f) \\ \alpha_1 = \frac{1}{24}\left(7-\frac{1}{n_2}\right)wt(f) \\ \alpha_2 = \frac{1}{n_2}wt(f) \\ \alpha_3 = \frac{1}{8}\left(1+\frac{1}{n_2}\right)wt\left(f\right) \end{cases}.$$

There does not exist any n_2 which can satisfy the restrictions $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, $wt(z_3^{n_3-2}) < \alpha_1$ and $n_2 \geq 3$ at the same time.

Therefore, in Case 2, there does not exist negative weight derivation D in the form of $D = cz_3^k\frac{\partial}{\partial z_1}\,(k \geq 1, c \neq 0)$.

In Case 3, the weights are

$$\begin{cases} \alpha_0 = \frac{1}{2}\left(1-\frac{1}{n_2}\right)wt(f) \\ \alpha_1 = \frac{1}{24}\left(5-\frac{1}{n_2}\right)wt(f) \\ \alpha_2 = \frac{1}{n_2}wt(f) \\ \alpha_3 = \frac{1}{6}\left(1+\frac{1}{n_2}\right)wt\left(f\right) \end{cases}.$$

There does not exist any n_2 which can satisfy the restrictions $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, $wt(z_3^{n_3-2}) < \alpha_1$ and $n_2 \geq 3$ at the same time.

Therefore, in Case 3, there does not exist negative weight derivation D in the form of $D = cz_3^k \frac{\partial}{\partial z_1}\,(k \geq 1, c \neq 0)$.

Therefore, we complete the proof. □

Lemma 2.53 (Case (xv) of Proposition 2.38). *Let* $(V, 0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4\colon f(z_0, z_1, z_2, z_3) = 0\}$ *be an isolated singularity defined by* $f(z_0, z_1, z_2, z_3) = z_0^2 z_3 + z_1^{n_1} z_0 + z_2^{n_2} z_1 + z_3^{n_3} (n_1 \geq 3, n_2 \geq 3)$ *of weight type* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ *where* $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$. *Let* $H_1(V)$ *be the* 1*-st Hessian algebra. Then there does not exist any negative weight derivation of* $H_1(V)$ *in the form of* $D = cz_2^k \frac{\partial}{\partial z_1}\,(k \geq 1, c \neq 0)$.

Proof. Assume that there exists some D in the form of $D = cz_2^k \frac{\partial}{\partial z_1}\,(k \geq 1, c \neq 0)$. Therefore, $\alpha_0 \geq \alpha_1 > \alpha_2 \geq \alpha_3$. From the weight relationship $n_3\alpha_3 = n_2\alpha_2 + \alpha_1 > (n_2 + 1)\,\alpha_3$, we have $n_3 > n_2 + 1$. Since $n_2 \geq 3$, we obtain $n_3 > 4$, which is equivalent to $n_3 \geq 5$.

Regardless of difference of constants and monomials in the ideal generated by elements of Hess(f), we get the equations below.

$$\text{Hess}\,(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_3 & z_1^{n_1-1} & 0 & z_0 \\ * & 0 & z_2^{n_2-1} & 0 \\ * & * & z_2^{n_2-2} z_1 & 0 \\ * & * & * & 0 \end{bmatrix}.$$

By $D\left(z_1^{n_1-1}\right) = c\,(n_1-1)\,z_1^{n_1-2} z_2^k \in (z_3)$, we obtain $c = 0$. There does not exist any negative weight derivation in the form of $D = cz_2^k \frac{\partial}{\partial z_1}\,(k \geq 1, c \neq 0)$.

Therefore, we complete the proof. □

Lemma 2.54 (Case (xvi) of Proposition 2.38). *Let* $(V, 0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4\colon f(z_0, z_1, z_2, z_3) = 0\}$ *be an isolated singularity defined by* $f(z_0, z_1, z_2, z_3) = z_0^2 z_2 + z_1^{n_1} z_0 + z_2^{n_2} + z_3^{n_3} z_1 (n_1 \geq 3, n_3 \geq 3)$ *of weight type* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ *where* $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$. *Let* $H_1(V)$ *be the* 1*-st Hessian algebra. Then there does not exist any negative weight derivation of* $H_1(V)$ *in the form of* $D = cz_3^k \frac{\partial}{\partial z_1}\,(k \geq 1, c \neq 0)$.

Proof. Assume that there exists some D in the form of $D = cz_3^k \frac{\partial}{\partial z_1}\,(k \geq 1, c \neq 0)$. From the weight relationship $n_2\alpha_2 = 2\alpha_0 + \alpha_2 \geq 3\alpha_2$, we have $n_2 \geq 3$.

Regardless of difference of constants and monomials in the ideal generated by elements of Hess(f), we get the equations below.

$$\text{Hess}\,(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_2 & z_1^{n_1-1} & z_0 & 0 \\ * & 0 & 0 & z_3^{n_3-1} \\ * & * & 0 & 0 \\ * & * & * & z_3^{n_3-2} z_1 \end{bmatrix}.$$

By $D\left(z_1^{n_1-1}\right) = c\,(n_1-1)\,z_1^{n_1-2}z_3^k \in (z_2)$, we obtain $c=0$. There does not exist any negative weight derivation in the form of $D = cz_3^k\frac{\partial}{\partial z_1}\,(k\geq 1, c\neq 0)$.

Therefore, we complete the proof. □

Lemma 2.55 (Case $(xvii)$ of Proposition 2.38). *Let $(V,0)=\{(z_0,z_1,z_2,z_3)\in \mathbb{C}^4\colon f(z_0,z_1,z_2,z_3)=0\}$ be an isolated singularity defined by $f(z_0,z_1,z_2,z_3)=z_0^2z_1+z_1^{n_1}+z_2^{n_2}z_3+z_3^{n_3}z_0(n_2\geq 3, n_3\geq 3)$ of weight type $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ where $\alpha_0\geq\alpha_1\geq\alpha_2\geq\alpha_3$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there exists negative weight derivation of $H_1(V)$ in the form of $D=cz_3^k\frac{\partial}{\partial z_2}\,(k\geq 1, c\neq 0)$ if and only if f in the form of $f=z_0^2z_1+z_1^3+z_2^3z_3+z_3^3z_0$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D|D=cz_3\frac{\partial}{\partial z_2}, c\neq 0\right\}$.*

Proof. Assume that there exists some D in the form of $D=cz_3^k\frac{\partial}{\partial z_2}\,(k\geq 1, c\neq 0)$. From the weight relationship $n_1\alpha_1 = 2\alpha_0+\alpha_1\geq 3\alpha_1$, we have $n_1\geq 3$.

Regardless of difference of constants and monomials in the ideal generated by elements of Hess(f), we get the equations below.

$$\text{Hess }(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03}\\ f_{10} & f_{11} & f_{12} & f_{13}\\ f_{20} & f_{21} & f_{22} & f_{23}\\ f_{30} & f_{31} & f_{32} & f_{33}\end{bmatrix} = \begin{bmatrix} z_1 & z_0 & 0 & z_3^{n_3-1}\\ * & 0 & 0 & 0\\ * & * & z_2^{n_2-2}z_3 & z_2^{n_2-1}\\ * & * & * & 0\end{bmatrix}.$$

From the weight relaionship

$$\begin{cases} 2\alpha_0+\alpha_1 = wt(f)\\ n_1\alpha_1 = wt(f)\\ n_2\alpha_2+\alpha_3 = wt(f)\\ n_3\alpha_3+\alpha_0 = wt(f)\end{cases},$$

we obtain

$$\begin{cases} \alpha_0 = \frac{1}{2}\left(1-\frac{1}{n_1}\right)wt(f)\\ \alpha_1 = \frac{1}{n_1}wt(f)\\ \alpha_2 = \frac{1}{n_2}\left(1-\frac{1}{2n_3}-\frac{1}{2n_1n_3}\right)wt(f)\\ \alpha_3 = \frac{1}{n_3}\left(\frac{1}{2}+\frac{1}{2n_1}\right)wt\,(f)\end{cases}.$$

It is easy to verify that $D\left(z_2^{n_2-1}\right) = c\,(n_2-1)\,z_2^{n_2-2}z_3^k \in \left(z_1,z_0,z_3^{n_3-1},z_2^{n_2-2}z_3\right)$. The only restriction of D we need to verify is that $D\left(z_2^{n_2-2}z_3\right) = c\,(n_2-2)\,z_2^{n_2-3}z_3^{k+1} \in \left(z_1,z_0,z_3^{n_3-1}\right)$. By our assumption that $c\neq 0$, we have $k\geq n_3-2$, which is equivalent to $wt(z_3^k)\geq wt(z_3^{n_3-2})$. Since D is negatively weighted, we have $wt(z_3^k)<\alpha_2$. Therefore, we have $wt(z_3^{n_3-2})<\alpha_2$.

Substituting the weights of α_2 and α_3 for it, we get

$$n_2 < \frac{1}{1-\frac{2}{n_3}}\left(\frac{1}{\frac{1}{2}+\frac{1}{2n_1}}-\frac{1}{n_3}\right) < \frac{2-\frac{1}{n_3}}{1-\frac{2}{n_3}} = 2+\frac{3}{n_3-2}.$$

If $n_2=3$, we obtain $3<2+\frac{3}{n_3-2}$, which is equivalent to $n_3<5$. Note that $n_3\geq 3$, we get $n_3=3$ or $n_3=4$ when $n_2=3$.

If $n_2=4$, we obtain $4<2+\frac{3}{n_3-2}$, which is equivalent to $n_3<\frac{7}{2}$. Note that $n_3\geq 3$, we get $n_3=3$ when $n_2=4$.

If $n_2 \geq 5$, we obtain $5 < 2 + \frac{3}{n_3-2}$, which is equivalent to $n_3 < 3$. Note that $n_3 \geq 3$, we get a contradiction when $n_2 \geq 5$.

There are 3 cases left:

Case 1: $n_2 = 3$, $n_3 = 3$;

Case 2: $n_2 = 3$, $n_3 = 4$;

Case 3: $n_2 = 4$, $n_3 = 3$.

In Case 1, the weights are

$$\begin{cases} \alpha_0 = \frac{1}{2}\left(1 - \frac{1}{n_1}\right) wt(f) \\ \alpha_1 = \frac{1}{n_1} wt(f) \\ \alpha_2 = \frac{1}{18}\left(5 - \frac{1}{n_1}\right) wt(f) \\ \alpha_3 = \frac{1}{6}\left(1 + \frac{1}{n_1}\right) wt(f) \end{cases}.$$

By $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, $wt(z_3^{n_3-2}) < \alpha_2$ and $n_1 \geq 3$, we get $3 \leq n_1 \leq \frac{19}{5}$. Therefore, we have $n_1 = 3$. The restrictions of k are $k \geq n_3 - 2 = 1$ and $k\alpha_3 < \alpha_2$. Therefore, $1 \leq k < \frac{7}{6}$ and we have $k = 1$.

Therefore, in Case 1, there exists negative weight derivation D in the form of $D = cz_3^k \frac{\partial}{\partial z_2}\,(k \geq 1, c \neq 0)$ if and only if f in the form of $f = z_0^2 z_1 + z_1^3 + z_2^3 z_3 + z_3^3 z_0$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D | D = cz_3 \frac{\partial}{\partial z_2}, c \neq 0\right\}$.

In Case 2, the weights are

$$\begin{cases} \alpha_0 = \frac{1}{2}\left(1 - \frac{1}{n_1}\right) wt(f) \\ \alpha_1 = \frac{1}{n_1} wt(f) \\ \alpha_2 = \frac{1}{24}\left(7 - \frac{1}{n_1}\right) wt(f) \\ \alpha_3 = \frac{1}{8}\left(1 + \frac{1}{n_1}\right) wt(f) \end{cases}.$$

There does not exist any n_1 which can satisfy the restrictions $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, $wt(z_3^{n_3-2}) < \alpha_2$ and $n_1 \geq 3$ at the same time.

Therefore, in Case 2, there does not exist negative weight derivation D in the form of $D = cz_3^k \frac{\partial}{\partial z_2}\,(k \geq 1, c \neq 0)$.

In Case 3, the weights are

$$\begin{cases} \alpha_0 = \frac{1}{2}\left(1 - \frac{1}{n_1}\right) wt(f) \\ \alpha_1 = \frac{1}{n_1} wt(f) \\ \alpha_2 = \frac{1}{24}\left(5 - \frac{1}{n_1}\right) wt(f) \\ \alpha_3 = \frac{1}{6}\left(1 + \frac{1}{n_1}\right) wt(f) \end{cases}.$$

There does not exist any n_1 which can satisfy the restrictions $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, $wt(z_3^{n_3-2}) < \alpha_2$ and $n_1 \geq 3$ at the same time.

Therefore, in Case 3, there does not exist negative weight derivation D in the form of $D = cz_3^k \frac{\partial}{\partial z_2}\,(k \geq 1, c \neq 0)$.

Therefore, we complete the proof. □

LEMMA 2.56 (Case (*xviii*) of Proposition 2.38). *Let* $(V, 0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 \colon f(z_0, z_1, z_2, z_3) = 0\}$ *be an isolated singularity defined by* $f(z_0, z_1, z_2, z_3) = z_0^2 z_1 + z_1^{n_1} + z_2^{n_2} z_0 + z_3^{n_3} z_2 (n_2 \geq 3, n_3 \geq 3)$ *of weight type* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ *where* $\alpha_0 \geq$

$\alpha_1 \geq \alpha_2 \geq \alpha_3$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there does not exist any negative weight derivation of $H_1(V)$ in the form of $D = cz_3^k \frac{\partial}{\partial z_2}$ $(k \geq 1, c \neq 0)$.

Proof. Assume that there exists some D in the form of $D = cz_3^k \frac{\partial}{\partial z_2}$ $(k \geq 1, c \neq 0)$. From the weight relationship $n_1\alpha_1 = 2\alpha_0 + \alpha_1 \geq 3\alpha_1$, we have $n_1 \geq 3$.

Regardless of difference of constants and monomials in the ideal generated by elements of Hess(f), we get the equations below.

$$\text{Hess } (f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_1 & z_0 & z_2^{n_2-1} & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & z_3^{n_3-1} \\ * & * & * & z_3^{n_3-2}z_2 \end{bmatrix}.$$

By $D\left(z_2^{n_2-1}\right) = c\,(n_2-1)\,z_2^{n_2-2}z_3^k \in (z_1, z_0)$, we obtain $c = 0$. There does not exist any negative weight derivation in the form of $D = cz_3^k \frac{\partial}{\partial z_2}$ $(k \geq 1, c \neq 0)$.

Therefore, we complete the proof. □

2.3. Type (III). Next we will discuss the case

$$f = z_0^{n_0}z_1 + z_1^{n_1}z_2 + z_2^{n_2}z_3 + z_3^{n_3}z_0$$

where $mult\,(f) \geq 3$. The weight order of $\alpha_0, \alpha_1, \alpha_2$ and α_3 is not determined. All results of this subsection are summarized in Proposition 2.57.

PROPOSITION 2.57 (Type (III) of Proposition 2.1). *Let $(V, 0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 : f(z_0, z_1, z_2, z_3) = 0\}$ be an isolated singularity defined by $f(z_0, z_1, z_2, z_3) = z_0^{n_0}z_1 + z_1^{n_1}z_2 + z_2^{n_2}z_3 + z_3^{n_3}z_0$ of weight type $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ where $mult\,(f) \geq 3$. Let $H_1(V)$ be the 1-st Hessian algebra. There exists negative weight derivation of $H_1(V)$ if and only if f is in one of the following forms after renumbering the variables z_0, z_1, z_2 and z_3 so that $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$:*

(i) $f = z_0^2z_2 + z_1^3z_3 + z_2^3z_1 + z_3^3z_0$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D | D = cz_3\frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(ii) $f = z_0^2z_3 + z_1^3z_2 + z_2^3z_0 + z_3^{n_3}z_1\,(n_3 \geq 4)$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D | D = cz_2\frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(iii) $f = z_0^2z_3 + z_1^4z_2 + z_2^3z_0 + z_3^{n_3}z_1\,(n_3 \geq 5)$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D | D = cz_2\frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(iv) $f = z_0^2z_3 + z_1^3z_2 + z_2^4z_0 + z_3^{n_3}z_1\,(n_3 \geq 6)$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D | D = cz_2^2\frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(v) $f = z_0^3z_3 + z_1^3z_2 + z_2^3z_0 + z_3^{n_3}z_1\,(n_3 \geq 24)$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D | D = cz_2z_3^k\frac{\partial}{\partial z_1}, c \neq 0, 1 \leq k < \frac{n_3-3}{20}, k \in \mathbb{Z}\right\}$.

Proof. After renumbering, the problem is divided into 6 cases, each of which satisfies $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$:

(i) $f = z_0^{n_0}z_1 + z_1^{n_1}z_2 + z_2^{n_2}z_3 + z_3^{n_3}z_0\,(n_0 \geq 2)$;
(ii) $f = z_0^{n_0}z_1 + z_1^{n_1}z_3 + z_2^{n_2}z_0 + z_3^{n_3}z_2\,(n_0 \geq 2)$;
(iii) $f = z_0^{n_0}z_2 + z_1^{n_1}z_3 + z_2^{n_2}z_1 + z_3^{n_3}z_0\,(n_0 \geq 2)$;
(iv) $f = z_0^{n_0}z_3 + z_1^{n_1}z_2 + z_2^{n_2}z_0 + z_3^{n_3}z_1\,(n_0 \geq 2)$;
(v) $f = z_0^{n_0}z_2 + z_1^{n_1}z_0 + z_2^{n_2}z_3 + z_3^{n_3}z_1\,(n_0 \geq 2)$;
(vi) $f = z_0^{n_0}z_3 + z_1^{n_1}z_0 + z_2^{n_2}z_1 + z_3^{n_3}z_2\,(n_0 \geq 2)$.

There are $4! = 24$ cases of weight relations.

Case (i) contains the original weight relationship $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3, \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_0, \alpha_2 \geq \alpha_3 \geq \alpha_0 \geq \alpha_1$ and $\alpha_3 \geq \alpha_0 \geq \alpha_1 \geq \alpha_2$.

Case (ii) contains the original weight relationship $\alpha_0 \geq \alpha_1 \geq \alpha_3 \geq \alpha_2, \alpha_1 \geq \alpha_2 \geq \alpha_0 \geq \alpha_3, \alpha_2 \geq \alpha_3 \geq \alpha_1 \geq \alpha_0$ and $\alpha_3 \geq \alpha_0 \geq \alpha_2 \geq \alpha_1$.

Case (iii) contains the original weight relationship $\alpha_0 \geq \alpha_2 \geq \alpha_1 \geq \alpha_3, \alpha_1 \geq \alpha_3 \geq \alpha_2 \geq \alpha_0, \alpha_2 \geq \alpha_0 \geq \alpha_3 \geq \alpha_1$ and $\alpha_3 \geq \alpha_1 \geq \alpha_0 \geq \alpha_2$.

Case (iv) contains the original weight relationship $\alpha_0 \geq \alpha_2 \geq \alpha_3 \geq \alpha_1, \alpha_1 \geq \alpha_3 \geq \alpha_0 \geq \alpha_2, \alpha_2 \geq \alpha_0 \geq \alpha_1 \geq \alpha_3$ and $\alpha_3 \geq \alpha_1 \geq \alpha_2 \geq \alpha_0$.

Case (v) contains the original weight relationship $\alpha_0 \geq \alpha_3 \geq \alpha_1 \geq \alpha_2, \alpha_1 \geq \alpha_0 \geq \alpha_2 \geq \alpha_3, \alpha_2 \geq \alpha_1 \geq \alpha_3 \geq \alpha_0$ and $\alpha_3 \geq \alpha_2 \geq \alpha_0 \geq \alpha_1$.

Case (vi) contains the original weight relationship $\alpha_0 \geq \alpha_3 \geq \alpha_2 \geq \alpha_1, \alpha_1 \geq \alpha_0 \geq \alpha_3 \geq \alpha_2, \alpha_2 \geq \alpha_1 \geq \alpha_0 \geq \alpha_3$ and $\alpha_3 \geq \alpha_2 \geq \alpha_1 \geq \alpha_0$.

The discussion about the 6 cases is rather trivial and occupies a certain space. One can check the following lemmas (from Lemma 2.58 to Lemma 2.63) for more details. $\square$

Lemma 2.58 (Case (i) of Proposition 2.57). *Let $(V, 0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 \colon f(z_0, z_1, z_2, z_3) = 0\}$ be an isolated singularity defined by $f(z_0, z_1, z_2, z_3) = z_0^{n_0} z_1 + z_1^{n_1} z_2 + z_2^{n_2} z_3 + z_3^{n_3} z_0 \, (n_0 \geq 2)$ of weight type $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ where $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there does not exist negative weight derivation of $H_1(V)$.*

Proof. If there exists some negative weight derivation D, D must be in the form of $D = p_0(z_1, z_2, z_3)\frac{\partial}{\partial z_0} + p_1(z_2, z_3)\frac{\partial}{\partial z_1} + c z_3^k \frac{\partial}{\partial z_2}$.

When $n_0 \geq 3$ holds, we obtain

$$2\alpha_i + \alpha_j \leq 3\alpha_0 < wt(f) = n_i \alpha_i + \alpha_j$$

for $(i, j) = (1, 2), (2, 3)$ or $(3, 0)$. Then $n_1 > 2$, $n_2 > 2$ and $n_3 > 2$. Thus $n_1 \geq 3$, $n_2 \geq 3$ and $n_3 \geq 3$. Regardless of difference of constants, we get

$$\mathrm{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0^{n_0-2} z_1 & z_0^{n_0-1} & 0 & z_3^{n_3-1} \\ * & z_1^{n_1-2} z_2 & z_1^{n_1-1} & 0 \\ * & * & z_2^{n_2-2} z_3 & z_2^{n_2-1} \\ * & * & * & z_3^{n_3-2} z_0 \end{bmatrix}.$$

From

$$D\left(z_0^{n_0-2} z_1\right) = p_0\left(z_1, z_2, z_3\right)(n_0 - 2) z_0^{n_0-3} z_1 + p_1\left(z_2, z_3\right) z_0^{n_0-2} = 0,$$

we obtain

$$p_0\left(z_1, z_2, z_3\right)(n_0 - 2) z_1 + p_1\left(z_2, z_3\right) z_0 = 0.$$

Therefore, $p_1\left(z_2, z_3\right) = 0$ and $p_0\left(z_1, z_2, z_3\right) = 0$. So $D = c z_3^k \frac{\partial}{\partial z_2}$.

From

$$D\left(z_1^{n_1-2} z_2\right) = c z_3^k z_1^{n_1-2} \in \left(z_0^{n_0-2} z_1, z_0^{n_0-1}, z_3^{n_3-1}\right),$$

we obtain

$$c z_3^k z_1^{n_1-2} \in \left(z_3^{n_3-1}\right).$$

If $c \neq 0$, it is clear that $wt\left(z_3^k\right) \geq wt\left(z_3^{n_3-1}\right)$. However, we can also see

$$wt\left(z_3^{n_3-1}\right) = wt(f) - \alpha_0 - \alpha_3 \geq n_0\alpha_0 + \alpha_1 - \alpha_0 - \alpha_0 = (n_0-2)\alpha_0 + \alpha_1 \geq \alpha_0 + \alpha_1 > \alpha_0,$$

while $wt\left(z_3^k\right) < \alpha_2 \leq \alpha_0$. We obtain $wt\left(z_3^{n_3-1}\right) > wt\left(z_3^k\right)$. This leads to a contradiction. Thus $c = 0$ and $D = 0$. Therefore, for any $f = z_0^{n_0}z_1 + z_1^{n_1}z_2 + z_2^{n_2}z_3 + z_3^{n_3}z_0$, when $n_0 \geq 3$, there does not exist negative weight derivation of $H_1(V)$.

When $n_0 = 2$ holds, f is in the form of $f = z_0^2 z_1 + z_1^{n_1}z_2 + z_2^{n_2}z_3 + z_3^{n_3}z_0$. From

$$\alpha_i + \alpha_j \leq 2\alpha_0 < wt(f) = n_i\alpha_i + \alpha_j$$

for $(i,j) = (1,2),(2,3)$ or $(3,0)$, we get $n_1 > 1$, $n_2 > 1$ and $n_3 > 1$. Thus $n_1 \geq 2$, $n_2 \geq 2$ and $n_3 \geq 2$. Regardless of difference of constants and monomials in the ideal generated by elements of Hess(f), we get the equations below.

$$\operatorname{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_1 & z_0 & 0 & z_3^{n_3-1} \\ * & z_1^{n_1-2}z_2 & z_1^{n_1-1} & 0 \\ * & * & z_2^{n_2-2}z_3 & z_2^{n_2-1} \\ * & * & * & 0 \end{bmatrix}.$$

Since z_1 and z_0 is an element of Hess(f), there does not exist nonzero element in $H_1(V)$ which is divided by z_1 or z_0. So D is in the form of $D = cz_3^k\frac{\partial}{\partial z_2}$.

If $c \neq 0$,

$$D\left(z_1^{n_1-2}z_2\right) = cz_3^k z_1^{n_1-2} \in \left(z_1, z_0, z_3^{n_3-1}\right)$$

is equivalent to

$$z_3^k z_1^{n_1-2} \in \left(z_1, z_3^{n_3-1}\right).$$

Since

$$wt\left(z_3^{n_3-1}\right) = wt(f) - \alpha_0 - \alpha_3 = 2\alpha_0 + \alpha_1 - \alpha_0 - \alpha_3 \geq \alpha_0$$

and

$$wt\left(z_3^k\right) < \alpha_2 \leq \alpha_0,$$

we obtain $wt\left(z_3^{n_3-1}\right) > wt\left(z_3^k\right)$ and $n_3 - 1 > k$. We can see that z_3^k cannot be divided by $z_3^{n_3-1}$ so $z_3^k z_1^{n_1-2} \in (z_1)$. Therefore, $D\left(z_1^{n_1-2}z_2\right) \in \left(z_1, z_0, z_3^{n_3-1}\right)$ is equivalent to $z_3^k z_1^{n_1-2} \in (z_1)$, or $n_1 \geq 3$.

Note that $D\left(z_2^{n_2-2}z_3\right)$ should satisfy

$$D\left(z_2^{n_2-2}z_3\right) = c\left(n_2-2\right)z_2^{n_2-3}z_3^{k+1} \in \left(z_1, z_0, z_3^{n_3-1}, z_1^{n_1-2}z_2, z_1^{n_1-1}\right),$$

which is equivalent to

$$c\left(n_2-2\right)z_2^{n_2-3}z_3^{k+1} \in \left(z_3^{n_3-1}\right).$$

From $c \neq 0$, we have $k+1 \geq n_3 - 1$, from which we obtain $k \geq n_3 - 2$.

If $k \geq n_3 - 1$, we have $z_3^k \in \left(z_1, z_0, z_3^{n_3-1}\right)$. Thus $D = 0$, which is equivalent to $c = 0$. If D is negatively weighted, k has to be equal to $n_3 - 2$. If $\alpha_0 = \alpha_2$, we have $n_1 = 2$, which contradicts to the conclusion that $n_1 \geq 3$. So we get $\alpha_0 > \alpha_2 \geq \alpha_3$.

From

$$n_1\alpha_2 + \alpha_3 \leq n_1\alpha_1 + \alpha_2 = n_2\alpha_2 + \alpha_3,$$

we obtain $n_2 \geq n_1 \geq 3$. Since

$$D\left(z_2^{n_2-1}\right) = cz_3^k\left(n_2-1\right)z_2^{n_2-2} \in \left(z_1, z_0, z_3^{n_3-1}, z_1^{n_1-2}z_2, z_1^{n_1-1}, z_2^{n_2-2}z_3\right),$$

the relation

$$cz_3^k\left(n_2-1\right)z_2^{n_2-2} \in \left(z_3^{n_3-1}, z_2^{n_2-2}z_3\right)$$

is obtained.

From the weight relationship $\alpha_0 + \alpha_3 < 2\alpha_0 < wt(f) = n_3\alpha_3 + \alpha_0$, we obtain $n_3 > 1$. Thus $n_3 \geq 2$ and $z_3^{n_3-1}$ is divided by z_3. Therefore, cz_3^k is divided by z_3. From the assumption $c \neq 0$, we obtain $k = n_3 - 2 \geq 1$ and $n_3 \geq 3$. If that is the case, we get $z_3^k\left(n_2-1\right)z_2^{n_2-2} \in \left(z_2^{n_2-2}z_3\right)$. Therefore the condition $D\left(z_2^{n_2-1}\right) = cz_3^k\left(n_2-1\right)z_2^{n_2-2} \in \left(z_1, z_0, z_3^{n_3-1}, z_1^{n_1-2}z_2, z_1^{n_1-1}, z_2^{n_2-2}z_3\right)$ is satisfied.

The only thing we need to verify is $D = cz_3^{n_3-2}\frac{\partial}{\partial z_2}(c \neq 0)$ is negatively weighted. If it is negatively weighted, we obtain $\alpha_0 > \alpha_2 > \alpha_3$.

From $\alpha_2 > wt\left(z_3^{n_3-2}\right) = \left(1 - \frac{2}{n_3}\right)wt\left(z_3^{n_3}\right) = \left(1 - \frac{2}{n_3}\right)\left(wt(f) - \alpha_0\right) = \left(1 - \frac{2}{n_3}\right)\left(\alpha_1 + \alpha_0\right) > 2\left(1 - \frac{2}{n_3}\right)\alpha_2$, we obtain $n_3 < 4$. Thus $n_3 = 3$. From $wt(f) = 2\alpha_0 + \alpha_1 > 2\alpha_0$, we obtain $\alpha_0 < \frac{1}{2}wt(f)$. From $wt(f) = 2\alpha_0 + \alpha_1 \leq 3\alpha_0$, we obtain $\alpha_0 \geq \frac{1}{3}wt(f)$. Thus $\alpha_3 = \frac{1}{3}\left(wt(f) - \alpha_0\right) \in \left(\frac{1}{6}wt(f), \frac{2}{9}wt(f)\right]$. From $wt(f) = n_2\alpha_2 + \alpha_3 > \left(n_2+1\right)\alpha_3 > \frac{n_2+1}{6}wt(f)$, we obtain $n_2 < 5$. Therefore, $n_2 \leq 4$. From $n_1\alpha_1 + \alpha_2 = n_2\alpha_2 + \alpha_3 < n_2\alpha_1 + \alpha_2$, we obtain $n_1 < n_2 \leq 4$. Therefore, $n_1 \leq 3$. From $n_1 \geq 3$, we obtain $n_1 = 3$. From $3 = n_1 < n_2 \leq 4$, we obtain $n_2 = 4$. Thus f can only be in the form of $f = z_0^2z_1 + z_1^3z_2 + z_2^4z_3 + z_3^3z_0$.

From the weight relationship

$$\begin{cases} 2\alpha_0 + \alpha_1 = wt(f) \\ 3\alpha_1 + \alpha_2 = wt(f) \\ 4\alpha_2 + \alpha_3 = wt(f) \\ 3\alpha_3 + \alpha_0 = wt(f) \end{cases},$$

we obtain

$$\begin{cases} \alpha_0 = \frac{26}{71}wt(f) \\ \alpha_1 = \frac{19}{71}wt(f) \\ \alpha_2 = \frac{14}{71}wt(f) \\ \alpha_3 = \frac{15}{71}wt(f) \end{cases}.$$

We can see that $\alpha_2 < \alpha_3$, which is in contradiction to $\alpha_2 > \alpha_3$.

Therefore, there does not exist negative weight derivation of $H_1(V)$ when $n_0 = 2$ for any $f = z_0^{n_0}z_1 + z_1^{n_1}z_2 + z_2^{n_2}z_3 + z_3^{n_3}z_0$.

In conclusion, there does not exist negative weight derivation of $H_1(V)$ for any $f = z_0^{n_0}z_1 + z_1^{n_1}z_2 + z_2^{n_2}z_3 + z_3^{n_3}z_0\,(n_0 \geq 2)$. ☐

LEMMA 2.59 (Case (ii) of Proposition 2.57). *Let* $(V, 0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 : f(z_0, z_1, z_2, z_3) = 0\}$ *be an isolated singularity defined by* $f(z_0, z_1, z_2, z_3) = z_0^{n_0}z_1 + z_1^{n_1}z_3 + z_2^{n_2}z_0 + z_3^{n_3}z_2\,(n_0 \geq 2)$ *of weight type* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ *where* $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq$

α_3. Let $H_1(V)$ be the 1-st Hessian algebra. Then there does not exist negative weight derivation of $H_1(V)$.

Proof. If there exists some negative weight derivation D, D must be in the form of $D = p_0(z_1, z_2, z_3)\frac{\partial}{\partial z_0} + p_1(z_2, z_3)\frac{\partial}{\partial z_1} + cz_3^k\frac{\partial}{\partial z_2}$.

When $n_0 \geq 3$ holds, from $2\alpha_i + \alpha_j \leq 3\alpha_0 < wt(f) = n_i\alpha_i + \alpha_j$, we get $n_i > 2$ for $(i,j) = (1,3), (2,0)$ or $(3,2)$. Then $n_1 > 2$, $n_2 > 2$ and $n_3 > 2$. Thus $n_1 \geq 3$, $n_2 \geq 3$ and $n_3 \geq 3$. Regardless of difference of constants, we obtain

$$\mathrm{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0^{n_0-2}z_1 & z_0^{n_0-1} & z_2^{n_2-1} & 0 \\ * & z_1^{n_1-2}z_3 & 0 & z_1^{n_1-1} \\ * & * & z_2^{n_2-2}z_0 & z_3^{n_3-1} \\ * & * & * & z_3^{n_3-2}z_2 \end{bmatrix}.$$

Since

$$D\left(z_0^{n_0-2}z_1\right) = p_0\left(z_1, z_2, z_3\right)(n_0-2)z_0^{n_0-3}z_1 + p_1\left(z_2, z_3\right)z_0^{n_0-2} = 0,$$

we obtain the equation

$$p_0\left(z_1, z_2, z_3\right)(n_0-2)z_1 + p_1\left(z_2, z_3\right)z_0 = 0.$$

Thus $p_1\left(z_2, z_3\right) = 0$ and $p_0\left(z_1, z_2, z_3\right) = 0$. So $D = cz_3^k\frac{\partial}{\partial z_2}$.

From

$$D\left(z_2^{n_2-1}\right) = cz_3^k\left(n_2-1\right)z_2^{n_2-2} \in \left(z_0^{n_0-2}z_1, z_0^{n_0-1}, z_1^{n_1-2}z_3\right),$$

we obtain $c = 0$. Thus $D = 0$, which contradicts to the assumption that D is negatively weighted. Therefore, when $n_0 \geq 3$, for any $f = z_0^{n_0}z_1 + z_1^{n_1}z_3 + z_2^{n_2}z_0 + z_3^{n_3}z_2$, there does not exist any negative weight derivation of $H_1(V)$.

When $n_0 = 2$ holds, f is in the form of $f = z_0^2z_1 + z_1^{n_1}z_3 + z_2^{n_2}z_0 + z_3^{n_3}z_2$. From

$$\alpha_i + \alpha_j \leq 2\alpha_0 < wt(f) = n_i\alpha_i + \alpha_j$$

for $(i,j) = (1,3), (2,0)$ or $(3,2)$, we get $n_1 > 1$, $n_2 > 1$ and $n_3 > 1$. Thus $n_1 \geq 2$, $n_2 \geq 2$ and $n_3 \geq 2$.

Regardless of difference of constants and monomials in the ideal generated by elements of Hess(f), we get the equations below.

$$\mathrm{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_1 & z_0 & z_2^{n_2-1} & 0 \\ * & z_1^{n_1-2}z_3 & 0 & 0 \\ * & * & 0 & z_3^{n_3-1} \\ * & * & * & z_3^{n_3-2}z_2 \end{bmatrix}$$

Since z_0 and z_1 are in the ideal generated by elements of Hess(f), the negative weight derivation is in the form of $D = cz_3^k\frac{\partial}{\partial z_2}$. If $\alpha_0 = \alpha_1 = \alpha_2$, we have $n_2 = 2$. Thus z_2 is in the ideal generated by elements of Hess(f). There does not exist any nonzero monomial in $H_1(V)$ that is divided by z_2. Thus $D = 0$.

Otherwise we obtain $\alpha_0 > \alpha_2 \geq \alpha_3$.

From

$$n_1\alpha_1 + \alpha_3 = 2\alpha_0 + \alpha_1 > \alpha_0 + \alpha_1 + \alpha_3 \geq 2\alpha_1 + \alpha_3,$$

we obtain $n_1 > 2$. Therefore, $n_1 \geq 3$.

From

$$D\left(z_2^{n_2-1}\right) = (n_2-1)\, c z_3^k z_2^{n_2-2} \in \left(z_1, z_0, z_1^{n_1-2} z_3\right) = (z_1, z_0)\,,$$

we obtain $c = 0$. Thus $D = 0$.

Therefore, when $n_0 = 2$, for any $f = z_0^{n_0} z_1 + z_1^{n_1} z_3 + z_2^{n_2} z_0 + z_3^{n_3} z_2$, there does not exist negative weight derivation of $H_1(V)$.

In conclusion, there does not exist negative weight derivation of $H_1(V)$ for any $f = z_0^{n_0} z_1 + z_1^{n_1} z_3 + z_2^{n_2} z_0 + z_3^{n_3} z_2\, (n_0 \geq 2)$. □

LEMMA 2.60 (Case (iii) of Proposition 2.57). *Let $(V, 0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 \colon f(z_0, z_1, z_2, z_3) = 0\}$ be an isolated singularity defined by $f(z_0, z_1, z_2, z_3) = z_0^{n_0} z_2 + z_1^{n_1} z_3 + z_2^{n_2} z_1 + z_3^{n_3} z_0\, (n_0 \geq 2)$ of weight type $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ where $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there exists negative weight derivation of $H_1(V)$ if and only if f is in the form of $f = z_0^2 z_2 + z_1^3 z_3 + z_2^3 z_1 + z_3^3 z_0$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D | D = c z_3 \frac{\partial}{\partial z_1}, c \neq 0\right\}$.*

Proof. If there exists some negative weight derivation D, D must be in the form of $D = p_0(z_1, z_2, z_3) \frac{\partial}{\partial z_0} + p_1(z_2, z_3) \frac{\partial}{\partial z_1} + c z_3^k \frac{\partial}{\partial z_2}$.

When $n_0 \geq 3$ holds, from $2\alpha_i + \alpha_j \leq 3\alpha_0 < wt(f) = n_i \alpha_i + \alpha_j$, we get $n_i > 2$ for $(i, j) = (1, 3), (2, 1)$ or $(3, 0)$. Then $n_1 > 2$, $n_2 > 2$ and $n_3 > 2$, which is equivalent to $n_1 \geq 3$, $n_2 \geq 3$ and $n_3 \geq 3$. Regardless of difference of constants, we obtain

$$\mathrm{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0^{n_0-2} z_2 & 0 & z_0^{n_0-1} & z_3^{n_3-1} \\ * & z_1^{n_1-2} z_3 & z_2^{n_2-1} & z_1^{n_1-1} \\ * & * & z_2^{n_2-2} z_1 & 0 \\ * & * & * & z_3^{n_3-2} z_0 \end{bmatrix}.$$

From

$$D\left(z_0^{n_0-2} z_2\right) = p_0(z_1, z_2, z_3)\, (n_0-2) z_0^{n_0-3} z_2 + c z_3^k z_0^{n_0-2} = 0,$$

we obtain

$$p_0(z_1, z_2, z_3)\, (n_0-2) z_2 + c z_3^k z_0 = 0.$$

Therefore, $c = 0$, $p_0(z_1, z_2, z_3) = 0$ and $D = p_1(z_2, z_3) \frac{\partial}{\partial z_1}$. From

$$D\left(z_1^{n_1-2} z_3\right) = p_1(z_2, z_3)\, (n_1-2)\, z_1^{n_1-3} z_3 \in \left(z_0^{n_0-2} z_2, z_0^{n_0-1}, z_3^{n_3-1}\right),$$

we obtain

$$p_1(z_2, z_3)\, z_1^{n_1-3} \in \left(z_3^{n_3-2}\right).$$

If $p_1(z_2, z_3) \neq 0$, we get $p_1(z_2, z_3)$ contains factor $z_3^{n_3-2}$. Then we obtain the equation $\alpha_0 \geq \alpha_1 > wt(p_1(z_2, z_3)) \geq wt\left(z_3^{n_3-2}\right) = \frac{n_3-2}{n_3}(wt(f) - \alpha_0) = \frac{n_3-2}{n_3}(n_0\alpha_0 + \alpha_2 - \alpha_0) \geq \frac{n_3-2}{n_3}(2\alpha_0 + \alpha_2) > \frac{n_3-2}{n_3} 2\alpha_0$. Then $n_3 < 4$, therefore, $n_3 = 3$.

Since $p_1(z_2, z_3)$ contains factor $z_3^{n_3-2} = z_3$, we get $\alpha_3 \leq wt(p_1(z_2, z_3)) < \alpha_1 \leq \alpha_0$. From the relation $n_0\alpha_0 + \alpha_2 = 3\alpha_3 + \alpha_0 < 3\alpha_0 + \alpha_2$, we get $n_0 < 3$, which is in contradiction to the assumption $n_0 \geq 3$. Thus $p_1(z_2, z_3) = 0$ and $D = 0$.

Therefore, when $n_0 \geq 3$, for any $f = z_0^{n_0} z_2 + z_1^{n_1} z_3 + z_2^{n_2} z_1 + z_3^{n_3} z_0$, there does not exist any negative weight derivation of $H_1(V)$.

When $n_0 = 2$ holds, we obtain

$$f = z_0^2 z_2 + z_1^{n_1} z_3 + z_2^{n_2} z_1 + z_3^{n_3} z_0.$$

From the relation

$$\alpha_i + \alpha_j \leq 2\alpha_0 < wt(f) = n_i\alpha_i + \alpha_j$$

for $(i,j) = (1,3), (2,1)$ or $(3,0)$, we obtain $n_1 > 1$, $n_2 > 1$ and $n_3 > 1$. Therefore, $n_1 \geq 2$, $n_2 \geq 2$ and $n_3 \geq 2$.

Thus

$$\mathrm{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_2 & 0 & z_0 & z_3^{n_3-1} \\ * & z_1^{n_1-2} z_3 & 0 & z_1^{n_1-1} \\ * & * & z_2^{n_2-2} z_1 & 0 \\ * & * & * & 0 \end{bmatrix}$$

regardless of difference of constants and useless monomials. It is clear that z_0 and z_2 are in the ideal generated by elements of Hess(f). There does not exist nonzero element in $H_1(V)$ which is divided by z_0 and z_2. Thus $D = c_1 z_3^{k_1} \frac{\partial}{\partial z_1}$.

If $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3$, we obtain $n_1 = n_2 = n_3 = 2$. f_{13} and f_{22} are in proportion to z_1. z_1 is in the ideal generated by elements of Hess(f). There does not exist nonzero element in $H_1(V)$ which can be divided by z_1. Thus $D = 0$.

Otherwise we have $\alpha_0 > \alpha_3$. In this case,

$$\alpha_0 + n_3\alpha_3 = 2\alpha_0 + \alpha_2 > \alpha_0 + \alpha_2 + \alpha_3 \geq \alpha_0 + 2\alpha_3.$$

Thus $n_3 > 2$, which is equivalent to $n_3 \geq 3$.

If $n_1 = 2$, f_{13} is in proportion to z_1. In this case, z_1 is in the ideal generated by elements of Hess(f). There does not exist any nonzero element in $H_1(V)$ which is divided by z_1. Thus $D = 0$.

If $n_1 \geq 3$, we obtain

$$D\left(z_1^{n_1-2} z_3\right) = (n_1 - 2)\, c_1 z_3^{k_1+1} z_1^{n_1-3} \in \left(z_2, z_0, z_3^{n_3-1}\right),$$

which is equivalent to

$$c_1 z_3^{k_1+1} z_1^{n_1-3} \in \left(z_2, z_0, z_3^{n_3-1}\right).$$

If $D \neq 0$, it is clear that $c_1 \neq 0$. Therefore, we have $k_1 + 1 \geq n_3 - 1$. So $k_1 \geq n_3 - 2$. If $k_1 \geq n_3 - 1$, $z_3^{k_1}$ is in the ideal generated by elements of Hess(f) and $D = 0$. This leads to a contradiction. Thus $k_1 < n_3 - 1 \leq k_1 + 1$. Therefore, we get $n_3 - 1 = k_1 + 1$ and $k_1 = n_3 - 2$. D is in the form of $D = c_1 z_3^{n_3-2} \frac{\partial}{\partial z_1}$.

In the following discussions, we assume $c_1 \neq 0$.

Consider the restriction

$$D\left(z_1^{n_1-1}\right) = (n_1 - 1)\, c z_3^{n_3-2} z_1^{n_1-2} \in \left(z_2, z_0, z_3^{n_3-1}, z_1^{n_1-2} z_3, z_2^{n_2-2} z_1\right).$$

Since $n_3 \geq 3$, we obtain $z_3^{n_3-2} z_1^{n_1-2} \in \left(z_1^{n_1-2} z_3\right)$. The restriction is satisfied.

Consider the restriction

$$D\left(z_2^{n_2-2} z_1\right) = c_1 z_3^{n_3-2} z_2^{n_2-2} \in \left(z_2, z_0, z_3^{n_3-1}, z_1^{n_1-2} z_3, z_1^{n_1-1}\right).$$

It is equivalent to

$$z_3^{n_3-2} z_2^{n_2-2} \in \left(z_2, z_1^{n_1-2} z_3\right).$$

If $n_2 = 2$, f_{22} is in proportion to z_1. Therefore, z_1 is in the ideal generated by elements of $\mathrm{Hess}(f)$. There does not exist nonzero element in $H_1(V)$ which is divided by z_1.

If $n_2 \geq 3$, we obtain $z_3^{n_3-2} z_2^{n_2-2} \in (z_2)$. The restriction is satisfied.

Therefore, the restriction $D\left(z_2^{n_2-2} z_1\right) = c_1 z_3^{n_3-2} z_2^{n_2-2} \in \left(z_2, z_0, z_3^{n_3-1}, z_1^{n_1-2} z_3, z_1^{n_1-1}\right)$ is satisfied if and only if $n_2 \geq 3$.

$D = c_1 z_3^{n_3-2} \frac{\partial}{\partial z_1}$ is negative weight if and only if $wt\left(z_3^{n_3-2}\right) < \alpha_1$.

If $\alpha_0 = \alpha_1 = \alpha_2$, we have $n_2 = 2$, which contradicts to the assumption $n_2 \geq 3$. Thus we have $\alpha_0 > \alpha_2$. Since $2\alpha_0 < 2\alpha_0 + \alpha_2 = wt(f) < 3\alpha_0$, we obtain $\alpha_0 \in \left(\frac{1}{3} wt(f), \frac{1}{2} wt(f)\right)$. Therefore we have $wt\left(z_3^{n_3}\right) = wt(f) - \alpha_0 \in \left(\frac{1}{2} wt(f), \frac{2}{3} wt(f)\right)$, which is equivalent to $\alpha_3 \in \left(\frac{1}{2n_3} wt(f), \frac{2}{3n_3} wt(f)\right)$.

Therefore,

$$wt\left(z_3^{n_3-2}\right) = \frac{n_3-2}{n_3} wt\left(z_3^{n_3}\right) = \left(1 - \frac{2}{n_3}\right) wt\left(z_3^{n_3}\right) \geq \frac{1}{3} wt\left(z_3^{n_3}\right) > \frac{1}{6} wt(f).$$

The conclusion

$$\alpha_1 > wt\left(z_3^{n_3-2}\right) > \frac{1}{6} wt(f)$$

and

$$\frac{1}{n_1}\left(1 - \frac{1}{2n_3}\right) wt(f) > \frac{1}{n_1}\left(wt(f) - \alpha_3\right) = \alpha_1 > wt\left(z_3^{n_3-2}\right) > \left(1 - \frac{2}{n_3}\right) \frac{1}{2} wt(f)$$

follows.

From $\frac{1}{n_1}\left(1 - \frac{1}{2n_3}\right) wt(f) > \left(1 - \frac{2}{n_3}\right) \frac{1}{2} wt(f)$, we obtain an upper bound for n_3:

$$n_3 < 2 + \frac{3}{n_1 - 2}.$$

If $n_1 = 3$, we have $3 \leq n_3 < 5$. Thus $n_3 = 3$ or $n_3 = 4$.

If $n_1 = 4$, we have $3 \leq n_3 < \frac{7}{2}$. Thus $n_3 = 3$.

If $n_1 \geq 5$, we have $3 \leq n_3 < 2 + \frac{3}{n_1-2} \leq 3$, which leads to a contradiction.

When $n_1 = 3$ and $n_3 = 3$, from the weight relationship

$$\begin{cases} 2\alpha_0 + \alpha_2 = wt(f) \\ 3\alpha_1 + \alpha_3 = wt(f) \\ n_2\alpha_2 + \alpha_1 = wt(f) \\ 3\alpha_3 + \alpha_0 = wt(f) \end{cases},$$

we obtain

$$\begin{cases} \alpha_0 = \dfrac{9n_2 - 7}{18n_2 - 1} wt(f) \\ \alpha_1 = \dfrac{5n_2 - 1}{18n_2 - 1} wt(f) \\ \alpha_2 = \dfrac{13}{18n_2 - 1} wt(f) \\ \alpha_3 = \dfrac{3n_2 + 2}{18n_2 - 1} wt(f) \end{cases}.$$

From $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, we obtain $9n_2-7 \geq 5n_2-1 \geq 13 \geq 3n_2+2$. Thus $\frac{14}{5} \leq n_2 \leq \frac{11}{3}$. Note that $n_2 \geq 3$, we get $3 \leq n_2 \leq \frac{11}{3}$. Therefore $n_2 = 3$. $wt\left(z_3^{n_3-2}\right) < \alpha_1$ is equivalent to $\frac{3n_2+2}{18n_2-1}wt(f) < \frac{5n_2-1}{18n_2-1}wt(f)$. The restriction is satisfied when $n_2 = 3$.

Thus when the conditions $n_0 = 2, n_1 = 3$ and $n_3 = 3$ in $f = z_0^{n_0}z_2+z_1^{n_1}z_3+z_2^{n_2}z_1+z_3^{n_3}z_0$ are satisfied at the same time, there exist a negative weight derivation for $H_1(V)$ if and only if $n_2 = 3$. In other words, f is in the form of $f = z_0^2z_2+z_1^3z_3+z_2^3z_1+z_3^3z_0$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D|D = cz_3\frac{\partial}{\partial z_1}, c \neq 0\right\}$.

When $n_1 = 3$ and $n_3 = 4$, from the weight relationship

$$\begin{cases} 2\alpha_0 + \alpha_2 = wt(f) \\ 3\alpha_1 + \alpha_3 = wt(f) \\ n_2\alpha_2 + \alpha_1 = wt(f) \\ 4\alpha_3 + \alpha_0 = wt(f) \end{cases},$$

we obtain

$$\begin{cases} \alpha_0 = \dfrac{12n_2-9}{24n_2-1}wt(f) \\ \alpha_1 = \dfrac{7n_2-1}{24n_2-1}wt(f) \\ \alpha_2 = \dfrac{17}{24n_2-1}wt(f) \\ \alpha_3 = \dfrac{3n_2+2}{24n_2-1}wt(f) \end{cases}.$$

From $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, we obtain the restriction $12n_2-9 \geq 7n_2-1 \geq 17 \geq 3n_2+2$. Thus $\frac{18}{7} \leq n_2 \leq 5$. Note that $n_2 \geq 3$, we get $3 \leq n_2 \leq 5$. Then $wt\left(z_3^{n_3-2}\right) < \alpha_1$ is equivalent to $\frac{2(3n_2+2)}{24n_2-1}wt(f) < \frac{7n_2-1}{24n_2-1}wt(f)$. Thus we have $n_2 > 5$, which contradicts to $n_2 \leq 5$.

Thus when $n_0 = 2$, $n_1 = 3$ and $n_3 = 4$ in $f = z_0^{n_0}z_2 + z_1^{n_1}z_3 + z_2^{n_2}z_1 + z_3^{n_3}z_0$ hold at the same time, for any $n_2 \geq 3$, there does not exist negative weight derivation for $H_1(V)$.

When $n_1 = 4$ and $n_3 = 3$, from the weight relationship

$$\begin{cases} 2\alpha_0 + \alpha_2 = wt(f) \\ 4\alpha_1 + \alpha_3 = wt(f) \\ n_2\alpha_2 + \alpha_1 = wt(f) \\ 3\alpha_3 + \alpha_0 = wt(f) \end{cases},$$

we obtain

$$\begin{cases} \alpha_0 = \dfrac{12n_2-10}{24n_2-1}wt(f) \\ \alpha_1 = \dfrac{5n_2-1}{24n_2-1}wt(f) \\ \alpha_2 = \dfrac{19}{24n_2-1}wt(f) \\ \alpha_3 = \dfrac{4n_2+3}{24n_2-1}wt(f) \end{cases}.$$

From $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, we obtain the restriction $12n_2-10 \geq 5n_2-1 \geq 19 \geq 4n_2+3$. Thus $4 \leq n_2 \leq 4$. Therefore, $n_2 = 4$. Then $wt\left(z_3^{n_3-2}\right) < \alpha_1$ is equivalent to $\frac{4n_2+3}{24n_2-1}wt(f) < \frac{5n_2-1}{24n_2-1}wt(f)$. When $n_2 = 4$, the inequation is false.

Thus when $n_0 = 2$, $n_1 = 4$ and $n_3 = 3$ in $f = z_0^{n_0}z_2 + z_1^{n_1}z_3 + z_2^{n_2}z_1 + z_3^{n_3}z_0$ hold at the same time, for any $n_2 \geq 3$, there does not exist negative weight derivation of $H_1(V)$.

Therefore, for any $f = z_0^{n_0}z_2 + z_1^{n_1}z_3 + z_2^{n_2}z_1 + z_3^{n_3}z_0\,(n_0 \geq 2)$, there exists negative weight derivation of $H_1(V)$ if and only if f is in the form of $f = z_0^2z_2 + z_1^3z_3 + z_2^3z_1 + z_3^3z_0$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D|D = cz_3\frac{\partial}{\partial z_1}, c \neq 0\right\}$. □

Lemma 2.61 (Case (iv) of Proposition 2.57). *Let $(V,0) = \{(z_0,z_1,z_2,z_3) \in \mathbb{C}^4 \colon f(z_0,z_1,z_2,z_3) = 0\}$ be an isolated singularity defined by $f(z_0,z_1,z_2,z_3) = z_0^{n_0}z_3 + z_1^{n_1}z_2 + z_2^{n_2}z_0 + z_3^{n_3}z_1\,(n_0 \geq 2)$ of weight type $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ where $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$. Let $H_1(V)$ be the 1-st Hessian algebra. Then there exists negative weight derivation of $H_1(V)$ if and only if f is in one of the following forms:*

(i) $f = z_0^2z_3 + z_1^3z_2 + z_2^3z_0 + z_3^{n_3}z_1\,(n_3 \geq 4)$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D|D = cz_2\frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(ii) $f = z_0^2z_3 + z_1^4z_2 + z_2^3z_0 + z_3^{n_3}z_1\,(n_3 \geq 5)$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D|D = cz_2\frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(iii) $f = z_0^2z_3 + z_1^3z_2 + z_2^4z_0 + z_3^{n_3}z_1\,(n_3 \geq 6)$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D|D = cz_2^2\frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(iv) $f = z_0^3z_3 + z_1^3z_2 + z_2^3z_0 + z_3^{n_3}z_1\,(n_3 \geq 24)$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D|D = cz_2z_3^k\frac{\partial}{\partial z_1}, c \neq 0, 1 \leq k < \frac{n_3-3}{20}, k \in \mathbb{Z}\right\}$.

Proof. If there exists some negative weight derivation D, D must be in the form of $D = p_0(z_1,z_2,z_3)\frac{\partial}{\partial z_0} + p_1(z_2,z_3)\frac{\partial}{\partial z_1} + cz_3^k\frac{\partial}{\partial z_2}$.

When $n_0 \geq 3$ holds, we obtain

$$2\alpha_i + \alpha_j \leq 3\alpha_0 < wt(f) = n_i\alpha_i + \alpha_j$$

for $(i,j) = (1,2)$, $(2,0)$ or $(3,1)$. Thus we obtain $n_1 > 2$, $n_2 > 2$ and $n_3 > 2$, which is equivalent to $n_1 \geq 3$, $n_2 \geq 3$ and $n_3 \geq 3$. Regardless of difference of constants, we obtain

$$\mathrm{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0^{n_0-2}z_3 & 0 & z_2^{n_2-1} & z_0^{n_0-1} \\ * & z_1^{n_1-2}z_2 & z_1^{n_1-1} & z_3^{n_3-1} \\ * & * & z_2^{n_2-2}z_0 & 0 \\ * & * & * & z_3^{n_3-2}z_1 \end{bmatrix}.$$

From

$$D\left(z_0^{n_0-2}z_3\right) = p_0\left(z_1,z_2,z_3\right)(n_0-2)z_0^{n_0-3}z_3 = 0,$$

we obtain

$$p_0\left(z_1,z_2,z_3\right) = 0.$$

From

$$D\left(z_2^{n_2-1}\right) = cz_3^k\left(n_2-1\right)z_2^{n_2-2} \in \left(z_0^{n_0-2}z_3, z_1^{n_1-2}z_2\right),$$

we obtain $c = 0$. Thus $D = p_1\left(z_2,z_3\right)\frac{\partial}{\partial z_1}$.

From

$$D\left(z_1^{n_1-2}z_2\right)=p_1\left(z_2,z_3\right)\left(n_1-2\right)z_1^{n_1-3}z_2\in\left(z_0^{n_0-2}z_3,z_2^{n_2-1},z_0^{n_0-1}\right),$$

we obtain

$$p_1\left(z_2,z_3\right)z_1^{n_1-3}z_2\in\left(z_2^{n_2-1}\right).$$

If $p_1\left(z_2,z_3\right)\neq 0$, $p_1\left(z_2,z_3\right)$ contains factor $z_2^{n_2-2}$.

From

$$D\left(z_1^{n_1-1}\right)=p_1\left(z_2,z_3\right)\left(n_1-1\right)z_1^{n_1-2}\in\left(z_0^{n_0-2}z_3,z_2^{n_2-1},z_0^{n_0-1},z_1^{n_1-2}z_2\right),$$

we obtain

$$p_1\left(z_2,z_3\right)\left(n_1-1\right)z_1^{n_1-2}\in\left(z_2^{n_2-1},z_1^{n_1-2}z_2\right).$$

Since $p_1\left(z_2,z_3\right)$ contains factor $z_2^{n_2-2}, p_1\left(z_2,z_3\right)$ contains factor z_2. In the way $p_1\left(z_2,z_3\right)\left(n_1-1\right)\ z_1^{n_1-2}\ \in\ \left(z_1^{n_1-2}z_2\right)$, the condition $p_1\left(z_2,z_3\right)\left(n_1-1\right)z_1^{n_1-2}\in\left(z_2^{n_2-1},z_1^{n_1-2}z_2\right)$ is satisfied.

From

$$D\left(z_3^{n_3-2}z_1\right)=p_1\left(z_2,z_3\right)z_3^{n_3-2}\in\left(z_0^{n_0-2}z_3,z_2^{n_2-1},z_0^{n_0-1},z_1^{n_1-2}z_2,z_1^{n_1-1},z_3^{n_3-1},z_2^{n_2-2}z_0\right),$$

we obtain

$$p_1\left(z_2,z_3\right)z_3^{n_3-2}\in\left(z_2^{n_2-1},z_3^{n_3-1}\right).$$

Note $wt\left(z_2^{n_2-1}\right)=wt(f)-\alpha_0-\alpha_2=n_0\alpha_0+\alpha_3-\alpha_0-\alpha_2>2\alpha_0-\alpha_0=\alpha_0\geq\alpha_1>wt\left(p_1\left(z_2,z_3\right)\right)$, $p_1\left(z_2,z_3\right)$ do not contain factor $z_2^{n_2-1}$. Thus $p_1\left(z_2,z_3\right)z_3^{n_3-2}\in\left(z_3^{n_3-1}\right)$. Then $p_1\left(z_2,z_3\right)$ contains the factor z_3. Thus $p_1\left(z_2,z_3\right)$ contains the factor $z_2^{n_2-2}z_3$. Then we obtain $\left(n_2-2\right)\alpha_2+\alpha_3\leq wt\left(p_1\left(z_2,z_3\right)\right)<\alpha_1\leq\alpha_0$. However, we also notice that $\left(n_2-2\right)\alpha_2+\alpha_3=\left(n_2-2\right)\frac{n_0\alpha_0+\alpha_3-\alpha_0}{n_2}+\alpha_3=\left(n_2-2\right)\frac{(n_0-1)\alpha_0+\alpha_3}{n_2}+\alpha_3>\left(n_0-1\right)\frac{(n_2-2)\alpha_0}{n_2}\geq 2\frac{(n_2-2)\alpha_0}{n_2}$. From the two inequations above, we obtain $2\frac{(n_2-2)\alpha_0}{n_2}<\alpha_0$. Therefore, $n_2<4$.

Note that $n_2\geq 3$, we obtain $n_2=3$. From $\left(n_2-2\right)\alpha_2+\alpha_3=\left(n_2-2\right)\frac{n_0\alpha_0+\alpha_3-\alpha_0}{n_2}+\alpha_3=\left(n_2-2\right)\frac{(n_0-1)\alpha_0+\alpha_3}{n_2}+\alpha_3>\left(n_0-1\right)\frac{(n_2-2)\alpha_0}{n_2}$, we obtain

$$\alpha_2+\alpha_3>\left(n_0-1\right)\frac{\alpha_0}{3}.$$

From

$$\left(n_2-2\right)\alpha_2+\alpha_3\leq wt\left(p_1\left(z_2,z_3\right)\right)<\alpha_1\leq\alpha_0,$$

we get

$$\alpha_2+\alpha_3<\alpha_0.$$

From the inequation $\left(n_0-1\right)\frac{\alpha_0}{3}<\alpha_0$, we obtain $n_0<4$. Considering the assumption $n_0\geq 3$, we obtain $n_0=3$. From another fact that

$$n_1\alpha_1+\alpha_2=3\alpha_2+\alpha_0=3\alpha_2+\frac{n_1\alpha_1+\alpha_2-\alpha_3}{3},$$

we obtain

$$2n_1\alpha_1 = 7\alpha_2 - \alpha_3 < 7\alpha_1.$$

Therefore, $n_1 < \frac{7}{2}$.

Since $n_1 \geq 3$, we deduce $n_1 = 3$ and f is of the form $f = z_0^3 z_3 + z_1^3 z_2 + z_2^3 z_0 + z_3^{n_3} z_1$. From the weight relationship

$$\begin{cases} 3\alpha_0 + \alpha_3 = 3\alpha_1 + \alpha_2 \\ 3\alpha_0 + \alpha_3 = 3\alpha_2 + \alpha_0 \\ 3\alpha_0 + \alpha_3 = n_3\alpha_3 + \alpha_1 \end{cases},$$

we obtain

$$\begin{cases} \alpha_1 = \dfrac{7\alpha_0 + 2\alpha_3}{9} \\ \alpha_2 = \dfrac{2\alpha_0 + \alpha_3}{3} \\ \alpha_3 = \dfrac{20}{9n_3 - 7}\alpha_0 \end{cases}.$$

Substituting the first two equations into $(n_2 - 2)\,\alpha_2 + \alpha_3 < \alpha_1$, we obtain $\alpha_0 > 10\alpha_3$. Substituting the last solution, we obtain $n_3 > 23$, which is equivalent to $n_3 \geq 24$.

When $n_3 > 23$, we obtain $\alpha_3 = \frac{20}{9n_3-7}\alpha_0 < \frac{1}{10}\alpha_0 < \alpha_0$. From the three equations that $\alpha_0 - \alpha_1 = \frac{2\alpha_0 - 2\alpha_3}{9} > 0$, $\alpha_1 - \alpha_2 = \frac{\alpha_0 - \alpha_3}{9} > 0$ and $\alpha_2 - \alpha_3 = \frac{2\alpha_0 - 2\alpha_3}{3} > 0$, we obtain $\alpha_0 > \alpha_1 > \alpha_2 > \alpha_3$. Thus $p_1(z_2, z_3) = c_1 z_2 z_3 (c_1 \neq 0)$ is qualified if and only if $n_3 \geq 24$.

So when $n_0 \geq 3$, there exists negative weight derivation if and only if f is in the form of $f = z_0^3 z_3 + z_1^3 z_2 + z_2^3 z_0 + z_3^{n_3} z_1$ and $n_3 \geq 24$. If $n_3 \geq 24, D = c_1 z_2 z_3 \frac{\partial}{\partial z_1}\,(c_1 \neq 0)$ satisfies the restriction. Regardless of difference of constants, $\mathrm{Hess}(f)$ is in the form of

$$\mathrm{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0 z_3 & 0 & z_2^2 & z_0^2 \\ * & z_1 z_2 & z_1^2 & z_3^{n_3-1} \\ * & * & z_2 z_0 & 0 \\ * & * & * & z_3^{n_3-2} z_1 \end{bmatrix}.$$

Considering $p_1(z_2, z_3)$ which is divided by $z_2 z_3$ and the elements of $\mathrm{Hess}(f)$, all the possible forms of $p_1(z_2, z_3)$ are $p_1(z_2, z_3) = c_1 z_2 z_3^{k_1}\,(k_1 \leq n_3 - 2, c_1 \neq 0)$ which satisfies the "negatively weighted" restriction $\alpha_2 + k_1\alpha_3 < \alpha_1$. From $\alpha_1 = \frac{7\alpha_0 + 2\alpha_3}{9}$ and $\alpha_2 = \frac{2\alpha_0 + \alpha_3}{3}$, we obtain $(9k_1 + 1)\alpha_3 < \alpha_0$. Substituting $\alpha_3 = \frac{20}{9n_3 - 7}\alpha_0$, we obtain $1 \leq k_1 < \frac{n_3 - 3}{20} < n_3 - 2$.

In conclusion, when $n_0 = 3$, there exists negative weight derivation of $H_1(V)$ if and only if f is in the form of $f = z_0^3 z_3 + z_1^3 z_2 + z_2^3 z_0 + z_3^{n_3} z_1$ and $n_3 \geq 24$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{ D | D = c z_2 z_3^k \frac{\partial}{\partial z_1}, c \neq 0, 1 \leq k < \frac{n_3 - 3}{20}, k \in \mathbb{Z} \right\}$.

When $n_0 = 2$ holds, we obtain

$$f = z_0^2 z_3 + z_1^{n_1} z_2 + z_2^{n_2} z_0 + z_3^{n_3} z_1.$$

From

$$\alpha_2 + n_1\alpha_1 = 2\alpha_0 + \alpha_3 > 2\alpha_0 \geq \alpha_1 + \alpha_2,$$

we obtain $n_1 > 1$, which is equivalent to $n_1 \geq 2$.

From

$$\alpha_0 + n_2\alpha_2 = 2\alpha_0 + \alpha_3 > 2\alpha_0 \geq \alpha_0 + \alpha_2,$$

we obtain $n_2 > 1$, which is equivalent to $n_2 \geq 2$.

From

$$\alpha_1 + n_3\alpha_3 = 2\alpha_0 + \alpha_3 > 2\alpha_0 \geq \alpha_0 + \alpha_3,$$

we obtain $n_3 > 1$, which is equivalent to $n_3 \geq 2$.

Regardless of difference of constants and useless monomials, we have

$$\mathrm{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_3 & 0 & z_2^{n_2-1} & z_0 \\ * & z_1^{n_1-2}z_2 & z_1^{n_1-1} & 0 \\ * & * & 0 & 0 \\ * & * & * & z_3^{n_3-2}z_1 \end{bmatrix}.$$

If $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3$, we obtain $n_1 = n_2 = n_3 = 2$ and $f = z_0^2z_3 + z_1^2z_2 + z_2^2z_0 + z_3^2z_1$. Thus z_0, z_1 and z_2 are in the ideal generated by elements of $\mathrm{Hess}(f)$. There does not exist any nonzero element in $H_1(V)$ which is divided by z_0, z_1 or z_2. Thus $D = 0$.

If $\alpha_0 > \alpha_3$, from $\alpha_1+n_3\alpha_3 = 2\alpha_0+\alpha_3 > \alpha_0+2\alpha_3 \geq \alpha_1+2\alpha_3$, we get $n_3 > 2$, which is equivalent to $n_3 \geq 3$. Regardless of difference of constants and useless monomials, we have

$$\mathrm{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_3 & 0 & z_2^{n_2-1} & z_0 \\ * & z_1^{n_1-2}z_2 & z_1^{n_1-1} & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}.$$

Thus z_0 and z_3 are in the ideal generated by elements of $\mathrm{Hess}(f)$. There does not exist any nonzero element in $H_1(V)$ which is divided by z_0 or z_3. If there exists negative weight derivation D, D must be in the form of $D = c_1z_2^{k_1}\frac{\partial}{\partial z_1} + c_2\frac{\partial}{\partial z_2}$. From

$$D\left(z_2^{n_2-1}\right) = (n_2-1)\,c_2z_2^{n_2-2} \in \left(z_3, z_1^{n_1-2}z_2\right),$$

we obtain $c_2z_2^{n_2-2} \in \left(z_1^{n_1-2}z_2\right)$.

If $n_1 = 2$, it is clear that z_1 and z_2 are in the ideal generated by elements of $\mathrm{Hess}(f)$. There does not exist any nonzero element which is divided by z_1 or z_2. Thus $D = 0$.

If $n_1 \geq 3$, we have $c_2 = 0$ and $D = c_1z_2^{k_1}\frac{\partial}{\partial z_1}$. If $c_1 \neq 0$, from

$$D\left(z_1^{n_1-2}z_2\right) = (n_1-2)\,c_1z_2^{k_1+1}z_1^{n_1-3} \in \left(z_3, z_2^{n_2-1}, z_0\right),$$

we can get $z_2^{k_1+1} \in \left(z_2^{n_2-1}\right)$, which is equivalent to $k_1 + 1 \geq n_2 - 1$. However, $z_2^{k_1}$ is not in the ideal $\left(z_2^{n_2-1}\right)$. Otherwise $D = 0$, which is equivalent to $c_1 = 0$. Thus $k_1 < n_2 - 1 \leq k_1 + 1$, from which we get $k_1 = n_2 - 2$. Therefore, $D\left(z_1^{n_1-2}z_2\right) \in \left(z_3, z_2^{n_2-1}, z_0\right)$ if and only if D is in the form of $D = c_1z_2^{n_2-2}\frac{\partial}{\partial z_1}$.

When $D = c_1z_2^{n_2-2}\frac{\partial}{\partial z_1}$ and $c_1 \neq 0$, we obtain

$$D\left(z_1^{n_1-1}\right) = (n_1-1)\,c_1z_2^{n_2-2}z_1^{n_1-2} \in \left(z_3, z_2^{n_2-1}, z_0, z_1^{n_1-2}z_2\right),$$

which is equivalent to

$$z_2^{n_2-2} z_1^{n_1-2} \in \left(z_1^{n_1-2} z_2\right).$$

Thus $n_2 - 2 \geq 1$ and $n_2 \geq 3$.

From the relation

$$wt(f) = 2\alpha_0 + \alpha_3 > 2\alpha_0,$$

we obtain

$$\alpha_0 < \frac{1}{2} wt(f).$$

From the relation

$$wt(f) = 2\alpha_0 + \alpha_3 < 3\alpha_0,$$

we obtain

$$\alpha_0 > \frac{1}{3} wt(f).$$

Therefore, we have

$$wt\left(z_2^{n_2}\right) = wt\left(f\right) - \alpha_0 \in \left(\frac{1}{2} wt(f), \frac{2}{3} wt(f)\right),$$

which is equivalent to

$$\alpha_2 \in \left(\frac{1}{2n_2} wt(f), \frac{2}{3n_2} wt(f)\right).$$

So we get

$$wt\left(z_2^{n_2-2}\right) \in \left(\frac{n_2-2}{2n_2} wt(f), \frac{2\left(n_2-2\right)}{3n_2} wt(f)\right).$$

We obtain a lower bound of α_1 from

$$\alpha_1 > wt\left(z_2^{n_2-2}\right) > \frac{1-\frac{2}{n_2}}{2} wt(f).$$

If $n_2 = 3$, we have $\alpha_1 > \frac{1}{6} wt(f)$ and $\alpha_2 > \frac{1}{6} wt(f)$. Thus $wt(f) = n_1\alpha_1 + \alpha_2 > \frac{n_1+1}{6} wt(f)$. Thus $n_1 < 5$. Note that $n_1 \geq 3$, we have $n_1 = 3$ or $n_1 = 4$.

If $n_2 = 4$, we have $\alpha_1 > \frac{1}{4} wt(f)$. Thus $wt(f) = n_1\alpha_1 + \alpha_2 > n_1\alpha_1 > \frac{n_1}{4} wt(f)$ and $n_1 < 4$. Note that $n_1 \geq 3$, we have $n_1 = 3$.

If $n_2 = 5$, we have $\alpha_1 > \frac{3}{10} wt(f)$ and $\alpha_2 > \frac{1}{10} wt(f)$. However, we also notice that $wt(f) = n_1\alpha_1 + \alpha_2 \geq 3\alpha_1 + \alpha_2 > wt(f)$. This leads to a contradiction.

If $n_2 \geq 6$, we have $\alpha_1 > \frac{1-\frac{2}{n_2}}{2} wt(f) \geq \frac{1}{3} wt(f)$. However, we also notice that $wt(f) = n_1\alpha_1 + \alpha_2 > 3\alpha_1 > wt(f)$. This leads to a contradiction.

In conclusion, there are 3 possibilities: $(n_1, n_2) = (3,3), (4,3)$ or $(3,4)$.

When $(n_1, n_2) = (3,3)$, from the weight relationship

$$\left\{\begin{array}{r} 2\alpha_0 + \alpha_3 = wt(f) \\ 3\alpha_1 + \alpha_2 = wt(f) \\ 3\alpha_2 + \alpha_0 = wt(f) \\ n_3\alpha_3 + \alpha_1 = wt(f) \end{array}\right.,$$

we obtain

$$\begin{cases} \alpha_0 = \frac{9n_3-7}{18n_3-1} wt(f) \\ \alpha_1 = \frac{5n_3-1}{18n_3-1} wt(f) \\ \alpha_2 = \frac{3n_3+2}{18n_3-1} wt(f) \\ \alpha_3 = \frac{13}{18n_3-1} wt(f) \end{cases}.$$

From $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, we obtain

$$\frac{9n_3-7}{18n_3-1} wt(f) \geq \frac{5n_3-1}{18n_3-1} wt(f) \geq \frac{3n_3+2}{18n_3-1} wt(f) \geq \frac{13}{18n_3-1} wt(f).$$

Thus $n_3 \geq \frac{11}{3}$, which is equivalent to $n_3 \geq 4$. From the negative weight restriction, we obtain $(n_2-2)\,\alpha_2 < \alpha_1$, in other words, $\alpha_2 < \alpha_1$. When $n_3 \geq 4$, the restriction holds. Thus when $(n_1, n_2) = (3,3)$ in $f = z_0^2 z_3 + z_1^{n_1} z_2 + z_2^{n_2} z_0 + z_3^{n_3} z_1$, there exist negative weight derivations if and only if $n_3 \geq 4$ and all the negative weight derivations are in the form of $D = c_1 z_2 \frac{\partial}{\partial z_1}\,(c_1 \neq 0)$.

When $(n_1, n_2) = (4,3)$, from the relations

$$\begin{cases} 2\alpha_0 + \alpha_3 = wt(f) \\ 4\alpha_1 + \alpha_2 = wt(f) \\ 3\alpha_2 + \alpha_0 = wt(f) \\ n_3\alpha_3 + \alpha_1 = wt(f) \end{cases},$$

we obtain

$$\begin{cases} \alpha_0 = \dfrac{12n_3-10}{24n_3-1} wt(f) \\ \alpha_1 = \dfrac{5n_3-1}{24n_3-1} wt(f) \\ \alpha_2 = \dfrac{4n_3+3}{24n_3-1} wt(f) \\ \alpha_3 = \dfrac{19}{24n_3-1} wt(f) \end{cases}.$$

From $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, we obtain

$$\frac{12n_3-10}{24n_3-1} wt(f) \geq \frac{5n_3-1}{24n_3-1} wt(f) \geq \frac{4n_3+3}{24n_3-1} wt(f) \geq \frac{19}{24n_3-1} wt(f).$$

When $n_3 \geq 3$, we get $n_3 \geq 4$. From the negative weight restriction, we have $(n_2-2)\,\alpha_2 < \alpha_1$, in other words, $\alpha_2 < \alpha_1$. When $n_3 \geq 4$, we get $n_3 > 4$, in other words, $n_3 \geq 5$. Thus when $(n_1, n_2) = (4,3)$ in $f = z_0^2 z_3 + z_1^{n_1} z_2 + z_2^{n_2} z_0 + z_3^{n_3} z_1$, there exist negative weight derivations if and only if $n_3 \geq 5$. In fact, all the negative weight derivations are in the form of $D = c_1 z_2 \frac{\partial}{\partial z_1}\,(c_1 \neq 0)$.

When $(n_1, n_2) = (3,4)$, from the relations

$$\begin{cases} 2\alpha_0 + \alpha_3 = wt(f) \\ 3\alpha_1 + \alpha_2 = wt(f) \\ 4\alpha_2 + \alpha_0 = wt(f) \\ n_3\alpha_3 + \alpha_1 = wt(f) \end{cases},$$

we obtain

$$\begin{cases} \alpha_0 = \frac{12n_3-9}{24n_3-1}wt(f) \\ \alpha_1 = \frac{7n_3-1}{24n_3-1}wt(f) \\ \alpha_2 = \frac{3n_3+2}{24n_3-1}wt(f) \\ \alpha_3 = \frac{17}{24n_3-1}wt(f) \end{cases}.$$

From $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$, we obtain

$$\frac{12n_3-9}{24n_3-1}wt(f) \geq \frac{7n_3-1}{24n_3-1}wt(f) \geq \frac{3n_3+2}{24n_3-1}wt(f) \geq \frac{17}{24n_3-1}wt(f).$$

When $n_3 \geq 3$, we get $n_3 \geq 5$. From the negative weight restriction, we have $(n_2-2)\,\alpha_2 < \alpha_1$, in other words, $2\alpha_2 < \alpha_1$. When $n_3 \geq 5$, we get $n_3 > 5$, in other words, $n_3 \geq 6$. Thus when $(n_1, n_2) = (4, 3)$ in $f = z_0^2 z_3 + z_1^{n_1} z_2 + z_2^{n_2} z_0 + z_3^{n_3} z_1$, there exist negative weight derivations of $H_1(V)$ if and only if $n_3 \geq 6$. In fact, all the negative weight derivations of $H_1(V)$ are in the form of $D = c_1 z_2^2 \frac{\partial}{\partial z_1}$ $(c_1 \neq 0)$.

Therefore, for any $f = z_0^{n_0} z_3 + z_1^{n_1} z_2 + z_2^{n_2} z_0 + z_3^{n_3} z_1$ $(n_0 \geq 2)$, there exists negative weight derivation of $H_1(V)$ if and only if f is in one of the following forms:

(1) $f = z_0^2 z_3 + z_1^3 z_2 + z_2^3 z_0 + z_3^{n_3} z_1$ $(n_3 \geq 4)$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D | D = c z_2 \frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(2) $f = z_0^2 z_3 + z_1^4 z_2 + z_2^3 z_0 + z_3^{n_3} z_1$ $(n_3 \geq 5)$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D | D = c z_2 \frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(3) $f = z_0^2 z_3 + z_1^3 z_2 + z_2^4 z_0 + z_3^{n_3} z_1$ $(n_3 \geq 6)$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D | D = c z_2^2 \frac{\partial}{\partial z_1}, c \neq 0\right\}$;

(4) $f = z_0^3 z_3 + z_1^3 z_2 + z_2^3 z_0 + z_3^{n_3} z_1$ $(n_3 \geq 24)$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{D | D = c z_2 z_3^k \frac{\partial}{\partial z_1}, c \neq 0, 1 \leq k < \frac{n_3-3}{20}, k \in \mathbb{Z}\right\}$. ☐

LEMMA 2.62 (Case (v) of Proposition 2.57). *Let* $(V, 0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 \colon f(z_0, z_1, z_2, z_3) = 0\}$ *be an isolated singularity defined by* $f(z_0, z_1, z_2, z_3) = z_0^{n_0} z_2 + z_1^{n_1} z_0 + z_2^{n_2} z_3 + z_3^{n_3} z_1$ $(n_0 \geq 2)$ *of weight type* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ *where* $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$. *Let* $H_1(V)$ *be the* 1*-st Hessian algebra. Then there does not exist any negative weight derivations of* $H_1(V)$.

Proof. If there exists some negative weight derivation D, D must be in the form of $D = p_0(z_1, z_2, z_3) \frac{\partial}{\partial z_0} + p_1(z_2, z_3) \frac{\partial}{\partial z_1} + c z_3^k \frac{\partial}{\partial z_2}$.

When $n_0 \geq 3$ holds, we obtain

$$2\alpha_i + \alpha_j \leq 3\alpha_0 < wt(f) = n_i \alpha_i + \alpha_j$$

for $(i, j) = (1, 0), (2, 3)$ or $(3, 1)$. Therefore, $n_1 > 2$, $n_2 > 2$ and $n_3 > 2$. Therefore, we have $n_1 \geq 3$, $n_2 \geq 3$ and $n_3 \geq 3$. Regardless of difference of constants, we obtain

$$\mathrm{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0^{n_0-2} z_2 & z_1^{n_1-1} & z_0^{n_0-1} & 0 \\ * & z_1^{n_1-2} z_0 & 0 & z_3^{n_3-1} \\ * & * & z_2^{n_2-2} z_3 & z_2^{n_2-1} \\ * & * & * & z_3^{n_3-2} z_1 \end{bmatrix}.$$

From the equation

$$D\left(z_0^{n_0-2} z_2\right) = p_0\left(z_1, z_2, z_3\right)(n_0-2) z_0^{n_0-3} z_2 + c z_3^k z_0^{n_0-2} = 0,$$

we obtain

$$p_0(z_1,z_2,z_3)(n_0-2)z_2+cz_3^kz_0=0.$$

Therefore, we have $c=0$ and $p_0(z_1,z_2,z_3)=0$.

So $D=p_1(z_2,z_3)\frac{\partial}{\partial z_1}$. From

$$D\left(z_1^{n_1-1}\right)=p_1(z_2,z_3)(n_1-1)z_1^{n_1-2}\in\left(z_0^{n_0-2}z_2\right),$$

we obtain $p_1(z_2,z_3)=0$ and $D=0$.

When $n_0\geq 3$, there does not exist negative weight derivation of $H_1(V)$ for any $f=z_0^{n_0}z_2+z_1^{n_1}z_0+z_2^{n_2}z_3+z_3^{n_3}z_1$.

When $n_0=2$ holds, we obtain

$$f=z_0^2z_2+z_1^{n_1}z_0+z_2^{n_2}z_3+z_3^{n_3}z_1.$$

From

$$n_1\alpha_1+\alpha_0=2\alpha_0+\alpha_2>2\alpha_0\geq\alpha_1+\alpha_0,$$

we obtain $n_1>1$, which is equivalent to $n_1\geq 2$.

From

$$n_2\alpha_2+\alpha_3=2\alpha_0+\alpha_2>2\alpha_0\geq\alpha_2+\alpha_3,$$

we obtain $n_2>1$, which is equivalent to $n_2\geq 2$.

From

$$n_3\alpha_3+\alpha_1=2\alpha_0+\alpha_2>2\alpha_0\geq\alpha_3+\alpha_1,$$

we obtain $n_3>1$, which is equivalent to $n_3\geq 2$.

Regardless of difference of constants and useless monomials, we have

$$\mathrm{Hess}(f)=\begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03}\\ f_{10} & f_{11} & f_{12} & f_{13}\\ f_{20} & f_{21} & f_{22} & f_{23}\\ f_{30} & f_{31} & f_{32} & f_{33}\end{bmatrix}=\begin{bmatrix} z_2 & z_1^{n_1-1} & z_0 & 0\\ * & 0 & 0 & z_3^{n_3-1}\\ * & * & z_2^{n_2-2}z_3 & 0\\ * & * & * & z_3^{n_3-2}z_1\end{bmatrix}.$$

If such negative weight derivation D exists, we have $D=c_1z_3^{k_1}\frac{\partial}{\partial z_1}$. If $\alpha_0=\alpha_1=\alpha_2$, we obtain $n_1=2$ and $f=z_0^2z_2+z_1^2z_0+z_2^{n_2}z_3+z_3^{n_3}z_1$. Thus z_0,z_1 and z_2 are in the ideal generated by elements of Hess(f). There does not exist any nonzero element in $H_1(V)$ which is divided by z_0,z_1 or z_2. Thus $D=0$.

Otherwise we obtain $\alpha_0>\alpha_2$. If $n_1=2$, similarly z_0,z_1 and z_2 are in the ideal generated by elements of Hess(f). There does not exist nonzero element in $H_1(V)$ which can be divided by z_0,z_1 or z_2. Thus $D=0$.

If $n_1\geq 3$, we obtain $D\left(z_1^{n_1-1}\right)=(n_1-1)c_1z_3^{k_1}z_1^{n_1-2}\in(z_2)$. Thus $c_1=0$ and $D=0$.

Therefore, when $n_0=2$, there does not exist negative weight derivation of $H_1(V)$ for any $f=z_0^{n_0}z_2+z_1^{n_1}z_0+z_2^{n_2}z_3+z_3^{n_3}z_1$.

In conclusion, there does not exist negative weight derivation of $H_1(V)$ for any $f=z_0^{n_0}z_2+z_1^{n_1}z_0+z_2^{n_2}z_3+z_3^{n_3}z_1\ (n_0\geq 2)$. ∎

LEMMA 2.63 (Case (vi) of Proposition 2.57). *Let* $(V,0) = \{(z_0,z_1,z_2,z_3) \in \mathbb{C}^4 \colon f(z_0,z_1,z_2,z_3) = 0\}$ *be an isolated singularity defined by* $f(z_0,z_1,z_2,z_3) = z_0^{n_0}z_3 + z_1^{n_1}z_0 + z_2^{n_2}z_1 + z_3^{n_3}z_2$ $(n_0 \geq 2)$ *of weight type* $(\alpha_0,\alpha_1,\alpha_2,\alpha_3; d)$ *where* $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$*. Let* $H_1(V)$ *be the* 1*-st Hessian algebra. Then there does not exist negative weight derivation of* $H_1(V)$*.*

Proof. If there exists some negative weight derivation D, D must be in the form of $D = p_0(z_1,z_2,z_3)\frac{\partial}{\partial z_0} + p_1(z_2,z_3)\frac{\partial}{\partial z_1} + cz_3^k\frac{\partial}{\partial z_2}$.

When $n_0 \geq 3$ holds, we obtain

$$2\alpha_i + \alpha_j \leq 3\alpha_0 < wt(f) = n_i\alpha_i + \alpha_j$$

for $(i,j) = (1,0),(2,1)$ or $(3,2)$. Therefore, $n_1 > 2$, $n_2 > 2$ and $n_3 > 2$. Therefore, $n_1 \geq 3$, $n_2 \geq 3$ and $n_3 \geq 3$. Regardless of difference of constants, we obtain

$$\mathrm{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_0^{n_0-2}z_3 & z_1^{n_1-1} & 0 & z_0^{n_0-1} \\ * & z_1^{n_1-2}z_0 & z_2^{n_2-1} & 0 \\ * & * & z_2^{n_2-2}z_1 & z_3^{n_3-1} \\ * & * & * & z_3^{n_3-2}z_2 \end{bmatrix}.$$

From

$$D\left(z_0^{n_0-2}z_3\right) = p_0\left(z_1,z_2,z_3\right)(n_0-2)z_0^{n_0-3}z_3 = 0,$$

we obtain $p_0\left(z_1,z_2,z_3\right) = 0$.

From

$$D\left(z_1^{n_1-1}\right) = p_1\left(z_2,z_3\right)(n_1-1)\,z_1^{n_1-2} \in \left(z_0^{n_0-2}z_3\right),$$

we obtain $p_1\left(z_2,z_3\right) = 0$.

From

$$D\left(z_2^{n_2-1}\right) = (n_2-1)\,cz_3^kz_2^{n_2-2} \in \left(z_0^{n_0-2}z_3, z_1^{n_1-1}, z_0^{n_0-1}, z_1^{n_1-2}z_0\right),$$

we obtain $c = 0$.

So $D = 0$.

When $n_0 \geq 3$, there does not exist negative weight derivation of $H_1(V)$ for any $f = z_0^{n_0}z_3 + z_1^{n_1}z_0 + z_2^{n_2}z_1 + z_3^{n_3}z_2$.

When $n_0 = 2$ holds, f is in the form of $f = z_0^2z_3 + z_1^{n_1}z_0 + z_2^{n_2}z_1 + z_3^{n_3}z_2$. We obtain

$$\alpha_i + \alpha_j \leq 2\alpha_0 < wt(f) = n_i\alpha_i + \alpha_j$$

for $(i,j) = (1,0),(2,1)$ or $(3,2)$. Therefore, $n_1 > 1$, $n_2 > 1$ and $n_3 > 1$, which is equivalent to $n_1 \geq 2$, $n_2 \geq 2$ and $n_3 \geq 2$.

Regardless of difference of constants and useless monomials, we have

$$\mathrm{Hess}(f) = \begin{bmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{10} & f_{11} & f_{12} & f_{13} \\ f_{20} & f_{21} & f_{22} & f_{23} \\ f_{30} & f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} z_3 & z_1^{n_1-1} & 0 & z_0 \\ * & 0 & z_2^{n_2-1} & 0 \\ * & * & z_2^{n_2-2}z_1 & z_3^{n_3-1} \\ * & * & * & z_3^{n_3-2}z_2 \end{bmatrix}.$$

Thus $D = c_1z_2^{k_1}\frac{\partial}{\partial z_1} + c_2\frac{\partial}{\partial z_2}$.

From

$$D\left(z_1^{n_1-1}\right)=\left(n_1-1\right)c_1z_2^{k_1}z_1^{n_1-2}\in\left(z_3\right),$$

we obtain $c_1=0$ and $D=c_2\frac{\partial}{\partial z_2}$.

From

$$D\left(z_2^{n_2-1}\right)=\left(n_2-1\right)c_2z_2^{n_2-2}\in\left(z_3,z_1^{n_1-1},z_0\right),$$

we obtain $c_2=0$ and $D=0$.

When $n_0=2$, there does not exist negative weight derivation of $H_1\left(V\right)$ for any $f=z_0^{n_0}z_3+z_1^{n_1}z_0+z_2^{n_2}z_1+z_3^{n_3}z_2$.

In conclusion, there does not exist negative weight derivation of $H_1\left(V\right)$ for any $f=z_0^{n_0}z_3+z_1^{n_1}z_0+z_2^{n_2}z_1+z_3^{n_3}z_2\left(n_0\geq 2\right)$. □

3. Type B Fewnomial Case. In this section, we will discuss the Type B fewnomial case where $mult\left(f\right)\geq 3$. The overall conclusion is written in Proposition 3.1.

PROPOSITION 3.1 (Type B fewnomial case of Theorem B). *Let* $(V,0)=\{(z_0,z_1,z_2,z_3)\in\mathbb{C}^4:\ f(z_0,z_1,z_2,z_3)=0\}$ *be an isolated singularity defined by the Type B fewnomial* $f(z_0,z_1,z_2,z_3)$ *of weight type* $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ *where* $mult\,(f)\geq 3$. *Let* $H_1(V)$ *be the* 1*-st Hessian algebra. There exists negative weight derivation of* $H_1\left(V\right)$ *if and only if* f *is in one of the following forms after renumbering the variables* z_0, z_1, z_2 *and* z_3 *so that* $\alpha_0\geq\alpha_1\geq\alpha_2\geq\alpha_3$ *(we combine the cases that can be transformed into each other by simply renumbering the variables, which is caused by the equal weights of asymmetrical variables):*

(i) $f=z_0^3+z_1^3+z_2^3z_3+z_3^3z_0$. *In this case, the set of negative weight derivations of* $H_1\left(V\right)$ *is* $\left\{D|D=cz_3\frac{\partial}{\partial z_2},c\neq 0\right\}$;

(ii) $f=z_0^3+z_1^3z_3+z_2^4+z_3^3z_0$. *In this case, the set of negative weight derivations of* $H_1\left(V\right)$ *is* $\left\{D|D=cz_3\frac{\partial}{\partial z_1},c\neq 0\right\}$;

(iii) $f=z_0^3+z_1^3z_2+z_2^3z_0+z_3^{n_3}\left(n_3\geq 5\right)$. *In this case, the set of negative weight derivations of* $H_1\left(V\right)$ *is* $\left\{D|D=cz_2\frac{\partial}{\partial z_1},c\neq 0\right\}$.

Therefore, if $mult\left(f\right)\geq 4$, *there does not exist any negative weight derivation of* $H_1\left(V\right)$.

Proof. By the definition of Type B fewnomial, after renumbering, we may assume $f(z_0,z_1,z_2,z_3)=f\left(z_{i_1},z_{j_1},\ z_{j_2},z_{j_3}\right)=g(z_{i_1})+h(z_{j_1},z_{j_2},z_{j_3})$ where i_1,j_1,j_2 and j_3 are any permutations of $0,1,2$ and 3. $g=g\left(z_{i_1}\right)$ is Type (I) and is equals to $z_{i_1}^{n_{i_1}}$. $h=h\left(z_{j_1},z_{j_2},z_{j_3}\right)$ is Type (II) or Type (III). From $mult\left(f\right)\geq 3$, we can get $wt(f)>2\max\left\{\alpha_{i_1},\alpha_{j_1},\alpha_{j_2},\alpha_{j_3}\right\}=2\alpha_0$.

We renumber z_{j_1}, z_{j_2} and z_{j_3} again to satisfy the weight relationship $\alpha_{i_1}\geq\alpha_{i_2}\geq\alpha_{i_3}$. In the Type (II) case, h is in the form of $h=z_{j_1}^{n_{j_1}}z_{j_2}+z_{j_2}^{n_{j_2}}z_{j_3}+z_{j_3}^{n_{j_3}}$, $h=z_{j_1}^{n_{j_1}}z_{j_3}+z_{j_2}^{n_{j_2}}+z_{j_3}^{n_{j_3}}z_{j_2}$, $h=z_{j_1}^{n_{j_1}}z_{j_3}+z_{j_2}^{n_{j_2}}z_{j_1}+z_{j_3}^{n_{j_3}}$, $h=z_{j_1}^{n_{j_1}}+z_{j_2}^{n_{j_2}}z_{j_3}+z_{j_3}^{n_{j_3}}z_{j_1}$, $h=z_{j_1}^{n_{j_1}}z_{j_2}+z_{j_2}^{n_{j_2}}+z_{j_3}^{n_{j_3}}z_{j_1}$ or $h=z_{j_1}^{n_{j_1}}+z_{j_2}^{n_{j_2}}z_{j_1}+z_{j_3}^{n_{j_3}}z_{j_2}$. In the Type (III) case, h is in the form of $h=z_{j_1}^{n_{j_1}}z_{j_2}+z_{j_2}^{n_{j_2}}z_{j_3}+z_{j_3}^{n_{j_3}}z_{j_1}$ or $h=z_{j_1}^{n_{j_1}}z_{j_3}+z_{j_2}^{n_{j_2}}z_{j_1}+z_{j_3}^{n_{j_3}}z_{j_2}$.

If f contains the monomial in proportion to $z_r^{n_r}$ where $r\in\{i_1,j_1,j_2,j_3\}$, from $n_r\alpha_r=wt(f)>2\max\left\{\alpha_{i_1},\alpha_{j_1},\alpha_{j_2},\alpha_{j_3}\right\}\geq 2\alpha_r$, we get $n_r>2$, which is equivalent to $n_r\geq 3$. So $n_{i_1}\geq 3$. If f contains the monomial in proportion to $z_r^{n_r}z_s$ where $r,s\in\{i_1,j_1,j_2,j_3\}$ and $r\neq s$, from $n_r\alpha_r+\alpha_s=wt(f)>2\max\left\{\alpha_{i_1},\alpha_{i_1},\alpha_{i_2},\alpha_{i_2}\right\}\geq\alpha_r+\alpha_s$, we get $n_r>1$ and $n_r\geq 2$.

Since f_{i_1} does not contain the variable z_{j_1}, z_{j_2} or z_{j_3}, it is clear that $f_{i_1j_1}, f_{i_1j_2}, f_{i_1j_3} = 0$ and $f_{i_1i_1} = n_{i_1}(n_{i_1}-1)z_{i_1}^{n_{i_1}-2}$ only contains the variable z_{i_1}. Since f_{j_1}, f_{j_2} and f_{j_3} do not contain the variable z_{i_1}, it is clear that $f_{j_1j_1}, f_{j_1j_2}, f_{j_1j_3}, f_{j_2j_2}, f_{j_2j_3}$ and $f_{j_3j_3}$ do not contain the variable z_{i_1}.

When $n_0 = 2$, we have the following discussions.

From $wt(f) > 2\alpha_0$, we know f cannot contain the monomial in proportion to z_0^2. So it has to contain the monomial in proportion to $z_0^2 z_{s'}$, in which $s' \in \{1,2,3\}$. From the structure of f, we know f does not contain the monomial in proportion to $z_{s'}^2 z_0$. Thus the ideal $(f_{i_1i_1}, f_{i_1j_1}, f_{i_1j_2}, f_{i_1j_3}, f_{j_1j_1}, f_{j_1j_2}, f_{j_1j_3}, f_{j_2j_2}, f_{j_2j_3}, f_{j_3j_3})$ contains z_0 and $z_{s'}$ which are in proportion to $f_{0s'}$ and f_{00} respectively, which means that the nonzero elements of $H_1(V)$ cannot be divided by z_0 and $z_{s'}$. Obviously we have $0 \neq i_1$ and $s' \neq i_1$. Since $n_{i_1}\alpha_{i_1} = wt(f) > 2\alpha_0 \geq 2\alpha_{i_1}$, it is clear that $n_{i_1} > 2$, which means $n_{i_1} \geq 3$. In the following paragraph, we refer to $z_{j'}$ as the variable different from z_0, $z_{s'}$ and z_{i_1}.

If $n_{i_1} = 3$, the ideal $(f_{i_1i_1}, f_{i_1j_1}, f_{i_1j_2}, f_{i_1j_3}, f_{j_1j_1}, f_{j_1j_2}, f_{j_1j_3}, f_{j_2j_2}, f_{j_2j_3}, f_{j_3j_3})$ contains z_{i_1} which is in proportion to $f_{i_1i_1}$, which means that the nonzero element of $H_1(V)$ cannot be divided by z_{i_1}. In this case, if there exists some negative weight derivation D, D must be in the form of $D = c'\frac{\partial}{\partial z_{j'}}$.

If $c' \neq 0$, we have $(f_{i_1i_1}, f_{i_1j_1}, f_{i_1j_2}, f_{i_1j_3}, f_{j_1j_1}, f_{j_1j_2}, f_{j_1j_3}, f_{j_2j_2}, f_{j_2j_3}, f_{j_3j_3}) = (1)$ because at least one of $f_{j_1j'}$, $f_{j_2j'}$ and $f_{j_3j'}$ is in proportion to a power of $z_{j'}$ and we can use D to reduce the power to 0. However, 1 is not in the ideal $(f_{i_1i_1}, f_{i_1j_1}, f_{i_1j_2}, f_{i_1j_3}, f_{j_1j_1}, f_{j_1j_2}, f_{j_1j_3}, f_{j_2j_2}, f_{j_2j_3}, f_{j_3j_3})$ since $wt(f_{rs}) \geq wt(f_{00}) = wt(f) - 2\alpha_0 > 0$ for $r, s \in \{i_1, j_1, j_2, j_3\}$ when f_{rs} is not equal to 0. This leads to a contradiction. Thus $D = 0$.

If $n_{i_1} > 3$, which is equivalent to $n_{i_1} \geqslant 4$, the ideal $(f_{i_1i_1}, f_{i_1j_1}, f_{i_1j_2}, f_{i_1j_3}, f_{j_1j_1}, f_{j_1j_2}, f_{j_1j_3}, f_{j_2j_2}, f_{j_2j_3}, f_{j_3j_3}) = \left(z_0, z_{s'}, z_{i_1}^{k_{i_1}}, z_{j'}^{k'}\right)$. Therefore, we get $k_{i_1} = n_{i_1} - 2 \geq 2$. We have $k' \geq 1$ because at least one of $f_{0j'}, f_{s'j'}$ and $f_{j'j'}$ is in proportion to a power of $z_{j'}$ where we can choose the one with the smaller power or smallest power if more than one of them satisfies the restriction and 1 is not in the ideal $(f_{i_1i_1}, f_{i_1j_1}, f_{i_1j_2}, f_{i_1j_3}, f_{j_1j_1}, f_{j_1j_2}, f_{j_1j_3}, f_{j_2j_2}, f_{j_2j_3}, f_{j_3j_3})$ since $wt(f_{rs}) \geq wt(f_{00}) = wt(f) - 2\alpha_0 > 0$ for $r, s \in \{i_1, j_1, j_2, j_3\}$ when f_{rs} is not equal to 0.

When $\alpha_{i_1} \leq \alpha_{j'}$, if there exists some negative weight derivation D, D must be in the form of $D = c_{j'}z_{i_1}^{w_{j'}}\frac{\partial}{\partial z_{j'}} + c_{i_1}\frac{\partial}{\partial z_{i_1}}$. From $D\left(z_{i_1}^{k_{i_1}}\right) = c_{i_1}k_{i_1}z_{i_1}^{k_{i_1}-1} \in \left(z_0, z_{s'}, z_{i_1}^{k_{i_1}}, z_{j'}^{k_{j'}}\right)$, we get $c_{i_1} = 0$. Since $D\left(z_{j'}^{k_{j'}}\right) = k_{j'}c_{j'}z_{i_1}^{w_{j'}}z_{j'}^{k_{j'}-1} \in \left(z_0, z_{s'}, z_{i_1}^{k_{i_1}}, z_{j'}^{k_{j'}}\right)$, it is clear that $c_{i_1} = 0$ or $z_{i_1}^{w_{j'}}$ is divided by $z_{i_1}^{k_{i_1}}$, both of which imply $D = 0$ in the sense of $H_1(V)$.

When $\alpha_{i_1} > \alpha_{j'}$, if there exists some negative weight derivation D, D must be in the form of $D = c_{j'}\frac{\partial}{\partial z_{j'}} + c_{i_1}z_{j'}^{w_{i_1}}\frac{\partial}{\partial z_{i_1}}$. From $D\left(z_{j'}^{k_{j'}}\right) = c_{j'}k_{j'}z_{j'}^{k_{j'}-1} \in \left(z_0, z_{s'}, z_{i_1}^{k_{i_1}}, z_{j'}^{k_{j'}}\right)$, we obtain $c_{j'} = 0$. Therefore, $D = c_{i_1}z_{j'}^{w_{i_1}}\frac{\partial}{\partial z_{i_1}}$. Since $D\left(z_{i_1}^{k_{i_1}}\right) = k_{i_1}c_{i_1}z_{j'}^{w_{i_1}}z_{i_1}^{k_{i_1}-1} \in \left(z_0, z_{s'}, z_{i_1}^{k_{i_1}}, z_{j'}^{k_{j'}}\right)$, it is clear that $c_{i_1} = 0$ or $z_{j'}^{w_{i_1}}$ is divided by $z_{j'}^{k_{j'}}$, both of which imply $D = 0$ in the sense of $H_1(V)$. Thus $D = 0$.

In conclusion, if $n_0 = 2$, there does not exist negative weight derivation for any f in Type B.

When $n_0 \geq 3$, we have the following discussions.

We figure out what case we can exclude first.

From the weight relationship $\alpha_{j_1} \geq \alpha_{j_2} \geq \alpha_{j_3}$, if there exists some negative weight derivation D, D must be in the form of $D = p_{i_1}(z_{j_1}, z_{j_2}, z_{j_3})\frac{\partial}{\partial z_{i_1}} + p_{j_1}(z_{i_1}, z_{j_2}, z_{j_3})\frac{\partial}{\partial z_{j_1}} + p_{j_2}(z_{i_1}, z_{j_3})\frac{\partial}{\partial z_{j_2}} + p_{j_3}(z_{i_1})\frac{\partial}{\partial z_{j_3}}$. Note that $D(f_{i_1i_1}) = n_{i_1}(n_{i_1}-1)(n_{i_1}-2)p_{i_1}(z_{j_1}, z_{j_2}, z_{j_3})z_{i_1}^{n_{i_1}-3} \in (f_{j_1j_1}, f_{j_1j_2}, f_{j_1j_3}, f_{j_2j_2}, f_{j_2j_3}, f_{j_3j_3})$. Any nonzero element of the set $\{f_{j_1j_1}, f_{j_1j_2}, f_{j_1j_3}, f_{j_2j_2}, f_{j_2j_3}, f_{j_3j_3}\}$ cannot be divided by z_{i_1}.

If $p_{i_1}(z_{j_1}, z_{j_2}, z_{j_3}) \neq 0$, there exists some nonzero $f_{rs}\,(r, s \in \{j_1, j_2, j_3\})$ satisfying $wt(f_{rs}) \leq wt(p_{i_1}(z_{j_1}, z_{j_2}, z_{j_3}))$. Therefore, we have $\alpha_0 \leq 3\alpha_0 - \alpha_r - \alpha_s \leq wt(f_{rs}) \leq wt(p_{i_1}(z_{j_1}, z_{j_2}, z_{j_3})) < \alpha_{i_1} \leq \alpha_0$. This leads to a contradiction.

Thus $p_{i_1}(z_{j_1}, z_{j_2}, z_{j_3}) = 0$. Therefore, $D = p_{j_1}(z_{i_1}, z_{j_2}, z_{j_3})\frac{\partial}{\partial z_{j_1}} + p_{j_2}(z_{i_1}, z_{j_3})\frac{\partial}{\partial z_{j_2}} + p_{j_3}(z_{i_1})\frac{\partial}{\partial z_{j_3}}$.

Since $f_{i_1i_1}$ is in proportion to $z_{i_1}^{n_{i_1}-2}$, the monomial in $p_{j_1}(z_{i_1}, z_{j_2}, z_{j_3}), p_{j_2}(z_{i_1,}, z_{j_3})$ or $p_{j_3}(z_{i_1})$ is 0 in the sense of $H_1(V)$ if it is divided by $z_{i_1}^{n_{i_1}-2}$.

Thus we obtain $D(f_{rs}) \in (f_{j_1j_1}, f_{j_1j_2}, f_{j_1j_3}, f_{j_2j_2}, f_{j_2j_3}, f_{j_3j_3})$.

We define

$$\begin{cases} p_{j_1}(z_{i_1}, z_{j_2}, z_{j_3}) = \sum_{k=0}^{n_{i_1}-3} z_{i_1}^k p_{j_1}^{(k)}(z_{j_2}, z_{j_3}) \\ p_{j_2}(z_{i_1}, z_{j_3}) = \sum_{k=0}^{n_{i_1}-3} z_{i_1}^k c_{j_2}^{(k)} z_{j_3}^{k_{j_2}^{(k)}} \\ p_{j_3}(z_{i_1}) = \sum_{k=0}^{n_{i_1}-3} z_{i_1}^k c_{j_3}^{(k)} \end{cases}.$$

If we define $D^{(k)} = p_{j_1}^{(k)}(z_{j_2}, z_{j_3})\frac{\partial}{\partial z_{j_1}} + c_{j_2}^{(k)} z_{j_3}^{k_{j_2}^{(k)}}\frac{\partial}{\partial z_{j_2}} + c_{j_3}^{(k)}\frac{\partial}{\partial z_{j_3}}$, we get $D = \sum_{k=0}^{n_{i_1}-3} z_{i_1}^k D^{(k)}$.

On the one hand, $D(f_{rs}) = \sum_{k=0}^{n_{i_1}-3} z_{i_1}^k D^{(k)}(f_{rs})$.

On the other hand, from $D(f_{rs}) \in (f_{j_1j_1}, f_{j_1j_2}, f_{j_1j_3}, f_{j_2j_2}, f_{j_2j_3}, f_{j_3j_3})$, we have

$$\begin{aligned} D(f_{rs}) =& \varphi_{j_1j_1}(z_{i_1}, z_{j_1}, z_{j_2}, z_{j_3}) f_{j_1j_1} + \varphi_{j_1j_2}(z_{i_1}, z_{j_1}, z_{j_2}, z_{j_3}) f_{j_1j_2} \\ &+ \varphi_{j_1j_3}(z_{i_1}, z_{j_1}, z_{j_2}, z_{j_3}) f_{j_1j_3} + \varphi_{j_2j_2}(z_{i_1}, z_{j_1}, z_{j_2}, z_{j_3}) f_{j_2j_2} \\ &+ \varphi_{j_2j_3}(z_{i_1}, z_{j_1}, z_{j_2}, z_{j_3}) f_{j_2j_3} + \varphi_{j_3j_3}(z_{i_1}, z_{j_1}, z_{j_2}, z_{j_3}) f_{j_3j_3}. \end{aligned}$$

If

$$\begin{cases} \varphi_{j_1j_1}\left(z_{i_1},z_{j_1},z_{j_2},z_{j_3}\right)=\sum\limits_{k=0}^{n_{i_1}-3} z_{i_1}^k\varphi_{j_1j_1}^{(k)}\left(z_{j_1},z_{j_2},z_{j_3}\right) \\ \varphi_{j_1j_2}\left(z_{i_1},z_{j_1},z_{j_2},z_{j_3}\right)=\sum\limits_{k=0}^{n_{i_1}-3} z_{i_1}^k\varphi_{j_1j_2}^{(k)}\left(z_{j_1},z_{j_2},z_{j_3}\right) \\ \varphi_{j_1j_3}\left(z_{i_1},z_{j_1},z_{j_2},z_{j_3}\right)=\sum\limits_{k=0}^{n_{i_1}-3} z_{i_1}^k\varphi_{j_1j_3}^{(k)}\left(z_{j_1},z_{j_2},z_{j_3}\right) \\ \varphi_{j_2j_2}\left(z_{i_1},z_{j_1},z_{j_2},z_{j_3}\right)=\sum\limits_{k=0}^{n_{i_1}-3} z_{i_1}^k\varphi_{j_2j_2}^{(k)}\left(z_{j_1},z_{j_2},z_{j_3}\right) \\ \varphi_{j_2j_3}\left(z_{i_1},z_{j_1},z_{j_2},z_{j_3}\right)=\sum\limits_{k=0}^{n_{i_1}-3} z_{i_1}^k\varphi_{j_2j_3}^{(k)}\left(z_{j_1},z_{j_2},z_{j_3}\right) \\ \varphi_{j_3j_3}\left(z_{i_1},z_{j_1},z_{j_2},z_{j_3}\right)=\sum\limits_{k=0}^{n_{i_1}-3} z_{i_1}^k\varphi_{j_3j_3}^{(k)}\left(z_{j_1},z_{j_2},z_{j_3}\right) \end{cases},$$

we have

$$\begin{aligned} & D\left(f_{rs}\right) \\ =& \left(\sum_{k=0}^{n_1-3} z_{i_1}^k\varphi_{j_1j_1}^{(k)}\left(z_{j_1},z_{j_2},z_{j_3}\right)\right)f_{j_1j_1}+\left(\sum_{k=0}^{n_{i_1}-3} z_{i_1}^k\varphi_{j_1j_2}^{(k)}\left(z_{j_1},z_{j_2},z_{j_3}\right)\right)f_{j_1j_2} \\ & +\left(\sum_{k=0}^{n_{i_1}-3} z_{i_1}^k\varphi_{j_1j_3}^{(k)}\left(z_{j_1},z_{j_2},z_{j_3}\right)\right)f_{j_1j_3}+\left(\sum_{k=0}^{n_{i_1}-3} z_{i_1}^k\varphi_{j_2j_2}^{(k)}\left(z_{j_1},z_{j_2},z_{j_3}\right)\right)f_{j_2j_2} \\ & +\left(\sum_{k=0}^{n_{i_1}-3} z_{i_1}^k\varphi_{j_2j_3}^{(k)}\left(z_{j_1},z_{j_2},z_{j_3}\right)\right)f_{j_2j_3}+\left(\sum_{k=0}^{n_{i_1}-3} z_{i_1}^k\varphi_{j_3j_3}^{(k)}\left(z_{j_1},z_{j_2},z_{j_3}\right)\right)f_{j_3j_3} \\ =& \sum_{k=0}^{n_{i_1}-3} z_{i_1}^k\left(\varphi_{j_1j_1}^{(k)}\left(z_{j_1},z_{j_2},z_{j_3}\right)f_{j_1j_1}+\varphi_{j_1j_2}^{(k)}\left(z_{j_1},z_{j_2},z_{j_3}\right)f_{j_1j_2}+\varphi_{j_1j_3}^{(k)}\left(z_{j_1},z_{j_2},z_{j_3}\right)f_{j_1j_3}\right. \\ & \left.+\varphi_{j_2j_2}^{(k)}\left(z_{j_1},z_{j_2},z_{j_3}\right)f_{j_2j_2}+\varphi_{j_2j_3}^{(k)}\left(z_{j_1},z_{j_2},z_{j_3}\right)f_{j_2j_3}+\varphi_{j_3j_3}^{(k)}\left(z_{j_1},z_{j_2},z_{j_3}\right)f_{j_3j_3}\right). \end{aligned}$$

So

$$\begin{aligned} & D^{(k)}\left(f_{rs}\right) \\ =&\varphi_{j_1j_1}^{(k)}\left(z_{j_1},z_{j_2},z_{j_3}\right)f_{j_1j_1}+\varphi_{j_1j_2}^{(k)}\left(z_{j_1},z_{j_2},z_{j_3}\right)f_{j_1j_2}+\varphi_{j_1j_3}^{(k)}\left(z_{j_1},z_{j_2},z_{j_3}\right)f_{j_1j_3} \\ & +\varphi_{j_2j_2}^{(k)}\left(z_{j_1},z_{j_2},z_{j_3}\right)f_{j_2j_2}+\varphi_{j_2j_3}^{(k)}\left(z_{j_1},z_{j_2},z_{j_3}\right)f_{j_2j_3}+\varphi_{j_3j_3}^{(k)}\left(z_{j_1},z_{j_2},z_{j_3}\right)f_{j_3j_3}. \end{aligned}$$

If there exists such negative weight derivation D which is not equal to 0, there exists $k \in \{0,1,\ldots,n_{i_1}-3\}$ so that $D^{(k)} \neq 0$. The reverse is obvious.

Thus to judge the existence of negative weight derivation D of $H_1(V)$, we only need to consider the derivation D in the form of $D = p_{j_1}\left(z_{j_2},z_{j_3}\right)\frac{\partial}{\partial z_{j_1}} + c_{j_2} z_{j_3}^{k_{j_2}}\frac{\partial}{\partial z_{j_2}} + c_{j_3}\frac{\partial}{\partial z_{j_3}}$.

Since $wt(f) = n_0\alpha_0 \geq 3\alpha_0$, f does not contain any monomial which has multiplicity less than 3. If f contains a monomial in proportion to $z_{j_3}^3, z_{j_3}^2 z_{j_1}, z_{j_3}^2 z_{j_2}, z_{j_1}^2 z_{j_3}$ or $z_{j_2}^2 z_{j_3}, z_{j_3}$ is in proportion to $f_{j_3j_3}, f_{j_1j_3}, f_{j_2j_3}, f_{j_1j_1}$ or $f_{j_2j_2}$ respectively. Thus z_{j_3} is in the ideal $(f_{i_1i_1}, f_{i_1j_1}, f_{i_1j_2}, f_{i_1j_3}, f_{j_1j_1}, f_{j_1j_2}, f_{j_1j_3}, f_{j_2j_2}, f_{j_2j_3}, f_{j_3j_3})$. There does not exist nonzero element in $H_1(V)$ which is divided by z_{j_3}, from which we get $c_{j_3} = 0$.

In other cases, z_{j_3} is not in the ideal $(f_{i_1i_1}, f_{i_1j_1}, f_{i_1j_2}, f_{i_1j_3}, f_{j_1j_1}, f_{j_1j_2}, f_{j_1j_3}, f_{j_2j_2}, f_{j_2j_3}, f_{j_3j_3})$.

Considering the structure of f, there exists $m_{j_3} \in \mathbb{N}^*$ so that $z_{j_3}^{m_{j_3}}$ is in the ideal $(f_{i_1i_1}, f_{i_1j_1}, f_{i_1j_2}, f_{i_1j_3}, f_{j_1j_1}, f_{j_1j_2}, f_{j_1j_3}, f_{j_2j_2}, f_{j_2j_3}, f_{j_3j_3})$, while $z_{j_3}^{m_{j_3}-1}$ is not. In fact, $z_{j_3}^{m_{j_3}}$ is in proportion to $f_{j_1j_3}, f_{j_2j_3}$ or $f_{j_3j_3}$. Since $D\left(z_{j_3}^{m_{j_3}}\right) = c_{j_3} m_{j_3} z_{j_3}^{m_{j_3}-1}$ is in the ideal $(f_{i_1i_1}, f_{i_1j_1}, f_{i_1j_2}, f_{i_1j_3}, f_{j_1j_1}, f_{j_1j_2}, f_{j_1j_3}, f_{j_2j_2}, f_{j_2j_3}, f_{j_3j_3})$, we can get $c_{j_3} = 0$.

In conclusion, if there exists some negative weight derivation D, D must be in the form of $D = p_{j_1}(z_{j_2}, z_{j_3}) \frac{\partial}{\partial z_{j_1}} + c_{j_2} z_{j_3}^{k_{j_2}} \frac{\partial}{\partial z_{j_2}}$.

If f contains the monomial in proportional to $z_{j_1}^2 z_{j_2}$, there does not exist any monomial in f in proportion to $z_{j_2}^{n_2} z_{j_1}$. Thus $f_{j_1j_2}$ and $f_{j_1j_1}$ are in proportion to z_{j_1} and z_{j_2} respectively. There does not exist nonzero element in $H_1(V)$ which is divided by z_{j_1} or z_{j_2}. Therefore, $D = 0$.

If f contains the monomial in proportion to $z_{j_1}^{n_{j_1}} z_{j_2}\,(n_{j_1} \geq 3)$, from $D(f_{j_1j_1}) = 0$, we get $(n_{j_1} - 2)\, p_{j_1}(z_{j_2}, z_{j_3})\, z_{j_1}^{n_{j_1}-3} z_{j_2} + c_{j_2} z_{j_3}^{k_{j_2}} z_{j_1}^{n_{j_1}-2} = 0$, which is equivalent to $(n_{j_1} - 2)\, p_{j_1}(z_{j_2}, z_{j_3})\, z_{j_2} + c_{j_2} z_{j_3}^{k_{j_2}} z_{j_1} = 0$. Therefore, $c_{j_2} = 0$ and $p_{j_1}(z_{j_2}, z_{j_3}) = 0$. Thus $D = 0$.

The derivation $D = p_{j_1}(z_{j_2}, z_{j_3}) \frac{\partial}{\partial z_{j_1}} + c_{j_2} z_{j_3}^{k_{j_2}} \frac{\partial}{\partial z_{j_2}}$ does not contain the variable z_{i_1}, so we only need to consider the function h. There are only 5 cases left. If h is in Type (II), h is in the form of $h = z_{j_1}^{n_{j_1}} z_{j_3} + z_{j_2}^{n_{j_2}} + z_{j_3}^{n_{j_3}} z_{j_2}, h = z_{j_1}^{n_{j_1}} z_{j_3} + z_{j_2}^{n_{j_2}} z_{j_1} + z_{j_3}^{n_{j_3}}, h = z_{j_1}^{n_{j_1}} + z_{j_2}^{n_{j_2}} z_{j_3} + z_{j_3}^{n_{j_3}} z_{j_1}$ or $h = z_{j_1}^{n_{j_1}} + z_{j_2}^{n_{j_2}} z_{j_1} + z_{j_3}^{n_{j_3}} z_{j_2}$. If h is in Type (III), h is in the form of $h = z_{j_1}^{n_{j_1}} z_{j_3} + z_{j_2}^{n_{j_2}} z_{j_1} + z_{j_3}^{n_{j_3}} z_{j_2}$.

It is clear that for any nonzero f_{rs} where $r, s \in \{j_1, j_2, j_3\}$, $D(f_{rs})$ cannot be divided by $f_{i_1i_1}$. We also notice that $D(f_{i_1i_1}) = 0$. So we do not consider the element $f_{i_1i_1}$ in the following inclusion relations.

We have the following discussions in the case $h = z_{j_1}^{n_{j_1}} z_{j_3} + z_{j_2}^{n_{j_2}} + z_{j_3}^{n_{j_3}} z_{j_2}$.

From $mult(f) \geq 3$, we have $n_{j_2} \geq 3, n_{j_1} \geq 2$ and $n_{j_3} \geq 2$. Regardless of difference of constants, we have

$$\mathrm{Hess}(h) = \begin{bmatrix} h_{j_1j_1} & h_{j_1j_2} & h_{j_1j_3} \\ h_{j_2j_1} & h_{j_2j_2} & h_{j_2j_3} \\ h_{j_3j_1} & h_{j_3j_2} & h_{j_3j_3} \end{bmatrix} = \begin{bmatrix} z_{j_1}^{n_{j_1}-2} z_{j_3} & 0 & z_{j_1}^{n_{j_1}-1} \\ * & z_{j_2}^{n_{j_2}-2} & z_{j_3}^{n_{j_3}-1} \\ * & * & z_{j_3}^{n_{j_3}-2} z_{j_2} \end{bmatrix}.$$

If $n_{j_1} = 2$, we have $(h_{j_1j_1}, h_{j_1j_2}, h_{j_1j_3}, h_{j_2j_2}, h_{j_2j_3}, h_{j_3j_3}) = \left(z_{j_3}, z_{j_1}, z_{j_2}^{n_{j_2}-2}, z_{j_3}^{n_{j_3}-2} z_{j_2}\right)$. If there exists some negative weight derivation D, D must be in the form of $D = c_{j_2} \frac{\partial}{\partial z_{j_2}}$. If $n_{j_2} = 3$ or $n_{j_3} = 2$, we have $(h_{j_1j_1}, h_{j_1j_2}, h_{j_1j_3}, h_{j_2j_2}, h_{j_2j_3}, h_{j_3j_3}) = (z_{j_3}, z_{j_1}, z_{j_2})$. There does not exist any nonzero element in $H_1(V)$ which is divided by z_{j_2}. Thus $D = 0$. If $n_{j_2} \geq 4$ and $n_{j_3} \geq 3$, we can get $z_{j_2}^{n_{j_2}-2}$ is in the ideal $(h_{j_1j_1}, h_{j_1j_2}, h_{j_1j_3}, h_{j_2j_2}, h_{j_2j_3}, h_{j_3j_3})$, while $z_{j_2}^{n_{j_2}-3}$ is not. From $D\left(z_{j_2}^{n_{j_2}-2}\right) = (n_{j_2} - 2)\, c_{j_2} z_{j_2}^{n_{j_2}-3} \in (h_{j_1j_1}, h_{j_1j_2}, h_{j_1j_3}, h_{j_2j_2}, h_{j_2j_3}, h_{j_3j_3})$, we get $c_{j_2} = 0$. In conclusion, if $n_0 \geq 3$ and $n_{j_1} = 2$, we have $D = 0$.

If $n_{j_1} \geq 3$, from $D\left(z_{j_1}^{n_{j_1}-2} z_{j_3}\right) = (n_{j_1}-2)\, p_{j_1}(z_{j_2}, z_{j_3})\, z_{j_1}^{n_{j_1}-3} z_{j_3} = 0$, we get $p_{j_1}(z_{j_2}, z_{j_3}) = 0$. Therefore, if there exists some negative weight derivation D, D must be in the form of $D = c_{j_2} z_{j_3}^{k_{j_2}} \frac{\partial}{\partial z_{j_2}}$. From $D\left(z_{j_2}^{n_{j_2}-2}\right) = (n_{j_2}-2)\, c_{j_2} z_{j_3}^{k_{j_2}} z_{j_2}^{n_{j_2}-3} \in \left(z_{j_1}^{n_{j_1}-2} z_{j_3}, z_{j_1}^{n_{j_1}-1}\right)$, we get $c_{j_2} = 0$. Therefore, if $n_0 \geq 3$ and $n_{j_1} \geq 3$, we have $D = 0$.

In conclusion, if $n_0 \geq 3$, there does not exist negative weight derivation when f is in the form of $f = z_{i_1}^{n_{i_1}} + z_{j_1}^{n_{j_1}} z_{j_3} + z_{j_2}^{n_{j_2}} + z_{j_3}^{n_{j_3}} z_{j_2}$.

We have the following discussions in the case $h = z_{j_1}^{n_{j_1}} z_{j_3} + z_{j_2}^{n_{j_2}} z_{j_1} + z_{j_3}^{n_{j_3}}$.

From $mult(f) \geq 3$, we obtain $n_{j_3} \geq 3, n_{j_1} \geq 2$ and $n_{j_2} \geq 2$.

Regardless of difference of constants, we have

$$\text{Hess }(h) = \begin{bmatrix} h_{j_1 j_1} & h_{j_1 j_2} & h_{j_1 j_3} \\ h_{j_2 j_1} & h_{j_2 j_2} & h_{j_2 j_3} \\ h_{j_3 j_1} & h_{j_3 j_2} & h_{j_3 j_3} \end{bmatrix} = \begin{bmatrix} z_{j_1}^{n_{j_1}-2} z_{j_3} & z_{j_2}^{n_{j_2}-1} & z_{j_1}^{n_{j_1}-1} \\ * & z_{j_2}^{n_{j_2}-2} z_{j_1} & 0 \\ * & * & z_{j_3}^{n_{j_3}-2} \end{bmatrix}.$$

If $n_{j_1} = 2$, we have $(h_{j_1 j_1}, h_{j_1 j_2}, h_{j_1 j_3}, h_{j_2 j_2}, h_{j_2 j_3}, h_{j_3 j_3}) = \left(z_{j_3}, z_{j_1}, z_{j_2}^{n_{j_2}-1}\right)$. If there exists some negative weight derivation D, D must be in the form of $D = c_{j_2} \frac{\partial}{\partial z_{j_2}}$. If $n_{j_2} = 2$, we have $(h_{j_1 j_1}, h_{j_1 j_2}, h_{j_1 j_3}, h_{j_2 j_2}, h_{j_2 j_3}, h_{j_3 j_3}) = (z_{j_3}, z_{j_1}, z_{j_2})$. There does not exist any nonzero element in $H_1(V)$ which is divided by z_{j_2}. Thus $D = 0$. If $n_{j_2} \geq 3$, $z_{j_2}^{n_{j_2}-1}$ is in the ideal $(h_{j_1 j_1}, h_{j_1 j_2}, h_{j_1 j_3}, h_{j_2 j_2}, h_{j_2 j_3}, h_{j_3 j_3})$, while $z_{j_2}^{n_{j_2}-2}$ is not. From $D\left(z_{j_2}^{n_{j_2}-1}\right) = (n_{j_2}-1)\, c_{j_2} z_{j_2}^{n_{j_2}-2} \in (h_{j_1 j_1}, h_{j_1 j_2}, h_{j_1 j_3}, h_{j_2 j_2}, h_{j_2 j_3}, h_{j_3 j_3})$, we get $c_{j_2} = 0$. Thus $D = 0$. Therefore, we have $D = 0$ if $n_{j_1} = 2$.

If $n_{j_1} \geq 3$, from $D\left(z_{j_1}^{n_{j_1}-2} z_{j_3}\right) = (n_{j_1}-2)\, p_{j_1}(z_{j_2}, z_{j_3})\, z_{j_1}^{n_{j_1}-3} z_{j_3} = 0$, we get $p_{j_1}(z_{j_2}, z_{j_3}) = 0$. Therefore, $D = c_{j_2} z_{j_3}^{k_{j_2}} \frac{\partial}{\partial z_{j_2}}$. From $D\left(z_{j_2}^{n_{j_2}-1}\right) = (n_{j_2}-1)\, c_{j_2} z_{j_3}^{k_{j_2}} z_{j_2}^{n_{j_2}-2} \in \left(z_{j_1}^{n_{j_1}-2} z_{j_3}\right)$, we get $c_{j_2} = 0$. Therefore, we have $D = 0$ if $n_{j_1} \geq 3$.

In conclusion, if $n_0 \geq 3$, there does not exist negative weight derivation when f is in the form of $f = z_{i_1}^{n_{i_1}} + z_{j_1}^{n_{j_1}} z_{j_3} + z_{j_2}^{n_{j_2}} z_{j_1} + z_{j_3}^{n_{j_3}}$.

We have the following discussions in the case $h = z_{j_1}^{n_{j_1}} + z_{j_2}^{n_{j_2}} z_{j_3} + z_{j_3}^{n_{j_3}} z_{j_1}$.

From $mult(f) \geq 3$, we obtain $n_{j_1} \geq 3, n_{j_2} \geq 2$ and $n_{j_3} \geq 2$.

Regardless of difference of constants, we have

$$\text{Hess}(h) = \begin{bmatrix} h_{j_1 j_1} & h_{j_1 j_2} & h_{j_1 j_3} \\ h_{j_2 j_1} & h_{j_2 j_2} & h_{j_2 j_3} \\ h_{j_3 j_1} & h_{j_3 j_2} & h_{j_3 j_3} \end{bmatrix} = \begin{bmatrix} z_{j_1}^{n_{j_1}-2} & 0 & z_{j_3}^{n_{j_3}-1} \\ * & z_{j_2}^{n_{j_2}-2} z_{j_3} & z_{j_2}^{n_{j_2}-1} \\ * & * & z_{j_3}^{n_{j_3}-2} z_{j_1} \end{bmatrix}.$$

If $n_{j_1} = 3, z_{j_1}$ is in the ideal $(h_{j_1 j_1}, h_{j_1 j_2}, h_{j_1 j_3}, h_{j_2 j_2}, h_{j_2 j_3}, h_{j_3 j_3})$. There does not exist nonzero element in $H_1(V)$ which can be divided by z_{j_1}. Thus if there exists some negative weight derivation D, D must be in the form of $D = c_{j_2} z_{j_3}^{k_{j_2}} \frac{\partial}{\partial z_{j_2}}$.

If $n_{j_2} = 2$, we have $(h_{j_1 j_1}, h_{j_1 j_2}, h_{j_1 j_3}, h_{j_2 j_2}, h_{j_2 j_3}, h_{j_3 j_3}) = (z_{j_3}, z_{j_1}, z_{j_2})$. There does not exist nonzero element in $H_1(V)$ which can be divided by z_{j_2}. Thus $D = 0$.

If $n_{j_2} \geq 3$, we have $3\alpha_{j_1} = wt(f) = n_{j_2}\alpha_{j_2} + \alpha_{j_3} > 2\alpha_{j_2} + \alpha_{j_3}$. Thus we get $\alpha_{j_1} > \alpha_{j_3}$, otherwise from $\alpha_{j_1} \geq \alpha_{j_2} \geq \alpha_{j_3}$, we have $\alpha_{j_1} = \alpha_{j_2} = \alpha_{j_3}$ and $3\alpha_{j_1} = 2\alpha_{j_2} + \alpha_{j_3}$, which leads to a contradiction. From $3\alpha_{j_1} = wt(f) = n_{j_3}\alpha_{j_3} + \alpha_{j_1} < (n_{j_3}+1)\,\alpha_{j_1}$, we get $n_{j_3} > 2$. Therefore, $n_{j_3} \geq 3$. Regardless of difference of constants and useless polynomials, we have

$$\text{Hess }(h) = \begin{bmatrix} h_{j_1j_1} & h_{j_1j_2} & h_{j_1j_3} \\ h_{j_2j_1} & h_{j_2j_2} & h_{j_2j_3} \\ h_{j_3j_1} & h_{j_3j_2} & h_{j_3j_3} \end{bmatrix} = \begin{bmatrix} z_{j_1} & 0 & z_{j_3}^{n_{j_3}-1} \\ * & z_{j_2}^{n_{j_2}-2} z_{j_3} & z_{j_2}^{n_{j_2}-1} \\ * & * & 0 \end{bmatrix}.$$

It is obvious that $D\left(z_{j_3}^{n_{j_3}-1}\right) = 0 \in (h_{j_1j_1}, h_{j_1j_2}, h_{j_1j_3}, h_{j_2j_2}, h_{j_2j_3}, h_{j_3j_3})$. From $D\left(z_{j_2}^{n_{j_2}-1}\right) = (n_{j_2}-1)\, c_{j_2} z_{j_3}^{k_{j_2}} z_{j_2}^{n_{j_2}-2} \in \left(z_{j_1}, z_{j_3}^{n_{j_3}-1}, z_{j_2}^{n_{j_2}-2} z_{j_3}\right)$, it is clear that $c_{j_2} = 0$ or $k_{j_2} \geq 1$.

Note that $D\left(z_{j_2}^{n_{j_2}-2} z_{j_3}\right) = (n_{j_2}-2)\, c_{j_2} z_{j_3}^{k_{j_2}+1} z_{j_2}^{n_{j_2}-3} \in \left(z_{j_1}, z_{j_3}^{n_{j_3}-1}\right)$. If $c_{j_2} \neq 0, z_{j_3}^{k_{j_2}+1}$ can be divided by $z_{j_3}^{n_{j_3}-1}$. $z_{j_3}^{k_{j_2}}$ cannot be divided by $z_{j_3}^{n_{j_3}-1}$, otherwise $D = 0$ in the sense of derivation of $H_1(V)$, which is equivalent to $c_{j_2} = 0$. Thus $k_{j_2}+1 \geq n_{j_3}-1 > k_{j_2}$, which is equivalent to $k_{j_2} = n_{j_3}-2$. Thus $D = c_{j_2} z_{j_3}^{n_{j_3}-2} \frac{\partial}{\partial z_{j_2}}$ and the only thing we need to check is whether $k_{j_2} = n_{j_3}-2$ satisfies the "negatively weighted" restriction.

Solving the equations

$$\begin{cases} 3\alpha_{j_1} = wt(f) \\ n_{j_2}\alpha_{j_2} + \alpha_{j_3} = wt(f) \\ n_{j_3}\alpha_{j_3} + \alpha_{j_1} = wt(f) \end{cases},$$

we get

$$\begin{cases} \alpha_{j_1} = \frac{1}{3} wt(f) \\ \alpha_{j_2} = \frac{1}{n_{j_2}}\left(1 - \frac{2}{3n_{j_3}}\right) wt(f) \\ \alpha_{j_3} = \frac{2}{3n_{j_3}} wt(f) \end{cases}.$$

From the "negatively weighted" restriction of $D = c_{j_2} z_{j_3}^{n_{j_3}-2} \frac{\partial}{\partial z_{j_2}}$, we get $(n_{j_3}-2)\,\alpha_{j_3} < \alpha_{j_2}$. Thus $(n_{j_3}-2)\,\frac{2}{3n_{j_3}} wt(f) < \frac{1}{n_{j_2}}\left(1 - \frac{2}{3n_{j_3}}\right) wt(f)$, which is equivalent to $n_{j_2} < \frac{1}{2}\left(\frac{4}{n_{j_3}-2} + 3\right)$. Since $n_{j_2} \geq 3$, we have $n_{j_3} < \frac{10}{3}$. From $n_{j_3} \geq 3$, we get $n_{j_3} = 3$. From $n_{j_2} < \frac{1}{2}\left(\frac{4}{n_{j_3}-2} + 3\right) = \frac{7}{2}$ and $n_{j_2} \geq 3$, we get $n_{j_2} = 3$.

In this special case, h is in the form of $h = z_{j_1}^3 + z_{j_2}^3 z_{j_3} + z_{j_3}^3 z_{j_1}$. Regardless of difference of constants and useless polynomials, we have

$$\text{Hess}(h) = \begin{bmatrix} h_{j_1j_1} & h_{j_1j_2} & h_{j_1j_3} \\ h_{j_2j_1} & h_{j_2j_2} & h_{j_2j_3} \\ h_{j_3j_1} & h_{j_3j_2} & h_{j_3j_3} \end{bmatrix} = \begin{bmatrix} z_{j_1} & 0 & z_{j_3}^2 \\ * & z_{j_2}z_{j_3} & z_{j_2}^2 \\ * & * & 0 \end{bmatrix}.$$

The weights of α_0, α_1, α_2 and α_3 are

$$\begin{cases} \alpha_{j_1} = \frac{1}{3} wt(f) \\ \alpha_{j_2} = \frac{7}{27} wt(f) \\ \alpha_{j_3} = \frac{2}{9} wt(f) \end{cases}.$$

Under this circumstance, the set of negative weight derivations of $H_1(V)$ is $\left\{D | D = c z_{j_3} \frac{\partial}{\partial z_{j_2}}, c \neq 0\right\}$.

If $n_{j_1} \geq 4$, from $D\left(z_{j_1}^{n_{j_1}-2}\right) = (n_{j_1}-2)\,p_{j_1}(z_{j_2},z_{j_3})\,z_{j_1}^{n_{j_1}-3} = 0$, we get $p_{j_1}(z_{j_2},z_{j_3}) = 0$. Therefore, if there exists some negative weight derivation D, D must be in the form of $D = c_{j_2} z_{j_3}^{k_{j_2}} \frac{\partial}{\partial z_{j_2}}$.

We assume $c_{j_2} \neq 0$. Note that $D\left(z_{j_2}^{n_{j_2}-2} z_{j_3}\right) = (n_{j_2}-2)\,c_{j_2} z_{j_3}^{k_{j_2}+1} z_{j_2}^{n_{j_2}-3} \in \left(z_{j_1}^{n_{j_1}-2}, z_{j_3}^{n_{j_3}-1}\right)$. If $c_{j_2} \neq 0$, we have $z_{j_3}^{k_{j_2}+1}$ is divided by $z_{j_3}^{n_{j_3}-1}$. Thus $k_{j_2} \geq n_{j_3}-2$.

If $\alpha_{j_1} = \alpha_{j_2} = \alpha_{j_3}$, all the nonzero elements in Hess(h) are of the same weight. Since D is negatively weighted, we have $D\left(z_{j_2}^{n_{j_2}-2} z_{j_3}\right) = 0$, from which we get $c_{j_2} = 0$. This leads to a contradiction. Thus we have $c_{j_2} = 0$ and $D = 0$.

If $\alpha_{j_1} > \alpha_{j_3}$, we have $n_{j_1}\alpha_{j_1} = wt(f) = n_{j_3}\alpha_{j_3} + \alpha_{j_1} < (n_{j_3}+1)\,\alpha_{j_1}$. Thus $n_{j_3} > n_{j_1} - 1$. In other words, $n_{j_3} \geq n_{j_1} \geq 4$. Since f is quasi-homogeneous, we have $\alpha_{j_1} = \frac{1}{n_{j_1}} wt(f)$ and $\alpha_{j_3} = \frac{1}{n_{j_3}}(wt(f) - \alpha_{j_1}) = \frac{1}{n_{j_3}}\left(1 - \frac{1}{n_{j_1}}\right) wt(f)$. From $wt\left(z_{j_3}^{n_{j_3}-2}\right) = \left(1 - \frac{2}{n_{j_3}}\right)\left(1 - \frac{1}{n_{j_1}}\right) wt(f) = \left(1 - \frac{2}{n_{j_3}}\right)(n_{j_1}-1)\,\alpha_{j_1} \geq \left(1 - \frac{2}{4}\right) \times 3\alpha_{j_1} = \frac{3}{2}\alpha_{j_1} > \alpha_{j_1}$ and $wt\left(z_{j_3}^{k_{j_2}}\right) < \alpha_{j_2} \leq \alpha_{j_1}$, we have $wt\left(z_{j_3}^{k_{j_2}}\right) < wt\left(z_{j_3}^{n_{j_3}-2}\right)$, which leads to $k_{j_2} < n_{j_3} - 2$ and it is in contradiction to $k_{j_2} \geq n_{j_3} - 2$. Thus $c_{j_2} = 0$ and $D = 0$.

Therefore, when $n_0 \geq 3$ and $n_{j_1} \geq 4$, there does not exist negative weight derivation of $H_1(V)$.

In conclusion, when $n_0 \geq 3$, there exist negative weight derivations for $f = z_{i_1}^{n_{i_1}} + z_{j_1}^{n_{j_1}} + z_{j_2}^{n_{j_2}} z_{j_3} + z_{j_3}^{n_{j_3}} z_{j_1}$ in Type B if and only if f is in the form of $f = z_{i_1}^{n_{i_1}} + z_{j_1}^{3} + z_{j_2}^{3} z_{j_3} + z_{j_3}^{3} z_{j_1}\,(n_{i_1} \geq 3)$. If such condition satisfies, the set of negative weight derivations of f is $\left\{D | D = c z_{j_3} \frac{\partial}{\partial z_{j_2}}, c \neq 0\right\}$.

We have the following discussions in the case $h = z_{j_1}^{n_{j_1}} + z_{j_2}^{n_{j_2}} z_{j_1} + z_{j_3}^{n_{j_3}} z_{j_2}$.

From $mult(f) \geq 3$, we have $n_{j_1} \geq 3, n_{j_2} \geq 2$ and $n_{j_3} \geq 2$.

Regardless of difference of constants, we have

$$\mathrm{Hess}(h) = \begin{bmatrix} h_{j_1j_1} & h_{j_1j_2} & h_{j_1j_3} \\ h_{j_2j_1} & h_{j_2j_2} & h_{j_2j_3} \\ h_{j_3j_1} & h_{j_3j_2} & h_{j_3j_3} \end{bmatrix} = \begin{bmatrix} z_{j_1}^{n_{j_1}-2} & z_{j_2}^{n_{j_2}-1} & 0 \\ * & z_{j_2}^{n_{j_2}-2} z_{j_1} & z_{j_3}^{n_{j_3}-1} \\ * & * & z_{j_3}^{n_{j_3}-2} z_{j_2} \end{bmatrix}.$$

From $D\left(z_{j_1}^{n_{j_1}-2}\right) = (n_{j_1}-2)\,p_{j_1}(z_{j_2},z_{j_3})\,z_{j_1}^{n_{j_1}-3} = 0$, we get $p_{j_1}(z_{j_2},z_{j_3}) = 0$. Therefore, D is in the form of $D = c_{j_2} z_{j_3}^{k_{j_2}} \frac{\partial}{\partial z_{j_2}}$.

From $D\left(z_{j_2}^{n_{j_2}-1}\right) = (n_{j_2}-1)\,c_{j_2} z_{j_3}^{k_{j_2}} z_{j_2}^{n_{j_2}-2} \in \left(z_{j_1}^{n_{j_1}-2}\right)$, we get $c_{j_2} = 0$. Therefore, we have $D = 0$.

In conclusion, if $n_0 \geq 3$, there does not exist negative weight derivation for any $f = z_{i_1}^{n_{i_1}} + z_{j_1}^{n_{j_1}} + z_{j_2}^{n_{j_2}} z_{j_1} + z_{j_3}^{n_{j_3}} z_{j_2}$ in Type B.

We have the following discussions in the case $h = z_{j_1}^{n_{j_1}} z_{j_3} + z_{j_2}^{n_{j_2}} z_{j_1} + z_{j_3}^{n_{j_3}} z_{j_2}$.

From $mult(f) \geq 3$, we have $n_{j_1} \geq 2, n_{j_2} \geq 2$ and $n_{j_3} \geq 2$.

Regardless of difference of constants, we have

$$\mathrm{Hess}(h) = \begin{bmatrix} h_{j_1j_1} & h_{j_1j_2} & h_{j_1j_3} \\ h_{j_2j_1} & h_{j_2j_2} & h_{j_2j_3} \\ h_{j_3j_1} & h_{j_3j_2} & h_{j_3j_3} \end{bmatrix} = \begin{bmatrix} z_{j_1}^{n_{j_1}-2} z_{j_3} & z_{j_2}^{n_{j_2}-1} & z_{j_1}^{n_{j_1}-1} \\ * & z_{j_2}^{n_{j_2}-2} z_{j_1} & z_{j_3}^{n_{j_3}-1} \\ * & * & z_{j_3}^{n_{j_3}-2} z_{j_2} \end{bmatrix}.$$

If $n_{j_1}=2$, z_{j_3} and z_{j_1} are in the ideal of $(h_{j_1j_1},h_{j_1j_2},h_{j_1j_3},h_{j_2j_2},h_{j_2j_3},h_{j_3j_3})$. Thus $D=c_{j_2}\frac{\partial}{\partial z_{j_2}}$. If $n_{j_2}=2$, z_{j_2} is in the ideal of $(h_{j_1j_1},h_{j_1j_2},h_{j_1j_3},h_{j_2j_2},h_{j_2j_3},h_{j_3j_3})$. Thus $D=0$. If $n_{j_2}\geq 3$, we have $D\left(z_{j_2}^{n_{j_2}-1}\right)=c_{j_2}\left(n_{j_2}-1\right)z_{j_2}^{n_{j_2}-2}\in\left(z_{j_3}\right)$. Thus $c_{j_2}=0$ and $D=0$. In conclusion, we have $D=0$ if $n_{j_1}=2$.

If $n_{j_1}\geq 3$, from $D\left(z_{j_1}^{n_{j_1}-2}z_{j_3}\right)=\left(n_{j_1}-2\right)p_{j_1}\left(z_{j_2},z_{j_3}\right)z_{j_1}^{n_{j_1}-3}z_{j_3}=0$, we get $p_{j_1}\left(z_{j_2},z_{j_3}\right)=0$. From $D\left(z_{j_2}^{n_{j_2}-1}\right)=\left(n_{j_2}-1\right)c_{j_2}z_{j_3}^{k_{j_2}}z_{j_2}^{n_{j_2}-2}\in\left(z_{j_1}^{n_{j_1}-2}z_{j_3}\right)$, we get $c_{j_2}=0$. In conclusion, we have $D=0$ if $n_{j_1}\geq 3$.

In conclusion, if $n_0\geq 3$, there does not exist negative weight derivation for any $f=z_{i_1}^{n_{i_1}}+z_{j_1}^{n_{j_1}}z_{j_3}+z_{j_2}^{n_{j_2}}z_{j_1}+z_{j_3}^{n_{j_3}}z_{j_2}$ in Type B case.

We can conclude that in Type B case, when $n_0\geq 3$, there exists negative weight derivation if and only if f is in the form of $f=z_{i_1}^{n_{i_1}}+z_{j_1}^3+z_{j_2}^3z_{j_3}+z_{j_3}^3z_{j_1}\left(n_{i_1}\geq 3\right)$. If such condition satisfies, the set of negative weight derivations of f is $\left\{D|D=cz_{j_3}\frac{\partial}{\partial z_{j_2}},c\neq 0\right\}$.

Therefore, in Type B case, when $n_0\geq 2$, there exists negative weight derivation if and only if f is in the form of $f=z_{i_1}^{n_{i_1}}+z_{j_1}^3+z_{j_2}^3z_{j_3}+z_{j_3}^3z_{j_1}\left(n_{i_1}\geq 3\right)$. If such condition satisfies, the set of negative weight derivations of f is $\left\{D|D=cz_{j_3}\frac{\partial}{\partial z_{j_2}},c\neq 0\right\}$.

Next we will discuss the relations between z_{i_1}, z_{j_1}, z_{j_2} and z_{j_3} and z_0, z_1, z_2 and z_3.

The solution of the equations

$$\begin{cases} n_{i_1}\alpha_{i_1}=wt(f)\\ 3\alpha_{j_1}=wt(f)\\ 3\alpha_{j_2}+\alpha_{j_3}=wt(f)\\ 3\alpha_{j_3}+\alpha_{j_1}=wt(f)\end{cases}$$

is

$$\begin{cases} \alpha_{i_1}=\frac{1}{n_{i_1}}wt(f)\\ \alpha_{j_1}=\frac{1}{3}wt(f)\\ \alpha_{j_2}=\frac{7}{27}wt(f)\\ \alpha_{j_3}=\frac{2}{9}wt(f)\end{cases}.$$

When $n_{i_1}=3$, we have $\alpha_{j_1}=\alpha_{i_1}>\alpha_{j_2}>\alpha_{j_3}$, which means $(j_1,i_1,j_2,j_3)=(0,1,2,3)$ or $(i_1,j_1,j_2,j_3)=(0,1,2,3)$; when $n_{i_1}=4$, we have $\alpha_{j_1}>\alpha_{j_2}>\alpha_{i_1}>\alpha_{j_3}$, which means $(j_1,j_2,i_1,j_3)=(0,1,2,3)$; when $n_{i_1}\geq 5$, we have $\alpha_{j_1}>\alpha_{j_2}>\alpha_{j_3}>\alpha_{i_1}$, which means $(j_1,j_2,j_3,i_1)=(0,1,2,3)$.

Therefore, if $(V,0)=\{(z_0,z_1,z_2,z_3)\in\mathbb{C}^4\colon f(z_0,z_1,z_2,z_3)=0\}$ is an isolated singularity defined by the Type B fewnomial $f(z_0,z_1,z_2,z_3)$ of weight type $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ where $mult\,(f)\geq 3$, let $H_1(V)$ be the 1-st Hessian algebra and let D be a derivation of $H_1(V)$, then after renumbering the variables z_0, z_1, z_2 and z_3 to satisfy the weight relationship $\alpha_0\geq\alpha_1\geq\alpha_2\geq\alpha_3$, there exists negative weight derivation if and only if f is in one of the following forms (we combine the cases that can be transformed into each other by simply renumbering the variables, which is caused by the equal weights of asymmetrical variables):

(i) $f=z_0^3+z_1^3+z_2^3z_3+z_3^3z_0$. In this case, the set of negative weight derivations of $H_1\left(V\right)$ is $\left\{D|D=cz_3\frac{\partial}{\partial z_2},c\neq 0\right\}$;

(ii) $f=z_0^3+z_1^3z_3+z_2^4+z_3^3z_0$. In this case, the set of negative weight derivations of $H_1\left(V\right)$ is $\left\{D|D=cz_3\frac{\partial}{\partial z_1},c\neq 0\right\}$;

(iii) $f = z_0^3 + z_1^3 z_2 + z_2^3 z_0 + z_3^{n_3}\ (n_3 \geq 5)$. In this case, the set of negative weight derivations of $H_1(V)$ is $\left\{ D | D = cz_2 \frac{\partial}{\partial z_1}, c \neq 0 \right\}$.

Therefore, if $mult\,(f) \geq 4$, there does not exist any negative weight derivation of $H_1(V)$. □

4. Type C Fewnomial Case. In this section, we will discuss the Type C fewnomial case where $mult\,(f) \geq 3$. The overall conclusion is written in Proposition 4.1.

PROPOSITION 4.1 (Type C fewnomial case of Theorem B). *Let* $(V, 0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 \colon f(z_0, z_1, z_2, z_3) = 0\}$ *be an isolated singularity defined by the Type C fewnomial* $f(z_0, z_1, z_2, z_3)$ *of weight type* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ *where* $mult\,(f) \geq 3$. *Let* $H_1(V)$ *be the* 1*-st Hessian algebra. Then there does not exist negative weight derivation of* $H_1(V)$.

Proof. By the definition of Type C fewnomial, after renumbering, we may assume $f(z_0, z_1, z_2, z_3) = f(z_{i_1}, z_{i_2}, z_{j_1}, z_{j_2}) = g(z_{i_1}, z_{i_2}) + h(z_{j_1}, z_{j_2})$ where $g(z_{i_1}, z_{i_2})$ and $h(z_{j_1}, z_{j_2})$ are Type (I), Type (II) or Type (III) fewnomial. Here we assume $\alpha_{i_1} \geq \alpha_{i_2}$, $\alpha_{j_1} \geq \alpha_{j_2}$ and $\alpha_{i_1} \geq \alpha_{j_1}$ where i_1, i_2, j_1 and j_2 are any permutations of $0, 1, 2$ and 3.

g has the following possible 4 forms:

Type (I): $g = z_{i_1}^{n_{i_1}} + z_{i_2}^{n_{i_2}}$;

Type (II): $g = z_{i_1}^{n_{i_1}} z_{i_2} + z_{i_2}^{n_{i_2}}$ or $g = z_{i_1}^{n_{i_1}} + z_{i_2}^{n_{i_2}} z_{i_1}$;

Type (III): $g = z_{i_1}^{n_{i_1}} z_{i_2} + z_{i_2}^{n_{i_2}} z_{i_1}$.

h has the following possible 4 forms:

Type (I): $h = z_{j_1}^{n_{j_1}} + z_{j_2}^{n_{j_2}}$;

Type (II): $h = z_{j_1}^{n_{j_1}} z_{j_2} + z_{j_2}^{n_{j_2}}$ or $h = z_{j_1}^{n_{j_1}} + z_{j_2}^{n_{j_2}} z_{j_1}$;

Type (III): $h = z_{j_1}^{n_{j_1}} z_{j_2} + z_{j_2}^{n_{j_2}} z_{j_1}$.

It is obvious that $f_{i_1 j_1}, f_{i_1 j_2}, f_{i_2 j_1}$ and $f_{i_2 j_2}$ are equal to 0. If any of $f_{i_1 i_1}, f_{i_1 i_2}, f_{i_2 i_2}$ is not equal to 0, it does not contain the factor z_{j_1} or z_{j_2}. If any of $f_{j_1 j_1}, f_{j_1 j_2}, f_{j_2 j_2}$ is not equal to 0, it does not contain the factor z_{i_1} or z_{i_2}.

We divide the proposition into 2 cases:

Case (i): f contains the monomial $z_{i_1}^{n_{i_1}} z_{i_2}$;

Case (ii): f contains the monomial $z_{i_1}^{n_{i_1}}$.

The calculation is lengthy. One can refer to the following two lemmas (Lemma 4.2 and Lemma 4.3 respectively) for more details. By the two lemmas, we complete the proof. □

In Lemma 4.2, we will discuss Case (i) of Proposition 4.1. That is, for the Type C fewnomial $f(z_0, z_1, z_2, z_3)$ satisfying $mult(f) \geq 3$, f takes the form of $f = z_0^{n_0} z_i + \cdots$ after we renumber the variables z_0, z_1, z_2 and z_3 to satisfy the weight relationship $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$.

LEMMA 4.2 (Case (i) of Proposition 4.1). *Let* $(V, 0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 \colon f(z_0, z_1, z_2, z_3) = 0\}$ *be an isolated singularity defined by the Type C fewnomial* $f(z_0, z_1, z_2, z_3)$ *of weight type* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ *where* $mult\,(f) \geq 3$. *Let* $H_1(V)$ *be the* 1*-st Hessian algebra. We renumber the variables* z_0, z_1, z_2 *and* z_3 *to satisfy the weight relationship* $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$. *If we get* $f = z_0^{n_0} z_i + \cdots$ *after renumbering, there does not exist negative weight derivation of* $H_1(V)$.

Proof. By the definition of Type C, after renumbering, f can be written in the form of

$$f(z_{i_1}, z_{i_2}, z_{j_1}, z_{j_2}) = g(z_{i_1}, z_{i_2}) + h(z_{j_1}, z_{j_2})$$

where $\alpha_{i_1} \geq \alpha_{i_2}$, $\alpha_{j_1} \geq \alpha_{j_2}$ and $\alpha_{i_1} \geq \alpha_{j_1}$. In this form, i_1, i_2, j_1 and j_2 are a permutation of 0, 1, 2 and 3. There is no harm to let $i_1 = 0$ and $i_2 = i$. By $mult\,(f) \geq 3$, we get $n_{i_1} \geq 2$. It is clear that $f_{i_1j_1} = 0$, $f_{i_1j_2} = 0$, $f_{i_2j_1} = 0$ and $f_{i_2j_2} = 0$.

If such negative weight derivation D exists, D can be written in the form of

$$D = p_{i_1}(z_{i_2}, z_{j_1}, z_{j_2})\frac{\partial}{\partial z_{i_1}} + p_{i_2}(z_{j_1}, z_{j_2})\frac{\partial}{\partial z_{i_2}} + p_{j_1}(z_{i_2}, z_{j_2})\frac{\partial}{\partial z_{j_1}} + c_{j_2}z_{i_2}^{k_{j_2}}\frac{\partial}{\partial z_{j_2}}.$$

It is easy to see that f contains the monomial $z_{i_1}^{n_{i_1}}z_{i_2}$. Therefore, regardless of difference of constants, $f_{i_1i_1} = z_{i_1}^{n_{i_1}-2}z_{i_2}$.

If $n_{i_1} \geq 3$, from $D\,(f_{i_1i_1}) = 0$, we get $(n_{i_1}-2)\,p_{i_1}\,(z_{i_2}, z_{j_1}, z_{j_2})\,z_{i_1}^{n_{i_1}-3}z_{i_2} + p_{i_2}(z_{j_1}, z_{j_2})z_{i_1}^{n_{i_1}-2} = 0$. Therefore, $(n_{i_1}-2)\,p_{i_1}\,(z_{i_2}, z_{j_1}, z_{j_2})\,z_{i_2} + p_{i_2}(z_{j_1}, z_{j_2})z_{i_1} = 0$. We can get $p_{i_1}\,(z_{i_2}, z_{j_1}, z_{j_2}) = 0$ and $p_{i_2}(z_{j_1}, z_{j_2}) = 0$.

If $n_{i_1} = 2$, it is clear that $f_{i_1i_1} = z_{i_2}$ regardless of difference of constants. If f contains the monomial $z_{i_2}^{n_{i_2}}$, we can get $f_{i_1i_2} = z_{i_1}$ regardless of difference of constants. If f contains the monomial $z_{i_2}^{n_{i_2}}z_{i_1}$, we can get $n_{i_2} \geq 2$. Therefore, $f_{i_1i_2} = z_{i_1} + z_{i_2}^{n_{i_2}-1}$ regardless of difference of constants. In both cases, both z_{i_1} and z_{i_2} are in the ideal $(f_{i_1i_1}, f_{i_1i_2}, f_{i_1j_1}, f_{i_1j_2}, f_{i_2i_2}, f_{i_2j_1}, f_{i_2j_2}, f_{j_1j_1}, f_{j_1j_2}, f_{j_2j_2})$. Any nonzero element in $H_1\,(V)$ cannot be divided by z_{i_1} or z_{i_2}. We can get $p_{i_1}\,(z_{i_2}, z_{j_1}, z_{j_2}) = 0$ and $p_{i_2}(z_{j_1}, z_{j_2}) = 0$.

Therefore, D is in the form of $D = p_{j_1}(z_{i_2}, z_{j_2})\frac{\partial}{\partial z_{j_1}} + c_{j_2}z_{i_2}^{k_{j_2}}\frac{\partial}{\partial z_{j_2}}$.

f contains either the monomial $z_{j_1}^{n_{j_1}}$ or the monomial $z_{j_1}^{n_{j_1}}z_{j_2}$.

If f contains the monomial $z_{j_1}^{n_{j_1}}$, we can get $n_{j_1} \geq 3$. We have the following discussions.

Regardless of difference of constants, $f_{j_1j_1} = z_{j_1}^{n_{j_1}-2}$. From $D\,(f_{j_1j_1}) = (n_{j_1}-2)\,p_{j_1}\,(z_{i_2}, z_{j_2})\;z_{j_1}^{n_{j_1}-3} \in (f_{i_1i_1}, f_{i_1i_2}, f_{i_2i_2})$, we get $p_{j_1}\,(z_{i_2}, z_{j_2})\,z_{j_1}^{n_{j_1}-3} \in (f_{i_1i_1}, f_{i_1i_2}, f_{i_2i_2})$. It is clear that any monomial of the element in the ideal $(f_{i_1i_1}, f_{i_1i_2}, f_{i_2i_2})$ has the property that the total weight with respect to z_{i_1} and z_{i_2} is not less than $wt\,(f_{i_1i_1})$.

If $p_{j_1}\,(z_{i_2}, z_{j_2}) \neq 0$, we can get $wt\,(p_{j_1}\,(z_{i_2}, z_{j_2})) \geq wt\,(f_{i_1i_1}) = wt\,(f) - 2\alpha_{i_1}$. Note that $wt\,(p_{j_1}\,(z_{i_2}, z_{j_2})) < \alpha_{j_1}$. Therefore, $\alpha_{j_1} > wt\,(f) - 2\alpha_{i_1}$. We can get $n_{i_1}\alpha_{i_1} + \alpha_{i_2} = wt\,(f) < \alpha_{j_1} + 2\alpha_{i_1} \leq 3\alpha_{i_1} < 3\alpha_{i_1} + \alpha_{i_2}$. Therefore, $n_{i_1} < 3$. Note that $n_{i_1} \geq 2$, we can get $n_{i_1} = 2$. It is clear that $f_{i_1i_1} = z_{i_2}$ regardless of difference of constants. If f contains the monomial $z_{i_2}^{n_{i_2}}$, we can get $f_{i_1i_2} = z_{i_1}$ regardless of difference of constants. If f contains the monomial $z_{i_2}^{n_{i_2}}z_{i_1}$, we can get $n_{i_2} \geq 2$. Therefore, $f_{i_1i_2} = z_{i_1} + z_{i_2}^{n_{i_2}-1}$ regardless of difference of constants. In both cases, we have $(f_{i_1i_1}, f_{i_1i_2}, f_{i_2i_2}) = (z_{i_1}, z_{i_2})$.

Therefore, we can remove the monomials in $p_{j_1}\,(z_{i_2}, z_{j_2})$ and $c_{j_2}z_{i_2}^{k_{j_2}}$ that is divided by z_{i_1} and z_{i_2}. If $c_{j_2} \neq 0$, it is clear that $k_{j_2} = 0$. If $p_{j_1}\,(z_{i_2}, z_{j_2}) \neq 0$, $p_{j_1}\,(z_{i_2}, z_{j_2})$ can be written in the form of $p_{j_1}\,(z_{j_2}) = c_{j_1}z_{j_2}^{k_{j_1}}$. Therefore, D can be written in the form of $D = c_{j_1}z_{j_2}^{k_{j_1}}\frac{\partial}{\partial z_{j_1}} + c_{j_2}\frac{\partial}{\partial z_{j_2}}$. We consider the relation $p_{j_1}\,(z_{i_2}, z_{j_2})\,z_{j_1}^{n_{j_1}-3} \in (f_{i_1i_1}, f_{i_1i_2}, f_{i_2i_2})$ again. It is equivalent to $c_{j_1}z_{j_2}^{k_{j_1}}z_{j_1}^{n_{j_1}-3} \in (z_{i_1}, z_{i_2})$. We can get $c_{j_1} = 0$ and $p_{j_1}\,(z_{i_2}, z_{j_2}) = 0$, which contradicts to $p_{j_1}\,(z_{i_2}, z_{j_2}) \neq 0$. Therefore, we have $p_{j_1}\,(z_{i_2}, z_{j_2}) = 0$ and D is in the form of $D = c_{j_2}z_{i_2}^{k_{j_2}}\frac{\partial}{\partial z_{j_2}}$.

If $h = z_{j_1}^{n_{j_1}} + z_{j_2}^{n_{j_2}}$, we can get $n_{j_2} \geq 3$. It is clear that $f_{j_1 j_1} = z_{j_1}^{n_{j_1}-2}$, $f_{j_1 j_2} = 0$ and $f_{j_2 j_2} = z_{j_2}^{n_{j_2}-2}$ regardless of difference of constants. By $D\left(z_{j_2}^{n_{j_2}-2}\right) = c_{j_2}(n_{j_2}-2)\, z_{i_2}^{k_{j_2}} z_{j_2}^{n_{j_2}-3} \in (f_{i_1 i_1}, f_{i_1 i_2}, f_{i_1 j_1}, f_{i_1 j_2}, f_{i_2 i_2}, f_{i_2 j_1}, f_{i_2 j_2}, f_{j_1 j_1}, f_{j_1 j_2}, f_{j_2 j_2})$, we can get $D\left(z_{j_2}^{n_{j_2}-2}\right) = c_{j_2}(n_{j_2}-2)\, z_{i_2}^{k_{j_2}}\ z_{j_2}^{n_{j_2}-3} \in (f_{i_1 i_1}, f_{i_1 i_2}, f_{i_2 i_2})$. If $c_{j_2} \neq 0$, we have $z_{i_2}^{k_{j_2}} z_{j_2}^{n_{j_2}-3} \in (f_{i_1 i_1}, f_{i_1 i_2}, f_{i_2 i_2})$. It is clear that any monomial of the element in the ideal $(f_{i_1 i_1}, f_{i_1 i_2}, f_{i_2 i_2})$ has the property that the total weight with respect to z_{i_1} and z_{i_2} is not less than $wt(f_{i_1 i_1})$. Therefore, $wt\left(z_{i_2}^{k_{j_2}}\right) \geq wt(f_{i_1 i_1}) = wt(f) - 2\alpha_{i_1}$. However, since D is negatively weighted, we have $wt\left(z_{i_2}^{k_{j_2}}\right) < \alpha_{j_2}$. Therefore, $wt(f) < 2\alpha_{i_1} + \alpha_{j_2}$. We can get $n_{i_1}\alpha_{i_1} + \alpha_{i_2} = wt(f) < 2\alpha_{i_1} + \alpha_{j_2} \leq 3\alpha_{i_1} < 3\alpha_{i_1} + \alpha_{i_2}$. Therefore, $n_{i_1} < 3$. Note that $n_{i_1} \geq 2$, we can get $n_{i_1} = 2$. It is clear that $f_{i_1 i_1} = z_{i_2}$ regardless of difference of constants. If f contains the monomial $z_{i_2}^{n_{i_2}}$, we can get $f_{i_1 i_2} = z_{i_1}$ regardless of difference of constants. If f contains the monomial $z_{i_2}^{n_{i_2}} z_{i_1}$, we can get $n_{i_2} \geq 2$. Therefore, $f_{i_1 i_2} = z_{i_1} + z_{i_2}^{n_{i_2}-1}$ regardless of difference of constants. In both cases, we have $(f_{i_1 i_1}, f_{i_1 i_2}, f_{i_2 i_2}) = (z_{i_1}, z_{i_2})$. Therefore, we have $k_{j_2} = 0$. Apply D to $z_{j_2}^{n_{j_2}-2}$ $n_{j_2} - 3$ times and we get z_{j_2} is in the ideal $(f_{i_1 i_1}, f_{i_1 i_2}, f_{i_1 j_1}, f_{i_1 j_2}, f_{i_2 i_2}, f_{i_2 j_1}, f_{i_2 j_2}, f_{j_1 j_1}, f_{j_1 j_2}, f_{j_2 j_2})$. Therefore, we have $D = 0$, which is equivalent to $c_{j_2} = 0$. We get a contradiction. Therefore, $c_{j_2} = 0$ and $D = 0$. In other words, such negative weight derivation D does not exist when $h = z_{j_1}^{n_{j_1}} + z_{j_2}^{n_{j_2}}$.

If $h = z_{j_1}^{n_{j_1}} + z_{j_2}^{n_{j_2}} z_{j_1}$, we can get $n_{j_2} \geq 2$. When $n_{j_2} = 2$, we have $f_{j_1 j_2} = z_{j_2}$ regardless of difference of constants. Therefore, z_{j_2} is in the ideal $(f_{i_1 i_1}, f_{i_1 i_2}, f_{i_1 j_1}, f_{i_1 j_2}, f_{i_2 i_2}, f_{i_2 j_1}, f_{i_2 j_2}, f_{j_1 j_1}, f_{j_1 j_2}, f_{j_2 j_2})$. Therefore, $D = 0$. When $n_{j_2} \geq 3$, we have $f_{j_1 j_1} = z_{j_1}^{n_{j_1}-2}$, $f_{j_1 j_2} = z_{j_2}^{n_{j_2}-1}$ and $f_{j_2 j_2} = z_{j_2}^{n_{j_2}-2} z_{j_1}$ regardless of difference of constants. If $k_{j_2} = 0$, we can apply D to $z_{j_2}^{n_{j_2}-2}$ $n_{j_2} - 3$ times and we get z_{j_2} is in the ideal $(f_{i_1 i_1}, f_{i_1 i_2}, f_{i_1 j_1}, f_{i_1 j_2}, f_{i_2 i_2}, f_{i_2 j_1}, f_{i_2 j_2}, f_{j_1 j_1}, f_{j_1 j_2}, f_{j_2 j_2})$. Therefore, $D = 0$. We only need to consider the case when $k_{j_2} \geq 1$. From the weight relationship, we have $D\left(z_{j_2}^{n_{j_2}-1}\right) = c_{j_2}(n_{j_2}-1)\, z_{i_2}^{k_{j_2}} z_{j_2}^{n_{j_2}-2} \in \left(f_{i_1 i_1}, f_{i_1 i_2}, f_{i_2 i_2}, z_{j_1}^{n_{j_1}-2}\right)$. Therefore, $D\left(z_{j_2}^{n_{j_2}-1}\right) = c_{j_2}(n_{j_2}-1)\, z_{i_2}^{k_{j_2}} z_{j_2}^{n_{j_2}-2} \in (f_{i_1 i_1}, f_{i_1 i_2}, f_{i_2 i_2})$. If $c_{j_2} \neq 0$, we have $z_{i_2}^{k_{j_2}} z_{j_2}^{n_{j_2}-2} \in (f_{i_1 i_1}, f_{i_1 i_2}, f_{i_2 i_2})$. It is clear that any monomial of the element in the ideal $(f_{i_1 i_1}, f_{i_1 i_2}, f_{i_2 i_2})$ has the property that the total weight with respect to z_{i_1} and z_{i_2} is not less than $wt(f_{i_1 i_1})$. Therefore, $wt\left(z_{i_2}^{k_{j_2}}\right) \geq wt(f_{i_1 i_1}) = wt(f) - 2\alpha_{i_1}$. However, since D is negatively weighted, we have $wt\left(z_{i_2}^{k_{j_2}}\right) < \alpha_{j_2}$. We can get $n_{i_1}\alpha_{i_1} + \alpha_{i_2} = wt(f) < 2\alpha_{i_1} + \alpha_{j_2} \leq 3\alpha_{i_1} < 3\alpha_{i_1} + \alpha_{i_2}$. Therefore, $n_{i_1} < 3$. Note that $n_{i_1} \geq 2$, we can get $n_{i_1} = 2$. It is clear that $f_{i_1 i_1} = z_{i_2}$ regardless of difference of constants. If f contains the monomial $z_{i_2}^{n_{i_2}}$, we can get $f_{i_1 i_2} = z_{i_1}$ regardless of difference of constants. If f contains the monomial $z_{i_2}^{n_{i_2}} z_{i_1}$, we can get $n_{i_2} \geq 2$. Therefore, $f_{i_1 i_2} = z_{i_1} + z_{i_2}^{n_{i_2}-1}$ regardless of difference of constants. In both cases, we have $(f_{i_1 i_1}, f_{i_1 i_2}, f_{i_2 i_2}) = (z_{i_1}, z_{i_2})$. Therefore, we have $D = 0$, which is in contradiction to our assumption that D is negatively weighted. Therefore, $c_{j_2} = 0$ and $D = 0$. In other words, such negative weight derivation D does not exist when $h = z_{j_1}^{n_{j_1}} + z_{j_2}^{n_{j_2}} z_{j_1}$.

In conclusion, there does not exist any negative weight derivation when f contains the monomial $z_{j_1}^{n_{j_1}}$.

If f contains the monomial $z_{j_1}^{n_{j_1}} z_{j_2}$, we can get $n_{j_1} \geq 2$. We have the following discussions.

If $n_{j_1}=2$, it is clear that $f_{j_1j_1}=z_{j_2}$ regardless of difference of constants. If f contains the monomial $z_{j_2}^{n_{j_2}}$, we can get $f_{j_1j_2}=z_{j_1}$ regardless of dirrerence of constants. If f contains the monomial $z_{j_2}^{n_{j_2}}z_{j_1}$, we can get $n_{j_2}\geq 2$. Therefore, $f_{j_1j_2}=z_{j_1}+z_{j_2}^{n_{j_2}-1}$ regardless of difference of constants. In both cases, we have $(f_{j_1j_1},f_{j_1j_2},f_{j_2j_2})=(z_{j_1},z_{j_2})$. Therefore, $D=0$. Such negative weight derivation D does not exist.

If $n_{j_1}\geq 3$, regardless of difference of constants, we have $f_{j_1j_1}=z_{j_1}^{n_{j_1}-2}z_{j_2}$ regardless of difference of constants. Note that $D(f_{j_1j_1})=(n_{j_1}-2)\,p_{j_1}(z_{i_2},z_{j_2})\,z_{j_1}^{n_{j_1}-3}z_{j_2}+c_{j_2}z_{i_2}^{k_{j_2}}z_{j_1}^{n_{j_1}-2}\in(f_{i_1i_1},f_{i_1i_2},f_{i_2i_2})$. It is clear that any monomial of the element in the ideal $(f_{i_1i_1},f_{i_1i_2},f_{i_2i_2})$ has the property that the total weight with respect to z_{i_1} and z_{i_2} is not less than $wt\,(f_{i_1i_1})$.

If $p_{j_1}(z_{i_2},z_{j_2})\neq 0$, we can get $wt\,(p_{j_1}(z_{i_2},z_{j_2}))\geq wt\,(f_{i_1i_1})=wt\,(f)-2\alpha_{i_1}$. Note that $wt\,(p_{j_1}(z_{i_2},z_{j_2}))<\alpha_{j_1}$. Therefore, $\alpha_{j_1}>wt\,(f)-2\alpha_{i_1}$. We can get $n_{i_1}\alpha_{i_1}+\alpha_{i_2}=wt\,(f)<\alpha_{j_1}+2\alpha_{i_1}\leq 3\alpha_{i_1}<3\alpha_{i_1}+\alpha_{i_2}$. Therefore, $n_{i_1}<3$. Note that $n_{i_1}\geq 2$, we can get $n_{i_1}=2$. It is clear that $f_{i_1i_1}=z_{i_2}$ regardless of difference of constants. If f contains the monomial $z_{i_2}^{n_{i_2}}$, we can get $f_{i_1i_2}=z_{i_1}$ regardless of difference of constants. If f contains the monomial $z_{i_2}^{n_{i_2}}z_{i_1}$, we can get $n_{i_2}\geq 2$. Therefore, $f_{i_1i_2}=z_{i_1}+z_{i_2}^{n_{i_2}-1}$ regardless of difference of constants. In both cases, we have $(f_{i_1i_1},f_{i_1i_2},f_{i_2i_2})=(z_{i_1},z_{i_2})$. Therefore, we can remove the monomials in $p_{j_1}(z_{i_2},z_{j_2})$ and $c_{j_2}z_{i_2}^{k_{j_2}}$ that cannot be divided by z_{i_1} and z_{i_2}. If $c_{j_2}\neq 0$, it is clear that $k_{j_2}=0$. It is clear that $p_{j_1}(z_{i_2},z_{j_2})$ can be written in the form of $p_{j_1}(z_{j_2})=c_{j_1}z_{j_2}^{k_{j_1}}$. Therefore, D can be written in the form of $D=c_{j_1}z_{j_2}^{k_{j_1}}\frac{\partial}{\partial z_{j_1}}+c_{j_2}\frac{\partial}{\partial z_{j_2}}$. We consider the $(n_{j_1}-2)\,p_{j_1}(z_{i_2},z_{j_2})\,z_{j_1}^{n_{j_1}-3}z_{j_2}+c_{j_2}z_{i_2}^{k_{j_2}}z_{j_1}^{n_{j_1}-2}\in(f_{i_1i_1},f_{i_1i_2},f_{i_2i_2})$ again. It is equivalent to $(n_{j_1}-2)\,c_{j_1}z_{j_2}^{k_{j_1}+1}z_{j_1}^{n_{j_1}-3}+c_{j_2}z_{j_1}^{n_{j_1}-2}\in(z_{i_1},z_{i_2})$. We can get $c_{j_1}=0$ and $c_{j_2}=0$, which is contraditory to $p_{j_1}(z_{i_2},z_{j_2})\neq 0$. Therefore, it is clear that $p_{j_1}(z_{i_2},z_{j_2})=0$. D is in the form of $D=c_{j_2}z_{i_2}^{k_{j_2}}\frac{\partial}{\partial z_{j_2}}$.

If $c_{j_2}\neq 0$, we can get $wt\left(z_{i_2}^{k_{j_2}}\right)\geq wt\,(f_{i_1i_1})=wt\,(f)-2\alpha_{i_1}$. Note that $wt\left(z_{i_2}^{k_{j_2}}\right)<\alpha_{j_2}$. Therefore, $\alpha_{j_2}>wt\,(f)-2\alpha_{i_1}$. We can get $n_{i_1}\alpha_{i_1}+\alpha_{i_2}=wt\,(f)<\alpha_{j_2}+2\alpha_{i_1}\leq 3\alpha_{i_1}<3\alpha_{i_1}+\alpha_{i_2}$. Therefore, $n_{i_1}<3$. Note that $n_{i_1}\geq 2$, we can get $n_{i_1}=2$. It is clear that $f_{i_1i_1}=z_{i_2}$ regardless of difference of constants. If f contains the monomial $z_{i_2}^{n_{i_2}}$, we can get $f_{i_1i_2}=z_{i_1}$ regardless of difference of constants. If f contains the monomial $z_{i_2}^{n_{i_2}}z_{i_1}$, we can get $n_{i_2}\geq 2$. Therefore, $f_{i_1i_2}=z_{i_1}+z_{i_2}^{n_{i_2}-1}$ regardless of difference of constants. In both cases, we have $(f_{i_1i_1},f_{i_1i_2},f_{i_2i_2})=(z_{i_1},z_{i_2})$. If $c_{j_2}\neq 0$, it is clear that $k_{j_2}=0$. Otherwise, it is equivalent to $D=0$, which contradicts to D is negatively weighted. Therefore, D can be written in the form of $D=c_{j_2}\frac{\partial}{\partial z_{j_2}}$. We consider the $(n_{j_1}-2)\,p_{j_1}(z_{i_2},z_{j_2})\,z_{j_1}^{n_{j_1}-3}z_{j_2}+c_{j_2}z_{i_2}^{k_{j_2}}z_{j_1}^{n_{j_1}-2}\in(f_{i_1i_1},f_{i_1i_2},f_{i_2i_2})$ again. It is equivalent to $c_{j_2}z_{j_1}^{n_{j_1}-2}\in(z_{i_1},z_{i_2})$. We can get $c_{j_2}=0$, which is contraditory to $c_{j_2}\neq 0$. Therefore, $c_{j_2}=0$ and $D=0$.

In conclusion, there does not exist any negative weight derivation when f contains the monomial $z_{j_1}^{n_{j_1}}z_{j_2}$.

Therefore, we complete the proof. □

In Lemma 4.3, we will discuss Case (ii) of Proposition 4.1. That is, for the Type

C fewnomial $f(z_0, z_1, z_2, z_3)$ satisfying $mult(f) \geq 3$, f takes the form of $f = z_0^{n_0} + \cdots$ after we renumber the variables z_0, z_1, z_2 and z_3 to satisfy the weight relationship $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$.

LEMMA 4.3 (Case (ii) of Proposition 4.1). *Let* $(V, 0) = \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 \colon f(z_0, z_1, z_2, z_3) = 0\}$ *be an isolated singularity defined by the Type C fewnomial* $f(z_0, z_1, z_2, z_3)$ *of weight type* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ *where* $mult(f) \geq 3$*. Let* $H_1(V)$ *be the* 1*-st Hessian algebra. We renumber the variables* z_0*,* z_1*,* z_2 *and* z_3 *to satisfy the weight relationship* $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3$*. If we get* $f = z_0^{n_0} + \cdots$ *after renumbering, there does not exist negative weight derivation of* $H_1(V)$*.*

Proof. By the definition of Type C, after renumbering, f can be written in the form of

$$f(z_{i_1}, z_{i_2}, z_{j_1}, z_{j_2}) = g(z_{i_1}, z_{i_2}) + h(z_{j_1}, z_{j_2})$$

where $\alpha_{i_1} \geq \alpha_{i_2}$, $\alpha_{j_1} \geq \alpha_{j_2}$ and $\alpha_{i_1} \geq \alpha_{j_1}$. In this form, i_1, i_2, j_1 and j_2 are a permutation of 0, 1, 2 and 3. There is no harm to let $i_1 = 0$. By $mult(f) \geq 3$, we get $n_{i_1} \geq 3$. It is clear that $f_{i_1 j_1} = 0$, $f_{i_1 j_2} = 0$, $f_{i_2 j_1} = 0$ and $f_{i_2 j_2} = 0$.

If such negative weight derivation D exists, D can be written in the form of

$$D = p_{i_1}(z_{i_2}, z_{j_1}, z_{j_2})\frac{\partial}{\partial z_{i_1}} + p_{i_2}(z_{j_1}, z_{j_2})\frac{\partial}{\partial z_{i_2}} + p_{j_1}(z_{i_2}, z_{j_2})\frac{\partial}{\partial z_{j_1}} + c_{j_2} z_{i_2}^{k_{j_2}}\frac{\partial}{\partial z_{j_2}}.$$

It is easy to see that f contains the monomial $z_{i_1}^{n_{i_1}}$. Therefore, regardless of difference of constants, $f_{i_1 i_1} = z_{i_1}^{n_{i_1}-2}$.

Since $n_{i_1} \geq 3$, from $D(f_{i_1 i_1}) = 0$, we get $(n_{i_1} - 2)\, p_{i_1}(z_{i_2}, z_{j_1}, z_{j_2})\, z_{i_1}^{n_{i_1}-3} = 0$. Therefore, we have $p_{i_1}(z_{i_2}, z_{j_1}, z_{j_2}) = 0$.

Therefore, D is in the form of $D = p_{i_2}(z_{j_1}, z_{j_2})\frac{\partial}{\partial z_{i_2}} + p_{j_1}(z_{i_2}, z_{j_2})\frac{\partial}{\partial z_{j_1}} + c_{j_2} z_{i_2}^{k_{j_2}}\frac{\partial}{\partial z_{j_2}}$.

If $g = z_{i_1}^{n_{i_1}} + z_{i_2}^{n_{i_2}}$, we can get $n_{i_2} \geq 3$. It is clear that $f_{i_1 i_1} = z_{i_1}^{n_{i_1}-2}$, $f_{i_1 i_2} = 0$ and $f_{i_2 i_2} = z_{i_2}^{n_{i_2}-2}$ regardless of difference of constants. By $D\left(z_{i_2}^{n_{i_2}-2}\right) = (n_{i_2} - 2)\, p_{i_2}(z_{j_1}, z_{j_2})\, z_{i_2}^{n_{i_2}-3} \in (z_{i_1}^{n_{i_1}-2}, f_{j_1 j_1}, f_{j_1 j_2}, f_{j_2 j_2})$, we can get $D\left(z_{i_2}^{n_{i_2}-2}\right) = (n_{i_2} - 2)\, p_{i_2}(z_{j_1}, z_{j_2})\, z_{i_2}^{n_{i_2}-3} \in (f_{j_1 j_1}, f_{j_1 j_2}, f_{j_2 j_2})$. It is clear that any monomial of the element in the ideal $(f_{j_1 j_1}, f_{j_1 j_2}, f_{j_2 j_2})$ has the property that the total weight with respect to z_{j_1} and z_{j_2} is not less than $wt(f_{j_1 j_1})$. If $p_{i_2}(z_{j_1}, z_{j_2}) \neq 0$, we have $wt(p_{i_2}(z_{j_1}, z_{j_2})) \geq wt(f_{j_1 j_1}) = wt(f) - 2\alpha_{j_1}$. However, since D is negatively weighted, we have $wt(p_{i_2}(z_{j_1}, z_{j_2})) < \alpha_{i_2}$. We can get $wt(f) < 2\alpha_{j_1} + \alpha_{i_2} \leq 3\alpha_{i_1} \leq n_{i_1}\alpha_{i_1} = wt(f)$. This leads to a contradiction. Therefore, if $g = z_{i_1}^{n_{i_1}} + z_{i_2}^{n_{i_2}}$, we have $p_{i_2}(z_{j_1}, z_{j_2}) = 0$ and D is in the form of $D = p_{j_1}(z_{i_2}, z_{j_2})\frac{\partial}{\partial z_{j_1}} + c_{j_2} z_{i_2}^{k_{j_2}}\frac{\partial}{\partial z_{j_2}}$.

If $g = z_{i_1}^{n_{i_1}} + z_{i_2}^{n_{i_2}} z_{i_1}$, we can get $n_{i_2} \geq 2$. It is clear that $f_{i_1 i_1} = z_{i_1}^{n_{i_1}-2}$, $f_{i_1 i_2} = z_{i_2}^{n_{i_2}-1}$ and $f_{i_2 i_2} = z_{i_2}^{n_{i_2}-2} z_{i_1}$ regardless of difference of constants. By $D\left(z_{i_2}^{n_{i_2}-1}\right) = (n_{i_2} - 1)\, p_{i_2}(z_{j_1}, z_{j_2})\, z_{i_2}^{n_{i_2}-2} \in (z_{i_1}^{n_{i_1}-2}, f_{j_1 j_1}, f_{j_1 j_2}, f_{j_2 j_2})$, we can get $D\left(z_{i_2}^{n_{i_2}-1}\right) = (n_{i_2} - 1)\, p_{i_2}(z_{j_1}, z_{j_2})\, z_{i_2}^{n_{i_2}-2} \in (f_{j_1 j_1}, f_{j_1 j_2}, f_{j_2 j_2})$. It is clear that any monomial of the element in the ideal $(f_{j_1 j_1}, f_{j_1 j_2}, f_{j_2 j_2})$ has the property that the total weight with respect to z_{j_1} and z_{j_2} is not less than $wt(f_{j_1 j_1})$. If $p_{i_2}(z_{j_1}, z_{j_2}) \neq 0$, we have $wt(p_{i_2}(z_{j_1}, z_{j_2})) \geq wt(f_{j_1 j_1}) = wt(f) - 2\alpha_{j_1}$. However, since D is negatively weighted, we have $wt(p_{i_2}(z_{j_1}, z_{j_2})) < \alpha_{i_2}$. We can get $wt(f) < 2\alpha_{j_1} + \alpha_{i_2} \leq$

$3\alpha_{i_1} \le n_{i_1}\alpha_{i_1} = wt(f)$. Contradiction. Therefore, if $g = z_{i_1}^{n_{i_1}} + z_{i_2}^{n_{i_2}} z_{i_1}$, we have $p_{i_2}(z_{j_1}, z_{j_2}) = 0$ and D is in the form of $D = p_{j_1}(z_{i_2}, z_{j_2})\frac{\partial}{\partial z_{j_1}} + c_{j_2} z_{i_2}^{k_{j_2}} \frac{\partial}{\partial z_{j_2}}$.

Therefore, D is in the form of $D = p_{j_1}(z_{i_2}, z_{j_2})\frac{\partial}{\partial z_{j_1}} + c_{j_2} z_{i_2}^{k_{j_2}} \frac{\partial}{\partial z_{j_2}}$.

f contains either the monomial $z_{j_1}^{n_{j_1}}$ or the monomial $z_{j_1}^{n_{j_1}} z_{j_2}$.

If f contains the monomial $z_{j_1}^{n_{j_1}}$, we can get $n_{j_1} \ge 3$. We have the following discussions.

Regardless of difference of constants, we have $f_{j_1 j_1} = z_{j_1}^{n_{j_1}-2}$. From $D(f_{j_1 j_1}) = (n_{j_1} - 2)\ p_{j_1}(z_{i_2}, z_{j_2})\, z_{j_1}^{n_{j_1}-3} \in (f_{i_1 i_1}, f_{i_1 i_2}, f_{i_1 j_1}, f_{i_1 j_2}, f_{i_2 i_2}, f_{i_2 j_1}, f_{i_2 j_2}, f_{j_1 j_1}, f_{j_1 j_2}, f_{j_2 j_2})$, we get $p_{j_1}(z_{i_2}, z_{j_2})\, z_{j_1}^{n_{j_1}-3} \in (f_{i_1 i_1}, f_{i_1 i_2}, f_{i_2 i_2})$. It is clear that any monomial of the element in the ideal $(f_{i_1 i_1}, f_{i_1 i_2}, f_{i_2 i_2})$ has the property that the total weight with respect to z_{i_1} and z_{i_2} is not less than $wt(f_{i_1 i_1})$.

If $p_{j_1}(z_{i_2}, z_{j_2}) \ne 0$, we can get $wt(p_{j_1}(z_{i_2}, z_{j_2})) \ge wt(f_{i_1 i_1}) = wt(f) - 2\alpha_{i_1}$. Note that $wt(p_{j_1}(z_{i_2}, z_{j_2})) < \alpha_{j_1}$. Therefore, $\alpha_{j_1} > wt(f) - 2\alpha_{i_1}$. We can get $n_{i_1}\alpha_{i_1} = wt(f) < \alpha_{j_1} + 2\alpha_{i_1} \le 3\alpha_{i_1}$. Therefore, $n_{i_1} < 3$. Note that $n_{i_1} \ge 3$, we get a contradiction. Therefore, it is clear that $p_{j_1}(z_{i_2}, z_{j_2}) = 0$. D is in the form of $D = c_{j_2} z_{i_2}^{k_{j_2}} \frac{\partial}{\partial z_{j_2}}$.

If $h = z_{j_1}^{n_{j_1}} + z_{j_2}^{n_{j_2}}$, we can get $n_{j_2} \ge 3$. It is clear that $f_{j_1 j_1} = z_{j_1}^{n_{j_1}-2}$, $f_{j_1 j_2} = 0$ and $f_{j_2 j_2} = z_{j_2}^{n_{j_2}-2}$ regardless of difference of constants. By $D\left(z_{j_2}^{n_{j_2}-2}\right) = c_{j_2}(n_{j_2} - 2)\, z_{i_2}^{k_{j_2}} z_{j_2}^{n_{j_2}-3} \in (f_{i_1 i_1}, f_{i_1 i_2}, f_{i_1 j_1}, f_{i_1 j_2}, f_{i_2 i_2}, f_{i_2 j_1}, f_{i_2 j_2}, f_{j_1 j_1}, f_{j_1 j_2}, f_{j_2 j_2})$, we can get $D\left(z_{j_2}^{n_{j_2}-2}\right) = c_{j_2}(n_{j_2} - 2)\, z_{i_2}^{k_{j_2}}\ z_{j_2}^{n_{j_2}-3} \in (f_{i_1 i_1}, f_{i_1 i_2}, f_{i_2 i_2})$. If $c_{j_2} \ne 0$, we have $z_{i_2}^{k_{j_2}} z_{j_2}^{n_{j_2}-3} \in (f_{i_1 i_1}, f_{i_1 i_2}, f_{i_2 i_2})$. It is clear that any monomial of the element in the ideal $(f_{i_1 i_1}, f_{i_1 i_2}, f_{i_2 i_2})$ has the property that the total weight with respect to z_{i_1} and z_{i_2} is not less than $wt(f_{i_1 i_1})$. Therefore, $wt\left(z_{i_2}^{k_{j_2}}\right) \ge wt(f_{i_1 i_1}) = wt(f) - 2\alpha_{i_1}$. However, since D is negatively weighted, we have $wt\left(z_{i_2}^{k_{j_2}}\right) < \alpha_{j_2}$. We can get $n_{i_1}\alpha_{i_1} = wt(f) < 2\alpha_{i_1} + \alpha_{j_2} \le 3\alpha_{i_1}$. Therefore, $n_{i_1} < 3$. Note that $n_{i_1} \ge 3$, we get a contradiction. Therefore, $c_{j_2} = 0$ and $D = 0$. In other words, such negative weight derivation D does not exist when $h = z_{j_1}^{n_{j_1}} + z_{j_2}^{n_{j_2}}$.

If $h = z_{j_1}^{n_{j_1}} + z_{j_2}^{n_{j_2}} z_{j_1}$, we can get $n_{j_2} \ge 2$. When $n_{j_2} = 2$, we have $f_{j_1 j_2} = z_{j_2}$ regardless of difference of constants. Therefore, z_{j_2} is in the ideal $(f_{i_1 i_1}, f_{i_1 i_2}, f_{i_1 j_1}, f_{i_1 j_2}, f_{i_2 i_2}, f_{i_2 j_1}, f_{i_2 j_2}, f_{j_1 j_1}, f_{j_1 j_2}, f_{j_2 j_2})$. Therefore, we obtain $D = 0$. In other words, such negative weight derivation D does not exist when $n_{j_2} = 2$. When $n_{j_2} \ge 3$, we have $f_{j_1 j_1} = z_{j_1}^{n_{j_1}-2}$, $f_{j_1 j_2} = z_{j_2}^{n_{j_2}-1}$ and $f_{j_2 j_2} = z_{j_2}^{n_{j_2}-2} z_{j_1}$ regardless of difference of constants. If $k_{j_2} = 0$ and $c_{j_2} \ne 0$, we can apply D to $z_{j_2}^{n_{j_2}-1}$ $n_{i_2} - 2$ times and we get z_{j_2} is in the ideal $(f_{i_1 i_1}, f_{i_1 i_2}, f_{i_1 j_1}, f_{i_1 j_2}, f_{i_2 i_2}, f_{i_2 j_1}, f_{i_2 j_2}, f_{j_1 j_1}, f_{j_1 j_2}, f_{j_2 j_2})$. Any nonzero element in $H_1(V)$ cannot be divided by z_{j_2}. Therefore, $D = 0$ when $k_{j_2} = 0$. We only need to consider the case when $k_{j_2} \ge 1$. From $D\left(z_{j_2}^{n_{j_2}-1}\right) = c_{j_2}(n_{j_2} - 1)\, z_{i_2}^{k_{j_2}} z_{j_2}^{n_{j_2}-2} \in \left(f_{i_1 i_1}, f_{i_1 i_2}, f_{i_2 i_2}, z_{j_1}^{n_{j_1}-2}\right)$, we get $D\left(z_{j_2}^{n_{j_2}-1}\right) = c_{j_2}(n_{j_2} - 1)\, z_{i_2}^{k_{j_2}} z_{j_2}^{n_{j_2}-2} \in (f_{i_1 i_1}, f_{i_1 i_2}, f_{i_2 i_2})$. If $c_{j_2} \ne 0$, we have $z_{i_2}^{k_{j_2}} z_{j_2}^{n_{j_2}-2} \in (f_{i_1 i_1}, f_{i_1 i_2}, f_{i_2 i_2})$. It is clear that any monomial of the element in the ideal $(f_{i_1 i_1}, f_{i_1 i_2}, f_{i_2 i_2})$ has the property that the total weight with respect to z_{i_1} and z_{i_2} is not less than $wt(f_{i_1 i_1})$. Therefore, $wt\left(z_{i_2}^{k_{j_2}}\right) \ge wt(f_{i_1 i_1}) = wt(f) - 2\alpha_{i_1}$.

However, since D is negatively weighted, we have $wt\left(z_{i_2}^{k_{j_2}}\right) < \alpha_{j_2}$. We can get $n_{i_1}\alpha_{i_1} = wt(f) < 2\alpha_{i_1} + \alpha_{j_2} \leq 3\alpha_{i_1}$. Therefore, $n_{i_1} < 3$. Note that $n_{i_1} \geq 3$, we get a contradiction. Therefore, $c_{j_2} = 0$ and $D = 0$. In other words, such negative weight derivation D does not exist when $n_{j_2} \geq 3$. Therefore, such negative weight derivation D does not exist when $h = z_{j_1}^{n_{j_1}} + z_{j_2}^{n_{j_2}} z_{j_1}$.

In conclusion, there does not exist any negative weight derivation when f contains the monomial $z_{j_1}^{n_{j_1}}$.

If f contains the monomial $z_{j_1}^{n_{j_1}} z_{j_2}$, we can get $n_{j_1} \geq 2$. We have the following discussions.

If $n_{j_1} = 2$, we can get $f_{j_1 j_1} = z_{j_2}$ regardless of difference of constants. If f contains the monomial $z_{j_2}^{n_{j_2}}$, we can get $f_{j_1 j_2} = z_{j_1}$. If f contains the monomial $z_{j_2}^{n_{j_2}} z_{j_1}$, we can get $n_{j_2} \geq 2$. Therefore, $f_{j_1 j_2} = z_{j_1} + z_{j_2}^{n_{j_2}-1}$ regardless of difference of constants. In both cases, we have $(f_{j_1 j_1}, f_{j_1 j_2}, f_{j_2 j_2}) = (z_{j_1}, z_{j_2})$. Therefore, $D = 0$. Such negative weight derivation D does not exist.

If $n_{j_1} \geq 3$, regardless of difference of constants, we have $f_{j_1 j_1} = z_{j_1}^{n_{j_1}-2} z_{j_2}$ regardless of difference of constants. Note that $D(f_{j_1 j_1}) = (n_{j_1} - 2)\, p_{j_1}(z_{i_2}, z_{j_2})\, z_{j_1}^{n_{j_1}-3} z_{j_2} + c_{j_2} z_{i_2}^{k_{j_2}} z_{j_1}^{n_{j_1}-2} \in (f_{i_1 i_1}, f_{i_1 i_2}, f_{i_2 i_2})$. It is clear that any monomial of the element in the ideal $(f_{i_1 i_1}, f_{i_1 i_2}, f_{i_2 i_2})$ has the property that the total weight with respect to z_{i_1} and z_{i_2} is not less than $wt(f_{i_1 i_1})$.

If $p_{j_1}(z_{i_2}, z_{j_2}) \neq 0$, we can get $wt(p_{j_1}(z_{i_2}, z_{j_2})) \geq wt(f_{i_1 i_1}) = wt(f) - 2\alpha_{i_1}$. Note that $wt(p_{j_1}(z_{i_2}, z_{j_2})) < \alpha_{j_1}$. Therefore, $\alpha_{j_1} > wt(f) - 2\alpha_{i_1}$. We can get $n_{i_1}\alpha_{i_1} = wt(f) < \alpha_{j_1} + 2\alpha_{i_1} \leq 3\alpha_{i_1}$. Therefore, $n_{i_1} < 3$. Note that $n_{i_1} \geq 3$, we get a contradiction. Therefore, it is clear that $p_{j_1}(z_{i_2}, z_{j_2}) = 0$. D is in the form of $D = c_{j_2} z_{i_2}^{k_{j_2}} \frac{\partial}{\partial z_{j_2}}$.

If $c_{j_2} \neq 0$, we can get $wt\left(z_{i_2}^{k_{j_2}}\right) \geq wt(f_{i_1 i_1}) = wt(f) - 2\alpha_{i_1}$. Note that $wt\left(z_{i_2}^{k_{j_2}}\right) < \alpha_{j_2}$. Therefore, $\alpha_{j_2} > wt(f) - 2\alpha_{i_1}$. We can get $n_{i_1}\alpha_{i_1} = wt(f) < \alpha_{j_2} + 2\alpha_{i_1} \leq 3\alpha_{i_1}$. Therefore, $n_{i_1} < 3$. Note that $n_{i_1} \geq 3$, we get a contradiction. Therefore, $c_{j_2} = 0$ and $D = 0$.

In conclusion, there does not exist any negative weight derivation when f contains the monomial $z_{j_1}^{n_{j_1}} z_{j_2}$.

Therefore, we complete the proof. ◻

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