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Complex structures of a twenty dimensional family of Calabi-Yau 3-folds

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In this paper, we classify all isomorphic classes of a family of Calabi-Yau 3-folds with 20 parameters. In addition, we show that the isomorphisms form a finite group. The invariants under the action of this group are calculated by introducing the so-called DS-graph.

1. Introduction

A Calabi-Yau manifold is a manifold with trivial canonical bundle. It plays an important role in theoretical physics. In super-string theory, it is conjectured that the extra dimensions of space-time take the form of a compact complex 3-dimensional Calabi-Yau manifold. Non-singular quintic 3-folds are Calabi-Yau manifolds. It is a fundamental important question in both mathematics and theoretical physics to determine when two given quintic 3-folds have the same complex structure. Such a question seems to be out of reach by 19th century invariant theory or modern geometric invariant theory. Candelas, Ossa, Green and Parkers [1] studied the following one dimensional family of Calabi-Yau 3-folds

$$X_t := \{(x_i) \in \mathbb{CP}^4 : x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + t x_1 x_2 x_3 x_4 x_5 = 0\}$$

in detail by means of the period map. Chen et al. [3, 4] generalize it to the families

$$X_t^{(n)} := \{(x_i) \in \mathbb{CP}^{n-1} : x_1^n + x_2^n + \ldots + x_n^n + t x_1 x_2 \cdots x_n = 0\}$$

with $n \geqslant 3$. The purpose of this paper is to study the complex structures of a distinguished class of 20-dimensional family of Calabi-Yau 3-folds $\{V(f_{\mathbf{t}})\}$

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with parameter $\mathbf{t} := (t_{i,j})$ for $i, j = 1, 2, 3, 4, 5$ and $i \neq j$, where

$$f_{\mathbf{t}} := f_{\mathbf{t}}(x) := x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + \sum_{i \neq j} t_{i,j} x_i^4 x_j.$$

Our main goal is to completely distinguish these complex structures and find out the moduli and modular group of this family by introducing a novel simple method.

Let G be a finite group $G = S_5 \ltimes \mathbb{Z}_5^5$. The group operation is given by

$$\begin{aligned}(\tau, b_1, b_2, \ldots, b_5) &\cdot (\sigma, a_1, a_2, \ldots, a_5)\\ &:= (\tau\sigma, b_{\sigma(1)} + a_1, b_{\sigma(2)} + a_2, \ldots, b_{\sigma(5)} + a_5)\end{aligned}$$

for $(\tau, b_1, b_2, \ldots, b_5), (\sigma, a_1, a_2, \ldots, a_5) \in G$. We define the group action on variables $t_{i,j}$ as follows. Fix η a 5-th primary root of 1. Then

$$(\sigma, a_1, a_2, \ldots, a_5) \cdot t_{i,j} := \eta^{a_i - a_j} t_{\sigma(i),\sigma(j)}.$$

When restrict the parameter $\mathbf{t}$ to a specific affine open subset, the main result of this paper states that the Calabi-Yau 3-fold $V(f_{\mathbf{t}})$ is biholomorphic to $V(f_{\mathbf{t}'})$ if and only if there exists some element $g \in G$ such that $g \cdot t_{i,j} = t'_{i,j}$ for $i, j = 1, 2, \ldots, 5$ and $i \neq j$. Moreover, we construct a basis of $\mathbb{C}[t_{ij}]^G$ expressed as the polynomials $\sum_{\mathrm{sym}} M(V, E_d, E_s)$, where (V, E_d, E_s) ranges over all DS-graphs. See the precise statement in Section 3.

2. Isomorphic class of family $\{V(f_{\mathbf{t}})\}$

In this section, we investigate the complex structures of the family $\{V(f_{\mathbf{t}})\}$.

Let $x = (x_1, x_2, \ldots, x_5)$. The derivative with respect to x_i will be shortly denoted by ∂_i. Let $I_{\mathbf{t}}$ be the ideal of $\mathbb{C}[x]$ generated by $\partial_{i,j} f_{\mathbf{t}} : i, j = 1, 2, \ldots, 5$. We introduce the algebra

$$A_{\mathbf{t}} := \mathbb{C}[x]/I_{\mathbf{t}}.$$

Observe that $A_{\mathbf{t}}$ is an invariant of the $V(f_{\mathbf{t}})$. We require the parameters $t_{i,j}$ verify the main assumption that ideal $I_{\mathbf{t}}$ is generated by x_i^3 and $\sum_{k \neq i} t_{i,k} x_i^2 x_k$ for $i = 1, 2, \ldots, 5$. We shall emphasize that this assumption holds generically. To see this, we deduce the following lemma.

Lemma 1. *Let (i, j, k, l, m) be an ordering of $\{1, 2, 3, 4, 5\}$. Suppose that the coefficients $t_{i,j}$ satisfy the following conditions*

1) $t_{i,j}t_{j,k}t_{k,i} + t_{j,i}t_{k,j}t_{i,k} \neq 0$.
2) $t_{l,i}, t_{l,j}, t_{l,k}$ *do not vanish simultaneously.*
3) $t_{m,i}, t_{m,j}, t_{m,k}$ *do not vanish simultaneously.*

Then the ideal $I_{\mathbf{t}}$ *is finite generated by* x_i^3 *and* $\sum_{k\neq i} t_{i,k}x_i^2x_k$ *for* $i = 1, 2, \ldots, 5$.

Proof. By direct computation, the second derivatives $\partial_{i,j}f_{\mathbf{t}}$ are expressed as

$$\partial_{i,j}f_{\mathbf{t}} = 4t_{i,j}x_i^3 + 4t_{j,i}x_j^3 \text{ for } i \neq j,$$

and

$$\partial_{i,i}f_{\mathbf{t}} = 20x_i^3 + 12\sum_{k\neq i} t_{i,k}x_i^2x_k.$$

The first condition of this lemma and the expressions of $\partial_{i,j}f_{\mathbf{t}}, \partial_{j,k}f_{\mathbf{t}}, \partial_{k,i}f_{\mathbf{t}}$ imply x_i^3, x_j^3, x_k^3 are contained in $I_{\mathbf{t}}$. The rest conditions yield that x_l^3, x_m^3 are also contained in $I_{\mathbf{t}}$. The proof of our lemma is completed by applying the expressions of $\partial_{i,i}(f_{\mathbf{t}})$ with $i = 1, 2, \ldots, 5$. $\square$

Let $T = \mathbb{C}^5$ be a $\mathbb{C}$-vector space spanned by x_i for $i = 1, 2, \ldots, 5$. For $\xi \in T$, we denote $\bar{\xi} \in A_{\mathbf{t}}$. Define subset of T by $S_{\mathbf{t}} := \{\xi \in T : \bar{\xi}^3 = 0\}$. Now we wish to determine the set $S_{\mathbf{t}}$.

Lemma 2. *Assume that the main assumption of* $\mathbf{t}$ *holds. Then the set* $S_{\mathbf{t}}$ *is independent on the coefficients* $\mathbf{t}$*. Moreover, it is given by coordinate axes. That is* $S_{\mathbf{t}} = \{\lambda x_i : \lambda \in \mathbb{C}, i = 1, 2, \ldots, 5\}$.

Proof. The fact $\bar{\xi}^3 = 0$ means that ξ^3 is generated by x_i^3 and $\sum_{k\neq i} t_{i,k}x_i^2x_k$. Write $\xi = \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 + \alpha_4x_4 + \alpha_5x_5$. Then this means

$$(\sum_{i=1}^{5} \alpha_ix_i)^3 = \sum_{i=1}^{5} \beta_ix_i^3 + \sum_{i=1}^{5} \gamma_i \sum_{k\neq i} t_{i,k}x_i^2x_k \tag{1}$$

with some coefficients β_i, γ_i. By comparing coefficients, we see that at most two α_i's are nonzero. Without loss generality, we assume that α_1 and α_2 are nonzero. Thus, Equation (1) becomes

$$
\begin{aligned}
(\alpha_1 x_1 + \alpha_2 x_2)^3 = \beta_1 x_1^3 + \beta_2 x_2^3 \\
&+ \gamma_1 (t_{1,2} x_1^2 x_2 + t_{1,3} x_1^2 x_3 + t_{1,4} x_1^2 x_4 + t_{1,5} x_1^2 x_5) \\
&+ \gamma_2 (t_{2,1} x_2^2 x_1 + t_{2,3} x_2^2 x_3 + t_{2,4} x_2^2 x_4 + t_{2,5} x_2^2 x_5).
\end{aligned}
$$

This implies $t_{1,k} = t_{2,k} = 0$ for $k = 3, 4, 5$. The fact $x_i^3 \in I_{\mathbf{t}}$ implies $t_{1,2} = t_{2,1} = 0$. Hence, we obtain $(\alpha_1 x_1 + \alpha_2 x_2)^3 = \beta_1 x_1^3 + \beta_2 x_2^3$. Therefore, we have $\alpha_1 = 0$ or $\alpha_2 = 0$. In conclusion, we find that $\xi = \lambda x_i$ for some constant λ. □

Theorem 3. *Assume that* $\mathbf{t}$ *and* $\mathbf{t}'$ *satisfy the main assumption. Then the linear isomorphism from* $V(f'_{\mathbf{t}})$ *to* $V(f_{\mathbf{t}})$ *is generated by permutations on the subscripts of* $t_{i,j}$ *and the scaling* $x_i \mapsto x_i \eta_i$ *such that* $\eta_i^5 = 1$ *and* $t'_{i,j} = t_{i,j} \eta_i \eta_j^{-1}$.

Proof. Suppose that $V(f_{\mathbf{t}})$ is isomorphic to $V(f'_{\mathbf{t}})$. So we have $A_{\mathbf{t}} \cong A_{\mathbf{t}'}$. Hence, $S_{\mathbf{t}}$ maps isomorphically to $S_{\mathbf{t}'}$. Denote by ϕ the isomorphism from $V(f'_{\mathbf{t}})$ to $V(f_{\mathbf{t}})$. From Lemma 2, we can assume that $\phi(x_i) := \lambda_i x_i$ for some constant λ_i with $i = 1, 2, \ldots, 5$. By assumption, we have $f_{\mathbf{t}'}(\phi(x)) = \lambda_0^5 f_{\mathbf{t}}(x)$ for some complex number λ_0. Then

$$
\begin{aligned}
f_{\mathbf{t}'}(\phi(x)) &= \sum_{i=1}^{5} \phi(x_i)^5 + \sum_{i \neq j} t'_{i,j} \phi(x_i)^4 \phi(x_j) \\
&= \sum_{i=1}^{5} \lambda_i^5 x_i^5 + \sum_{i \neq j} t'_{i,j} \lambda_i^4 \lambda_j x_i^4 x_j \\
&= \sum_{i=1}^{5} \lambda_0^5 x_i^5 + \lambda_0^5 \sum_{i \neq j} t_{i,j} x_i^4 x_j.
\end{aligned}
$$

This implies $\lambda_i^5 = \lambda_0^5$ and $\lambda_i^4 \lambda_j t'_{i,j} = \lambda_0^5 t_{i,j}$. Put $\eta_i := \lambda_i / \lambda_0$. Then $t'_{i,j} = \eta_i \eta_j^{-1} t_{i,j}$. Conversely, it is easy to verify that permutation on the subscripts and scaling defined above map isomorphically from $V(f_{\mathbf{t}'})$ to $V(f_{\mathbf{t}})$. □

Let G be a finite group $G = S_5 \ltimes \mathbb{Z}_5^5$. Fix η a 5-th primary root of 1. According to previous theorem, it is natural to define the group action on variables $t_{i,j}$ as

$$g \cdot t_{i,j} := \eta^{a_i - a_j} t_{\sigma(i),\sigma(j)}$$

for $g = (\sigma, a_1, a_2, \ldots, a_5) \in G$. Applying the previous theorem, we obtain the following corollary.

Corollary 4. *Assume that* $\mathbf{t}$ *and* $\mathbf{t}'$ *satisfy the main assumption. Then* $V(f_{\mathbf{t}})$ *and* $V(f_{\mathbf{t}'})$ *are isomorphic if and only if there exists some element* $g \in G$*, such that* $g \cdot t_{i,j} = t'_{i,j}$ *for* $i, j = 1, 2, \ldots, 5$ *and* $i \neq j$*.*

3. Directed graph

In this section, we establish a relationship between invariants of $V(f_{\mathbf{t}})$ and some directed graphs. The canonical directed graph consists of the set of vertices V and the set of directed edges E. In order to investigate the invariances of the family $\{V(f_{\mathbf{t}})\}$, we introduce the new kind of directed graphs which shall be called DS-graph. It consists of the set of vertices V and the set of dashed edges E_d and the set of (multiple) solid edges E_s such that

1) the couple (V, E_s) is the union of distinct loops;
2) the couple (V, E_d) is a directed graph containing no loops.

Now we fix V to be a set of five vertices. Let (V, E_d, E_s) be a DS-graph. Associate it with a monomial $M(V, E_d, E_s)$ in 20 variables, write $t_{i,j}$ with $i, j = 1, 2, \ldots, 5$ and $i \neq j$. That is

$$M(V, E_d, E_s) := \prod_{(i,j)\in E_d} t_{i,j}^5 \prod_{(i,j)\in E_s} t_{i,j}.$$

As usual, two DS-graphs will be viewed isomorphically if they differ exactly by a permutation on the vertices. Denote by DS(5) the set of all isomorphism class of DS-graphs associated to the set of vertices V. For a monomial m in 20 variables $t_{i,j}$, we denote $\sum_{\mathrm{sym}} m$ the symmetric sum of m, namely,

$$\sum_{\mathrm{sym}} m := \sum_{\sigma \in S_5} \sigma(m).$$

We are in the position to describe a basis of the quotient $\mathbb{C}[t_{i,j}]^G$.

Theorem 5. *The quotient* $\mathbb{C}[t_{ij}]^G$ *is generated by the polynomials* $\sum_{\mathrm{sym}} M(V, E_d, E_s)$ *with* $(V, E_d, E_s) \in \mathrm{DS}(5)$*.*

Proof. Let P be a polynomial in $\mathbb{C}[t_{ij}]^G$. Since P is invariant under symmetric group, one may write

$$P = \sum_{\text{sym}} P_1 + \ldots + \sum_{\text{sym}} P_k,$$

where all P_s's are monomials. By assumption, P_i is invariant under action of subgroup $\{id\} \times \mathbb{Z}_5^5$. We see that P_s can be split into three parts

$$P_s = \prod_{i=1}^{5} (t_{i,j}^5)^{e_{i,j}} \cdot \prod_{\{i_j\}} (t_{i_1,i_2} t_{i_2,i_3} \cdots t_{i_{n-1},i_n} t_{i_n,i_1}) \cdot R. \tag{2}$$

Geometrically, the second part is represented by loops. Hence, we may assume that $R = \lambda t_{i_1,j_1} t_{i_2,j_2} \ldots t_{i_m,j_m}$ contains no loops. That means

$$i_1 \notin \{j_1, j_2, j_3, \ldots, j_m\}.$$

However, R is not invariant under $\{id\} \times \mathbb{Z}_5^5$ if $m \geqslant 1$. Thus, we have $R = \lambda$. Suppose that all power $e_{i,j}$'s in the expression of P_s equal 1 and the second part consists of distinct loops. Then clearly the polynomial P_s is given by some DS-graph (V, E_d, E_s). That is $P_s = M(V, E_d, E_s)$. To complete the proof, it suffices to reduce the powers in (2). Define invariant polynomials

$$S_1 := \sum_{\text{sym}} (t_{i,j}^5)^2 - \left(\sum_{\text{sym}} t_{i,j}^5 \right)^2,$$

and

$$S_2 := \sum_{\text{sym}} (t_{i_1,i_2} t_{i_2,i_3} \cdots t_{i_{n-1},i_n} t_{i_n,i_1})^2 - \left(\sum_{\text{sym}} t_{i_1,i_2} t_{i_2,i_3} \cdots t_{i_{n-1},i_n} t_{i_n,i_1} \right)^2.$$

We find that both S_1 and S_2 satisfy the previous condition. Hence they are generated by polynomials represented by DS-graphs. It yields that $\sum_{\text{sym}} (t_{i,j}^5)^2$ and $\sum_{\text{sym}} (t_{i_1,i_2} t_{i_2,i_3} \cdots t_{i_{n-1},i_n} t_{i_n,i_1})^2$ are also represented by DS-graphs. This completes the proof. $\square$

References

[1] B. H. Lian, K. Liu, and S. T. Yau, "Mirror principle I," *Asian Journal of Mathematics* **1** (1997), no. 4, 729–763.

[2] P. Candelas, X. C. D. L. Ossa, P. S. Green, and L. Parkes, "A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory," *Nuclear Physics B* **359** (1991), no. 1, 21–74.

[3] H. Chen and S. S.-T. Yau, "Moduli and modular groups of a class of Calabi-Yau n-dimensional manifolds, $n \geq 3$," *Journal of Differential Geometry* **54** (2000), no. 2000, 1–12.

[4] H. Chen, C. Seeley, and S. S.-T. Yau, "Algebraic determination of isomorphism classes of the moduli algebras of $\tilde{E}_6$ singularities," *Mathematische Annalen* **318** (2000), no. 4, 637–666.

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