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Equivalences between isolated hypersurface singularities

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1. Introduction

Let $\mathcal{O}_{n+1}$ be the ring of germs of holomorphic functions $(\mathbb{C}^{n+1}, 0) \to \mathbb{C}$. There are many important equivalence relations that have been defined on the elements of $\mathcal{O}_{n+1}$. $\mathcal{R}$-, $\mathcal{RL}$-, and $\mathcal{K}$-equivalence are well known in function theory. Each of these equivalence relations can be defined in terms of a Lie group action on $\mathcal{O}_{n+1}$. For instance two functions are defined to be $\mathcal{R}$-equivalent if they are the same up to a holomorphic change of coordinates in the domain. In this case the Lie group acting on $\mathcal{O}_{n+1}$ is the group of all holomorphic change of coordinates preserving the origin. Simple complete characterizations of when two functions are $\mathcal{R}$-, $\mathcal{RL}$-, or $\mathcal{K}$-equivalent were given by Yau [9] and by Mather and Yau [6].

$\mathcal{D}$-, $\mathcal{A}$-, and $\mathcal{B}$-equivalence come from singularity theory. These equivalence relations are defined on the basis of algebra isomorphisms. For example, we can associate a $\mathbb{C}$-algebra $\mathcal{O}_{n+1}/\Delta(f)$, the Milnor algebra, to any $f \in \mathcal{O}_{n+1}$, where $\Delta(f)$ is the ideal in $\mathcal{O}_{n+1}$ generated by the partial derivatives of f. We say that two functions are $\mathcal{D}$-equivalent if their associated Milnor algebras are isomorphic.

It is an interesting question to determine the relationships between these six equivalences. The goal of this paper is to study these relationships. For a holomorphic function f with a critical point at the origin, we determine when the equivalence classes of f with respect to two different equivalence relations coincide.

The purpose of this paper is two-fold. On the one hand, we give a necessary and sufficient condition for $\mathcal{RL}$-equivalence to coincide with $\mathcal{K}$-equivalence (cf. Theorem 5.1). This leads us to define the new notion of almost quasi-homogeneous functions. We suspect that the singularities defined by almost quasi-homogeneous functions may form a distinguished class of singularities which have some special properties shared by quasi-homogeneous ones.

In Sect. 6, we discuss the relationship between $\mathcal{D}$- and $\mathcal{K}$-equivalence. Perhaps the most striking result here is Theorem 6.9, which provides us a lot of examples

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with two holomorphic functions having isolated critical points at the origin and the same Jacobian ideals, but their zero sets are not biholomorphically equivalent. We also give an example (cf. Example 6.8) such that

$$\begin{array}{ccccc} \mathscr{R}(f) & \subsetneqq & \mathscr{R}\mathscr{L}(f) & \subsetneqq & \mathscr{K}(f) \\ & & \rotatebox{90}{$\subsetneqq$} & & \\ & & \mathscr{Q}(f) & & \end{array}$$

with $\mathscr{Q}(f) \nsubseteq \mathscr{K}(f)$ and $\mathscr{K}(f) \nsubseteq \mathscr{Q}(f)$. This answers a question raised by G.M. Greuel, who asked whether such functions exist. The computation of this example is extremely difficult, if not impossible, by hand. We have developed a computer program which allows us to check all of the equivalence relations. The examples show the effectiveness of our criteria in checking whether the equivalence classes coincide or not.

In Sect. 7, we explain how the computer programs work, and how they can be used to compute generators of the modules and ideals discussed in this paper, including $a(f)$, which is an important notion in $\mathscr{Q}$-equivalence.

The results mentioned above, together with results obtained in [6, 8–10], complete the solution of the problem of determining the equivalence between isolated hypersurface singularities began in the sixties. On the other hand, since the problem is completely solved and the method here can prove the previous results as well, we also give a complete and self-contained account of the relationship between these equivalences. Our methods are elementary and easy to comprehend. Development of important tools such as computations of the tangent spaces to the orbit manifolds by Mather [4, 5] and Shoshitaishvili [8] have been included here to assist the reader.

2. The hierarchy of equivalence relations

Let $\mathcal{O}_{n+1}$ denote the ring of germs at the origin of holomorphic functions $f:(\mathbb{C}^{n+1},0)\to\mathbb{C}$. $\mathcal{O}_{n+1}$ has a unique maximal ideal m_{n+1} consisting of the germs of holomorphic functions which vanish at the origin. Let G_{n+1} be the set of germs at the origin of biholomorphisms ϕ: $(\mathbb{C}^{n+1},0)\to(\mathbb{C}^{n+1},0)$. G_{n+1} can be made into a group by using composition of map germs for the group operation.

Definition 2.1. Two germs of holomorphic functions $f,g:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ are called *right equivalent* if there exists a $\phi\in G_{n+1}$ such that $f=g\circ\phi$. We use the notation $f \overset{R}{\sim} g$ to denote right equivalence.

The group $\mathscr{R}=G_{n+1}$ acts on m_{n+1} by composition on the right. The right equivalence classes are the orbits of this group action. The orbit of $f\in m_{n+1}$ is denoted by

$$\mathscr{R}(f)=\{g\in m_{n+1} \mid g \overset{R}{\sim} f\}$$

Definition 2.2. Two germs of holomorphic functions $f,g:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ are called *right–left equivalent* if there exist $\phi\in G_{n+1}$ and $\psi\in G_1$ such that $f=\psi\circ g\circ\phi$. The notation $f \overset{RL}{\sim} g$ is used to indicate right-left equivalence.

Right–left equivalence also arises from a group action. The group $\mathcal{RL} = G_1 \times G_{n+1}$ acts on m_{n+1} by composing on the left with the G_1 component and on the right with the component from G_{n+1}. These orbits are denoted by

$$\mathcal{RL}(f) = \{g \in m_{n+1} \mid g \overset{RL}{\sim} f\}$$

Definition 2.3. Suppose f, $g:(\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ are holomorphic map germs. f and g are called *contact equivalent* if and only if there exists a germ of a biholomorphism $H:(\mathbb{C}^{n+2},0) \to (\mathbb{C}^{n+2},0)$ such that
a) $H(\mathbb{C}^{n+1} \times \{0\},0) = (\mathbb{C}^{n+1} \times \{0\},0)$
b) $H(\text{graph } f) = \text{graph } g$

The notation $f \overset{C}{\sim} g$ is used to indicate contact equivalence.

The contact group $\mathcal{K}$ was first defined in the C^∞ category by Mather [4]. For each pair of positive integers (n,p) he associated a group $\mathcal{K}$ of germs of C^∞ mappings. Much later Mather and Yau [6] defined the holomorphic analog associated with the pair $(n+1,1)$ which we are interested in here.

Definition 2.4. The *contact group* $\mathcal{K}$ consists of those germs of biholomorphisms $H:\mathbb{C}^{n+2} \to \mathbb{C}^{n+2}$ for which there exists a holomorphic map $h:(\mathbb{C}^{n+1},0) \to (\mathbb{C}^{n+1},0)$ such that the following diagram commutes

$$\begin{array}{ccccc}
(\mathbb{C}^{n+1},0) & \xrightarrow{\iota} & (\mathbb{C}^{n+2},0) & \xrightarrow{\pi} & (\mathbb{C}^{n+1},0) \\
\downarrow h & & \downarrow H & & \downarrow h \\
(\mathbb{C}^{n+1},0) & \xrightarrow{\iota} & (\mathbb{C}^{n+2},0) & \xrightarrow{\pi} & (\mathbb{C}^{n+1},0)
\end{array}$$

where $\iota(z_0,\ldots,z_n) = (z_0,\ldots,z_n,0)$ and $\pi(z_0,\ldots,z_n,w) = (z_0,\ldots,z_n)$. The group operation is composition.

This condition can be stated alternately. It says that $H(z_0,\ldots,z_{n+1})$ can be written in the form $(h(z_0,\ldots,z_n), k(z_0,\ldots,z_{n+1}))$ where $h:(\mathbb{C}^{n+1},0) \to (\mathbb{C}^{n+1},0)$ is the germ of a biholomorphism and $k:(\mathbb{C}^{n+2},0) \to (\mathbb{C},0)$ is the germ of a holomorphic map with the property that $k(z_0,\ldots,z_n,0) = 0$.

We can now give the action of the group $\mathcal{K}$ on m_{n+1}. If $H \in \mathcal{K}$ and $f \in m_{n+1}$, then $g = Hf$ is defined by the equation $g = k \circ (\mathrm{id}, f) \circ h^{-1}$. It is easy to check that elements of $\mathcal{O}_{n+1}$ are contact equivalent if and only if they lie in the same $\mathcal{K}$-orbit.

The $\mathcal{K}$-orbits are denoted by

$$\mathcal{K}(f) = \{g \in m_{n+1} \mid g \overset{C}{\sim} f\}$$

Contact equivalence is important because it turns out to be very geometric. The following proposition, due to Mather [5] in the C^∞ category and later appearing in [6], explains its significance.

Proposition 2.5. *Let $(V,0)$ and $(W,0)$ be germs of hypersurfaces in $\mathbb{C}^{n+1}$ defined by f, $g \in m_{n+1}$ respectively. Then f and g are in the same $\mathcal{K}$-orbit if and only if the germs $(V,0)$ and $(W,0)$ are biholomorphically equivalent.*

Proof. First, suppose f and g are in the same $\mathscr{K}$-orbit. Let H be an element of $\mathscr{K}$ such that H (graph f) = graph g. Then the following set germ equalities hold.

$$h^{-1}(W)=h^{-1}(\iota^{-1}\operatorname{graph} g)=\iota^{-1}(H^{-1}\operatorname{graph} g)=\iota^{-1}(\operatorname{graph} f)=V$$

This shows that h provides a biholomorphic equivalence between $(V,0)$ and $(W,0)$.

Now suppose that $(V,0)$ and $(W,0)$ are biholomorphically equivalent. Let $h:(\mathbb{C}^{n+1},0)\to(\mathbb{C}^{n+1},0)$ be a germ of a biholomorphic mapping such that $h(V)=W$. Then there is a unit $u\in\mathcal{O}_{n+1}$ for which $f=u(g\circ h)$. Define $H:(\mathbb{C}^{n+2},0)\to(\mathbb{C}^{n+2},0)$ by $H(z,w)=(h(z),u^{-1}(z)w)$ where $z\in\mathbb{C}^{n+1}$ and $w\in\mathbb{C}$. Then $H=\mathscr{K}$ and $H(z,f(z))=(h(z),u^{-1}(z)f(z))=(h(z),g\circ h(z))$ for $z=(z_0,\dots,z_n)$, so H (graph f) = graph g.

For any $f\in m_{n+1}$ we define the *Jacobian ideal* $\Delta(f)\subset\mathcal{O}_{n+1}$ to be the ideal generated by the partial derivatives of f. The $\mathbb{C}$-algebra $\mathcal{O}_{n+1}/\Delta(f)$ will be called the *Milnor algebra* associated to f. When $f=0$ defines an isolated singularity at the origin, then the dimension of $\mathcal{O}_{n+1}/\Delta(f)$, considered as a $\mathbb{C}$-vector space, is the topological invariant μ, the Milnor number of the singularity.

Definition 2.6. Two holomorphic germs f, $g:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ are *$\mathscr{Q}$-equivalent* if there is a $\mathbb{C}$-algebra isomorphism of Milnor algebras $\mathcal{O}_{n+1}/\Delta(f)\simeq\mathcal{O}_{n+1}/\Delta(g)$. We also introduce the notation

$$\mathscr{Q}(f)=\{g\in m_{n+1}\,|\,\mathcal{O}_{n+1}/\Delta(f)\simeq\mathcal{O}_{n+1}/\Delta(g)\}$$

The $\mathbb{C}$-algebra $\mathcal{O}_{n+1}/(f,\Delta(f))$ is called the *moduli algebra.* This name is a natural choice because, considered as a $\mathbb{C}$-vector space, it is the base space for the miniversal deformation of the singularity defined by $f=0$.

Definition 2.7. Two holomorphic germs f, $g:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ are *$\mathscr{A}$-equivalent* if there is a $\mathbb{C}$-algebra isomorphism of moduli algebras $\mathcal{O}_{n+1}/(f,\Delta(f))\simeq\mathcal{O}_{n+1}/(g,\Delta(g))$. We will use the following notation for the $\mathscr{A}$-equivalence classes.

$$\mathscr{A}(f)=\{g\in m_{n+1}\,|\,\mathcal{O}_{n+1}/(f,\Delta(f))\simeq\mathcal{O}_{n+1}/(g,\Delta(g))\}$$

Definition 2.8. Two holomorphic germs f, $g:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ are *$\mathscr{B}$-equivalent* if there is a $\mathbb{C}$-algebra isomorphism $\mathcal{O}_{n+1}/(f,m_{n+1}\Delta(f))\simeq\mathcal{O}_{n+1}/(g,m_{n+1}\Delta(g))$. The $\mathscr{B}$-equivalence classes are denoted by

$$\mathscr{B}(f)=\{g\in m_{n+1}\,|\,\mathcal{O}_{n+1}/(f,m_{n+1}\Delta(f))\simeq\mathcal{O}_{n+1}/(g,m_{n+1}\Delta(g))\}$$

Proposition 2.9. *The diagram shown below gives some of the relationships between the different equivalence classes.*

$$\begin{array}{ccccccc}\mathscr{R}(f)&\subseteq&\mathscr{RL}(f)&\subseteq&\mathscr{K}(f)&\subseteq&\mathscr{A}(f)\\ &&\cap\!|&&\cap\!|&&\\ &&\mathscr{Q}(f)&&\mathscr{B}(f).&&\end{array}$$

Proof. The inclusions $\mathscr{R}(f)\subseteq\mathscr{RL}(f)\subseteq\mathscr{K}(f)$ hold because there are corresponding embeddings of the groups which respect the group actions. The embedding $\mathscr{R}\hookrightarrow\mathscr{RL}$ is given by $g\mapsto(\mathrm{id},g)$, while $\mathscr{RL}\hookrightarrow\mathscr{K}$ is defined by $(v,h)\mapsto H$, where $Hf=(\mathrm{id},v\circ f)\circ h$.

To establish that $\mathscr{RL}(f)\subseteq\mathscr{Q}(f)$ we will use the following lemma.

Lemma 2.10. *Suppose* $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ *is a germ of a holomorphic function and* $\gamma=(\psi,\phi)$ *is an element of* $\mathscr{RL}$. *Let* $\phi^*:\mathcal{O}_{n+1}\to\mathcal{O}_{n+1}$ *be the pullback map given by* $f\mapsto f\circ\phi$. *Then* $\phi^*\Delta(f)=\Delta(\gamma f)$.

Proof. According to the chain rule,

$$\frac{\partial\gamma f}{\partial z_i}=\frac{d\psi}{dz}(f\circ\phi)\sum_{j=0}^{n}\left(\frac{\partial f}{\partial z_j}\circ\phi\right)\frac{\partial\phi}{\partial z_i}$$

This shows that $\Delta(\gamma f)\subseteq\phi^*\Delta(f)$. For the reverse inclusion, we use the hypothesis that both ψ and ϕ are biholomorphic at the origin. This means that the derivative $d\psi/dz$ and the Jacobian matrix $D\phi=(\partial\phi^j/\partial z_i)$ have inverses. Let (c_j^i) be the inverse of $D\phi$. Then

$$\phi^*\frac{\partial f}{\partial z_j}=\left(\frac{d\psi}{dz}(f\circ\phi)\right)^{-1}\sum_{i=0}^{n}c_j^i\frac{\partial\gamma f}{\partial z_i}$$

verifying the opposite inclusion. □

Suppose that $g\in\mathscr{RL}(f)$. Then there exists $\gamma\in\mathscr{RL}$, $\gamma=(\psi,\phi)$ for which $g=\gamma f$. Now ϕ induces an isomorphism $\phi^*:\mathcal{O}_{n+1}\to\mathcal{O}_{n+1}$. According to Lemma 2.10, $\phi^*\Delta(f)=\Delta(g)$. This means that ϕ^* induces an isomorphism of the quotient rings, so $g\in\mathscr{Q}(f)$. This proves the inclusion $\mathscr{RL}(f)\subseteq\mathscr{Q}(f)$.

The inclusions $\mathscr{K}(f)\subseteq\mathscr{A}(f)$ and $\mathscr{K}(f)\subseteq\mathscr{B}(f)$ follow from the next lemma in a similar manner.

Lemma 2.11. *Suppose* f, $g:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ *are germs of holomorphic functions which are contact equivalent, that is,* $g=u(f\circ\phi)$ *for some* u *a unit and* ϕ *a biholomorphic change of coordinates. Then the following equations hold.*

a) $\phi^*(f,\Delta(f))=(g,\Delta(g))$.

b) $\phi^*(f,m_{n+1}\Delta(f))=(g,m_{n+1}\Delta(g))$.

Proof. Let $g'=f\circ\phi$. Applying the product rule to $g=ug'$ we get

$$\frac{\partial g}{\partial z_i}=\frac{\partial u}{\partial z_i}g'+u\frac{\partial g'}{\partial z_i}\tag{2.12}$$

showing that $(g,\Delta(g))\subseteq(g',\Delta(g'))$. Performing the same computation except with g and g' interchanged, and u replaced by u^{-1} proves the opposite inclusion. Thus $(g,\Delta(g))=(g',\Delta(g'))$.

Using Lemma 2.10, $(g',\Delta(g'))=\phi^*(f,\Delta(f))$. Combining the two equations proves a). The proof of part b) is very similar. □

3. Finite determinacy

For any f, $g\in\mathcal{O}_{n+1}$, we say that f and g have the same k-jet at the origin if their derivatives at the origin agree up to order $\leq k$. The k-jet $f^{(k)}$ is the equivalence class of all $g\in\mathcal{O}_{n+1}$ which have the same k-jet as f.

Definition 3.1. Let $f\in\mathcal{O}_{n+1}$ and let $\mathscr{G}$ be a group which actrs on $\mathcal{O}_{n+1}$. f is *k-determined* relative to $\mathscr{G}$ if for any $g\in\mathcal{O}_{n+1}$ such that $g^{(k)}=f^{(k)}$, the $\mathscr{G}$-orbit of f

contains g. We say that f is finitely determined relative to $\mathscr{G}$ if f is k-determined for some positive integer k.

The following theorem shows that the notion of finite determinacy can be expressed in both algebraic and geometric terms. We will use the notation $f^{-1}m_1$ to represent the *module* consisting of all elements of the form $\sum_{i=1}^{\infty} a_i f^i$ where $\sum_{i=1}^{\infty} a_i t^i$ is a convergent power series vanishing at zero.

Theorem 3.2. *Let $(V,0)$ be the germ of a hypersurface in $\mathbb{C}^{n+1}$ defined by $f=0$. The following conditions are equivalent.*
a) *$V\backslash\{0\}$ is nonsingular.*
b) *$\mathcal{O}_{n+1}/(f,\Delta(f))$ is a finite dimensional $\mathbb{C}$-vector space.*
c) *$\mathcal{O}_{n+1}/(f,m_{n+1}\Delta(f))$ is a finite dimensional $\mathbb{C}$-vector space.*
d) *$\mathcal{O}_{n+1}/f^{-1}m_1+m_{n+1}\Delta(f)$ is a finite dimensional $\mathbb{C}$-vector space.*
e) *$\mathcal{O}_{n+1}/\Delta(f)$ is a finite dimensional $\mathbb{C}$-vector space.*
f) *$\mathcal{O}_{n+1}/m_{n+1}\Delta(f)$ is a finite dimensional $\mathbb{C}$-vector space.*
g) *f is finitely determined relative to $\mathscr{K}$.*
h) *f is finitely determined relative to $\mathscr{R}\mathscr{L}$.*
i) *f is finitely determined relative to $\mathscr{R}$.*

Proof. The chain of inclusions $m_{n+1}\Delta(f)\subset f^{-1}m_1+m_{n+1}\Delta(f)\subset(f,m_{n+1}\Delta(f))\subset(f,\Delta(f))$ implies that f)$\Rightarrow$d)$\Rightarrow$c)$\Rightarrow$b). Similarly, the inclusions $m_{n+1}\Delta(f)\subset\Delta(f)\subset(f,\Delta(f))$ show that f)$\Rightarrow$e)$\Rightarrow$b).

When b) holds, then there exists some positive integer N so that $m_{n+1}^N\subset(f,\Delta(f))$. But then $V(f,\Delta(f))=\{0\}$. This shows $V\backslash\{0\}$ is nonsingular, and so b)$\Rightarrow$a).

To prove a)$\Rightarrow$d), we use Hilbert's *Nullstellensatz*. If a) holds, then $m_{n+1}\Delta(f)$ must be m_{n+1}-primary. This means that for some positive integer N, $m_{n+1}^N\subset m_{n+1}\Delta(f)$. But then $\mathcal{O}_{n+1}/m_{n+1}\Delta(f)$ is a finite dimensional $\mathbb{C}$-vector space.

This shows that the first six conditions are equivalent. The work of Mather [4], Theorem 3.5, p. 293 shows that g)$\Leftrightarrow$b), h)$\Leftrightarrow$d), and i)$\Leftrightarrow$e). His paper uses notation that is somewhat different from that which is used here because the results were proved in the C^∞ category, and they were stated somewhat more generally. Nevertheless in Section 9, p. 307–308 he shows that they are also valid in the complex analytic category. □

The hypothesis that f is finitely determined simplifies the diagram in Proposition 2.9 showing the relationship between the different types of germ equivalence. The notions of $\mathscr{K}$-, $\mathscr{A}$-, and $\mathscr{B}$-equivalence turn out to be exactly the same. This is the content of the following theorem of Mather and Yau [6].

Theorem 3.3. *Suppose f, $g:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ are holomorphic function with isolated critical points at the origin. The following statements are equivalent.*
a) *f, g are $\mathscr{K}$-equivalent*
b) *f, g are $\mathscr{A}$-equivalent*
c) *f, g are $\mathscr{B}$-equivalent*

We will not prove this theorem here, but many of the techniques used in its proof appear in this paper. The rest of this section is devoted to the computation of the tangent spaces to the k-jet orbits. These results will be used frequently in the sections which follow.

Let J^k be the set of k-jets at the origin of elements of $\mathcal{O}_{n+1}$. J^k has a natural complex analytic structure obtained by using the Taylor series coefficients as coordinates. For each of the groups $\mathcal{R}$, $\mathcal{RL}$, and $\mathcal{K}$, let $\mathcal{R}^k$, $\mathcal{RL}^k$, and $\mathcal{K}^k$ denote the respective sets of k-jets at the origin. They are complex Lie groups which act on J^k.

For any $f \in \mathcal{O}_{n+1}$ we use the notations $\mathcal{R}^k(f)$, $\mathcal{RL}^k(f)$, and $\mathcal{K}^k(f)$ to stand for the orbits of $f^{(k)}$ with respect to $\mathcal{R}^k$, $\mathcal{RL}^k$, and $\mathcal{K}^k$.

Theorem 3.4. *Let $f:(\mathbb{C}^{n+1},0)\to\mathbb{C}$ be the germ of a holomorphic function with an isolated singularity at the origin. Then $\mathcal{R}^k(f)$, $\mathcal{RL}^k(f)$, and $\mathcal{K}^k(f)$ are complex analytic manifolds. The following $\mathbb{C}$-vector space isomorphisms exist between their tangent spaces at $f^{(k)}$ and subspaces of J^k.*

a) $T_f(\mathcal{R}^k(f)) \simeq m_{n+1}\Delta(f)J^k$
b) $T_f(\mathcal{RL}^k(f)) \simeq (f^{-1}m_1 + m_{n+1}\Delta(f))J^k$
c) $T_f(\mathcal{K}^k(f)) \simeq (f, m_{n+1}\Delta(f))J^k$

Proof. It is well known that the orbits of a Lie group $\mathcal{G}$ acting analytically on a complex manifold M are submanifolds so we only need to compute the tangent spaces.

Let π_z be the map which takes any element of J^k and sends it to the values of its k-th order Taylor approximation at z. Each of the three isomorphisms a)–c) arises from the map which takes a tangent vector v and assigns it to $\phi(z) = v(\pi_z) \in J^k$. It is clear that this map is a homomorphism. It is injective because the images of v on the coordinate functions of J^k are just the Taylor coefficients at the origin of $\phi(z)$. If $v(\pi_z) = 0$ for all z, then v applied to the coordinate functions must be zero, so v would have to equal zero as well. We must calculate the image of this map for each group $\mathcal{R}^k$, $\mathcal{RL}^k$, and $\mathcal{K}^k$ separately. It should be understood that all of these computations are to be performed modulo m_{n+1}^{k+1}.

We start with the case when $\mathcal{G} = \mathcal{R}^k$. For any tangent vector v there exists a germ of a holomorphic curve γ_t in $\mathcal{R}^k(f)$ through $f^{(k)}$ such that

$$v(g) = \left.\frac{dg\circ\gamma_t}{dt}\right|_{t=0}$$

for all germs of maps $g:\mathcal{R}^k(f)\to\mathbb{C}$. Now γ_t can be lifted to a curve $\tilde{\gamma}_t$ in $\mathcal{R}^k$, such that $\gamma_t = f\circ\tilde{\gamma}_t$, and

$$\begin{aligned} v(\pi_z) &= \left.\frac{df\circ\tilde{\gamma}_t}{dt}\right|_{t=0} \\ &= \sum_{i=0}^{n} \frac{\partial f}{\partial z_i}(\tilde{\gamma}_0)\tilde{\gamma}_t'^{(i)}\Big|_{t=0} \end{aligned}$$

But $\tilde{\gamma}_0$ is the identity and $\tilde{\gamma}_t(0) = 0$, so each component of $\tilde{\gamma}_t'|_{t=0}$ is in m_{n+1}. Therefore $v(\pi_z) \in m_{n+1}\Delta(f)$.

On the other hand, suppose that $\phi \in m_{n+1}\Delta(f)$, $\phi = \sum_{i=0}^{n} \xi_i \frac{\partial f}{\partial z_i}$ with $\xi_i \in m_{n+1}$. Define a holomorphic family of k-jets by $\gamma_t(z_0, \ldots, z_n) = f(z_0 + t\xi_0(z_0, \ldots, z_n), \ldots, z_n + t\xi_n(z_0, \ldots, z_n))$. In a neighborhood of the origin, this is a curve in $\mathscr{R}^k(f)$ through $f^{(k)}$ whose tangent vector at the origin maps to ϕ.

When $\mathscr{G} = \mathscr{R}\mathscr{L}^k$ the argument is similar, but the lifting of the curve γ_t is given by the equation $\gamma_t = \tilde{\psi}_t \circ f \circ \tilde{\gamma}_t$. Expand $\tilde{\psi}_t$ in a Taylor series $\tilde{\psi}_t = \sum_{i=0}^{k} \tilde{\psi}_t^{(i)} z^i$. Then $\gamma_t = \sum_{i=0}^{k} \tilde{\psi}_t^{(i)} (f \circ \tilde{\gamma}_t)^i$. Here is the computation of $v(\pi_z)$ in this case.

$$v(\pi_z) = \sum_{i=0}^{k} \frac{d\tilde{\psi}_t^{(i)}}{dt}\bigg|_{t=0} (f \circ \tilde{\gamma}_0)^i + \sum_{i=1}^{k} \tilde{\psi}_0^{(i)} i (f \circ \tilde{\gamma}_0)^{i-1} \frac{df \circ \tilde{\gamma}_t}{dt}\bigg|_{t=0}$$

The first term is clearly in $f^{-1}m_1$ and second one is in $m_{n+1}\Delta(f)$ according the computation made in the previous paragraph. Now we prove that all elements of $(f^{-1}m_1 + m_{n+1}\Delta(f))J^k$ are images of tangent vectors. Suppose $\phi = \sum_{i=1}^{k} \xi^{(i)} f^i + \eta$, $\eta \in m_{n+1}\Delta(f)$. We only have to check that $\sum_{i=0}^{n} \xi^{(i)} f^i$ is the image of a tangent vector, because we have already proved that η corresponds to a tangent vector of $\mathscr{R}(f) \subseteq \mathscr{R}\mathscr{L}(f)$. Define γ_t by $\sum_{i=1}^{k} (1 + t\xi^{(i)}) f^i$. This is a curve in $\mathscr{R}\mathscr{L}^k(f)$ through $f^{(k)}$ with a tangent vector that maps into $\sum_{i=0}^{k} \xi^{(i)} f^i$.

Finally when $\mathscr{G} = \mathscr{K}^k$ we can use Proposition 2.5 to the lift the curve γ_t to $\gamma_t = \tilde{u}_t(f \circ \tilde{\gamma}_t)$. $v(\pi_z)$ can be computed as follows

$$v(\pi_z) = \frac{d\tilde{u}_t}{dt}\bigg|_{t=0} f \circ \tilde{\gamma}_0 + \tilde{u}_0 \sum_{i=0}^{n} \frac{\partial f}{\partial z_i}(\tilde{\gamma}_0) \frac{d\tilde{\gamma}_t}{dt}\bigg|_{t=0}.$$

Using the same reasoning as in the computation of the tangent space of $\mathscr{R}^k(f)$, $v(\pi_z) \in (f, m_{n+1}\Delta(f))$. To see that all members of $(f, m_{n+1}\Delta(f))$ are images of tangent vectors, we take a general element $\phi = uf + \eta$ where $\eta \in m_{n+1}\Delta(f)$. As above, it is only necessary to show that uf corresponds to a tangent vector. For this, define γ_t by $(1 + tu)f$. This is a curve in $\mathscr{K}^k(f)$ through $f^{(k)}$ with a tangent vector that maps to uf. □

We want to look at the jet version of $\mathscr{Q}$-equivalence as well. Let $\mathscr{Q}^k(f) = \{g^{(k)} \mid \mathcal{O}_{n+1}/\Delta(f) + m_{n+1}{}^k \simeq \mathcal{O}_{n+1}/\Delta(g) + m_{n+1}{}^k\}$, $a^k(f) = \{g^{(k)} \mid \Delta(g) \subseteq \Delta(f) + m_{n+1}{}^k\}$, and $A^k(f) = (a^k(f) + m_{n+1}\Delta(f))J^k$. The following result, due to Shoshitaishvili [8], gives the structue of the $\mathscr{Q}^k$-equivalence classes.

Theorem 3.5. *Let $f:(\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function with an isolated singularity at the origin. Then $\mathscr{Q}^k(f)$ is a complex analytic manifold and its tangent space at $f^{(k)}$ is isomorphic to the vector space $A^k(f)$.*

Proof. $m_{n+1}{}^2 J^k$ is the subspace of J^k consisting of k-jets of holomorphic functions with singularities at the origin. Let L be a vector subspace of $a^k(f)$ complementary to $(a^k(f) \cap m_{n+1}\Delta(f))J^k$ and M be a vector subspace of $m_{n+1}{}^2J^k$ complementary to $A^k(f)$. Within $\mathscr{R}^k$ let T be a local transversal through the identity to the subgroup which fixes $f^{(k)}$.

Define a mapping $\phi: L \times M \times T \to m_{n+1}{}^2J^k$ by $\phi(l, m, t) = (l + m + f^{(k)}) \circ t$. From the way L and M were chosen, $L \times M$ is transverse to the tangent space of $\mathscr{R}^k(f)$ at $f^{(k)}$. For this reason, the differential ϕ_* at $(0, 0, \mathrm{id})$ is an isomorphism. The inverse function theorem implies that locally the map ϕ is biholomorphic.

We want to prove that there exist neighborhoods U of $f^{(k)}$ in $m_{n+1}{}^2J^k$, V of 0 in L, and W of the identity in T for which $U \cap \mathscr{Q}^k(f) = \phi(V \times \{0\} \times W)$. This will show that $\mathscr{Q}^k(f)$ is a manifold in the neighborhood of $f^{(k)}$ and also that its tangent space is $A^k(f)$. The following lemma is used to construct U.

Lemma 3.6. *There exists a neighborhood U' of the origin in $L \times M$ for which $(f^{(k)} + U') \cap \mathscr{Q}^k(f) = (f^{(k)} + U') \cap (f^{(k)} + L)$.*

Proof. Let $\mathscr{E}^k$ be the vector space of k-jets of holomorphic mappings $(\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$. The gradient map $\nabla: m_{n+1}{}^2J^k \to \mathscr{E}^{k-1}$ given by $h \mapsto \nabla h$ is linear and one-one. We are going to use this embedding of $\mathscr{Q}^k(f)$ in $\mathscr{E}^{k-1}$ to study its the local structure.

The theory developed by Mather [4], [5] is more general than has been presented here, and in particular it applies to the elements of $\mathscr{E}^k$. Proposition 2.5 also generalizes, as found in Mather [5], Theorem 2.1. In this formulation it implies that if f and g are $\mathscr{Q}^k$-equivalent, then ∇f and ∇g are $\mathscr{K}^{k-1}$-equivalent.

Thus the image of $\mathscr{Q}^k(f)$ under the gradient map is contained in $\mathscr{K}^{k-1}(\nabla f)$. Let $\tilde{A}^{k-1}$ be the tangent space to $\mathscr{K}^{k-1}(\nabla f)$ at ∇f. The computation carried out in Theorem 3.4c generalizes to this case as well, see Mather [4], Theorem 7.4a. Each tangent vector in $\tilde{A}^{k-1}$ can be written in the form $Hf \cdot v + B\nabla f$ modulo $m_{n+1}{}^k$ where Hf is the Hessian matrix, v is any $n+1$-tuple of elements of m_{n+1}, and B is any $(n+1) \times (n+1)$ matrix of elements of $\mathcal{O}_{n+1}$. By using the product rule to rewrite this expression, it can be seen that any tangent vector is of the form $\nabla(d_v f) + B'\nabla f$ modulo $m_{n+1}{}^k$, where $d_v f$ is the directional derivative of f along v.

It follows that $\nabla(m_{n+1}{}^2J^k) \cap \tilde{A}^{k-1} = \nabla(A^k(f))$ and also that $\tilde{A}^{k-1} = \tilde{a}^{k-1} + \tilde{\imath}^{k-1}$, where $\tilde{a}^{k-1} = \{(B\nabla f)^{(k-1)} | B \text{ and } n \times n \text{ matrix}\}$ and $\tilde{\imath}^{k-1} = \nabla(m_{n+1}\Delta(f))J^{k-1}$.

Because $\nabla(L) \subset \tilde{a}^{k-1}$ and $\nabla(L) \cap \tilde{\imath}^{k-1} = 0$, we can find a subspace $\tilde{L} \subset \tilde{a}^{k-1}$ which contains $\nabla(L)$ and which is complementary to $\tilde{a}^{k-1} \cap \tilde{\imath}^{k-1}$. And because $\nabla(M) \cap \tilde{A}^{k-1} = 0$, there is a subspace $\tilde{M} \subset \mathscr{E}^k$ which contains $\nabla(M)$ and which is complementary to $\tilde{A}^{k-1}$.

Let $\tilde{N}$ be the submanifold $\nabla f^{(k)} + \tilde{L} \times \tilde{M}$. Because $\mathscr{K}^{k-1}(\nabla f)$ is algebraic, there exists a neighborhood $\tilde{U}$ of the origin in $\mathscr{E}^{k-1}$ such that $(\nabla f^{(k)} + \tilde{U}) \cap N \cap \mathscr{K}^{k-1}(\nabla f)$ is a manifold of dimension $\dim_{\mathbb{C}} L$. On the other hand, from the definition of $\tilde{a}^{k-1}$, there are sufficiently small neighborhoods $\tilde{V}$ of the origin in $\tilde{L}$ such that $(\nabla f^{(k)} + \tilde{V}) \subset N \cap \mathscr{K}^{k-1}(\nabla f)$. Choosing $\tilde{U}$ small enough, we will have $\nabla f^{(k)} + \tilde{V} = (\nabla f^{(k)} + \tilde{U}) \cap N \cap \mathscr{K}^{k-1}(\nabla f)$. Choosing $U' = \nabla^{-1}\tilde{U}$ completes the proof of this lemma. □

Choose a product neighborhood $U = V \times V' \times W$ small enough so that ϕ is biholomorphic and so that Lemma 3.6 is satisfied by $U' = V \times V'$. Suppose $(l,m,t) \in L \times M \times T$. If $\phi(l,m,t) = (f^{(k)} + l + m) \circ t \in \mathscr{Q}^k(f)$, then $f^{(k)} + l + m \in \mathscr{Q}^k(f)$, because $\mathscr{R}^k(f) \subseteq \mathscr{Q}^k(f)$. According to Lemma 3.6, $m = 0$. We have shown $U \cap \mathscr{Q}^k(f) = \phi(V \times \{0\} \times W)$, completing the proof of the theorem. □

4. Weighted homogeneity and $\mathscr{R}$-orbit equivalence

In this section we investigate the conditions when the $\mathscr{R}$-orbit of a holomorphic function with an isolated critical point at the origin is the same as the $\mathscr{R}\mathscr{L}$-, $\mathscr{K}$-, and $\mathscr{Q}$-orbits. It turns out that these orbits coincide precisely when the function is analytically equivalent to a weighted homogeneous polynomial.

The following lemma will be very useful for the results to come. We want to emphasize that this lemma is very general and does not require that the singularity be isolated. A similar result appears in Shoshitaishvili [8, Lemma 2]. That result is somewhat stronger, but is restricted to the case of an isolated singularity. Our lemma is powerful enough for our applications. Moreover the proof is extremely elementary.

Lemma 4.1. *Suppose $f, g \in \mathcal{O}_{n+1}$, f is weighted homogeneous, and $\Delta(f) = \Delta(g)$. Then $g \in m_{n+1}\Delta(g)$.*

Proof. Suppose that f is weighted homogeneous of degree d with weights $a_0, \ldots, a_n$. By definition $f(t^{a_0}z_0, \ldots, t^{a_n}z_n) = t^d f(z_0, \ldots, z_n)$ for all t. It is easy to check that $\frac{\partial f}{\partial z_j}$ either vanishes or is weighted homogeneous of degree $d - a_j$. And, since $\Delta(f) = \Delta(g)$ there exist elements $\alpha_{ij}, \beta_{ij} \in \mathcal{O}_{n+1}$ for which

$$\frac{\partial f}{\partial z_i} = \sum_{j=0}^{n} \alpha_{ij} \frac{\partial g}{\partial z_j}$$

$$\frac{\partial g}{\partial z_i} = \sum_{j=0}^{n} \beta_{ij} \frac{\partial f}{\partial z_j}$$

We are going to use these facts in the computation below

$$\begin{aligned}
\frac{d}{dt} g(t^{a_0}z_0, \ldots, t^{a_n}z_n) &= \sum_{i=0}^{n} a_i t^{a_i - 1} z_i \frac{\partial g}{\partial z_i}(t^{a_0}z_0, \ldots, t^{a_n}z_n) \\
&= \sum_{i=0}^{n} \sum_{j=0}^{n} a_i z_i t^{a_i - 1} \beta_{ij}(t^{a_0}z_0, \ldots, t^{a_n}z_n) \frac{\partial f}{\partial z_j}(t^{a_0}z_0, \ldots, t^{a_n}z_n) \\
&= \sum_{i=0}^{n} \sum_{j=0}^{n} a_i z_i t^{d - a_j + a_i - 1} \beta_{ij}(t^{a_0}z_0, \ldots, t^{a_n}z_n) \frac{\partial f}{\partial z_j}(z_0, \ldots, z_n) \\
&= \sum_{k=0}^{n} \left[\sum_{i=0}^{n} \sum_{j=0}^{n} a_i z_i t^{d - a_j + a_i - 1} \beta_{ij}(t^{a_0}z_0, \ldots, t^{a_n}z_n) \alpha_{jk}(z_0, \ldots, z_n) \right] \\
&\quad \cdot \frac{\partial g}{\partial z_k}(z_0, \ldots, z_n)
\end{aligned}$$

Then integrate back to find that

$$g(z_0,\ldots,z_n)=\int_0^1\frac{d}{dt}g(t^{a_0}z_0,\ldots,t^{a_n}z_n)dt$$

$$=\sum_{k=0}^{n} b_k(z_0,\ldots,z_n)\frac{\partial g}{\partial z_k}(z_0,\ldots,z_n)$$

where

$$b_k(z_0,\ldots,z_n)=\sum_{i=0}^{n}\sum_{j=0}^{n} a_i z_i \alpha_{jk}(z_0,\ldots,z_n)\int_0^1 t^{t-a_j+a_i-1}\beta_{ij}(t^{a_0}z_0,\ldots,t^{a_n}z_n)dt\in m_{n+1}$$

This proves that $g\in m_{n+1}\Delta(g)$. □

We can now begin examining the conditions for when the $\mathscr{R}$-orbits coincide with other orbits. The first result is originally due to Shoshitaishvili [8].

Theorem 4.2. *Suppose $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ is a holomorphic function with an isolated critical point at the origin. The following statements are equivalent.*

a) $\mathscr{Q}(f)=\mathscr{R}(f)$

b) $m_{n+1}\Delta(f)=a(f)+m_{n+1}\Delta(f)$

c) $a(f)\subseteq m_{n+1}\Delta(f)$

d) *f is right equivalent to a weighted homogeneous polynomial.*

Proof. We start by showing that a)⇒b). Because f defines an isolated singularity, $m_{n+1}{}^k\subset\Delta(f)$ for all large k. This means that $\mathscr{Q}^k(f)$ is precisely $\mathscr{Q}(f)J^k$. It follows that $\mathscr{Q}^k(f)=\mathscr{R}^k(f)$ for all k large enough. Therefore their tangent spaces must coincide. Using the results of Theorems 3.4 and 3.5 we see that $(a(f)+m_{n+1}\Delta(f))J^k=m_{n+1}\Delta(f)J^k$ for all large k. This means that $a(f)+m_{n+1}\Delta(f)=m_{n+1}\Delta(f)$.

The implication b)⇒c) is obvious. As for c)⇒d), $f\in a(f)$ implies $f\in m_{n+1}\Delta(f)$. By Saito's theorem [7], f is right equivalent to a weighted homogeneous polynomial.

The final implication, d)⇒a), takes more proof. Assume that f is right equivalent to a weighted homogeneous polynomial f' and $g\in\mathscr{Q}(f)$. Then we only need to show that $g\in\mathscr{R}(f)$.

Since $\mathscr{R}(f)=\mathscr{R}(f')$, we can assume without loss of generality that f is a weighted homogeneous polynomial. The following lemma allows us to also assume that $\Delta(f)=\Delta(g)$.

Lemma 4.3. *Suppose f, $g:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ are holomorphic functions with isolated critical points at the origin and $\mathscr{O}_{n+1}/\Delta(f)\simeq\mathscr{O}_{n+1}/\Delta(g)$. Then there exists a $g'\in\mathscr{R}(g)$ such that $\Delta(f)=\Delta(g')$.*

Proof. Suppose $\phi:\mathscr{O}_{n+1}/\Delta(f)\to\mathscr{O}_{n+1}/\Delta(g)$ is a $\mathbb{C}$-algebra isomorphism. We are going to construct a local system $z_0,\ldots,z_n$ of holomorphic coordinates on $\mathbb{C}^{n+1}$, centered at the origin. Let $k=\dim_{\mathbb{C}}(\Delta(f)\cap m_{n+1}+m_{n+1}{}^2)/m_{n+1}{}^2$. Choose elements $z_0,\ldots,z_{k-1}\in\Delta(f)\cap m_{n+1}$ which are linearly independent modulo $m_{n+1}{}^2$. Then pick $n-k$ more functions $z_k,\ldots,z_n\in m_{n+1}$ to form a basis modulo $m_{n+1}{}^2$. By the inverse function theorem, $z_0, \ldots, z_n$ form a holomorphic local system of coordinates. We can now define a lifting $\tilde{\phi}: \mathcal{O}_{n+1} \to \mathcal{O}_{n+1}$ of ϕ by specifying its image on each of the coordinate functions. For each $i = k, \ldots, n$ pick $w_i = \tilde{\phi}(z_i) \in \mathcal{O}_{n+1}$ so that its projection in $\mathcal{O}_{n+1}/\Delta(g)$ is $\phi(\bar{z}_i)$. Since ϕ is an isomorphism of the quotient rings, the $w_k, \ldots, w_n$ must be linearly independent modulo $m_{n+1}{}^2$. Then choose $w_0, \ldots, w_{k-1} \in \Delta(g) \cap m_{n+1}$ so that the $w_0, \ldots, w_n$ complete a basis modulo $m_{n+1}{}^2$. By its construction this map makes the diagram

$$\begin{array}{ccc} \mathcal{O}_{n+1} & \xrightarrow{\tilde{\phi}} & \mathcal{O}_{n+1} \\ \Big| & & \Big| \\ \mathcal{O}_{n+1}/\Delta(f) & \xrightarrow{\phi} & \mathcal{O}_{n+1}/\Delta(g) \end{array}$$

commute. Furthermore, the w_i form a local system of coordinates so that $\tilde{\phi}$ must be biholomorphic at the origin. According to Lemma 2.10, $\Delta(f) = \phi^* \Delta(g) = \Delta(g \circ \phi)$. Let $g' = g \circ \phi \in \mathcal{R}(g)$ and the proof of our lemma is complete. $\square$

We will assume from now on that

$$\Delta(f) = \Delta(g) \tag{4.4}$$

where f is a weighted homogeneous polynomial defining an isolated critical point at the origin. It follows from Lemma 4.1 that

$$g \in m_{n+1} \Delta(g) \tag{4.5}$$

$$f \in m_{n+1} \Delta(f) \tag{4.6}$$

We will also assume that $f \neq g$, because otherwise there is nothing more to prove.

Let L be the complex line in $\mathcal{O}_{n+1}$ joining f to g. Every element of L is of the form $h = (1-w)f + wg$ for some $w \in \mathbb{C}$. Because of (4.4), $m_{n+1}\Delta(h) \subseteq m_{n+1}\Delta(f)$. Let L_0 be the set of $h \in L$ for which

$$m_{n+1}\Delta(h) = m_{n+1}\Delta(f) \tag{4.7}$$

Lemma 4.8. *L_0 is a connected complex manifold.*

Proof. Since f defines an isolated critical point at the origin, there exists an integer k such that $m_{n+1}{}^k \subseteq m_{n+1}\Delta(f)$. For any such k

$$m_{n+1}\Delta(h)J^k = m_{n+1}\Delta(f)J^k \tag{4.9}$$

holds if and only if (4.7) holds.

The $\mathbb{C}$-vector space $m_{n+1}\Delta(h)J^k$ is generated by the elements $v_i(h) = \left(z^p \dfrac{\partial h}{\partial z_q} \right)^{(k)}$, $i = (p, q)$, where p runs through the non-negative multi-indices with degree between 1 and k and $q = 0, 1, \ldots, k$. Let d be the dimension of the $\mathbb{C}$-vector space $m_{n+1}\Delta(f)J^k$. By choosing a basis of this space, we may represent each $v_i(h)$ as a row vector of length d.

Together the $v_i(h)$ form a matrix with d columns. Because $v_i(h) = (1-w)v_i(f) + wv_i(g)$, each coefficient of the matrix is a linear function of w. Equation (4.9) will hold if and only if at least one of the $d \times d$ minors has a nonzero determinant. Since it holds for $w = 0$, at least one of the minors must have a determinant which does not vanish identically. Therefore it is a polynomial in w of degree $\leq d$. Hence there are at most d values at which (4.9) fails to hold.

Therefore we have shown that L_0 is equal to L with at most a finite number of points deleted. Since L is a complex line, this implies that L_0 must be connected. □

Since f has an isolated critical point at the origin, f is finitely determined with respect to $\mathcal{R}$. Therefore it is enough to show that $g^{(k)} \in \mathcal{R}^k(f)$ for every positive integer k. We are going to show $L_0 J^k \subset \mathcal{R}^k(f)$ by using the following result proved by Mather [5, Lemma 3.1, pp. 234–236]. This lemma will be used repeatedly, so for convenience, we will give the proof here.

Lemma 4.10. *Let $\alpha: G \times U \to U$ be a C^∞ action of a Lie group G on a C^∞-manifold U, and let V be a connected C^∞-submanifold of U. Then necessary and sufficient conditions for V to be contained in a single orbit of α are that*

a) $T_v(Gv) \supseteq T_v V$, *if* $v \in V$

b) $\dim T_v(Gv)$ *is independent of the choice of* $v \in V$

Proof. Necessity is trivial. Now we prove sufficiency. For each $v \in U$, let $\alpha_v: G \to U$ be the mapping defined by $g \mapsto \alpha(g, v)$. Then $T_v(Gv) = \alpha_{v*}(T_{\mathrm{id}}G)$. Provide $T_{\mathrm{id}}G$ with a Hilbert norm and for each $v \in V$, let L_v be the orthogonal complement of $\ker \alpha_{v*}$ in $T_{\mathrm{id}}G$. Define $L = \bigcup_{v \in V} (v \times L_v) \subset V \times T_{\mathrm{id}}G$. Condition b) implies that L is a subvector bundle over V of $V \times T_{\mathrm{id}}G$. Let $L_0 = \bigcup_{v \in V} (\alpha_{v*}^{-1}(T_v V) \cap L_v)$. Condition a) shows that L_0 is a subvector bundle of L and the mapping $\bigcup_{v \in V} \alpha_{v*}: L_0 \to TV$ is an isomorphism of C^∞-vector bundles. Let $\beta: TV \to L_0$ be the inverse of this mapping, and let $\pi: V \times T_{\mathrm{id}}G \to T_{\mathrm{id}}G$ denote the projection map. Then $\pi \circ \beta: TV \to T_{\mathrm{id}}G$ is a C^∞-mapping, and $\alpha_{v*}(\pi \circ \beta(\eta)) = \eta$ for any $\eta \in T_v V$.

To prove that V is contained in a single orbit of α, it is enough to show that any two points v_1, v_2 of V are contained in the same orbit. Since V is connected, there is a smooth curve $\gamma: [0, 1] \to V$ joining v_1 to v_2. We only need to show that for any $t_0 \in [0, 1]$, there is an $\varepsilon > 0$ such that if $t_0 - \varepsilon < t < t_0 + \varepsilon$, then $\gamma(t)$ is contained in the same orbit as $\gamma(t_0)$.

Let $\gamma'(t) \in T_{\gamma(t)}V$ denote the derivative of $\gamma(t)$ with respect to t, and define $X(t) = \pi \circ \beta(\gamma'(t)) \in T_{\mathrm{id}}G$. $X(t)$ is a C^∞ function of t and

$$\alpha_{\gamma(t)*}(X(t)) = \gamma'(t) \tag{4.11}$$

From the existence theory for ordinary differential equations, it follows that there exists a curve $t \mapsto \mu(t)$ in G defined for $t_0 - \varepsilon < t < t_0 + \varepsilon$ for a suitable $\varepsilon > 0$ such that $\mu(t_0) = I$ and

$$\frac{d\mu(t)}{dt} = \tilde{X}_t(\mu(t)) \tag{4.12}$$

where $\tilde{X}_t$ is the unique right invariant vector field on G which extends $X(t)$.

We now show that $\mu(t)^{-1}\gamma(t) = \gamma(t_0)$ for $t_0 - \varepsilon < t < t_0 + \varepsilon$. This will imply that $\gamma(t)$ is in the same orbit as $\gamma(t_0)$ for all t within this range and finish the proof the lemma. The derivative with respect to t is

$$\frac{d}{dt}\mu(t)^{-1}\gamma(t) = \frac{d\mu(t)^{-1}}{dt}\gamma(t) + \mu(t)^{-1}\frac{d\gamma(t)}{dt}$$

$$= \mu(t)^{-1}\left(-\frac{d\mu(t)}{dt}\mu(t)^{-1}\gamma(t) + \frac{d\gamma(t)}{dt}\right)$$

By (4.12) and the fact that $\tilde{X}_t$ is right invariant, the quantity inside the brackets becomes $-X(t)\gamma(t) + \gamma'(t)$. According to (4.11), this is zero. Since $\mu(t_0) = I$, this shows that $\mu(t)^{-1}\gamma(t) = \gamma(t_0)$ for $t_0 - \varepsilon < t < t_0 + \varepsilon$. This completes the proof of the lemma. □

We will now apply this lemma. Take the action of α to be the action of $G = \mathscr{R}^k$ on $U = J^k$. We can deduce from Lemma 4.8 that $V = L_0 J^k$ is a connected submanifold of $U = J^k$. By Theorem 3.4a, $T_h(\mathscr{R}^k h) = m_{n+1}\Delta(h)J^k$, for any $h \in \mathcal{O}_{n+1}$. If $h^{(k)} \in L_0 J^k$, then (4.7) holds, and we obtain

$$T_h(\mathscr{R}^k h) = m_{n+1}\Delta(f)J^k \tag{4.13}$$

which verifies condition b) of Lemma 4.10. The tangent space $T_h(L_0 J^k)$ is the one dimensional complex subspace of J^k spanned by $g - f$. By (4.5) and (4.6), $g - f \in m_{n+1}\Delta(f)J^k$. Hence $T_h(L_0 J^k) \subset T_h(\mathscr{R}^k h)$, which shows that condition a) holds as well.

Therefore we may apply Lemma 4.10 to conclude that $L_0 J^k$ is contained in a single orbit of the action of $\mathscr{R}^k$ of J^k. This proves our result. □

Theorem 4.14. *Let $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ be a holomorphic function with an isolated critical point at the origin. Then the following statements are equivalent*

a) $\mathscr{R}(f) = \mathscr{K}(f)$

b) $m_{n+1}\Delta(f) = (f, m_{n+1}\Delta(f))$

c) *f is right equivalent to a weighted homogeneous polynomial.*

Proof. To prove a)$\Rightarrow$b), we use the computation of the tangent spaces performed in Theorem 3.4a, c. Since $\mathscr{R}(f) = \mathscr{K}(f)$, $\mathscr{R}^k(f) = \mathscr{K}^k(f)$ for all k. We can equate their tangent spaces, getting $m_{n+1}\Delta(f)J^k = (f, m_{n+1}\Delta(f))J^k$ for all k. But then $m_{n+1}\Delta(f) = (f, m_{n+1}\Delta(f))$.

Condition b) is equivalent to saying that $(f) \subseteq m_{n+1}\Delta(f)$. According to Saito's theorem [7], f is right equivalent to a weighted homogeneous polynomial. Therefore b)$\Rightarrow$c).

Finally, for c)$\Rightarrow$a), it suffices to prove that $\mathscr{K}(f) \subseteq \mathscr{R}(f)$. We may assume without loss of generality that f is actually weighted homogeneous. Therefore $f = \sum_{i=0}^{n} a_i \frac{\partial f}{\partial z_i}$, where $a_i \in m_{n+1}$. Suppose $g = \mathscr{K}(f)$. Then there exist $u \in \mathcal{O}_{n+1}$, $u(0) \neq 0$ and $\phi \in m_{n+1}$ such that $g = u(f \circ \phi)$. Making use of Lemma 2.11 and the fact that $f \in \Delta(f)$, we find that $\phi^*\Delta(f) = \Delta(g)$. This means that ϕ^* induces a $\mathbb{C}$-algebra isomorphism $\mathcal{O}_{n+1}/\Delta(g) \simeq \mathcal{O}_{n+1}/\Delta(f)$. Therefore g is right equivalent to f by Theorem 4.2. □

Theorem 4.15. *Let* $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ *be a holomorphic function with an isolated critical point at the origin. Then the following statements are equivalent*

a) $\mathscr{R}(f)=\mathscr{RL}(f)$

b) $m_{n+1}\Delta(f)=f^{-1}m_1+m_{n+1}\Delta(f)$

c) f *is right equivalent to a weighted homogeneous polynomial.*

Proof. To prove a)$\Rightarrow$b), we use the computation of the tangent spaces performed in Theorem 3.4a, b. Since $\mathscr{R}(f)=\mathscr{RL}(f)$, $\mathscr{R}^k(f)=\mathscr{RL}^k(f)$ for all k. We can equate their tangent spaces, getting $m_{n+1}\Delta(f)J^k=(f^{-1}m_1+m_{n+1}\Delta(f))J^k$ for all k. But then $m_{n+1}\Delta(f)=f^{-1}m_1+m_{n+1}\Delta(f)$.

Condition b) implies that $f^{-1}m_1\subseteq m_{n+1}\Delta(f)$, so $f\in m_{n+1}\Delta(f)$. Then, as before, Saito's theorem [7] implies that c) holds.

The implication c)$\Rightarrow$a) can be proved by using Theorem 4.14 and the fact that $\mathscr{R}(f)\subseteq\mathscr{RL}(f)\subseteq\mathscr{K}(f)$. □

5. $\mathscr{RL}$-orbit equivalence

In this section we investigate the conditions when the $\mathscr{RL}$-orbit of a holomorphic function with an isolated critical point at the origin is the same as the $\mathscr{K}$- and $\mathscr{Q}$-orbits.

Theorem 5.1. *Suppose* $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ *is a holomorphic function with an isolated critical point at the origin. Then the following statements are equivalent.*

a) $\mathscr{RL}(f)=\mathscr{K}(f)$

b) $f^{-1}m_1+m_{n+1}\Delta(f)=(f,m_{n+1}\Delta(f))$

c) $m_{n+1}(f)\subseteq m_{n+1}\Delta(f)$

Proof. a)$\Rightarrow$b) is proved by using the computation of the tangent spaces performed in Theorem 3.4b, c. Since $\mathscr{RL}(f)=\mathscr{K}(f)$, $\mathscr{RL}^k(f)=\mathscr{K}^k(f)$ for all k. We can equate their tangent spaces, getting $(f^{-1}m_1+m_{n+1}\Delta(f))J^k=(f,m_{n+1}\Delta(f))J^k$ for all k. But then $f^{-1}m_1+m_{n+1}\Delta(f)=(f,m_{n+1}\Delta(f))$.

Assume that b) holds. Then $z_jf\in f^{-1}m_1+m_{n+1}\Delta(f)$, so there exists a convergent power series $a(t)=\sum_{i=1}^{\infty}a_it^i$, $a_i\in\mathbb{C}$ such that $z_jf=\sum_{i=1}^{\infty}a_if^i+\sum_{i=0}^{n}b_i\frac{\partial f}{\partial z_i}$ where $b_i\in m_{n+1}$ for $0\leqq i\leqq n$. There are two cases to consider, depending on whether or not a_1 is nonzero. If $a_1\neq 0$, then $u=a_1-z_j+\sum_{i=2}^{\infty}a_if^{i-1}$ is a unit element in $\mathcal{O}_{n+1}$ and $f=u^{-1}\left(-\sum_{i=0}^{n}b_i\frac{\partial f}{\partial z_i}\right)\in m_{n+1}\Delta(f)$. In particular $m_{n+1}(f)\subset m_{n+1}\Delta(f)$ when $a_1\neq 0$.

On the other hand, if $a_1=0$, then we have $z_jf=\left(\sum_{i=2}^{\infty}a_if^{i-1}\right)f+\sum_{i=0}^{n}b_i\frac{\partial f}{\partial z_i}$. Since f has a critical point at the origin, $f\in m_{n+1}{}^2$. Therefore $m_{n+1}(f)\subseteq m_{n+1}{}^2(f)+m_{n+1}\Delta(f)$. Using Nakayama's Lemma, it follows that $m_{n+1}(f)\subseteq m_{n+1}\Delta(f)$. Therefore we have proved in either case that b)$\Rightarrow$c).

Finally, to prove c)$\Rightarrow$a), it is sufficient to prove that $\mathscr{K}(f)\subseteq\mathscr{RL}(f)$. Suppose $g\in\mathscr{K}(f)$. Then there exists $u\in\mathcal{O}_{n+1}, u(0)\neq 0$, such that $g=u(f\circ h)$ where

$h:(\mathbb{C}^{n+1},0)\to(\mathbb{C}^{n+1},0)$ is a germ at the origin of a biholomorphic mapping. Now $f'=u(0)^{-1}(f\circ h)$ is holomorphic function with the property that $\mathscr{K}(f)=\mathscr{K}(f')$ and $\mathscr{RL}(f)=\mathscr{RL}(f')$. Thus by replacing f by f', we may assume without loss of generality that $g=uf$ where $u(0)=1$.

Using Lemma 2.11

$$(f, m_{n+1}\Delta(f)) = (g, m_{n+1}\Delta(g)) \tag{5.2}$$

We will also assume that $f \neq g$, because otherwise there is nothing more to prove.

Let L be the complex line in $\mathcal{O}_{n+1}$ joining f to g. Since every $h\in L$ can be written in the form $h=(1-w)f+wg$ for some $w\in\mathbb{C}$, we have $(h, m_{n+1}\Delta(h))\subseteq(f, m_{n+1}\Delta(f))$. Let L_0 be the set of $h\in L$ for which the two ideals are equal. Using an argument similar to the one used to prove Lemma 4.8, we find that L_0 is a connected manifold.

The hypothesis that $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ has an isolated critical point at the origin implies that f is finitely determined with respect to $\mathscr{RL}$. Hence it is enough to prove that $g^{(k)}\in\mathscr{RL}^k(f)$ for every positive integer k. In what follows let k be a fixed positive integer.

We want to apply Lemma 4.10. In this case $G=\mathscr{RL}^k$, $U=J^k$, and $V=L_0$. We have to check that conditions a) and b) of the lemma are applicable.

Suppose $h\in L_0$. Then $h=(1-w)f+wg=(1-w+wu)f$ for some $w\in\mathbb{C}$. Since $u(0)=1$, $1-w+wu$ is a unit in $\mathcal{O}_{n+1}$. the following lemma can be applied to h.

Lemma 5.3. *Suppose f, $h:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ are holomorphic functions with isolated critical points at the origin and $h=uf$ where $u\in\mathcal{O}_{n+1}$ is a unit. If $m_{n+1}(f)\subseteq m_{n+1}\Delta(f)$, then $m_{n+1}(h)\subseteq m_{n+1}\Delta(h)$.*

Proof. Using the hypothesis that $m_{n+1}(f)\subseteq m_{n+1}\Delta(f)$ and (2.12), it is easy to see that

$$m_{n+1}\Delta(h)\subseteq m_{n+1}\Delta(f) \tag{5.4}$$

Our first step is to show that these ideals are actually equal. We can do this by proving that $\dim_{\mathbb{C}}\mathcal{O}_{n+1}/m_{n+1}\Delta(f)=\dim_{\mathbb{C}}\mathcal{O}_{n+1}/m_{n+1}\Delta(h)$.

The exact sequence

$$0\to\Delta(f)/m_{n+1}\Delta(f)\to\mathcal{O}_{n+1}/m_{n+1}\Delta(f)\to\mathcal{O}_{n+1}/\Delta(f)\to 0$$

shows that

$$\dim_{\mathbb{C}}\mathcal{O}_{n+1}/m_{n+1}\Delta(f)=\dim_{\mathbb{C}}\mathcal{O}_{n+1}/\Delta(f)+\dim_{\mathbb{C}}\Delta(f)/m_{n+1}\Delta(f)$$

We are going to show that the right hand side of this equation depends only on the analytic type of the singularity, and not on the defining equation $f=0$. The first term on the right hand side is the Milnor number, which is a topological invariant of the singularity. We will now prove that the second term is equal to $n+1$.

Consider the map $\phi:\mathbb{C}^{n+1}\to\Delta(f)/m_{n+1}\Delta(f)$ defined by $(a_0,a_1,\dots,a_n)\mapsto \sum_{i=0}^{n} a_i\frac{\partial f}{\partial z_i}+m_{n+1}\Delta(f)$. This map is obviously surjective. Suppose that ϕ is not injective, then there exists a nonzero vector $(a_0,a_1,\dots,a_n)$ in $\mathbb{C}^{n+1}$ such that

$\sum_{i=0}^{n} a_i \frac{\partial f}{\partial z_i} \in m_{n+1}\Delta(f)$. Without loss of generality, we shall assume $a_0 \neq 0$. Then there exist $b_0, \ldots, b_n \in m_{n+1}$ such that $\sum_{i=0}^{n} a_i \frac{\partial f}{\partial z_i} = \sum_{j=0}^{n} b_j \frac{\partial f}{\partial z_j}$. Rearranging terms we find that $\frac{\partial f}{\partial z_0} = (a_0 - b_0)^{-1} \sum_{j=1}^{n} (-a_j + b_j) \frac{\partial f}{\partial z_j}$. This means that $\Delta(f)$ is generated by less than $n+1$ elements, so the critical point of f at the origin cannot be isolated. We have shown that ϕ is an isomorphism, and $\dim_{\mathbb{C}} \Delta(f)/m_{n+1}\Delta(f) = n+1$.

This proves that $\dim_{\mathbb{C}} \mathcal{O}_{n+1}/m_{n+1}\Delta(f)$ depends only on the singularity and not on f. Since $f = 0$ and $h = 0$ define the same singularity, $\dim_{\mathbb{C}} \mathcal{O}_{n+1}/m_{n+1}\Delta(f) = \dim_{\mathbb{C}} \mathcal{O}_{n+1}/m_{n+1}\Delta(h)$. Combined with (5.4), we see that $m_{n+1}\Delta(h) = m_{n+1}\Delta(f)$.

Since $m_{n+1}(f) \subseteq m_{n+1}\Delta(f)$ and $h = uf$, we have $m_{n+1}(h) \subseteq m_{n+1}\Delta(h)$. This completes the proof of the lemma.

We can now use Lemma 5.3 and (5.2) to show that

$$\begin{aligned} f^{-1}m_1 + m_{n+1}\Delta(f) &= (f, m_{n+1}\Delta(f)) \\ &= (h, m_{n+1}\Delta(h)) \\ &= h^{-1}m_1 + m_{n+1}\Delta(h) \end{aligned}$$

In particular we can see that

$$(f^{-1}m_1 + m_{n+1}\Delta(f))J^k = (h^{-1}m_1 + m_{n+1}\Delta(h))J^k \tag{5.5}$$

for any $h \in L_0$. Combining this with the computation of the tangent space in Theorem 3.4b, $T_h(\mathscr{RL}^k h) = (f^{-1}m_1 + m_{n+1}\Delta(f))J^k$ for any $h \in L_0$. This shows that condition b) holds.

The tangent space of L_0 at any h is the one dimensional complex subspace of J^k spanned by $(g-f)^{(k)}$. According to (5.5), $(g-f)^{(k)} \in (f^{-1}m_1 + m_{n+1}\Delta(f))J^k$, proving that $T_h(L_0) \subset T_h(\mathscr{RL}^k h)$. Thus condition a) holds as well.

We can now apply Lemma 4.10. We deduce that L_0 is contained in a single orbit of the action of $\mathscr{RL}^k$ on J^k, and so in particular $g^{(k)} \in \mathscr{RL}^k(f)$. □

In [7], Saito proved for any f with an isolated critical point at the origin, $f \in m_{n+1}\Delta(f)$ if and only if up to a biholomorphic change of coordinates f is a weighted homogeneous polynomial. Any f satisfying $f \in m_{n+1}\Delta(f)$ is called a *quasi-homogeneous* function. Theorems 4.2, 4.15, and 4.14 show that the following conditions are equivalent: f is quasi-homogeneous, $\mathscr{R}(f) = \mathscr{RL}(f)$, $\mathscr{R}(f) = \mathscr{K}(f)$, and $\mathscr{R}(f) = \mathscr{Q}(f)$. Theorem 5.1 suggests the following definition.

Definition 5.6. Suppose $f:(\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ is a function with an isolated critical point at the origin. f is said to be an *almost quasi-homogeneous* function if $m_{n+1}(f) \subseteq m_{n+1}\Delta(f)$.

The previous theorem leads us to expect that the singularities defined by almost quasi-homogeneous functions may form a distinguished class of singularities which have some special properties.

We can also give a criterion for when the $\mathscr{RL}$ and $\mathscr{Q}$ orbits coincide. This result is originally due to Shoshitaishvili [8].

Theorem 5.7. *Suppose $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ is a holomorphic function with an isolated critical point at the origin. Then the following statements are equivalent.*
a) $\mathscr{RL}(f)=\mathscr{Q}(f)$
b) $f^{-1}m_1+m_{n+1}\Delta(f)=a(f)+m_{n+1}\Delta(f)$

Proof. a)$\Rightarrow$b) is proved by using the computation of the tangent spaces performed in Theorems 3.4b and 3.5. Since $\mathscr{RL}(f)=\mathscr{Q}(f)$, $\mathscr{RL}^k(f)=\mathscr{Q}^k(f)$ for all k. We can equate their tangent spaces, getting $(f^{-1}m_1+m_{n+1}\Delta(f))J^k=(a(f)+m_{n+1}\Delta(f))J^k$ for all k. But then $f^{-1}m_1+m_{n+1}\Delta(f)=a(f)+m_{n+1}\Delta(f)$.

For b)$\Rightarrow$a), there are two cases. If $f\in\Delta(f)$, then we can use Theorem 4.2 to handle this case. We will assume that $f\notin\Delta(f)$ from now on. We need to prove that if $g\in\mathscr{Q}(f)$, then $g\in\mathscr{RL}(f)$. Using Lemma 4.3, we may also assume without loss of generality that $\Delta(f)=\Delta(g)$. A final assumption is that $f\neq g$.

Let L be the complex line in $\mathcal{O}_{n+1}$ joining f to g. Since every $h\in L$ can be written in the form $h=(1-w)f+wg$ for some $w\in\mathbb{C}$, we have $\Delta(h)\subseteq\Delta(f)$. Let L_0 be the set of $h\in L$ for which the two ideals are equal. Using an argument similar to the one used to prove Lemma 4.8, we find that L_0 is a connected manifold. The following lemma applies to all elements of L_0.

Lemma 5.8. *Let $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ be a holomorphic function with an isolated critical point at the origin. Suppone $f^{-1}m_1+m_{n+1}\Delta(f)=a(f)+m_{n+1}\Delta(f)$ and $f\notin\Delta(f)$. Then for any $h\in m_{n+1}$ such that $\Delta(f)=\Delta(h)$, we have $f^{-1}m_1+m_{n+1}\Delta(f)=h^{-1}m_1+m_{n+1}\Delta(h)$.*

Proof. Since $\Delta(f)=\Delta(h)$, we have $h\in a(f)$. It follows that

$$\begin{aligned}h^{-1}m_1+m_{n+1}\Delta(h)&\subseteq a(f)+m_{n+1}\Delta(f)\\&=f^{-1}m_1+m_{n+1}\Delta(f)\end{aligned}$$

and there exists a power series $p(t)=\sum_{k=1}^{\infty}a_kt^k$ in $\mathbb{C}[\![t]\!]$ and $\eta\in m_{n+1}\Delta(f)$ such that

$$h=\sum_{k=1}^{\infty}a_kf^k+\eta \tag{5.9}$$

We claim that $a_1\neq0$. Suppose not. Then differentiating (5.9) we get

$$\frac{\partial h}{\partial z_i}=\left(\sum_{k=2}^{\infty}ka_kf^{k-1}\right)\frac{\partial f}{\partial z_i}+\frac{\partial\eta}{\partial z_i} \tag{5.10}$$

Since $\sum_{k=2}^{\infty}ka_kf^{k-1}\in m_{n+1}$, it follows that $\Delta(f)=\Delta(h)\subseteq m_{n+1}\Delta(f)+\Delta(\eta)$. By Nakayama's Lemma, we have $\Delta(f)\subseteq\Delta(\eta)$. On the other hand, since

$$\frac{\partial\eta}{\partial z_i}=\frac{\partial h}{\partial z_i}-\left(\sum_{k=2}^{\infty}ka_kf^{k-1}\right)\frac{\partial f}{\partial z_i} \tag{5.11}$$

we have $\Delta(\eta)\subseteq\Delta(f)$. This implies that $\Delta(\eta)=\Delta(f)$. But $\eta\in m_{n+1}\Delta(\eta)$, and so Lemma 4.1 implies that $f\in m_{n+1}\Delta(f)$. This is a contradiction, so it must have been true that $a_1\neq0$.

Hence there exists a series $r(t)$ belonging to the maximal ideal in $\mathbb{C}\{t\}$ such that $r(p(t)) = t$. According to (5.9), $r(h - \eta) = f$. Therefore we have $f = r(h) + \xi$ where $\xi \in m_{n+1}\Delta(f) = m_{n+1}\Delta(h)$. It follows that $f \in h^{-1}m_1 + m_{n+1}\Delta(h)$, and in particular $f^{-1}m_1 + m_{n+1}\Delta(f) \subseteq h^{-1}m_1 + m_{n+1}\Delta(h)$. This completes the proof of the lemma. □

This lemma implies that for any $h \in L_0$,

$$f^{-1}m_1 + m_{n+1}\Delta(f) = h^{-1}m_1 + m_{n+1}\Delta(h) \tag{5.12}$$

Since f is finitely determined with respect to $\mathscr{RL}$, it is enough to prove that $g^{(k)} \in \mathscr{RL}^k(f)$ for every positive integer k. In what follows let k be a fixed positive integer.

As before we must verify that Lemma 4.10 can be applied to our situation. In this case $G = \mathscr{RL}^k$, $U = J^k$, and $V = L_0J^k$. We have to check that conditions a) and b) of the lemma are applicable.

Using Theorem 3.4b and (5.12) we find that

$$\begin{aligned} T_h(\mathscr{RL}^k h) &= (h^{-1}m_1 + m_{n+1}\Delta(h))J^k \\ &= (f^{-1}m_1 + m_{n+1}\Delta(f))J^k \end{aligned}$$

for any $h \in \mathcal{O}_{n+1}$. This verifies condition b). We also know that $T_h(L_0J^k)$ is the one dimensional complex subspace of J^k spanned by $g - f$. Applying Lemma 5.8 with h replaced by g, we get $(g-f)^{(k)} \in (f^{-1}m_1 + m_{n+1}\Delta(f))J^k$. This shows that $T_h(L_0) \subset T_h(\mathscr{RL}^k h))$ which verifies condition a).

Therefore we can apply Lemma 4.10 to conclude that L_0J^k is contained in $\mathscr{RL}^k(f)$. Since $g^{(k)} \in L_0J^k$, this is all we need to prove. □

We will finish this section by giving an example of a function which is almost quasi-homogeneous, but not quasi-homogeneous and also an example of a function which is not almost quasi-homogeneous.

Example 5.13. Let $f(x, y) = x^5 + y^5 + x^3y^3$. Then
a) f is not quasi-homogeneous.
b) f is almost quasi-homogeneous.
c) $(f, m_{n+1}\Delta(f)) = f^{-1}m_1 + m_{n+1}\Delta(f) = a(f) + m_{n+1}\Delta(f)$
In particular, we have

$$\begin{array}{ccc} \mathscr{R}(f) \subsetneqq & \mathscr{RL}(f) & = \mathscr{K}(f) \\ & \| & \\ & \mathscr{Q}(f) & \end{array}$$

Proof. Assume that $f \in m_{n+1}\Delta(f)$. Then there exist power series $a(x, y) = \sum_{i+j\geq 1} a_{ij}x^iy^j$ and $b(x, y) = \sum_{i+j\geq 1} b_{ij}x^iy^j$ *for which* $f - a\frac{\partial f}{\partial z} - b\frac{\partial f}{\partial y} = 0$. If we multiply this equation out and equate each of the coefficients to zero we get a system of linear equations involving the a_{ij} and b_{ij}. This system includes the equations $a_{10} = \frac{1}{5}$, $b_{01} = \frac{1}{5}$, and $3a_{10} + 3a_{01} = 1$, obtained by equating the coefficients of x^5, y^5, and x^3y^3 respectively. However these equations are inconsistent with each other. This is a contradiction, and therefore f is not quasi-homogeneous.

To prove that f is almost quasi-homogeneous, we make the following calculation

$$xf=\left(\frac{1}{5}x^2\frac{\partial f}{\partial x}+\frac{1}{5}xy\frac{\partial f}{\partial y}\right)-\frac{1}{5}x^4y^3\in m\Delta(f)+m^7$$

Now it is easy to see that $m^7\subseteq m\Delta(f)+m^8$. According to Nakayama's Lemma,

(5.14) $$m^7\subseteq m\Delta(f)$$

This shows that $xf\in m\Delta(f)$. We can prove in a similar manner that $yf\in m\Delta(f)$, which verifies that f is almost quasi-homogeneous.

Since f is almost quasi-homogeneous, we can use Theorem 4.15 to see that

$$\begin{aligned}(f,m\Delta(f))&=f^{-1}m_1+m\Delta(f)\\&\subseteq a(f)+m\Delta(f)\end{aligned}$$

All we need to prove is $a(f)\subseteq(f,m\Delta(f))$. Let $g\in a(f)$. Then $\left(\frac{\partial g}{\partial x},\frac{\partial g}{\partial y}\right)\subseteq\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)$. This implies that the order of g is at least 5 and that

$$\frac{\partial g}{\partial x}=a_0x^4+a_1y^4+\text{terms of order}\geqq 5$$

Integrating back, we find that

$$g=\tfrac{1}{5}a_0x^5+a_1xy^4+a_2y^5+\text{terms of order}\geqq 6$$

Now differentiating by y, we find that

$$\frac{\partial g}{\partial y}=4a_1xy^3+5a_2y^4+\text{terms of order}\geqq 5$$

Since $\frac{\partial g}{\partial y}\in\Delta(f)$, it must be true that $a_1=0$. Now expand g up to order 6.

$$\begin{aligned}g=\tfrac{1}{5}a_0x^5+a_2y^5+a_3x^6+a_4x^5y+a_5x^4y^2+a_6x^3y^3+a_7x^2y^4+a_8xy^5+a_9y^6\\+\text{terms of order}\geqq 7\end{aligned}$$

By differentiating and expressing $\frac{\partial g}{\partial x}$ in terms of the partials of f and solving the resulting system of linear equations we find that $a_5=0$ and $\frac{1}{5}a_0=a_6$. Doing the same thing for $\frac{\partial g}{\partial y}$ shows that $a_7=0$ and $a_2=a_6$. As a result we can write

$$\begin{aligned}g&=\tfrac{1}{5}a_0(x^5+y^5+x^3y^3)+a_3x^6+a_4x^5y+a_8xy^5+a_9y^6+\text{terms of order}\geqq 7\\&=\tfrac{1}{5}a_0f+\tfrac{1}{5}(a_3x^2+a_4xy)\frac{\partial f}{\partial x}+\frac{1}{5}(a_8xy+a_9y^2)\frac{\partial f}{\partial y}+\text{terms of order}\geqq 7\end{aligned}$$

Using (5.14) we can see that $g\in(f)+m\Delta(f)$. Therefore we have shown that part c) holds.

The relationships in the diagram follow from Theorems 4.15, 5.1, and 5.7. □

Example 5.15. Let $f(x, y) = (y + x^4)(y^2 + x^9)$. Then f is not almost quasi-homogeneous. In particular

$$\mathscr{R}(f) \subsetneqq \mathscr{RL}(f) \subsetneqq \mathscr{K}(f)$$

Proof. We are going to show that $xf \notin m_{n+1}\Delta(f)$. Assume on the contrary that $xf \in m_{n+1}\Delta(f)$. Then there exist power series $a(x, y) = \sum_{i,j \geqq 1} a_{ij}x^i y^j$ and $b(x, y) = \sum_{i,j \geqq 1} b_{ij}x^i y^j$ in m_{n+1}, such that

$$xf = a(x, y)\frac{\partial f}{\partial x} + b(x, y)\frac{\partial f}{\partial y} \tag{5.16}$$

By comparing the coefficients of y^3, xy^3, x^4y^2, x^5y^2, x^8y, x^9y, and x^{14} on both sides respectively, we get the following equations.

$$\begin{aligned} b_{01} &= 0 \\ 3b_{11} &= 1 \\ 4a_{10} + 3b_{40} + 2b_{01} &= 0 \\ 4a_{20} + 3b_{50} + 2b_{11} &= 1 \\ 2b_{40} &= 0 \\ 9a_{10} + 2b_{50} + b_{01} &= 0 \\ 13a_{20} + b_{50} &= 1 \end{aligned}$$

This system of equations is inconsistent and leads to a contradiction. This means f is not an almost quasi-homogeneous polynomial. We will consider this example again in the next section. □

6. Relationship between $\mathscr{Q}$ and $\mathscr{K}$ equivalence

There are still two more natural questions. The first is whether $\mathscr{K}(f) \subseteq \mathscr{Q}(f)$, that is, whether the Milnor algebra isomorphism type is an invariant of the corresponding singularity. The second is whether $\mathscr{Q}(f) \subseteq \mathscr{K}(f)$, that is, whether the analytic type of an isolated singularity is determined by the Milnor algebras which are associated to it. The following proposition gives an answer to the first question.

Proposition 6.1. *Suppose $f:(\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is a holomorphic function with an isolated critical point at the origin with $\mathscr{K}(f) \subseteq \mathscr{Q}(f)$. Then $f \in \Delta(f) + m_{n+1}\Delta^2(f)$, where $\Delta^2(f)$ is the ideal in $\mathcal{O}_{n+1}$ generated by all second partial derivatives of f.*

Proof. Using the computation of the tangent spaces to the manifolds $\mathscr{K}(f)$ and $\mathscr{Q}(f)$ found in Theorems 3.4c and 3.5, $\mathscr{K}(f) \subseteq \mathscr{Q}(f)$ implies that $(f, m_{n+1}\Delta(f))J^k \subseteq (a(f) + m_{n+1}\Delta(f))J^k$ for all k. Since both ideals contain some power of the maximal ideal m_{n+1}, we have

(6.2) $$(f, m_{n+1}\Delta(f)) \subseteq a(f) + m_{n+1}\Delta(f)$$

Then $(1+z_0)f \in a(f) + m_{n+1}\Delta(f)$ and there exist $g \in a(f)$ and $\xi_j \in m_{n+1}$ such that

$$(1+z_0)f = g + \sum_{j=0}^{n} \xi_j \frac{\partial f}{\partial z_j}$$

Differentiating with respect to z_0,

$$f + (1+z_0)\frac{\partial f}{\partial z_0} = \frac{\partial g}{\partial z_0} + \sum_{j=0}^{n} \frac{\partial \xi_j}{\partial z_0}\frac{\partial f}{\partial z_j} + \sum_{j=0}^{n} \xi_j \frac{\partial^2 f}{\partial z_0 \partial z_j}$$

By definition of $a(f)$, $\frac{\partial g}{\partial z_0} \in \Delta(f)$. Therefore $f \in \Delta(f) + m_{n+1}\Delta^2(f)$. □

The following remark, due to Mather, shows that it is *not* true in general that $\mathscr{K}(f) \subseteq \mathscr{Q}(f)$.

Remark 6.3. There exists a polynomial $f(x, y)$ such that $f \notin \Delta(f) + \Delta^2(f)$. In particular $\mathscr{K}(f) \nsubseteq \mathscr{Q}(f)$.

Proof. We are going to show that we can find polynomials of the form

(6.4) $$f(x, y) = \sum_{i=1}^{6} x^{a_i} y^{b_i}$$

for which $f \notin \Delta(f) + \Delta^2(f)$. We will restrict the exponents a_i and b_i so that they satisfy the condition

(6.5) $$\max(a_i - a_j, b_i - b_j) \geqq 3$$

for each $i \neq j$.

For the moment assume that $f \in \Delta(f) + \Delta^2(f)$ and write f in terms of its first and second partial derivatives. Then if we equate the coefficients of the $x^{a_i}y^{b_i}$ terms on both sides we get six linear equations $i = 1, \ldots, 6$

(6.6) $$a_i x_1 + b_i x_2 + a_i(a_i - 1)x_3 + a_i b_i x_4 + b_i(b_i - 1)x_5 = 1$$

which must have a common solution $x_1, \ldots, x_5 \in \mathbb{C}$. Condition (6.5) assures that no cross terms arising from an $x^{a_j}y^{b_j}$ term contribute to the final $x^{a_i}y^{b_j}$ term of $i \neq j$.

Therefore it cannot be true that $f \in \Delta(f) + \Delta^2(f)$ if the matrix

(6.7) $$\begin{bmatrix} a_1 & b_1 & a_1(a_1-1) & a_1b_1 & b_1(b_1-1) & 1 \\ a_2 & b_2 & a_2(a_2-1) & a_2b_2 & b_2(b_2-1) & 1 \\ a_3 & b_3 & a_3(a_3-1) & a_3b_3 & b_3(b_3-1) & 1 \\ a_4 & b_4 & a_4(a_4-1) & a_4b_4 & b_4(b_4-1) & 1 \\ a_5 & b_5 & a_5(a_5-1) & a_5b_5 & b_5(b_5-1) & 1 \\ a_6 & b_6 & a_6(a_6-1) & a_6b_6 & b_6(b_6-1) & 1 \end{bmatrix}$$

is nonsingular. Let $\Delta(\tilde{a},\tilde{b})$ denote its determinant, where $\tilde{a}=(a_1,\ldots,a_6)$ and $\tilde{b}=(b_1,\ldots,b_6)$.

Since the functions $1, a, b, a(a-1), ab, b(b-1)$ are linearly independent, the determinant $\Delta(\tilde{a},\tilde{b})$ cannot vanish on any open subset of $\mathbb{C}^6\times\mathbb{C}^6$. Consequently there exist non-negative vectors $\tilde{a},\tilde{b}\in\mathbb{Q}_+^6$ such that $\Delta(\tilde{a},\tilde{b})\neq 0$ and such that condition (6.5) holds. But then $\Delta(\lambda\tilde{a},\lambda\tilde{b})$ is a polynomial in λ which does not vanish identically. We can choose $\lambda\geqq 1$, so that $\lambda\tilde{a}$, $\lambda\tilde{b}\in\mathbb{Z}_+^6$ and $\Delta(\lambda\tilde{a},\lambda\tilde{b})\neq 0$.

Let a_i and b_i be the i-th components of $\lambda\tilde{a}$ and $\lambda\tilde{b}$, respectively. Using these as the exponents in (6.4), we obtain an $f(x,y)$ which is not contained in $\Delta(f)+\Delta^2(f)$. □

Example 6.8. Let $f(x,y)=x^{15}+x^{12}y^3+x^9y^6+x^6y^{10}+x^3y^{13}+y^{17}$. Then the following relationships hold

$$\begin{array}{ccccc}\mathscr{R}(f) & \subsetneqq & \mathscr{RL}(f) & \subsetneqq & \mathscr{K}(f)\\ & & \cap\!\!\!\!\!\!\neq & & \\ & & \mathscr{Q}(f) & & \end{array}$$

with $\mathscr{Q}(f)\nsubseteq\mathscr{K}(f)$ and $\mathscr{K}(f)\nsubseteq\mathscr{Q}(f)$.

Proof. With Remark 6.3 in mind, we looked for positive integers a_i, b_j, $1\leqq i,j\leqq 6$ which satisfy (6.5) and for which the matrix (6.7) is nonsingular. A computer program that calculates ranks of matrices with exact rational entries aided our investigation. The polynomial f was the lowest degree example that we could find. Our selection procedure guarantees that $\mathscr{K}(f)\nsubseteq\mathscr{Q}(f)$, and it follows that $\mathscr{RL}(f)\subsetneqq\mathscr{K}(f)$ as well. f is not quasi-homogeneous because $f\notin\Delta(f)+\Delta^2(f)$. This means that $\mathscr{R}(f)\subsetneqq\mathscr{RL}(f)$.

We used computer programs described in the next section to check the remaining inclusions. It was found that $a(f)\nsubseteq(f,m\Delta(f))$. This shows that $\mathscr{RL}(f)\subsetneqq\mathscr{Q}(f)$ and $\mathscr{Q}(f)\nsubseteq\mathscr{K}(f)$.

The computations in this example are complex. The Milnor number of the singularity is 209, and the smallest power of the maximal ideal contained within $\Delta(f)$ is m^{30}. $a(f)+m\Delta(f)$ modulo m^{31} has dimension 317, while $(f,m\Delta(f))$ modulo m^{31} has dimension 329. All of the generators we found for $a(f)$ which were not contained in $(f,m\Delta(f))$ were extremely complicated. Some of their coefficients were rational numbers with over 30 digits in both the numerator and denominator. □

We now turn to the second question and give a general method for constructing functions F for which $\mathscr{Q}(F)\nsubseteq\mathscr{K}(F)$.

Theorem 6.9. *Suppose* $F(x_1,\ldots,x_n,y_1,\ldots,y_n)=f(x_1,\ldots,x_n)+f(y_1,\ldots,y_n)$ *where* $n>1$ *and* $f(x_1,\ldots,x_n)$ *is a function with an isolated critical point at the origin which is not quasi-homogeneous. Then* $\mathscr{Q}(F)\nsubseteq\mathscr{K}(F)$.

Proof. Suppose $\mathscr{Q}(F)\subseteq\mathscr{K}(F)$. Then using the computation of the tangent spaces to the manifolds $\mathscr{Q}(F)$ and $\mathscr{K}(F)$ performed in Theorems 3.5 and 3.4c, $\mathscr{Q}(F)\subseteq\mathscr{K}(F)$

implies that $(a(F)+m_{2n}\Delta(F))J^k \subseteq (F, m_{2n}\Delta(F))J^k$ for all k. Since both ideals contain some power of the maximal ideal m_{2n}, we have

$$a(F)+m_{2n}\Delta(F) \subseteq (F, m_{2n}\Delta(F)) \tag{6.10}$$

In the following, let x stand for $x_1, \ldots, x_n$ and y for $y_1, \ldots, y_n$. According to the definition of $a(F)$, $f(x) \in a(F)$. Using (6.10), $f(x) \in (F, m_{2n}\Delta(F))$, so there exist $b(x, y)$, $c_j(x, y)$, $d_j(x, y) \in \mathbb{C}\{x, y\}$ with $c_j(0,0) = d_j(0,0) = 0$ such that

$$f(x) = b(x, y)(f(x)+f(y)) + \sum_{j=1}^{n} c_j(x, y)\frac{\partial f}{\partial x_j}(x) + \sum_{j=1}^{n} d_j(x, y)\frac{\partial f}{\partial y_j}(y) \tag{6.11}$$

Now $b(x, y)$ must be a unit in $\mathbb{C}\{x, y\}$. Otherwise we can rearrange the terms in (6.11) and set $y=0$ to find that

$$(1-b(x,0))f(x) = \sum_{j=1}^{n} c_j(x,0)\frac{\partial f}{\partial x_j}(x) \tag{6.12}$$

Here we have used the fact that $f(y)$ has a singularity at the origin. This equation implies that $f(x) \in m_{n+1}\Delta(f)$. Since $f=0$ defines an isolated singularity, Saito's theorem [7] implies that f is quasi-homogeneous. This is a contradiction to our hypothesis, so it must be true that $b(x, y)$ is a unit.

Next rearrange the terms in (6.11) and set $x=0$. We get

$$-b(0, y)f(y) = \sum_{j=1}^{n} d_j(0, y)\frac{\partial f}{\partial y_j}(y) \tag{6.13}$$

where we have again used the fact that $f(x)$ has a singularity at the origin. Since $b(0, y)$ is a unit in $\mathbb{C}\{y\}$, it follows that $f(y) \in m_{n+1}\Delta(f)$. As before, this contradicts our hypothesis that f is not quasi-homogeneous. Therefore we conclude that $\mathscr{Q}(F) \nsubseteq \mathscr{K}(F)$. □

Corollary 6.14. *Suppose* $F(x_1, \ldots, x_n, y_1, \ldots, y_n) = f(x_1, \ldots, x) + f(y_1, \ldots, y_n)$ *where* $n>1$ *and* $f(x_1, \ldots, x_n)$ *is a function with an isolated critical point at the origin which is not quasi-homogeneous. Then there exists a* $G \in \mathbb{C}\{x, y\}$ *such that* $\Delta(G) = \Delta(F)$ *but* $G \notin \mathscr{K}(F)$.

Proof. According to Theorem 6.9, there exists $H \in \mathbb{C}\{x, y\}$ such that $\mathcal{O}_{n+1}/\Delta(H) \simeq \mathcal{O}_{n+1}/\Delta(F)$ but $H \notin \mathscr{K}(F)$. Using Lemma 4.3, we can find $G \in \mathscr{R}(H)$ such that $\Delta(F) = \Delta(G)$. Since H is not in $\mathscr{R}(F)$, it follows that G is not in $\mathscr{K}(F)$.

These arguments can be modified to work in the C^∞ category as well. The following remark summarizes this extension to the C^∞ case.

Remark 6.15. Suppose $F(x_1, \ldots, x_n, y_1, \ldots, y_n) = f(x_1, \ldots, x_n) + f(y_1, \ldots, y_n)$ where $n>1$ and $f(x_1, \ldots, x_n)$ is a function with an isolated critical point at the origin which is not quasi-homogeneous. Then there exists $G(x_1, \ldots, x_n, y_1, \ldots, x_n) \in \mathbb{C}\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ such that $\Delta(F) = \Delta(G)$ and the zero set $V(F)$ defined by $F=0$ is not C^∞-diffeomorphism equivalent to the zero set $V(G)$ defined by $G=0$ although the two sets are homeomorphic.

When F has real coefficients this is also a consequence of Ephraim's Theorem [3] and Corollary 6.14.

Corollary 6.16. *For any $n > 1$, there exists a one parameter family of non- quasi-homogeneous isolated singularities in which the Milnor algebras corresponding to each singularity are the same, but in which the diffeomorphism types are different.*

Example 6.17. Let $F(x, y, z, w) = x^5 + y^5 + z^5 + w^5 + x^3y^3 + z^3w^3$. Then the following relationships hold

$$\begin{array}{ccccc} \mathscr{R}(F) & \subsetneqq & \mathscr{RL}(F) & \subsetneqq & \mathscr{K}(F) \\ & & \cap\!\!\!\!\!\!\not\;\; & & \\ & & \mathscr{Q}(F) & & \end{array}$$

and $\mathscr{Q}(F) \nsubseteq \mathscr{K}(F)$.

Proof. Observe that $F(x, y, z, w) = f(x, y) + f(z, w)$ where $f(x, y) = x^5 + y^5 + x^3y^3$. We have already shown in Example 5.13 that $f(x, y)$ is not a quasi-homogeneous function. By Theorem 6.9, we have $\mathscr{Q}(F) \nsubseteq \mathscr{K}(F)$.

We now claim that F is also not almost quasi-homogeneous, that is, $m_{n+1}(F) \nsubseteq m_{n+1}\Delta(F)$. We are going to show that $xF \notin m_{n+1}\Delta(F)$. Assume the opposite is true. Then there exist power series $a(x, y, z, w)$, $b(x, y, z, w)$, $c(x, y, z, w)$ and $d(x, y, z, w)$ in m_{n+1} such that

$$(6.18)\quad xF = a(x, y, z, w)\frac{\partial F}{\partial x} + b(x, y, z, w)\frac{\partial F}{\partial y} + c(x, y, z, w)\frac{\partial F}{\partial z} + d(x, y, z, w)\frac{\partial F}{\partial w}$$

Comparing the coefficients of xz^5, xw^5, xz^3w^3 on both sides, we get the following equations respectively.

$$\begin{aligned} 5c_{1010} &= 1 \\ 5d_{0101} &= 1 \\ 3c_{1010} + 3d_{0101} &= 1. \end{aligned}$$

It turns out that these linear equations form a inconsistent system. This means that $xF \notin m_{n+1}\Delta(F)$ and so F is not almost quasi-homogeneous. It follows from Theorems 4.15 and 5.1 that $\mathscr{R}(F) \subsetneqq \mathscr{RL}(F) \subsetneqq \mathscr{K}(F)$. It is also clear that $\mathscr{RL}(F) \subsetneqq \mathscr{Q}(F)$, because otherwise $\mathscr{Q}(F) = \mathscr{RL}(F) \subsetneqq \mathscr{K}(F)$ which contradicts the fact that $\mathscr{Q}(F) \nsubseteq \mathscr{K}(F)$. □

We will give two more examples which we have computed.

Example 6.19. Let $f(x, y) = (y + x^4)(y^2 + x^9)$. Then the following relationships hold

$$\begin{array}{ccccc} \mathscr{R}(f) & \subsetneqq & \mathscr{RL}(f) & \subsetneqq & \mathscr{K}(f) \\ & & \cap\!\!\!\!\!\!\not\;\; & & \\ & & \mathscr{Q}(F) & & \end{array}$$

and $\mathscr{Q}(f) = \mathscr{K}(f)$.

Proof. This example was discussed in the previous section, but we did not consider the inclusions involving $\mathscr{Q}(f)$.

We have used computer programs to check these inclusions and have found that $a(f) \nsubseteq f^{-1}m_1 + m\Delta(f)$. One generator of $a(f)$ that is not contained in $f^{-1}m_1 + m\Delta(f)$ is $x^2y^3 + x^6y^2 + x^{11}y + \frac{91}{90}x^{15} + \frac{35}{768}x^{16}$. On the other hand the programs showed that $a(f) + m\Delta(f) = (f, m\Delta(f))$.

Despite the simple form of the polynomial f, the computing problem was still fairly complex. The Milnor number of this singularity is 23, but the smallest power of the maximal ideal contained in $\Delta(f)$ is m^{16}. The dimension of the $\mathbb{C}$-vector space $a(f)$ modulo m^{17} is 113 and the dimension of $f^{-1}m_1 + m\Delta(f)$ modulo m^{17} is 129. □

Example 6.20. Let $f(x, y, z) = x^8 + y^8 + z^8 + x^2y^2z^2$. Then the following relationships hold

$$\begin{array}{ccccc} \mathscr{R}(f) & \subsetneqq & \mathscr{RL}(f) & \subsetneqq & \mathscr{K}(f) \\ & & \not\cap & & \\ & & \mathscr{Q}(f) & & \end{array}$$

and $\mathscr{Q}(f) = \mathscr{K}(f)$.

Proof. As in the preceding examples, this singularity was analyzed using computer programs discussed in the next section. It was found that $a(f) + m\Delta(f) = (f, m\Delta(f))$, but $a(f) \nsubseteq f^{-1}m_1 + m\Delta(f)$. One generator of $a(f)$ that is not contained in $f^{-1}m_1 + m\Delta(f)$ is $x^3y^2z^2 + \frac{10}{9}x^9 + xy^8 + xz^8$. These inclusions are enough to verify the diagram given above.

The Milnor number of this singularity is 215, with the smallest power of the maximal ideal contained in $\Delta(f)$ being m^{17}. The dimension of the $\mathbb{C}$-vector space $a(f)$ modulo m^{18} is 706. □

7. Computational methods

In this section we will describe the computer programs that have been used to check the examples in Sect. 6. Earlier versions of some of the programs have already been discussed by Benson and Yau in [2], but many new programs have been added since that paper.

The programs are written in the C language using the techniques described by Benson in [1]. These techniques and the associated libraries of subroutines have made it possible to develop programs rapidly. In most cases it was possible to implement the algorithms in a compact and readable form.

Our approach has been to develop programs that work on objects of four different types: ideals in power series rings $\mathbb{Q}[\![x_1, \ldots, x_n]\!]$, prepared ideals in power series rings $\mathbb{Q}[\![x_1, \ldots, x_n]\!]$, submodules of jet spaces $\mathbb{Q}[\![x_1, \ldots, x_n]\!]/m_{n+1}^k$, and finite dimensional $\mathbb{Q}$-algebras. We have developed a description file format for specifying each type of object and the programs are able to read and write description files for these objects.

Each of the programs either creates a new description file based on input from

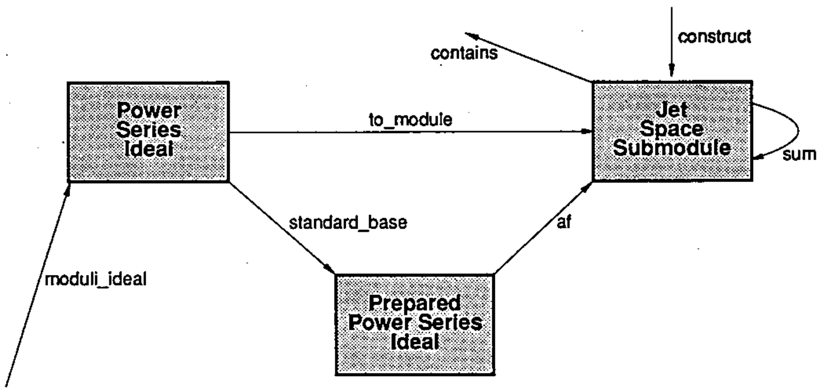

Fig. 7.1. Organization of software

the terminal, reads a description file and computes a new one, or reads a description file and does some analysis.

Figure 7.1 gives a graphical description of how these programs relate to one another. The programs are shown as edges in a directed graph with the description formats as vertices. The starting vertex of each edge indicates the format of the program's input, while the terminating vertex shows the program's output format. When the endpoint is not on a vertex, the meaning is that the input or output is from or to the terminal in a format different from those used for the description files.

We will give brief descriptions of each of the programs used for computing examples in this paper.

Moduli-ideal. Creates a description file for the following types of ideals: m_{n+1}^k, $m_{n+1}^k \Delta(f)$, and $(f, m_{n+1}\Delta(f))$. It prompts the user for the number and names of the variables, the type of ideal, the value of k, and the polynomial, if necessary.
Construct module. Creates a description file for a module. The user is prompted for the number and names of the variables and the generators.
To-module ⟨*ideal-file*⟩ *k*. Creates a module description file corresponding to an ideal specified by ⟨ideal-file⟩. k is the power of the maximal ideal to be used.
Contains ⟨*module-1*⟩⟨*module-2*⟩. Determines whether the module given by the file ⟨module-1⟩ contains the module given by the file ⟨module-2⟩. The algorithm used is to reduce each generator of ⟨module-2⟩ by the set of generators of ⟨module-1⟩. ⟨module-1⟩ contains ⟨module-2⟩ if and only if all of the reductions are zero.
Sum ⟨*module-1*⟩⟨*module-2*⟩. Creates a description file for the module which is the sum of the modules specified by ⟨module-1⟩ and ⟨module-2⟩.

Standard-base. Finds a prepared ideal description corresponding to an ideal description file read from the standard input. The ideal must contain a power of the maximal ideal. A prepared ideal description consists of a standard base for the ideal along with a basis of monomials for the quotient algebra.
af. Finds a description file for the module $a(f)$ modulo m_{n+1}^k, where k is one more than the minimum power of the maximal ideal contained in $\Delta(f)$. The program reads a prepared description file for the ideal $\Delta(f)$ from the standard input. Then it finds all monomials of degree less than or equal to $k-1$ which are in $a(f)$. These form part of a basis for a (f) modulo m_{n+1}^k. The remaining generators are found

by taking a generic linear combination of the other monomials of degree less than or equal to $k-1$ and determining conditions on the coefficients for the derivatives to be contained in $\Delta(f)$. These conditions form a homogeneous system of linear equations involving the coefficients of this polynomial. The program solves this system and resubstitutes the solutions back into the polynomial. The remaining generators can then be read from this polynomial. It is clear that the lexicographically smallest monomial in this linear combination cannot have a derivative which is in the monomial basis for quotient algebra of $\Delta(f)$. The program takes this in account when it forms the linear combination in order to reduce the amount of computation.

To show how these programs work together, we give the UNIX commands that were used to determine whether $a(f) \subset f^{-1}m_1 + m_{n+1}\Delta(f)$:

```
% moduli-ideal|standard-base|af > af-module
% moduli-ideal > m-delta-ideal
% to-module m-delta-ideal k > m-delta-module
% construct module > f-powers-module
% sum m-delta-module f-powers-module > f-powers-m-delta-module
% contains f-powers-m-delta-module af-module
```

A SUN-3 computer running the UNIX operating system has been used for program development. Most computations have not taken more than a few minutes of CPU time. However $a(f)$ in Example 6.8 with its many digit fractions was an exception. 4 hours and 52 minutes of computer time was used.

We will be happy to share these programs and related ones with other researchers upon request.

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