Journal of Algebra 618 (2023) 165–194

Contents lists available at ScienceDirect

Journal of Algebra

journal homepage: www.elsevier.com/locate/jalgebra

Higher Nash blow-up local algebras of singularities and its derivation Lie algebras ☆

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A R T I C L E   I N F O

Article history:
Received 8 October 2021
Available online 14 December 2022
Communicated by Steven Dale Cutkosky

MSC:
14B05
32S05

Keywords:
Derivations
Hessian algebra
Weighted homogeneous isolated hypersurface singularity

A B S T R A C T

In this paper, we introduce new invariants to a singularity $(V, 0)$, i.e., the derivation Lie algebras $\mathcal{L}_k(V)$ of the higher Nash blow-up local algebra $\mathcal{M}_k(V)$. A new conjecture about the non-existence of negative weighted derivations of $\mathcal{L}_k(V)$ for weighted homogeneous isolated hypersurface singularities is proposed. We verify this conjecture partially. Moreover, we compute the Lie algebra $\mathcal{L}_2(V)$ for binomial isolated singularities. We also formulate a sharp upper estimate conjecture for the dimension of $\mathcal{L}_k(V)$ for weighted homogeneous isolated hypersurface singularities and verify this conjecture for a large class of singularities.

☆ Guorui Ma is co-first author of this paper. Both Yau and Zuo are supported by NSFC Grant 11961141005. Zuo is supported by NSFC Grant 12271280 and Tsinghua University Initiative Scientific Research Program. Yau is supported by Tsinghua University Education Foundation fund (042202008).
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https://doi.org/10.1016/j.jalgebra.2022.11.016

1. Introduction

In classification theory of isolated singularities, one always wants to find various invariants associated to isolated singularities. Hopefully with enough invariants found, one can distinguish between different isolated singularities up to contact equivalence. However, not many effective invariants are known. Moreover, most of known invariants, for example, the geometric genus, are hard to compute in general. In this article, we shall introduce a new numerical invariant to isolated hypersurface singularities. The invariant can be calculated easily comparing with other invariants of isolated singularities.

The algebra of germs of holomorphic functions at the origin of $\mathbb{C}^n$ is denoted as $\mathcal{O}_n$. Clearly, $\mathcal{O}_n$ can be naturally identified with the algebra of convergent power series in n indeterminates with complex coefficients. As a ring $\mathcal{O}_n$ has a unique maximal ideal $\mathfrak{m}$, the set of germs of holomorphic functions which vanish at the origin. Let $(V, 0)$ be an isolated hypersurface singularity defined by a holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. The multiplicity $mult(f)$ of the singularity $(V, 0)$ is defined to be the order of the lowest nonvanishing term in the power series expansion of f at 0.

For any isolated hypersurface singularity $(V, 0) \subset (\mathbb{C}^n, 0)$ defined by f, Yau considers the Lie algebra of derivations of moduli algebra

$$A(V) := \mathcal{O}_n/(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}), \text{i.e.}, L(V) = \mathrm{Der}(A(V), A(V)).$$

It is known that $L(V)$ is a finite dimensional solvable Lie algebra ([38], [39]). $L(V)$ is called the Yau algebra of V in [25] and [40] in order to distinguish from Lie algebras of other types appearing in singularity theory ([1], [2]). The Yau algebra plays an important role in singularities [31]. In this paper, we will introduce a new derivation Lie algebra which is a generalization of Yau algebra.

The classical Nash blow-up of an algebraic variety can be viewed as the parameter space of the tangent spaces of smooth points and their limits. There is a natural question to ask whether we can get a smooth variety by Nash blow-ups. There are lots of works on it, such as González-Sprinberg [14], Hironaka [16], Nobile [28], Rebassoo [30], and Spivakovsky [32], etc.

As we know, A. Nobile has the following famous theorem.

Theorem 1.1 *([28]). Let X be a variety over an algebraically closed field of characteristic zero, then the Nash blow-up of X is an isomorphism if and only if X is non-singular.*

It is known that Nobile's theorem fails over fields of prime characteristic. This theorem was recently proved in positive characteristic adding the condition of normality.

Theorem 1.2 *(Theorem 3.10 in [12]). Let X be a normal variety over an algebraically closed field $\mathbb{K}$ of dimension d. Suppose that $\mathbb{K}$ has prime characteristic p. If $\mathbf{Nash}_1(X) \cong X$, then X is non-singular.*

The conception Nash blow-up was generalized which was called higher Nash blow-up. The precise definition and related properties of higher Nash blow-up can be found in section 2. However, the higher Nash blow-up is hard to compute in general. To deal with this problem, especially in the hypersurface case. D. Duarte introduced higher-order Jacobian matrix of a polynomial which can be also found in section 2.

Let $F \in \mathbb{C}[x_1, \cdots, x_s]$, the higher-order Jacobian matrix $\mathrm{Jac}_k(F)$ is useful to make explicit computations concerning the higher Nash blow-up of the hypersurface $V(F)$. The ideal whose blow-up is the higher Nash blow-up of order k of a hypersurface $V(F)$, corresponds to the ideal generated by the maximal minors of $\mathrm{Jac}_k(F)$ [11]. From this fact, it is natural for us to introduce the following higher Nash blow-up local algebra for an isolated hypersurface singularity.

Definition 1.3. With the notations above. Let $F \in \mathbb{C}[x_1, \cdots, x_s]$ and $\mathrm{Jac}_k(F)$ be the Jacobian matrix with order k. Let $\mathcal{J}_k(F) \subset \mathbb{C}[x_1, \ldots, x_s]$ be the ideal generated by the maximal minors of $\mathrm{Jac}_k(F)$. Then we define a new higher Nash blow-up local algebra of V to be: $\mathcal{M}_k(V) := \mathcal{O}_s/\langle F, \mathcal{J}_k(F)\rangle$. We use $d_k(V)$ to denote the dimension of $\mathcal{M}_k(V)$.

Remark 1.4. If $(V, 0)$ is an isolated hypersurface singularity, then it is easy to see that the $\mathcal{M}_1(V)$ is exactly the moduli algebra $A(V)$, moreover, $\mathcal{M}_k(V)$ is Artinian (cf. Corollary 2.2, [11]).

We propose the following conjecture.

Conjecture 1.5. *Let $(V, 0)$ be an isolated hypersurface singularity defined by a polynomial $F(x_1, \cdots, x_s)$. Then $\mathcal{M}_k(V)$ is contact invariant of $(V, 0)$, i.e. it depends only on the isomorphism class of the germ $(V, 0)$.*

In the following, we verify the Conjecture 1.5 for $s = 2, k = 2$.

Theorem A. *Suppose $(V, 0) = \{(x_1, x_2) \in \mathbb{C}^2 \ : \ F(x_1, x_2) = 0\}$ and $(W, 0) = \{(x_1, x_2) \in \mathbb{C}^2 \ : \ G(x_1, x_2) = 0\}$ are isolated hypersurface singularities. If $(V, 0)$ is biholomorphically equivalent to $(W, 0)$, then $\mathcal{M}_2(V)$ is isomorphic to $\mathcal{M}_2(W)$.*

Based on Theorem A, it is natural for us to introduce the following new derivation Lie algebras.

Definition 1.6. The derivation Lie algebras $\mathcal{L}_k(V)$ is defined to be the Lie algebra of derivations of the local Artinian algebra $\mathcal{M}_k(V)$, i.e., $\mathcal{L}_k(V) = Der(\mathcal{M}_k(V))$. Its dimension is denoted as $\rho_k(V)$.

It follows from Theorem A that the derivation Lie algebra of $M_2(V)$ and $\rho_2(V)$ are new invariants for a curve singularity $(V, 0)$.

Let $R = \mathcal{O}_n/(f_1, f_2, \cdots, f_n)$ be a complete intersection local Artinian algebra and $f_1, f_2, \cdots, f_n$ are weighted homogeneous polynomials with the same type $(w_1, w_2, \cdots, w_n; d)$. Motivated by rational homotopy theory, Halperin conjectured that there are no negative weight derivations on R, which is one of the most important open problems in rational homotopy theory (see [13], [27]). This problem is also important in differential geometry [5].

Halperin Conjecture. *(Equivalent Form) [6] Let $x_1, \cdots, x_n$ be weighted variables and $f_1, \cdots, f_n$ be weighted homogeneous polynomials. Suppose that R is an Artinian algebra of the form $\mathcal{O}_n/(f_1, f_2, \cdots, f_n)$. Then there is no non-zero negative weight derivation on R.*

Chen, Yau, and Zuo have made essential progress in solving the Halperin conjecture. They developed a new theory of introducing new weight type associated with a negative weight derivation on the weighted polynomial ring and have given the necessary properties for solving these two conjectures. They gave a positive answer to the Halperin conjecture [8] when the degrees of $f_1, \cdots, f_n$ are bounded below by a constant C depending only on the weights of $x_1, \cdots, x_n$. Moreover, this bound C is improved in several special cases. Their theorems about the Halperin Conjecture are more or less optimal in the sense that only the lower bound constant can be improved. For recent studies about various derivation Lie algebras of singularities, the interested reader can refer to [8–10], [17–24], [26], [35,36].

Assuming that $f(x_1, x_2)$ defines a weighted homogeneous singularity, it follows from Lemma 4.1 that $\mathcal{M}_2(V)$ and $\mathcal{L}_2(V)$ are also naturally graded. We believe that this fact is also true in general. It is natural to propose the following new conjecture:

Conjecture 1.7. *Let $(V, 0) = \{(x_1, \ldots, x_n) \in \mathbb{C}^n : f(x_1, \ldots, x_n) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial f of weight type $(\alpha_1, \ldots, \alpha_n; d)$. Assume that $d \geq 2\alpha_1 \geq \cdots \geq 2\alpha_n > 0$ without loss of generality. Let $\mathcal{M}_k(V)$ be the k-th local algebra as above. Then $\mathcal{M}_k(V)$ and $\mathcal{L}_k(V)$ are also naturally graded. Furthermore, there is no non-zero negative weight derivation on the $\mathcal{M}_k(V)$, i.e., $\mathcal{L}_k(V)$ is non-negatively graded.*

The following result verifies Conjecture 1.7 partially.

Theorem B. *Let $(V, 0) = \{(x_1, x_2) \in \mathbb{C}^2 : G(x_1, x_2) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial G with $mult(G) = m$ of weight type $(\alpha_1, \alpha_2; d)$. Assume that $d \geq 2\alpha_1 \geq 2\alpha_2 > 0$ without loss of generality. Let $\mathcal{L}_2(V)$ be the derivation Lie algebra of the second moduli algebra $\mathcal{M}_2(V)$ and $D \in \mathcal{L}_2(V)$. If D is of negative weight, then $D \equiv 0$.*

For the estimation of the numerical invariant $\rho_k(V)$, we proposed the following new conjecture.

Conjecture 1.8. *Assume that* $\rho_k(\{x_1^{a_1}+\cdots+x_n^{a_n}=0\}) = h_k(a_1,\cdots,a_n)$*. Let* $(V,0) = \{(x_1,x_2,\cdots,x_n)\in\mathbb{C}^n : f(x_1,x_2,\cdots,x_n)=0\}, (n\geq 2)$ *be an isolated singularity defined by the weighted homogeneous polynomial* $f(x_1,x_2,\cdots,x_n)$ *of weight type* $(w_1,w_2,\cdots,w_n;1)$*. Then* $\rho_k(V)\leq h_k(1/w_1,\cdots,1/w_n)$*.*

Remark 1.9. Conjecture 1.8 claims that if a weighted homogeneous singularity has the same weight type as a Brieskorn singularity $x_1^{a_1}+\cdots+x_n^{a_n}$, then the invariant ρ_k of the Brieskorn singularity is greater than or equal to the same invariant of the other one.

Theorem C verifies Conjecture 1.8 partially.

Theorem C. *Let* $(V,0)$ *be a binomial singularity defined by the weighted homogeneous polynomial* $f(x_1,x_2)$ *(see Corollary 3.6) with weight type* $(w_1,w_2;1)$*. Then*

$$\rho_2(V)\leq h_2(\frac{1}{w_1},\frac{1}{w_2}) = \begin{cases} 6; & w_1=\frac{1}{2}, w_2=\frac{1}{2}\\ \frac{4}{w_2}-3; & w_1=\frac{1}{2}, w_2\geq\frac{1}{3}\\ \frac{4}{w_1}-3; & w_1\geq\frac{1}{3}, w_2=\frac{1}{2}\\ \frac{4}{w_1w_2}-4(\frac{1}{w_1}+\frac{1}{w_2})+7; & w_1\geq\frac{1}{3},\ w_2\geq\frac{1}{3} \text{ and } w_1=w_2\\ \frac{4}{w_1w_2}-4(\frac{1}{w_1}+\frac{1}{w_2})+6; & w_1\geq\frac{1}{3},\ w_2\geq\frac{1}{3} \text{ and } w_1\neq w_2. \end{cases}$$

Sections 2 and 3 contain some preliminaries on higher Nash blow-up and higher order Jacobian matrix, and derivation Lie algebra. The proofs of Theorem A, Theorem B and Theorem C are given in section 4.

Acknowledgments. We would like to thank the anonymous referee for careful reading the manuscript and giving numerous helpful suggestions.

2. Higher Nash blow-up and higher order Jacobian matrix

We review some standard results about higher Nash blow-up and higher order Jacobian matrix including their definition and some basic properties in this section. Readers can easily find them with more details in the corresponding references.

Firstly, we just recall the basic definition and properties of higher Nash blow-ups.

Definition 2.1 *(Higher Nash blow-up [34]).* Let X be a variety of dimension d, $x\in X$, $x^{(n)} := Spec(\mathcal{O}_{X,x}/m_x^{n+1})$ its n-th infinitesimal neighborhood and $\mathbf{Hilb}_{\binom{d+n}{n}}(X)$ is the Hilbert scheme of $\binom{d+n}{n}$ points of X. If X is smooth at x, then $x^{(n)}$ is an Artinian subscheme of X of length $\binom{d+n}{n}$. Therefore, it corresponds to a point

$$[x^{(n)}]\in\mathbf{Hilb}_{\binom{d+n}{n}}(X),$$

which induced the following morphism of schemes,

$$\sigma_n : X_{sm} \longrightarrow \mathbf{Hilb}_{\binom{d+n}{n}}(X).$$

The graph of σ_n is canonically isomorphic to X_{sm}, where X_{sm} denotes the smooth locus of X. We define the n-th Nash blow-up of X (called the higher Nash blow-up of order n of X), denoted by $\mathbf{Nash}_n(X)$, to be the closure of the graph Γ_{σ_n} with reduced scheme structure in $X \times_k \mathbf{Hilb}_{\binom{d+n}{n}}(X)$, together with the projection

$$\pi_n : \mathbf{Nash}_n(X) \longrightarrow X,$$

which is projective and birational. Moreover, it is an isomorphism over X_{sm}.

A. Oneto and E. Zatini introduced the following definition.

Definition 2.2 *(Nash Blow-up associated to a coherent sheaf [29]).* Let X be a reduced Noetherian scheme, $\mathcal{M}$ a coherent $\mathcal{O}_X$-module locally free of constant rank r on an open dense subscheme $U \subset X$, and $\mathbf{Grass}_r(\mathcal{M})$ the Grassmaniann of $\mathcal{M}$ of rank r, which is a projective X-scheme. Then the fiber product $\mathbf{Grass}_r(\mathcal{M}) \times_X U$ is isomorphic to U by the projection. The Nash blow-up of X associated with $\mathcal{M}$, is defined to be the closure of $\mathbf{Grass}_r(\mathcal{M}) \times_X U$, (denoted by $\mathbf{Nash}(X, \mathcal{M})$) together with the natural morphism

$$\pi_{\mathcal{M}} : \mathbf{Nash}(X, \mathcal{M}) \longrightarrow X,$$

which is projective and birational.

Remark 2.3 *([34]).* When X is a variety and $\mathcal{M} = \Omega_{X/k}$, the $\mathbf{Nash}(X, \Omega_{X/k})$ is the classical Nash blow-up of X. Moreover,

$$\mathbf{Nash}_n(X) \cong \mathbf{Nash}(X, \mathcal{P}^n_X) \cong \mathbf{Nash}(X, \mathcal{P}^n_{X,+}),$$

where $\mathcal{P}^n_X := \mathcal{O}_{X \times_k X}/\mathcal{I}^{n+1}_\Delta$ (which is the structure sheaf of $\Delta^{(n)}$ and called the sheaf of principal parts of order n of X) and $\mathcal{P}^n_{X,+} := \mathcal{I}_\Delta/\mathcal{I}^{n+1}_\Delta$ ($\mathcal{I}_\Delta$ is the ideal sheaf defining the diagonal $\Delta \subset X \times_k X$, $\Delta^{(n)} \subset X \times_k X$ is the n-th infinitesimal neighborhood of the diagonal). When X is a variety, these are coherent. For more details, see [15] and [29].

Higher Nash blow-ups have lots of general properties. We just recall the following two theorems.

Theorem 2.4 *(Compatibility with étale morphisms [34]). Let $Y \longrightarrow X$ be an étale morphism of varieties. Then for every n, there exists a canonical isomorphism*

$$\mathbf{Nash}_n(Y) \cong \mathbf{Nash}_n(X) \times_X Y.$$

Theorem 2.5 *(Stable under group actions [34]). Let X be a variety of dimension d and G an algebraic group over k acting on X. The subscheme $\mathbf{Nash}_n(X) \subset X \times_k \mathbf{Hilb}_{\binom{d+n}{n}}(X)$ is stable under the induced natural action of G on $X \times_k \mathbf{Hilb}_{\binom{d+n}{n}}(X)$,*

$$G \times_k X \times_k \mathbf{Hilb}_{\binom{d+n}{n}}(X) \longrightarrow X \times_k \mathbf{Hilb}_{\binom{d+n}{n}}(X)$$
$$(g, x, [Z]) \longmapsto (gx, [gZ]).$$

Thus, the G-action on X naturally lifts to $\mathbf{Nash}_n(X)$.

Similarly, the higher version of Noble's result with some restriction also holds.

Theorem 2.6 *(cf. Theorem 4.13, [11]). Let $F \in \mathbb{C}[x_1, \ldots, x_s]$ be an irreducible polynomial and $X = V(F) \subset \mathbb{C}^s$. Suppose X is normal. Let $(\mathbf{Nash}_n(X), \pi_n)$ be the n-th Nash blow-up of X. Then π_n is an isomorphism if and only if X is non-singular.*

In order to compute higher Nash blow-up, especially in the hypersurface case, we recall the following definition which was given in [11], where readers can find more details.

We recall the definition of a higher-order Jacobian matrix of a polynomial. First recall the multi-index notation. Let $\alpha = (\alpha_1, \ldots, \alpha_s), \beta = (\beta_1, \ldots, \beta_s) \in \mathbb{N}^s$:

1). $\alpha \leq \beta \Leftrightarrow \alpha_i \leq \beta_i \quad \forall i \in \{1, \ldots, s\}$.
2). $|\alpha| = \alpha_1 + \cdots + \alpha_s$.
3). $\alpha! = \alpha_1! \cdot \alpha_2! \cdots \alpha_s!$.
4). $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2}\cdots\binom{\alpha_s}{\beta_s} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$.
5). $\partial^\alpha = \partial^{\alpha_1}\partial^{\alpha_2}\cdots\partial^{\alpha_s}$.

Using this notation, the general Leibniz rule states that

$$\partial^\alpha(g \cdot f) = \sum_{\{\beta | \beta \leq \alpha\}} \binom{\alpha}{\beta} \partial^{\alpha-\beta} f \partial^\beta g$$

for any $f, g \in \mathbb{C}[x_1, \ldots, x_s]$. If we define $\partial^{\alpha-\beta} f = 0$ when $\alpha_i < \beta_i$ for some $1 \leq i \leq s$ then the general Leibniz rule can also be written as:

$$\partial^\alpha(g \cdot f) = \sum_{\{\beta | 0 \leq |\beta| \leq |\alpha|\}} \binom{\alpha}{\beta} \partial^{\alpha-\beta} f \partial^\beta g. \tag{1}$$

Let $F \in \mathbb{C}[x_1, \ldots, x_s]$ and $p = (a_1, \ldots, a_s) \in X = \mathbf{V}(F) \subset \mathbb{C}^s$. Let $\mathfrak{a}_p = \langle x_1 - a_1, \ldots, x_s - a_s\rangle \subset \mathbb{C}[x_1, \ldots, x_s]$. Fix $n \in \mathbb{N}$. Let $N = \binom{n+s}{s}$ and consider the following linear map:

$$\begin{aligned}\theta : \mathfrak{a}_p &\to \mathbb{C}^{N-1}\\ f &\mapsto \left(\frac{\partial^\alpha f}{\alpha!} \mid 1 \leq |\alpha| \leq n\right)_{|p}.\end{aligned}$$

We arrange this vector increasingly using graded lexicographical order, where $\alpha_1 < \alpha_2 < \cdots < \alpha_s$.

Let $\mathfrak{b} = \langle F \rangle$. Notice that $\mathfrak{b} \subset \mathfrak{a}_p$. Let $g \cdot F \in \mathfrak{b}$, where $g \in \mathbb{C}[x_1, \ldots, x_s]$. Using the general Leibniz rule (1) and the fact $F(p) = 0$, we can write $\theta(gF)$ as follows (recall that we defined $\partial^{\alpha-\beta} F = 0$ if $\alpha_i < \beta_i$ for some i):

$$\theta(gF) = \sum_{\{\beta|0\leq|\beta|\leq n-1\}} \partial^\beta g(p) \cdot \left(\binom{\alpha}{\beta} \frac{\partial^{\alpha-\beta} F}{\alpha!} | 1 \leq |\alpha| \leq n\right)_{|p}.$$

Let $\beta \in \mathbb{N}^s$ be such that $0 \leq |\beta| \leq n-1$. We define

$$r_\beta := \beta! \cdot \left(\binom{\alpha}{\beta} \frac{\partial^{\alpha-\beta} F}{\alpha!} | 1 \leq |\alpha| \leq n\right). \tag{2}$$

As before, we arrange r_β using graded lexicographical order on α. There are $M = \binom{n+s-1}{s}$ such vectors.

Definition 2.7 *(cf. Definition 1.2, [11]).* Let $\mathrm{Jac}_n(F)$ be the matrix whose rows are the M vectors r_β defined in (2). We arrange these rows using graded lexicographical order on β, where $\beta_1 < \beta_2 < \ldots < \beta_s$. In particular, $\mathrm{Jac}_n(F)$ is a $M \times (N-1)$-matrix. We call $\mathrm{Jac}_n(F)$ the Jacobian matrix of order n or the higher-order Jacobian matrix.

Remark 2.8. The higher-order Jacobian matrix was later rediscovered and further developed in [3] and [4].

The following example gives an intuitional illustration of the above definition of higher-order version of the Jacobian matrix of a polynomial. Since we only care about isolated hypersurface singularities in this paper, it is important to show the computation for the hypersurface case.

Example 2.9 *([11]).* Let $F(x, y) = x^3 - y^2 \in \mathbb{C}[x, y]$. Let $p = (a, b) \in X = \mathbf{V}(F)$. By above definition, we obtain the Jacobian matrix of order 2 of F as follows

$$\mathrm{Jac}_2(F) = \begin{pmatrix} 3x^2 & -2y & 3x & 0 & -1 \\ F & 0 & 3x^2 & -2y & 0 \\ 0 & F & 0 & 3x^2 & -2y \end{pmatrix}.$$

Theorem 2.10 *(Generalized Jacobian criterion [11]). Let $F \in \mathbb{C}[x_1, \ldots, x_s]$ be a reduced non-constant polynomial. Let $p \in V(F) \subset \mathbb{C}^s$. For $n \in \mathbb{N}$, let $M = \binom{n+s-1}{s}$. Then p is non-singular if and only if $rank(Jac_n(F))|_p = M$.*

3. Derivation Lie algebras and fewnomial singularities

In this section, we shall briefly introduce the basic definitions and important results which are helpful to solve the problem. The following basic concepts and results will be used to compute the derivation Lie algebras of isolated hypersurface singularities.

Recall that a derivation of commutative associative algebra A is defined as a linear endomorphism D of A satisfying the Leibniz rule: $D(ab) = D(a)b + aD(b)$. Thus for such an algebra A one can consider the Lie algebra of its derivations $\mathrm{Der}(A, A)$ with the bracket defined by the commutator of linear endomorphisms.

Definition 3.1. Let J be an ideal in an analytic algebra S. Then $\mathrm{Der}_J S \subseteq \mathrm{Der}_{\mathbb{C}} S$ is Lie subalgebra of all $\sigma \in \mathrm{Der}_{\mathbb{C}} S$ for which $\sigma(J) \subset J$.

We shall use the following well-known result to compute the derivations.

Theorem 3.2 *([36]). Let J be an ideal in $R = \mathbb{C}\{x_1, \cdots, x_n\}$. Then there is a natural isomorphism of Lie algebras*

$$(Der_J R)/(J \cdot Der_{\mathbb{C}} R) \cong Der_{\mathbb{C}}(R/J).$$

Definition 3.3. A polynomial $f \in \mathbb{C}[x_1, x_2, \cdots, x_n]$ is called quasi-homogeneous (or weighted homogeneous) if there exist positive rational numbers $w_1, \ldots, w_n$ (called weights of indeterminates x_j) and d such that, for each monomial $\prod x_j^{k_j}$ appearing in f with non-zero coefficient, one has $\sum w_j k_j = d$. The number d is called the quasi-homogeneous degree (w-degree) of f with respect to weights w_j and is denoted by $\deg f$. The collection $(w; d) = (w_1, \cdots, w_n; d)$ is called the quasi-homogeneity type (qh-type) of f.

Definition 3.4. An isolated hypersurface singularity in $\mathbb{C}^n$ is fewnomial if it can be defined by a n-nomial in n variables and it is a weighted homogeneous fewnomial isolated singularity if it can be defined by a weighted homogeneous fewnomial. 3-nomial isolated hypersurface singularity is also called trinomial singularity.

Proposition 3.5 *(cf. Theorem 9.1, [10]). Let f be a weighted homogeneous fewnomial isolated singularity with mult$(f) \geq 3$. Then f analytically equivalent to a linear combination of the following three series:*

Type A. $x_1^{a_1} + x_2^{a_2} + \cdots + x_{n-1}^{a_{n-1}} + x_n^{a_n}$, $n \geq 1$,
Type B. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n}$, $n \geq 2$,
Type C. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_1$, $n \geq 2$.

Proposition 3.5 has an immediate corollary.

Corollary 3.6. *Each binomial isolated singularity is analytically equivalent to one from the three series: A)* $x_1^{a_1} + x_2^{a_2}$, *B)* $x_1^{a_1}x_2 + x_2^{a_2}$, *C)* $x_1^{a_1}x_2 + x_2^{a_2}x_1$.

4. Proof of main theorems

We need the following lemma in order to give a proof of Theorem A.

Lemma 4.1. *Let* $(W, 0)$ *be an isolated hypersurface singularity defined by a polynomial* $G(x_1, x_2)$. *Then*

$$\mathcal{M}_2(W) = \mathcal{O}_2 \Big/ \Big\langle G,\ \left(\frac{\partial G}{\partial x_1}\right)^3,\ \left(\frac{\partial G}{\partial x_2}\right)^3,\ \left(\frac{\partial G}{\partial x_1}\right)^2 \frac{\partial G}{\partial x_2},\ \frac{\partial G}{\partial x_1}\left(\frac{\partial G}{\partial x_2}\right)^2,$$
$$\left(\frac{\partial G}{\partial x_2}\right)^2 \cdot \frac{\partial^2 G}{\partial x_1^2} + \left(\frac{\partial G}{\partial x_1}\right)^2 \cdot \frac{\partial^2 G}{\partial x_2^2} - 2\frac{\partial G}{\partial x_1} \cdot \frac{\partial G}{\partial x_2} \cdot \frac{\partial^2 G}{\partial x_1 \partial x_2} \Big\rangle.$$

Proof. Note that

$$Jac_2(G) = \begin{bmatrix} \frac{\partial G}{\partial x_1} & \frac{\partial G}{\partial x_2} & \frac{1}{2}\frac{\partial^2 G}{\partial x_1^2} & \frac{\partial^2 G}{\partial x_1 \partial x_2} & \frac{1}{2}\frac{\partial^2 G}{\partial x_2^2} \\ G & 0 & \frac{\partial G}{\partial x_1} & \frac{\partial G}{\partial x_2} & 0 \\ 0 & G & 0 & \frac{\partial G}{\partial x_1} & \frac{\partial G}{\partial x_2} \end{bmatrix}.$$

Let G_{ijk} be the maximal minors corresponding to the i, j, k-columns of $Jac_2(G)$. Hence,

$$G_{123} = \frac{1}{2} \cdot G^2 \cdot \frac{\partial^2 G}{\partial x_1^2} - G \cdot \left(\frac{\partial G}{\partial x_1}\right)^2;$$

$$G_{124} = G^2 \cdot \frac{\partial^2 G}{\partial x_1 \partial x_2} - 2 \cdot G \cdot \frac{\partial G}{\partial x_1} \cdot \frac{\partial G}{\partial x_2};$$

$$G_{125} = \frac{1}{2} \cdot G^2 \cdot \frac{\partial^2 G}{\partial x_1^2} - G \cdot \left(\frac{\partial G}{\partial x_2}\right)^2;$$

$$G_{134} = \left(\frac{\partial G}{\partial x_1}\right)^3 - \frac{1}{2} \cdot G \cdot \frac{\partial G}{\partial x_1} \cdot \frac{\partial^2 G}{\partial x_1^2};$$

$$G_{135} = \left(\frac{\partial G}{\partial x_1}\right)^2 \cdot \frac{\partial G}{\partial x_2} - \frac{1}{2} \cdot G \cdot \frac{\partial G}{\partial x_2} \cdot \frac{\partial^2 G}{\partial x_1^2};$$

$$G_{145} = \frac{\partial G}{\partial x_1} \cdot \left(\frac{\partial G}{\partial x_2}\right)^2 + \frac{1}{2} \cdot G \cdot \frac{\partial G}{\partial x_1} \cdot \frac{\partial^2 G}{\partial x_2^2} - G \cdot \frac{\partial G}{\partial x_2} \cdot \frac{\partial^2 G}{\partial x_1 \partial x_2};$$

$$G_{234} := \left(\frac{\partial G}{\partial x_1}\right)^2 \cdot \frac{\partial G}{\partial x_2} + \frac{1}{2} \cdot G \cdot \frac{\partial G}{\partial x_2} \cdot \frac{\partial^2 G}{\partial x_1^2} - G \cdot \frac{\partial G}{\partial x_1} \cdot \frac{\partial^2 G}{\partial x_1 \partial x_2};$$

$$G_{235} := \frac{\partial G}{\partial x_1} \cdot \left(\frac{\partial G}{\partial x_2}\right)^2 - \frac{1}{2} \cdot G \cdot \frac{\partial G}{\partial x_1} \cdot \frac{\partial^2 G}{\partial x_2^2};$$

$$G_{245} := \left(\frac{\partial G}{\partial x_2}\right)^3 - \frac{1}{2} \cdot G \cdot \frac{\partial G}{\partial x_2} \cdot \frac{\partial^2 G}{\partial x_2^2};$$

$$G_{345} := \frac{1}{2} \cdot \left(\frac{\partial G}{\partial x_2}\right)^2 \cdot \frac{\partial^2 G}{\partial x_1^2} + \frac{1}{2} \cdot \left(\frac{\partial G}{\partial x_1}\right)^2 \cdot \frac{\partial^2 G}{\partial x_2^2} - \frac{\partial G}{\partial x_1} \cdot \frac{\partial G}{\partial x_2} \cdot \frac{\partial^2 G}{\partial x_1 \partial x_2}.$$

Hence, $\mathcal{J}_2(G)$ is generated by $G_{123}, G_{124}, \cdots, G_{345}$. That implies that

$$(G, \mathcal{J}_2(G)) = \Big\langle G, \quad \left(\frac{\partial G}{\partial x_1}\right)^3, \quad \left(\frac{\partial G}{\partial x_2}\right)^3, \quad \left(\frac{\partial G}{\partial x_1}\right)^2 \frac{\partial G}{\partial x_2}, \quad \frac{\partial G}{\partial x_1}\left(\frac{\partial G}{\partial x_2}\right)^2,$$
$$\left(\frac{\partial G}{\partial x_2}\right)^2 \cdot \frac{\partial^2 G}{\partial x_1^2} + \left(\frac{\partial G}{\partial x_1}\right)^2 \cdot \frac{\partial^2 G}{\partial x_2^2} - 2\frac{\partial G}{\partial x_1} \cdot \frac{\partial G}{\partial x_2} \cdot \frac{\partial^2 G}{\partial x_1 \partial x_2}\Big\rangle.$$

Thus the lemma has been proved. □

Proof of Theorem A. Let $h : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be the germ at the origin of a biholomorphic mapping, such that $h(V) = W$. Then there exist $u \in \mathcal{O}$ such that $F = u \cdot (G \circ h)$ and $u(0) \neq 0$. Write $h = (h_1, h_2)$, where $h_i : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$.

By Leibniz rule, we obtain the following two relations

$$\frac{\partial F}{\partial x_i} = \frac{\partial u}{\partial x_i} \cdot (G \circ h) + u \cdot \sum_{j=1}^{2} (\frac{\partial G}{\partial x_j} \circ h) \cdot \frac{\partial h_j}{\partial x_i};$$

$$\frac{\partial^2 F}{\partial x_i \partial x_l}$$
$$= \frac{\partial \left[\frac{\partial u}{\partial x_i} \cdot (G \circ h) + u \cdot \sum_{j=1}^{2} (\frac{\partial G}{\partial x_j} \circ h) \cdot \frac{\partial h_j}{\partial x_i}\right]}{\partial x_l}$$
$$= \frac{\partial^2 u}{\partial x_i \partial x_l} \cdot (G \circ h) + \frac{\partial u}{\partial x_i} \cdot \sum_{j=1}^{2} (\frac{\partial G}{\partial x_j} \circ h) \cdot \frac{\partial h_j}{\partial x_l} + \frac{\partial u}{\partial x_l} \cdot \left[\sum_{j=1}^{2} (\frac{\partial G}{\partial x_j} \circ h) \cdot \frac{\partial h_j}{\partial x_i}\right]$$
$$+ u \cdot \sum_{j=1}^{2} \left\{ \left[\sum_{r=1}^{2} \left(\frac{\partial^2 G}{\partial x_j \partial x_r} \circ h\right) \cdot \frac{\partial h_r}{\partial x_l}\right] \cdot \frac{\partial h_j}{\partial x_i} + \left(\frac{\partial G}{\partial x_j} \circ h\right) \cdot \frac{\partial h_j}{\partial x_i \partial x_l} \right\}.$$

Hence, the elements $\left(\frac{\partial F}{\partial x_1}\right)^3, \left(\frac{\partial F}{\partial x_2}\right)^3, \left(\frac{\partial F}{\partial x_1}\right)^2 \cdot \frac{\partial F}{\partial x_2}, \frac{\partial F}{\partial x_1} \cdot \left(\frac{\partial F}{\partial x_2}\right)^2$ belong to the ideal:

$$\Big\langle G \circ h, \left(\frac{\partial G}{\partial x_1} \circ h\right)^3, \left(\frac{\partial G}{\partial x_2} \circ h\right)^3, \left(\frac{\partial G}{\partial x_1} \circ h\right)^2 \cdot \left(\frac{\partial G}{\partial x_2} \circ h\right), \left(\frac{\partial G}{\partial x_1} \circ h\right) \cdot \left(\frac{\partial G}{\partial x_2} \circ h\right)^2 \Big\rangle.$$

Now, we compute that (after module $\left\langle G\circ h, \left(\frac{\partial G}{\partial x_1}\circ h\right)^3, \left(\frac{\partial G}{\partial x_2}\circ h\right)^3, \left(\frac{\partial G}{\partial x_1}\circ h\right)^2\cdot\left(\frac{\partial G}{\partial x_2}\circ h\right), \left(\frac{\partial G}{\partial x_1}\circ h\right)\cdot\left(\frac{\partial G}{\partial x_2}\circ h\right)^2\right\rangle$)

$$\begin{aligned}
&\left(\frac{\partial F}{\partial x_2}\right)^2\cdot\frac{\partial^2 F}{\partial x_1^2}+\left(\frac{\partial F}{\partial x_1}\right)^2\cdot\frac{\partial^2 F}{\partial x_2^2}-2\frac{\partial F}{\partial x_1}\cdot\frac{\partial F}{\partial x_2}\cdot\frac{\partial^2 F}{\partial x_1\partial x_2}\\
=&u^2\cdot\left[\left(\frac{\partial G}{\partial x_1}\circ h\right)\cdot\frac{\partial h_1}{\partial x_2}+\left(\frac{\partial G}{\partial x_2}\circ h\right)\cdot\frac{\partial h_2}{\partial x_2}\right]^2\cdot\\
&u\cdot\left\{\left[\left(\frac{\partial^2 G}{\partial x_1^2}\circ h\right)\cdot\frac{\partial h_1}{\partial x_1}+\left(\frac{\partial^2 G}{\partial x_1\partial x_2}\circ h\right)\cdot\frac{\partial h_2}{\partial x_1}\right]\cdot\frac{\partial h_1}{\partial x_1}+\right.\\
&\left.\left[\left(\frac{\partial^2 G}{\partial x_1\partial x_2}\circ h\right)\cdot\frac{\partial h_1}{\partial x_1}+\left(\frac{\partial^2 G}{\partial x_2^2}\circ h\right)\cdot\frac{\partial h_2}{\partial x_1}\right]\cdot\frac{\partial h_2}{\partial x_1}\right\} \qquad (3)\\
&+u^2\cdot\left[\left(\frac{\partial G}{\partial x_1}\circ h\right)\cdot\frac{\partial h_1}{\partial x_1}+\left(\frac{\partial G}{\partial x_2}\circ h\right)\cdot\frac{\partial h_2}{\partial x_1}\right]^2\cdot\\
&u\cdot\left\{\left[\left(\frac{\partial^2 G}{\partial x_1^2}\circ h\right)\cdot\frac{\partial h_1}{\partial x_2}+\left(\frac{\partial^2 G}{\partial x_1\partial x_2}\circ h\right)\cdot\frac{\partial h_2}{\partial x_2}\right]\cdot\frac{\partial h_1}{\partial x_2}+\right.\\
&\left.\left[\left(\frac{\partial^2 G}{\partial x_1\partial x_2}\circ h\right)\cdot\frac{\partial h_1}{\partial x_2}+\left(\frac{\partial^2 G}{\partial x_2^2}\circ h\right)\cdot\frac{\partial h_2}{\partial x_2}\right]\cdot\frac{\partial h_2}{\partial x_2}\right\}\\
&-2\cdot u^2\cdot\left[\left(\frac{\partial G}{\partial x_1}\circ h\right)\cdot\frac{\partial h_1}{\partial x_1}+\left(\frac{\partial G}{\partial x_2}\circ h\right)\cdot\frac{\partial h_2}{\partial x_1}\right]\cdot\left[\left(\frac{\partial G}{\partial x_1}\circ h\right)\cdot\frac{\partial h_1}{\partial x_2}+\left(\frac{\partial G}{\partial x_2}\circ h\right)\cdot\frac{\partial h_2}{\partial x_2}\right]\cdot\\
&u\cdot\left\{\left[\left(\frac{\partial^2 G}{\partial x_1^2}\circ h\right)\cdot\frac{\partial h_1}{\partial x_2}+\left(\frac{\partial^2 G}{\partial x_1\partial x_2}\circ h\right)\cdot\frac{\partial h_2}{\partial x_2}\right]\cdot\frac{\partial h_1}{\partial x_1}+\right.\\
&\left.\left[\left(\frac{\partial^2 G}{\partial x_1\partial x_2}\circ h\right)\cdot\frac{\partial h_1}{\partial x_2}+\left(\frac{\partial^2 G}{\partial x_2^2}\circ h\right)\cdot\frac{\partial h_2}{\partial x_2}\right]\cdot\frac{\partial h_2}{\partial x_1}\right\}. \qquad (4)
\end{aligned}$$

Let

$$a_i:=\frac{\partial G}{\partial x_i}\circ h;\qquad a_{ij}:=\frac{\partial^2 G}{\partial x_i\partial x_j}\circ h;\qquad b_{ij}:=\frac{\partial h_i}{\partial x_j}\qquad for\quad i,j\in\{1,2\}.$$

Then Equation (3) can be written as follows

$$\begin{aligned}
&\left(\frac{\partial F}{\partial x_2}\right)^2\cdot\frac{\partial^2 F}{\partial x_1^2}+\left(\frac{\partial F}{\partial x_1}\right)^2\cdot\frac{\partial^2 F}{\partial x_2^2}-2\frac{\partial F}{\partial x_1}\cdot\frac{\partial F}{\partial x_2}\cdot\frac{\partial^2 F}{\partial x_1\partial x_2}\\
=&u^3\cdot(a_1\cdot b_{12}+a_2\cdot b_{22})^2\cdot[(a_{11}\cdot b_{11}+a_{12}\cdot b_{21})\cdot b_{11}+(a_{12}\cdot b_{11}+a_{22}\cdot b_{21})\cdot b_{21}]+
\end{aligned}$$

$$
\begin{aligned}
&u^3 \cdot (a_1 \cdot b_{11} + a_2 \cdot b_{21})^2 \cdot [(a_{11} \cdot b_{12} + a_{12} \cdot b_{22}) \cdot b_{12} + (a_{12} \cdot b_{12} + a_{22} \cdot b_{22}) \cdot b_{22}] - \\
&u^3 \cdot 2 \cdot (a_1 \cdot b_{11} + a_2 \cdot b_{21}) \cdot (a_1 \cdot b_{12} + a_2 \cdot b_{22}) \cdot [(a_{11} \cdot b_{12} + a_{12} \cdot b_{22}) \cdot b_{11} + \\
&\qquad\qquad\qquad\qquad\qquad\qquad (a_{12} \cdot b_{12} + a_{22} \cdot b_{22}) \cdot b_{21}] \\
=&u^3 \cdot (a_1^2 a_{22} + a_2^2 a_{11} - 2a_1 a_2 a_{12}) \cdot (b_{11}^2 b_{22}^2 + b_{12}^2 b_{21}^2 - 2b_{11} b_{12} b_{21} b_{22}).
\end{aligned}
$$

It follows from Lemma 4.1 that

$$
\mathcal{M}_2(W) = \mathcal{O}_2 \Big/ \Big\langle G, \ \left(\frac{\partial G}{\partial x_1}\right)^3, \ \left(\frac{\partial G}{\partial x_2}\right)^3, \ \left(\frac{\partial G}{\partial x_1}\right)^2 \frac{\partial G}{\partial x_2}, \ \frac{\partial G}{\partial x_1}\left(\frac{\partial G}{\partial x_2}\right)^2, \\
\left(\frac{\partial G}{\partial x_2}\right)^2 \cdot \frac{\partial^2 G}{\partial x_1^2} + \left(\frac{\partial G}{\partial x_1}\right)^2 \cdot \frac{\partial^2 G}{\partial x_2^2} - 2\frac{\partial G}{\partial x_1} \cdot \frac{\partial G}{\partial x_2} \cdot \frac{\partial^2 G}{\partial x_1 \partial x_2} \Big\rangle.
$$

From this, we can define

$$
h^* : \mathcal{O}_2 \longrightarrow \mathcal{O}_2
$$

by $h^* u = u \circ h$. It is clear that h^* induces $\mathbb{C}$-algebra isomorphisms $\mathcal{M}_2(V) \cong \mathcal{M}_2(W)$. □

Proof of Theorem B.

First, we recall some useful lemmas.

Lemma 4.2 *([33]). Let $\mathcal{M} = \bigoplus_{i=0}^t \mathcal{M}_i$ be a graded commutative Artinian local algebra. Let $\mathcal{L}(\mathcal{M})$ be the algebra of derivations of $\mathcal{M}$. Then $\mathcal{L}(\mathcal{M})$ is a graded Lie algebra $\bigoplus_{k=-t}^t \mathcal{L}_k$ where $\mathcal{L}_k = \{D \in \mathcal{L}(\mathcal{M}) : D(\mathcal{M}_i) \subset \mathcal{M}_{i+k}$ for all $i\}$.*

Lemma 4.3 *([33] [37]). Let $(\mathcal{M}, m)$ be a commutative local Artinian algebra and $D \in \mathcal{L}(\mathcal{M})$ be any derivation of $\mathcal{M}$. Then D preserves the m-adic filtration of $\mathcal{M}$, i.e., $D(m) \subset m$.*

Lemma 4.4 *(Lemma 2.1 in [7]). Let f be a weighted homogeneous polynomial with isolated singularity in $x_1, \ldots, x_n$ variables of type $(\alpha_1, \ldots, \alpha_n; d)$. Assume that $\alpha_1 \geq \cdots \geq \alpha_n$. Then f must be one of the following two cases:*
Case (1).

$$
f = x_1^m + a_1(x_2, \ldots, x_n) x_1^{m-1} + \cdots + a_{m-1}(x_2, \ldots, x_n) x_1 + a_m(x_2, \ldots, x_n).
$$

Case (2).

$$
f = x_1^m x_i + a_1(x_2, \ldots, x_n) x_1^{m-1} + \cdots + a_{m-1}(x_2, \ldots, x_n) x_1 + a_m(x_2, \ldots, x_n),
$$

where $i \neq 1$.

Now, we begin to prove Theorem B.

Proof. According to Lemma 4.1, let

$$f_1 = \left(\frac{\partial G}{\partial x_1}\right)^3, \qquad f_2 = \left(\frac{\partial G}{\partial x_2}\right)^3, \qquad f_3 = \left(\frac{\partial G}{\partial x_1}\right)^2 \cdot \frac{\partial G}{\partial x_2}, \qquad f_4 = \frac{\partial G}{\partial x_1} \cdot \left(\frac{\partial G}{\partial x_2}\right)^2,$$
$$f_5 = \left(\frac{\partial G}{\partial x_2}\right)^2 \cdot \frac{\partial^2 G}{\partial x_1^2} + \left(\frac{\partial G}{\partial x_1}\right)^2 \cdot \frac{\partial^2 G}{\partial x_2^2} - 2\frac{\partial G}{\partial x_1} \cdot \frac{\partial G}{\partial x_2} \cdot \frac{\partial^2 G}{\partial x_1 \partial x_2}.$$

By Lemma 4.3, we obtain that D is of the form $D = cx_2^k \frac{\partial}{\partial x_1}$, with $k\alpha_2 < \alpha_1$, $c \in \mathbb{C}$, $k \in \mathbb{Z}^+$.

We only need to consider $\alpha_1 \neq \alpha_2$, by the assumption, we have $\alpha_1 > \alpha_2$. Hence, it is easy to see that $wt(f_2) > wt(f_1), wt(f_3) > wt(f_1), wt(f_4) > wt(f_1)$. Therefore, the minimal weight degree of the element in $\{f_i, 1 \leq i \leq 5\}$ is equal to $wt(f_1)$ or $wt(f_5)$.

By Lemma 4.4, we have the following two cases.

Case (1).

$$G(x_1, x_2) = x_1^m + a_1(x_2)x_1^{m-1} + \cdots + a_m(x_2).$$

If $\alpha_1 \geq 2\alpha_2$, then $wt(f_1) \leq wt(f_5)$. If $wt(G) \geq wt(f_1)$, then $D(f_1) = 0$ (since D is of negative weight) which implies that $D = 0$. If $wt(G) < wt(f_1)$, then $D(G) = 0$ which implies that $D = 0$ again.

If $\alpha_1 < 2\alpha_2$, then $wt(f_5) < wt(f_1)$. If $wt(G) < wt(f_5)$, then $D(G) = 0$ which implies that $D = 0$. If $wt(G) \geq wt(f_5)$, then $D(f_5) = 0$. Let i be the first index such that $a_i(x_2) \not\equiv 0$. By the assumption of $d \geq 2\alpha_1 > 2\alpha_2$, we obtain that $m \geq 2$ and $mult(a_i) \geq 2$.

Then we obtain that

$$\begin{aligned}
f_5 &= \left(\frac{\partial a_i}{\partial x_2}x_1^{m-i} + \cdots + \frac{\partial a_m}{\partial x_2}\right)^2 \cdot [m(m-1)x_1^{m-2} + \cdots + 2a_{m-2}(x_2)] \\
&+ [mx_1^{m-1} + \cdots + a_{m-1}(x_2)]^2 \cdot \left(\frac{\partial^2 a_i}{\partial x_2^2}x_1^{m-i} + \cdots + \frac{\partial^2 a_m}{\partial x_2^2}\right) \\
&- 2[mx_1^{m-1} + \cdots + a_{m-1}(x_2)] \cdot \left(\frac{\partial a_i}{\partial x_2}x_1^{m-i} + \cdots + \frac{\partial a_m}{\partial x_2}\right) \\
&\cdot \left((m-i)\frac{\partial a_i}{\partial x_2}x_1^{m-i-1} + \cdots + \frac{\partial a_{m-1}}{\partial x_2}\right) \\
&= m^2 \frac{\partial^2 a_i}{\partial x_2^2}x_1^{3m-i-2} + \textit{Lower terms}.
\end{aligned}$$

Hence $0 = D(f_5) = cm^2(3m-i-2)x_2^k \frac{\partial^2 a_i}{\partial x_2^2} x_1^{3m-i-3} + \textit{Lower terms}$, which implies that $c = 0$, i.e., $D \equiv 0$.

Case (2).

$$G(x_1, x_2) = x_2 x_1^m + a_1(x_2) x_1^{m-1} + \cdots + a_m(x_2).$$

If $\alpha_1 \geq 2\alpha_2$, then $wt(f_1) \leq wt(f_5)$. If $wt(G) \geq wt(f_1)$, then $D(f_1) = 0$ which implies that $D = 0$. If $wt(G) < wt(f_1)$, then $D(G) = 0$ which implies that $D = 0$ again.

If $\alpha_1 < 2\alpha_2$, then $wt(f_5) < wt(f_1)$. If $wt(G) < wt(f_5)$, then $D(G) = 0$ which implies that $D = 0$. If $wt(G) \geq wt(f_5)$, then $D(f_5) = 0$. Let i be the first index such that $a_i(x_2) \not\equiv 0$. By the assumption of $d \geq 2\alpha_1 > 2\alpha_2$, we obtain that $m \geq 2$ and $mult(a_i) \geq 2$.

Then we obtain that

$$\begin{aligned}
f_5 = &\left(x_1^m + \frac{\partial a_i}{\partial x_2} x_1^{m-i} + \cdots + \frac{\partial a_m}{\partial x_2} \right)^2 \cdot [m(m-1)x_1^{m-2}x_2 + \cdots + 2a_{m-2}(x_2)] \\
&+ [mx_1^{m-1}x_2 + \cdots + a_{m-1}(x_2)]^2 \cdot \left(\frac{\partial^2 a_i}{\partial x_2^2} x_1^{m-i} + \cdots + \frac{\partial^2 a_m}{\partial x_2^2} \right) \\
&- 2[mx_1^{m-1}x_2 + \cdots + a_{m-1}(x_2)] \cdot \left(x_1^m + \frac{\partial a_i}{\partial x_2} x_1^{m-i} + \cdots + \frac{\partial a_m}{\partial x_2} \right) \cdot \\
&\left(mx_1^{m-1} + (m-i)\frac{\partial a_i}{\partial x_2} x_1^{m-i-1} + \cdots + \frac{\partial a_{m-1}}{\partial x_2} \right) \\
=&[m(m-1) - 2m^2] x_1^{3m-2} x_2 + Lower\ terms \\
=& - m(m+1) x_1^{3m-2} x_2 + Lower\ terms.
\end{aligned}$$

Hence $0 = D(f_5) = -cm(m+1)(3m-2)x_2^{k+1}x_1^{3m-3} + Lower\ terms$, which implies that $c = 0$, i.e., $D \equiv 0$. □

To prove Theorem C, we need to prove the following propositions first.

Proposition 4.5. *Let $(V, 0)$ be a binomial isolated singularity of type A which is defined by $f = x_1^{a_1} + x_2^{a_2}$ $(a_1 \geq 2, a_2 \geq 2)$ with weight type $(\frac{1}{a_1}, \frac{1}{a_2}; 1)$. Then*

$$\rho_2(V) = \begin{cases} 4a_1a_2 - 4(a_1 + a_2) + 7; & a_1 \geq 3,\ a_2 \geq 3,\ and\ a_1 = a_2 \\ 6; & a_1 = 2, a_2 = 2 \\ 4a_2 - 3; & a_1 = 2, a_2 \geq 3 \\ 4a_1 - 3; & a_1 \geq 3, a_2 = 2 \\ 4a_1a_2 - 4(a_1 + a_2) + 6; & a_1 \geq 3,\ a_2 \geq 3,\ and\ a_1 \neq a_2. \end{cases}$$

Proof. It follows from Lemma 4.1 that the local algebra

$$\mathcal{M}_2(V) = \mathcal{O}_2 \Big/ \Big\langle f, \left(\frac{\partial f}{\partial x_1}\right)^3, \left(\frac{\partial f}{\partial x_2}\right)^3, \left(\frac{\partial f}{\partial x_1}\right)^2 \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_1} \left(\frac{\partial f}{\partial x_2}\right)^2,$$

$$\left(\frac{\partial f}{\partial x_2}\right)^2\frac{\partial^2 f}{\partial x_1^2}+\left(\frac{\partial f}{\partial x_1}\right)^2\frac{\partial^2 f}{\partial x_2^2}-2\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}\frac{\partial^2 f}{\partial x_1\partial x_2}\Big\rangle,$$

has a monomial basis of the form:

(1) When $a_1 \geq 3,\ a_2 \geq 3$, and $a_1 = a_2$,
$\{x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq a_1-1, 0 \leq i_2 \leq 2a_2-3; x_1^{a_1-3}x_2^{i_2}, 2a_2-2 \leq i_2 \leq 2a_2-1; x_1^{a_1-2}x_2^{2a_2-2};$
$x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq a_1-4, 2a_2-2 \leq i_2 \leq 3a_2-4\}$.
(2) When $a_1 = 2, a_2 = 2,\ \{1, x_1, x_2, x_2^2, x_1x_2\}$.
(3) When $a_1 = 2,\ a_2 \geq 3, \{x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq 1, 0 \leq i_2 \leq a_2-1; x_2^{i_2}, a_2 \leq i_2 \leq 2a_2-3\}$.
(4) if $a_1 \geq 3,\ a_2 = 2, \{x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq a_1-1, 0 \leq i_2 \leq 1; x_1^{i_1}, a_1 \leq i_1 \leq 2a_1-3\}$.
(5) When $a_1 \geq 3,\ a_2 \geq 3$, and $a_1 \neq a_2$,
$\{x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq a_1-1, 0 \leq i_2 \leq 2a_2-3; x_1^{a_1-3}x_2^{i_2}, 2a_2-2 \leq i_2 \leq 2a_2-1;$

$$x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq a_1-4, 2a_2-2 \leq i_2 \leq 3a_2-4\}.$$

We divide the proof into five cases.

Case (1): When $a_1 \geq 5,\ a_2 \geq 5$, and $a_1 = a_2$, we obtain the following basis of Lie algebra $\mathcal{L}_2(V)$:

$$x_2^{i_2+a_2-1}\partial_1 - x_1^{a_1-1}x_2^{i_2}\partial_2, 0 \leq i_2 \leq a_2-2; x_1^{a_1-2}x_2^{i_2}\partial_1 + x_1^{a_1-3}x_2^{i_2+1}\partial_2, 0 \leq i_2 \leq a_2-1;$$
$$x_1^{i_1}x_2^{i_2}\partial_1 + x_1^{i_1-1}x_2^{i_2+1}\partial_2, 1 \leq i_1 \leq a_1-3, 0 \leq i_2 \leq 2a_2-4; x_1^{a_1-1}x_2^{i_2}\partial_1 + x_1^{a_1-2}x_2^{i_2+1}\partial_2,$$
$$0 \leq i_2 \leq a_2-2; x_2^{i_2}\partial_1, 2a_2-2 \leq i_2 \leq 3a_2-4;$$
$$x_1^{i_1}x_2^{i_2}\partial_1, 1 \leq i_1 \leq a_1-4, 2a_2-3 \leq i_2 \leq 3a_2-4;$$
$$x_1^{a_1-3}x_2^{i_2}\partial_1, 2a_2-3 \leq i_2 \leq 2a_2-1; x_1^{a_1-2}x_2^{i_2}\partial_1, a_2 \leq i_2 \leq 2a_2-2;$$
$$x_1^{a_1-1}x_2^{i_2}\partial_1, a_2-1 \leq i_2 \leq 2a_2-3;$$
$$x_1^{i_1}x_2^{i_2}\partial_2, 0 \leq i_1 \leq a_1-4, 2a_2-2 \leq i_2 \leq 3a_2-4; x_1^{a_1-3}x_2^{i_2}\partial_2, a_2+1 \leq i_2 \leq 2a_2-1;$$
$$x_1^{a_1-2}x_2^{i_2}\partial_2, a_2 \leq i_2 \leq 2a_2-2; x_1^{a_1-1}x_2^{i_2}\partial_2, a_2-1 \leq i_2 \leq 2a_2-3.$$

Therefore we obtain a formula

$$\rho_2(V) = 4a_1a_2 - 4(a_1+a_2) + 7. \tag{5}$$

Remark 4.6. When $a_1 = 3, a_2 = 3$, the Lie algebra $\mathcal{L}_2(V)$ has the following basis:

$$x_2^2\partial_1 - x_1^2\partial_2; x_2^3\partial_1 - x_1^2x_2\partial_2; x_2^4\partial_1; x_2^5\partial_1; x_1\partial_1 + x_2\partial_2; x_1x_2\partial_1 + x_2^2\partial_2; x_1x_2^2\partial_1 + x_2^3\partial_2;$$
$$x_1x_2^3\partial_1; x_1x_2^4\partial_1;$$

$$x_1^2\partial_1 + x_1x_2\partial_2; x_1^2x_2\partial_1 + x_1x_2^2\partial_2; x_1^2x_2^2\partial_1; x_1^2x_2^3\partial_1; x_2^4\partial_2; x_2^5\partial_2; x_1x_2^3\partial_2; x_1x_2^4\partial_2; x_1^2x_2^2\partial_2;$$
$$x_1^2x_2^3\partial_2.$$
$$\rho_2(V) = 19.$$

Thus, this case also satisfies the formula (5) above.

It is easy to check that the case of $a_1 = 4, a_2 = 4$ also satisfies the formula (5) above, it has the following basis:

$$x_2^3\partial_1 - x_1^3\partial_2; x_2^4\partial_1 - x_1^3x_2\partial_2; x_2^5\partial_1 - x_1^3x_2^2\partial_2; x_2^6\partial_1; x_2^7\partial_1; x_2^8\partial_1; x_1\partial_1 + x_2\partial_2;$$
$$x_1x_2\partial_1 + x_2^2\partial_2; x_1x_2^2\partial_1 + x_2^3\partial_2;$$
$$x_1x_2^3\partial_1 + x_2^4\partial_2; x_1x_2^4\partial_1 + x_2^5\partial_2; x_1x_2^5\partial_1; x_1x_2^6\partial_1; x_1x_2^7\partial_1; x_1^2\partial_1 + x_1x_2\partial_2; x_1^2x_2\partial_1 + x_1x_2^2\partial_2;$$
$$x_1^2x_2^2\partial_1 + x_1x_2^3\partial_2;$$
$$x_1^2x_2^3\partial_1+x_1x_2^4\partial_2; x_1^2x_2^4\partial_1; x_1^2x_2^5\partial_x; x_1^2x_2^6\partial_1; x_1^3\partial_1+x_1^2x_2\partial_2; x_1^3x_2\partial_1+x_1^2x_2^2\partial_2; x_1^3x_2^2\partial_1+x_1^2x_2^3\partial_2;$$
$$x_1^3x_2^3\partial_1; x_1^3x_2^4\partial_1; x_1^3x_2^5\partial_1; x_2^6\partial_2; x_2^7\partial_2; x_2^8\partial_2; x_1x_2^5\partial_2; x_1x_2^6\partial_2; x_1x_2^7\partial_2; x_1^2x_2^4\partial_2; x_1^2x_2^5\partial_2; x_1^2x_2^6\partial_2;$$
$$x_1^3x_2^3\partial_2;$$
$$x_1^3x_2^4\partial_2; x_1^3x_2^5\partial_2.$$

Therefore, we obtain the following formula

$$\rho_2(V) = 39.$$

Thus, this case also satisfies the formula (5) above.

Case (2): When $a_1 = 2$, $a_2 = 2$, we obtain the following basis of $\mathcal{L}_2(V)$:

$$x_2\partial_1 - x_1\partial_2; x_2^2\partial_1; x_1\partial_1 + x_2\partial_2; x_1x_2\partial_1; x_2^2\partial_2; x_1x_2\partial_2.$$

Therefore, we obtain the following formula

$$\rho_2(V) = 6.$$

Case (3): When $a_1 = 2$, $a_2 \geq 3$, we obtain the following basis of $\mathcal{L}_2(V)$:

$$x_2^{a_2-1}\partial_1 - \frac{2x_1}{a_2}\partial_2; x_1x_2^{i_2}\partial_1 + \frac{2x_2^{i_2+1}}{a_2}\partial_2, 0 \leq i_2 \leq a_2 - 3; x_2^{i_2}\partial_1, a_2 \leq i_2 \leq 2a_2 - 3;$$
$$x_1x_2^{i_2}\partial_1, a_2 - 2 \leq i_2 \leq a_2 - 1; x_2^{i_2}\partial_2, a_2 - 1 \leq i_2 \leq 2a_2 - 3; x_1x_2^{i_2}\partial_2, 1 \leq i_2 \leq a_2 - 1.$$

Therefore, we obtain the following formula

$$\rho_2(V) = 4a_2 - 3.$$

Case (4): When $a_1 \geq 3$, $a_2 = 2$, we obtain the following basis of $\mathcal{L}_2(V)$:

$$x_2\partial_1 - \frac{a_1 x_1^{a_1-1}}{2}\partial_2; x_1^{i_1}\partial_1 + \frac{a_1 x_1^{i_1-1}}{2}x_2\partial_2, 1 \leq i_1 \leq a_1 - 2; x_1^{i_1}x_2\partial_1, 1 \leq i_1 \leq a_1 - 1;$$
$$x_1^{a_1-1}\partial_1; x_1^{i_1}x_2^2\partial_1, 0 \leq i_1 \leq a_1 - 3; x_1^{i_1}x_2^2\partial_2, 0 \leq i_1 \leq a_1 - 3; x_1^{i_1}x_2\partial_2, a_1 - 2 \leq i_1 \leq a_1 - 1.$$

Therefore, we obtain the following formula

$$\rho_2(V) = 4a_1 - 3.$$

Case (5): When $a_1 \geq 5$, $a_2 \geq 4$, and $a_1 \neq a_2$, we obtain the following basis of $\mathcal{L}_2(V)$:

$$x_2^{i_2+a_2-1}\partial_1 - \frac{a_1}{a_2}x_1^{a_1-1}x_2^{i_2}\partial_2, 0 \leq i_2 \leq a_2 - 2;$$
$$x_1^{a_1-2}x_2^{i_2}\partial_1 + \frac{a_1}{a_2}x_1^{a_1-3}x_2^{i_2+1}\partial_2, 0 \leq i_2 \leq a_2 - 1;$$
$$x_1^{i_1}x_2^{i_2}\partial_1 + \frac{a_1}{a_2}x_1^{i_1-1}x_2^{i_2+1}\partial_2, 1 \leq i_1 \leq a_1 - 3, 0 \leq i_2 \leq 2a_2 - 4;$$
$$x_1^{a_1-1}x_2^{i_2}\partial_1 + \frac{a_1}{a_2}x_1^{a_1-2}x_2^{i_2+1}\partial_2,$$
$$0 \leq i_2 \leq a_2 - 3; x_2^{i_2}\partial_1, 2a_2 - 2 \leq i_2 \leq 3a_2 - 4;$$
$$x_1^{i_1}x_2^{i_2}\partial_1, 1 \leq i_1 \leq a_1 - 4, 2a_2 - 3 \leq i_2 \leq 3a_2 - 4;$$
$$x_1^{a_1-3}x_2^{i_2}\partial_1, 2a_2 - 3 \leq i_2 \leq 2a_2 - 1; x_1^{a_1-2}x_2^{i_2}\partial_1, a_2 \leq i_2 \leq 2a_2 - 3;$$
$$x_1^{a_1-1}x_2^{i_2}\partial_1, a_2 - 2 \leq i_2 \leq 2a_2 - 3;$$
$$x_1^{i_1}x_2^{i_2}\partial_2, 0 \leq i_1 \leq a_1 - 4, 2a_2 - 2 \leq i_2 \leq 3a_2 - 4; x_1^{a_1-3}x_2^{i_2}\partial_2, a_2 + 1 \leq i_2 \leq 2a_2 - 1;$$
$$x_1^{a_1-2}x_2^{i_2}\partial_2, a_2 - 1 \leq i_2 \leq 2a_2 - 3; x_1^{a_1-1}x_2^{i_2}\partial_2, a_2 - 1 \leq i_2 \leq 2a_2 - 3.$$

Therefore, we obtain the following formula

$$\rho_2(V) = 4a_1a_2 - 4(a_1 + a_2) + 6. \tag{6}$$

Remark 4.7. When $a_1 = 4$, $a_2 \geq 5$, we obtain the following basis of $\mathcal{L}_2(V)$:

$$x_2^{i_2+a_2-1}\partial_1 - \frac{4}{a_2}x_1^3x_2^{i_2}\partial_2, 0 \leq i_2 \leq a_2 - 2; x_1^2x_2^{i_2}\partial_1 + \frac{4}{a_2}x_1x_2^{i_2+1}\partial_2, 0 \leq i_2 \leq a_2 - 1;$$
$$x_1x_2^{i_2}\partial_1 + \frac{4}{a_2}x_2^{i_2+1}\partial_2, 0 \leq i_2 \leq 2a_2 - 4; x_1^3x_2^{i_2}\partial_1 + \frac{4}{a_2}x_1^2x_2^{i_2+1}\partial_2, 0 \leq i_2 \leq a_2 - 3;$$
$$x_2^{i_2}\partial_1, 2a_2 - 2 \leq i_2 \leq 3a_2 - 4;$$
$$x_1x_2^{i_2}\partial_1, 2a_2 - 3 \leq i_2 \leq 2a_2 - 1; x_1^2x_2^{i_2}\partial_1, a_2 \leq i_2 \leq 2a_2 - 3; x_1^3x_2^{i_2}\partial_1, a_2 - 2 \leq i_2 \leq 2a_2 - 3;$$
$$x_2^{i_2}\partial_2, 2a_2 - 2 \leq i_2 \leq 3a_2 - 4; x_1x_2^{i_2}\partial_2, a_2 + 1 \leq i_2 \leq 2a_2 - 1;$$
$$x_1^2x_2^{i_2}\partial_2, a_2 - 1 \leq i_2 \leq 2a_2 - 3; x_1^3x_2^{i_2}\partial_2, a_2 - 1 \leq i_2 \leq 2a_2 - 3.$$

Therefore, we obtain the following formula

$$\rho_2(V) = 12a_2 - 10.$$

Similarly, when $a_1 \geq 5, a_2 = 4$, we obtain the following formula

$$\rho_2(V) = 12a_1 - 10.$$

When $a_1 = 3, a_2 \geq 4$, the Lie algebra $\mathcal{L}_2(V)$ has the following basis:

$$x_2^{i_2+a_2-1}\partial_1 - \frac{3}{a_2}x_1^2x_2^{i_2}\partial_2, 0 \leq i_2 \leq a_2 - 2; x_1x_2^{i_2}\partial_1 + \frac{3}{a_2}x_2^{i_2+1}\partial_2, 0 \leq i_2 \leq a_2 - 1;$$
$$x_1^2x_2^{i_2}\partial_1 + \frac{3}{a_2}x_1x_2^{i_2+1}\partial_2, 0 \leq i_2 \leq a_2 - 3; x_2^{i_2}\partial_1, 2a_2 - 2 \leq i_2 \leq 2a_2 - 1;$$
$$x_1x_2^{i_2}\partial_1, a_2 \leq i_2 \leq 2a_2 - 3;$$
$$x_1^2x_2^{i_2}\partial_1, a_2 - 2 \leq i_2 \leq 2a_2 - 3; x_2^{i_2}\partial_2, a_2 + 1 \leq i_2 \leq 2a_2 - 1;$$
$$x_1x_2^{i_2}\partial_2, a_2 - 1 \leq i_2 \leq 2a_2 - 3; x_1^2x_2^{i_2}\partial_2, a_2 - 1 \leq i_2 \leq 2a_2 - 3.$$

Therefore, we obtain the following formula

$$\rho_2(V) = 8a_2 - 6.$$

Similarly, when $a_1 \geq 4, a_2 = 3$, we obtain the following formula

$$\rho_2(V) = 8a_1 - 6.$$

Therefore, when $a_1 \geq 3, a_2 \geq 3, a_1 \neq a_2$, the formula (6) holds. □

Proposition 4.8. *Let $(V, 0)$ be a binomial isolated singularity of type B which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}$ ($a_1 \geq 1, a_2 \geq 2$) with weight type $(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}; 1)$. Then*

$$\rho_2(V) = \begin{cases} 4a_1a_2 - 4a_2 + 7; & a_1 \geq 2,\ a_2 \geq 3,\ \textit{and } a_1 = a_2 - 1 \\ 6; & a_1 = 1, a_2 \geq 2 \\ 8a_1 - 3; & a_1 \geq 2, a_2 = 2 \\ 4a_2 + 6; & a_1 = 2,\ a_2 \geq 4 \\ 4a_1a_2 - 4a_2 + 6; & a_1 \geq 3,\ a_2 \geq 3,\ \textit{and } a_1 \neq a_2 - 1. \end{cases}$$

Furthermore, we have

$$\rho_2(V) \leq \begin{cases} 6; & a_1 = 1, a_2 = 2 \\ 4a_2 - 3; & \frac{a_1a_2}{a_2-1} = 2, a_2 \geq 3 \\ 8a_1 - 3; & a_1 \geq 2, a_2 = 2 \\ 4a_1a_2 - 4a_2 + 7; & \frac{a_1a_2}{a_2-1} \geq 3,\ a_2 \geq 3,\ \textit{and } a_1 = a_2 - 1 \\ 4a_1a_2 - 4a_2 + 6; & \frac{a_1a_2}{a_2-1} \geq 3,\ a_2 \geq 3,\ \textit{and } a_1 \neq a_2 - 1. \end{cases}$$

Proof. It follows from Lemma 4.1 that the local algebra

$$\mathcal{M}_2(V) = \mathcal{O}_2 \Big/ \Big\langle f, \left(\frac{\partial f}{\partial x_1}\right)^3, \left(\frac{\partial f}{\partial x_2}\right)^3, \left(\frac{\partial f}{\partial x_1}\right)^2 \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_1}\left(\frac{\partial f}{\partial x_2}\right)^2,$$
$$\left(\frac{\partial f}{\partial x_2}\right)^2 \frac{\partial^2 f}{\partial x_1^2} + \left(\frac{\partial f}{\partial x_1}\right)^2 \frac{\partial^2 f}{\partial x_2^2} - 2\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}\frac{\partial^2 f}{\partial x_1 \partial x_2}\Big\rangle,$$

has a monomial basis of the form:

(1) When $a_1 \geq 3$, $a_2 \geq 3$, and $a_1 = a_2 - 1$,
$\{x_1^{i_1}x_2^{i_2}, 1 \leq i_1 \leq a_1 - 1, 0 \leq i_2 \leq 2a_2 - 2; x_1^{a_1-3}x_2^{i_2}, 2a_2 - 1 \leq i_2 \leq 2a_2; x_1^{a_1-2}x_2^{2a_2-1};$
$x_2^{i_2}, 0 \leq i_2 \leq 2a_2 - 2; x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq a_1 - 4, 2a_2 - 1 \leq i_2 \leq 3a_2 - 3;$
$x_1^{i_1}, a_1 \leq i_1 \leq 3a_1 - 1\}$.
(2) When $a_1 = 1, a_2 \geq 2$, $\{1, x_1, x_2, x_2^2, x_1^2\}$.
(3) When $a_1 \geq 2$, $a_2 = 2$,
$\{x_1^{i_1}x_2^{i_2}, 1 \leq i_1 \leq a_1 - 1, 0 \leq i_2 \leq 2; x_2^{i_2}, 0 \leq i_2 \leq 2; x_1^{i_1}x_2^3, 0 \leq i_1 \leq a_1 - 3;$
$x_1^{i_1}, a_1 \leq i_1 \leq 3a_1 - 1\}$.
(4) When $a_1 = 2$, $a_2 \geq 4$,
$\{x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq 1, 0 \leq i_2 \leq a_2 + 1; x_2^{i_2}, a_2 + 2 \leq i_2 \leq 2a_2 - 2; x_1^{i_1}, 2 \leq i_1 \leq 5\}$.
(5) When $a_1 \geq 3$, $a_2 \geq 3$, and $a_1 \neq a_2 - 1$.
$\{x_1^{i_1}x_2^{i_2}, 1 \leq i_1 \leq a_1 - 1, 0 \leq i_2 \leq 2a_2 - 2; x_1^{a_1-3}x_2^{i_2}, 2a_2 - 1 \leq i_2 \leq 2a_2;$
$x_2^{i_2}, 0 \leq i_2 \leq 2a_2 - 2; x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq a_1 - 4, 2a_2 - 1 \leq i_2 \leq 3a_2 - 3;$
$x_1^{i_1}, a_1 \leq i_1 \leq 3a_1 - 1\}$.

Remark 4.9. It is easy to check that, when $a_1 = 2, a_2 = 3$, the monomial base is consistent with the list in (1) if we look x_1^{-1} as 0.

Case (1): When $a_1 \geq 3$, $a_2 \geq 3$, and $a_1 = a_2 - 1$, we obtain the following basis of $\mathcal{L}_2(V)$:

$$x_2^{a_2-2+i_2}\partial_1 - x_1^{a_1-1}x_2^{i_2}\partial_2, 1 \leq i_2 \leq a_2 - 1;$$
$$x_1^{i_1}x_2^{i_2}\partial_1, 0 \leq i_1 \leq a_1 - 4, 2a_2 - 2 \leq i_2 \leq 3a_2 - 3;$$
$$x_1^{i_1}x_2^{2a_2-3}\partial_1 - \frac{1}{a_2(a_2-2)}x_1^{2a_1}\partial_2, 1 \leq i_1 \leq a_1 - 3;$$
$$x_1^{i_1}x_2^{i_2}\partial_1 + x_1^{i_1-1}x_2^{i_2+1}\partial_2, 1 \leq i_1 \leq a_1 - 3,$$

$$0 \le i_2 \le 2a_2 - 4; x_1^{a_1-3} x_2^{i_2}\partial_1, 2a_2 - 2 \le i_2 \le 2a_2;$$
$$x_1^{a_1-2} x_2^{i_2}\partial_1 + x_1^{a_1-3} x_2^{i_2+1}\partial_2, 0 \le i_2 \le a_2;$$
$$x_1^{a_1-2} x_2^{i_2}\partial_1, a_2 + 1 \le i_2 \le 2a_2 - 2; x_1^{a_1-1} x_2^{i_2}\partial_1 + x_1^{a_1-2} x_2^{i_2+1}\partial_2, 0 \le i_2 \le a_2 - 1;$$
$$x_1^{a_1-2} x_2^{2a_2-1}\partial_1;$$
$$x_1^{a_1-1} x_2^{i_2}\partial_1, a_2 \le i_2 \le 2a_2 - 2; x_1^{i_1}\partial_1, 2a_1 \le i_1 \le 3a_1 - 1; x_1^{a_1}\partial_1 + x_1^{a_1-1} x_2\partial_2;$$
$$x_1^{a_1+1+i_1}\partial_1 - x_1^{i_1} x_2^{a_2}\partial_2, 0 \le i_1 \le a_1 - 2;$$
$$\frac{1}{a_2(a_2-2)} x_1^{2a_1+i_1} + x_1^{i_1} x_2^{2a_2-2}\partial_2, 0 \le i_1 \le a_1 - 4;$$
$$x_1^{i_1} x_2^{i_2}\partial_2, 0 \le i_1 \le a_1 - 4, 2a_2 - 1 \le i_2 \le 3a_2 - 3; x_1^{a_1-3} x_2^{i_2}\partial_2, a_2 + 2 \le i_2 \le 2a_2;$$
$$x_1^{a_1-2} x_2^{i_2}\partial_2, a_2 + 1 \le i_2 \le 2a_2 - 1; x_1^{a_1-1} x_2^{i_2}\partial_2, a_2 \le i_2 \le 2a_2 - 2;$$
$$x_1^{3a_1-3}\partial_2; x_1^{3a_1-2}\partial_2; x_1^{3a_1-1}\partial_2.$$

Therefore, we obtain the following formula

$$\rho_2(V) = 4a_1a_2 - 4a_2 + 7. \tag{7}$$

Remark 4.10. (1) When $a_1 = 3, a_4 = 4$, one looks $x_1^{i_1} x_2^{i_2}\partial_1, 0 \le i_1 \le a_1 - 4, 2a_2 - 2 \le i_2 \le 3a_2 - 3$ as 0 above. We use this similar convention throughout the paper.

(2) It is easy to check that the case of $a_1 = 2, a_2 = 3$ also satisfies the formula (7) above, it has the following basis:

$$x_2^2\partial_1 - x_1x_2\partial_2; x_2^3\partial_1 - x_1x_2^2\partial_2; x_2^4\partial_1; x_2^5\partial_1; x_1\partial_1 + x_2\partial_2; x_1x_2\partial_1 + x_2^2\partial_2; x_1x_2^2\partial_1 + x_2^3\partial_2;$$
$$x_1x_2^3\partial_1; x_1x_2^4\partial_1;$$
$$x_1^2\partial_1 + x_1x_2\partial_2; x_1^3\partial_1 - x_2^3\partial_2; x_1^4\partial_1; x_1^5\partial_1; x_2^4\partial_2; x_2^5\partial_2; x_1x_2^3\partial_2; x_1x_2^4\partial_2; x_1^4\partial_2; x_1^5\partial_2.$$
$$\rho_2(V) = 19.$$

Case (2): When $a_1 = 1$, $a_2 \ge 2$, we obtain the following basis of $\mathcal{L}_2(V)$:

$$x_2\partial_1 - x_2\partial_2; x_2^2\partial_1; x_1^2\partial_1; x_1\partial_1 + x_2\partial_2; x_2^2\partial_2; x_1^2\partial_2.$$

Therefore, we obtain the following formula

$$\rho_2(V) = 6.$$

Case (3): When $a_1 \ge 3$, $a_2 = 2$, we obtain the following basis of $\mathcal{L}_2(V)$:

$$x_2\partial_1 - a_1x_1^{a_1-1}x_2\partial_2; x_2^2\partial_1; x_2^3\partial_1; x_1^{i_1} x_2^{i_2}\partial_1 + a_1x_1^{i_1-1} x_2^{i_2+1}\partial_2, 1 \le i_1 \le a_1 - 2, 0 \le i_2 \le 1;$$
$$x_1^{i_1} x_2^{i_2}\partial_1, 1 \le i_1 \le a_1 - 3, 2 \le i_2 \le 3; x_1^{a_1+1+i_1}\partial_1 - a_1x_1^{i_1} x_2^2\partial_2, 0 \le i_1 \le a_1 - 3; x_1^{i_1}\partial_1,$$

$$2a_1-1\le i_1\le 3a_1-1; x_1^{a_1-2}x_2^2\partial_1; x_1^{a_1-1}\partial_1+a_1x_1^{a_1-2}x_2\partial_2; x_1^{a_1-1}x_2\partial_1;$$
$$x_1^{a_1-1}x_2^2\partial_1; x_1^{a_1}\partial_1+a_1x_1^{a_1-1}x_2\partial_2;$$
$$x_1^{i_1}\partial_2, 2a_1\le i_1\le 3a_1-1; x_1^{i_1}x_2^3\partial_2, 0\le i_1\le a_1-3, x_1^{a_1-2}x_2^2\partial_2; x_1^{a_1-1}x_2^2\partial_2.$$

Therefore, we obtain the following formula

$$\rho_2(V)=8a_1-3. \tag{8}$$

When $a_1=2, a_2=2$, the Lie algebra $\mathcal{L}_2(V)$ has the following basis:

$$x_2\partial_1-2x_1x_2\partial_2; x_2^2\partial_1; x_1\partial_1+2x_2\partial_2; x_1x_2\partial_1; x_1x_2^2\partial_1; x_1^2\partial_1+2x_1x_2\partial_2; x_1^3\partial_1; x_1^4\partial_1;$$
$$x_1^5\partial_1; x_2^2\partial_2; x_1x_2^2\partial_2;$$

$x_1^4\partial_2; x_1^5\partial_2$. Thus, we obtain the following formula

$$\rho_2(V)=13.$$

It also satisfies the formula (8) above.

Case (4): When $a_1=2$, $a_2\ge 4$, we obtain the following basis of $\mathcal{L}_2(V)$:

$$x_2^{a_2-1}\partial_1-\frac{2}{a_2-1}x_1x_2\partial_2; x_2^{a_2}\partial_1-\frac{2}{a_2-1}x_1x_2^2\partial_2; x_2^{i_2}\partial_1, a_2+1\le i_2\le 2a_2-2;$$
$$x_1x_2^{i_2}\partial_1+\frac{2}{a_2-1}x_2^{i_2+1}\partial_2, 0\le i_2\le a_2-2; x_1x_2^{i_2}\partial_1, a_2-1\le i_2\le a_2+1; x_1^2\partial_1+\frac{2}{a_2-1}x_1x_2\partial_2;$$
$$x_1^{i_1}\partial_1, 3\le i_1\le 5; x_2^{i_2}\partial_2; a_2\le i_2\le 2a_2-2; x_1^4\partial_2; x_1^5\partial_2; x_1x_2^{i_2}\partial_2; 3\le i_2\le a_2+1.$$

Therefore, we obtain the following formula

$$\rho_2(V)=4a_2+6.$$

Case (5): When $a_1\ge 3$, $a_2\ge 4$, and $a_1\ne a_2-1$, we obtain the following basis of $\mathcal{L}_2(V)$:

$$x_2^{a_2-2+i_2}\partial_1-\frac{a_1}{a_2-1}x_1^{a_1-1}x_2^{i_2}\partial_2, 1\le i_2\le a_2-1;$$
$$x_1^{i_1}x_2^{i_2}\partial_1, 0\le i_1\le a_1-4, 2a_2-2\le i_2\le 3a_2-3;$$
$$x_1^{i_1}x_2^{2a_2-3}\partial_1-\frac{a_1}{a_2(a_2-1)(a_2-2)}x_1^{2a_1}\partial_2, 1\le i_1\le a_1-3;$$
$$x_1^{i_1}x_2^{i_2}\partial_1+\frac{a_1}{a_2-1}x_1^{i_1-1}x_2^{i_2+1}\partial_2, 1\le i_1\le a_1-3,$$
$$0\le i_2\le 2a_2-4; x_1^{a_1-3}x_2^{i_2}\partial_1, 2a_2-2\le i_2\le 2a_2;$$
$$x_1^{a_1-2}x_2^{i_2}\partial_1+\frac{a_1}{a_2-1}x_1^{a_1-3}x_2^{i_2+1}\partial_2, 0\le i_2\le a_2;$$

$$x_1^{a_1-2}x_2^{i_2}\partial_1, a_2+1 \le i_2 \le 2a_2-2; x_1^{a_1-1}x_2^{i_2}\partial_1 + \frac{a_1}{a_2-1}x_1^{a_1-2}x_2^{i_2+1}\partial_2, 0 \le i_2 \le a_2-2;$$
$$x_1^{a_1-1}x_2^{i_2}\partial_1, a_2-1 \le i_2 \le 2a_2-2;$$
$$x_1^{i_1}\partial_1, 2a_1-1 \le i_1 \le 3a_1-1; x_1^{a_1}\partial_1 + \frac{a_1}{a_2-1}x_1^{a_1-1}x_2\partial_2;$$
$$x_1^{a_1+1+i_1}\partial_1 - \frac{a_1}{a_2-1}x_1^{i_1}x_2^{a_2}\partial_2, 0 \le i_1 \le a_1-3;$$
$$(\frac{1}{a_2(a_2-2)}x_1^{2a_1+i_1} + x_1^{i_1}x_2^{2a_2-2})\partial_2, 0 \le i_1 \le a_1-4;$$
$$x_1^{i_1}x_2^{i_2}\partial_2, 0 \le i_1 \le a_1-4, 2a_2-1 \le i_2 \le 3a_2-3; x_1^{a_1-3}x_2^{i_2}\partial_2, a_2+2 \le i_2 \le 2a_2;$$
$$x_1^{a_1-2}x_2^{i_2}\partial_2, a_2 \le i_2 \le 2a_2-2; x_1^{a_1-1}x_2^{i_2}\partial_2, a_2 \le i_2 \le 2a_2-2;$$
$$x_1^{3a_1-3}\partial_2; x_1^{3a_1-2}\partial_2; x_1^{3a_1-1}\partial_2.$$

Therefore we obtain the following formula

$$\rho_2(V) = 4a_1a_2 - 4a_2 + 6. \tag{9}$$

When $a_1 \ge 3, a_2 = 3$, this case also satisfies the formula (9) above, it has the following basis:

$$x_2^{1+i_2}\partial_1 - \frac{a_1}{2}x_1^{a_1-1}x_2^{i_2}\partial_2, 1 \le i_2 \le 2; x_1^{i_1}x_2^{i_2}\partial_1, 0 \le i_1 \le a_1-4, 4 \le i_2 \le 6;$$
$$x_1^{i_1}x_2^3\partial_1 - \frac{a_1}{2(a_1-2)}x_1^{2a_1}\partial_2, 1 \le i_1 \le a_1-2; x_1^{a_1-2}x_2^4\partial_1;$$
$$x_1^{i_1}x_2^{i_2}\partial_1 + \frac{a_1}{2}x_1^{i_1-1}x_2^{i_2+1}\partial_2, 1 \le i_1 \le a_1-2;$$
$$0 \le i_2 \le 2; x_1^{a_1-3}x_2^{i_2}\partial_1, 4 \le i_2 \le 6; x_1^{a_1-1}x_2^{i_2}\partial_1 + \frac{a_1}{2}x_1^{a_1-2}x_2^{i_2+1}\partial_2, 0 \le i_2 \le 1;$$
$$x_1^{a_1-1}x_2^{i_2}\partial_1, 2 \le i_2 \le 4; x_1^{i_1}\partial_1, 2a_1-1 \le i_1 \le 3a_1-1; x_1^{a_1}\partial_1 + \frac{a_1}{2}x_1^{a_1-1}x_2\partial_2;$$
$$x_1^{a_1+1+i_1}\partial_1 - \frac{a_1}{2}x_1^{i_1}x_2^3\partial_2, 0 \le i_1 \le a_1-3; (\frac{1}{3}x_1^{2a_1+i_1} + x_1^{i_1}x_2^4)\partial_2, 0 \le i_1 \le a_1-3;$$
$$x_1^{i_1}x_2^{i_2}\partial_2, 0 \le i_1 \le a_1-3, 5 \le i_2 \le 6; x_1^{a_1-2}x_2^{i_2}\partial_2, 3 \le i_2 \le 4; x_1^{a_1-1}x_2^{i_2}\partial_2, 3 \le i_2 \le 4;$$
$$x_1^{3a_1-2}\partial_2; x_1^{3a_1-1}\partial_2.$$

Thus, we obtain

$$\rho_2(V) = 12a_1 - 6.$$

Therefore, the first part of Proposition 4.8 has been proved.

It follows from Proposition 4.5 that

$$h_2(a_1,a_2)=\begin{cases}6; & a_1=2, a_2=2\\ 4a_2-3; & a_1=2, a_2\geq 3\\ 4a_1-3; & a_1\geq 3, a_2=2\\ 4a_1a_2-4(a_1+a_2)+7; & a_1\geq 3,\ a_2\geq 3, \text{ and } a_1=a_2\\ 4a_1a_2-4(a_1+a_2)+6; & a_1\geq 3,\ a_2\geq 3, \text{ and } a_1\neq a_2.\end{cases}$$

After plugging into the weight type $(\frac{a_2-1}{a_1a_2},\frac{1}{a_2};1)$ of binomial isolated singularity of type B, we obtain

$$h_2(\frac{1}{w_1},\frac{1}{w_2})=\begin{cases}6; & a_1=1, a_2=2\\ 4a_2-3; & \frac{a_1a_2}{a_2-1}=2, a_2\geq 3\\ 8a_1-3; & a_1\geq 2, a_2=2\\ 4a_1a_2-4a_2+7; & \frac{a_1a_2}{a_2-1}\geq 3,\ a_2\geq 3, \text{ and } a_1=a_2-1\\ 4a_1a_2-4a_2+6; & \frac{a_1a_2}{a_2-1}\geq 3,\ a_2\geq 3, \text{ and } a_1\neq a_2-1.\end{cases}$$

The Conjecture 1.8 claims

$$h_2(\frac{1}{w_1},\frac{1}{w_2})\geq\begin{cases}4a_1a_2-4a_2+7; & a_1\geq 2,\ a_2\geq 3, \text{ and } a_1=a_2-1\\ 6; & a_1=1, a_2\geq 2\\ 8a_1-3; & a_1\geq 2, a_2=2\\ 4a_2+6; & a_1=2,\ a_2\geq 4\\ 4a_1a_2-4a_2+6; & a_1\geq 3,\ a_2\geq 3, \text{ and } a_1\neq a_2-1.\end{cases}\tag{10}$$

It is easy to check that (10) holds. Thus the second part of Proposition 4.8 has been proved. □

Proposition 4.11. *Let $(V,0)$ be a binomial isolated singularity of type C which is defined by $f=x_1^{a_1}x_2+x_2^{a_2}x_1$ $(a_1\geq a_2\geq 1)$ with weight type $(\frac{a_2-1}{a_1a_2-1},\frac{a_1-1}{a_1a_2-1};1)$. Then*

$$\rho_2(V)=\begin{cases}6a_1; & a_1\geq 1, a_2=1\\ 4a_1a_2+3; & a_1\geq 2,\ a_2\geq 2,\ \textit{and } a_1=a_2\\ 4a_1a_2+2; & a_1\geq 2,\ a_2\geq 2,\ \textit{and } a_1\neq a_2.\end{cases}$$

Furthermore, we have

$$\rho_2(V)\leq\begin{cases}6; & \frac{a_1a_2-1}{a_2-1}=2, \frac{a_1a_2-1}{a_1-1}=2\\ \frac{4(a_1a_2-1)}{a_1-1}-3; & \frac{a_1a_2-1}{a_2-1}=2, \frac{a_1a_2-1}{a_1-1}\geq 3\\ \frac{4(a_1a_2-1)}{a_2-1}-3; & \frac{a_1a_2-1}{a_2-1}\geq 3, \frac{a_1a_2-1}{a_1-1}=2\\ 4a_1a_2+3; & a_1\geq 2, \textit{and } a_1=a_2\\ 4a_1a_2+2; & \frac{a_1a_2-1}{a_2-1}\geq 3, \frac{a_1a_2-1}{a_1-1}\geq 3,\ \textit{and } a_1\neq a_2.\end{cases}$$

Proof. It follows from Lemma 4.1 that the local algebra

$$\mathcal{M}_2(V) = \mathcal{O}_2 \Big/ \Big\langle f, \left(\frac{\partial f}{\partial x_1}\right)^3, \left(\frac{\partial f}{\partial x_2}\right)^3, \left(\frac{\partial f}{\partial x_1}\right)^2 \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_1}\left(\frac{\partial f}{\partial x_2}\right)^2,$$
$$\left(\frac{\partial f}{\partial x_2}\right)^2 \frac{\partial^2 f}{\partial x_1^2} + \left(\frac{\partial f}{\partial x_1}\right)^2 \frac{\partial^2 f}{\partial x_2^2} - 2\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}\frac{\partial^2 f}{\partial x_1 \partial x_2}\Big\rangle,$$

has a monomial basis of the form:

(1) When $a_1 \geq 3$, $a_2 \geq 3$, and $a_1 = a_2$,
$\{x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq a_1 - 1; 0 \leq i_2 \leq 2a_2; x_1^{i_1}, a_1 \leq i_1 \leq 3a_1 - 1; x_2^{i_2}, 2a_2 + 1 \leq i_2 \leq 4a_2 - 4;$
$x_1^{i_1}x_2^{i_2}, 1 \leq i_1 \leq a_1 - 2, 2a_2 + 1 \leq i_2 \leq 3a_2 - 3; x_1 x_2^{3a_2-2}\}$.
(2) When $a_1 = 2$, $a_2 = 2$,
$\{x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq 1; 0 \leq i_2 \leq 4; x_1^{i_1}, 2 \leq i_1 \leq 5; x_2^5\}$.
(3) When $a_1 \geq 1$, $a_2 = 1$, $\{x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq a_1 - 1; 0 \leq i_2 \leq 2; x_1^{i_1}, a_1 \leq i_1 \leq 3a_1 - 1\}$.
(4) When $a_1 \geq 3$, $a_2 \geq 3$, and $a_1 \neq a_2$,
$\{x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq a_1 - 1; 0 \leq i_2 \leq 2a_2; x_1^{i_1}, a_1 \leq i_1 \leq 3a_1 - 1; x_2^{i_2},$
$2a_2 + 1 \leq i_2 \leq 4a_2 - 4; x_1^{i_1}x_2^{i_2}, 1 \leq i_1 \leq a_1 - 2, 2a_2 + 1 \leq i_2 \leq 3a_2 - 3\}$.
(5) When $a_1 \geq 3$, $a_2 = 2$,
$\{x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq a_1 - 1; 0 \leq i_2 \leq 3; x_1^{i_1}, a_1 \leq i_1 \leq 3a_1 - 1; x_2^{i_2}, 4 \leq i_2 \leq 5\}$.

Case (1): When $a_1 \geq 3$, $a_3 \geq 3$, and $a_1 = a_2$, we obtain the following basis of $\mathcal{L}_2(V)$:

$$x_2^{i_2+a_2-2}\partial_1 + a_1(a_2-2)x_1^{a_1-2}x_2^{i_2}\partial_2, a_2 + 2 \leq i_2 \leq 2a_2 - 2; x_2^{i_2}\partial_1, 3a_2 - 3 \leq i_2 \leq 4a_2 - 4;$$
$$x_1^{i_1}x_2^{i_2}\partial_1 + x_1^{i_1-1}x_2^{i_2+1}\partial_2, 2 \leq i_1 \leq a_1 - 1, 0 \leq i_2 \leq 2a_2 - 4;$$
$$x_1 x_2^{i_2}\partial_1 + x_2^{i_2+1}\partial_2, 0 \leq i_2 \leq a_2 - 2;$$
$$x_1 x_2^{i_2+a_2-2}\partial_1 - x_1^{a_1-1}x_2^{i_2}\partial_2; 1 \leq i_2 \leq a_2; x_1^2 x_2^{2a_2-3}\partial_1 + x_1 x_2^{2a_2-2}\partial_2;$$
$$x_1^{i_1}x_2^{2a_2-3}\partial_1 - \frac{x_1^{i_1+2a_1-3}}{a_1(a_2-1)}\partial_2,$$
$$3 \leq i_1 \leq a_1 - 1; x_1 x_2^{i_2}\partial_1, 2a_2 - 1 \leq i_2 \leq 3a_2 - 2;$$
$$x_1^{i_1}x_2^{i_2}\partial_1, 2 \leq i_1 \leq a_1 - 2, 2a_2 - 2 \leq i_2 \leq 3a_2 - 3;$$
$$x_1^{a_1-1}x_2^{i_2}\partial_1, 2a_2 - 2 \leq i_2 \leq 2a_2; x_1^{a_1}\partial_1 + x_1^{a_1-1}x_2\partial_2; x_1^{i_1+a_1}\partial_1 - x_1^{i_1}x_2^{a_2}\partial_2, 1 \leq i_1 \leq a_1 - 1;$$
$$x_1^{i_1}\partial_1; 2a_1 \leq i_1 \leq 3a_1 - 1; (x_1^{a_1-1}x_2^{i_2} + x_2^{i_2+a_2-1})\partial_2; 1 \leq i_2 \leq a_2; x_2^{i_2}\partial_2; 2a_2 \leq i_2 \leq 4a_2 - 4;$$
$$x_1^{i_1}x_2^{i_2}\partial_2; 1 \leq i_1 \leq a_1 - 2; 2a_2 - 1 \leq i_2 \leq 3a_2 - 3; x_1 x_2^{3a_2-2}\partial_2;$$
$$\left(\frac{x_1^{i_1+2a_2-2}}{a_2(a_1-2)} + x_1^{i_1}x_2^{2a_2-2}\right)\partial_2; 2 \leq i_2 \leq a_1 - 2;$$

$$x_1^{a_1-1}x_2^{i_2}\partial_2, a_2+1 \le i_2 \le 2a_2; x_1^{i_1}\partial_2; 3a_1-3 \le i_1 \le 3a_1-1.$$

Therefore we obtain the following formula

$$\rho_2(V) = 4a_1a_2 + 3.$$

Case (2): When $a_1 = 2$, $a_2 = 2$, we obtain the following basis of $\mathcal{L}_2(V)$:

$$x_2^4\partial_1; x_2^5\partial_1; x_1\partial_1 + x_2\partial_2; x_1x_2\partial_1 - x_1x_2\partial_2; x_1x_2^2\partial_1 - x_1x_2^2\partial_2; x_1x_2^3\partial_1; x_1x_2^4\partial_1;$$
$$x_1^2\partial_1 + x_1x_2\partial_2; x_1^3\partial_1 - x_1x_2^2\partial_2;$$
$$x_1^4\partial_1; x_1^5\partial_1; (x_1x_2 + x_2^2)\partial_2; (x_1x_2^2 + x_2^3)\partial_2; x_2^4\partial_2; x_2^5\partial_2; x_1x_2^3\partial_2; x_1x_2^4\partial_2; x_1^4\partial_2; x_2^5\partial_2.$$

Therefore, we obtain the following formula

$$\rho_2(V) = 19.$$

Case (3): When $a_1 \ge 3$, $a_2 = 1$, we obtain the following basis of $\mathcal{L}_2(V)$:

$$x_1^{i_1}x_2^2\partial_1, 0 \le i_1 \le a_1-1; (x_1^{i_1} + x_1^{i_1+a_1-1})\partial_1, a_1+2 \le i_1 \le 2a_1;$$
$$(x_1^{i_1} + x_1^{i_1+3a_1-3})\partial_1, 1 \le i_1 \le 2;$$
$$(x_1^{i_1} - x_1^{2a_1-2+i_1})\partial_1, 3 \le i_1 \le a_1+1; x_1^{i_1}x_2^{i_2}\partial_2, 0 \le i_1 \le a_1-1, 1 \le i_2 \le 2;$$
$$(x_1^2 - 2x_1^{3a_1-1} - 3x_1^{2a_1})\partial_2;$$
$$(x_1^{i_1} + 2x_1^{a_1-1+i_1} + x_1^{2a_1-2+i_1})\partial_2, 3 \le i_1 \le a_1+1.$$

Therefore, we obtain the following formula

$$\rho_2(V) = 6a_1. \tag{11}$$

Remark 4.12. It is easy to check that the cases $a_1 = 1, a_2 = 1$, and $a_1 = 2, a_2 = 1$ also satisfy the formula (11) above. Their basis are as follows:
for $a_1 = 1, a_2 = 1$,

$$x_2^2\partial_1; x_1\partial_1; x_1^2\partial_1; x_2\partial_2; x_2^2\partial_2; x_1^2\partial_2,$$

for $a_1 = 2, a_2 = 1$,

$$x_2^2\partial_1; (-x_1^5 + x_1)\partial_1; x_1x_2^2\partial_1; (x_1^5 + x_1^2)\partial_1; (-x_1^5 + x_1^3)\partial_1;$$
$$(x_1^5 + x_1^4)\partial_1; x_2\partial_2; x_2^2\partial_2; x_1x_2\partial_2; x_1x_2^2\partial_2; (-2x_1^5 - 3x_1^4 + x_1^2)\partial_2; (x_1^5 + 2x_1^4 + x_1^3)\partial_2.$$

Case (4): When $a_1 \geq 4$, $a_2 \geq 4$, and $a_1 \neq a_2$, we obtain the following basis of $\mathcal{L}_2(V)$:

$$x_2^{i_2+a_2-2}\partial_1 + \frac{a_1(a_1-1)(a_1-2)}{a_2-1}x_1^{a_1-2}x_2^{i_2}\partial_2, a_2+2 \leq i_2 \leq 2a_2-2;$$
$$x_2^{i_2}\partial_1, 3a_2-3 \leq i_2 \leq 4a_2-4;$$
$$x_1^{i_1}x_2^{i_2}\partial_1 + \frac{a_1-1}{a_2-1}x_1^{i_1-1}x_2^{i_2+1}\partial_2, 2 \leq i_1 \leq a_1-1, 0 \leq i_2 \leq 2a_2-4; x_1x_2^{i_2}\partial_1 + \frac{a_1-1}{a_2-1}x_2^{i_2+1}\partial_2,$$
$$0 \leq i_2 \leq a_2-2; x_1x_2^{i_2+a_2-2}\partial_1 - \frac{a_1-1}{a_2-1}x_1^{a_1-1}x_2^{i_2}\partial_2, 1 \leq i_2 \leq a_2;$$
$$x_1^2x_2^{2a_2-3}\partial_1 + \frac{a_1-1}{a_2-1}x_1x_2^{2a_2-2}\partial_2;$$
$$x_1^{i_1}x_2^{2a_2-3}\partial_1 - \frac{(a_1-1)x_1^{i_1+2a_1-3}}{a_2(a_2-1)(a_2-2)}\partial_2, 3 \leq i_1 \leq a_1-1;$$
$$x_1x_2^{i_2}\partial_1, 2a_2-1 \leq i_2 \leq 3a_2-3; x_1^{i_1}x_2^{i_2}\partial_1, 2 \leq i_1 \leq a_1-2,$$
$$2a_2-2 \leq i_2 \leq 3a_2-3; x_1^{a_1-1}x_2^{i_2}\partial_1, 2a_2-2 \leq i_2 \leq 2a_2; x_1^{a_1}\partial_1 + \frac{a_1-1}{a_2-1}x_1^{a_1-1}x_2\partial_2; x_1^{i_1+a_1}\partial_1$$
$$-\frac{a_1-1}{a_2-1}x_1^{i_1}x_2^{a_2}\partial_2, 1 \leq i_1 \leq a_1-1; x_1^{i_1}\partial_1, 2a_1 \leq i_1 \leq 3a_1-1;$$
$$(x_1^{a_1-1}x_2^{i_2} + x_2^{i_2+a_2-1})\partial_2, 1 \leq i_2 \leq a_2;$$
$$x_2^{i_2}\partial_2, 2a_2 \leq i_2 \leq 4a_2-4; x_1^{i_1}x_2^{i_2}\partial_2, 1 \leq i_1 \leq a_1-2, 2a_2-1 \leq i_2 \leq 3a_2-3; x_1x_2^{3a_2-2}\partial_2;$$
$$(\frac{x_1^{i_1+2a_2-2}}{a_2(a_1-2)} + x_1^{i_1}x_2^{2a_2-2})\partial_2, 2 \leq i_1 \leq a_1-2; x_1^{a_1-1}x_2^{i_2}\partial_2, a_2+1 \leq i_2 \leq 2a_2;$$
$$x_1^{i_1}\partial_2, 3a_1-3 \leq i_1 \leq 3a_1-1.$$

Therefore, we obtain

$$\rho_2(V) = 4a_1a_2 + 2. \tag{12}$$

Remark 4.13. It is easy to check that the case $a_1 = 3, a_2 \geq 4$ also satisfies the formula (12) above. The basis is as follows:

$$x_2^{i_2+a_2-2}\partial_1 + \frac{6}{a_2-1}x_1x_2^{i_2}\partial_2, a_2+2 \leq i_2 \leq 2a_2-2; x_2^{i_2}\partial_1, 3a_2-3 \leq i_2 \leq 4a_2-4;$$
$$x_1^2x_2^{i_2}\partial_1 + \frac{2}{a_2-1}x_1x_2^{i_2+1}\partial_2, 0 \leq i_2 \leq 2a_2-4; x_1x_2^{i_2}\partial_1 + \frac{2}{a_2-1}x_2^{i_2+1}\partial_2,$$
$$0 \leq i_2 \leq a_2-2; x_1x_2^{i_2+a_2-2}\partial_1 - \frac{2}{a_2-1}x_1^2x_2^{i_2}\partial_2; 1 \leq i_2 \leq a_2-1;$$
$$x_1^2x_2^{2a_2-3}\partial_1 + \frac{2}{a_2-1}x_1x_2^{2a_2-2}\partial_2;$$
$$x_1x_2^{i_2}\partial_1, 2a_2-2 \leq i_2 \leq 3a_2-3; x_1^2x_2^{i_2}\partial_1, 2a_2-2 \leq i_2 \leq 2a_2; x_1^3\partial_1 + \frac{2}{a_2-1}x_1^2x_2\partial_2; x_1^4\partial_1$$

$$-\frac{2}{a_2-1}x_1x_2^{a_2}\partial_2; x_1^{i_1}\partial_1, 5 \le i_1 \le 8; (x_1^2x_2^{i_2} + x_2^{i_2+a_2-1})\partial_2, 1 \le i_2 \le a_2-1;$$
$$x_2^{i_2}\partial_2, 2a_2-1 \le i_2 \le 4a_2-4; x_1x_2^{i_2}\partial_2, 2a_2-1 \le i_2 \le 3a_2-3;$$
$$x_1^2x_2^{i_2}\partial_2, a_2 \le i_2 \le 2a_2; x_1^{i_1}\partial_2, 6 \le i_1 \le 8.$$

Thus, we obtain

$$\rho_2(V) = 12a_2 + 2.$$

Similarly, when $a_1 \ge 4$, $a_2 = 3$, we obtain

$$\rho_2(V) = 12a_1 + 2.$$

Therefore, when $a_1 \ge 3$, $a_2 \ge 3$, and $a_1 \ne a_2$, the formula (12) holds.

Case (5): When $a_1 \ge 3$, $a_2 = 2$, we obtain the following basis of $\mathcal{L}_2(V)$:

$$x_1^{i_1}x_2^{i_2}\partial_1, 1 \le i_1 \le a_1-1, 2 \le i_2 \le 3; x_2^4\partial_1; x_2^5\partial_1;$$
$$x_1^{i_1}x_2^{i_2-1}\partial_1 + (a_1-1)x_1^{i_1-1}x_2^{i_2}\partial_2, 2 \le i_1 \le a_1-1,$$
$$1 \le i_2 \le 2; x_1\partial_1 + (a_1-1)x_2\partial_2; x_1x_2\partial_1 - (a_1-1)x_1^{a_1-1}x_2\partial_2; x_1^{a_1}\partial_1 + (a_1-1)x_1^{a_1-1}x_2\partial_2;$$
$$x_1^{i_1+a_1}\partial_1 - (a_1-1)x_1^{i_1}x_2^2\partial_2, 1 \le i_1 \le a_1-2; x_1^{i_1}\partial_1, 2a_1-1 \le i_1 \le 3a_1-1; (x_1^{a_1-1}x_2 + x_2^2)\partial_2;$$
$$x_1^{i_1}x_2^3\partial_2, 1 \le i_1 \le a_1-2; x_2^{i_2}\partial_2, 3 \le i_2 \le 5; x_1^{a_1-1}x_2^{i_2}\partial_2; 2 \le i_2 \le 3;$$
$$x_1^{i_1}\partial_2, 2a_1 \le i_1 \le 3a_1-1.$$

Thus, we obtain the following formula

$$\rho_2(V) = 8a_1 + 2.$$

Similarly, it is easy to check that, for $a_1 = 2, a_2 \ge 3$, we get $\rho_2(V) = 8a_2 + 2$.
Therefore, the first part of Proposition 4.5 has been proved.
It follows from Proposition 4.5 that

$$h_2(a_1,a_2) = \begin{cases} 6; & a_1 = 2, a_2 = 2 \\ 4a_2 - 3; & a_1 = 2, a_2 \ge 3 \\ 4a_1 - 3; & a_1 \ge 3, a_2 = 2 \\ 4a_1a_2 - 4(a_1+a_2) + 7; & a_1 \ge 3,\ a_2 \ge 3, \text{ and } a_1 = a_2 \\ 4a_1a_2 - 4(a_1+a_2) + 6; & a_1 \ge 3,\ a_2 \ge 3, \text{ and } a_1 \ne a_2. \end{cases}$$

After plugging into the weight type $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}; 1)$ of the binomial isolated singularity of type C, we have

$$h_2(\frac{1}{w_1}, \frac{1}{w_2}) = \begin{cases} 6; & \frac{a_1a_2-1}{a_2-1} = 2, \frac{a_1a_2-1}{a_1-1} = 2 \\ \frac{4(a_1a_2-1)}{a_1-1} - 3; & \frac{a_1a_2-1}{a_2-1} = 2, \frac{a_1a_2-1}{a_1-1} \geq 3 \\ \frac{4(a_1a_2-1)}{a_2-1} - 3; & \frac{a_1a_2-1}{a_2-1} \geq 3, \frac{a_1a_2-1}{a_1-1} = 2 \\ 4a_1a_2 + 3; & a_1 \geq 2, a_1 = a_2 \\ 4a_1a_2 + 2; & \frac{a_1a_2-1}{a_2-1} \geq 3, \frac{a_1a_2-1}{a_1-1} \geq 3, \text{ and } a_1 \neq a_2. \end{cases}$$

Conjecture 1.8 claims that

$$h_2(\frac{1}{w_1}, \frac{1}{w_2}) \geq \begin{cases} 6a_1; & a_1 \geq 1, a_2 = 1 \\ 4a_1a_2 + 3; & a_1 \geq 2,\ a_2 \geq 2, \text{ and } a_1 = a_2 \\ 4a_1a_2 + 2; & a_1 \geq 2,\ a_2 \geq 2, \text{ and } a_1 \neq a_2. \end{cases} \tag{13}$$

It is easy to check that (13) holds. □

Proof of Theorem C. Let $f \in \mathbb{C}\{x_1, x_2\}$ be a weighted homogeneous fewnomial isolated singularity. It has the following three cases:

Type A. $x_1^{a_1} + x_2^{a_2}$,

Type B. $x_1^{a_1}x_2 + x_2^{a_2}$,

Type C. $x_1^{a_1}x_2 + x_2^{a_2}x_1$.

Theorem C is an immediate corollary of Propositions 4.5, 4.8, and 4.11. □

Data availability

No data was used for the research described in the article.

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