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ABSTRACT

In this paper, we introduce a series of new invariants for singularities. A new conjecture about the non-existence of negative weight derivations of the new k-th singular locus moduli algebras for weighted homogeneous isolated hypersurface singularities is proposed. We verify this conjecture in some cases.

I. INTRODUCTION

Many highly non-trivial physical questions such as the Coulomb branch spectrum and the Seiberg–Witten solution[1,2] can be easily found by studying the mini-versal deformation of the isolated singularity. In Ref. 3, we classify three-dimensional isolated weighted homogeneous rational complete intersection singularities, which define many new four-dimensional $\mathcal{N}=2$ superconformal field theories. In Ref. 4, we classify threefold isolated quotient as Gorenstein singularities. These singularities are rigid, i.e., there is no non-trivial deformation, and we conjecture that they define 4d $\mathcal{N}=2$ SCFTs that do not have a Coulomb branch. In this article, we will introduce a series of new invariants for isolated singularities. These new invariants are very useful in the classification theory of isolated singularities.

This article has two purposes. On the one hand, we introduce a series of new local Artinian algebras associated with singularities in Sec. II. These algebras and their dimensions are natural new invariants of singularities. On the other hand, motivated by the famous Halperin conjecture, we investigate the derivation Lie algebras of these new local Artinian algebras. Again, these Lie algebras and their dimensions are new invariants of singularities. We generalize the Halperin conjecture and verify the conjecture in several particular cases.

We first recall the Halperin conjecture. A classic result of Borel[5] states that the Serre spectral sequence for the rational cohomology of the universal bundle $G/H \to B_H \to B_G$ collapses if G/H is a homogeneous space of equal rank pairs (G,H) of compact connected Lie groups. Halperin made a very general conjecture on the collapsing of the Serre spectral sequence on a general fibration, which is one of the most important open problems in rational homotopy theory (see Ref. 6, p. 516, and Ref. 7).

Recall that a finite simply-connected cell complex C is called elliptic if all but finitely many cohomology and homotopy groups of C are finite. If C is elliptic, then C has a non-negative Euler characteristic.

Halperin conjecture (Ref. 6 p. 516). Suppose F is a rational elliptic space with a non-zero Euler-Poincaré characteristic, and $F \to E \to B$ is a Serre fibration of simply-connected spaces. Then the (rational) Serre spectral sequence for this fibration collapses at E_2.

In fact, for an elliptic space with a positive Euler characteristic, all cohomology must be concentrated in even degrees, i.e., cohomology is zero in an odd degree. Furthermore, the cohomology algebra is a complete intersection, i.e., it has the same number of generators as relations.

It was shown that the above conjecture is equivalent to the following conjecture about the non-existence of negative weight derivations (Ref. 7, Theorem A on p. 329).

Suppose that A is a weighted zero-dimensional complete intersection algebra, i.e., a commutative algebra of the form

$$A = \mathbb{C}[z_1, z_2, \ldots, z_n]/(g_1, g_2, \ldots, g_n),$$

where the $g_1, g_2, \ldots, g_n$ is a regular sequence of length n. Here the variables have strictly positive integral weights, denoted by $\mathrm{wt}(z_i) = \alpha_i$ for $1 \le i \le n$, and the equations are weighted homogeneous with respect to these weights. They are arranged for future convenience in the decreasing order of the weighted degrees: $d_i := \mathrm{wt}(g_i)$ for $i = 1, 2, \ldots, n$ with $d_1 \ge d_2 \ge \cdots \ge d_n$. Consequently, the algebra A is naturally graded, and one may speak of its weighted homogeneous degree k derivations (k is an integer). A linear map $D : A \to A$ is a derivation if $D(ab) = D(a)b + aD(b)$, for any $a, b \in A$. We use $\mathrm{Der}^k(A)$ to denote the set of weighted degree k derivations of A, i.e., D belongs to $Der^k(A)$ if $D : A^* \to A^{*+k}$. One of the most important open problems in rational homotopy theory (see Ref. 6) is related to the vanishing of the above derivations in strictly negative degrees.

Halperin conjecture.[7] Let

$$A = \mathbb{C}[z_1, \ldots, z_n]/(g_1, g_2, \ldots, g_n),$$

be a weighted zero-dimensional complete intersection algebra. Here the variables z_i have strictly positive integral weights, denoted by $\mathrm{wt}(z_i) = \alpha_i$, $1 \le i \le n$, and g_i are weighted homogeneous polynomials with respect to these weights. Then $\mathrm{Der}^{<0}(A) = 0$.

Assuming that all the weights α_i are even, this has the following topological interpretation. If a space X has $H^*(X, \mathbb{C}) = A$ as graded algebras, then it is known that the vanishing of $\mathrm{Der}^{<0}(A) = 0$ implies the collapsing at the E_2-term of the Serre spectral sequence with the $\mathbb{C}$-coefficients of any orientable fibration having X as a fiber. The above collapsing properties also imply vanishing properties when $\mathbb{C}$ is replaced by $\mathbb{Q}$ and X in a rational space (see, e.g., Ref. 7). The Halperin conjecture has been verified in several particular cases, as follows:

(1) Equal weights ($\alpha_1 = \alpha_2 = \cdots = \alpha_n$), see Ref. 8.
(2) $n = 2, 3$, see Refs. 9 and 10.
(3) "fibered" algebras, see Ref. 11.
(4) Assuming $\mathbb{C}[z_1, z_2, \ldots, z_n]/(g_1, g_2, \ldots, g_{n-1})$ is reduced, see Ref. 12.
(5) Homogeneous spaces of equal rank compact connected Lie groups $[A = H^*(G/K)]$, see Ref. 13.

For recent progress on the Halperin conjecture and its generalizations, interested readers can refer to Refs. 14 and 15.

Moreover, in the classification theory of isolated singularities, one always wants to find various invariants associated with isolated singularities. Hopefully, with enough invariants found, one can distinguish between different isolated singularities up to a certain equivalence. However, not many effective invariants are known. Moreover, most of the known invariants, for example, the cohomological invariant geometric genus, are hard to compute in general. In this article, we shall introduce two series of new numerical invariants for isolated hypersurface singularities. One is the dimension of the new local Artinian algebras, i.e., $\delta_{\mathrm{Sing}^k X}$ (see Definition II.1), and another is the dimension of its derivation Lie algebra, i.e., $\rho_{\mathrm{Sing}^k X}$ (see Definition II.7). These new invariants can be calculated easily and compared with other invariants of isolated singularities.

Let $\mathcal{O}_n$ be the germs of holomorphic functions at the origin. Clearly, $\mathcal{O}_n$ can be naturally identified with the algebra of convergent power series in n indeterminates with complex coefficients. As a ring, $\mathcal{O}_n$ has a unique maximal ideal $\mathfrak{m}$, the set of germs of holomorphic functions that vanish at the origin. Let $(X, 0)$ be an isolated hypersurface singularity defined by the germ of the holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. The multiplicity mult (f) of the singularity $(X, 0)$ is defined to be the order of the lowest nonvanishing term in the power series expansion of f at 0.

For any isolated hypersurface singularity $(X, 0) \subset (\mathbb{C}^n, 0)$ defined by f, the algebra

$$A(X) := \mathcal{O}_n \Big/ \left(f, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right),$$

is called the moduli algebra of $(X, 0)$.[16] The second author firstly introduced the Lie algebra of derivations of moduli algebra $A(X)$, i.e., $L(X) = \mathrm{Der}(A(X), A(X))$. It is known that $L(X)$ is a finite-dimensional solvable Lie algebra.[17] $L(X)$ is called the Yau algebra of X in Refs. 18 and 19 to distinguish from Lie algebras of other types appearing in the theory of singularities.[20] The Yau algebra plays an important role in singularities.[21] In this paper, we will introduce a series of new derivation Lie algebras that are natural generalizations of the Yau algebra.

Theorem A and Theorem B in Sec. II are our main results.

This paper is organized as follows: in Sec. II, we will introduce a series of new local Artinian algebras $M_{\mathrm{Sing}^k X}$ (see Definition II.1) associated with singularities. Motivated by the Halperin conjecture, a new conjecture (Conjecture II.8) is proposed. Our main results are to verify this conjecture in several particular cases. In Sec. III, we give the proof of the main results.

II. SINGULAR LOCUS ALGEBRAS

In this paper, we focus on weighted homogeneous isolated hypersurface singularities. However, our theory also applies to generally isolated hypersurface singularities at the expense of introducing some complications in the notation and the formulation. In what follows, we only consider weighted homogeneous singularity.

Let $(X,0) = \{(z_1,\ldots,z_n) \in \mathbb{C}^n : f(z_1,\ldots,z_n) = 0\}$ be an isolated hypersurface singularity with f a weighted homogeneous polynomial. We use $\mathrm{Sing}^1 X$ to denote the singular locus of X. $\mathrm{Sing}^1 X$ is a zero-dimensional scheme. We will construct a series of zero-dimensional schemes called higher singular loci, which have different scheme structures as follows: in addition, we use the tool of new moduli algebras, which are so called the higher singular locus moduli algebras (see Definition II.1), to investigate the singularity $(X,0)$.

To determine $\mathrm{Sing}^1 X$, we need to consider the following Jacobian matrix at the singular point 0. By abusing notation, we shall omit the point 0. Let

$$\mathrm{Jac}^1 := \begin{bmatrix} f_1 & f_2 & \cdots & f_n \end{bmatrix},$$

where $f_i := \frac{\partial f}{\partial z_i}$. Hence, the first singular locus is defined as

$$\mathrm{Sing}^1 X = \{\mathbf{z} | f_1(\mathbf{z}) = \cdots = f_n(\mathbf{z}) = 0\} = \{0\}.$$

Denote $I_{\mathrm{Sing}^1 X} := (f_1,\ldots,f_n)$, which is just the Jacobian ideal of f. $\mathrm{Sing}^1 X$ is also an analytic space. It is just a point geometrically. We may also consider its singular locus due to different scheme structures. To determine $\mathrm{Sing}^2 X := \mathrm{Sing}^1(\mathrm{Sing}^1 X)$, we need to consider the following Jacobian matrix at 0. Let

$$\mathrm{Jac}^2 := \begin{bmatrix} f_{11} & f_{21} & \cdots & f_{n1} \\ f_{12} & f_{22} & \cdots & f_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ f_{1n} & f_{2n} & \cdots & f_{nn} \end{bmatrix},$$

where $f_{ij} := \frac{\partial^2 f}{\partial z_i \partial z_j}$. Hence, the second singular locus is defined as

$$\mathrm{Sing}^2 X = \{\mathbf{z} | f_1(\mathbf{z}) = \cdots = f_n(\mathbf{z}) = h(\mathbf{z}) = 0\},$$

where $h := \det[f_{ij}]_{1\le i,j\le n}$. Denote $I_{\mathrm{Sing}^2 X} := (f_1,\ldots,f_n,h)$. $\mathrm{Sing}^2 X$ is still an analytic space and just a point geometrically as well. We may also consider its singular locus due to different scheme structures. To determine $\mathrm{Sing}^3 X := \mathrm{Sing}^1(\mathrm{Sing}^1(\mathrm{Sing}^1 X))$, which means the singular locus of $\mathrm{Sing}^2 X$, we need to consider the following Jacobian matrix at 0. Let

$$\mathrm{Jac}^3 := \begin{bmatrix} f_{11} & f_{21} & \cdots & f_{n1} & h_1 \\ f_{12} & f_{22} & \cdots & f_{n2} & h_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{1n} & f_{2n} & \cdots & f_{nn} & h_n \end{bmatrix},$$

where $h_i = \frac{\partial h}{\partial z_i}$. Hence, the third singular locus is defined as

$$\mathrm{Sing}^3 X = \left\{\mathbf{z} | f_1(\mathbf{z}) = \cdots = f_n(\mathbf{z}) = h(z) = {}^3h^1(\mathbf{z}) = \cdots = {}^3h^n(\mathbf{z}) = 0\right\},$$

where ${}^3h^j$, $1 \le j \le n$, are defined as follows: let ${}^3\bar{h}_i := \begin{bmatrix} f_{i1} & f_{i2} & \cdots & f_{in} \end{bmatrix}^T$ be the i-th column of Jac^3 for $1 \le i \le n$ and ${}^3\bar{h}_{n+1} := \begin{bmatrix} h_1 & h_2 & \cdots & h_n \end{bmatrix}^T$ be the $(n+1)$-th column of Jac^3. Set ${}^3H^i := \begin{bmatrix} {}^3\bar{h}_1 & {}^3\bar{h}_2 & \cdots & {}^3\hat{\bar{h}}_i & \cdots & {}^3\bar{h}_{n+1} \end{bmatrix}$ for $1 \le i \le n$ by deleting the column $\bar{h}_i$. We define ${}^3h^i := \det({}^3H^i)$. We use $I_{\mathrm{Sing}^3 X} := (f_1,\ldots,f_n,h,{}^3h^1,\ldots,{}^3h^n)$ to denote the ideal generated by those polynomials defining $\mathrm{Sing}^3 X$.

In general, if we have defined the Jac^k with $k \ge 2$, we define Jac^{k+1} as follows: Assume that Jac^k has $(l_k+1)n \times n$-submatrices. We use ${}^kH^i$, $0 \le i \le l_k$ to denote all those submatrices. There is a special $n \times n$-submatrices

$$\begin{bmatrix} f_{11} & f_{21} & \cdots & f_{n1} \\ f_{12} & f_{22} & \cdots & f_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ f_{1n} & f_{2n} & \cdots & f_{nn} \end{bmatrix}.$$

We always use ${}^{k}H^{0}$ to denote this submatrix [that is why we use $(l_k + 1)$ in stead of l_k above]. Set ${}^{k}h^{i} = \det({}^{k}H^{i})$ (note that ${}^{k}h^{0} = h$). In general, the k-th singular locus is defined as

$$\mathrm{Sing}^{k}X = \left\{ \mathbf{z} | f_1(\mathbf{z}) = \cdots = f_n(\mathbf{z}) = h(\mathbf{z}) = {}^{k}h^{1}(\mathbf{z}) = \cdots = {}^{k}h^{l_k}(\mathbf{z}) = 0 \right\}.$$

Therefore

$$I_{\mathrm{Sing}^{k}X} := \left(f_1, \ldots, f_n, h, {}^{k}h^{1}, \ldots, {}^{k}h^{l_k} \right).$$

Hence, Jac^{k+1} is defined as

$$\mathrm{Jac}^{k+1} := \begin{bmatrix} f_{11} & f_{21} & \cdots & f_{n1} & h_1 & ({}^{k}h^{1})_1 & \ldots & ({}^{k}h^{l_k})_1 \\ f_{12} & f_{22} & \cdots & f_{n2} & h_2 & ({}^{k}h^{1})_2 & \ldots & ({}^{k}h^{l_k})_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{1n} & f_{2n} & \cdots & f_{nn} & h_n & ({}^{k}h^{1})_n & \ldots & ({}^{k}h^{l_k})_n \end{bmatrix},$$

where $({}^{k}h^{j})_i := \frac{\partial^{k}h^{j}}{\partial z_i}$ for $1 \le i \le n$, $1 \le j \le l_k$. In fact, it is easy to see that Jac^{k+1} is obtained by just adding some ordered columns to the Jac^{k}. We only emphasized the first $(n+1)$ columns of Jac^{k+1} here because these columns will play a key role in the subsequent discussion. For other newly added columns, we do not emphasize their order here. Hence, the $(k+1)$-th singular locus is defined as

$$\mathrm{Sing}^{k+1}X = \left\{ \mathbf{z} | f_1(\mathbf{z}) = \cdots = f_n(\mathbf{z}) = h(\mathbf{z}) = {}^{k+1}h^{1}(\mathbf{z}) = \cdots = {}^{k+1}h^{l_{k+1}}(\mathbf{z}) = 0 \right\},$$

where ${}^{k+1}h^{j}$, $1 \le j \le l_{k+1}$ are defined similarly as above. We use

$$I_{\mathrm{Sing}^{k+1}X} := \left(f_1, \ldots, f_n, h, {}^{k+1}h^{1}, \ldots, {}^{k+1}h^{l_{k+1}} \right),$$

to denote the ideal generated by those polynomials defined $\mathrm{Sing}^{k+1}X$.

Definition II.1. Let $(X, 0)$ be an isolated hypersurface singularity. We define the (first order) singular locus moduli algebra,

$$M_{\mathrm{Sing}^{1}X} := \mathcal{O}_n / I_{\mathrm{Sing}^{1}X},$$

which is just the Moduli algebra.

Furthermore, we define the k-th singular locus moduli algebras as follows:

$$M_{\mathrm{Sing}^{k}X} := \mathcal{O}_n / I_{\mathrm{Sing}^{k}X}.$$

The $\mathbb{C}$-dimension of $M_{\mathrm{Sing}^{k}X}$ as a vector space is denoted as $\delta_{\mathrm{Sing}^{k}X}$.

Remark II.2. $M_{\mathrm{Sing}^{1}X}$ is consistent with the usual moduli algebra $A(X)$, thus $M_{\mathrm{Sing}^{k}X}$ are natural generalizations of moduli algebra. In fact, $M_{\mathrm{Sing}^{2}X}$ is consistent with the algebra that was studied by the second and third authors in Ref. 22. Those algebras are Artinian local algebras.

We recall some basic knowledge of the module of Kähler differential and Fitting ideal briefly. For more details, readers can refer to Refs. 23 and 24.

Definition II.3. Let S be a $\mathbb{C}$-algebra, the module of Kähler differentials of S over $\mathbb{C}$, denoted by $\Omega_{S/\mathbb{C}}$, is the S-module generated by the set $\{df : f \in S\}$ satisfies the relations

$$d(ss') = sd(s') + s'd(s),$$
$$d(rs + r's') = rd(s) + r'd(s'),$$

for all $r, r' \in \mathbb{C}$ and $s, s' \in S$.

Let $S = \mathbb{C}[z_1,\ldots,z_n]/I$ be a finitely generated $\mathbb{C}$-algebra. If $I = (g_1,\ldots,g_s)$, then $\Omega_{S/\mathbb{C}} = \mathrm{coker}(d : I/I^2 \to \oplus_i Sdz_i)$, where $\oplus_i Sdz_i = S\otimes_{\mathbb{C}}\Omega_{\mathbb{C}[z_1,\ldots,z_n]/\mathbb{C}}$ is a free S-module on generators dz_i. We view I/I^2 as a homomorphic image of a free S-module with generators e_i mapping to the classes of the g_i, the composition

$$\mathcal{J} : \bigoplus Se_i \to I/I^2 \to \bigoplus_i Sdz_i,$$

is a map of free S-modules. Hence, $\Omega_{S/\mathbb{C}}$ is the cokernel of the Jacobian matrix $\mathcal{J} = [\frac{\partial g_j}{\partial z_i}]$. The reader can find more details and examples in Chap. 16.1 of Ref. 23.

The Fitting ideal was first defined by Fitting in Ref. 25. The reader can also find this definition in Chap. 20 of Ref. 23 and Sec. 1 of the lecture notes Ref. 24.

Definition II.4. Let M be a finite generated $\mathbb{C}$-module. We represent M as the cokernel of an $\mathbb{C}$-linear map between free $\mathbb{C}$-modules of finite rank,

$$F \xrightarrow{\varphi} G \to M \to 0,$$

where F is a free $\mathbb{C}$-module of rank q and G is a free $\mathbb{C}$-module of rank p. For each j we associate to M, the ideal $F_j(M)$, called j-th Fitting ideal of M, generated by the $(p-j)\times(p-j)$ minors of the matrix representing φ. If $j \geq p$, we set $F_j(M) = \mathbb{C}$. If $p - j > q$, we set $F_j(M) = 0$.

Remark II.5. $F_j(M)$ depends only on the $\mathbb{C}$-module M and not on the choice of a presentation.

Proposition II.6 (Ref. 23, Corollary 16.20). Let $R = \mathbb{C}[z_1,\ldots,z_n]/I$ be an affine ring, and suppose that I has pure codimension c. Suppose that $I = (g_1,\ldots,g_s)$. If J is the ideal of R generated by $c\times c$ minors of the Jacobian matrix $\left[\frac{\partial g_j}{\partial z_i}\right]$, then J defines the singular locus of R in the sense that a prime P of R contains J if and only if R_P is not a regular local ring.

J is the $(n-c)$-th Fitting ideal of $\Omega_{R/\mathbb{C}}$. Hence, J depends only on R and $\mathbb{C}$.

In our setting, $I_{\mathrm{Sing}^k X}$ has pure codimension n for all $k \geq 2$ when $\mathrm{Sing}^k \neq \emptyset$. By the above-mentioned discussion, all higher singular locus moduli algebras are invariants of isolated hypersurface singularities whose dimensions are numerical invariants of isolated hypersurface singularities.

Inspired by Yau algebra, we give the following definition:

Definition II.7. We define $L_{\mathrm{Sing}^k X}$ be the derivation Lie algebra of $M_{\mathrm{Sing}^k X}$ for $k \geq 1$, i.e., $L_{\mathrm{Sing}^k X} = \mathrm{Der}(M_{\mathrm{Sing}^k X}, M_{\mathrm{Sing}^k X})$. The dimension of $L_{\mathrm{Sing}^k X}$ as a $\mathbb{C}$-vector space is denoted as $\rho_{\mathrm{Sing}^k X}$.

By the definition, the $L_{\mathrm{Sing}^1 X}$ is the same as the Yau algebra $L(X)$. Therefore, the $L_{\mathrm{Sing}^k X}$ is a natural generalization of the Yau algebra.

Motivated by the Halperin conjecture, we proposed the following conjecture, called the generalized Yau conjecture.

Conjecture II.8. Let $(X,0) = \{(z_1,\ldots,z_n) \in \mathbb{C}^n : f(z_1,\ldots,z_n) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial f of weight type $(\alpha_1,\ldots,\alpha_n; d)$. Assume that $d \geq 2\alpha_1 \geq \cdots \geq 2\alpha_n > 0$ without loss of generality.[26] Then for any $k \geq 1$, there is no non-zero negative weight derivation on $M_{\mathrm{Sing}^k X}$, i.e., $L_{\mathrm{Sing}^k X}$ is non-negatively graded.

This Conjecture II.8 is a generalization of Conjecture 1.5 in Ref. 27. It was investigated only for small k before. This Conjecture was verified in low dimensions when $k = 1$ or 2 (cf. Theorem II.9 and Theorem II.10) and, in general, in n dimensions for $k = 1$ under a certain condition (cf. Theorem II.11).

In this paper, we obtain the following two main results, which verify Conjecture II.8 for any k in low dimensions.

Theorem A. *Let $(X,0) = \{(z_1,\ldots,z_n) \in \mathbb{C}^n : f(z_1,\ldots,z_n) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial f of weight type $(\alpha_1,\ldots,\alpha_n; d)$. Assume that $n \leq 4$ and $d \geq 2\alpha_1 \geq \cdots \geq 2\alpha_n > 0$ without loss of generality. Then for any $k \geq 1$, there is no non-zero negative weight derivation on $M_{\mathrm{Sing}^k X}$, i.e., $L_{\mathrm{Sing}^k X}$ is non-negatively graded.*

For $k = 1$ or 2, Theorem A has already been proved in Refs. 27–29. The following Theorem II.9 verified the Conjecture II.8 for $k = 1$ and $n \leq 4$.

Theorem II.9 (Refs. 28 and 29). *Let $(X,0) = \{(z_1,\ldots,z_n) \in \mathbb{C}^n : f(z_1,\ldots,z_n) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial f of weight type $(\alpha_1,\ldots,\alpha_n; d)$, $n \leq 4$. Assume that $d \geq 2\alpha_1 \geq \cdots \geq 2\alpha_n > 0$ without loss of generality. Then there is no non-zero negative weight derivation on $M_{\mathrm{Sing}^1 X}$, i.e., $L_{\mathrm{Sing}^1 X}$ is non-negatively graded.*

The following Theorem II.10 verified the Conjecture II.8 for $k = 2$ and $n \le 4$.

Theorem II.10 (Ref. 27). *Let $(X, 0) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : f(z_1, \dots, z_n) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial f of weight type $(\alpha_1, \dots, \alpha_n;\ d)$, $n \le 4$. Assume that $d \ge 2\alpha_1 \ge \dots \ge 2\alpha_n > 0$ without loss of generality. Then there is no non-zero negative weight derivation on $M_{\mathrm{Sing}^2 X}$, i.e., $L_{\mathrm{Sing}^2 X}$ is non-negatively graded.*

In this article, we obtain the following two results, which verify Conjecture II.8 for any k, in general, and n dimension under a certain condition.

Theorem B. *Let $(X, 0) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : f(z_1, \dots, z_n) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial f of weight type $(\alpha_1, \dots, \alpha_n;\ d)$. Assume that $d \ge 2\alpha_1 \ge \dots \ge 2\alpha_n > 0$ without loss of generality. If $\alpha_1 \le 2\alpha_n$, then for any $k \ge 1$, there is no non-zero negative weight derivation on $M_{\mathrm{Sing}^k X}$, i.e., $L_{\mathrm{Sing}^k X}$ is non-negatively graded.*

For $k = 1$, Theorem B has been proved in Ref. 30. In fact, the following Theorem II.11 verified the Conjecture II.8 for $k = 1$ under the condition $\alpha_1 \le 2\alpha_n$.

Theorem II.11 (Ref. 30). *Let $(X, 0) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : f(z_1, \dots, z_n) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial f of weight type $(\alpha_1, \dots, \alpha_n;\ d)$. Assume that $d \ge 2\alpha_1 \ge \dots \ge 2\alpha_n > 0$ without loss of generality. If $\alpha_1 \le 2\alpha_n$, then there is no non-zero negative weight derivation on $M_{\mathrm{Sing}^1 X}$, i.e., $L_{\mathrm{Sing}^1 X}$ is non-negatively graded.*

III. PROOF OF THEOREMS

First, we recall the following known results, which will be used to prove our main results frequently.

Proposition III.1 (Ref. 31, Proposition 2.6). Let $A = \oplus_{i=0}^{k} A_i$ be a graded commutative Artinian local algebra with $A_0 = \mathbb{C}$. Suppose the maximal ideal of A is generated by A_j for some $j > 0$. Then $L(A)$ is a graded Lie algebra without negative weight.

Lemma III.2 (Ref. 32). Let $(A, \mathfrak{m})$ be a commutative local Artinian algebra [$\mathfrak{m}$ is the unique maximal ideal of A and $D \in L(A)$ be the derivation of A]. Then D preserves the m-adic filtration of A, i.e., $D(\mathfrak{m}) \subset \mathfrak{m}$.

Lemma III.3 (Ref. 29 Lemma 2.1). Let f be a weighted homogeneous polynomial with isolated singularity in $z_1, \dots, z_n$ variables of type $(\alpha_1, \dots, \alpha_n;\ d)$. Assume $\mathrm{wt}(z_1) = \alpha_1 \ge \mathrm{wt}(z_2) = \alpha_2 \ge \dots \ge \mathrm{wt}(z_n) = \alpha_n$. Then f must be as in one of the following two cases:

Case (1):

$$f = z_1^m + a_1(z_2, \dots, z_n) z_1^{m-1} + \dots + a_{m-1}(z_2, \dots, z_n) z_1 + a_m(z_2, \dots, z_n).$$

Case (2):

$$f = z_1^m z_i + a_1(z_2, \dots, z_n) z_1^{m-1} + \dots + a_{m-1}(z_2, \dots, z_n) z_1 + a_m(z_2, \dots, z_n),$$

with $2 \le i \le n$.

Lemma III.4 (Ref. 28 Lemma 1.2). Let f be a weighted homogeneous polynomial in $z_1, \dots, z_n$ that defines an isolated singularity at the origin. Then there is a term of the form $z_i^{a_i}$ or $z_i^{a_i} z_j$ in f for any i ($a_i \ge 2$ in the case $z_i^{a_i}$ and $a_i \ge 1$ otherwise).

Lemma III.5 (Ref. 33). Let I be an ideal in $R = \mathbb{C}[z_1, \dots, z_n]$. Then there is a natural isomorphism of Lie algebras,

$$(Der_I R)/(I \cdot Der_{\mathbb{C}} R) \cong Der_{\mathbb{C}}(R/I).$$

First, we need to deal with the $\mathrm{mult}(f) = 2$ case, i.e., $d < 2\alpha_1 + \alpha_n$. By Lemma III.3, f must be as in one of the following cases:

Case (1):

$$f = z_1^2 + a_1(z_2, \dots, z_n) z_1 + a_2(z_2, \dots, z_n).$$

Case (2):

$$f = z_1 z_i + a_1(z_2, \dots, z_n),$$

for $i = 2, \dots, n$.

If $\alpha_1 = \cdots = \alpha_n$, then the polynomial f of both Case (1) and Case (2) is a homogeneous polynomial. Hence, there exists no non-zero negative weight derivation on any higher singular locus moduli algebra $M_{\mathrm{Sing}^k X}$.

As follows, we assume that not all the α_i are equal, i.e., f is not a homogeneous polynomial. We consider the following three propositions:

Proposition III.6. Let $(X,0) = \{(z_1,z_2) \in \mathbb{C}^2 : f(z_1,z_2) = 0\}$ *be an isolated singularity defined by the weighted homogeneous polynomial* f *of weight type* $(\alpha_1,\alpha_2;\ d < 2\alpha_1 + \alpha_2)$. *Assume that* $d \geq 2\alpha_1 \geq \alpha_2 > 0$ *without loss of generality. Then for any* $k \geq 1$, *there is no non-zero negative weight derivation on* $M_{\mathrm{Sing}^k X}$.

Proof. As in the discussion earlier, we just need to consider the following two cases. Let $D_k = cz_2^r \frac{\partial}{\partial z_1}$ be a negative derivation of $M_{\mathrm{Sing}^k X}$ by Lemma III.2.

Case (1). $f = z_1^2 + a_1(z_2)z_1 + a_2(z_2)$.

For any $M_{\mathrm{Sing}^k X}$ with $k \geq 3$, let F be an element of the smallest weighted degree in $I_{\mathrm{Sing}^k X}$. If $\frac{\partial f}{\partial z_1} = 2z_1 + a_1(z_2)$ is such F, then $D_k(\frac{\partial f}{\partial z_1}) = 2cz_2^r = 0$, which implies that $c = 0$, Hence, $D_k = 0$. If $\frac{\partial f}{\partial z_1}$ is not such F, i.e., $\mathrm{wt}(F) < \alpha_1$, then F must be the form lz_2^s with $l \in \mathbb{C}$ and $s\alpha_2 < \alpha_1$. If $D_k(\frac{\partial f}{\partial z_1}) = 0$, then $c = 0$. Hence, $D_k = 0$. If $D_k(\frac{\partial f}{\partial z_1}) = 2cz_2^r \neq 0$, then $r \geq s$, i.e., $z_2^r \in I_{\mathrm{Sing}^k X}$. Hence, $D_k = 0$ as a derivation of $M_{\mathrm{Sing}^k X}$.

Case (2). $f = z_1z_2 + a_1(z_2)$.

Since $\alpha_1 + \alpha_2 = d \geq 2\alpha_1$, we obtain that $\alpha_1 = \alpha_2$, i.e., f is a homogeneous polynomial. Hence, there exists no non-zero negative weight derivation on any higher singular locus moduli algebra $M_{\mathrm{Sing}^k X}$. □

Proposition III.7. Let $(X,0) = \{(z_1,z_2,z_3) \in \mathbb{C}^3 : f(z_1,z_2,z_3) = 0\}$ *be an isolated singularity defined by the weighted homogeneous polynomial* f *with* $\mathrm{mult}(f) = 2$ *of weight type* $(\alpha_1,\alpha_2,\alpha_3;\ d)$. *Assume that* $d \geq 2\alpha_1 = 2\alpha_2 \geq 2\alpha_3 > 0$. *Then for any* $k \geq 1$, *there is no non-zero negative weight derivation on* $M_{\mathrm{Sing}^k X}$.

Proof. As in the discussion earlier, we just need to consider the following three cases. Let $D_k = p(z_2,z_3)\frac{\partial}{\partial z_1} + cz_3^r \frac{\partial}{\partial z_2}$ be a negative weight derivation of $M_{\mathrm{Sing}^k X}$ by Lemma III.2.

Case (1). $f = z_1^2 + a_1(z_2,z_3)z_1 + a_2(z_2,z_3)$.

Consider the following coordinate transformation:

$$\begin{cases} z_1 = z_1' - \dfrac{1}{2}a_1(z_2',z_3'), \\ z_2 = z_2', \\ z_3 = z_3'. \end{cases}$$

By abusing the notation, we shall use the original symbol z_i to represent the symbol z_i' after the coordinate transformation. We can assume that

$$f = z_1^2 + g(z_2,z_3).$$

We obtain that $M_{\mathrm{Sing}^k X} = M_{\mathrm{Sing}^k \tilde{X}}$, where $(\tilde{X},0) = \{(z_2,z_3) \in \mathbb{C}^2 : g(z_2,z_3) = 0\}$ be the isolated hypersurface singularity defined by $g(z_2,z_3)$. By Proposition III.6, we obtain the conclusion.

Case (2). $f = z_1z_2 + a_1(z_2,z_3)$.

Since $\alpha_1 + \alpha_2 = d \geq 2\alpha_1$, we obtain that $\alpha_1 = \alpha_2 \geq \alpha_3$. (If $\alpha_1 = \alpha_2 = \alpha_3$, then f is a homogeneous polynomial. Hence, we assume that $\alpha_1 = \alpha_2 > \alpha_3$) For any $M_{\mathrm{Sing}^k X}$ with $k \geq 3$. Since $\frac{\partial f}{\partial z_1} = z_2 \in I_{\mathrm{Sing}^k X}$, hence, $D_k = p(z_3)\frac{\partial}{\partial z_1} + cz_3^r \frac{\partial}{\partial z_2}$.

We consider $D_k\left(\frac{\partial f}{\partial z_1}\right) = cz_3^r$. Since

$$h = \begin{vmatrix} 0 & 1 & 0 \\ 1 & \dfrac{\partial^2 a_1}{\partial (z_2)^2} & \dfrac{\partial^2 a_1}{\partial z_2 \partial z_3} \\ 0 & \dfrac{\partial^2 a_1}{\partial z_2 \partial z_3} & \dfrac{\partial^2 a_1}{\partial (z_3)^2} \end{vmatrix} = -\frac{\partial^2 a_1}{\partial (z_3)^2}.$$

Hence, $D_k\left(\frac{\partial f}{\partial z_1}\right) = cz_3^r = 0$ or $D_k\left(\frac{\partial f}{\partial z_1}\right)$ is a linear combination of $h = {}^kh^0, {}^kh^1, \ldots, {}^kh^{l_k}$. If $D_k\left(\frac{\partial f}{\partial z_1}\right) = cz_3^r = 0$, then $c = 0$, i.e., $D_k = 0$. If $D_k\left(\frac{\partial f}{\partial z_1}\right)$ is a linear combination of $h = {}^kh^0, {}^kh^1, \ldots, {}^kh^{l_k}$, then ${}^kh^i = z_3^{l_i}$ with $0 \le l_i \le r$ for $0 \le i \le l_k$. Hence, f must be the form

$$f = z_1z_2 + az_2^2 + bz_2z_3^p + dz_3^q,$$

with $d \neq 0$.

Consider the following coordinate transformation:

$$\begin{cases} z_1' = z_1 + az_2 + bz_3^p, \\ z_2' = z_2, \\ z_3' = d^{\frac{1}{q}} z_3. \end{cases}$$

By abusing the notation, we shall use the original symbol z_i to represent the symbol z_i' after the coordinate transformation. We can assume that $f = z_1z_2 + z_3^q$. We compute the higher singular locus moduli algebra as follows:

$$\mathrm{Jac}^3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & qz_3^{q-1} & -q(q-1)z_3^{q-2} \end{bmatrix},$$

$$\mathrm{Jac}^4 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & qz_3^{q-1} & -q(q-1)z_3^{q-2} & q(q-1)(q-2)z_3^{q-3} \end{bmatrix},$$

$$\cdots \quad \cdots \quad \cdots$$

$$\mathrm{Jac}^q = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & qz_3^{q-1} & -q(q-1)z_3^{q-2} & \cdots & (-1)^{q+1}q!z_3 \end{bmatrix}.$$

Hence,

$$\begin{aligned} M_{\mathrm{Sing}^3X} &= \mathbb{C}[z_1, z_2, z_3]/(z_1, z_2, z_3^{q-2}), \\ M_{\mathrm{Sing}^4X} &= \mathbb{C}[z_1, z_2, z_3]/(z_1, z_2, z_3^{q-3}), \\ &\cdots \quad \cdots \quad \cdots \\ M_{\mathrm{Sing}^kX} &= M_{\mathrm{Sing}^qX} = \mathbb{C}[z_1, z_2, z_3]/(z_1, z_2, z_3), \end{aligned}$$

for $k \ge q + 1$. It is directly to check that all the higher singular locus moduli algebra have no non-zero negative weight derivation.

Case (3). $f = z_1z_3 + a_1(z_2, z_3)$.

Since $\alpha_1 + \alpha_3 = d \ge 2\alpha_1$, we obtain that $\alpha_1 = \alpha_2 = \alpha_3$, i.e., f is a homogeneous polynomial. Hence, there exists no non-zero negative weight derivation on any higher singular locus moduli algebra M_{Sing^kX}. □

Proposition III.8. Let $(X, 0) = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : f(z_1, z_2, z_3, z_4) = 0\}$ *be an isolated singularity defined by the weighted homogeneous polynomial* f *with* $\mathrm{mult}(f) = 2$ *of weight type* $(\alpha_1, \alpha_2, \alpha_3, \alpha_4;\ d)$*. Assume that* $d \ge 2\alpha_1 = 2\alpha_2 \ge 2\alpha_3 \ge 2\alpha_4 > 0$*. Then for any* $k \ge 1$*, there is no non-zero negative weight derivation on* M_{Sing^kX}.

Proof. As in the discussion earlier, we just need to consider the following four cases. Let $D_k = p_1(z_2, z_3, z_4)\frac{\partial}{\partial z_1} + p_2(z_3, z_4)\frac{\partial}{\partial z_2} + cz_4^r\frac{\partial}{\partial z_3}$ be a negative weight derivation of M_{Sing^kX} by Lemma III.2.

Case (1). $f = z_1^2 + a_1(z_2, z_3, z_4)z_1 + a_2(z_2, z_3, z_4)$.

Consider the following coordinate transformation:

$$\begin{cases} z_1 = z_1' - \dfrac{1}{2}a_1(z_2', z_3', z_4'), \\ z_2 = z_2', \\ z_3 = z_3', \\ z_4 = z_4'. \end{cases}$$

By abusing the notation, we shall use the original symbol z_i to represent the symbol z_i' after the coordinate transformation. We can assume that

$$f = z_1^2 + g(z_2, z_3, z_4).$$

We obtain that $M_{\mathrm{Sing}^k X} = M_{\mathrm{Sing}^k \tilde{X}}$, where $(\tilde{X}, 0) = \{(z_2, z_3, z_4) \in \mathbb{C}^3 : g(z_2, z_3, z_4) = 0\}$ be the isolated hypersurface singularity defined by $g(z_2, z_3, z_4)$. By Proposition III.7, we obtain the conclusion.

Case (2). $f = z_1 z_2 + a_1(z_2, z_3, z_4)$.

Rewrite f as follows:

$$\begin{aligned} f &= z_1 z_2 + a_1(z_2, z_3, z_4), \\ &= z_1 z_2 + z_2 g_1(z_2, z_3, z_4) + g_2(z_3, z_4), \\ &= z_2(z_1 + g_1(z_2, z_3, z_4)) + g_2(z_3, z_4). \end{aligned}$$

Consider the following coordinate transformation:

$$\begin{cases} z_1' = z_1 + g_1(z_2, z_3, z_4), \\ z_2' = z_2, \\ z_3' = z_3, \\ z_4' = z_4. \end{cases}$$

By abusing the notation, we shall use the original symbol z_i to represent the symbol z_i' after the coordinate transformation. Hence, $f = z_1 z_2 + g(z_3, z_4)$. We obtain that $M_{\mathrm{Sing}^k X} = M_{\mathrm{Sing}^k \tilde{X}}$, where $(\tilde{X}, 0) = \{(z_3, z_4) \in \mathbb{C}^2 : g(z_3, z_4) = 0\}$ be the isolated hypersurface singularity defined by $g(z_3, z_4)$. By Proposition III.6, we obtain the conclusion.

Case (3). $f = z_1 z_3 + a_1(z_2, z_3, z_4)$.

Rewrite f as follows:

$$\begin{aligned} f &= z_1 z_3 + a_1(z_2, z_3, z_4), \\ &= z_1 z_3 + z_3 g_1(z_2, z_3, z_4) + g_2(z_2, z_4), \\ &= z_3(z_1 + g_1(z_2, z_3, z_4)) + g_2(z_2, z_4). \end{aligned}$$

Consider the following coordinate transformation:

$$\begin{cases} z_1' = z_1 + g_1(z_2, z_3, z_4), \\ z_2' = z_2, \\ z_3' = z_3, \\ z_4' = z_4. \end{cases}$$

By abusing the notation, we shall use the original symbol z_i to represent the symbol z_i' after the coordinate transformation. Hence, $f = z_1 z_3 + g(z_2, z_4)$. We obtain that $M_{\mathrm{Sing}^k X} = M_{\mathrm{Sing}^k \tilde{X}}$, where $(\tilde{X}, 0) = \{(z_2, z_4) \in \mathbb{C}^2 : g(z_2, z_4) = 0\}$ be the isolated hypersurface singularity defined by $g(z_2, z_4)$. By Proposition III.6, we obtain the conclusion.

Case (4). $f = z_1 z_4 + a_1(z_2, z_3, z_4)$.

Since $\alpha_1 + \alpha_4 = d \geq 2\alpha_1$, we obtain that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$, i.e., f is a homogeneous polynomial. Hence, there exists no non-zero negative weight derivation on any higher singular locus moduli algebra $M_{\mathrm{Sing}^k X}$. □

Proof of Theorem A. If $\mathrm{mult}(f) = 2$, then by Proposition III.6, Proposition III.7, and Proposition III.8, we obtain the conclusion. Now, we assume $\mathrm{mult}(f) \geq 3$. Since $\mathrm{mult}(f) \geq 3$ and $d \geq 2\alpha_1 \geq \cdots \geq 2\alpha_n > 0$ by assumption, it follows from Lemma III.3 that the weighted degree of f satisfied $d \geq 2\alpha_1 + \alpha_n$. To prove Theorem A, we first need to prove the following lemma.

Lemma III.9. With the above notations, let f be a weighted homogeneous polynomial with an isolated singularity at the origin of weight type $(\alpha_1,\ldots,\alpha_n;\ d)$. Assume $\mathrm{wt}(z_1)=\alpha_1\geq \mathrm{wt}(z_2)=\alpha_2\geq\cdots\geq \mathrm{wt}(z_n)=\alpha_n$ and $d\geq 2\alpha_1+\alpha_n$. Then

$$\mathrm{wt}({}^k h^j) > \mathrm{wt}(h),$$

for $k\geq 3$, $1\leq j\leq l_k$.

Proof. We compute directly that

$$\begin{aligned}\mathrm{wt}({}^3 h^j\ ) &= (2n-1)d - 4\sum_{s=1}^{n}\alpha_s + \alpha_j,\\ &= \mathrm{wt}(h) + (n-1)d - 2\sum_{s=1}^{n}\alpha_s + \alpha_j,\end{aligned}$$

for $1\leq j\leq n$. Since $d\geq 2\alpha_1+\alpha_n$, we obtain that

$$\begin{aligned}(n-1)d - 2\sum_{s=1}^{n}\alpha_s + \alpha_n &= (d-2\alpha_1-\alpha_n) + \sum_{s=2}^{n-1}(d-2\alpha_s),\\ &> 0.\end{aligned}$$

Hence,

$$\mathrm{wt}({}^3 h^1\ )\geq\cdots\geq \mathrm{wt}({}^3 h^j\ )\geq\cdots\geq \mathrm{wt}({}^3 h^n\ ) > \mathrm{wt}(h).$$

We assume that $\mathrm{wt}({}^l h^j) > \mathrm{wt}(h)$ holds for all $3\leq l\leq k-1$. Now we consider the case $l=k$. For any ${}^k h^j$ with $1\leq j\leq l_k$, there must be at least one column of the corresponding minor ${}^k H^j$ that is the s-th column of Jac^k where $s\geq n+1$. Hence, if there are more than two such columns of ${}^k H^j$, we obtain that $\mathrm{wt}({}^l h^j) > \mathrm{wt}(h)$ by induction hypothesis with the assumption $d\geq 2\alpha_1+\alpha_n$. If there is only one such column of ${}^k H^j$, we obtain that $\mathrm{wt}({}^l h^j) > \mathrm{wt}(h)$ by the same discussion of $\mathrm{wt}({}^3 h^j) > \mathrm{wt}(h)$. □

Let $I_{\mathrm{Sing}^2 X} := (f_1, f_2, \ldots, f_n, h)$ be the ideal corresponding to $\mathrm{Sing}^2 M$. For any $k>2$, we set $I_{\mathrm{Sing}^k X} = (I_{\mathrm{Sing}^2 X}, I'_k)$ be the ideal corresponding to $\mathrm{Sing}^k M$, where I'_k is generated by the rest generators of $I_{\mathrm{Sing}^k X}$ except those of $I_{\mathrm{Sing}^2 X}$. For $k>2$, if D_k is a non-zero negative weight derivation of $M_{\mathrm{Sing}^k X}$, by Lemma III.9, we obtain that $D_k(I_{\mathrm{Sing}^2 X})\subset I_{\mathrm{Sing}^2 X}$, i.e., D_k is also a non-zero negative weight derivation of $M_{\mathrm{Sing}^2 X}$ by Lemma III.5. By Theorem II.10, we obtain that such D_k does not exist. Hence, we finish the Proof of Theorem A. □

Proposition III.10. Let $(X,0)=\{(z_1,\ldots,z_n)\in\mathbb{C}^n : f(z_1,\ldots,z_n)=0\}$ be an isolated singularity defined by the weighted homogeneous polynomial f of weight type $(\alpha_1,\ldots,\alpha_n;\ d)$. Assume that $\mathrm{mult}(f)=2$, and $d\geq 2\alpha_1\geq 2\alpha_2\geq\cdots\geq 2\alpha_n>0$. If $\alpha_1\leq 2\alpha_n$, then for any $k\geq 1$, there is no non-zero negative weight derivation on $M_{\mathrm{Sing}^k X}$, i.e., $L_{\mathrm{Sing}^k X}$ is non-negatively graded.

Proof. We prove Proposition III.10 by induction, assuming that Proposition III.10 has been proved for f which has less than $(n-1)$ variables. Now we consider the n variables case. As in the discussion earlier, we just need to consider the following three cases. Let $D_k = p_1(z_2,\ldots,z_n)\frac{\partial}{\partial z_1} + p_2(z_3,\ldots,z_n)\frac{\partial}{\partial z_2} + \cdots + c z_n^r \frac{\partial}{\partial z_{n-1}}$ be a negative weight derivation of $M_{\mathrm{Sing}^k X}$ by Lemma III.2.

Case (1). $f = z_1^2 + a_1(z_2,\ldots,z_n)z_1 + a_2(z_2,\ldots,z_n)$.

Consider the following coordinate transformation:

$$\begin{cases} z_1 = z'_1 - \dfrac{1}{2}a_1(z'_2,\ldots,z'_n),\\ z_2 = z'_2,\\ z_3 = z'_3,\\ \ldots\\ z_n = z'_n.\end{cases}$$

By abusing the notation, we shall use the original symbol z_i to represent the symbol z'_i after the coordinate transformation. We can assume that

$$f = z_1^2 + g(z_2,\ldots,z_n).$$

We obtain that $M_{\mathrm{Sing}^k X} = M_{\mathrm{Sing}^k \tilde{X}}$, where $(\tilde{X},0)=\{(z_2,\ldots,z_n)\in\mathbb{C}^{n-1} : g(z_2,\ldots,z_n)=0\}$ be the isolated hypersurface singularity defined by $g(z_2,\ldots,z_n)$. By induction, we obtain the conclusion.

Case (2). $f = z_1 z_i + a_1(z_2, \ldots, z_4)$ with $2 \le i \le n-1$.

Rewrite f as follows:

$$\begin{aligned} f &= z_1 z_i + a_1(z_2, \ldots, z_n), \\ &= z_1 z_i + z_i g_1(z_2, \ldots, z_n) + g_2(z_2, \ldots, \hat{z}_i, \ldots, z_n), \\ &= z_i(z_1 + g_1(z_2, \ldots, z_n)) + g_2(z_2, \ldots, \hat{z}_i, \ldots, z_n). \end{aligned}$$

Consider the following coordinate transformation:

$$\begin{cases} z_1' = z_1 + g_1(z_2, \ldots, z_n), \\ z_2' = z_2, \\ \ldots \\ z_n' = z_n. \end{cases}$$

By abusing the notation, we shall use the original symbol z_i to represent the symbol z_i' after the coordinate transformation. Hence, $f = z_1 z_i + g(z_2, \ldots, \hat{z}_i, \ldots, z_n)$. We obtain that $M_{\mathrm{Sing}^k X} = M_{\mathrm{Sing}^k \tilde{X}}$, where $(\tilde{X}, 0) = \{(z_2, \ldots, \hat{z}_i, \ldots, z_n) \in \mathbb{C}^{n-2} : g(z_2, \ldots, \hat{z}_i, \ldots, z_n) = 0\}$ be the isolated hypersurface singularity defined by $g(z_2, \ldots, \hat{z}_i, \ldots, z_n)$. By induction, we obtain the conclusion.

Case (4). $f = z_1 z_n + a_1(z_2, \ldots, z_n)$.

Since $\alpha_1 + \alpha_n = d \ge 2\alpha_1$, we obtain that $\alpha_1 = \alpha_2 = \cdots = \alpha_n$, i.e., f is a homogeneous polynomial. Hence, there exists no non-zero negative weight derivation on any higher singular locus moduli algebra $M_{\mathrm{Sing}^k X}$. □

Proof of Theorem B. When $\mathrm{mult}(f) = 2$, by Proposition III.10, we obtain the conclusion. Now, we assume that $\mathrm{mult}(f) \ge 3$. First, we consider the following lemma:

Lemma III.11. Let $(X, 0) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : f(z_1, \ldots, z_n) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial f with $\mathrm{mult}(f) \ge 3$ *of weight type $(\alpha_1, \ldots, \alpha_n; d)$. Assume that $d \ge 2\alpha_1 \ge \cdots \ge 2\alpha_n > 0$ without loss of generality. If $\alpha_1 \le 2\alpha_n$, then there is no non-zero negative weight derivation on $M_{\mathrm{Sing}^2 X}$, i.e., $L_{\mathrm{Sing}^2 X}$ is non-negatively graded.*

Proof. Let D_2 be a negative weight derivation of $M_{\mathrm{Sing}^2 X}$. By Lemma III.2, we can assume that D_2 has to be the following form:

$$D_2 = p_1(z_2, \ldots, z_n)\frac{\partial}{\partial z_1} + p_2(z_3, \ldots, z_n)\frac{\partial}{\partial z_2} + \cdots + c z_n^r \frac{\partial}{\partial z_{n-1}}.$$

We compute the commutator $[\frac{\partial}{\partial z_i}, D]$ for $1 \le i \le n$ as follows:

$$\begin{aligned} &\left[\frac{\partial}{\partial z_1}, D\right] = 0, \\ &\left[\frac{\partial}{\partial z_2}, D\right] = \frac{\partial p_1}{\partial z_2}\frac{\partial}{\partial z_1}, \\ &\ldots \quad \ldots \quad \ldots \\ &\left[\frac{\partial}{\partial z_i}, D\right] = \frac{\partial p_1}{\partial z_i}\frac{\partial}{\partial z_1} + \cdots + \frac{\partial p_{i-1}}{\partial z_i}\frac{\partial}{\partial z_{i-1}}, \\ &\ldots \quad \ldots \quad \ldots \\ &\left[\frac{\partial}{\partial z_n}, D\right] = \frac{\partial p_1}{\partial z_n}\frac{\partial}{\partial z_1} + \cdots + \frac{\partial p_{n-2}}{\partial z_n}\frac{\partial}{\partial z_{n-2}} + \frac{\partial c z_n^r}{\partial z_n}\frac{\partial}{\partial z_{n-1}}. \end{aligned}$$

Since

$$\begin{aligned} \mathrm{wt}(h) &= nd - 2\sum_{i=1}^{n}\alpha_i, \\ &= (d - \alpha_n) + (d - 2\alpha_1 + \alpha_n) + (n-2)d - 2\sum_{i=2}^{n-1}\alpha_i, \\ &\ge \mathrm{wt}\left(\frac{\partial f}{\partial z_n}\right), \end{aligned}$$

then

$$
\begin{aligned}
&D\left(\frac{\partial f}{\partial z_1}\right) = 0,\\
&D\left(\frac{\partial f}{\partial z_2}\right) \in \left(\frac{\partial f}{\partial z_1}\right),\\
&\dots\quad\dots\quad\dots\\
&D\left(\frac{\partial f}{\partial z_n}\right) \in \left(\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_{n-1}}\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\frac{\partial(Df)}{\partial z_1} = 0,\\
&\frac{\partial(Df)}{\partial z_2} \in \left(\frac{\partial f}{\partial z_1}\right),\\
&\dots\quad\dots\quad\dots\\
&\frac{\partial(Df)}{\partial z_n} \in \left(\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_{n-1}}\right).
\end{aligned}
$$

We obtain that $\frac{\partial(Df)}{\partial z_2} = g(z_2,\dots,z_n)\frac{\partial f}{\partial z_1}$. If $g(z_2,\dots,z_n) \neq 0$, then $\mathrm{wt}(g) \geq \alpha_n$. This implies that $\mathrm{wt}\left(\frac{\partial(Df)}{\partial z_2}\right) = \mathrm{wt}(D) + d - \alpha_2 \leq \mathrm{wt}(D) + d - \alpha_n < d - \alpha_n \leq d - \alpha_1 + \alpha_n \leq \mathrm{wt}\left(g(z_2,\dots,z_n)\frac{\partial f}{\partial z_1}\right)$. This is a contradiction. Hence, $g \equiv 0$, i.e., $\frac{\partial(Df)}{\partial z_2} = 0$. For any $3 \leq i \leq n$, we assume that

$$
\frac{\partial(Df)}{\partial z_i} = q_1(z_2,\dots,z_n)\frac{\partial f}{\partial z_1} + \dots + q_{i-1}(z_i,\dots,z_n)\frac{\partial f}{\partial z_{i-1}}.
$$

If there exists $1 \leq j \leq i-1$ such that $q_j \not\equiv 0$, then $\mathrm{wt}\left(\frac{\partial(Df)}{\partial z_i}\right) = \mathrm{wt}(D) + d - \alpha_i \leq \mathrm{wt}(D) + d - \alpha_n < d - \alpha_n \leq d - \alpha_1 + \alpha_n \leq \mathrm{wt}\left(q_1(z_2,\dots,z_n)\frac{\partial f}{\partial z_1} + \dots + q_{i-1}(z_i,\dots,z_n)\frac{\partial f}{\partial z_{i-1}}\right)$. This is a contradiction. Hence, $q_j \equiv 0$, for all $1 \leq j \leq i-1$, i.e., $\frac{\partial(Df)}{\partial z_i} = 0$. Since Df does not depend on $z_1,\dots,z_n$, we conclude that Df is a constant, which means that $\mathrm{wt}(D) = -d$. Moreover, $\mathrm{wt}(D) \geq -\alpha_1$. This is a contradiction. Hence, there is no non-zero negative weight derivation on $M_{\mathrm{Sing}^2 X}$. □

By the assumption of $\mathrm{mult}(f) \geq 3$, we obtain that $d \geq 2\alpha_1 + \alpha_n$. By Lemma III.9, we obtain that $\mathrm{wt}({}^k h^j) \geq \mathrm{wt}(h)$. Let $I_{\mathrm{Sing}^2 X} := (f_1, f_2, \dots, f_n, h)$ be the ideal corresponding to $\mathrm{Sing}^2 M$. For any $k > 2$, we set $I_{\mathrm{Sing}^k X} = (I_{\mathrm{Sing}^2 X}, I_k')$ for the ideal corresponding to $\mathrm{Sing}^k M$, where I_k' is generated by the rest of the generators of $I_{\mathrm{Sing}^k X}$ except those of $I_{\mathrm{Sing}^2 X}$. For $k > 2$, if D_k is a non-zero negative weight derivation of $M_{\mathrm{Sing}^k X}$, by Lemma III.9, we obtain that $D_k(I_{\mathrm{Sing}^2 X}) \subset I_{\mathrm{Sing}^2 X}$, i.e., D_k is also a non-zero negative weight derivation of $M_{\mathrm{Sing}^2 X}$ by Lemma III.5. By Theorem III.11, we obtain that such D_k does not exist. Hence, we finish the Proof of Theorem B. □

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

All authors contributed equally to this work.

Guorui Ma: Conceptualization (equal); Investigation (equal); Writing – original draft (equal); Writing – review & editing (equal). **Stephen S.-T. Yau**: Supervision (equal); Writing – review & editing (equal). **Huaiqing Zuo**: Conceptualization (equal); Investigation (equal); Supervision (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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