Geometriae Dedicata (2023) 217:34
https://doi.org/10.1007/s10711-023-00768-0

ORIGINAL PAPER

k-th Milnor numbers and k-th Tjurina numbers of weighted homogeneous singularities

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Received: 1 September 2022 / Accepted: 5 January 2023

Abstract

Let $(V, 0) \subset (\mathbb{C}^n, 0)$ be a weighted homogeneous isolated hypersurface singularity. In this paper, we give explicit formulas of its k-th Milnor numbers and the k-th Tjurina numbers in terms of its weight type. Moreover, we propose a sharp lower bound conjecture for the k-th Tjurina numbers and verify this conjecture for binomial singularities. We also give a new characterization for the simple hypersurface singularities.

Keywords Moduli algebra · Isolated singularity · Weighted homogeneous · Derivation Lie algebra

Mathematics Subject Classification 14B05 · 32S05

1 Introduction

Let $\mathcal{O}_n = \mathbb{C}\{x_1, \ldots, x_n\}$ be the analytic algebra of convergent power series. For any isolated hypersurface singularity $(V, 0) \subset (\mathbb{C}^n, 0)$ where $V = V(f) = \{f = 0\}$, $f \in \mathcal{O}_n$, the algebra $A(V) = \mathcal{O}_n/(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$ is an invariant of $(V, 0)$. This algebra is called the

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Published online: 20 January 2023

moduli algebra of V; its dimension $\tau(V)$ is called Tjurina number. The order of the lowest nonvanishing term in the power series expansion of f at 0 is called the multiplicity of the singularity $(V, 0)$ denoted as $mult(f)$. A polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ is weighted homogeneous, if there exist n positive rational numbers $w_1, \dots, w_n$, called weights of the variables $x_1, \dots, x_n$, such that $\sum a_i w_i = d$ is the same for all nonzero terms of f. The d is called the weight degree of f with respect to weights $w_1, ..., w_n$, and is denoted by w-deg f. The weight type of f is denoted by $(w_1, ..., w_n; d)$. Without loss of generality, we can assume that w-deg$f = 1$. The Milnor number of an isolated hypersurface singularity is denoted as $\mu := \dim \mathbb{C}\{x_1, \dots, x_n\} \big/ (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$. The Tjurina number and Milnor number are probably the most important invariants of singularities. In [16], Milnor and Orlik proved that the Milnor number μ of a weighted homogeneous hypersurface singularity of type $(w_1, \dots, w_n; 1)$ can be given by its weight: $\mu = (\frac{1}{w_1} - 1)(\frac{1}{w_2} - 1) \cdots (\frac{1}{w_n} - 1)$. According to the beautiful theorem of Saito [18], f becomes a weighted homogeneous polynomial after a biholomorphic change of coordinates if and only if $\mu = \tau$.

By Mather-Yau theorem [17], the complex structure of an isolated hypersurface singularity is uniquely determined by its moduli algebra. Motivated by this, Yau has considered further the Lie algebra of derivations of the moduli algebra $A(V)$, denoted as $L(V) := \mathrm{Der}_{\mathbb{C}}(A(V), A(V))$. This Lie algebra $L(V)$ has been proven to be finite dimensional and was called Yau algebra. The dimension $\lambda(V)$ of $L(V)$ was called Yau number [6, 14, 23]. The Yau algebra plays an important role in singularity theory [19]. Yau and his collaborators have been systematically studying the Yau algebra and its generalizations since the 1980s [3–5, 8–12, 15, 20–22, 24, 25].

Recall that we have the following well-known generalized Mather-Yau theorem ([7], Theorem 2.26).

Theorem 1.1 *Let $m = (x_1, \dots, x_n)$ be the maximal ideal of $\mathcal{O}_n$. Let $f, g \in m \subset \mathcal{O}_n$. The following are equivalent:*

1. $(V(f), 0) \cong (V(g), 0)$;
2. *For all* $k \geq 0$, $\mathcal{O}_n/(f, m^k J(f)) \cong \mathcal{O}_n/(g, m^k J(g))$ *as* $\mathbb{C}$*-algebra;*
3. *There is some* $k \geq 0$ *such that* $\mathcal{O}_n/(f, m^k J(f)) \cong \mathcal{O}_n/(g, m^k J(g))$ *as* $\mathbb{C}$*-algebra,*

where $J(f) = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$.

In particular, if $k = 0, 1$ above, then the claim of the equivalence of (1) and (3) is exactly the same as those in the Mather-Yau theorem [17].

For each integer $k \geq 0$, we call $\tau^k(V) = \dim \frac{\mathcal{O}_n}{(f, m^k J(f))}$ the k-th Tjurina number and $\mu^k(V) = \dim \frac{\mathcal{O}_n}{m^k J(f)}$ the k-th Milnor number, respectively. These invariants are generalizations of the Tjurina numbers and Milnor numbers.

For a weighted homogeneous singularity $(V, 0)$, Milnor and Orlik obtained a formula for μ [16]. In this paper, we shall generalize the formula to $\mu^k(V)$ and $\tau^k(V)$. We shall show that $\tau^k(V)$ and $\mu^k(V)$ can also be computed by just the weight type. Thus, both $\tau^k(V)$ and $\mu^k(V)$ are topological invariants for binomial plane curve singularity, since the weight type is a topological invariant. We obtain the formulas of $\tau^k(V)$ and $\mu^k(V)$ for binomial singularities as follows.

Theorem A *Let $(V, 0)$ be a binomial isolated singularity defined by f. Then $\tau^k(V)$ depends only on the weight type of $(V, 0)$. We have:*

(1) if $f = x_1^{a_1} + x_2^{a_2}$ $(2 \leq a_1 \leq a_2)$ with weight type $(\frac{1}{a_1}, \frac{1}{a_2}; 1)$, then

$$\tau^k(V) = \begin{cases} a_1 a_2 - (a_1 + a_2) + 1 + \frac{k^2+3k}{2}; & 0 \leq k < a_1, \\ a_1 k + \frac{(2a_2 - a_1)(a_1 - 1)}{2}; & k \geq a_1; \end{cases}$$

(2) if $f = x_1^{a_1}x_2 + x_2^{a_2}$ $(2 \le a_1 + 1 \le a_2)$ *with weight type* $(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}; 1)$*, then*

$$\tau^k(V) = \begin{cases} a_1a_2 - a_2 + 1 + \frac{k^2+3k}{2}; & k < a_1 + 1, \\ (a_1+1)k + \frac{(2a_2-a_1)(a_1-1)}{2} + 1; & k \ge a_1 + 1. \end{cases}$$

(3) if $f = x_1^{a_1}x_2 + x_2^{a_2}$ $(a_1 + 1 \ge a_2 \ge 2)$ *with weight type* $(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}; 1)$*, then*

$$\tau^k(V) = \begin{cases} a_1a_2 - a_2 + 1 + \frac{k^2+3k}{2}; & 0 \le k < a_2, \\ a_2k + a_1a_2 + \frac{a_2}{2} - \frac{a_2^2}{2}; & a_2 \le k; \end{cases}$$

(4) if $f = x_1^{a_1}x_2 + x_2^{a_2}x_1$ $(1 \le a_1 \le a_2)$ *with weight type* $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}; 1)$*, then*

$$\tau^k(V) = \begin{cases} a_1a_2 + \frac{k^2+3k}{2}; & 0 \le k < a_1, a_1 \ge 2, \\ (a_1+1)k + a_1a_2 + \frac{a_1}{2} - \frac{a_1^2}{2}; & k \ge a_1 \ge 2, \\ 2k+1; & k \ge 0, a_1 = 1. \end{cases}$$

Theorem B *Let* $(V, 0)$ *be a binomial isolated singularity which is defined by* f*. Then* $\tau^k(V)$ *depends only on the weight type of* $(V, 0)$*. We have:*

(1) if $f = x_1^{a_1} + x_2^{a_2}$ $(2 \le a_1 \le a_2)$ *with weight type* $(\frac{1}{a_1}, \frac{1}{a_2}; 1)$*, then*

$$\mu^k(V) = \begin{cases} a_1a_2 - (a_1 + a_2) + 1 + k^2 + k; & 0 \le k < a_1, \\ (a_1 - \frac{1}{2})k + \frac{(2a_2-a_1)(a_1-1)}{2} + \frac{k^2}{2}; & k \ge a_1; \end{cases}$$

(2) if $f = x_1^{a_1}x_2 + x_2^{a_2}(2 \le a_1 + 1 \le a_2)$ *with weight type* $(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}; 1)$*, then*

$$\mu^k(V) = \begin{cases} a_1a_2 - a_2 + 1 + k^2 + k; & 0 \le k < a_1 + 1, \\ (a_1 + \frac{1}{2})k + \frac{k^2}{2} + \frac{(2a_2-a_1)(a_1-1)}{2} + 1; & k \ge a_1 + 1; \end{cases}$$

(3) if $f = x_1^{a_1}x_2 + x_2^{a_2}$ $(a_1 + 1 \ge a_2 \ge 2)$ *with weight type* $(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}; 1)$*, then*

$$\mu^k(V) = \begin{cases} a_1a_2 - a_2 + 1 + k + k^2; & 0 \le k < a_2, \\ (a_2 - \frac{1}{2})k + \frac{k^2}{2} + a_1a_2 + \frac{a_2}{2} - \frac{a_2^2}{2}; & a_2 \le k; \end{cases}$$

(4) if $f = x_1^{a_1}x_2 + x_2^{a_2}x_1$ $(1 \le a_1 \le a_2)$ *with weight type* $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}; 1)$*, then*

$$\mu^k(V) = \begin{cases} k^2 + k + a_1a_2; & 0 \le k < a_1, a_1 \ge 2, \\ (a_1 + \frac{1}{2})k + \frac{k^2}{2} + a_1a_2 + \frac{a_1}{2} - \frac{a_1^2}{2}; & k \ge a_1 \ge 2, \\ \frac{k^2}{2} + \frac{3k}{2} + 1; & k \ge 0, a_1 = 1. \end{cases}$$

Secondly, we conjecture the following sharp lower bound for $\tau^k(V)$ and verify it in the case of binomial singularities.

Conjecture 1.1 For each $k \ge 0$, assume that $\tau^k(\{x_1^{a_1} + \cdots + x_n^{a_n} = 0\}) = \ell^k(a_1, \ldots, a_n)$. Let $(V, 0) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n : f(x_1, x_2, \ldots, x_n) = 0\}$, $n \ge 2$, be an isolated singularity, defined by the weighted homogeneous polynomial $f(x_1, x_2, \ldots, x_n)$ of weight type $(w_1, w_2, \ldots, w_n; 1)$. Then

$$\tau^k(V) \ge \ell^k(1/w_1, \ldots, 1/w_n).$$

Remark 1.1 Conjecture 1.1 says that, with the same weight type, the Brieskorn singularity has the Tjurina number τ^k being smaller than or equal to that of any other weighted homogeneous singularity.

In this paper, we verify Conjecture 1.1 for binomial singularities. See Theorem C below.

Theorem C *Let $(V, 0)$ be a binomial singularity (see Corollary 2.1) defined by the weighted homogeneous polynomial $f(x_1, x_2)$ with the weight type $(w_1, w_2; 1)$. Then*

$$\tau^k(V) \geq \ell^k\left(\frac{1}{w_1}, \frac{1}{w_2}\right) = \begin{cases} \frac{1}{w_1 w_2} - (\frac{1}{w_1} + \frac{1}{w_2}) + \frac{k^2+3k}{2} + 1; & 0 \leq k < \frac{1}{w_1}, \quad 2 \leq \frac{1}{w_1} \leq \frac{1}{w_2}, \\ \frac{k}{w_1} + \frac{(\frac{2}{w_2} - \frac{1}{w_1})(\frac{1}{w_1} - 1)}{2}; & k \geq \frac{1}{w_1}, \quad 2 \leq \frac{1}{w_1} \leq \frac{1}{w_2}. \end{cases}$$

Recall that the Yau algebras of simple singularities and simple elliptic singularities were computed and a number of elaborate applications to deformation theory were presented in [2] and [19]. However, the Yau algebra can not characterize the simple singularties completely. In [6], it has been shown that if X and Y are two simple singularities except for the pair A_6 and D_5, then $L(X) \cong L(Y)$ as Lie algebras if and only if X and Y are analytically isomorphic. Therefore, it is natural to find new Lie algebras which can be used to distinguish singularities completely at least for the simple singularities. In [9], we introduced the k-th Yau algebra in the following way.

Based on Theorem 1.1, it is natural for us to introduce a new series of k-th Yau algebras $L^k(V)$, which are defined to be the Lie algebra of derivations of the k-th moduli algebra $A^k(V) = \mathcal{O}_n/(f, m^k J(f))$, $k \geq 0$, i.e., $L^k(V) = \mathrm{Der}_{\mathbb{C}}(A^k(V), A^k(V))$. Its dimension is denoted by $\lambda^k(V)$. This number $\lambda^k(V)$ is a new numerical analytic invariant of the singularities. We call it the k-th Yau number. In particular, $L^0(V)$ is exactly the Yau algebra, thus $L^0(V) = L(V)$, $\lambda^0(V) = \lambda(V)$. Therefore, it is reasonable to believe that these new Lie algebras $L^k(V)$ and the numerical invariants $\lambda^k(V)$ should also play important roles in the study of singularities.

Recall that the simple (i.e., ADE) singularities play a significant role in singularity theory [1]. They consist of two series $A_k : \{x_1^{k+1} + x_2^2 = 0\} \subset \mathbb{C}^2, k \geq 1$, $D_k : \{x_1^2 x_2 + x_2^{k-1} = 0\} \subset \mathbb{C}^2$, $k \geq 4$, and three exceptional singularities E_6, E_7, E_8 defined in $\mathbb{C}^2$ by polynomials $x_1^3 + x_2^4$, ${x_1}^3 + x_1 {x_2}^3$ and $x_1^3 + x_2^5$, respectively. Notice that each simple singularity belongs to one of these three series: A) $x_1^a + x_2^b$, B) $x_1^a x_2 + x_2^b$, and C) $x_1^a x_2 + x_2^b x_1$. These are called binomial singularities, a special kind of fewnomial singularities, see Definition 2.1.

Since $L(V)$ can not characterize the simple singularities completely [6], there is a natural question: whether the simple singularities (or which classes of more general singularities) can be characterized completely by the Lie algebra $L^k(V)$ for $k \in \mathbb{N}_{\geq 1}$? In [11], we proposed the following conjecture.

Conjecture 1.2 If X and Y are two ADE singularities, then $L^k(X) \cong L^k(Y)$ for any $k \in \mathbb{N}_{\geq 1}$ if and only if X and Y are analytically isomorphic.

Conjecture 1.2 has been confirmed positively for $k = 1$ in [11]. We shall validate this conjecture for $k = 2$, from which we could see that the k-th Yau algebra $L^k(V)$, $k \geq 1$, is more subtle compared to the Yau algebra $L(V)$.

Theorem D *If X and Y are two simple hypersurface singularities, then $L^2(X) \cong L^2(Y)$ as Lie algebras, if and only if X and Y are analytically isomorphic.*

2 Fewnomial singularities

Since we would also deal with new Lie algebras of fewnomial isolated singularities, we recall the definition of fewnomial isolated singularities. The concepts related to fewnomial are introduced in [13].

Definition 2.1 We say that a polynomial $f \in \mathbb{C}[z_1, z_2, \ldots, z_n]$ is fewnomial, if the number of monomials in f does not exceed n.

Obviously, the number of monomials in f may depend on the system of coordinates. In order to be rigorous, we shall only allow linear transformations of coordinates, and say that f (or rather its germ at the origin) is a k-nomial, if k is the smallest natural number such that f becomes a k-nomial after (possibly) a linear transformation of coordinates. An isolated hypersurface singularity V is called k-nomial, if there exists an isolated hypersurface singularity Y analytically isomorphic to V, which can be defined by a k-nomial and k is the smallest such number. It was shown that a singularity defined by a fewnomial f can be isolated, only if f is a n-nomial in n variables with its multiplicity at least 3.

Definition 2.2 We say that an isolated hypersurface singularity V is fewnomial , if it can be defined by a fewnomial polynomial f. V is called weighted homogeneous fewnomial isolated singularity, if it can be defined by a weighted homogeneous fewnomial polynomial f. A 2-nomial (resp. 3-nomial) isolated hypersurface singularity is also called binomial (resp. trinomial) singularity.

The following proposition and its corollary tells us that each simple singularity belongs to one of the three types of series.

Proposition 2.1 ([25]) *Let f be a weighted homogeneous fewnomial isolated hypersurface singularity with multiplicity at least 3. Then f is analytically equivalent to a linear combination of the following three series:*

Type A. $x_1^{a_1} + x_2^{a_2} + \cdots + x_{n-1}^{a_{n-1}} + x_n^{a_n}$, $n \geq 1$,

Type B. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}$, $n \geq 2$,

Type C. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1$, $n \geq 2$.

Corollary 2.1 ([25]) *Each binomial isolated singularity is analytically equivalent to one of the three series:*

Type A. $f = x_1^{a_1} + x_2^{a_2}$,

Type B. $f = x_1^{a_1}x_2 + x_2^{a_2}$,

Type C. $f = x_1^{a_1}x_2 + x_2^{a_2}x_1$.

In many situations, it is necessary to display explicitly all the basis of $A(V)$. And there always exist such basis consisting of monomials. This kind of basis is called monomial basis and will often be used in the sequel. Recall that the monomial basis of moduli algebras of simple singularities (A_k, D_k, E_6, E_7, E_8) are given in [1].

2.1 An algorithm to compute the k-th Tjurina number τ^k of binomial isolated singularities

Proposition 2.2 *Let $(V,0)$ be a binomial isolated singularity of type A, defined by $f = x_1^{a_1} + x_2^{a_2}$ $(2 \le a_1 \le a_2)$ with weight type $(\frac{1}{a_1}, \frac{1}{a_2}; 1)$. Then*

$$\tau^k(V) = \begin{cases} a_1a_2 - (a_1+a_2) + 1 + \frac{k^2+3k}{2}; & 0 \le k < a_1, \\ a_1k + \frac{(2a_2-a_1)(a_1-1)}{2}; & k \ge a_1. \end{cases}$$

Proof The Jacobian ideal of f is $J(f) := (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}) = (a_1x_1^{a_1-1}, a_2x_2^{a_2-1})$, and maximal ideal of $\mathcal{O}_n$ is $m = (x_1, x_2)$.

To compute the k-th Tjurina number, i.e., the dimension of the $\mathbb{C}$-module $A^k(f) = F(V)/\bar{m}^k\bar{J}(f)$, we first consider the local algebra $F(V) := \mathbb{C}\{x_1, x_2\}/(f) = \mathbb{C}\{\bar{x}_1, \bar{x}_2\}$;

then by a quotient again, we get $A^k(f) = F(V)/\bar{m}^k\bar{J}(f)$. Throughout this paper, we draw a bar over a term to denote its image in the corresponding quotient map.

According to the identity $\bar{f} = 0$ in $\mathbb{C}\{\bar{x}_1, \bar{x}_2\} = \mathbb{C}\{x_1, x_2\}/(f)$, we have the identity $\bar{x}_1^{a_1} = -\bar{x}_2^{a_2}(*)$. And then we have the following direct sum decomposition of $\mathbb{C}$-module $F(V)$:

$$\mathbb{C}\{\bar{x}_1, \bar{x}_2\} = \mathbb{C}\{\bar{x}_2\} + \mathbb{C}\{\bar{x}_2\}\bar{x}_1 + \mathbb{C}\{\bar{x}_2\}\bar{x}_1^2 + \cdots + \mathbb{C}\{\bar{x}_2\}\bar{x}_1^{a_1-1} = \sum_{i=0}^{a_1-1} \mathbb{C}\{\bar{x}_2\}\bar{x}_1^i. \quad (\sharp)$$

The right hand side of ($\sharp$) contains only finitely many terms, since all the higher order terms can be included in the first few terms, say

$$\mathbb{C}\{\bar{x}_2\}\bar{x}_1^{a_1} = \mathbb{C}\{\bar{x}_2\}\bar{x}_2^{a_2} \subset \mathbb{C}\{\bar{x}_2\},$$

by the identity $\bar{x}_1^{a_1} = -\bar{x}_2^{a_2}(*)$.

Next, in the local algebra $F(V)$, we give a simplification of $\bar{m}^k\bar{J}(f)$. There we use the notations $\bar{J}_1 = (\frac{\partial \bar{f}}{\partial \bar{x}_1})$, $\bar{J}_2 = (\frac{\partial \bar{f}}{\partial \bar{x}_2})$. And then when $k \ge 1$ we have

$$\begin{aligned} \bar{m}^k\bar{J}(f) &= (\bar{x}_1^k, \bar{x}_1^{k-1}\bar{x}_2, \ldots, \bar{x}_1\bar{x}_2^{k-1}, \bar{x}_2^k)(\bar{J}_1, \bar{J}_2) \\ &= \bar{x}_1\bar{J}_1(\bar{x}_1^{k-1}, \bar{x}_1^{k-2}\bar{x}_2, \ldots, \bar{x}_2^{k-1}) + \bar{x}_2^k\bar{J}_1 \\ &\quad + \bar{x}_2\bar{J}_2(\bar{x}_1^{k-1}, \bar{x}_1^{k-2}\bar{x}_2, \ldots, \bar{x}_1\bar{x}_2^{k-2}, \bar{x}_2^{k-1}) + \bar{x}_1^k\bar{J}_2 \\ &= \bar{x}_1(\bar{x}_1^{a_1-1})(\bar{x}_1, \bar{x}_2)^{k-1} + \bar{x}_2^k\bar{J}_1 + \bar{x}_2(\bar{x}_2^{a_2-1})(\bar{x}_1, \bar{x}_2)^{k-1} + \bar{x}_1^k\bar{J}_2. \end{aligned}$$

By the equality $\bar{x}_1^{a_1} = -\bar{x}_2^{a_2}(*)$ in the local algebra $F(V)$, when $k \ge 1$ we then have

$$\begin{aligned} \bar{m}^k\bar{J}(f) &= (\bar{x}_2^{a_2})(\bar{x}_1, \bar{x}_2)^{k-1} + \bar{x}_2^k\bar{J}_1 + \bar{x}_1^k\bar{J}_2 \\ &= \sum_{i=0}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + (\bar{x}_2^k\bar{x}_1^{a_1-1}) + (\bar{x}_1^k\bar{x}_2^{a_2-1}) \\ &= \sum_{i=0}^{k}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + (\bar{x}_2^k\bar{x}_1^{a_1-1}). \end{aligned}$$

When $k \geq a_1 \geq 2$, we have

$$\begin{aligned}\bar{m}^k \bar{J}(f) &\overset{*}{=} \sum_{i=0}^{a_1-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + \sum_{a_1}^{k}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + (\bar{x}_2^k\bar{x}_1^{a_1-1}) \\ &= \left[\sum_{i=0}^{a_1-2}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + (\bar{x}_2^{k+a_2-a_1}\bar{x}_1^{a_1-1})\right] + \sum_{a_1}^{k}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + (\bar{x}_2^k\bar{x}_1^{a_1-1}).\end{aligned}$$

By the assumption $a_1 \leq a_2$ in $f = x_1^{a_1} + x_2^{a_2}$, when $k \geq a_1 \geq 2$, we have inclusions of ideals:

$$\begin{aligned}&(\bar{x}_2^{k+a_2-a_1}\bar{x}_1^{a_1-1}) \subset (\bar{x}_2^k\bar{x}_1^{a_1-1}), \\ &\sum_{a_1}^{k}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) \subset \sum_{i=0}^{a_1-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i),\end{aligned}$$

and here the inclusion of $\sum_{a_1}^{k}(\bar{x}_2^{k+2a_2-1-i}\bar{x}_1^{i-a_1})$ arises from the fact that we have

$$\sum_{i=a_1}^{k}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) = \sum_{h=1}^{d-1}\sum_{i=ha_1}^{(h+1)a_1-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + \sum_{da_1}^{k}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i),$$

where $d \in \mathbb{N}_{\geq 1}$ is such that $k - (d-1)a_1 \geq a_1$, and $k - da_1 < a_1$.

Furthermore, by the identity $\bar{x}_1^{a_1} = \bar{x}_2^{a_2}(*)$, one has

- $\sum_{i=a_1}^{k}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) \overset{*}{=} \sum_{h=1}^{d-1}\sum_{i=ha_1}^{(h+1)a_1-1}(\bar{x}_2^{k+(h+1)a_2-1-i}\bar{x}_1^{i-ha_1}) + \sum_{i=da_1}^{k}(\bar{x}_2^{k+(d+1)a_2-1-i}\bar{x}_1^{i-da_1})$

$$= \sum_{h=1}^{d-1}\sum_{i=0}^{a_1-1}(\bar{x}_2^{k+(h+1)a_2-1-i-ha_1}\bar{x}_1^i) + \sum_{i=0}^{k-da_1}(\bar{x}_2^{k+(d+1)a_2-1-i-da_1}\bar{x}_1^i);$$

- $1 \leq h \leq d-1, a_1 \leq a_2, k - da_1 < a_1$

$$\Rightarrow \sum_{h=1}^{d-1}\sum_{i=0}^{a_1-1}(\bar{x}_2^{k+(h+1)a_2-1-i-ha_1}\bar{x}_1^i) \subset \sum_{i=0}^{a_1-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i),$$

$$\sum_{i=0}^{k-da_1}(\bar{x}_2^{k+(d+1)a_2-1-i-da_1}\bar{x}_1^i) \subset \sum_{i=0}^{a_1-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i).$$

Consequently, we have the simplification of $\bar{m}^k\bar{J}(f)$ when $k \geq a_1 \geq 2$:

$$\bar{m}^k\bar{J}(f) = \sum_{i=0}^{a_1-2}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + (\bar{x}_2^k\bar{x}_1^{a_1-1}).$$

And then, when $k \geq a_1$, we have for $f = x_1^{a_1} + x_2^{a_2}$ $(2 \leq a_1 \leq a_2)$:

$$\begin{aligned}\tau^k(f) &= \dim\frac{\mathbb{C}\{x_1, x_2\}}{(f) + m^kJ(f)} = \dim\frac{\mathbb{C}\{\bar{x}_1, \bar{x}_2\}}{\bar{m}^k\bar{J}(f)} \\ &= \dim\frac{\sum_{i=0}^{a_1-1}\mathbb{C}\{\bar{x}_2\}\bar{x}_1^i}{\sum_{i=0}^{a_1-2}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + (\bar{x}_2^k\bar{x}_1^{a_1-1})} = \sum_{i=0}^{a_1-2}(k + a_2 - 1 - i) + k \\ &= a_1k + a_2a_1 - a_2 - \frac{a_1^2}{2} + \frac{a_1}{2}.\end{aligned}$$

If $k < a_1 \leq a_2$, we still have

$$\mathbb{C}\{\bar{x}_1, \bar{x}_2\} = \sum_{i=0}^{a_1-1} \mathbb{C}\{\bar{x_2}\}\bar{x}_1^i,$$

$$\bar{m}^k \bar{J}(f) = \sum_{i=0}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + (\bar{x}_2^k\bar{x}_1^{a_1-1}) + (\bar{x}_2^{a_2-1}\bar{x}_1^k).$$

If there is the case that $0 \leq k = a_1 - 1 < a_1 \leq a_2$, then

$$(\bar{x}_2^k\bar{x}_1^{a_1-1}) = (\bar{x}_2^{a_1-1}\bar{x}_1^{a_1-1}), \quad (\bar{x}_2^{a_2-1}\bar{x}_1^k) = (\bar{x}_2^{a_2-1}\bar{x}_1^{a_1-1}),$$
$$\Rightarrow (\bar{x}_2^{a_2-1}\bar{x}_1^k) \subset (\bar{x}_2^k\bar{x}_1^{a_1-1}).$$

Then there we have

$$\begin{aligned}\tau^k &= \dim \frac{\sum_{i=0}^{a_1-1} \mathbb{C}\{\bar{x_2}\}\bar{x}_1^i}{\sum_{i=0}^{a_1-2}(\bar{x}_2^{a_1+a_2-2-i}\bar{x}_1^i) + (\bar{x}_2^{a_1-1}\bar{x}_1^{a_1-1})} \\ &= \left[\sum_{i=0}^{a_1-2}(a_1 + a_2 - 2 - i)\right] + (a_1 - 1) \\ &= a_1a_2 - (a_1 + a_2) + 1 + \frac{(a_1-1)^2 + 3(a_1-1)}{2}.\end{aligned}$$

If $0 \leq k < a_1 - 1 < a_2$, then
we can take an expression of $\bar{m}^k \bar{J}(f)$:

$$\bar{m}^k \bar{J}(f) = \sum_{i=0}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + (\bar{x}_2^{a_2-1}\bar{x}_1^k) + (\bar{x}_2^k\bar{x}_1^{a_1-1}) + \sum_{i=k+1}^{a_1-2}(\bar{x}_2^{a_2-1}\bar{x}_1^i).$$

There we choose an equivalent expression of $\bar{m}^k \bar{J}(f)$, and then we do the quotient:

$$\begin{aligned}\tau^k(f) &= \dim \frac{\sum_{i=0}^{a_1-1} \mathbb{C}\{\bar{x_2}\}\bar{x}_1^i}{\sum_{i=0}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + (\bar{x}_2^{a_2-1}\bar{x}_1^k) + (\bar{x}_2^k\bar{x}_1^{a_1-1}) + \sum_{i=k+1}^{a_1-2}(\bar{x}_2^{a_2-1}\bar{x}_1^i)} \\ &= \left[\sum_{i=0}^{k-1}(k + a_2 - 1 - i)\right] + (a_2 - 1) + k + (a_1 - k - 2)(a_2 - 1) \\ &= a_1a_2 - (a_1 + a_2) + 1 + \frac{k^2 + 3k}{2}.\end{aligned}$$

Consequently, we get the whole formula of $\tau^k(f)$, that is:

$$\tau^k(V) = \begin{cases} a_1a_2 - (a_1 + a_2) + 1 + \frac{k^2+3k}{2}; & 0 \leq k < a_1, \\ a_1k + a_2a_1 - a_2 - \frac{a_1^2}{2} + \frac{a_1}{2}; & k \geq a_1. \end{cases}$$

□

Algorithm. Let f be any binomial isolated singularity. The proof of Proposition 2.2 provides an algorithm for computing the k-th Tjurina number $\tau^k(f)$ of the binomial isolated singularity f.

Step 1. Do a direct sum decomposition of the local algebra $F(V) = \frac{\mathbb{C}\{x_1,x_2\}}{f} =: \mathbb{C}\{\bar{x}_1, \bar{x}_2\}$ using the equality $\bar{f} = 0$ in $F(V)$.

Step 2. Simplify the expression of $\bar{m}^k \bar{J}(f)$ in $\mathbb{C}\{\bar{x}_1, \bar{x}_2\}$, i.e., the image of $\bar{m}^k \bar{J}(f)$ in $F(V)$. Usually, we first give the simplification under the assumption that $k \geq h_0$, where h_0 is some positive integer.

Step 3. After having a simplified expression of $\bar{m}^k \bar{J}(f)$ under the condition that $k \geq h_0$, we do the quotient $F(V)/\bar{m}^k \bar{J}(f)$ and determine its dimension, which is exactly the k-th Tjurina number for $k \geq h_0$. As one can see, till this step, we may only have parts of the results due to some restrictions of k, say $k \geq h_0$, needed in the simplifications above. To complete the results for all possible values of k, we need the last step.

Step 4. Complete the formulas of $\tau^k(f)$ for all possible values of k. In general, in this step, conversely to the idea in Step 2 to some extent, we add some terms to the simplified expression of $\bar{m}^k \bar{J}(f)$ keeping it an equivalent expression of $\bar{m}^k \bar{J}(f)$. Adding some terms is an operation which makes $\bar{m}^k \bar{J}(f)$ seem more complicated but quite useful when we do the quotient $F(V)/\bar{m}^k \bar{J}(f)$.

In the next part, following the steps in the Algorithm, we shall give the formulas for the other two series of binomial isolated singularities: (B) $f = x_1^{a_1} x_2 + x_2^{a_2}$ and (C) $f = x_1^{a_1} x_2 + x_2^{a_2} x_1$, respectively.

Lemma 2.1 *For (B) $f = x_1^{a_1} x_2 + x_2^{a_2}, a_1 \geq 1$, and (C) $f = x_1^{a_1} x_2 + x_2^{a_2} x_1, a_1 \geq 2$, their local algebras $F(V) = \frac{\mathbb{C}\{x_1,x_2\}}{f} = \mathbb{C}\{\bar{x}_1, \bar{x}_2\}$, as modules over $\mathbb{C}$, have the same forms direct sum decomposition*

$$\mathbb{C}\{\bar{x}_1, \bar{x}_2\} = \sum_{i=0}^{a_1-1} \mathbb{C}\{\bar{x}_2\}\bar{x}_1^i + \mathbb{C}\{\bar{x}_1\}\bar{x}_1^{a_1}.$$

Proof (1) For $f = x_1^{a_1} x_2 + x_2^{a_2}$, there is an identity $\bar{x}_1^{a_1}\bar{x}_2 = -\bar{x}_2^{a_2} (*)$ due to the equality $\bar{f} = 0$ in $\mathbb{C}\{\bar{x}_1, \bar{x}_2\}$. Thus we have a series of equalities:

$$\begin{aligned}
&\mathbb{C}\{\bar{x}_2\}\bar{x}_1^{a_1} = \mathbb{C}\bar{x}_1^{a_1} + \mathbb{C}\{\bar{x}_2\}\bar{x}_2\bar{x}_1^{a_1} \overset{*}{=} \mathbb{C}\bar{x}_1^{a_1} + \mathbb{C}\{\bar{x}_2\}\bar{x}_2^{a_2},\\
&\mathbb{C}\{\bar{x}_2\}\bar{x}_1^{a_1+1} = \mathbb{C}\bar{x}_1^{a_1+1} + \mathbb{C}\{\bar{x}_2\}\bar{x}_2^{a_2}\bar{x}_1,\\
&\ldots,\\
&\mathbb{C}\{\bar{x}_2\}\bar{x}_1^{2a_1-1} = \mathbb{C}\bar{x}_1^{2a_1-1} + \mathbb{C}\{\bar{x}_2\}\bar{x}_2^{a_2}\bar{x}_1^{a_1-1},\\
&\mathbb{C}\{\bar{x}_2\}\bar{x}_1^{2a_1} = \mathbb{C}\bar{x}_1^{2a_1} + \mathbb{C}\{\bar{x}_2\}\bar{x}_2^{a_2}\bar{x}_1^{a_1} \overset{*}{=} \mathbb{C}\bar{x}_1^{2a_1} + \mathbb{C}\{\bar{x}_2\}\bar{x}_2^{2a_2-1},\\
&\ldots.
\end{aligned}$$

By the equalities above, we observe a periodic result that all the second parts of $\mathbb{C}\{\bar{x}_2\}\bar{x}_1^l$, $l \geq a_1$, are contained in $\sum_{i=0}^{a_1-1} \mathbb{C}\{\bar{x}_2\}\bar{x}_1^i$. The reason is that if $l = l_1 (\text{mod } a_1), 0 \leq l_1 < a_1$, that is, if there exists such a positive integer $d \in \mathbb{N}_{\geq 0}$ that $0 \leq l - (d-1)a_1 \geq a_1$, and $l - da_1 < a_1$, we then have

$$\mathbb{C}\{\bar{x}_2\}\bar{x}_1^l \overset{*}{=} \mathbb{C}\bar{x}_1^l + \mathbb{C}\{\bar{x}_2\}\bar{x}_2^{da_2-(d-1)}\bar{x}_1^{l-da_1},$$

and an inclusion of the second parts of $\mathbb{C}\{\bar{x}_2\}\bar{x}_1^l$:

$$\mathbb{C}\{\bar{x}_2\}\bar{x}_2^{da_2-(d-1)}\bar{x}_1^{l-da_1} \subset \sum_{i=0}^{a_1-1} \mathbb{C}\{\bar{x}_2\}\bar{x}_1^i.$$

Thus we have

$$\mathbb{C}\{\bar{x}_1,\bar{x}_2\}=\sum_{i=0}^{a_1-1}\mathbb{C}\{\bar{x}_2\}\bar{x}_1^i+\sum_{i=a_1}^{\infty}\mathbb{C}\{\bar{x}_2\}\bar{x}_1^i=\sum_{i=0}^{a_1-1}\mathbb{C}\{\bar{x}_2\}\bar{x}_1^i+\sum_{i=0}^{\infty}\mathbb{C}\bar{x}_1^{a_1+i}$$
$$=\sum_{i=0}^{a_1-1}\mathbb{C}\{\bar{x}_2\}\bar{x}_1^i+\mathbb{C}\{\bar{x}_1\}\bar{x}_1^{a_1}.$$

(2) For $f=x_1^{a_1}x_2+x_2^{a_2}x_1(a_1\geq 2)$, there is an identity $\bar{x}_1^{a_1}\bar{x}_2=-\bar{x}_2^{a_2}\bar{x}_1(*)$ due to the equality $\bar{f}=0$ in $\mathbb{C}\{\bar{x}_1,\bar{x}_2\}$. Thus we have a series of equalities:

$$\begin{aligned}
&\mathbb{C}\{\bar{x}_2\}\bar{x}_1^{a_1}=\mathbb{C}\bar{x}_1^{a_1}+\mathbb{C}\{\bar{x}_2\}\bar{x}_2\bar{x}_1^{a_1}\overset{*}{=}\mathbb{C}\bar{x}_1^{a_1}+\mathbb{C}\{\bar{x}_2\}\bar{x}_2^{a_2}\bar{x}_1,\\
&\mathbb{C}\{\bar{x}_2\}\bar{x}_1^{a_1+1}=\mathbb{C}\bar{x}_1^{a_1+1}+\mathbb{C}\{\bar{x}_2\}\bar{x}_2^{a_2}\bar{x}_1^2,\\
&\ldots,\\
&\mathbb{C}\{\bar{x}_2\}\bar{x}_1^{2a_1-2}=\mathbb{C}\bar{x}_1^{2a_1-2}+\mathbb{C}\{\bar{x}_2\}\bar{x}_2^{a_2}\bar{x}_1^{a_1-1},\\
&\mathbb{C}\{\bar{x}_2\}\bar{x}_1^{2a_1-1}=\mathbb{C}\bar{x}_1^{2a_1-1}+\mathbb{C}\{\bar{x}_2\}\bar{x}_2^{a_2}\bar{x}_1^{a_1}\overset{*}{=}\mathbb{C}\bar{x}_1^{2a_1-1}+\mathbb{C}\{\bar{x}_2\}\bar{x}_2^{2a_2-1}\bar{x}_1,\\
&\ldots.
\end{aligned}$$

Like in case (1), we observe a similar periodic result that all the second parts of $\mathbb{C}\{\bar{x}_2\}\bar{x}_1^l$, $l\geq a_1$, are contained in $\sum_{i=1}^{a_1-1}\mathbb{C}\{\bar{x}_2\}\bar{x}_1^i$, since if there exist such a positive integer $d\in\mathbb{N}_{\geq 0}$ that $l-(d-1)(a_1-1)\geq a_1-1$, and $l-d(a_1-1)<a_1-1$, one has

$$\mathbb{C}\{\bar{x}_2\}\bar{x}_1^l=\mathbb{C}\bar{x}_1^l+\mathbb{C}\{\bar{x}_2\}\bar{x}_2^{da_2-(d-1)}\bar{x}_1^{l-d(a_1-1)+1},$$

and the inclusion of the second parts of $\mathbb{C}\{\bar{x}_2\}\bar{x}_1^l$:

$$\mathbb{C}\{\bar{x}_2\}\bar{x}_2^{da_2-(d-1)}\bar{x}_1^{l-d(a_1-1)+1}\subset\sum_{i=1}^{a_1-1}\mathbb{C}\{\bar{x}_2\}\bar{x}_1^i.$$

Thus we have the expression of the local algebra of $f=x_1^{a_1}x_2+x_2^{a_2}x_1$:

$$\mathbb{C}\{\bar{x}_1,\bar{x}_2\}=\sum_{i=0}^{a_1-1}\mathbb{C}\{\bar{x}_2\}\bar{x}_1^i+\sum_{i=a_1}^{\infty}\mathbb{C}\{\bar{x}_2\}\bar{x}_1^i=\sum_{i=0}^{a_1-1}\mathbb{C}\{\bar{x}_2\}\bar{x}_1^i+\mathbb{C}\{\bar{x}_1\}\bar{x}_1^{a_1}.$$

□

Next we shall give, to some extent, simplified expressions of $\bar{m}^k\bar{J}(f)$ in their local algebras $\mathbb{C}\{\bar{x}_1,\bar{x}_2\}$ of the binomial singularity of types (B) and (C).

Remark 2.1 A weighted homogeneous polynomial f of type $(w_1,\ldots,w_n;d)$ satisfies the Euler identity:

$$d\cdot f=\sum_{i=1}^{n}w_ix_i\frac{\partial f}{\partial x_i}\quad\text{in }\mathbb{C}\{\boldsymbol{x}\}.$$

In our case, the identity:

$$d\cdot f=w_1x_1\frac{\partial f}{\partial x_1}+w_2x_2\frac{\partial f}{\partial x_2},$$

leads to equalities of elements in the local algebra $\mathbb{C}\{x_1, x_2\}/(f)$:

$$0 = \bar{f} \in (\bar{f}) \Rightarrow \bar{x}_1 \frac{\partial \bar{f}}{\partial \bar{x}_1} = \bar{x}_2 \frac{\partial \bar{f}}{\partial \bar{x}_2}, \tag{*}$$

as well as of ideals:

$$\bar{x}_1 \bar{J}_1 = \bar{x}_2 \bar{J}_2, \tag{**}$$

where we use the notation $(J_1, J_2) := ((\frac{\partial f}{\partial x_1}), (\frac{\partial f}{\partial x_2}))$, and then a notation for $\bar{J}(f)$, that is, image of $J(f)$ in the local algebra, i.e., $\bar{J}(f) = \bar{J}_1 + \bar{J}_2$.

Using the equality (**) here, we have a kind of simplifications of $\bar{m}^k \bar{J}(f)$ for general cases.

Lemma 2.2 *For each binomial isolated singularity f, the ideal $\bar{m}^k \bar{J}(f)$ has the following kind of simplified form, whenever $k \geq 1$, in its local algebra $F(V) = \mathbb{C}\{\bar{x}_1, \bar{x}_2\}$:*

$$\bar{m}^k \bar{J}(f) = \bar{x}_1 \bar{J}_1 (\bar{x}_1, \bar{x}_2)^{k-1} + \bar{x}_2^k \bar{J}_1 + \bar{x}_1^k \bar{J}_2.$$

Proof By Remark 2.1, one has

$$\begin{aligned} \bar{m}^k \bar{J}(f) &= (\bar{x}_1^k, \bar{x}_1^{k-1}\bar{x}_2, \ldots, \bar{x}_1\bar{x}_2^{k-1}, \bar{x}_2^k)(\bar{J}_1 + \bar{J}_2) \\ &= \bar{x}_1 \bar{J}_1 (\bar{x}_1^{k-1}, \ldots, \bar{x}_2^{k-1}) + \bar{x}_2^k \bar{J}_1 + \bar{x}_2 \bar{J}_2 (\bar{x}_1^{k-1}, \ldots, \bar{x}_2^{k-1}) + \bar{x}_1^k \bar{J}_2 \\ &= \bar{x}_1 \bar{J}_1 (\bar{x}_1, \bar{x}_2)^{k-1} + \bar{x}_2^k \bar{J}_1 + \bar{x}_2 \bar{J}_2 (\bar{x}_1, \bar{x}_2)^{k-1} + \bar{x}_1^k \bar{J}_2 \\ &\overset{(**)}{=} \bar{x}_1 \bar{J}_1 (\bar{x}_1, \bar{x}_2)^{k-1} + \bar{x}_2^k \bar{J}_1 + \bar{x}_1^k \bar{J}_2. \end{aligned}$$

□

Lemma 2.3 *For type (B) $f = x_1^{a_1} x_2 + x_2^{a_2}$, in the local algebra $F(V) = \mathbb{C}\{\bar{x}_1, \bar{x}_2\}$, $\bar{m}^k \bar{J}(f)$ has a simplification when $k \geq 1$:*

$$\sum_{i=0}^{k-1} (\bar{x}_2^{k+a_2-1-i} \bar{x}_1^i) + (\bar{x}_2^{k+1} \bar{x}_1^{a_1-1}) + (\bar{x}_1^{k+a_1} + a_2 \bar{x}_1^k \bar{x}_2^{a_2-1}),$$

and specifically we have

(1A). *If $3 \leq a_1 + 1 \leq a_2$ and $k \geq a_1 + 1$, there is*

$$\bar{m}^k \bar{J}(f) = \sum_{i=0}^{a_1-2} (\bar{x}_2^{k+a_2-1-i} \bar{x}_1^i) + (\bar{x}_2^{k+1} \bar{x}_1^{a_1-1}) + (\bar{x}_1^{k+a_1});$$

(1B). *If $2 = a_1 + 1 \leq a_2$ and $k \geq 1$, there is*

$$\bar{m}^k \bar{J}(f) = (\bar{x}_2^{k+1}) + (\bar{x}_1^{k+1});$$

(2A). *If $a_1 + 1 \geq a_2 \geq 2$ and $a_1 \geq k \geq a_2$, there is:*

$$\begin{aligned} \bar{m}^k \bar{J}(f) = \sum_{i=0}^{k-a_2} (\bar{x}_2^{2a_2-1} \bar{x}_1^i) + \sum_{i=k-a_2+1}^{k-1} (\bar{x}_2^{k+a_2-1-i} \bar{x}_1^i) \\ + \sum_{i=k}^{a_1-1} (\bar{x}_2^{a_2} \bar{x}_1^i) + (\bar{x}_1^{k+a_1} + a_2 \bar{x}_2^{a_2-1} \bar{x}_1^k); \end{aligned}$$

(2B). *If* $a_1 + 1 \geq a_2 \geq 2$ *and* $k > a_1 \geq a_2 - 1$, *there is:*

$$\bar{m}^k \bar{J}(f) = \sum_{i=0}^{k-1-da_1} (\bar{x}_2^{k+(d+1)a_2-da_1-(d+1)-i} \bar{x}_1^i) + \sum_{i=k-da_1}^{k-da_1+a_1-a_2} (\bar{x}_2^{(d+1)a_2-d} \bar{x}_1^i)$$
$$+ \sum_{i=k-da_1+a_1+1-a_2}^{a_1-1} (\bar{x}_2^{k+a_2+(d-1)(a_2-a_1)-d-i} \bar{x}_1^i) + (\bar{x}_1^{k+a_1}).$$

Proof The image of the Jacobian ideal of $f = x_1^{a_1} x_2 + x_2^{a_2}$ in its local algebra is

$$\bar{J}(f) = (\bar{x}_1^{a_1-1}\bar{x}_2, \bar{x}_1^{a_1} + a_2 \bar{x}_2^{a_2-1}).$$

By Lemma 2.2, when $k \geq 1$, we have

$$\begin{aligned}
\bar{m}^k \bar{J}(f) &= \bar{x}_1 \bar{J}_1 (\bar{x}_1, \bar{x}_2)^{k-1} + \bar{x}_2^k \bar{J}_1 + \bar{x}_1^k \bar{J}_2 \\
&= \bar{x}_1 (\bar{x}_1^{a_1-1}\bar{x}_2)(\bar{x}_1, \bar{x}_2)^{k-1} + \bar{x}_2^k (\bar{x}_1^{a_1-1}\bar{x}_2) + \bar{x}_1^k (\bar{x}_1^{a_1} + a_2 \bar{x}_2^{a_2-1}) \\
&\overset{*}{=} (\bar{x}_2^{a_2})(\bar{x}_1, \bar{x}_2)^{k-1} + \bar{x}_2^k (\bar{x}_1^{a_1-1}\bar{x}_2) + \bar{x}_1^k (\bar{x}_1^{a_1} + a_2 \bar{x}_2^{a_2-1}) \\
&= \sum_{i=0}^{k-1} (\bar{x}_2^{k+a_2-1-i} \bar{x}_1^i) + (\bar{x}_2^{k+1} \bar{x}_1^{a_1-1}) + (\bar{x}_1^{k+a_1} + a_2 \bar{x}_2^{a_2-1} \bar{x}_1^k).
\end{aligned}$$

(1). When $2 \leq a_1 + 1 \leq a_2$, we first have that

$$\bar{m}^k \bar{J}(f) = \sum_{i=0}^{k-1} (\bar{x}_2^{k+a_2-1-i} \bar{x}_1^i) + (\bar{x}_2^{k+1} \bar{x}_1^{a_1-1}) + (\bar{x}_1^{k+a_1} + a_2 \bar{x}_2^{a_2-1} \bar{x}_1^k).$$

(1A). If $a_1 + 1 \leq a_2$, $k \geq a_1 + 1$, firstly we have

$$\bar{m}^k \bar{J}(f) = \sum_{i=0}^{a_1-1} (\bar{x}_2^{k+a_2-1-i} \bar{x}_1^i) + \sum_{i=a_1}^{k-1} (\bar{x}_2^{k+a_2-1-i} \bar{x}_1^i) + (\bar{x}_2^{k+1} \bar{x}_1^{a_1-1})$$
$$+ (\bar{x}_1^{k+a_1} + a_2 \bar{x}_2^{a_2-1} \bar{x}_1^k).$$

Take $i = a_1 - 1$ in $(\bar{x}_2^{k+a_2-1-i} \bar{x}_1^i)$, we have

$$a_1 + 1 \leq a_2 \Rightarrow (\bar{x}_2^{k+a_2-a_1} \bar{x}_1^{a_1-1}) \subset (\bar{x}_2^{k+1} \bar{x}_1^{a_1-1}).$$

Then when $3 \leq a_1 + 1 \leq a_2$, we have

$$\sum_{i=0}^{a_1-1} (\bar{x}_2^{k+a_2-1-i} \bar{x}_1^i) + (\bar{x}_2^{k+1} \bar{x}_1^{a_1-1}) = \sum_{i=0}^{a_1-2} (\bar{x}_2^{k+a_2-1-i} \bar{x}_1^i) + (\bar{x}_2^{k+1} \bar{x}_1^{a_1-1}), \tag{1A.1}$$

and this leads to the equality:

$$\bar{m}^k \bar{J}(f) = \sum_{i=0}^{a_1-2} (\bar{x}_2^{k+a_2-1-i} \bar{x}_1^i) + (\bar{x}_2^{k+1} \bar{x}_1^{a_1-1}) + \sum_{i=a_1}^{k-1} (\bar{x}_2^{k+a_2-1-i} \bar{x}_1^i)$$
$$+ (\bar{x}_1^{k+a_1} + a_2 \bar{x}_2^{a_2-1} \bar{x}_1^k).$$

Considering $\sum_{i=a_1}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i)$ above, we use the similar argument in Lemma 2.1 about the decomposition of $\mathbb{C}\{\bar{x}_1, \bar{x}_2\}$. This argument shows that we can always lower the power of $\bar{x}_1$ in $(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i)$ of the sum $\sum_{i=a_1}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i)$ to a number less than or equal to $a_1 - 1$, by the identity $\bar{x}_1^{a_1}\bar{x}_2 = \bar{x}_2^{a_2}(*)$. After this reduction, we have

$$\sum_{i=a_1}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) \subset \sum_{i=0}^{a_1-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i). \tag{1A.2}$$

To explain (1A.2) explicitly, there is the equality:

$$\sum_{i=a_1}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) = \sum_{h=1}^{d-1}\sum_{i=ha_1}^{(h+1)a_1-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + \sum_{da_1}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i),$$

where $d \in \mathbb{N}_{\geq 1}$ is such that $k - 1 - (d-1)a_1 \geq a_1$, and $k - 1 - da_1 < a_1$.

Furthermore, by the identity $\bar{x}_1^{a_1}\bar{x}_2 = \bar{x}_2^{a_2}(*)$, one has

$$\begin{aligned}
&\sum_{i=a_1}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) \qquad (1A.3)\\
&\overset{*}{=} \sum_{h=1}^{d-1}\sum_{i=ha_1}^{(h+1)a_1-1}(\bar{x}_2^{k+(h+1)a_2-h-1-i}\bar{x}_1^{i-ha_1}) + \sum_{i=da_1}^{k-1}(\bar{x}_2^{k+(d+1)a_2-d-1-i}\bar{x}_1^{i-da_1})\\
&= \sum_{h=1}^{d-1}\sum_{i=0}^{a_1-1}(\bar{x}_2^{k+(h+1)a_2-h-1-i-ha_1}\bar{x}_1^i) + \sum_{i=0}^{k-1-da_1}(\bar{x}_2^{k+(d+1)a_2-d-1-i-da_1}\bar{x}_1^i).
\end{aligned}$$

And since there is

$$a_1 + 1 \leq a_2 \Rightarrow k + (h+1)a_2 - h - 1 - i - ha_1 > k + a_2 - 1 - i, \quad \forall 1 \leq h \leq d-1.$$

This fact leads to the inclusion:

$$\sum_{h=1}^{d-1}\sum_{i=0}^{a_1-1}(\bar{x}_2^{k+(h+1)a_2-h-1-i-ha_1}\bar{x}_1^i) \subset \sum_{i=0}^{a_1-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i). \tag{1A.4}$$

Take $h = 1$ in $\sum_{i=0}^{a_1-1}(\bar{x}_2^{k+(h+1)a_2-h-1-i-ha_1}\bar{x}_1^i)$, we have $\sum_{i=0}^{a_1-1}(\bar{x}_2^{k+2a_2-2-i-a_1}\bar{x}_1^i)$. Thus by the assumption $0 \leq k - 1 - da_1 < a_1$ and $d \geq 1$, we have

$$\begin{aligned}
&k + (d+1)a_2 - d - 1 - i - da_1 \geq k + 2a_2 - 2 - i - a_1, 0 \leq i \leq k - 1 - da_1 \qquad (1A.5)\\
&\Rightarrow \sum_{i=0}^{k-1-da_1}(\bar{x}_2^{k+(d+1)a_2-d-1-i-da_1}\bar{x}_1^i) \subset \sum_{h=1}^{d-1}\sum_{i=0}^{a_1-1}(\bar{x}_2^{k+(h+1)a_2-h-1-i-ha_1}\bar{x}_1^i).
\end{aligned}$$

By (1A.4) and (1A.5), we then have (1A.2):

$$\sum_{i=a_1}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) \subset \sum_{i=0}^{a_1-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i).$$

Furthermore,

we also have

$$a_2x_2^{a_2-1}x_1^k \in \sum_{i=0}^{a_1-2}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + (\bar{x}_2^{k+1}\bar{x}_1^{a_1-1}).$$

Thus we come to the result that for $f = x_1^{a_1}x_2 + x_2^{a_2}$, if $3 \le a_1+1 \le a_2$, $k \ge a_1+1$, one has

$$\bar{m}^k\bar{J}(f) = \sum_{i=0}^{a_1-2}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + (\bar{x}_2^{k+1}\bar{x}_1^{a_1-1}) + (\bar{x}_1^{k+a_1}).$$

(1B). If $2 = a_1+1 \le a_2, k \ge 1$, then for $f = \bar{x}_1\bar{x}_2 + \bar{x}_2^{a_2}$, according to Lemma 2.2, one has

$$\begin{aligned}\bar{m}^k\bar{J}(f) &= \bar{x}_1\bar{J}_1(\bar{x}_1,\bar{x}_2)^{k-1} + \bar{x}_2^k\bar{J}_1 + \bar{x}_1^k\bar{J}_2\\ &= \sum_{i=0}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + (\bar{x}_2^{k+1}\bar{x}_1^{a_1-1}) + (\bar{x}_1^{k+a_1} + a_2\bar{x}_2^{a_2-1}\bar{x}_1^k)\\ &= \sum_{i=0}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + (\bar{x}_2^{k+1}) + (\bar{x}_1^{k+1} + a_2\bar{x}_2^{a_2-1}\bar{x}_1^k).\end{aligned}$$

There similarly hold the inclusions:

$$\sum_{i=0}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) \subset (\bar{x}_2^{k+a_2-1}) \subset (\bar{x}_2^{k+1}),$$

$$a_2\bar{x}_2^{a_2-1}\bar{x}_1^k \in \sum_{i=0}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i).$$

And then when $2 = a_1+1 \le a_2, k \ge 1$, we have

$$\bar{m}^k\bar{J}(f) = (\bar{x}_2^{k+1}) + (\bar{x}_1^{k+1}).$$

(2). (2A). If $a_1+1 \ge a_2 \ge 2, a_1-1 \ge k \ge a_2 \ge 2$, we still have

$$\bar{m}^k\bar{J}(f) = \sum_{i=0}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + (\bar{x}_2^{k+1}\bar{x}_1^{a_1-1}) + (\bar{x}_1^{k+a_1} + a_2\bar{x}_2^{a_2-1}\bar{x}_1^k).$$

And the inequality $a_1-1 \ge k$ leads to the equality:

$$\sum_{i=0}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + (\bar{x}_2^{k+1}\bar{x}_1^{a_1-1}) = \sum_{i=0}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + \sum_{i=k}^{a_1-1}(\bar{x}_2^{a_2}\bar{x}_1^i) + (\bar{x}_2^{k+1}\bar{x}_1^{a_1-1}),$$

and $k \ge a_2$ leads to the following inclusion of ideals:

$$(\bar{x}_2^{k+1}\bar{x}_1^{a_1-1}) \subset (\bar{x}_2^{a_2}\bar{x}_1^{a_1-1}) \subset \sum_{i=k}^{a_1-1}(\bar{x}_2^{a_2}\bar{x}_1^i) \subset \sum_{i=0}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i). \tag{2A.1}$$

Thus we have

$$(\bar{x}_2^{k+1}\bar{x}_1^{a_1-1}) \subset \sum_{i=0}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i), \tag{2A.2}$$

$$\bar{m}^k \bar{J}(f) = \sum_{i=0}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + (\bar{x}_2^{k+1}\bar{x}_1^{a_1-1}) + (\bar{x}_1^{k+a_1} + a_2\bar{x}_2^{a_2-1}\bar{x}_1^k)$$
$$= \sum_{i=0}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + \sum_{i=k}^{a_1-1}(\bar{x}_2^{a_2}\bar{x}_1^i) + (\bar{x}_1^{k+a_1} + a_2\bar{x}_2^{a_2-1}\bar{x}_1^k).$$

In fact, this expression is still not simplified enough, we need to do more.

As one can see, by $\bar{x}_1^{a_1}\bar{x}_2 = \bar{x}_2^{a_2}(*)$, there is

$$(\bar{x}_2^{2a_2-1}) \stackrel{*}{=} (\bar{x}_2^{a_2}\bar{x}_1^{a_1}) \subset \sum_{i=k}^{a_1-1}(\bar{x}_2^{a_2}\bar{x}_1^i), \tag{2A.3}$$

and $k \geq a_2$ leads to the inclusion:

$$\sum_{i=0}^{k-a_2}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) \subset (\bar{x}_2^{2a_2-1}) \subset \sum_{i=k}^{a_1-1}(\bar{x}_2^{a_2}\bar{x}_1^i). \tag{2A.3'}$$

Thus if $a_1 - 1 \geq k \geq a_2 \geq 2$, we have

$$\bar{m}^k \bar{J}(f) = \sum_{i=0}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + \sum_{i=k}^{a_1-1}(\bar{x}_2^{a_2}\bar{x}_1^i) + (\bar{x}_1^{k+a_1} + a_2\bar{x}_2^{a_2-1}\bar{x}_1^k)$$
$$= \sum_{i=k-a_2+1}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + \sum_{i=k}^{a_1-1}(\bar{x}_2^{a_2}\bar{x}_1^i) + (\bar{x}_1^{k+a_1} + a_2\bar{x}_2^{a_2-1}\bar{x}_1^k).$$

To do quotient of $\mathbb{C}\{\bar{x}_1, \bar{x}_2\}$ by $\bar{m}^k \bar{J}(f)$ in the sequel, we add some terms to this expression just like we add $\sum_{i=k}^{a_1-1}(\bar{x}_2^{a_2}\bar{x}_1^i)$ above.

Thus when $a_1 - 1 \geq k \geq a_2 \geq 2$, we can take the following form of $\bar{m}^k \bar{J}(f)$:

$$\bar{m}^k \bar{J}(f) = \sum_{i=0}^{k-a_2}(\bar{x}_2^{2a_2-1}\bar{x}_1^i) + \sum_{i=k-a_2+1}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i)$$
$$+ \sum_{i=k}^{a_1-1}(\bar{x}_2^{a_2}\bar{x}_1^i) + (\bar{x}_1^{k+a_1} + a_2\bar{x}_2^{a_2-1}\bar{x}_1^k).$$

(2B). If $k \geq a_1 \geq a_2 - 1$, still we have

$$\bar{m}^k \bar{J}(f) = \sum_{i=0}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + (\bar{x}_2^{k+1}\bar{x}_1^{a_1-1}) + (\bar{x}_1^{k+a_1} + a_2\bar{x}_2^{a_2-1}\bar{x}_1^k).$$

By the similar argument used in the decomposition of $\mathbb{C}\{\bar{x}_1, \bar{x}_2\}$ in Lemma 2.1, we can always reduce the power of $\bar{x}_1$ in $\sum_{i=0}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i)$ as well as in $a_2\bar{x}_2^{a_2-1}\bar{x}_1^k$ to a power less than or equal to $a_1 - 1$ using the equality $\bar{x}_1^{a_1}\bar{x}_2 = \bar{x}_2^{a_2}(*)$. Firstly, we have

$$a_2\bar{x}_2^{a_2-1}\bar{x}_1^k \in \sum_{i=a_1}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i). \tag{$2B.6'$}$$

And suppose $d \in \mathbb{N}_{\geq 1}$ is such that $k-1-(d-1)a_1 \geq a_1$, and $k-1-da_1 < a_1$, we also have

$$\begin{aligned}\sum_{i=0}^{da_1-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) &= \sum_{h=0}^{d-1}\sum_{i=ha_1}^{(h+1)a_1-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) \\ &\overset{*}{=} \sum_{h=0}^{d-1}\sum_{i=ha_1}^{(h+1)a_1-1}(\bar{x}_2^{k+(h+1)a_2-h-1-i}\bar{x}_1^{i-ha_1}) \\ &= \sum_{h=0}^{d-1}\sum_{i=0}^{a_1-1}(\bar{x}_2^{k+(h+1)a_2-h-1-i-ha_1}\bar{x}_1^i); \\ \sum_{i=(d-1)a_1}^{da_1-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) &\overset{*}{=} \sum_{i=(d-1)a_1}^{da_1-1}(\bar{x}_2^{k+da_2-(d-1)-1-i}\bar{x}_1^{i-(d-1)a_1}) \\ &= \sum_{i=0}^{a_1-1}(\bar{x}_2^{k+da_2-(d-1)-1-i-(d-1)a_1}\bar{x}_1^i).\end{aligned} \tag{2B.1}$$

And by the assumption of d, $\forall d > h \geq 1$, we have

$$k+(h+1)a_2-h-1-i-ha_1 > k+da_2-(d-1)-1-i-(d-1)a_1,$$

which leads to the inclusion:

$$\sum_{i=0}^{da_1-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) \subset \sum_{i=(d-1)a_1}^{da_1-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i). \tag{2B.2}$$

Besides, we have

$$\begin{aligned}\sum_{i=da_1}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) &\overset{*}{=} \sum_{i=da_1}^{k-1}(\bar{x}_2^{k+(d+1)a_2-1-i-d}\bar{x}_1^{i-da_1}) \\ &= \sum_{i=0}^{k-1-da_1}(\bar{x}_2^{k+(d+1)a_2-1-i-d-da_1}\bar{x}_1^i).\end{aligned} \tag{2B.3}$$

Dividing the right hand side of (2B.1) into two parts, one has

$$\begin{aligned}&\sum_{i=(d-1)a_1}^{da_1-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) = \sum_{i=0}^{a_1-1}(\bar{x}_2^{k+da_2-(d-1)-1-i-(d-1)a_1}\bar{x}_1^i) \\ &= \sum_{i=0}^{k-1-da_1}(\bar{x}_2^{k+da_2-(d-1)-1-i-(d-1)a_1}\bar{x}_1^i) + \sum_{i=k-da_1}^{a_1-1}(\bar{x}_2^{k+da_2-(d-1)-1-i-(d-1)a_1}\bar{x}_1^i).\end{aligned} \tag{$2B.1'$}$$

By the assumption $a_1+1 \geq a_2$, there is

$$k+da_2-(d-1)-1-i-(d-1)a_1 \geq k+(d+1)a_2-1-i-d-da_1,$$

which leads to the inclusion:

$$\sum_{i=0}^{k-1-da_1}(\bar{x}_2^{k+da_2-(d-1)-1-i-(d-1)a_1}\bar{x}_1^i)\subset\sum_{i=0}^{k-1-da_1}(\bar{x}_2^{k+(d+1)a_2-1-i-d-da_1}\bar{x}_1^i)$$
$$=\sum_{i=da_1}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i). \tag{2B.4}$$

Take $i=k-1-da_1$ in the middle part of (2B.4), we get the ideal $(\bar{x}_2^{(d+1)a_2-d}\bar{x}_1^{k-1-da_1})$, and the inclusion:

$$\sum_{i=k-da_1}^{k-da_1+a_1-a_2}(\bar{x}_2^{k+da_2-(d-1)-1-i-(d-1)a_1}\bar{x}_1^i)\sum_{i=k-da_1}^{k-da_1+a_1-a_2}(\bar{x}_2^{(d+1)a_2-d}\bar{x}_1^i)\subset(\bar{x}_2^{(d+1)a_2-d}\bar{x}_1^{k-1-da_1}) \tag{2B.4'}$$
$$\subset\sum_{i=0}^{k-1-da_1}(\bar{x}_2^{k+(d+1)a_2-1-i-d-da_1}\bar{x}_1^i)=\sum_{i=da_1}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i).$$

Here the inclusion holds since there is

$$i\le k-da_1+a_1-a_2\Rightarrow k+da_2-(d-1)-1-i-(d-1)a_1\ge(d+1)a_2-d.$$

According to (2B.4) and $(2B.4)'$, we have

$$\sum_{i=(d-1)a_1}^{da_1-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i)+\sum_{i=da_1}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) \tag{2B.5}$$
$$=\sum_{i=k-da_1+a_1-a_2+1}^{a_1-1}(\bar{x}_2^{k+da_2-(d-1)-1-i-(d-1)a_1}\bar{x}_1^i)+\sum_{i=da_1}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i).$$

Take $i=a_1-1$ in $(\bar{x}_2^{k+da_2-(d-1)-1-i-(d-1)a_1}\bar{x}_1^i)$ above, we get the ideal $(\bar{x}_2^{k-(d-1)+d(a_2-a_1)}\bar{x}_1^{a_1-1})$, and then we have

$$a_1+1\ge a_2\Rightarrow k-(d-1)+d(a_2-a_1)\le k+1$$
$$\Rightarrow(\bar{x}_2^{k+1}\bar{x}_1^{a_1-1})\subset(\bar{x}_2^{k-(d-1)+d(a_2-a_1)}\bar{x}_1^{a_1-1}). \tag{2B.6}$$

According to the inclusions $(2B.1, 4, 6)'$ and $(2B.4, 5, 6)$, if $k \geq a_1 \geq a_2 - 1$, we have a satisfying simplified form of $\bar{m}^k \bar{J}(f)$:

$$\begin{aligned}\bar{m}^k \bar{J}(f) &= \sum_{i=0}^{da_1-1} (\bar{x}_2^{k+a_2-1-i} \bar{x}_1^i) + \sum_{i=da_1}^{k-1} (\bar{x}_2^{k+a_2-1-i} \bar{x}_1^i) + (\bar{x}_2^{k+1} \bar{x}_1^{a_1-1}) \\ &\quad + (\bar{x}_1^{k+a_1} + a_2 \bar{x}_2^{a_2-1} \bar{x}_1^k) \\ &= \sum_{i=(d-1)a_1}^{da_1-1} (\bar{x}_2^{k+a_2-1-i} \bar{x}_1^i) + \sum_{i=da_1}^{k-1} (\bar{x}_2^{k+a_2-1-i} \bar{x}_1^i) + (\bar{x}_1^{k+a_1}) \\ &= \sum_{i=k-da_1+a_1-a_2+1}^{a_1-1} (\bar{x}_2^{k+da_2-(d-1)-1-i-(d-1)a_1} \bar{x}_1^i) \\ &\quad + \sum_{i=0}^{k-1-da_1} (\bar{x}_2^{k+(d+1)a_2-da_1-(d+1)-i} \bar{x}_1^i) + (\bar{x}_1^{k+a_1}) \\ &= \sum_{i=k-da_1+a_1+1-a_2}^{a_1-1} (\bar{x}_2^{k+a_2+(d-1)(a_2-a_1)-d-i} \bar{x}_1^i) \\ &\quad + \sum_{i=0}^{k-1-da_1} (\bar{x}_2^{k+(d+1)a_2-da_1-(d+1)-i} \bar{x}_1^i) + (\bar{x}_1^{k+a_1}).\end{aligned}$$

Then to do the quotient of $\mathbb{C}\{\bar{x}_1, \bar{x}_2\}$ by $\bar{m}^k \bar{J}(f)$ in the sequel, by $(2B.4)'$, we can add some terms to this expression just like above in **(2A)**. Thus when $k \geq a_1 \geq a_2 - 1$, we have

$$\begin{aligned}\bar{m}^k \bar{J}(f) = &\sum_{i=0}^{k-1-da_1} (\bar{x}_2^{k+(d+1)a_2-da_1-(d+1)-i} \bar{x}_1^i) + \sum_{i=k-da_1}^{k-da_1+a_1-a_2} (\bar{x}_2^{(d+1)a_2-d} \bar{x}_1^i) \\ &+ \sum_{i=k-da_1+a_1+1-a_2}^{a_1-1} (\bar{x}_2^{k+a_2+(d-1)(a_2-a_1)-d-i} \bar{x}_1^i) + (\bar{x}_1^{k+a_1}).\end{aligned}$$

□

Lemma 2.4 *For* (C) $f = x_1^{a_1} x_2 + x_2^{a_2} x_1, 2 \leq a_1 \leq a_2$, *in the local algebra* $F(V) = \mathbb{C}\{\bar{x}_1, \bar{x}_2\}$, *if* $k \geq a_1$, $\bar{m}^k \bar{J}(f)$ *has the following simplification:*

$$\bar{m}^k \bar{J}(f) = \sum_{i=1}^{a_1-1} (\bar{x}_2^{k+a_2-i} \bar{x}_1^i) + (\bar{x}_2^{k+a_2} + a_1 \bar{x}_2^{k+1} \bar{x}_1^{a_1-1}) + (\bar{x}_1^{k+a_1}).$$

Proof The image of the Jacobian ideal of $f = x_1^{a_1} x_2 + x_2^{a_2} x_1$ in its local algebras is

$$\bar{J}(f) = (\bar{x}_2^{a_2} + a_1 \bar{x}_1^{a_1-1} \bar{x}_2, \bar{x}_1^{a_1} + a_2 \bar{x}_2^{a_2-1} \bar{x}_1).$$

According to the Lemma 2.2, if $k \geq 1$, firstly we have

$$\begin{aligned}\bar{m}^k \bar{J}(f) &= \bar{x}_1 \bar{J}_1 (\bar{x}_1, \bar{x}_2)^{k-1} + \bar{x}_2^k \bar{J}_1 + \bar{x}_1^k \bar{J}_2 \\ &= \bar{x}_1(\bar{x}_2^{a_2} + a_1\bar{x}_1^{a_1-1}\bar{x}_2)(\bar{x}_1, \bar{x}_2)^{k-1} + \bar{x}_2^k(\bar{x}_2^{a_2} + a_1\bar{x}_1^{a_1-1}\bar{x}_2) + \bar{x}_1^k(\bar{x}_1^{a_1} + a_2\bar{x}_2^{a_2-1}\bar{x}_1) \\ &\overset{*}{=} (\bar{x}_2^{a_2}\bar{x}_1)(\bar{x}_1, \bar{x}_2)^{k-1} + \bar{x}_2^k(\bar{x}_2^{a_2} + a_1\bar{x}_1^{a_1-1}\bar{x}_2) + \bar{x}_1^k(\bar{x}_1^{a_1} + a_2\bar{x}_2^{a_2-1}\bar{x}_1) \\ &= \sum_{i=0}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^{i+1}) + (\bar{x}_2^{k+a_2} + a_1\bar{x}_2^{k+1}\bar{x}_1^{a_1-1}) + (\bar{x}_1^{k+a_1} + a_2\bar{x}_2^{a_2-1}\bar{x}_1^{k+1}).\end{aligned}$$

If $2 \leq a_1 \leq a_2$, $k \geq a_1$, then there is $k-1 \geq a_1 - 1$, and we have inclusion of ideals:

$$\sum_{i=a_1-1}^{k-1} (\bar{x}_2^{k+a_2-1-i}\bar{x}_1^{i+1}) \overset{*}{=} \sum_{i=a_1-1}^{k-1} (\bar{x}_2^{k+2a_2-2-i}\bar{x}_1^{i+2-a_1}) \subset \sum_{i=0}^{a_1-2}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^{i+1}),$$

which leads to the equality:

$$\begin{aligned}\sum_{i=0}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^{i+1}) &= \sum_{i=0}^{a_1-2}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^{i+1}) + \sum_{i=a_1-1}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^{i+1}) \\ &= \sum_{i=0}^{a_1-2}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^{i+1}).\end{aligned}$$

Then we have

$$\begin{aligned}\bar{m}^k \bar{J}(f) &= \sum_{i=0}^{a_1-2}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^{i+1}) + (\bar{x}_2^{k+a_2} + a_1\bar{x}_2^{k+1}\bar{x}_1^{a_1-1}) + (\bar{x}_1^{k+a_1} + a_2\bar{x}_2^{a_2-1}\bar{x}_1^{k+1}) \\ &= \sum_{i=1}^{a_1-1}(\bar{x}_2^{k+a_2-i}\bar{x}_1^{i}) + (\bar{x}_2^{k+a_2} + a_1\bar{x}_2^{k+1}\bar{x}_1^{a_1-1}) + (\bar{x}_1^{k+a_1} + a_2\bar{x}_2^{a_2-1}\bar{x}_1^{k+1}).\end{aligned}$$

Furthermore, for $k \geq a_1$, we have the reduction of the power of $\bar{x}_1$ in $a_2\bar{x}_1^{k+1}\bar{x}_2^{a_2-1}$. This reduction leads to the inclusion:

$$a_2\bar{x}_2^{a_2-1}\bar{x}_1^{k+1} \in \sum_{i=0}^{a_1-1}(\bar{x}_2^{k+a_2-i})\bar{x}_1^i.$$

Thus for $f = x_1^{a_1}x_2 + x_2^{a_2}x_1 (2 \leq a_1 \leq a_2)$, we have when $k \geq a_1$:

$$\bar{m}^k \bar{J}(f) = \sum_{i=1}^{a_1-1}(\bar{x}_2^{k+a_2-i}\bar{x}_1^i) + (\bar{x}_2^{k+a_2} + a_1\bar{x}_1^{a_1-1}\bar{x}_2^{k+1}) + (\bar{x}_1^{k+a_1}).$$

□

We have given the formulas of the k-th Tjurina number for almost all but finitely many values of k. Subsequently, as described in the step 4, we will complete these formulas for all possible values of k.

Proposition 2.3 *Let $(V, 0)$ be a binomial isolated singularity of type B, defined by $f = x_1^{a_1}x_2 + x_2^{a_2}$ $(2 \leq a_1 + 1 \leq a_2)$ with weight type $(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}; 1)$. Then*

$$\tau^k(V) = \begin{cases} a_1a_2 - a_2 + 1 + \frac{k^2+3k}{2}; & k < a_1 + 1, \\ (a_1+1)k + \frac{(2a_2-a_1)(a_1-1)}{2} + 1; & k \geq a_1 + 1. \end{cases}$$

Proof **(1).** According to Lemmas 2.1 and 2.3 **(1A)**, for $f = x_1^{a_1}x_2 + x_2^{a_2}$, if $3 \le a_1 + 1 \le a_2$, and $k \ge a_1 + 1$, we have the simplified forms of $\mathbb{C}\{\bar{x}_1, \bar{x}_2\}$ and $m^k \bar{J}(f)$ respectively:

$$\mathbb{C}\{\bar{x}_1, \bar{x}_2\} = \sum_{i=0}^{a_1-1} \mathbb{C}\{\bar{x}_2\}\bar{x}_1^i + \mathbb{C}\{\bar{x}_1\}\bar{x}_1^{a_1},$$

$$\bar{m}^k \bar{J}(f) = \sum_{i=0}^{a_1-2} (\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + (\bar{x}_2^{k+1}\bar{x}_1^{a_1-1}) + (\bar{x}_1^{k+a_1}).$$

Thus we have

$$\tau^k(f) = \dim \frac{\mathbb{C}\{\bar{x}_1, \bar{x}_2\}}{\bar{m}^k \bar{J}(f)} = \sum_{i=0}^{a_1-2} (k + a_2 - 1 - i) + (k+1) + k$$

$$= \frac{(a_1 - 1)(2k + 2a_2 - a_1)}{2} + 2k + 1$$

$$= (a_1 + 1)k + a_1 a_2 - a_2 - \frac{a_1^2}{2} + \frac{a_1}{2} + 1.$$

Moreover, if $2 = a_1 + 1 \le a_2$, $k \ge 1$, according to Lemma 2.3 **(1A)**, we have

$$\frac{\mathbb{C}\{\bar{x}_1, \bar{x}_2\}}{\bar{m}^k \bar{J}(f)} = \frac{\mathbb{C}\{\bar{x}_2\} + \mathbb{C}\{\bar{x}_1\}\bar{x}_1}{(\bar{x}_2^{k+1}) + (\bar{x}_1^{k+1})},$$

$$\tau^k(f) = \dim \frac{\mathbb{C}\{\bar{x}_1, \bar{x}_2\}}{\bar{m}^k \bar{J}(f)}$$

$$= (k+1) + k = 2k + 1.$$

This formula fits well with the situation when $3 \le a_1 + 1 \le a_2$.

(2). In the case $2 \le a_1 + 1 \le a_2$, $1 \le k < a_1 + 1$, as told in **Step 4**, one can complete the formulas as follows.

According to Lemma 2.2, we have

$$\begin{aligned}\bar{m}^k \bar{J}(f) &= \bar{x}_1 \bar{J}_1(\bar{x}_1, \bar{x}_2)^{k-1} + \bar{x}_2^k \bar{J}_1 + \bar{x}_1^k \bar{J}_2 \\ &= \bar{x}_1(\bar{x}_1^{a_1-1}\bar{x}_2)(\bar{x}_1, \bar{x}_2)^{k-1} + \bar{x}_2^k(\bar{x}_1^{a_1-1}\bar{x}_2) + \bar{x}_1^k(\bar{x}_1^{a_1} + a_2\bar{x}_2^{a_2-1}) \\ &\overset{*}{=} (\bar{x}_2^{a_2})(\bar{x}_1, \bar{x}_2)^{k-1} + \bar{x}_2^k(\bar{x}_1^{a_1-1}\bar{x}_2) + \bar{x}_1^k(\bar{x}_1^{a_1} + a_2\bar{x}_2^{a_2-1}) \\ &= \sum_{i=0}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + (\bar{x}_2^{k+1}\bar{x}_1^{a_1-1}) + (\bar{x}_1^{k+a_1} + a_2\bar{x}_2^{a_2-1}\bar{x}_1^k).\end{aligned}$$

Then there is

$$\frac{F(V)}{\bar{m}^k \bar{J}(f)} = \frac{\sum_{i=0}^{a_1-1} \mathbb{C}\{\bar{x}_2\}\bar{x}_1^i + \mathbb{C}\{\bar{x}_1\}\bar{x}_1^{a_1}}{\sum_{i=0}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + (\bar{x}_2^{k+1}\bar{x}_1^{a_1-1}) + (\bar{x}_1^{k+a_1} + a_2\bar{x}_2^{a_2-1}\bar{x}_1^k)}.$$

Hence, as a module over $\mathbb{C}$, $\frac{F(V)}{\bar{m}^k \bar{J}(f)}$ has a subcollection of basis consisting of

$$\begin{aligned}
&1, \bar{x}_2, \bar{x}_2^2, \dots, \bar{x}_2^{k+a_2-2};\\
&\bar{x}_1, \bar{x}_2\bar{x}_1, \dots, \bar{x}_2^{k+a_2-3}\bar{x}_1;\\
&\bar{x}_1^2, \bar{x}_2\bar{x}_1^2, \dots, \bar{x}_2^{k+a_2-4}\bar{x}_1^2;\\
&\dots\\
&\bar{x}_1^{k-1}, \bar{x}_2\bar{x}_1^{k-1}, \dots, \bar{x}_2^{a_2-1}\bar{x}_1^{k-1};\\
&\bar{x}_1^{a_1-1}, \dots, \bar{x}_2^{k}\bar{x}_1^{a_1-1};
\end{aligned}$$

as well as

$$\begin{aligned}
&\bar{x}_1^{k}, \bar{x}_2\bar{x}_1^{k}, \dots, \bar{x}_2^{a_2-1}\bar{x}_1^{k}; \qquad (*)\\
&\bar{x}_1^{k+1}, \bar{x}_2\bar{x}_1^{k+1}, \dots, \bar{x}_2^{a_2-1}\bar{x}_1^{k+1};\\
&\dots\\
&\bar{x}_1^{a_1-2}, \bar{x}_2\bar{x}_1^{a_1-2}, \dots, \bar{x}_2^{a_2-1}\bar{x}_1^{a_1-2}.
\end{aligned}$$

Since in the quotient module $\frac{F(V)}{\bar{m}^k J(\bar{f})}$, we have $(\bar{x}_1^{k+a_1} + a_2\bar{x}_2^{a_2-1}\bar{x}_1^{k}) = 0$, which leads to the result that in $\mathbb{C}\{\bar{x}_1\}\bar{x}_1^{a_1} =< \bar{x}_1^{a_1}, \bar{x}_1^{a_1+1}, \bar{x}_1^{a_1+2}, \dots >$, we have $\bar{x}_1^{k+a_1+h} = -a_2\bar{x}_2^{a_2-1}\bar{x}_1^{k+h}$.

Thus for these generators in $\{\bar{x}_1^{k+a_1+h} \mid 1 \le k < a_1+1, h \ge 0.\}$, we observe that the ones in $\{\bar{x}_1^{k+a_1+h} = \bar{x}_2^{a_2-1}\bar{x}_1^{k+h} \mid 1 \le k < a_1+1, 0 \le h \le a_1-2-k.\}$ have been counted in the above list $(*)$; and the monomials in $\{\bar{x}_2^{a_2-1}\bar{x}_1^{k+h} \mid 1 \le k < a_1, h > a_1-2-k\}$ are all zeros since we have $a_2 - 1 \ge a_1 \ge k+1$, $k+h \ge a_1 - 1$, and there is $\bar{x}_2^{k+1}\bar{x}_1^{a_1-1} = 0$ in $\frac{F(V)}{\bar{m}^k \bar{J}(f)}$; the monomial in $\{\bar{x}_2^{a_2-1}\bar{x}_1^{k+h} \mid k = a_1, h = 0\}$ has also been counted in the above list (b), and all monomials in $\{\bar{x}_2^{a_2-1}\bar{x}_1^{k+h} \mid k = a_1, h > 0\} = \{\bar{x}_2^{2a_2-2}\bar{x}_1^{h} \mid h > 0\}$ are zeros since there is $\bar{x}_2^{k+a_2-1} = \bar{x}_2^{a_1+a_2-1} = 0$ and $2a_2 - 2 \ge a_1 + a_2 - 1$.

Thus the remaining part of the basis of the module $\frac{F(V)}{\bar{m}^k J(\bar{f})}$ consists of

$$\bar{x}_1^{a_1}, \bar{x}_1^{a_1+1}, \dots, \bar{x}_1^{k+a_1-1}.$$

Consequently, we have the dimension of this $\mathbb{C}$-module:

$$\begin{aligned}
\tau^k(f) = \dim \frac{F(V)}{\bar{m}^k J(\bar{f})} &= [(k+a_2-1) + (k+a_2-2) + \cdots + a_2]\\
&\quad + (k+1) + a_2(a_1-1-k) + k\\
&= \frac{(k+2a_2-1)k}{2} + (k+1) + a_2(a_1-k-1) + k\\
&= a_1a_2 - a_2 + 1 + \frac{k^2+3k}{2}.
\end{aligned}$$

This formula fits well with the situation when $k = 0$.

The complete formulas of τ^k for $f = x_1^{a_1}x_2 + x_2^{a_2}$, $2 \le a_1 + 1 \le a_2$, are

$$\tau^k(V) = \begin{cases} a_1a_2 - a_2 + 1 + \frac{k^2+3k}{2}; & k < a_1+1,\\ (a_1+1)k + \frac{(2a_2-a_1)(a_1-1)}{2} + 1; & k \ge a_1+1. \end{cases}$$

Remark 2.2 For $f = x_a^{a_1} x_2 + x_2^{a_2}$, $2 \le a_1 + 1 \le a_2, a_1 \ge 1$, $1 \le k < a_1 + 1$, we did an operation above, like what is told in **Algorithm.Step 4**. We added some terms in the equality of $\bar{m}^k \bar{J}(f)$. This operation made $\bar{m}^k \bar{J}(f)$ seem more complicated, but the operation is useful when we do the quotient $F(V)/\bar{m}^k \bar{J}(f)$. The form of $\bar{m}^k \bar{J}(f)$ we used above is

$$\bar{m}^k \bar{J}(f) = \sum_{i=0}^{k-1} (\bar{x}_2^{k+a_2-1-i} \bar{x}_1^i) + \sum_{i=k}^{a_1-2} (\bar{x}_2^{a_2} \bar{x}_1^i) + (\bar{x}_2^{k+1} \bar{x}_1^{a_1-1}) + (\bar{x}_1^{k+a_1} + a_2 \bar{x}_2^{a_2-1} \bar{x}_1^k).$$

□

Proposition 2.4 *Let $(V, 0)$ be a binomial isolated singularity of type B, defined by $f = x_1^{a_1} x_2 + x_2^{a_2}$ $(a_1 + 1 \ge a_2 \ge 2)$ with weight type $(\frac{a_2-1}{a_1 a_2}, \frac{1}{a_2}; 1)$. Then*

$$\tau^k(V) = \begin{cases} a_1 a_2 - a_2 + 1 + \frac{k^2+3k}{2}; \ 0 \le k < a_2, \\ a_2 k + a_1 a_2 + \frac{a_2}{2} - \frac{a_2^2}{2}; \ \ a_2 \le k; \end{cases}$$

Proof **(1)** If $a_1 \ge k \ge a_2$, according to Lemma 2.3.**(2).(2A)**, we have

$$\bar{m}^k \bar{J}(f) = \sum_{i=0}^{k-a_2} (\bar{x}_2^{2a_2-1} \bar{x}_1^i) + \sum_{i=k-a_2+1}^{k-1} (\bar{x}_2^{k+a_2-1-i} \bar{x}_1^i) + \sum_{i=k}^{a_1-1} (\bar{x}_2^{a_2} \bar{x}_1^i) \\ + (\bar{x}_1^{k+a_1} + a_2 \bar{x}_2^{a_2-1} \bar{x}_1^k).$$

Consequently, we have

$$\begin{aligned} \tau^k(f) &= \dim \frac{\mathbb{C}\{\bar{x}_1, \bar{x}_2\}}{\bar{m}^k \bar{J}(f)} = \dim \frac{\sum_{i=0}^{a_1-1} \mathbb{C}\{\bar{x}_2\}\bar{x}_1^i + \mathbb{C}\{\bar{x}_1\}\bar{x}_1^{a_1}}{\bar{m}^k \bar{J}(f)} \\ &= \left[\sum_{i=0}^{k-a_2} (2a_2 - 1)\right] + \left[\sum_{i=k-a_2+1}^{k-1} (k + a_2 - 1 - i)\right] + \left[\sum_{i=k}^{a_1-1} a_2\right] + k \\ &= (k - a_2 + 1)(2a_2 - 1) + [\sum_{i=0}^{a_2-2} (2a_2 - 2 - i)] + (a_1 - 1 - k + 1)a_2 + k \\ &= a_2 k + a_2 a_1 + \frac{a_2}{2} - \frac{a_2^2}{2}. \end{aligned}$$

(2) If $k > a_1 \ge a_2 - 1$, according to Lemma 2.3.**(2).(2B)**, we have

$$\bar{m}^k \bar{J}(f) = \sum_{i=0}^{k-1-da_1} (\bar{x}_2^{k+(d+1)a_2-da_1-(d+1)-i} \bar{x}_1^i) + \sum_{i=k-da_1}^{k-da_1+a_1-a_2} (\bar{x}_2^{(d+1)a_2-d} \bar{x}_1^i) \\ + \sum_{i=k-da_1+a_1+1-a_2}^{a_1-1} (\bar{x}_2^{k+a_2+(d-1)(a_2-a_1)-d-i} \bar{x}_1^i) + (\bar{x}_1^{k+a_1}).$$

Then we have

$$\begin{aligned}
\tau^k(f) &= \dim \frac{\sum_{i=0}^{a_1-1} \mathbb{C}\{\bar{x}_2\}\bar{x}_1^i + \mathbb{C}\{\bar{x}_1\}\bar{x}_1^{a_1}}{\bar{m}^k \bar{J}(f)} \\
&= \left[\sum_{i=0}^{k-da_1-1} (k + d(a_2 - a_1) - (d+1) + a_2 - i)\right] + (a_1 + 1 - a_2)(da_2 - d + a_2) \\
&\quad + \left[\sum_{i=k-da_1+a_1+1-a_2}^{a_1-1} (k + a_2 + (d-1)(a_2 - a_1) - d - i)\right] + k \\
&= \left[\sum_{i=1}^{a_2-1} (k + d(a_2 - a_1) - d + i)\right] + (a_1 + 1 - a_2)(da_2 - d + a_2) + k \\
&= a_2 k + a_1 a_2 + \frac{a_2}{2} - \frac{a_2^2}{2}.
\end{aligned}$$

(3) If $1 \le k < a_2$,
by Lemma 2.2, when $k \ge 1$, firstly we have

$$\begin{aligned}
\bar{m}^k \bar{J}(f) &= (\bar{x}_2^{a_2})(\bar{x}_1, \bar{x}_2)^{k-1} + \bar{x}_2^k(\bar{x}_1^{a_1-1}\bar{x}_2) + \bar{x}_1^k(\bar{x}_1^{a_1} + a_2\bar{x}_2^{a_2-1}) \\
&= \sum_{i=0}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + (\bar{x}_2^{k+1}\bar{x}_1^{a_1-1}) + (\bar{x}_1^{k+a_1} + a_2\bar{x}_2^{a_2-1}\bar{x}_1^k).
\end{aligned}$$

Since there is $1 \le k < a_2 \le a_1 + 1 \Rightarrow 1 \le k \le a_2 - 1 \le a_1 - 2$, following the **Algorithm Step 4**, we can take the expression of $\bar{m}^k \bar{J}(f)$:

$$\bar{m}^k \bar{J}(f) = \sum_{i=0}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^i) + \sum_{i=k}^{a_1-2}(\bar{x}_2^{a_2}\bar{x}_1^i) + (\bar{x}_2^{k+1}\bar{x}_1^{a_1-1}) + (\bar{x}_1^{k+a_1} + a_2\bar{x}_1^k\bar{x}_2^{a_2-1}).$$

Then we have

$$\begin{aligned}
\tau^k(f) = \dim \frac{\mathbb{C}\{\bar{x}_1, \bar{x}_2\}}{\bar{m}^k \bar{J}(f)} &= \dim \frac{\sum_{i=0}^{a_1-1} \mathbb{C}\{\bar{x}_2\}\bar{x}_1^i + \mathbb{C}\{\bar{x}_1\}\bar{x}_1^{a_1}}{\bar{m}^k \bar{J}(f)} \\
&= \sum_{i=0}^{k-1}(k + a_2 - 1 - i) + (a_1 - 2 - k + 1)a_2 + (k+1) + k \\
&= a_2 a_1 - a_2 + 1 + \frac{3k}{2} + \frac{k^2}{2}.
\end{aligned}$$

This formula fits well with the situation when $k = 0$.

Finally, we have

$$\tau^k(V) = \begin{cases} a_1a_2 - a_2 + 1 + \frac{k^2+3k}{2};\ 0 \le k < a_2, \\ a_2k + a_1a_2 + \frac{a_2}{2} - \frac{a_2^2}{2};\ \ a_2 \le k; \end{cases}$$

□

Proposition 2.5 *Let $(V,0)$ be a binomial isolated singularity of type C, defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_1$ ($1 \le a_1 \le a_2$) with weight type $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}; 1)$. Then*

$$\tau^k(V) = \begin{cases} a_1a_2 + \frac{k^2+3k}{2}; & 0 \le k < a_1, a_1 \ge 2, \\ (a_1+1)k + a_1a_2 + \frac{a_1}{2} - \frac{a_1^2}{2}; & k \ge a_1 \ge 2, \\ 2k+1; & k \ge 0, a_1 = 1. \end{cases}$$

Proof **(1).** By Lemma 2.4, if $2 \le a_1 < a_2$, $k \ge a_1$, we have

$$\frac{\mathbb{C}\{\bar{x}_1, \bar{x}_2\}}{\bar{m}^k \bar{J}(f)} = \frac{\sum_{i=0}^{a_1-1} \mathbb{C}\{\bar{x}_2\}\bar{x}_1^i + \mathbb{C}\{\bar{x}_1\}\bar{x}_1^{a_1}}{\sum_{i=1}^{a_1-1}(\bar{x}_2^{k+a_2-i}\bar{x}_1^i) + (\bar{x}_2^{k+a_2} + a_1\bar{x}_2^{k+1}\bar{x}_1^{a_1-1}) + (\bar{x}_1^{k+a_1})}.$$

Thus, when $2 \le a_1 \le a_2$ and $k \ge a_1$, we have

$$\begin{aligned} \tau^k(f) = \dim \frac{\mathbb{C}\{\bar{x}_1, \bar{x}_2\}}{\bar{m}^k \bar{J}(f)} &= (k+a_2) + \left[\sum_{i=1}^{a_1-1}(k+a_2-i)\right] + k \\ &= (a_1+1)k + a_1a_2 - \frac{a_1^2}{2} + \frac{a_1}{2}. \end{aligned}$$

(2).(2a) If $1 = a_1 \le a_2$, and $k \ge 1$, we know

$$\mathbb{C}\{\bar{x}_1, \bar{x}_2\} = \frac{\mathbb{C}\{x_1, x_2\}}{(x_1x_2 + x_2^{a_2}x_1)},$$

and by Lemma 2.2, we have

$$\bar{m}^k \bar{J}(f) = \bar{x}_1(\bar{x}_2^{a_2} + \bar{x}_2)(\bar{x}_1, \bar{x}_2)^{k-1} + \bar{x}_2^k(\bar{x}_2^{a_2} + \bar{x}_2) + \bar{x}_1^k(\bar{x}_1 + a_2\bar{x}_2^{a_2-1}\bar{x}_1).$$

And then there is:

$$\begin{aligned} &\tau^k(f) \\ &= \dim \frac{\mathbb{C}\{\bar{x}_1, \bar{x}_2\}}{\bar{m}^k \bar{J}(f)} \\ &= \dim \frac{\mathbb{C}\{x_1, x_2\}}{(x_1x_2 + x_2^{a_2}x_1) + \bar{x}_1(\bar{x}_2^{a_2} + \bar{x}_2)(\bar{x}_1, \bar{x}_2)^{k-1} + \bar{x}_2^k(\bar{x}_2^{a_2} + \bar{x}_2) + \bar{x}_1^k(\bar{x}_1 + a_2\bar{x}_2^{a_2-1}\bar{x}_1)}. \end{aligned}$$

Since in the local rings $\mathbb{C}\{x_1, x_2\}$ or $\mathbb{C}\{\bar{x}_1, \bar{x}_2\}$, we have

$$\begin{aligned} &(x_1x_2 + x_2^{a_2}x_1) = x_1x_2(1 + x_2^{a_2-1}), \\ &\bar{x}_1(\bar{x}_2^{a_2} + \bar{x}_2)(\bar{x}_1, \bar{x}_2)^{k-1} = \bar{x}_1\bar{x}_2(\bar{x}_2^{a_2-1} + 1)(\bar{x}_1, \bar{x}_2)^{k-1}, \\ &\bar{x}_2^k(\bar{x}_2^{a_2} + \bar{x}_2) = \bar{x}_2^{k+1}(\bar{x}_2^{a_2-1} + 1), \\ &\bar{x}_1^k(\bar{x}_1 + a_2\bar{x}_2^{a_2-1}\bar{x}_1) = \bar{x}_1^{k+1}(1 + a_2\bar{x}_2^{a_2-1}), \end{aligned}$$

and that $1 + x_2^{a_2-1}$, $\bar{x}_2^{a_2-1} + 1$, $1 + a_2\bar{x}_2^{a_2-1}$ are all units.

Consequently, when $1 = a_1 \leq a_2$ and $k \geq 1$, we have

$$\begin{aligned}\tau^k(f) &= \dim \frac{\mathbb{C}\{x_1, x_2\}}{(x_1x_2) + \bar{x}_1\bar{x}_2(\bar{x}_1, \bar{x}_2)^{k-1} + (\bar{x}_2^{k+1}) + (\bar{x}_1^{k+1})} \\ &= \dim \frac{\mathbb{C}\{x_1, x_2\}}{(x_1x_2) + (\bar{x}_2^{k+1}) + (\bar{x}_1^{k+1})} \\ &= 2(k+1) - 1 = 2k + 1.\end{aligned}$$

(2b) In the case $k = 0$, $1 = a_1 \leq a_2$, it is easy to get $\tau(f) = 1$.
(3a) For the case $2 \leq a_1 \leq a_2$ and $1 \leq k < a_1$, according to Lemma 2.2, firstly we have

$$\begin{aligned}&\bar{m}^k \bar{J}(f) \\ &= \bar{x}_1 \bar{J}_1(\bar{x}_1, \bar{x}_2)^{k-1} + \bar{x}_2^k \bar{J}_1 + \bar{x}_1^k \bar{J}_2 \\ &= \bar{x}_1(\bar{x}_2^{a_2} + a_1\bar{x}_1^{a_1-1}\bar{x}_2)(\bar{x}_1, \bar{x}_2)^{k-1} + \bar{x}_2^k(\bar{x}_2^{a_2} + a_1\bar{x}_1^{a_1-1}\bar{x}_2) + \bar{x}_1^k(\bar{x}_1^{a_1} + a_2\bar{x}_2^{a_2-1}\bar{x}_1) \\ &\overset{*}{=} (\bar{x}_2^{a_2}\bar{x}_1)(\bar{x}_1, \bar{x}_2)^{k-1} + \bar{x}_2^k(\bar{x}_2^{a_2} + a_1\bar{x}_2\bar{x}_1^{a_1-1}) + \bar{x}_1^k(\bar{x}_1^{a_1} + a_2\bar{x}_2^{a_2-1}\bar{x}_1) \\ &= \sum_{i=0}^{k-1}(\bar{x}_2^{k+a_2-1-i}\bar{x}_1^{i+1}) + (\bar{x}_2^{k+a_2} + a_1\bar{x}_2^{k+1}\bar{x}_1^{a_1-1}) + (\bar{x}_1^{k+a_1} + a_2\bar{x}_2^{a_2-1}\bar{x}_1^{k+1}) \\ &= \sum_{i=1}^{k}(\bar{x}_2^{k+a_2-i}\bar{x}_1^{i}) + (\bar{x}_2^{k+a_2} + a_1\bar{x}_2^{k+1}\bar{x}_1^{a_1-1}) + (\bar{x}_1^{k+a_1} + a_2\bar{x}_2^{a_2-1}\bar{x}_1^{k+1}).\end{aligned}$$

There we come to the situation mentioned in **Algorithm.Step4**. Thus we take an equivalent expression of $\bar{m}^k\bar{J}(f)$:

$$\begin{aligned}\bar{m}^k \bar{J}(f) = &\sum_{i=1}^{k}(\bar{x}_2^{k+a_2-i}\bar{x}_1^{i}) + \sum_{k+1}^{a_1-1}(\bar{x}_2^{a_2}\bar{x}_1^{i}) + (\bar{x}_2^{k+a_2} + a_1\bar{x}_2^{k+1}\bar{x}_1^{a_1-1}) \\ &+(\bar{x}_1^{k+a_1} + a_2\bar{x}_2^{a_2-1}\bar{x}_1^{k+1}).\end{aligned}$$

Considering $\mathbb{C}\{\bar{x}_2\} =< 1, \bar{x}_2, \bar{x}_2^2, \ldots >$, the submodule of $\mathbb{C}\{\bar{x}_1, \bar{x}_2\}$, we have

$$(\bar{x}_2^{k+a_2} + a_1x_2^{k+1}\bar{x}_1^{a_1-1}) = 0 \Rightarrow \bar{x}_2^{k+a_2} = \bar{x}_2^{k+1}\bar{x}_1^{a_1-1},$$

which means $\bar{x}_2^{k+a_2+h}, h \geq 0$ in $\mathbb{C}\{\bar{x}_2\} =< 1, \bar{x}_2, \bar{x}_2^2, \ldots >$ are either equal to zero or have been counted as generators. Thus $\mathbb{C}\{\bar{x}_2\} =< 1, \bar{x}_2, \bar{x}_2^2, \ldots >$ contributes to $\frac{F(V)}{\bar{m}^k\bar{J}(f)}$ a collection of basis:

$$1, \bar{x}_2, \bar{x}_2^2, \ldots, \bar{x}_2^{k+a_2-1}.$$

Similarly, in the the quotient module $\frac{F(V)}{\bar{m}^k\bar{J}(f)}$, we have $\bar{x}_2^{a_2-1}\bar{x}_1^{k+1} = x_1^{k+a_1}$. Thus elements like $x_1^{k+a_1+h}, h \geq 0$ in $\mathbb{C}\{\bar{x}_1\}\bar{x}_1^{a_1} =< \bar{x}_1^{a_1}, \bar{x}_1^{a_1+1}, \bar{x}_1^{a_1+2}, \ldots >$ are either equal to zero or have been counted as generators.

Consequently, $\mathbb{C}\{\bar{x}_1\}\bar{x}_1^{a_1} =< \bar{x}_1^{a_1}, \bar{x}_1^{a_1+1}, \bar{x}_1^{a_1+2}, \ldots >$ contributes to $\frac{F(V)}{\bar{m}^k\bar{J}(f)}$ a finite collection of basis:

$$\bar{x}_1^{a_1}, \bar{x}_1^{a_1+1}, \bar{x}_1^{a_1+2}, \ldots, \bar{x}_1^{k+a_1-1}.$$

Then if $1 \le k \le a_1 - 1, a_1 \ge 2,$ one has

$$\begin{aligned}\tau^k(f) &= \dim \frac{F(V)}{(\bar{m}^k \bar{J}(f))}\\ &= [(k + a_2 - 1) + \cdots + a_2] + a_2[(a_1 - 1) - (k + 1) + 1] + (k + a_2) + (k)\\ &= \frac{(k + 2a_2 - 1)k}{2} + a_2(a_1 - k - 1) + (k + a_2) + k\\ &= a_1a_2 + \frac{k^2}{2} + \frac{3k}{2}.\end{aligned}$$

(3b) If $k = 0, 2 \le a_1 \le a_2$, we have $\tau(f) = a_1a_2$.
Consequently, we get the complete formulas of τ^k for f= $x_a^{a_1} x_2 + x_2^{a_2} \bar{x}_1(1 \le a_1 \le a_2)$:

$$\tau^k(V) = \begin{cases} a_1a_2 + \frac{k^2}{2} + \frac{3k}{2}; & 0 \le k < a_1, a_1 \ge 2,\\ (a_1 + 1)k + a_1a_2 + \frac{a_1}{2} - \frac{a_1^2}{2}; & k \ge a_1 \ge 2,\\ 2k + 1; & k \ge 0, a_1 = 1.\end{cases}$$

□

2.2 From the *k*-th Tjurina number τ^k to the *k*-th Milnor number μ^k

Lemma 2.5 *Let A be a ring, and M be an A-module. If M_1, M_2 are submodules of M, then*

$$\frac{(M_1 + M_2)}{M_1} \cong \frac{(M_2)}{M_1 \cap M_2},$$

and there is an exact sequence:

$$0 \longrightarrow \frac{(M_1 + M_2)}{M_2} \longrightarrow \frac{M}{M_2} \longrightarrow \frac{M}{(M_1 + M_2)} \longrightarrow 0.$$

Corollary 2.2 *For $f \in \mathcal{O}_n = \mathbb{C}\{x_1, \dots, x_n\}$, there is an exact sequence:*

$$0 \longrightarrow \frac{(f) + m^k J(f)}{m^k J(f)} \longrightarrow \frac{\mathcal{O}_n}{m^k J(f)} \longrightarrow \frac{\mathcal{O}_n}{(f) + m^k J(f)} \longrightarrow 0,$$

which leads to an equality of the k-th Tjurina number and k-th Milnor number:

$$\dim \frac{\mathcal{O}_n}{m^k J(f)} = \dim \frac{\mathcal{O}_n}{(f) + m^k J(f)} + \dim \frac{(f) + m^k J(f)}{m^k J(f)},$$

i.e.,

$$\mu^k(f) = \tau^k(f) + \dim \frac{(f) + m^k J(f)}{m^k J(f)}.$$

Proof Use Lemma 2.5, and let A, M_1 and M_2 be $\mathcal{O}_n$, $m^k J(f)$ and (f) respectively. □

Proposition 2.6 *For an isolated singularity defined by a weighted homogeneous polynomial f of type $(w_1, \dots, w_n; d)$, we have its k-th Tjurina number and the k-th Milnor number satisfying*

$$\mu^k(f) = \tau^k(f) + \binom{n + k - 2}{n}.$$

Proof By Corrollary 2.2, we only need to check that:

$$\dim \frac{(f) + m^k J(f)}{m^k J(f)} = \binom{n+k-2}{n}.$$

A weighted homogeneous polynomial f of type $(w_1, \ldots, w_n; d)$ satisfies the Euler identity as mentioned in Remark 2.1:

$$d \cdot f = \sum_{i=1}^{n} w_i x_i \frac{\partial f}{\partial x_i} \quad \text{in } \mathbb{C}\{\boldsymbol{x}\},$$

by which we know f is contained in $J(f)$, and then we have

$$(f) \subset mJ(f) \Rightarrow m^{k-1}(f) \subset m^k J(f).$$

By Lemma 2.5, there is

$$\frac{(f) + m^k J(f)}{m^k J(f)} \cong \frac{(f)}{(f) \cap m^k J(f)}.$$

In $\frac{(f)}{(f)\cap m^k J(f)}$, all the nonzero elements are exactly the images of the elements that belong to the submodule $m^l(f)$, $0 \le l \le k-2$. Thus we have

$$\begin{aligned}\dim \frac{(f)}{(f) \cap m^k J(f)} &= \dim \sum_{l=0}^{k-2} m^l(f) = \dim \frac{(f) + m^k J(f)}{m^k J(f)} \\ &= \binom{n+k-2}{n}.\end{aligned}$$

Consequently, we have

$$\mu^k(f) = \tau^k(f) + \binom{n+k-2}{n}.$$

And for the binomial case in this paper, we have

$$\mu^k(f) = \tau^k(f) + \binom{2+k-2}{2} = \tau^k(f) + \frac{k(k-1)}{2}.$$

□

By Proposition 2.6, once we have known the formulas of the k-th Tjurina numbers $\tau^k(f)$, we have a quick way to give the formulas of the k-th Milnor numbers!

Proposition 2.7 *Let $(V, 0)$ be a binomial isolated singularity of type A, defined by $f = x_1^{a_1} + x_2^{a_2}$ $(2 \le a_1 \le a_2)$ with weight type $(\frac{1}{a_1}, \frac{1}{a_2}; 1)$. Then*

$$\mu^k(V) = \begin{cases} a_1a_2 - (a_1 + a_2) + 1 + k^2 + k; & 0 \le k < a_1, \\ (a_1 - \frac{1}{2})k + \frac{(2a_2-a_1)(a_1-1)}{2} + \frac{k^2}{2}; & k \ge a_1; \end{cases}$$

Proposition 2.8 *Let $(V, 0)$ be a binomial isolated singularity of type B, defined by $f = x_1^{a_1}x_2 + x_2^{a_2}$ $(2 \le a_1 + 1 \le a_2)$ with weight type $(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}; 1)$. Then*

$$\mu^k(V) = \begin{cases} a_1a_2 - a_2 + 1 + k^2 + k; & 0 \le k < a_1 + 1, \\ (a_1 + \frac{1}{2})k + \frac{k^2}{2} + \frac{(2a_2-a_1)(a_1-1)}{2} + 1; & k \ge a_1 + 1; \end{cases}$$

Proposition 2.9 *Let* $(V, 0)$ *be a binomial isolated singularity of type B, defined by* $f = x_1^{a_1}x_2 + x_2^{a_2}$ $(a_1 + 1 \geq a_2 \geq 2)$ *with weight type* $(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}; 1)$. *Then*

$$\mu^k(V) = \begin{cases} a_1a_2 - a_2 + 1 + k + k^2; & 0 \leq k < a_2, \\ (a_2 - \frac{1}{2})k + \frac{k^2}{2} + a_1a_2 + \frac{a_2}{2} - \frac{a_2^2}{2}; & a_2 \leq k; \end{cases}$$

Proposition 2.10 *Let* $(V, 0)$ *be a binomial isolated singularity of type C, defined by* $f = x_1^{a_1}x_2 + x_2^{a_2}x_1$ $(1 \leq a_1 \leq a_2)$ *with weight type* $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}; 1)$. *Then*

$$\mu^k(V) = \begin{cases} k^2 + k + a_1a_2; & 0 \leq k < a_1, a_1 \geq 2, \\ (a_1 + \frac{1}{2})k + \frac{k^2}{2} + a_1a_2 + \frac{a_1}{2} - \frac{a_1^2}{2}; & k \geq a_1 \geq 2, \\ \frac{k^2}{2} + \frac{3k}{2} + 1; & k \geq 0, a_1 = 1. \end{cases}$$

3 Proofs of theorems

Proof of Theorem A.

Theorem A follows from Propositions 2.2– 2.5 immediately.

Proof of Theorem B.

Theorem B follows from Propositions 2.7–2.10 immediately.

Proof of Theorem C.

Proof It follows from Propositions 2.2 to 2.5 that the inequality

$$\tau^k(V) \geq \ell^k\left(\frac{1}{w_1}, \frac{1}{w_2}\right) = \begin{cases} \frac{1}{w_1w_2} - (\frac{1}{w_1} + \frac{1}{w_2}) + \frac{k^2+3k}{2} + 1; & 0 \leq k < \frac{1}{w_1}, \quad 2 \leq \frac{1}{w_1} \leq \frac{1}{w_2}, \\ \frac{k}{w_1} + \frac{(\frac{2}{w_2} - \frac{1}{w_1})(\frac{1}{w_1} - 1)}{2}; & k \geq \frac{1}{w_1}, \quad 2 \leq \frac{1}{w_1} \leq \frac{1}{w_2}, \end{cases}$$

holds true. □

In the following proof of Theorem D, we shall distinguish a simple hypersurface singularity from the other by using the corresponding dimension of the Lie algebra $L^2(V)$, minimal number of generators of the nilradical of Lie algebra $L^2(V)$ or the dimension sequence of the derived series.

Proof of Theorem D.

Proof It is easy to see that $L^2(A_k)$ associated to the series

$$A_k : \{x_1^2 + x_2^{k+1} = 0\} \subset \mathbb{C}^2, k \geq 1,$$

has dimension

$$\lambda^2(A_k) = \begin{cases} k + 6; & k \geq 2, \\ 6; & k = 1. \end{cases}$$

In the case $k \geq 2$, the Lie algebra $L^2(A_k)$ has the following basis:

$$e_1 = (k+1)x_1\partial_1 + 2x_2\partial_2, \quad e_2 = x_2^k\partial_1, \quad e_3 = x_2^2\partial_2, e_4 = x_2^3\partial_2, \ldots, e_k = x_2^{k-1}\partial_2,$$
$$e_{k+1} = x_2^k\partial_1 + x_1\partial_2, \quad e_{k+2} = x_2^k\partial_2, \quad e_{k+3} = x_1x_2\partial_2, \quad e_{k+4} = x_2^{k+1}\partial_2,$$
$$e_{k+5} = x_1x_2\partial_1, e_{k+6} = x_2^{k+1}\partial_1.$$

Let $g(A_k)$ be the nilradical of Lie algebra $L^2(A_k)$, one has

$$g(A_k) =< e_2, e_3, e_4, \ldots, e_{k+6} > .$$

The nilradical $g(A_k)$ has the following multiplication table:

Case 1. If k is even and $k = 2l + 8$, $l \geq 1$, then

$$\begin{aligned}
&[e_2, e_3] = -ke_{k+6}, \quad [e_2, e_{k+1}] = e_{k+2}, \quad [e_2, e_{k+3}] = e_{k+4}, \quad [e_2, e_{k+5}] = e_{k+6},\\
&[e_3, e_4] = e_5, \quad [e_3, e_5] = 2e_6, \quad [e_3, e_6] = 3e_7, \quad \ldots, \quad [e_3, e_k] = (k-3)e_{k+2},\\
&[e_3, e_{k+1}] = -2e_{k+3} + ke_{k+6}, \quad [e_3, e_{k+2}] = (k-2)e_{k+4},\\
&[e_4, e_5] = e_7, \quad [e_4, e_6] = 2e_8, \quad [e_4, e_7] = 3e_9, \quad \ldots, \quad [e_4, e_{k-2}] = (k-6)e_k,\\
&[e_4, e_{k-1}] = (k-5)e_{k+2}, \quad [e_4, e_k] = (k-4)e_{k+4},\\
&[e_5, e_6] = e_9, \quad [e_5, e_7] = 2e_{10}, \quad [e_5, e_8] = 3e_{11}, \quad \ldots, \quad [e_5, e_{k-3}] = (k-8)e_k,\\
&[e_5, e_{k-2}] = (k-7)e_{k+2}, \quad [e_5, e_{k-1}] = (k-6)e_{k+4},\\
&\quad \vdots\\
&[e_{l+5}, e_{l+6}] = e_{2l+10}, \quad [e_{l+5}, e_{l+7}] = 2e_{2l+12},\\
&\quad [e_{2l+9}, e_{2l+13}] = -e_{2l+13}, \quad [e_{2l+9}, e_{2l+14}] = -e_{2l+12}.
\end{aligned}$$

Case 2. If k is odd and $k = 2l + 9$, $l \geq 0$, then

$$\begin{aligned}
&[e_2, e_3] = -ke_{k+6}, \quad [e_2, e_{k+1}] = e_{k+2}, \quad [e_2, e_{k+3}] = e_{k+4}, \quad [e_2, e_{k+5}] = e_{k+6},\\
&[e_3, e_4] = e_5, \quad [e_3, e_5] = 2e_6, \quad [e_3, e_6] = 3e_7, \quad \ldots, \quad [e_3, e_k] = (k-3)e_{k+2},\\
&[e_3, e_{k+1}] = -2e_{k+3} + ke_{k+6}, \quad [e_3, e_{k+2}] = (k-2)e_{k+4},\\
&[e_4, e_5] = e_7, \quad [e_4, e_6] = 2e_8, \quad [e_4, e_7] = 3e_9, \quad \ldots, \quad [e_4, e_{k-2}] = (k-6)e_k,\\
&[e_4, e_{k-1}] = (k-5)e_{k+2}, \quad [e_4, e_k] = (k-4)e_{k+4},\\
&[e_5, e_6] = e_9, \quad [e_5, e_7] = 2e_{10}, \quad [e_5, e_8] = 3e_{11}, \quad \ldots, \quad [e_5, e_{k-3}] = (k-8)e_k,\\
&[e_5, e_{k-2}] = (k-7)e_{k+2}, \quad [e_5, e_{k-1}] = (k-6)e_{k+4},\\
&\quad \vdots\\
&[e_{l+6}, e_{l+9}] = e_{2l+13}, \quad [e_{2l+10}, e_{2l+14}] = -e_{2l+12}, \quad [e_{2l+10}, e_{2l+15}] = -e_{2l+13}.
\end{aligned}$$

It follows that when $k \geq 9$, the minimal spanning set of the nilradical $g(A_k)/[g(A_k), g(A_k)]$ is

$$\{e_2, e_3, e_4, e_{k+1}, e_{k+5}\}.$$

It is easy to see that the Lie algebra $L^2(D_k)$ associated to the series

$$D_k : \{x_1^2 x_2 + x_2^{k-1} = 0\} \subset \mathbb{C}^2, k \geq 4,$$

has the following dimension:

$$\lambda^2(D_k) = \begin{cases} k + 10; & k \geq 5, \\ 13; & k = 4. \end{cases}$$

In the case $k \geq 5$, $L^2(D_k)$ has the following basis:

$$e_1 = (k-2)x_1\partial_1 + 2x_2\partial_2, \quad e_2 = (x_2^{k-2} + x_1^2)\partial_1, \quad e_3 = -x_1x_2\partial_1 + x_2^2\partial_2,$$
$$e_4 = x_2^{k-3}\partial_1, \quad e_5 = (x_2^{k-2} + x_1^2)\partial_2, \quad e_6 = x_2^2\partial_2, \quad e_7 = x_2^{k-2}\partial_1,$$
$$e_8 = x_2^3\partial_2, \quad e_9 = x_2^4\partial_2, \quad e_{10} = x_2^5\partial_2, \quad \ldots \quad , e_{k+2} = x_2^{k-3}\partial_2,$$
$$e_{k+3} = x_2^{k-2}\partial_1 + x_1x_2\partial_2, \quad e_{k+4} = x_2^{k-2}\partial_2, \quad e_{k+5} = x_1^3\partial_2, \quad e_{k+6} = x_1x_2^2\partial_2,$$
$$e_{k+7} = x_2^{k-1}\partial_2, \quad e_{k+8} = x_1^3\partial_1, \quad e_{k+9} = x_1x_2^2\partial_1, \quad e_{k+10} = x_2^{k-1}\partial_1.$$

Let $g(D_k)$ be the nilradical of the Lie algebra $L^2(D_k)$ associated to D_k, one has

$$g(D_k) =< e_2, e_3, e_4, \ldots, e_{k+10} > .$$

The nilradical $g(D_k)$ has the following multiplication table:
Case 1. If k is even and $k = 2l + 6$, $l \geq 1$, then

$$[e_2, e_3] = -ke_{k+10}, \quad [e_2, e_5] = 2e_{k+5}, \quad [e_2, e_6] = -(k-2)e_{k+10},$$
$$[e_3, e_4] = (k-2)e_7, \quad [e_3, e_5] = ke_{k+7} + e_{k+8}, \quad [e_3, e_6] = e_{k+9},$$
$$[e_3, e_7] = (k-1)e_{k+10}, [e_3, e_8] = e_9, \quad [e_3, e_9] = 2e_{10}, \quad [e_3, e_{10}] = 3e_{11}, \quad \ldots,$$
$$[e_3, e_{k+2}] = (k-5)e_{k+4}, \quad [e_3, e_{k+3}] = -2e_{k+6} + (k-2)e_{k+10},$$
$$[e_3, e_{k+4}] = (k-4)e_{k+7}, \quad [e_4, e_6] = -(k-3)e_7, \quad [e_4, e_8] = -(k-3)e_{k+10},$$
$$[e_4, e_{k+3}] = e_{k+4}, [e_4, e_{k+6}] = e_{k+7}, \quad [e_4, e_{k+9}] = e_{k+10}, \quad [e_5, e_6] = -(k-2)e_{k+7},$$
$$[e_5, e_{k+3}] = e_{k+5}, [e_6, e_7] = (k-2)e_{k+10}, [e_6, e_8] = e_9, \quad [e_6, e_9] = 2e_{10},$$
$$[e_6, e_{10}] = 3e_{11}, \quad \ldots, \quad [e_6, e_{k+2}] = (k-5)e_{k+4},$$
$$[e_6, e_{k+3}] = -e_{k+6} + (k-2)e_{k+10}, \quad [e_6, e_{k+4}] = (k-4)e_{k+7}, \quad [e_7, e_{k+3}] = e_{k+7},$$
$$[e_8, e_9] = e_{11}, \quad [e_8, e_{10}] = 2e_{12}, \quad [e_8, e_{11}] = 3e_{13}, \quad \ldots, \quad [e_8, e_k] = (k-8)e_{k+2},$$
$$[e_8, e_{k+1}] = (k-7)e_{k+4}, \quad [e_8, e_{k+2}] = (k-6)e_{k+7},$$
$$[e_9, e_{10}] = e_{13}, \quad [e_9, e_{11}] = 2e_{14}, \quad [e_9, e_{12}] = 3e_{15}, \quad \ldots, \quad [e_9, e_{k-1}] = (k-10)e_{k+2},$$
$$[e_9, e_k] = (k-9)e_{k+4}, \quad [e_9, e_{k+1}] = (k-8)e_{k+7},$$
$$[e_{10}, e_{11}] = e_{15}, \quad [e_{10}, e_{12}] = 2e_{16}, \quad [e_{10}, e_{13}] = 3e_{17}, \quad \ldots,$$
$$[e_{10}, e_{k-2}] = (k-12)e_{k+2}, [e_{10}, e_{k-1}] = (k-11)e_{k+4}, \quad [e_{10}, e_k] = (k-10)e_{k+7},$$
$$\vdots$$
$$[e_{l+7}, e_{l+8}] = e_{2l+10}, \quad [e_{l+7}, e_{l+9}] = 2e_{2l+13}.$$

Case 2. If k is odd and $k = 2l + 5$, $l \geq 1$, then

$$
\begin{aligned}
&[e_2, e_3] = -ke_{k+10}, \quad [e_2, e_5] = 2e_{k+5}, \quad [e_2, e_6] = -(k-2)e_{k+10}, \\
&[e_3, e_4] = (k-2)e_{k-4}, \quad [e_3, e_5] = ke_{k+7} + e_{k+8}, \quad [e_3, e_6] = e_{k+9}, \quad [e_3, e_7] = (k-1)e_{k+10}, \\
&[e_3, e_8] = e_9, \quad [e_3, e_9] = 2e_{10}, \quad [e_3, e_{10}] = 3e_{11}, \quad \ldots, \\
&[e_3, e_{k+1}] = (k-6)e_{k+2}, \\
&[e_3, e_{k+2}] = (k-5)e_{k+4}, \quad [e_3, e_{k+3}] = -2e_{k+6} + (k-2)e_{k+10}, \quad [e_3, e_{k+4}] = (k-4)e_{k+7,} \\
&[e_4, e_6] = -(k-3)e_7, \quad [e_4, e_8] = -(k-3)e_{k+10}, \quad [e_4, e_{k+3}] = e_{k+4}, \quad [e_4, e_{k+6}] = e_{k+7}, \\
&[e_4, e_{k+9}] = e_{k+10}, \quad [e_5, e_6] = -(k-2)e_{k+7}, \quad [e_5, e_{k+3}] = e_{k+5}, \quad [e_6, e_7] = (k-2)e_{k+10}, \\
&[e_6, e_8] = e_9, \quad [e_6, e_9] = 2e_{10}, \quad [e_6, e_{10}] = 3e_{11}, \quad \ldots, \quad [e_6, e_{k+1}] = (k-6)e_{k+2}, \\
&[e_6, e_{k+2}] = (k-5)e_{k+3}, \quad [e_6, e_{k+3}] = -e_{k+6} + (k-2)e_{k+10}, \quad [e_6, e_{k+4}] = (k-4)e_{k+7}, \\
&[e_7, e_{k+3}] = e_{k+7}, \quad [e_8, e_9] = e_{11}, \quad [e_8, e_{10}] = 2e_{12}, \quad [e_8, e_{11}] = 3e_{13}, \\
&\qquad \ldots, \quad [e_8, e_k] = (k-8)e_{k+2}, \\
&[e_8, e_{k+1}] = (k-7)e_{k+4}, \quad [e_8, e_{k+2}] = (k-6)e_{k+7}, \\
&[e_9, e_{10}] = e_{13}, \quad [e_9, e_{11}] = 2e_{14}, \quad [e_9, e_{12}] = 3e_{15}, \quad \ldots, \quad [e_9, e_{k-1}] = (k-10)e_{k+2}, \\
&[e_9, e_k] = (k-9)e_{k+4}, \quad [e_9, e_{k+1}] = (k-8)e_{k+7}, \\
&[e_{10}, e_{11}] = e_{15}, \quad [e_{10}, e_{12}] = 2e_{16}, \quad [e_{10}, e_{13}] = 3e_{17}, \quad \ldots, \quad [e_{10}, e_{k-2}] = (k-12)e_{k+2}, \\
&[e_{10}, e_{k-1}] = (k-11)e_{k+4}, \quad [e_{10}, e_k] = (k-10)e_{k+7}, \\
&\quad \vdots \\
&[e_{l+7}, e_{l+8}] = e_{2l+12}.
\end{aligned}
$$

It follows that when $k \geq 7$, the minimal spanning set of the nilradical $g(D_k)/[g(D_k), g(D_k)]$ is $\{e_2, e_3, e_4, e_5, e_6, e_8, e_{k+3}\}$. Therefore when $k \geq 7$, nilradicals of Lie algebras associated to the series A_k and D_k have different minimal numbers of generators.

Notice that the two pairs $(L^2(D_5), L^2(A_9))$ and $(L^2(D_6), L^2(A_{10}))$ have the same dimension. The minimal numbers of generators of the nilradicals of $L^2(D_5)$, $L^2(D_6)$, $L^2(A_9)$ and $L^2(A_{10})$ are 6, 7, 5 and 5 respectively. It is easy to see that $\lambda^2(E_6) = 17$, $\lambda^2(E_7) = 19$, $\lambda^2(E_8) = 20$. Next we need to distinguish the remaining pairs which have the same dimension. We only need to consider the following four cases:

(1) $L^2(E_6) \ncong L^2(D_7)$, $L^2(E_6) \ncong L^2(A_{11})$, $L^2(D_7) \ncong L^2(A_{11})$,
(2) $L^2(E_7) \ncong L^2(D_9)$, $L^2(E_7) \ncong L^2(A_{13})$, $L^2(D_9) \ncong L^2(A_{13})$,
(3) $L^2(E_8) \ncong L^2(D_{10})$, $L^2(E_8) \ncong L^2(A_{14})$, $L^2(D_{10}) \ncong L^2(A_{14})$,
(4) $L^2(D_4) \ncong L^2(A_7)$.

Now it is sufficient to prove the following proposition.

Proposition 3.1 *The Lie algebras $L^2(V)$ associated to simple hypersurface singularities in the following four cases are not isomorphic:*

(1) $L^2(E_6) \ncong L^2(D_7)$, $L^2(E_6) \ncong L^2(A_{11})$, $L^2(D_7) \ncong L^2(A_{11})$,

(2) $L^2(E_7) \ncong L^2(D_9)$, $L^2(E_7) \ncong L^2(A_{13})$, $L^2(D_9) \ncong L^2(A_{13})$,

(3) $L^2(E_8) \ncong L^2(D_{10})$, $L^2(E_8) \ncong L^2(A_{14})$, $L^2(D_{10}) \ncong L^2(A_{14})$,

(4) $L^2(D_4) \ncong L^2(A_7)$.

Proof Case (1). It is easy to see that $L^2(A_{11})$ is a 17-dimensional complex Lie algebra spanned by the following basis:

$$e_1 = 12x_1\partial_1 + 2x_2\partial_2, \quad e_2 = x_2^{11}\partial_1, \quad e_3 = x_2^2\partial_2, e_4 = x_2^3\partial_2, \ldots, e_{11} = x_2^{10}\partial_2,$$
$$e_{12} = x_2^{11}\partial_1 + x_1\partial_2, \quad e_{13} = x_2^{11}\partial_2, \quad e_{14} = x_1x_2\partial_2, \quad e_{15} = x_2^{12}\partial_2, \quad e_{16} = x_1x_2\partial_1,$$
$$e_{17} = x_2^{12}\partial_1.$$

The nilradical $g(A_{11})$ of the Lie algebra $L^2(A_{11})$ is spanned by

$$\{e_2, e_3, e_4, \ldots, e_{17}\}.$$

The nilradical $g(A_{11})$ has the following multiplication table:

$$[e_2, e_3] = -11e_{17}, [e_2, e_{12}] = e_{13}, [e_2, e_{14}] = e_{15}, [e_2, e_{16}] = e_{17}, [e_3, e_4] = e_5,$$
$$[e_3, e_5] = 2e_6, [e_3, e_6] = 3e_7, [e_3, e_7] = 4e_8, [e_3, e_8] = 5e_9, [e_3, e_9] = 6e_{10},$$
$$[e_3, e_{10}] = 7e_{11}, [e_3, e_{11}] = 8e_{13}, [e_3, e_{12}] = -2e_{14} + 11e_{17}, [e_3, e_{13}] = 9e_{15}, [e_4, e_5] = e_7,$$
$$[e_4, e_6] = 2e_8, [e_4, e_7] = 3e_9, [e_4, e_8] = 4e_{10}, [e_4, e_9] = 5e_{11}, [e_4, e_{10}] = 6e_{13},$$
$$[e_4, e_{11}] = 7e_{15}, [e_5, e_6] = e_9, [e_5, e_7] = 2e_{10}, [e_5, e_8] = 3e_{11}, [e_5, e_9] = 4e_{13},$$
$$[e_5, e_{10}] = 5e_{15}, [e_6, e_7] = e_{11}, [e_6, e_8] = 2e_{13}, [e_6, e_9] = 3e_{15}, [e_7, e_8] = e_{15},$$
$$[e_{12}, e_{16}] = -e_{14}, [e_{12}, e_{17}] = -e_{15}.$$

It follows from the multiplication table that the sequence of dimensions of derived series is $\{16, 11, 5, 0\}$.

It is easy to see that $L^2(D_7)$ is a 17-dimensional complex Lie algebra spanned by the following basis:

$$e_1 = 5x_1\partial_1 + 2x_2\partial_2, \quad e_2 = (x_2^5 + x_1^2)\partial_1, \quad e_3 = -x_1x_2\partial_1 + x_2^2\partial_2, \quad e_4 = x_2^4\partial_1,$$
$$e_5 = (x_2^5 + x_1^2)\partial_2, \quad e_6 = x_2^2\partial_2, \quad e_7 = x_2^5\partial_1, \quad e_8 = x_2^3\partial_2, \quad e_9 = x_2^4\partial_2, \quad e_{10} = x_2^5\partial_1 + x_1x_2\partial_2,$$
$$e_{11} = x_2^5\partial_2, \quad e_{12} = x_1^3\partial_2, \quad e_{13} = x_1x_2^2\partial_2, \quad e_{14} = x_2^6\partial_2, \quad e_{15} = x_1^3\partial_1, \quad e_{16} = x_1x_2^2\partial_1,$$
$$e_{17} = x_2^6\partial_1.$$

The nilradical $g(D_7)$ of the Lie algebra $L^2(D_7)$ is spanned by

$$\{e_2, e_3, e_4, \ldots, e_{17}\}.$$

The nilradical $g(D_7)$ has the following multiplication table:

$$[e_2, e_3] = -7e_{17}, [e_2, e_5] = 2e_{12}, [e_2, e_6] = -5e_{17}, [e_3, e_4] = 5e_7, [e_3, e_5] = 7e_{14} + e_{15},$$
$$[e_3, e_6] = e_{16}, [e_3, e_7] = 6e_{17}, [e_3, e_8] = e_9, [e_3, e_9] = 2e_{11}, [e_3, e_{10}] = -2e_{13} + 5e_{17},$$
$$[e_3, e_{11}] = 3e_{14}, [e_4, e_6] = -4e_7, [e_4, e_8] = -4e_{17}, [e_4, e_{10}] = e_{11}, [e_4, e_{13}] = e_{14},$$
$$[e_4, e_{16}] = e_{17}, [e_5, e_6] = -5e_{14}, [e_5, e_{10}] = e_{12}, [e_6, e_7] = 5e_{17}, [e_6, e_8] = e_9,$$
$$[e_6, e_9] = 2e_{11}, [e_6, e_{10}] = -e_{13} + 5e_{17}, [e_6, e_{11}] = 3e_{14}, [e_7, e_{10}] = e_{14}, [e_8, e_9] = e_{14}.$$

It follows from the multiplication table that the sequence of dimensions of derived series is $\{16, 9, 3, 1, 0\}$.

It is easy to see that $L^2(E_6)$ is a 17-dimensional complex Lie algebra spanned by the following basis:

$$e_1 = 4x_1\partial_1 + 3x_2\partial_2, \quad e_2 = x_2^2\partial_1, \quad e_3 = 2x_1x_2\partial_1 - x_2^2\partial_2, \quad e_4 = x_2^3\partial_1 + x_1^2\partial_2,$$
$$e_5 = x_1^2\partial_1 - x_1x_2\partial_2, \quad e_6 = x_2^2\partial_2, \quad e_7 = x_1x_2\partial_2, \quad e_8 = x_1x_2^2\partial_1, \quad e_9 = x_1^2\partial_2,$$
$$e_{10} = x_1x_2^2\partial_2, \quad e_{11} = x_1x_2^2\partial_1 - x_2^3\partial_2, \quad e_{12} = x_1^2x_2\partial_2, \quad e_{13} = x_1x_2^3\partial_2, \quad e_{14} = x_2^4\partial_2,$$
$$e_{15} = x_1^2x_2\partial_1, \quad e_{16} = x_1x_2^3\partial_1, \quad e_{17} = x_2^4\partial_1.$$

The nilradical $g(E_6)$ of the Lie algebra $L^2(E_6)$ is spanned by:

$$\{e_2, e_3, e_4, \dots, e_{17}\}.$$

The nilradical $g(E_6)$ has the following multiplication table:

$$[e_2, e_3] = 4e_4 + 4e_9, [e_2, e_4] = 2e_{10} - 2e_{15}, [e_2, e_5] = e_{11} - 3e_8, [e_2, e_6] = -2e_4 - 2e_9,$$
$$[e_2, e_7] = e_{11} - e_8, [e_2, e_8] = e_{17}, [e_2, e_9] = -2e_{10} + 2e_{15}, [e_2, e_{10}] = e_{14} - 2e_{16},$$
$$[e_2, e_{11}] = -3e_{17}, [e_2, e_{12}] = 2e_{13}, [e_2, e_{15}] = 2e_{16},$$
$$[e_3, e_4] = 6e_{12} - 3e_{17}, [e_3, e_5] = 3e_{10} - 4e_{15},$$
$$[e_3, e_6] = -2e_8, [e_3, e_7] = 3e_{10} - 2e_{15}, [e_3, e_8] = -2e_{16},$$
$$[e_3, e_9] = -6e_{12} - 2e_{17}, [e_3, e_{10}] = 2e_{13},$$
$$[e_3, e_{11}] = -e_{14}, [e_4, e_5] = -2e_{14} - 5e_{16}, [e_4, e_6] = 2e_{12} - 3e_{17},$$
$$[e_4, e_7] = -3e_{16}, [e_4, e_9] = -2e_{13},$$
$$[e_5, e_6] = e_{10}, [e_5, e_7] = -e_{12}, [e_5, e_8] = -e_{13}, [e_5, e_9] = -3e_{14},$$
$$[e_5, e_{11}] = 3e_{13}, [e_6, e_7] = -e_{10},$$
$$[e_6, e_8] = 2e_{16}, [e_6, e_9] = 2e_{12}, [e_6, e_{11}] = e_{14} - 2e_{16},$$
$$[e_7, e_8] = -e_{13}, [e_7, e_9] = -e_{14}, [e_7, e_{11}] = 3e_{13}.$$

It follows from the multiplication table that the sequence of dimensions of derived series is $\{16, 10, 0\}$.
Therefore the Lie algebras $L^2(D_7)$, $L^2(E_6)$ and $L^2(A_{11})$ have different sequences of dimensions of derived series. Therefore these three Lie algebras are pairwise non-isomorphic. Similarly we can prove the cases (2), (3), and (4). □

It follows from Proposition 3.1 that the Lie algebras $L^2(V)$ associated to the simple hypersurface singularities, in the corresponding four cases, are non-isomorphic. Therefore we completely characterize the simple hypersurface singularities by using their Lie algebras $L^2(V)$. □

Acknowledgements We would like to thank the anonymous referee for careful reading the manuscript and giving numerous helpful suggestions. Both Yau and Zuo are supported by NSFC Grant 11961141005. Zuo is supported by NSFC Grant 12271280 and Tsinghua University Initiative Scientific Research Program. Yau is supported by Tsinghua University Education Foundation fund (042202008).

Author Contributions All authors contributed to the study conception and design. Material preparation, data collection and analysis were performed by NH, ZL, SY, and HZ. The first draft of the manuscript was partially written by all authors and they commented on previous versions of the manuscript. All authors read and approved the final manuscript.

Funding Both Yau and Zuo are supported by NSFC Grant 11961141005. Zuo is supported by NSFC Grant 12271280 and Tsinghua University Initiative Scientific Research Program. Yau is supported by Tsinghua University Education Foundation fund (042202008) and Tsinghua University start-up fund.

Data availability All data generated or analysed during this study are included in this published article.

Declarations

Ethics approval This submitted work is original and this manuscript has not been submitted to more than one journal for simultaneous consideration. All authors certify that there are no ethics issues.

Conflict of interest All authors certify that they have no conflict of interest.

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