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DERIVATIONS OF LOCAL k-TH HESSIAN ALGEBRAS OF SINGULARITIES

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In our previous work, we introduced a series of new derivation Lie algebras $L_k(V)$ associated to an isolated hypersurface singularity $(V, 0)$. These are new analytic invariants of singularities. Here, we investigate $L_2(V)$ for fewnomial isolated singularities and obtain the formula of $\lambda_k(V)$ (i.e., the dimension of $L_k(V)$) for trinomial singularities. Furthermore, we prove the sharp upper estimate conjecture for $L_2(V)$. This is a continuation of our previous work (*Math. Z.* **298**:3-4 (2021), 1813–1829). We proposed two new conjectures for $\tau_k(V)$ and $\lambda_k(V)$ and we prove these conjectures for a large class of singularities.

1. Introduction

Let $\mathscr{O}_n$ be the ring of holomorphic function germs $f(x_1, \dots, x_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. The ring $\mathscr{O}_n$ has a unique maximal ideal $m = (x_1, \dots, x_n)$. For any $f \in \mathscr{O}_n$, we denote by $V = V(f)$ the germ at the origin of $\mathbb{C}^n$ of hypersurface $\{f = 0\} \subset \mathbb{C}^n$. If the origin is an isolated zero of the gradient of f, i.e., $\big(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\big)$, then $(V, 0)$ is a germ of isolated hypersurface singularity. According to Hilbert's Nullstellensatz for an isolated singularity $(V(f), 0)$ the algebra $A(V) = \mathscr{O}_n/\big(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\big)$ is finite-dimensional. This algebra $A(V)$ is called the Tyurina algebra of V and its dimension $\tau(V)$ is called Tyurina number.

Theorem 1.1 (Mather–Yau theorem, [25]). *Let V_1 and V_2 be two isolated hypersurface singularities, and $A(V_1)$ and $A(V_2)$ be the Tyurina algebras. Then $(V_1, 0) \cong (V_2, 0)$ if and only if $A(V_1) \cong A(V_2)$.*

Recall that the multiplicity (mult(f)) of the isolated hypersurface singularity $(V, 0)$ is defined to be the order of the lowest nonvanishing term in the power series expansion of f at 0.

Definition 1.2. A polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ is called weighted homogeneous if there exist positive rational numbers $w_1, \dots, w_n$ (i.e., weights of $x_1, \dots, x_n$) and d such that $\sum a_i w_i = d$ for each monomial $\prod x_i^{a_i}$ appearing in f with nonzero coefficient. The number d is called weighted homogeneous degree (w-deg) of f with respect to weights w_j, $1 \le j \le n$.

Assume that f is a weighted homogeneous polynomial, the weight type of f is denoted as $(w_1, \dots, w_n; d)$. Without loss of generality, we can assume that w-deg$f = 1$. It is known that the weight types of weighted homogeneous hypersurface singularities are topological invariants when $n = 2, 3$ (cf. [39; 27]). The Milnor number of the isolated hypersurface singularity $(V(f), 0)$ is defined by $\mu = \dim \mathscr{O}_n/\big(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\big)$. It is well-known that the Milnor number of weighted homogeneous hypersurface singularity depends only on

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the weight type: $\mu = \left(\frac{1}{w_1} - 1\right)\left(\frac{1}{w_2} - 1\right) \cdots \left(\frac{1}{w_n} - 1\right)$ (see [26]). Furthermore, f is a weighted homogeneous polynomial after a biholomorphic change of coordinates if and only if $\mu = \tau$ (see [28]).

Fewnomial singularities are an important class of weighted homogeneous isolated hypersurface singularity. These singularities were introduced by Elashvili and Khimshiashvili [11]. A weighted homogeneous polynomial $f(x_1, \dots, x_n)$ is called fewnomial if the number of variables coincides with the number of monomials [11; 22; 23]. An isolated hypersurface singularity is called fewnomial singularity if it is defined by a fewnomial polynomial (see also Definition 2.6). According to Ebeling and Takahashi [10], the fewnomial singularity is also called the invertible singularity and plays an important role in mirror symmetry.

For any isolated hypersurface singularity $(V(f), 0) \subset (\mathbb{C}^n, 0)$, in the early 1980s, the second author first investigated the Lie algebra of derivations of Tyurina algebra $A(V)$, i.e., $L(V) := \mathrm{Der}(A(V), A(V))$. He proved that $L(V)$ is a solvable Lie algebra (cf. [33; 34]). In order to distinguish it from Lie algebras of other types appearing in singularity theory [1; 2; 4], one calls $L(V)$ the Yau algebra and its dimension $\lambda(V)$ the Yau number of V (cf. [40; 22]). In the past years, the authors have introduced series of new Lie algebras which are generalizations of the Yau algebra. We believe that the Yau algebra and its generalizations will play an important role in singularity theory. Beginning from the eighties, Yau and his collaborators have been systematically studying various Lie algebras of isolated hypersurface singularities (see, e.g., [3; 4; 6–8; 13–17; 24; 31–38].

In [17], we introduced a series of new derivation Lie algebras associated to an isolated hypersurface singularity $(V, 0)$ defined by the holomorphic function $f(x_1, \dots, x_n)$.

Definition 1.3. Let $\mathrm{Hess}(f)$ be the Hessian matrix (f_{ij}) of the second order partial derivatives of f and $h(f)$ be the Hessian of f, i.e., the determinant of this matrix $\mathrm{Hess}(f)$. More generally, for each k satisfying $0 \le k \le n$ we denote by $h_k(f)$ the ideal in $\mathscr{O}_n$ generated by all $k \times k$-minors in the matrix $\mathrm{Hess}(f)$. In particular, the ideal $h_n(f) = (h(f))$ is a principal ideal. For each k as above, consider the k-th Hessian algebra of the polynomial f defined by

$$H_k(f) = \mathscr{O}_n/(f + J(f) + h_k(f)).$$

In particular, $H_0(f)$ is exactly the well-known Tyurina algebra $A(V)$. The dimension of $H_k(f)$ is denoted as τ_k and $\tau_0 = \tau$.

It is known that the isomorphism class of the local k-th Hessian algebra $H_k(f)$ is contact invariant of f, i.e., depends only on the isomorphism class of the germ $(V, 0)$ [9]. We define $L_k(V) := \mathrm{Der}(H_k(V), H_k(V))$ (see Definition 2.5). The dimension of $L_k(V)$ is denoted by $\lambda_k(V)$ which is a new numerical analytic invariant of an isolated hypersurface singularity.

In [8], we first studied $L_n(V)$ (note that we use a different notation $L^*(V)$ and $\lambda^*(V)$ instead of $L_n(V)$ and $\lambda_n(V)$ there). We used the $L_n(V)$ to investigate the complex analytic structures of singularities and obtained the following result.

Theorem 1.4 [8, Theorem A]. *The Torelli-type theorem holds for simple elliptic singularities $\widetilde{E}_8$. That is, $L_n(V_{t_1}) \cong L_n(V_{t_2})$ as Lie algebras, for $t_1 \ne t_2$ in $\mathbb{C} - \{t \in \mathbb{C} : 4t^3 + 27 = 0\}$, if and only if V_{t_1} and V_{t_2} are analytically isomorphic* (*i.e.*, $t_1^3 = t_2^3$). *In particular, $\widetilde{E}_8$ gives rise to a nontrivial one-parameter family of solvable* (*resp. nilpotent*) *Lie algebras of dimension* 12 (*resp.* 11).

We also compared the dimension of $L_n(V)$ (resp. $\tau_n(V)$) with Yau number (resp. Milnor number).

Theorem 1.5 [8, Theorem D]. *Let f be a weighted homogeneous polynomial in $\mathbb{C}[x_1, x_2, \dots, x_n]$ $(2 \le n \le 4)$ with respect to weight system $(w_1, w_2, \dots, w_n; 1)$ and with $\mathrm{mult}(f) \ge 3$. Suppose that f defines an isolated singularity $(V, 0)$, then*

$$\lambda_n(V) = \lambda(V).$$

Theorem 1.6 [8]. *Let V be an isolated singularity defined by a weighted homogeneous polynomial f. Then*

$$\tau_n(V) = \mu(V) - 1,$$

where $\tau_n(V)$ is the dimension of $H_n(V)$ and $\mu(V)$ is the Milnor number of V.

Remark 1.7. It follows from Theorems 1.5 and 1.6 that $\tau_2(V) = \mu(V) - 1$, $\lambda_2(V) = \lambda(V)$ for binomial singularities (see Definition 2.6) and $\tau_3(V) = \mu(V) - 1$, $\lambda_3(V) = \lambda(V)$ for trinomial singularities (see Definition 2.6).

It is interesting to bound the Yau number of weighted homogeneous singularities with a number which only depends on the weight type. In [38], Yau and Zuo firstly proposed the sharp upper estimate conjecture that bounds the Yau number. They also proved that this conjecture holds in the case of binomial singularities. Furthermore, in [14], this conjecture was verified for trinomial singularities. In [17] we generalized the sharp upper estimate conjecture in [38] and proposed the following more general conjecture.

Conjecture 1.8 [17]. *For each $0 \le k \le n$, assume that $\lambda_k(\{x_1^{a_1} + \cdots + x_n^{a_n} = 0\}) = h_k(a_1, \dots, a_n)$. Let $(V, 0) = \{(x_1, x_2, \dots, x_n) \in \mathbb{C}^n : f(x_1, x_2, \dots, x_n) = 0\}$, $(n \ge 2)$ be an isolated singularity defined by the weighted homogeneous polynomial $f(x_1, x_2, \dots, x_n)$ of weight type $(w_1, w_2, \dots, w_n; 1)$ and $\mathrm{mult}(f) \ge 4$. Then $\lambda_k(V) \le h_k(1/w_1, \dots, 1/w_n)$.*

Remark 1.9. The inequality in Conjecture 1.8 holds true under the condition of $\mathrm{mult}(f) = 3$ and $k = 0$ for binomial singularities [38] and trinomial singularities [14]. In [17], we gave some examples which show that the inequality does not hold for $k = 1$ and $\mathrm{mult}(f) = 3$ for binomial and trinomial singularities. In this article we will see that the inequality in Conjecture 1.8 holds true under the condition of $\mathrm{mult}(f) = 3$ and $k = 2$ for trinomial singularities (see Theorem A).

We find that Conjecture 1.8 is extremely difficult to verify in general. It was only proved when $k = 0, 1$ for binomial and trinomial singularities in [38; 14] and [17] respectively. Conjecture 1.8 was proved in [38] when $k = 2$ for binomial singularities. In this article we will prove that Conjecture 1.8 holds true when $k = 2$ for trinomial singularities (see Theorem A).

For a weighted homogeneous singularity $(V, 0)$, we believe that the $\tau_k(V)$ can also be bounded with a number which only depends on the weight type. We propose the following sharp upper estimate conjecture for $\tau_k(V)$.

Conjecture 1.10. *For each $0 \le k \le n$, assume that $\tau_k(\{x_1^{a_1} + \cdots + x_n^{a_n} = 0\}) = \ell_k(a_1, \dots, a_n)$. Let $(V, 0) = \{(x_1, x_2, \dots, x_n) \in \mathbb{C}^n : f(x_1, x_2, \dots, x_n) = 0\}$, $(n \ge 2)$ be an isolated singularity defined by the weighted homogeneous polynomial $f(x_1, x_2, \dots, x_n)$ of weight type $(w_1, w_2, \dots, w_n; 1)$. Then $\tau_k(V) \le \ell_k(1/w_1, \dots, 1/w_n)$.*

Thus a natural interesting question is whether there is any relation between the numerical invariants $\lambda_k(V)$ of isolated hypersurface singularities $(V, 0)$. We propose the following inequality conjecture.

Conjecture 1.11. *With the notation as above, let* $(V,0)$ *be an isolated hypersurface singularity. Then*

$$\lambda_0(V)=\lambda_n(V)>\lambda_{n-1}(V)>\cdots>\lambda_1(V).$$

Remark 1.12. On the one hand, Theorem 1.5 tells us that the equality in Conjecture 1.11 holds true when the $(V,0)$ is small-dimensional weighted homogeneous singularity. On the other hand, it follows from a beautiful result of Saito [29, Corollary 3.8] that the equality in Conjecture 1.11 holds when $(V,0)$ is not weighted homogeneous.

The main purpose of this paper is to verify Conjectures 1.8, 1.10, and 1.11 for binomial and trinomial singularities for small k. We obtain the following main results.

Theorem A. Let $(V,0)$ be a trinomial singularity defined by the weighted homogeneous polynomial $f(x_1,x_2,x_3)$ (see Proposition 2.9) with weight type $(w_1,w_2,w_3;1)$ and $\mathrm{mult}(f)\geq 3$. Then

$$\lambda_2(V)\leq h_2\Big(\frac{1}{w_1},\frac{1}{w_2},\frac{1}{w_3}\Big)=\frac{3}{w_1w_2w_3}+4\Big(\frac{1}{w_1}+\frac{1}{w_2}+\frac{1}{w_3}\Big)-4\Big(\frac{1}{w_1w_2}+\frac{1}{w_1w_3}+\frac{1}{w_2w_3}\Big).$$

Theorem B. Let $(V,0)$ be a binomial singularity defined by the weighted homogeneous polynomial $f(x_1,x_2)$ (see Corollary 2.8) with weight type $(w_1,w_2;1)$ and $\mathrm{mult}(f)\geq 3$. Then

$$\tau_k(V)\leq h_k\Big(\frac{1}{w_1},\frac{1}{w_2}\Big),\qquad k=1,2.$$

Theorem C. Let $(V,0)$ be a trinomial singularity defined by the weighted homogeneous polynomial $f(x_1,x_2,x_3)$ (see Proposition 2.9) with weight type $(w_1,w_2,w_3;1)$ and $\mathrm{mult}(f)\geq 3$. Then

$$\tau_k(V)\leq h_k\Big(\frac{1}{w_1},\frac{1}{w_2},\frac{1}{w_3}\Big),\qquad k=1,2,3.$$

Theorem D. Let $(V,0)$ be a binomial singularity defined by the weighted homogeneous polynomial $f(x_1,x_2)$ (see Corollary 2.8) with weight type $(w_1,w_2;1)$ and $\mathrm{mult}(f)\geq 3$. Then

$$\lambda_0(V)=\lambda_2(V)>\lambda_1(V).$$

Theorem E. Let $(V,0)$ be a trinomial singularity defined by the weighted homogeneous polynomial $f(x_1,x_2,x_3)$ (see Proposition 2.9) with weight type $(w_1,w_2,w_3;1)$ and $\mathrm{mult}(f)\geq 3$. Then

$$\lambda_0(V)=\lambda_3(V)>\lambda_2(V)>\lambda_1(V).$$

2. Preliminary

We recall some definitions and basic results for derivation Lie algebras in this section.

Recall that a derivation of commutative associative algebra A is defined as a linear endomorphism D of A satisfying the Leibniz rule: $D(ab)=D(a)b+aD(b)$. Thus for such an algebra A one can consider the Lie algebra of its derivations $\mathrm{Der}(A,A)$ (or $\mathrm{Der}\,A$) with the bracket defined by the commutator of linear endomorphisms.

Definition 2.1. Let A be an associative algebra over $\mathbb{C}$. The subalgebra of endomorphisms of A generated by the identity element and left and right multiplications by elements of A is called multiplication algebra $M(A)$ of A. The centroid $C(A)$ is defined as the set of endomorphisms of A which commute with all elements of $M(A)$. Obviously, $C(A)$ is a unital subalgebra of $\mathrm{End}(A)$.

Let $S = A \otimes B$ be a tensor product of finite-dimensional associative algebras with units. It follows from a general result (see [5, Proposition 1.2]) that $\operatorname{Der} S \cong (\operatorname{Der} A) \otimes C(B) + C(A) \otimes (\operatorname{Der} B)$. Since the centroid coincides with the algebra itself for commutative associative algebras with unit, one has the following result for commutative associative algebras A, B:

Theorem 2.2 [5]. *For commutative associative algebras A, B,*

$$\operatorname{Der} S \cong (\operatorname{Der} A) \otimes B + A \otimes (\operatorname{Der} B). \tag{2-1}$$

We will only use (2-1) since all algebras here are commutative associative algebras with unit.

Definition 2.3. Let J be an ideal in an analytic algebra S. Then $\operatorname{Der}_J S \subseteq \operatorname{Der}_{\mathbb{C}} S$ is Lie subalgebra of all $\sigma \in \operatorname{Der}_{\mathbb{C}} S$ for which $\sigma(J) \subset J$.

The following well-known result is useful to compute the derivations.

Theorem 2.4 [12]. *Let J be an ideal in $R = \mathbb{C}\{x_1, \ldots, x_n\}$. Then there is a natural isomorphism of Lie algebras*

$$(\operatorname{Der}_J R)/(J \cdot \operatorname{Der}_{\mathbb{C}} R) \cong \operatorname{Der}_{\mathbb{C}}(R/J).$$

Definition 2.5. Let $(V, 0)$ be an isolated hypersurface singularity. The series new derivation Lie algebra arising from the isolated hypersurface singularity $(V, 0)$ is defined as $L_k(V) := \operatorname{Der}(H_k(f), H_k(f))$, $0 \le k \le n$ (where $H_k(f) = \mathcal{O}_{n+1}/(f + J(f) + h_k(f))$ and $h_k(f)$ is the ideal in $\mathcal{O}_n$ generated by all $k \times k$-minors in the matrix $\operatorname{Hess}(f)$). Its dimension is denoted as $\lambda_k(V)$. In particular, $H_0(f)$ is exactly the well-known Tyurina algebra $A(V)$. Thus $L_k(V)$ is a generalization of Yau algebra $L(V)$ and $L_0(V) = L(V)$. These numbers $\lambda_k(V)$ are new numerical analytic invariants of an isolated hypersurface singularity.

Definition 2.6. An isolated hypersurface singularity in $\mathbb{C}^n$ is fewnomial if it can be defined by a n-nomial in n variables and it is a weighted homogeneous fewnomial isolated singularity if it can be defined by a weighted homogeneous fewnomial. A 2 (resp. 3)-nomial isolated hypersurface singularity is also called a binomial (resp. trinomial) singularity.

Proposition 2.7 [38]. *Let f be a weighted homogeneous fewnomial isolated singularity with* $\operatorname{mult}(f) \ge 3$. *Then f is analytically equivalent to a linear combination of the following three series*:

Type A. $x_1^{a_1} + x_2^{a_2} + \cdots + x_{n-1}^{a_{n-1}} + x_n^{a_n}$, $n \ge 1$,

Type B. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n}$, $n \ge 2$,

Type C. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_1$, $n \ge 2$.

Proposition 2.7 has an immediate corollary.

Corollary 2.8. *Each binomial isolated singularity is analytically equivalent to one from the three series*:

$$\text{(A)} \quad x_1^{a_1} + x_2^{a_2}, \qquad \text{(B)} \quad x_1^{a_1} x_2 + x_2^{a_2}, \qquad \text{(C)} \quad x_1^{a_1} x_2 + x_2^{a_2} x_1.$$

Ebeling and Takahashi [10] give the following classification of weighted homogeneous fewnomial singularities in case of three variables.

Proposition 2.9 [10]. *Let $f(x_1, x_2, x_3)$ be a weighted homogeneous fewnomial isolated singularity with* $\operatorname{mult}(f) \geq 3$. *Then f is analytically equivalent to the following five types*:

Type 1. $x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$,

Type 2. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}$,

Type 3. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_1$,

Type 4. $x_1^{a_1} + x_2^{a_2} + x_3^{a_3}x_1$,

Type 5. $x_1^{a_1}x_2 + x_2^{a_2}x_1 + x_3^{a_3}$.

In order to prove Theorems A–E, we need to use the following results from [38; 14; 18; 17].

Proposition 2.10 [38]. *Let $(V, 0)$ be a fewnomial isolated singularity of Type A which is defined by $f = x_1^{a_1} + x_2^{a_2} + \cdots + x_n^{a_n}$ $(a_i \geq 3,\ 1 \leq i \leq n)$ with weight type $\big(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}; 1\big)$. Then*

$$\mu(V) = (a_1 - 1)(a_2 - 1) \cdots (a_n - 1),$$

$$\lambda(V) = n \prod_{i=1}^{n} (a_i - 1) - \sum_{i}^{n} (a_1 - 1)(a_2 - 1) \cdots \widehat{(a_i - 1)} \cdots (a_n - 1),$$

where $\widehat{(a_i - 1)}$ means that $a_i - 1$ is omitted.

Proposition 2.11 [38]. *Let $(V, 0)$ be a binomial isolated singularity of Type B which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}$ $(a_1 \geq 2, a_2 \geq 3)$ with weight type $\big(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}; 1\big)$. Then*

$$\mu(V) = a_2(a_1 - 1) + 1,$$

$$\lambda(V) = 2a_1a_2 - 2a_1 - 3a_2 + 5,$$

$$\lambda(V) \leq 2\Big(\frac{a_1a_2}{a_2-1} - 1\Big)(a_2 - 1) - \frac{a_1a_2}{a_2-1} - a_2 + 2.$$

Proposition 2.12 [38]. *Let $(V, 0)$ be a binomial isolated singularity of Type C which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_1$ $(a_1 \geq 2, a_2 \geq 2)$ with weight type $\big(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}; 1\big)$. Then*

$$\mu(V) = a_1a_2.$$

If $\operatorname{mult}(f) \geq 4$, *i.e., $a_1, a_2 \geq 3$, then the Yau number satisfies*

$$\lambda(V) = 2a_1a_2 - 2a_1 - 2a_2 + 6.$$

If $\operatorname{mult}(f) = 3$, *i.e., $f = x_1^2x_2 + x_2^{a_2}x_1$, then the Yau number is $\lambda(V) = 2a_2$. Furthermore,*

$$\lambda(V) \leq 2\Big(\frac{a_1a_2-1}{a_1-1} - 1\Big)\Big(\frac{a_1a_2-1}{a_2-1} - 1\Big) - \frac{a_1a_2-1}{a_1-1} - \frac{a_1a_2-1}{a_2-1} + 2.$$

Proposition 2.13 [17]. *Let $(V, 0)$ be a fewnomial isolated singularity of Type A which is defined by $f = x_1^{a_1} + x_2^{a_2}$ $(a_1 \geq 3,\ a_2 \geq 3)$ with weight type $\big(\frac{1}{a_1}, \frac{1}{a_2}; 1\big)$. Then*

$$\tau_1(V) = (a_1 - 2)(a_2 - 2),$$

$$\lambda_1(V) = 2a_1a_2 - 5(a_1 + a_2) + 12.$$

Proposition 2.14 [17]. *Let $(V,0)$ be a binomial isolated singularity of Type B which is defined by $f=x_1^{a_1}+x_2^{a_2}x_1$ $(a_1\geq 3,\ a_2\geq 2)$ with weight type $\left(\frac{1}{a_1},\frac{a_1-1}{a_1a_2};1\right)$. Then*

$$\tau_1(V)=a_1a_2-2(a_1+a_2)+5,$$

$$\lambda_1(V)=\begin{cases}2a_1a_2-5(a_1+a_2)+15; & a_1\geq 4,\ a_2\geq 3,\\ a_2-2; & a_1=3,\ a_2\geq 3,\\ 0; & a_1\geq 3,\ a_2=2.\end{cases}$$

Proposition 2.15 [17]. *Let $(V,0)$ be a binomial isolated singularity of Type C which is defined by $f=x_1^{a_1}x_2+x_2^{a_2}x_1$ $(a_1\geq 2, a_2\geq 2)$ with weight type $\left(\frac{a_2-1}{a_1a_2-1},\frac{a_1-1}{a_1a_2-1};1\right)$. Then*

$$\tau_1(V)=\begin{cases}a_1a_2-2(a_1+a_2)+7; & a_1\geq 3,\ a_2\geq 3,\\ 1; & a_1=2,\ a_2\geq 2,\end{cases}$$

$$\lambda_1(V)=\begin{cases}2a_1a_2-5(a_1+a_2)+19; & a_1\geq 5,\ a_2\geq 5,\\ a_2+1; & a_1=3,\ a_2\geq 3,\\ 3a_2-2; & a_1=4,\ a_2\geq 5,\\ 9; & a_1=4,\ a_2=4,\\ 0; & a_1=2,\ a_2\geq 2.\end{cases}$$

Proposition 2.16 [14]. *Let $(V,0)$ be a fewnomial isolated singularity of Type 2 which is defined by $f=x_1^{a_1}x_2+x_2^{a_2}x_3+x_3^{a_3}$ $(a_1\geq 2, a_2\geq 2, a_3\geq 3)$ with weight type $\left(\frac{1-a_3+a_2a_3}{a_1a_2a_3},\frac{a_3-1}{a_2a_3},\frac{1}{a_3};1\right)$. Then*

$$\mu(V)=(a_1a_2a_3-1+a_3-a_2a_3),$$

$$\lambda(V)=\begin{cases}3a_1a_2a_3-2a_1a_3-4a_2a_3+6a_3+2a_1-2a_1a_2+2a_2-7; & a_1\geq 2,\ a_2\geq 3,\ a_3\geq 3,\\ 4a_1a_3-3a_3-2a_1-1; & a_1\geq 2,\ a_2=2,\ a_3\geq 3.\end{cases}$$

Furthermore,

$$\lambda(V)\leq 3a_1a_2a_3-4(a_2a_3+a_3-1)-\frac{(a_1a_2a_3-1+a_3-a_2a_3)(a_3-1)}{1-a_3+a_2a_3}-\frac{(a_1a_2a_3-1+a_3-a_2a_3)}{a_3-1}.$$

Proposition 2.17 [14]. *Let $(V,0)$ be a fewnomial isolated singularity of Type 3 which is defined by $f=x_1^{a_1}x_2+x_2^{a_2}x_3+x_3^{a_3}x_1$ $(a_1\geq 2, a_2\geq 2, a_3\geq 2)$ with weight type*

$$\left(\frac{1-a_3+a_2a_3}{1+a_1a_2a_3},\frac{1-a_1+a_1a_3}{1+a_1a_2a_3},\frac{1-a_2+a_1a_2}{1+a_1a_2a_3};1\right).$$

Then

$$\mu(V)=a_1a_2a_3,$$

$$\lambda(V)=\begin{cases}12; & a_1=2,\ a_2=2,\ a_3=2,\\ 3a_1a_2a_3-2(a_1a_2+a_1a_3+a_2a_3)+2(a_1+a_2+a_3)-1; & \text{otherwise}.\end{cases}$$

Furthermore,

$$\lambda(V)\leq 3a_1a_2a_3-\frac{a_1a_3(1-a_2+a_1a_2)}{1-a_1+a_1a_3}-\frac{a_1a_2(1-a_3+a_2a_3)}{1-a_2+a_1a_2}-\frac{a_2a_3(1-a_1+a_1a_3)}{1-a_3+a_2a_3}.$$

Proposition 2.18 [18]. *Let $(V,0)$ be a fewnomial isolated singularity of Type 4 which is defined by $f = x_1^{a_1} + x_2^{a_2} + x_3^{a_3}x_1$ $(a_1 \geq 3, a_2 \geq 3, a_3 \geq 2)$ with weight type $\left(\frac{1}{a_1}, \frac{1}{a_2}, \frac{a_1-1}{a_1a_3}; 1\right)$. Then*

$$\mu(V) = a_1a_2a_3 - (a_1a_2 + a_1a_3) + a_1 + a_2 - 1,$$
$$\lambda(V) = 3a_1a_2a_3 - 4a_1a_2 - 4a_1a_3 - 2a_2a_3 + 5a_1 + 6a_2 + 2a_3 - 7.$$

Proposition 2.19 [18]. *Let $(V,0)$ be a fewnomial isolated singularity of Type 5 which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_1 + x_3^{a_3}$ $(a_1 \geq 2, a_2 \geq 2, a_3 \geq 3)$ with weight type $\left(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}, \frac{1}{a_3}; 1\right)$. Then*

$$\mu(V) = a_1a_2(a_3-1),$$
$$\lambda(V) = \begin{cases} 3a_1a_2a_3 - 4a_1a_2 - 2(a_2a_3 + a_1a_3) + 2(a_1+a_2) + 6a_3 - 6; & a_1 \geq 3,\ a_2 \geq 3,\ a_3 \geq 3, \\ 4a_2a_3 - 6a_2; & a_1 = 2,\ a_2 \geq 2,\ a_3 \geq 3. \end{cases}$$

Remark 2.20. Types 4 and 5 are Thom–Sebastiani summations of Types 1, 2, and 3. It follows from Theorem 2.21 that Types 4, 5 satisfy Conjecture 1.8.

Theorem 2.21 [38]. *Let $(V_f,0) \subset (\mathbb{C}^n,0)$ and $(V_g,0) \subset (\mathbb{C}^m,0)$ be defined by weighted homogeneous polynomials $f(x_1,\dots,x_n) = 0$ of weight type $(w_1,\dots,w_n;1)$ and $g(y_1,\dots,y_m) = 0$ of weight type $(w_{n+1},\dots,w_{n+m};1)$ respectively. Let $\mu(V_f)$, $\mu(V_g)$, $A(V_f)$ and $A(V_g)$ be the Milnor numbers and Tyurina algebras of $(V_f,0)$ and $(V_g,0)$ respectively. Then*

$$\lambda(V_{f+g}) = \mu(V_f)\lambda(V_g) + \mu(V_g)\lambda(V_f).$$

Also, if both f and g satisfy Conjecture 1.8, then $f+g$ also satisfies the same conjecture.

Proposition 2.22 [17]. *Let $(V,0)$ be a fewnomial isolated singularity of Type 1 which is defined by $f = x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$ $(a_1 \geq 3, a_2 \geq 3, a_3 \geq 3)$ with weight type $\left(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}; 1\right)$. Then*

$$\tau_1(V) = a_1a_2a_3 - 2(a_1a_2 + a_1a_3 + a_2a_3) + 4(a_1+a_2+a_3) - 8,$$
$$\lambda_1(V) = 3a_1a_2a_3 + 16(a_1+a_2+a_3) - 7(a_1a_2 + a_1a_3 + a_2a_3) - 36.$$

Proposition 2.23 [17]. *Let $(V,0)$ be a fewnomial isolated singularity of Type 2 which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}$ $(a_1 \geq 2, a_2 \geq 2, a_3 \geq 3)$ with weight type $\left(\frac{1-a_3+a_2a_3}{a_1a_2a_3}, \frac{a_3-1}{a_2a_3}, \frac{1}{a_3}; 1\right)$. Then*

$$\tau_1(V) = a_1a_2a_3 - 2(a_1a_2 + a_1a_3 + a_2a_3) + 5(a_1+a_3) + 4a_2 - 12,$$

$$\lambda_1(V) = \begin{cases} 3a_1a_2a_3 - 7(a_1a_2 + a_1a_3 + a_2a_3) + 20(a_1+a_3) + 16a_2 - 55; & a_1 \geq 4,\ a_2 \geq 4,\ a_3 \geq 4, \\ 2a_1a_3 - a_1 - 3a_3 - 1; & a_1 \geq 3,\ a_2 = 3,\ a_3 \geq 4, \\ 2a_2a_3 - 5a_2 - a_3 + 5; & a_1 = 3,\ a_2 \geq 4,\ a_3 \geq 4, \\ 2a_1a_2 - 3a_1 - 5a_2 + 10; & a_1 \geq 3,\ a_2 \geq 3,\ a_3 = 3, \\ a_3 - 3; & a_1 = 2,\ a_2 \geq 2,\ a_3 \geq 3, \\ a_1 - 3; & a_1 \geq 3,\ a_2 = 2,\ a_3 \geq 3. \end{cases}$$

Proposition 2.24 [17]. *Let $(V,0)$ be a fewnomial isolated singularity of Type 3 which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_1$ $(a_1 \geq 2, a_2 \geq 2, a_3 \geq 2)$ with weight type*

$$\left(\frac{1-a_3+a_2a_3}{1+a_1a_2a_3}, \frac{1-a_1+a_1a_3}{1+a_1a_2a_3}, \frac{1-a_2+a_1a_2}{1+a_1a_2a_3}; 1\right).$$

Then

$$\tau_1(V)=\begin{cases} a_1a_2a_3-2(a_1a_2+a_1a_3+a_2a_3)+5(a_1+a_2+a_3)-14; & a_1\geq 3,\ a_2\geq 3,\ a_3\geq 3,\\ a_3-1; & a_1=2,\ a_2\geq 2,\ a_3\geq 2,\end{cases}$$

$$\lambda_1(V)=\begin{cases} 3a_1a_2a_3+20(a_1+a_2+a_3)-7(a_1a_2+a_1a_3+a_2a_3)-63; & a_1\geq 3,\ a_2\geq 3,\ a_3\geq 3,\\ a_3-2; & a_1=2,\ a_2\geq 2,\ a_3\geq 2.\end{cases}$$

Proposition 2.25 [17]. *Let $(V,0)$ be a fewnomial isolated singularity of Type 4 which is defined by $f=x_1^{a_1}+x_2^{a_2}+x_3^{a_3}x_1$ $(a_1\geq 3, a_2\geq 3, a_3\geq 2)$ with weight type $\left(\frac{1}{a_1},\frac{1}{a_2},\frac{a_1-1}{a_1a_3};1\right)$. Then*

$$\tau_1(V)=a_1a_2a_3-2(a_1a_2+a_1a_3+a_2a_3)+4(a_1+a_3)+5a_2-10,$$

$$\lambda_1(V)=\begin{cases} 3a_1a_2a_3+16(a_1+a_3)+20a_2-7(a_1a_2+a_1a_3+a_2a_3)-45; & a_1\geq 4,\ a_2\geq 3,\ a_3\geq 3,\\ 2a_2a_3-3a_2-5a_3+7; & a_1=3,\ a_2\geq 3,\ a_3\geq 2,\\ a_2-3; & a_1\geq 4,\ a_2\geq 3,\ a_3=2.\end{cases}$$

Proposition 2.26 [17]. *Let $(V,0)$ be a fewnomial isolated singularity of Type 5 which is defined by $f=x_1^{a_1}x_2+x_2^{a_2}x_1+x_3^{a_3}$ $(a_1\geq 2, a_2\geq 2, a_3\geq 3)$ with weight type $\left(\frac{a_2-1}{a_1a_2-1},\frac{a_1-1}{a_1a_2-1},\frac{1}{a_3};1\right)$. Then*

$$\tau_1(V)=\begin{cases} a_1a_2a_3-2(a_1a_2+a_1a_3+a_2a_3)+4(a_1+a_2)+7a_3-14; & a_1\geq 3,\ a_2\geq 3,\ a_3\geq 3,\\ a_3-2; & a_1=2,\ a_2\geq 2,\ a_3\geq 3,\end{cases}$$

$$\lambda_1(V)=\begin{cases} 3a_1a_2a_3+16(a_1+a_2)+26a_3-7(a_1a_2+a_1a_3+a_2a_3)-59; & a_1\geq 5,\ a_2\geq 5,\ a_3\geq 3,\\ a_3-3; & a_1=2,\ a_2\geq 2,\ a_3\geq 3,\\ 2a_2a_3-5a_2+2a_3-5; & a_1=3,\ a_2\geq 3,\ a_3\geq 3,\\ 6a_2a_3-14a_2-8a_3+17; & a_1=4,\ a_2\geq 4,\ a_3\geq 3.\end{cases}$$

3. Proof of Theorems

In order to prove the Theorems A–E, we need to prove the following propositions.

Proposition 3.1. *Let $(V,0)$ be a fewnomial isolated singularity of Type 1 which is defined by $f=x_1^{a_1}+x_2^{a_2}+x_3^{a_3}$ $(a_1\geq 3, a_2\geq 3, a_3\geq 3)$ with weight type $\left(\frac{1}{a_1},\frac{1}{a_2},\frac{1}{a_3};1\right)$. Then*

$$\lambda_2(V)=3a_1a_2a_3+4(a_1+a_2+a_3)-4(a_1a_2+a_1a_3+a_2a_3).$$

Proof. It is easy to see that the Hessian algebra

$$H_2(V)=\mathscr{O}_3/(f+J(f)+h_2(f))$$

has dimension $a_1a_2a_3-(a_1a_2+a_1a_3+a_2a_3)+4$ and has a monomial basis of the form

$$\big\{x_1^{i_1}x_2^{i_2}x_3^{i_3},\ 0\leq i_1\leq a_1-3,\ 0\leq i_2\leq a_2-2,\ 0\leq i_3\leq a_3-3;$$
$$x_1^{a_1-2}x_2^{i_2}x_3^{i_3},\ 0\leq i_2\leq a_2-3,\ 0\leq i_3\leq a_3-3;\ x_1^{i_1}x_2^{i_2}x_3^{a_3-2},\ 0\leq i_1\leq a_1-3,\ 0\leq i_2\leq a_2-3\big\},$$

with the following relations:

$$x_1^{a_1-1}=x_2^{a_2-1}=x_3^{a_3-1}=x_1^{a_1-2}x_2^{a_2-2}=x_1^{a_1-2}x_3^{a_3-2}=x_2^{a_2-2}x_3^{a_3-2}=0. \tag{3-1}$$

In order to compute a derivation D of $H_2(V)$, it suffices to indicate its values on the generators x_1, x_2, x_3 which can be written in terms of the basis. Thus we can write

$$Dx_j = \sum_{i_1=0}^{a_1-3}\sum_{i_2=0}^{a_2-2}\sum_{i_3=0}^{a_3-3} c^j_{i_1,i_2,i_3} x_1^{i_1}x_2^{i_2}x_3^{i_3} + \sum_{i_2=0}^{a_2-3}\sum_{i_3=0}^{a_3-3} c^j_{a_1-2,i_2,i_3} x_1^{a_1-2}x_2^{i_2}x_3^{i_3} + \sum_{i_1=0}^{a_1-3}\sum_{i_2=0}^{a_2-3} c^j_{i_1,i_2,a_3-2} x_1^{i_1}x_2^{i_2}x_3^{a_3-2}, \qquad j=1,2,3.$$

Using relations (3-1) and derivation D of $H_2(V)$, we obtain the following description of Lie algebras in question:

$$\begin{array}{llll}
x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_1, & 1\le i_1\le a_1-2, & 0\le i_2\le a_2-3, & 0\le i_3\le a_3-3;\\
x_2^{i_2}x_3^{a_3-2}\partial_1, & 0\le i_2\le a_2-3; & & \\
x_1^{i_1}x_2^{i_2}x_3^{a_3-2}\partial_1, & 1\le i_1\le a_1-3, & 0\le i_2\le a_2-3; & \\
x_2^{a_2-2}x_3^{i_3}\partial_1, & 0\le i_3\le a_3-3; & & \\
x_1^{i_1}x_2^{a_2-2}x_3^{i_3}\partial_1, & 1\le i_1\le a_1-3, & 0\le i_3\le a_3-3; & \\
x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2, & 0\le i_1\le a_1-3, & 1\le i_2\le a_2-2, & 0\le i_3\le a_3-3;\\
x_1^{i_1}x_3^{a_3-2}\partial_2, & 0\le i_1\le a_1-3; & & \\
x_1^{a_1-2}x_2^{i_2}x_3^{i_3}\partial_2, & 1\le i_2\le a_2-3, & 0\le i_3\le a_3-3; & \\
x_1^{a_1-2}x_3^{i_3}\partial_2, & 0\le i_3\le a_3-3; & & \\
x_1^{i_1}x_2^{i_2}x_3^{a_3-2}\partial_2, & 0\le i_1\le a_1-3, & 1\le i_2\le a_2-3; & \\
x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_3, & 0\le i_1\le a_1-3, & 0\le i_2\le a_2-3, & 1\le i_3\le a_3-2;\\
x_1^{i_1}x_2^{a_2-2}\partial_3, & 0\le i_1\le a_1-3; & & \\
x_1^{a_1-2}x_2^{i_2}x_3^{i_3}\partial_3, & 0\le i_2\le a_2-3, & 1\le i_3\le a_3-3; & \\
x_1^{a_1-2}x_2^{i_2}\partial_3, & 0\le i_2\le a_2-3; & & \\
x_1^{i_1}x_2^{a_2-2}x_3^{i_3}\partial_3, & 0\le i_1\le a_1-3, & 1\le i_3\le a_3-3. &
\end{array}$$

Therefore we have

$$\lambda_2(V) = 3a_1a_2a_3 + 4(a_1+a_2+a_3) - 4(a_1a_2+a_1a_3+a_2a_3). \qquad \square$$

Proposition 3.2. *Let $(V,0)$ be a fewnomial isolated singularity of Type 2 which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}$ $(a_1\ge 2, a_2\ge 2, a_3\ge 3)$ with weight type*

$$\Big(\frac{1-a_3+a_2a_3}{a_1a_2a_3}, \frac{a_3-1}{a_2a_3}, \frac{1}{a_3}; 1\Big).$$

Then

$$\lambda_2(V) = 3a_1a_2a_3 - 2a_1a_2 - 2a_1a_3 - 4a_2a_3 + 5a_3 + 2.$$

Furthermore, we need to show that when $a_1 \geq 2,\ a_2 \geq 2,\ a_3 \geq 3$, *then*

$$\begin{aligned}3a_1a_2a_3 - 2a_1a_2 - 2a_1a_3 - 4a_2a_3 + 5a_3 + 2 \\ \leq \frac{3a_1a_2^2a_3^3}{(1-a_3+a_2a_3)(a_3-1)} - 4\left(\frac{a_1a_2^2a_3^2}{(1-a_3+a_2a_3)(a_3-1)} + \frac{a_1a_2a_3^2}{1-a_3+a_2a_3} + \frac{a_2a_3^2}{a_3-1}\right) \\ + 4\left(\frac{a_1a_2a_3}{1-a_3+a_2a_3} + \frac{a_2a_3}{a_3-1} + a_3\right).\end{aligned}$$

Proof. It is easy to see that the Hessian algebra

$$H_2(V) = \mathcal{O}_3/(f + J(f) + h_2(f))$$

has dimension $a_1a_2a_3 - a_2a_3 - 2a_1 - 2a_2 + 6$ and has a monomial basis of the form

$$\begin{aligned}\{x_1^{i_1}x_2^{i_2}x_3^{i_3},\ 0 \leq i_1 \leq a_1-2;\ 0 \leq i_2 \leq a_2-2;\ 0 \leq i_3 \leq a_3-3;\ x_1^{a_1-1}x_3^{i_3},\ 0 \leq i_3 \leq a_3-3; \\ x_1^{i_1}x_2^{i_2}x_3^{i_3},\ 0 \leq i_1 \leq a_1-3,\ 0 \leq i_2 \leq a_2-3,\ a_3-2 \leq i_3 \leq a_3-1;\ x_1^{i_1}x_2^{a_2-2}x_3^{a_3-2},\ 0 \leq i_1 \leq a_1-3; \\ x_1^{i_1}x_2^{a_2-1}x_3^{i_3},\ 0 \leq i_1 \leq a_1-3,\ 0 \leq i_3 \leq a_3-2;\ x_1^{a_1-2}x_3^{a_3-2};\ x_1^{a_1-2}x_2^{a_2-1}\}.\end{aligned}$$

In order to compute a derivation D of $H_2(V)$ it suffices to indicate its values on the generators x_1, x_2, x_3 which can be written in terms of the basis. Thus we can write

$$\begin{aligned}Dx_j = \sum_{i_1=0}^{a_1-2}\sum_{i_2=0}^{a_2-2}\sum_{i_3=0}^{a_3-3} c^j_{i_1,i_2,i_3}x_1^{i_1}x_2^{i_2}x_3^{i_3} + \sum_{i_1=0}^{a_1-3}\sum_{i_3=0}^{a_3-2} c^j_{i_1,a_2-1,i_3}x_1^{i_1}x_2^{a_2-1}x_3^{i_3} + c^j_{a_1-2,a_2-1,0}x_1^{a_1-2}x_2^{a_2-1} \\ + \sum_{i_1=0}^{a_1-3}\sum_{i_2=0}^{a_2-3}\sum_{i_3=a_3-2}^{a_3-1} c^j_{i_1,i_2,i_3}x_1^{i_1}x_2^{i_2}x_3^{i_3} + \sum_{i_3=0}^{a_3-3} c^j_{a_1-1,0,i_3}x_1^{a_1-1}x_3^{i_3} + c^j_{a_1-2,0,a_3-2}x_1^{a_1-2}x_3^{a_3-2} \\ + \sum_{i_1=0}^{a_1-3} c^j_{i_1,a_2-2,a_3-2}x_1^{i_1}x_2^{a_2}x_3^{a_3-2}, \qquad j = 1, 2, 3.\end{aligned}$$

We obtain the following description of Lie algebras in question:

$$\begin{array}{llll}
x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_1, & 1 \leq i_1 \leq a_1-3, & 1 \leq i_2 \leq a_2-1, & 0 \leq i_3 \leq a_3-2; \\
x_1^{i_1}x_3^{a_3-2}\partial_1, & 1 \leq i_1 \leq a_1-2; & & \\
x_2^{a_2-1}\partial_1; & & & \\
x_2^{i_2}x_3^{i_3}\partial_1, & 0 \leq i_2 \leq a_2-3, & a_3-2 \leq i_3 \leq a_3-1; & \\
x_2^{i_2}x_3^{i_3}\partial_1, & a_2-2 \leq i_2 \leq a_2-1, & 1 \leq i_3 \leq a_3-2; & \\
(a_3(a_2-1)+1)x_1^{i_1}\partial_1 + a_1(a_3-1)x_1^{i_1-1}x_2\partial_2 + x_1^{i_1-1}x_3\partial_3, & & 1 \leq i_1 \leq a_1-2; & \\
x_1^{i_1-2}x_2^{a_2-1}\partial_1; & & & \\
x_1^{i_1}x_3^{i_3}\partial_1 + a_1x_1^{i_1-1}x_3^{i_3+1}\partial_3, & 1 \leq i_1 \leq a_1-2, & 1 \leq i_3 \leq a_3-3; & \\
x_1^{a_1-1}x_3^{i_3}\partial_1, & 0 \leq i_3 \leq a_3-3; & & \\
x_1^{a_1-2}x_2^{i_2}x_3^{i_3+1}\partial_1, & 1 \leq i_1 \leq a_1-2, & 0 \leq i_3 \leq a_3-3; &
\end{array}$$

$$
\begin{array}{llll}
x_1^{i_1}x_2^{i_2}x_3^{a_3-1}\partial_1, & 1\le i_1\le a_1-3, & 0\le i_2\le a_2-3; & \\
x_1^{i_1}x_3^{a_3-2}\partial_2-\frac{a_2}{a_1(a_3-1)}x_2^{a_2-1}x_1^{i_1}\partial_3, & 0\le i_1\le a_1-3; & & \\
x_1^{i_1}x_3^{a_3-1}\partial_2, & 0\le i_1\le a_1-3; & & \\
x_1^{a_1-2}x_3^{a_3-2}\partial_2; & & & \\
x_1^{i_1}x_2x_3^{i_3}\partial_2-(a_2-1)x_1^{i_1}x_3^{i_3+1}\partial_3, & 0\le i_1\le a_1-3, & 1\le i_3\le a_3-3; & \\
x_1^{i_1}x_3^{a_3-1}\partial_2, & 0\le i_1\le a_1-3; & & \\
x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2, & 0\le i_1\le a_1-3, & 1\le i_2\le a_2-3, & a_3-2\le i_3\le a_3-1; \\
x_1^{a_1-1}x_3^{i_3}\partial_2, & 0\le i_3\le a_3-3; & & \\
x_1^{a_1-2}x_2^{a_2-1}\partial_2; & & & \\
x_1^{a_1-2}x_2^{i_2}x_3^{i_3}\partial_2, & 1\le i_2\le a_2-2, & 0\le i_3\le a_3-3; & \\
x_1^{i_1}x_2^{a_2-1}\partial_2, & 0\le i_1\le a_1-3; & & \\
x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2, & 0\le i_1\le a_1-3, & 2\le i_2\le a_2-1, & 1\le i_3\le a_3-3; \\
x_1^{a_1-1}x_3^{i_3}\partial_3, & 0\le i_3\le a_3-3; & & \\
x_1^{i_1}x_2^{i_2}\partial_2+x_1^{i_1}x_2^{i_2-1}x_3\partial_3, & 0\le i_1\le a_1-3, & 2\le i_2\le a_2-2; & \\
x_1^{i_1}x_3^{a_3-1}\partial_3, & 0\le i_1\le a_1-3; & & \\
x_1^{i_1}x_2^{i_2}x_3^{a_3-2}\partial_2, & 0\le i_1\le a_1-3, & a_2-2\le i_2\le a_2-2; & \\
x_1^{a_1-2}x_3^{i_3}\partial_3, & 1\le i_3\le a_3-2; & & \\
x_1^{a_1-2}x_2^{a_2-1}\partial_3; & & & \\
x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_3, & 0\le i_1\le a_1-3, & 1\le i_2\le a_2-3, & 2\le i_3\le a_3-1; \\
x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_3, & 0\le i_1\le a_1-3, & a_2-2\le i_2\le a_2-1, & 1\le i_3\le a_3-2; \\
x_1^{a_1-2}x_2^{i_2}x_3^{i_3}\partial_3, & 1\le i_2\le a_2-2, & 0\le i_3\le a_3-3. &
\end{array}
$$

Therefore we have

$$\lambda_2(V)=3a_1a_2a_3-2a_1a_2-2a_1a_3-4a_2a_3+5a_3+2.$$

Next, we need to show that when $a_1\ge 2$, $a_2\ge 2$, $a_3\ge 3$, then

$$
\begin{aligned}
&3a_1a_2a_3-2a_1a_2-2a_1a_3-4a_2a_3+5a_3+2\\
&\qquad\le \frac{3a_1a_2^2a_3^3}{(1-a_3+a_2a_3)(a_3-1)}-4\left(\frac{a_1a_2^2a_3^2}{(1-a_3+a_2a_3)(a_3-1)}+\frac{a_1a_2a_3^2}{1-a_3+a_2a_3}+\frac{a_2a_3^2}{a_3-1}\right)\\
&\qquad\qquad+4\left(\frac{a_1a_2a_3}{1-a_3+a_2a_3}+\frac{a_2a_3}{a_3-1}+a_3\right).
\end{aligned}
$$

After simplification we get

$$a_3(a_1-2)+a_1(a_3-2)+a_2(a_1-1)+a_1a_2+a_1+a_2+\frac{a_1a_2(a_3-2)+2}{a_3-1}+\frac{a_1a_3(a_3-1)}{1+a_3(a_2-1)}+3\geq 0. \quad \square$$

Proposition 3.3. *Let* $(V,0)$ *be a fewnomial isolated singularity of Type* 3 *which is defined by* $f=x_1^{a_1}x_2+x_2^{a_2}x_3+x_3^{a_3}x_1$ $(a_1\geq 2, a_2\geq 2, a_3\geq 2)$ *with weight type*

$$\left(\frac{1-a_3+a_2a_3}{1+a_1a_2a_3},\frac{1-a_1+a_1a_3}{1+a_1a_2a_3},\frac{1-a_2+a_1a_2}{1+a_1a_2a_3};1\right).$$

Then

$$\lambda_2(V)=\begin{cases}3a_1a_2a_3-2(a_1a_2+a_1a_3+a_2a_3)+13; & a_1\geq 3,\ a_2\geq 3,\ a_3\geq 3,\\ 4a_2a_3-5(a_2+a_3)+16; & a_1=2,\ a_2\geq 3,\ a_3\geq 3,\\ 3a_3+3; & a_1=2,\ a_2=2,\ a_3\geq 3,\\ 9; & a_1=2,\ a_2=2,\ a_3=2.\end{cases}$$

Furthermore, we need to show that when $a_1\geq 3, a_2\geq 3, a_3\geq 3$, *then*

$$\begin{aligned}&3a_1a_2a_3-2(a_1a_2+a_1a_3+a_2a_3)+13\\ &\leq\frac{3(1+a_1a_2a_3)^3}{(1-a_3+a_2a_3)(1-a_1+a_1a_3)(1-a_2+a_1a_2)}+4\left(\frac{1+a_1a_2a_3}{1-a_3+a_2a_3}+\frac{1+a_1a_2a_3}{1-a_1+a_1a_3}+\frac{1+a_1a_2a_3}{1-a_2+a_1a_2}\right)\\ &\quad-4\left(\frac{(1+a_1a_2a_3)^2}{(1-a_3+a_2a_3)(1-a_1+a_1a_3)}+\frac{(1+a_1a_2a_3)^2}{(1-a_1+a_1a_3)(1-a_2+a_1a_2)}+\frac{(1+a_1a_2a_3)^2}{(1-a_3+a_2a_3)(1-a_2+a_1a_2)}\right).\end{aligned}$$

Proof. It is easy to see that the Hessian algebra

$$H_2(V)=\mathcal{O}_3/(f+J(f)+h_2(f))$$

has dimension $a_1a_2a_3-2(a_1+a_2+a_3)+8$ and has a monomial basis of the form

$$\begin{aligned}\{&x_1^{i_1}x_2^{i_2}x_3^{i_3}, 0\leq i_1\leq a_1-2, 0\leq i_2\leq a_2-2, 0\leq i_3\leq a_3-2; x_1^{a_1-1}x_3^{i_3}, 0\leq i_3\leq a_3-2;\\ &x_2^{i_2}x_3^{i_3}, 0\leq i_2\leq a_2-3, a_3-1\leq i_3\leq 2a_3-3; x_1^{i_1}x_2^{i_2}, 0\leq i_1\leq a_1-3, a_2\leq i_2\leq 2a_2-3;\\ &x_1^{i_1}x_2^{a_2-1}x_3^{i_3}, 0\leq i_1\leq a_1-3, 0\leq i_3\leq a_3-2; x_1^{a_1-2}x_2^{a_2-1}; x_2^{a_2-2}x_3^{a_3-1}\}.\end{aligned}$$

In order to compute a derivation D of $H_2(V)$ it suffices to indicate its values on the generators x_1, x_2, x_3 which can be written in terms of the basis. Thus we can write

$$\begin{aligned}Dx_j=&\sum_{i_1=0}^{a_1-2}\sum_{i_2=0}^{a_2-2}\sum_{i_3=0}^{a_3-2}c^j_{i_1,i_2,i_3}x_1^{i_1}x_2^{i_2}x_3^{i_3}+\sum_{i_2=0}^{a_2-3}\sum_{i_3=a_3-1}^{2a_3-3}c^j_{0,i_2,i_3}x_2^{i_2}x_3^{i_3}+c^j_{0,a_2-2,a_3-1}x_2^{a_2-2}x_3^{a_3-1}\\ &+\sum_{i_1=0}^{a_1-3}\sum_{i_3=0}^{a_3-2}c^j_{i_1,a_2-1,i_3}x_1^{i_1}x_2^{a_2-1}x_3^{i_3}+\sum_{i_1=0}^{a_1-3}\sum_{i_2=a_2}^{2a_3-3}c^j_{i_1,i_2,0}x_1^{i_1}x_2^{i_2}+\sum_{i_3=0}^{a_3-2}c^j_{a_1-1,0,i_3}x_1^{a_1-1}x_3^{i_3}\\ &+c^j_{a_1-2,a_2-1,0}x_1^{a_1-2}x_2^{a_2-1}, \qquad j=1,2,3.\end{aligned}$$

We obtain the following description of Lie algebras in question:

$$
\begin{array}{llll}
x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_1, & 1\le i_1\le a_1-2, & 0\le i_2\le a_2-2, & 0\le i_3\le a_3-2;\\
x_1^{a_1-1}x_3^{i_3}\partial_1, & 0\le i_3\le a_3-2; & & \\
x_1^{i_1}x_2^{i_2}\partial_1, & 0\le i_1\le a_1-3, & a_2\le i_2\le 2a_2-3; & \\
x_1^{i_1}x_2^{a_2-1}x_3^{i_3}\partial_1, & 0\le i_1\le a_1-3, & 0\le i_3\le a_3-2; & \\
x_2^{i_2}x_3^{i_3}\partial_1, & 0\le i_2\le a_2-3, & a_3-1\le i_3\le 2a_3-3; & \\
x_2^{a_2-2}x_3^{i_3}\partial_1, & 1\le i_3\le a_3-1, & & \\
x_1^{a_1-2}x_2^{a_2-1}\partial_1; & & & \\
x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2, & 1\le i_1\le a_1-2, & 1\le i_2\le a_2-2, & 1\le i_3\le a_3-2;\\
x_1^{a_1-1}x_3^{i_3}\partial_2, & a_3-3\le i_3\le a_3-2; & & \\
x_2^{i_2}x_3^{i_3}\partial_2, & 0\le i_2\le a_2-3, & a_3-1\le i_3\le 2a_3-3; & \\
x_2^{i_2}x_3^{i_3}\partial_2, & 2\le i_2\le a_2-2, & 1\le i_3\le a_3-2; & \\
x_1^{i_1}x_2^{i_2}\partial_2, & 0\le i_1\le a_1-3, & a_2\le i_2\le 2a_2-3; & \\
x_1^{i_1}x_2^{a_2-1}x_3^{i_3}\partial_2, & 0\le i_1\le a_1-3, & 0\le i_3\le a_3-2; & \\
x_1^{i_1}x_2^{i_2}\partial_2+\frac{a_3}{a_2-1}x_1x_2^{a_2-3}x_3\partial_3, & 1\le i_1\le a_1-3, & 2\le i_2\le a_2-2; & \\
x_1^{i_1}x_3^{a_3-2}\partial_2, & 1\le i_1\le a_1-2; & & \\
x_1^{a_1-2}x_2^{i_2}\partial_2, & 1\le i_2\le a_2-1; & & \\
x_2x_3^{a_3-2}\partial_2; & & & \\
x_2^{a_2-2}x_3^{a_3-1}\partial_2; & & & \\
x_1^{a_1-2}x_3^{i_3}\partial_3, & 1\le i_3\le a_3-2; & & \\
x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_3, & 1\le i_1\le a_1-2, & 1\le i_2\le a_2-2, & 2\le i_3\le a_3-2;\\
x_1^{a_1-1}x_3^{i_3}\partial_3, & 0\le i_3\le a_3-2; & & \\
x_2^{i_2}x_3^{i_3}\partial_3, & 0\le i_2\le a_2-3, & a_3-1\le i_3\le 2a_3-3; & \\
x_1^{i_1}x_2^{i_2}\partial_3, & 0\le i_1\le a_1-3, & a_2\le i_2\le 2a_2-3; & \\
x_2^{i_2}x_3^{i_3}\partial_3, & a_2-2\le i_2\le a_2-1, & 1\le i_3\le a_3-2; & \\
x_1^{a_1-2}x_2^{i_2}\partial_3, & a_2-2\le i_2\le a_2-1; & & \\
x_2^{a_2-2}x_3^{a_3-1}\partial_3; & & & \\
x_1^{i_1}x_2^{a_2-1}x_3^{i_3}\partial_3, & 1\le i_1\le a_1-3, & 1\le i_3\le a_3-2; & \\
x_1^{i_1}x_2^{a_2-2}x_3\partial_3, & 1\le i_1\le a_1-3; & & \\
x_1^{a_1-3}x_2^{a_2-1}\partial_3; & & & \\
x_1^{a_1-2}x_2^{i_2}x_3\partial_3, & 1\le i_2\le a_2-2. & &
\end{array}
$$

Therefore we have

$$\lambda_2(V) = 3a_1a_2a_3 - 2(a_1a_2 + a_1a_3 + a_2a_3) + 13.$$

In the case of $a_1 = 2$, $a_2 = a_3 \geq 3$, we obtain the following description of Lie algebras in question:

$$\begin{array}{lll}
x_2^{i_2}x_3^{i_3}\partial_1, & 0 \leq i_2 \leq a_2 - 3, & a_3 \leq i_3 \leq 2a_3 - 3; \\
x_2^{i_2}x_3^{i_3}\partial_2, & 0 \leq i_2 \leq a_2 - 3, & a_3 - 1 \leq i_3 \leq 2a_3 - 3; \\
x_2^{i_2}x_3^{i_3}\partial_2, & 1 \leq i_2 \leq a_1 - 2, & 0 \leq i_3 \leq a_3 - 2; \\
x_2^{i_2}x_3^{i_3}\partial_3, & 0 \leq i_2 \leq a_2 - 3, & a_3 - 1 \leq i_3 \leq 2a_3 - 3; \\
x_1x_3^{i_3}\partial_1, & 0 \leq i_3 \leq a_3 - 2; & \\
x_2^{a_2-2}x_3^{i_3}\partial_1, & a_3 - 2 \leq i_3 \leq a_3 - 1; & \\
x_1x_3^{i_3}\partial_2, & a_3 - 3 \leq i_3 \leq a_3 - 2; & \\
x_2^{a_2-2}x_3^{i_3}\partial_3, & 1 \leq i_3 \leq a_3 - 1; & \\
x_1x_3^{i_3}\partial_3, & 0 \leq i_3 \leq a_3 - 2; & \\
x_2^{a_2-1}\partial_1; & x_2^{a_2-2}x_3^{a_3-1}\partial_2; & a_3 - 2; \\
x_2^{a_2-1}\partial_2; & x_2^{a_2-1}\partial_3; & x_2^{a_2-2}x_3^{a_3-1}\partial_3.
\end{array}$$

Therefore we have

$$\lambda_2(V) = 4a_2a_3 - 5(a_2 + a_3) + 16.$$

In the case of $a_1 = a_2 = 2,\ a_3 \geq 3$, we obtain the following description of Lie algebras in question:

$$x_3^{a_3-1}\partial_1; \quad x_2\partial_1; \quad x_1x_3^{i_3}\partial_1, \quad 0 \leq i_3 \leq a_3 - 2; \quad x_3^{a_3-1}\partial_2; \quad x_2\partial_2; \quad x_1x_3^{a_3-2}\partial_2;$$
$$x_3^{a_3-1}\partial_3; \quad x_2\partial_3; \quad x_1\partial_3; \quad x_1^{i_1}x_3^{i_3}\partial_3, \quad 0 \leq i_1 \leq 1, \quad 1 \leq i_3 \leq a_3 - 2.$$

Therefore we have

$$\lambda_2(V) = 3a_3 + 3.$$

In the case of $a_1 = a_2 = a_3 = 2$, we obtain the following description of Lie algebras in question:

$$x_3\partial_1; \quad x_2\partial_1; \quad x_1\partial_1; \quad x_3\partial_2; \quad x_2\partial_2; \quad x_1\partial_2; \quad x_3\partial_3; \quad x_2\partial_3; \quad x_1\partial_3.$$

Therefore we have

$$\lambda_2(V) = 9.$$

Next, we need to show that when $a_1 \geq 3,\ a_2 \geq 3,\ a_3 \geq 3$, then

$$\begin{aligned}
&3a_1a_2a_3 - 2(a_1a_2 + a_1a_3 + a_2a_3) + 13 \\
&\leq \frac{3(1+a_1a_2a_3)^3}{(1-a_3+a_2a_3)(1-a_1+a_1a_3)(1-a_2+a_1a_2)} + 4\left(\frac{1+a_1a_2a_3}{1-a_3+a_2a_3} + \frac{1+a_1a_2a_3}{1-a_1+a_1a_3} + \frac{1+a_1a_2a_3}{1-a_2+a_1a_2}\right) \\
&\quad -4\left(\frac{(1+a_1a_2a_3)^2}{(1-a_3+a_2a_3)(1-a_1+a_1a_3)} + \frac{(1+a_1a_2a_3)^2}{(1-a_1+a_1a_3)(1-a_2+a_1a_2)} + \frac{(1+a_1a_2a_3)^2}{(1-a_3+a_2a_3)(1-a_2+a_1a_2)}\right).
\end{aligned}$$

After simplification we get

$$\frac{a_1a_2(a_1-2)(a_3-2)+\frac{1}{2}\big(a_1a_3(a_2-2)\big)+\frac{1}{2}\big(a_2(a_1a_3-4)\big)}{1-a_1+a_1a_3}$$
$$+\frac{a_2a_3(a_2-2)(a_1-2)+\frac{1}{2}\big(a_1a_2(a_3-2)\big)+\frac{1}{2}\big(a_3(a_1a_2-4)\big)}{1-a_2+a_1a_2}$$
$$+a_1+a_2+a_3+\frac{a_1a_3(a_2-2)(a_3-2)+\frac{1}{2}\big(a_2a_3(a_1-2)\big)+\frac{1}{2}\big(a_1(a_2a_3-4)\big)}{1-a_3+a_2a_3}-7\geq 0.$$

We also need to show that when $a_1=2, a_2\geq 3, a_3\geq 3$, then

$$4a_2a_3-5(a_2+a_3)+16$$
$$\leq \frac{3(1+2a_2a_3)^3}{(1-a_3+a_2a_3)(2a_3-1)(1+2a_2)}+4\bigg(\frac{1+2a_2a_3}{1-a_3+a_2a_3}+\frac{1+2a_2a_3}{2a_3-1}+\frac{1+2a_2a_3}{a_2+1}\bigg)$$
$$-4\bigg(\frac{(1+2a_2a_3)^2}{(1-a_3+a_2a_3)(2a_3-1)}+\frac{(1+2a_2a_3)^2}{(2a_3-1)(1+a_2)}+\frac{(1+2a_2a_3)^2}{(1-a_3+a_2a_3)(1+a_2)}\bigg).$$

After simplification we get

$$\frac{\frac{1}{2}\big(2a_3(a_2-2)\big)+\frac{1}{2}\big(a_2(2a_3-4)\big)}{2a_3-1}+\frac{\frac{1}{2}\big(2a_2(a_3-2)\big)+\frac{1}{2}\big(a_3(2a_2-4)\big)}{2a_2-1}$$
$$+\frac{2a_3(a_2-2)(a_3-2)+\frac{1}{2}\big(2(a_2a_3-4)\big)}{1-a_3+a_2a_3}+2(a_2+a_3)-8\geq 0.$$

Note that Conjecture 1.8 has equality for $a_1=a_2=a_3=2$. Similarly, it is easy to see that in the case of $a_1=a_2=2,\ a_3\geq 3$, Conjecture 1.8 holds. □

Proposition 3.4. *Let $(V,0)$ be a fewnomial isolated singularity of Type 4 which is defined by $f=x_1^{a_1}+x_2^{a_2}+x_3^{a_3}x_1$ $(a_1\geq 3,\ a_2\geq 3,\ a_3\geq 2)$ with weight type $\big(\frac{1}{a_1},\frac{1}{a_2},\frac{a_1-1}{a_1a_3};1\big)$. Then*

$$\lambda_2(V)=\begin{cases}3a_1a_2a_3-4a_1a_2-4a_1a_3-2a_2a_3+4a_1+5a_2; & a_1\geq 4,\ a_2\geq 3,\ a_3\geq 3,\\ 7a_2a_3-7a_2-13a_3+14; & a_1=3,\ a_2\geq 3,\ a_3\geq 2,\\ 2a_1a_2-5a_1+a_2+3; & a_1\geq 4,\ a_2\geq 3,\ a_3=2.\end{cases}$$

Furthermore, we need to show that when $a_1\geq 4, a_2\geq 3, a_3\geq 3$, then

$$3a_1a_2a_3-4a_1a_2-4a_1a_3-2a_2a_3+4a_1+5a_2$$
$$\leq\frac{3a_1^2a_2a_3}{a_1-1}+4\bigg(a_1+a_2+\frac{a_1a_3}{a_1-1}\bigg)-4\bigg(a_1a_2+\frac{a_1a_2a_3}{a_1-1}+\frac{a_1^2a_3}{a_1-1}\bigg).$$

Proof. It is easy to see that the Hessian algebra

$$H_2(V)=\mathcal{O}_3/(f+J(f)+h_2(f))$$

has dimension $a_1a_2a_3-a_1a_2-a_1a_3-2a_3+5$ and has a monomial basis of the form

$$\big\{x_1^{i_1}x_2^{i_2}x_3^{i_3},\ 0\le i_1\le a_1-2,\ 0\le i_2\le a_2-3,\ 0\le i_3\le a_3-2;\ x_2^{a_2-2}x_3^{a_3-2};\\ x_2^{i_2}x_3^{i_3},\ 0\le i_2\le a_2-3,\ a_3-1\le i_3\le 2a_3-3;\ x_1^{i_1}x_2^{a_2-2}x_3^{i_3},\ 0\le i_1\le a_1-3,\ 0\le i_3\le a_3-3\big\}.$$

In order to compute a derivation D of $H_2(V)$ it suffices to indicate its values on the generators x_1, x_2, x_3 which can be written in terms of the basis. Thus we can write

$$Dx_j=\sum_{i_1=0}^{a_1-2}\sum_{i_2=0}^{a_2-3}\sum_{i_3=0}^{a_3-2}c^j_{i_1,i_2,i_3}x_1^{i_1}x_2^{i_2}x_3^{i_3}+\sum_{i_2=0}^{a_2-3}\sum_{i_3=a_3-1}^{2a_3-3}c^j_{0,i_2,i_3}x_2^{i_2}x_3^{i_3}+\sum_{i_1=0}^{a_1-3}\sum_{i_3=0}^{a_3-3}c^j_{i_1,a_2-2,i_3}x_1^{i_1}x_2^{a_2-2}x_3^{i_3}\\ +c^j_{0,a_2-2,a_3-2}x_2^{a_2-2}x_3^{a_3-2},\qquad j=1,2,3.$$

We obtain the following description of Lie algebras in question:

$$\begin{array}{llll}
x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_1, & 1\le i_1\le a_1-2, & 0\le i_2\le a_2-3, & 0\le i_3\le a_3-2;\\
x_2^{a_2-2}x_3^{a_3-2}\partial_1; & & & \\
x_2^{i_2}x_3^{i_3}\partial_1, & 0\le i_2\le a_2-3, & a_3-1\le i_3\le 2a_3-3; & \\
x_1^{i_1}x_2^{a_2-2}x_3^{i_3}\partial_1, & 0\le i_1\le a_1-3, & 0\le i_3\le a_3-3; & \\
x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2, & 0\le i_1\le a_1-2, & 1\le i_2\le a_2-3, & 0\le i_3\le a_3-2;\\
x_2^{a_2-2}x_3^{a_3-2}\partial_2; & & & \\
x_2^{i_2}x_3^{i_3}\partial_2, & 0\le i_2\le a_2-3, & a_3-1\le i_3\le 2a_3-3; & \\
x_1^{i_1}x_2^{a_2-2}x_3^{i_3}\partial_2, & 0\le i_1\le a_1-3, & 0\le i_3\le a_3-3; & \\
x_1^{i_1}x_3^{a_3-2}\partial_2, & 1\le i_1\le a_1-3, & 1\le i_2\le a_2-3; & \\
x_1^{a_1-2}x_3^{i_3}\partial_2, & 0\le i_3\le a_3-2; & & \\
x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_3, & 1\le i_1\le a_1-2, & 0\le i_2\le a_2-3, & 1\le i_3\le a_3-2;\\
x_2^{a_2-2}x_3^{a_3-2}\partial_3; & & & \\
x_2^{i_2}x_3^{i_3}\partial_3, & 0\le i_2\le a_2-3, & a_3-1\le i_3\le 2a_3-3; & \\
x_1^{i_1}x_2^{a_2-2}x_3^{i_3}\partial_3, & 0\le i_1\le a_1-3, & 0\le i_3\le a_3-3; & \\
x_1^{a_1-2}x_2^{i_2}\partial_3, & 0\le i_2\le a_2-3. & &
\end{array}$$

Therefore we have

$$\lambda_2(V)=3a_1a_2a_3-4a_1a_2-4a_1a_3-2a_2a_3+4a_1+5a_2.$$

In the case of $a_1=3,\ a_2\ge 3,\ a_3\ge 2$, we obtain the following description of Lie algebras in question:

$$\begin{array}{llll}
x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2, & 0\le i_1\le 1, & 1\le i_2\le a_2-3, & 0\le i_3\le a_3-2;\\
x_1x_3^{a_3-2}\partial_2; & & & \\
x_2^{i_2}x_3^{i_3}\partial_1, & 0\le i_2\le a_2-3, & a_3-1\le i_3\le 2a_3-3; &
\end{array}$$

$$
\begin{array}{lll}
x_1x_2^{i_2}x_3^{i_3}\partial_1, & 0\le i_2\le a_2-3, & 0\le i_3\le a_3-2;\\
x_1x_2^{i_2}x_3^{i_3}\partial_3, & 0\le i_2\le a_2-3, & 0\le i_3\le a_3-2;\\
x_2^{i_2}x_3^{i_3}\partial_3, & 0\le i_2\le a_2-3, & a_3-1\le i_3\le 2a_3-3;\\
x_2^{a_2-2}x_3^{i_3}\partial_1, & 0\le i_3\le a_3-2; & \\
x_2^{a_2-2}x_3^{i_3}\partial_2, & 0\le i_3\le a_3-2; & \\
x_2^{a_2-2}x_3^{i_3}\partial_3, & 0\le i_3\le a_3-2 & \\
x_2^{i_2}x_3^{i_3}\partial_2, & 0\le i_2\le a_2-3, & a_3-1\le i_3\le 2a_3-3.
\end{array}
$$

Therefore we have

$$\lambda_2(V)=7a_2a_3-7a_2-13a_3+14.$$

In the case of $a_1\ge 4$, $a_2\ge 3$, $a_3=2$, we obtain the following description of Lie algebras in question:

$$
\begin{array}{lllll}
x_1^{i_1}x_2^{i_2}\partial_1, & 1\le i_1\le a_1-2, & 0\le i_2\le a_2-3; & & \\
x_2^{i_2}x_3^{i_3}\partial_2, & 1\le i_2\le a_2-3, & 0\le i_3\le 1; & & \\
x_1^{i_1}x_2^{i_2}\partial_2, & 1\le i_1\le a_1-2, & 1\le i_2\le a_2-3; & & \\
x_2^{i_2}x_3\partial_1, & 0\le i_2\le a_2-3; & & & \\
x_1^{a_1-1}x_2^{i_2}\partial_3, & 0\le i_2\le a_2-3; & & & \\
x_2^{i_2}x_3\partial_3, & 0\le i_2\le a_2-3; & & & \\
x_2^{a_2-2}\partial_1; & x_3\partial_2; & x_2^{a_2-2}\partial_2; & x_1^{a_1-2}\partial_2; & x_2^{a_2-2}\partial_3.
\end{array}
$$

Therefore we have

$$\lambda_2(V)=2a_1a_2-5a_1+a_2+3.$$

Next, we need to show that when $a_1\ge 4,\ a_2\ge 3,\ a_3\ge 3$, then

$$
\begin{aligned}
&3a_1a_2a_3-4a_1a_2-4a_1a_3-2a_2a_3+4a_1+5a_2\\
&\qquad\le \frac{3a_1^2a_2a_3}{a_1-1}+4\left(a_1+a_2+\frac{a_1a_3}{a_1-1}\right)-4\left(a_1a_2+\frac{a_1a_2a_3}{a_1-1}+\frac{a_1^2a_3}{a_1-1}\right).
\end{aligned}
$$

After simplification we get

$$a_2((a_1-1)(a_3-1)-a_3)\ge 0.$$

We also need to show that when $a_1=3, a_2\ge 3, a_3\ge 2$, then

$$7a_2a_3-7a_2-13a_3+14\le \frac{27a_2a_3}{2}+4\left(3+a_2+\frac{3a_3}{2}\right)-4\left(3a_2+\frac{3a_2a_3}{2}+\frac{9a_3}{2}\right).$$

After simplification we get

$$a_2(a_3-1)\ge 0.$$

Similarly, it is easy to see that in the case of $a_1\ge 4,\ a_2\ge 3,\ a_3=2$, Conjecture 1.8 holds. $\square$

Proposition 3.5. *Let* $(V,0)$ *be a fewnomial isolated singularity of Type* 5 *which is defined by* $f = x_1^{a_1}x_2 + x_2^{a_2}x_1 + x_3^{a_3}$ $(a_1 \geq 2, a_2 \geq 2, a_3 \geq 3)$ *with weight type* $\left(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}, \frac{1}{a_3}; 1\right)$. *Then*

$$\lambda_2(V) = \begin{cases} 3a_1a_2a_3 - 4a_1a_2 - 2(a_1a_3 + a_2a_3) + 5a_3 + 4; & a_1 \geq 3,\ a_2 \geq 3,\ a_3 \geq 3, \\ 4a_2a_3 - 10a_2 - a_3 + 8; & a_1 = 2,\ a_2 \geq 2,\ a_3 \geq 3., \end{cases}$$

Furthermore, we need to show that when $a_1 \geq 3, a_2 \geq 3, a_3 \geq 3$, *then*

$$3a_1a_2a_3 - 4a_1a_2 - 2(a_1a_3 + a_2a_3) + 5a_3 + 4$$
$$\leq \frac{3a_3(a_1a_2-1)^2}{(a_2-1)(a_1-1)} + 4\left(\frac{a_1a_2-1}{a_2-1} + \frac{a_1a_2-1}{a_1-1} + a_3\right) - 4\left(\frac{(a_1a_2-1)^2}{(a_2-1)(a_1-1)} + \frac{a_3(a_1a_2-1)}{a_1-1} + \frac{a_3(a_1a_2-1)}{a_2-1}\right).$$

Proof. It is easy to see that the Hessian algebra

$$H_2(V) = \mathscr{O}_3/(f + J(f) + h_2(f))$$

has dimension $a_1a_2a_3 - a_1a_2 - 2(a_1+a_2) - a_3 + 9$ and has a monomial basis of the form

$$\big\{x_1^{i_1}x_2^{i_2}x_3^{i_3}, 0 \leq i_1 \leq a_1-2, 0 \leq i_2 \leq a_2-1, 0 \leq i_3 \leq a_3-3;\ x_2^{a_2-2}x_3^{a_3-2};$$
$$x_2^{i_2}x_3^{i_3}, a_2 \leq i_2 \leq 2a_2-3, 0 \leq i_3 \leq a_3-3;\ x_1^{i_1}x_2^{i_2}x_3^{a_3-2}, 0 \leq i_1 \leq a_1-3, 0 \leq i_2 \leq a_2-3;$$
$$x_1^{a_1-1}x_3^{i_3}, 0 \leq i_3 \leq a_3-3;\ x_2^{i_2}x_3^{a_3-2}, a_2-2 \leq i_2 \leq a_2-1\big\}.$$

In order to compute a derivation D of $H_2(V)$ it suffices to indicate its values on the generators x_1, x_2, x_3 which can be written in terms of the basis. Thus we can write

$$Dx_j = \sum_{i_1=0}^{a_1-2}\sum_{i_2=0}^{a_2-1}\sum_{i_3=0}^{a_3-3} c^j_{i_1,i_2,i_3}x_1^{i_1}x_2^{i_2}x_3^{i_3} + \sum_{i_2=a_2}^{2a_2-3}\sum_{i_3=0}^{a_3-3} c^j_{0,i_2,i_3}x_2^{i_2}x_3^{i_3} + \sum_{i_1=0}^{a_1-3}\sum_{i_2=0}^{a_2-3} c^j_{i_1,i_2,a_3-2}x_1^{i_1}x_2^{i_2}x_3^{a_3-2}$$
$$+ \sum_{i_3=0}^{a_3-3} c^j_{a_1-1,0,i_3}x_1^{a_1-1}x_3^{i_3} + \sum_{i_2=a_2-2}^{a_2-1} c^j_{0,i_2,a_3-2}x_2^{i_2}x_3^{a_3-2} + c^j_{a_1-2,0,a_3-2}x_1^{a_1-2}x_3^{a_3-2}, \qquad j = 1,2,3.$$

We obtain the following description of Lie algebras in question:

$$\begin{array}{llll}
x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_1, & 1 \leq i_1 \leq a_1-2, & 0 \leq i_2 \leq a_2-1, & 0 \leq i_3 \leq a_3-3; \\
x_1^{a_1-2}x_3^{a_3-2}\partial_1; & & & \\
x_2^{i_2}x_3^{i_3}\partial_1, & a_2-2 \leq i_2 \leq 2a_2-3, & 0 \leq i_3 \leq a_3-3; & \\
x_1^{i_1}x_2^{i_2}x_3^{a_3-2}\partial_1, & 0 \leq i_1 \leq a_1-3, & 0 \leq i_2 \leq a_2-3; & \\
x_1^{a_1-1}x_3^{i_3}\partial_1, & 0 \leq i_3 \leq a_3-3; & & \\
x_2^{i_2}x_3^{a_3-2}\partial_1, & a_2-2 \leq i_2 \leq a_2-1; & & \\
x_1^{a_1-1}x_3^{i_3}\partial_2, & 0 \leq i_3 \leq a_3-3; & & \\
x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2, & 1 \leq i_1 \leq a_1-2, & 2 \leq i_2 \leq a_2-1, & 0 \leq i_3 \leq a_3-3; \\
x_1^{a_1-2}x_3^{a_3-2}\partial_2; & & & \\
x_2^{i_2}x_3^{i_3}\partial_2, & a_2-1 \leq i_2 \leq 2a_2-3, & 0 \leq i_3 \leq a_3-3; &
\end{array}$$

$$
\begin{array}{llll}
x_1^{i_1}x_2^{i_2}x_3^{a_3-2}\partial_2, & 1\le i_1\le a_1-3, & 0\le i_2\le a_2-3; & \\
x_1^{a_1-2}x_2x_3^{i_3}\partial_2, & 0\le i_3\le a_3-3; & & \\
x_2^{i_2}x_3^{a_3-2}\partial_2, & 0\le i_2\le a_2-1; & & \\
x_1^{a_1-1}x_3^{i_3}\partial_3, & 1\le i_3\le a_3-3; & & \\
x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_3, & 0\le i_1\le a_1-2, & 0\le i_2\le a_2-1, & 1\le i_3\le a_3-3; \\
x_2^{a_2-1}\partial_3; & & & \\
x_2^{i_2}x_3^{i_3}\partial_3, & a_2\le i_2\le 2a_2-3, & 0\le i_3\le a_3-3; & \\
x_1^{i_1}x_2^{i_2}x_3^{a_3-2}\partial_3, & 1\le i_1\le a_1-3, & 1\le i_2\le a_2-3; & \\
x_2^{i_2}x_3^{a_3-2}\partial_3, & 0\le i_2\le a_2-1; & & \\
x_1^{i_1}x_3^{a_3-2}\partial_3, & 1\le i_1\le a_1-2; & & \\
x_1^{a_1-2}x_2^{i_2}\partial_3, & 1\le i_2\le a_2-3; & & \\
x_1^{i_1}x_2^{i_2}\partial_3, & 1\le i_1\le a_1-2, & a_2-2\le i_2\le a_2-1. &
\end{array}
$$

Therefore we have

$$\lambda_2(V)=3a_1a_2a_3-4a_1a_2-2(a_1a_3+a_2a_3)+5a_3+4.$$

In the case of $a_1=2$, $a_2\ge 2$, $a_3\ge 3$, we obtain the following description of Lie algebras in question:

$$
\begin{array}{lll}
x_2^{i_2}x_3^{i_3}\partial_1, & a_2-1\le i_2\le 2a_2-3, & 0\le i_3\le a_3-3; \\
x_2^{i_2}x_3^{i_3}\partial_2, & a_2-1\le i_2\le 2a_2-3, & 0\le i_3\le a_3-3; \\
x_2^{i_2}x_3^{i_3}\partial_3, & 0\le i_2\le 2a_2-3, & 1\le i_3\le a_3-3; \\
x_1x_3^{i_3}\partial_1, & 0\le i_3\le a_3-3; & \\
x_3^{a_3-2}\partial_1; & x_3^{a_3-2}\partial_3; & \\
x_1x_3^{i_3}\partial_2, & 0\le i_3\le a_3-3; & \\
x_1x_3^{i_3}\partial_3, & 1\le i_3\le a_3-3; & \\
x_3^{a_3-2}\partial_2; & x_2^{a_2-1}\partial_3; & x_2^{2a_2-3}\partial_3.
\end{array}
$$

Therefore we have

$$\lambda_2(V)=4a_2a_3-10a_2-a_3+8.$$

Next, we need to show that when $a_1\ge 3$, $a_2\ge 3$, $a_3\ge 3$, then

$$
\begin{aligned}
&3a_1a_2a_3-4a_1a_2-2(a_1a_3+a_2a_3)+5a_3+4\\
&\quad\le \frac{3a_3(a_1a_2-1)^2}{(a_2-1)(a_1-1)}+4\left(\frac{a_1a_2-1}{a_2-1}+\frac{a_1a_2-1}{a_1-1}+a_3\right)-4\left(\frac{(a_1a_2-1)^2}{(a_2-1)(a_1-1)}+\frac{a_3(a_1a_2-1)}{a_1-1}+\frac{a_3(a_1a_2-1)}{a_2-1}\right).
\end{aligned}
$$

After simplification we get

$$[a_1(a_2-2)(a_1-3)+a_2(a_1-2)(a_2-3)+(a_1-3)+(a_2-3)](a_3-1)(a_1-1)(a_2-1) \\ +a_1(a_2-2)(a_1-3)+a_2(a_1-2)(a_2-3)+a_1+a_2-6\geq 0.$$

We also need to show that when $a_1=2,\ a_2\geq 2,\ a_3\geq 3$, then

$$4a_2a_3-10a_2-a_3+8 \\ \leq \frac{3a_3(2a_2-1)^2}{(a_2-1)}+4\left(\frac{2a_2-1}{a_2-1}+2a_2-1+a_3\right)-4\left(\frac{(2a_2-1)^2}{(a_2-1)}+a_3(2a_2-1)+\frac{a_3(2a_2-1)}{a_2-1}\right).$$

After simplification we get

$$2a_2(a_2-3)+a_3(7a_2-5)+4\geq 0.$$ □

4. Proof of Theorems

Proof of Theorem A. It is easy to see that Theorem A follows from Propositions 3.1–3.5. □

Proof of Theorem B. It follows from Propositions 2.10–2.15 and Remark 1.7 that the inequalities $\tau_k(V)\leq \ell_k\big(\frac{1}{w_1},\frac{1}{w_2}\big),\ k=1,2$, hold true. □

Proof of Theorem C. It follows from Propositions 2.10, 2.16–2.26, 3.1–3.5 and Remark 1.7 that the inequalities $\tau_k(V)\leq \ell_k\big(\frac{1}{w_1},\frac{1}{w_2},\frac{1}{w_3}\big),\ k=1,2,3$, hold true. □

Proof of Theorem D. It follows from Propositions 2.10–2.15 and Remark 1.7 that the inequality $\lambda_0(V)=\lambda_2(V)>\lambda_1(V)$ holds true. □

Proof of Theorem E. It follows from Propositions 2.10, 2.16–2.26, 3.1–3.5 and Remark 1.7 that the inequality $\lambda_0(V)=\lambda_3(V)>\lambda_2(V)>\lambda_1(V)$ holds true. □

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