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An obstruction for smoothing of Gorenstein surface singularities

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§0. Introduction

An isolated singularity of a complex analytic space V is called smoothable if there is a flat family $\phi : \mathscr{V} \to D$ over the unit disk D such that $\phi^{-1}(0)$ is isomorphic to V and the generic fibre of ϕ is non-singular. The first example of a non-smoothable singularity was due to R. Thom (cf. is a cone over $\mathbf{P}^1 \times \mathbf{P}^2$ embedded in $\mathbf{P}^5$ by Segre embedding). The proof uses complex cobordism theory and is based on the fact that the link of a smoothable singularity of V having dimension say k (which is the intersection of V and the boundary S^{2N-1} of a small ball about the singular point of V embedded in $\mathbf{C}^N$) defines the trivial element of $\pi_{2N-1}(MU(N-k))$ where $MU(N-k)$ is the Thom space of the universal $(N-k)$-bundle. Hartshorne used the Barth–Ogus type theorem to prove non-smoothability of cones over some algebraic varieties (cf. [Ha] which contains good overview of the subject around that period). E. Rees and E. Thomas [RT] made detailed calculation of the homotopy groups of the Thom spaces in question and used them to construct more examples of non-smoothable singularities. J. Wahl [W] constructed additional examples based on a new idea. His methods of detecting non-smoothability of Gorenstein surface singularities used the formula for the length β of Coker $(\Theta_{V/T} \otimes \mathcal{O}_V \to \Theta_V)$:

$$\beta = h^1(\Theta_{\tilde{V}}) + 10p_g + 2K^2$$

under the hypothesis of the existence of globalizing smoothing of V which was proven only recently by Looijenga [Lo]. Here $\Theta_{\tilde{V}}$ (resp. Θ_V, resp. $\Theta_{V/T}$) denotes the tangent sheaf of the desingularization $\tilde{V}$ of V, (resp. V, resp. relative derivations), p_g is the geometric genus $h^1(\mathcal{O}_{\tilde{V}})$ and K is the canonical class of $\tilde{V}$. The equivalent formula was also proved independently by the second author [Y2]. The example of Wahl [W] depends on the fact that under further hypothesis (cf. Theorem 4.3 of [W]) β is the dimension of the smoothing component. Hence singularities for which the expression is negative automatically cannot be smoothable. Using this, he proved that cusp with multiplicity m with r exceptional curves in the minimal resolution and $m > r + 9$ is non-smoothable. The point is that cusp singularities are taut by Laufer [La1]. Therefore $h^1(\Theta_{\tilde{V}})$ can be computed easily for cusp singularities. However, in general, it is difficult to compute $h^1(\Theta_{\tilde{V}})$. Recently Wahl and Looijenga [LW] pointed out that invariants of the linking pairing on the link of singularity can be used to detect non-smoothability although calculations of these invariants are not obvious.

The purpose of this work is to show that Rohlin's μ-invariant can be used to detect non-smoothability as well. The use is rather similar to the use of the signature defects by A. Durfee [D] to obtain a formula for the signature of the Milnor fibre. Under certain circumstances (cf. Proposition 2 below), the link of singularity has the same $\mathbf{Z}_2$ homology as S^3 and hence has a unique spin structure. It can be calculated from a resolution or from a smoothing. For example comparison of these two expressions leads to the congruence $K^2 + 8p_g \equiv 0 \pmod{16}$ which is a necessary condition for smoothability if $K \equiv 0 \pmod 2$. The main purpose of this paper is to prove the following theorem:

THEOREM 1. *Let $(V, 0)$ be a 2-dimensional smoothable Gorenstein singularity. Let $\tilde{V}$ be a resolution, K the canonical class and S be a smooth (real) surface in $\tilde{V}$ dual to K* mod 2. *Then there is a $\mathbf{Z}_2$-quadratic form on $H_1(S, \mathbf{Z}_2)$ with* Arf *invariant* Arf S *such that*

$$K^2 + 8p_g = S^2 + 8 \operatorname{Arf} S \pmod{16}.$$

As the corollary, one obtains the following result which in application is easier to use than Theorem 1.

THEOREM 2. *Let $(V, 0)$ be a 2-dimensional smoothable Gorenstein singularity. Let $\tilde{V}$ be a resolution of V and $E = \bigcup_i E_i$ be the irreducible decomposition of the exceptional set E in $\tilde{V}$. Define S to be such a union of exceptional curves that $S \cdot E_j \equiv E_j^2 \pmod 2$ for all exceptional curves E_j. Assume that*

(a) *The first betti number of the exceptional set E of $\tilde{V}$ is zero, i.e. weighted dual graph of $\tilde{V}$ is a tree and all exceptional curves are rational.*

(b) *The determinant of the intersection form of the exceptional set E is odd.*

Then

$$K^2 + 8p_g \equiv \sum_{E_i \subseteq S} (E_i^2) \pmod{16}.$$

We give examples of Gorenstein surface singularities violating the congruence and hence non-smoothable. These examples also provide a negative answer to the question of Seade ([S1]) on Arf invariant of quadratic form associated to the surface dual to the canonical class of resolution (cf. Remark 2.5). It seems that non-smoothability of these singularities cannot be detected by previously used means. For example, the Thom obstruction pointed out earlier is in group $\pi_{2N-1}(MU(N-k))$ where N is the complex dimension of the ambient space and k is the codimension of singular subspace. As pointed out in [RT] $\pi_{2N-1}(MU(N-k))$ is isomorphic, for $2k \leq N$, to Ω^U_{2k-1} which is the unitary cobordism group of dimension $2k-1$ and the latter group is trivial (cf. [RT] for the references). The inequality $2k \leq N$ is satisfied in our examples and hence the Thom obstruction for smoothability is trivial.

We would like to thank Professor M. Benson for his help on computer programs.

§1. Rohlin's μ-invariant

In this section, we shall collect the various definitions needed in this paper. Recall first that a Spin structure on a manifold M is a double cover $\tilde{P}$ of the principal SO-bundle P associated with the tangent bundle of M such that its restriction on any fibre of the canonical projection $P \to M$ is isomorphic to the non-trivial cover Spin $\to SO$. Spin manifolds M_1 and M_2 are Spin cobordant if there exist Spin manifold W such that $\partial W = M_1 \cup M_2$ and Spin structure on W restrict to given Spin structures on M_1 and M_2, Spin structures on a manifold M exist if and only if $w_2(M) = 0$ and the set of Spin structures on M has a structure of an affine space over $H^1(M, \mathbf{Z}_2)$ (cf. [Mi]). This follows from the exact sequence (low degree terms of the spectral sequence of fibration $P \to M$).

$$0 \to H^1(M, \mathbf{Z}_2) \to H^1(P, \mathbf{Z}_2) \xrightarrow{f} H^1(SO(n), \mathbf{Z}_2) \to H^2(M, \mathbf{Z}_2).$$

The image of the right homomorphism is $w_2(M)$. The homomorphism f is onto if and only if M admits a Spin structure and the set of Spin structures is the inverse image of the nontrivial element in $H^1(SO(n), \mathbf{Z}_2)$. In particular if M is a 3-dimensional $\mathbf{Z}_2$-homology sphere ($H_1(M, \mathbf{Z}_2) = H_2(M, \mathbf{Z}_2) = 0$), then M admits a unique Spin structure. Alternatively one can describe a Spin structure on M as a lifting of the classifying map $M \to BSO(n)$ to the map $M \to B$ Spin (n)

$$\begin{array}{ccc} & & B\,\mathrm{Spin}\,(n) \\ & \nearrow & \downarrow \\ M & \leftrightarrow & BSO(n) \end{array}$$

so that the diagram commutes.

Recall the definition of Rohlin's invariant. Let M be a closed oriented Spin 3-manifold. The group of Spin cobordism in dimension 3 is trivial. Hence there is a Spin 4-manifold W such that $\partial W = M$ and such that the Spin structure of W restricts to the given Spin structure on M. The Rohlin invariant μ is $(\sigma(W)/16) \bmod \mathbf{Z}$ (cf. [HNK] §7). Now let W be a 4-manifold which bounds a manifold ∂W which has a fixed Spin structure σ. The obstruction to extending σ to a Spin structure on W is an obstruction to extending to W given lifting $\partial W \to B\,\mathrm{Spin}\,(n-1) \to B\,\mathrm{Spin}\,(n)$. This obstruction is an element of $H^2(W, \partial W, \mathbf{Z}_2)$, (a relative Steifel Whitney class $w_2(W, \sigma)$). Note that if S is a closed nonsingular surface in W then $W - S$ admits a Spin structure extending a given Spin structure σ on the boundary ∂W if and only if S is dual to $w_2(W, \sigma)$ i.e. under Poincáre duality homomorphism $H^2(W, \partial W, \mathbf{Z}_2) \xrightarrow{\sim} H_2(W, \mathbf{Z}_2)$ the image of $w_2(W)$ is the image of the fundamental class of S in $H_2(W, \mathbf{Z}_2)$. Indeed the obstruction for existence of a Spin structure on $W - S$ extending the given one on ∂W is an element of $H^2(W - S, \partial W, \mathbf{Z}_2)$ which is the image of $w_2(W, \sigma)$ under the inclusion map $i^* : H^2(W, \partial W) \to H^2(W - S, \partial W)$. Under Poincáre duality homomorphism i^* corresponds to the restriction map $H_2(W) \to H_2(W, S)$ where $H^2(W - S, \partial W)$ is identified with $H_2(W - S)$ via isomorphisms $H^2(W - S, \partial W) \xrightarrow{\sim} H^2(W - T(S), \partial W) \xrightarrow{\sim} H_2(W - T(S), \partial T(S)) \xrightarrow{\sim} H_2(W, S)$. ($T(S)$) is a tubular neighborhood of S.) The kernel of $H_2(W) \to H_2(W, S)$ is generated by the fundamental class of S and our claim follows. Recall that for $S \subset W$ which is dual to $w_2(W)$ corresponds to $\mathbf{Z}_2$-quadratic form on $H_1(S, \mathbf{Z}_2)$ defined as follows. Perform, if necessary, surgery on W to assure that every element $\alpha \in H_1(S, \mathbf{Z}_2)$ can be represented by a closed curve for which there exists a surface $D_\alpha \subset W$ such that $\partial D_\alpha = \alpha$ and D_α is transversal to S along α. Then $q(\alpha)$ is the sum mod 2 of the obstruction to extending the normal vector field to α in S to a normal in W vector field to D_α and the number of intersections of D_α and S. $q(\alpha)$ is independent of the choice of the surgery on W, α mod 2 and D_α ([FK] Cor. 1) and is quadratic in α. The Arf invariant of S is defined to be the Arf invariant of q.

LEMMA. *If S is dual to $w_2(W)$ mod 2 and the* Spin *structure on ∂W is the restriction of* Spin *structure on $W - S$, then*

$$\mu(\partial W) = \frac{\sigma(W) - S^2}{8} \equiv \mathrm{Arf}\, S \mod 2$$

where $\sigma(W)$ denotes the signature of W.

Proof: Let $\bar{W}$ be a 4-manifold with a Spin structure such that $\partial W = \partial \bar{W}$ and such that the restriction of the Spin structure from $\bar{W}$ on $\partial \bar{W}$ is the same as the restriction of the Spin structure from $W - S$ on ∂W. Then $(W - S) \cup (\bar{W})$ inherits a Spin structure and hence S is dual to $w_2(W \cup \bar{W})$. By Rohlin's theorem and the additivity of the signature

$$\frac{\sigma(W) - \sigma(\bar{W}) - S^2}{8} \equiv \operatorname{Arf} S \mod 2.$$

Hence

$$\frac{\sigma(\bar{W})}{8} \equiv \frac{\sigma(W) - S^2}{8} + \operatorname{Arf} S \mod 2$$

and the left side of this congruence is the μ-invariant. Q.E.D.

Finally recall the analytic method to compute Arf S from [L] assuming that W is a complex manifold with boundary ∂W and S is a complex curve. Let K be the canonical divisor. Let $S \equiv K \bmod 2$ and D be such divisor that $K + S = 2D$. Then $\operatorname{Arf} S = \dim H^\circ(S, \mathcal{O}_S(D)) \bmod 2$. For example if W is resolution of the singularity $z_1^2 + z_2^3 + z_3^6 = 0$ which is a bundle over torus S, then $K = -S$, $D = 0$ and $\operatorname{Arf} S = \dim \mathrm{H}^\circ(S, \mathcal{O}_S) = 1$ (cf. [S2] example 4.4).

§2. A congruence for invariants of smoothable Gorenstein singularities with link a $\mathbf{Z}_2$-sphere

In this section we prove Theorem 1 and derive the corollaries on which the examples of non-smoothable singularities are based.

Proof of Theorem 1. A non-vanishing holomorphic form ω on $V - \{0\}$ defines a subbundle in the principle bundle associated to the tangent bundle consisting of the frames (v_1, v_2) such that $\omega(v_1, v_2) = 1$, which is a $SU(2)$ subbundle, i.e. a SU (2)-structure on $V - \{0\}$. Nonvanishing form on $V - \{0\}$ extends to a non-vanishing form on a nearby Milnor fibre (cf. [S1]) and produces a SU (2) structure on it as well. The boundary of the Milnor fibre and the boundary of $V - \{0\}$ which is identified with the boundary of the resolution have equivalent SU (2) = Spin (3) structures. Let S be a smooth surface dual to K. Then according to the lemma from the previous section, the μ-invariant calculated from the resolution is

$$\frac{-s - S^2}{8} + \operatorname{Arf} S \mod 2$$

because the signature of the resolution is $-s$ where s is the number of exceptional curves in the resolution. On the other hand the μ-invariant calculated from the Milnor fibre is

$$\tfrac{1}{8}(-K^2 - s - 8p_g) \quad \bmod 2$$

as follows from the second author's work [Y] because as pointed out earlier the Milnor fiber admits a Spin structure. Equating these two expressions leads to Theorem 1.

Proof of Theorem 2. The proof uses the following.

PROPOSITION 2. *Let $(V, 0)$ be a normal 2-dimensional singularity with link L. Let $\tilde{V}$ be a resolution of V. Then L is a $\mathbf{Z}_2$ homology sphere if and only if the following conditions are satisfied*:

(a) *The first betti number of the exceptional set E of $\tilde{V}$ is zero i.e. weighted dual graph of $\tilde{V}$ is a tree and all exceptional curves are rational.*
(b) *The determinant of the intersection form of the exceptional set E is odd.*

Proof. (cf. [NR]) Recall that $H_1(\tilde{V}, L; \mathbf{Z}_2)$ is isomorphic to the dual of $H^1(\tilde{V}, L; \mathbf{Z}_2)$. By Lefschetz duality $H^1(\tilde{V}, L; \mathbf{Z}_2)$ isomorphic to $H_3(\tilde{V}; \mathbf{Z}_2)$. Since $\tilde{V}$ is homotopy equivalent to E, we have $H_3(\tilde{V}; \mathbf{Z}_2) \cong H_3(E; \mathbf{Z}_2) = 0$. Hence we have $H_1(\tilde{V}, L; \mathbf{Z}_2) = 0$. Consider the long homology exact sequence

$$H_2(L; \mathbf{Z}_2) \to H_2(\tilde{V}; \mathbf{Z}_2) \xrightarrow{i} H_2(\tilde{V}, L; \mathbf{Z}_2) \to H_1(L; \mathbf{Z}_2)$$
$$\to H_1(\tilde{V}, \mathbf{Z}_2) \to 0.$$

Notice that the matrix of i is the intersection matrix of the exceptional set E. If L is a $\mathbf{Z}_2$-homology sphere, then $H_2(L; \mathbf{Z}_2) = 0 = H_1(L; \mathbf{Z}_2)$. It follows from the exact sequence that $b_1(\tilde{V}, \mathbf{Z}_2) = 0$ (hence condition (a) is satisfied) and the matrix of i is invertible (hence condition (b) is satisfied). Conversely if conditions (a) and (b) are satisfied, then $H_1(L; \mathbf{Z}_2)$ is zero by the above exact sequence. Poincare duality tells us that $H_2(L; \mathbf{Z}_2)$ is also zero. So L is a $\mathbf{Z}_2$-homology sphere. Q.E.D.

Now let us note that if the resolution graph of the singularity is assumed to be a tree, then S is a disjoint union of smooth curves. Indeed $(S \cdot E_i)$ is equal to the sum of E_i^2 and the number of vertices in the graph of resolution which are adjacent to E_i and belong to S. Hence the latter number should be even. But the end point of subgraph S is adjacent to one point. Hence S does not have end points, i.e. S is a disjoint union of points. Therefore $S^2 \equiv \Sigma\, E_i^2$ and Arf $S = 0$ because all curves are rational i.e. $H_1(S, \mathbf{Z}_2) = 0$ and Theorem 2 follows. (Q.E.D.)

REMARK 2.1. S consists of those exceptional curves on which K has odd multiplicity. This follows from the facts that $w_2 \equiv K (\text{mod } 2)$ and Wu formula $E_j^2 \equiv w_2 \cdot E_j (\text{mod } 2)$ for all exceptional curves E_j (adjunction formula: $K \cdot E_j = 2 - 2g_j + E_j^2$). In fact the proof of Theorem 2 asserts that under the hypothesis of Theorem 2, exceptional curves on which K has odd multiplicity cannot intersect each other.

REMARK 2.2. On page 483 of [S1], Seade poses the question if $\text{Arf } K \equiv p_g (\text{mod } 2)$ where $p_g = h^1(\mathcal{O}_{\tilde{V}})$ is the geometric genus and $\text{Arf } K$ is the Arf invariant of Rohlin's form associated with a smooth surface dual to the $w_2(\tilde{V})$. This congruence is equivalent to the congruence from Theorem 1. Indeed taking S dual to K as an integral class, Theorem 1 reduces to Seade's congruence. Conversely, let us apply Theorem 2 of Seade [S2] in the case $W = K$ and $W = S$. By considering the Adam's invariant we have

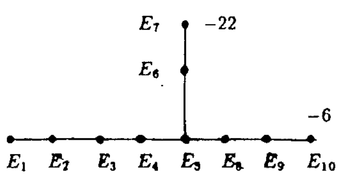
$$\tfrac{1}{2}\hat{A}(\tilde{V}) + \frac{S^2 - 8 \text{ Arf } S}{16} = \tfrac{1}{2}\hat{A}(\tilde{V}) + \frac{K^2 - 8 \text{ Arf } K}{16}.$$

Hence $S^2 - 8 \text{ Arf } S \equiv K^2 - 8 \text{ Arf } K \ (16)$ and equivalence of Theorem 1 and Seade congruence follows.

§3. Examples of non-smoothable singularities

In the following examples, we shall construct resolution $\tilde{V}$ explicitly. V is obtained by blowing down the exceptional set in $\tilde{V}$ by Grauet-Mumford criterion [Mu]. We shall first write down the weighted dual graph. Then we shall use the plumbing construction to write down the complex manifold with the exceptional set having the same weighted dual graph as the prescribed one.

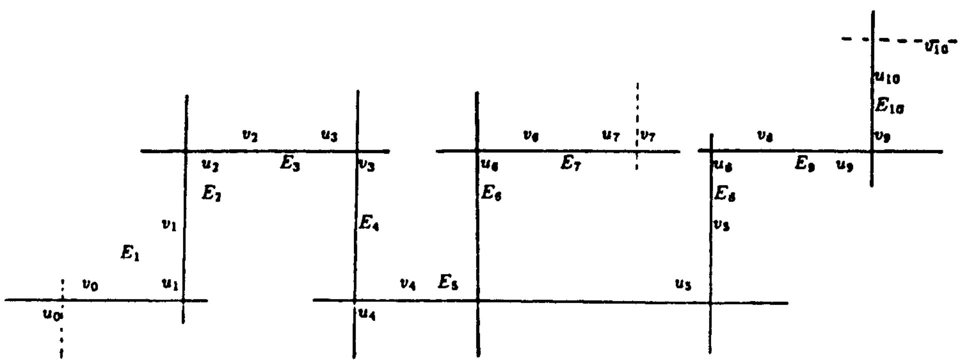
EXAMPLE 1. The weighted dual graph of the exceptional set is given as follows:



We use the convention that the vertex without a specified weight has the self-intersection number -2 and the genera of all vertices are zeros. Notice that this is a Gorenstein graph. It is easy to check that the determinant of the intersection matrix is odd and the canonical divisor K is given by

$$\begin{array}{lcccccccc} & & & & & -20 & & & \\ & & & & & -420 & & & \\ K = & -164 & -328 & -492 & -656 & -820 & -564 & -308 & -52 \end{array} \tag{3.1}$$

The manifold $\tilde{V}$ consists of coordinate patches glued as follows:



The coordinate transformations are given by

$$u_{10} = \frac{1}{v_9} \qquad v_{10} = u_9 v_9^6$$

$$u_9 = \frac{1}{v_8} \qquad v_9 = u_8 v_8^2$$

$$u_8 = \frac{1}{u_5(1-u_5)} \qquad v_8 = u_5 v_5^2 (1-u_5)$$

$$u_7 = \frac{1}{v_6} \qquad v_7 = u_6 v_6^{22}$$

$$u_6 = \frac{1}{u_5 v_5} \qquad v_6 = u_5 v_5^2 (1-u_5)$$

$$u_5 = \frac{1}{v_4} \qquad v_5 = u_4 v_4^2$$

$$u_4 = \frac{1}{v_3} \qquad v_4 = u_3 v_3^2$$

$$u_3 = \frac{1}{v_2} \qquad v_3 = u_2 v_2^2$$

$$u_2 = \frac{1}{v_1} \qquad v_2 = u_1 v_1^2$$

$$u_1 = \frac{1}{v_0} \qquad v_1 = u_0 v_0^2$$

A function

$$\begin{aligned}
x &= u_5^a v_5^b (1 - u_5)^c \\
&= u_8^{2a-b} v_8^a (1 + u_8^2 v_8)^{b-a-c} \\
&= u_9^{3a-2b} v_9^{2a-b} (1 + u_9^3 v_9^2)^{b-a-c} \\
&= u_{10}^{16a-11b} v_{10}^{3a-2b} (1 + u_{10}^{16} v_{10}^3)^{b-a-c} \\
&= u_6^{2c-b} v_6^c (1 + u_6^2 v_6)^{b-a-c} \\
&= u_7^{43c-22b} v_7^{2c-b} (1 + u_7^{43} v_7^2)^{b-a-c} \\
&= u_4^b v_4^{2b-a-c} (v_4 - 1)^c \\
&= u_3^{2b-a-c} v_3^{3b-2a-2c} (u_3 v_3^2 - 1)^c \\
&= u_2^{3b-2a-2c} v_2^{4b-3a-3c} (u_3^2 v_2^3 - 1)^c \\
&= u_1^{4b-3a-3c} v_1^{5b-4a-4c} (u_1^3 v_1^4 - 1)^c \\
&= u_0^{5b-4a-4c} v_0^{6b-5a-5c} (u_0^4 v_0^5 - 1)^c
\end{aligned}$$

is holomorphic in $\tilde{V}$ if and only if

$$\begin{cases} a \geq 0, \quad b \geq 0, \quad c \geq 0 \\ 16a - 11b \geq 0 \\ 43c - 22b \geq 0 \\ 6b - 5a - 5c \geq 0 \end{cases} \tag{3.2}$$

The divisor of x is $(x) =$

$$\begin{matrix} & & & & 2c-b & & & \\ & & & & c & & & \\ 5b-4a-4c & 4b-3a-3c & 3b-2a-2c & 2b-a-c & b & a & 2a-b & 3a-2b \end{matrix} \tag{3.3}$$

With the help of Max Benson's computer program, we find that the integral semi-group of (3.2) has seven generators. Therefore the minimal embedding dimension of V (the blown-down of $\tilde{V}$) is seven. By abusing the notation, we shall denote the divisor (x) of a holomorphic function x by (a, b, c). The actual formula for (x) is given by (3.3). The seven coordinate functions are as follows:

$$(x_1) = (148, 215, 110)$$

$$(x_2) = (473, 688, 352)$$

$$(x_3) = (55, 80, 41)$$

$$(x_4) = (86, 125, 64)$$

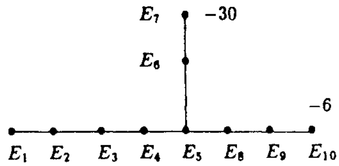
$$(x_5) = (117, 170, 87)$$

$$(x_6) = (176, 256, 131)$$

$$(x_7) = (207, 301, 154)$$

Recall that p_g is also equal to $\dim_{\mathbb{C}} H^\circ(\tilde{V} - E, \Omega^2)/H^\circ(\tilde{V}, \Omega^2)$ where Ω^2 is the sheaf of germs of holomorphic 2-forms on $\tilde{V}$ by a result of Laufer [La2]. This means that p_g is the number of independent meromorphic 2-forms that cannot be extended across the exceptional set. We can construct a meromorphic 2-form ω which has no zeros on $\tilde{V} - E$ and the divisor (ω) is exactly K as shown in (3.1). Notice that any meromorphic 2-form $\omega' \in H^\circ(\tilde{V} - E, \Omega^2)$ can be written as $\omega' = f\omega$ where f is a holomorphic function on $\tilde{V}$ because any homomorphic function on $\tilde{V} - E$ extends across E. Therefore to compute p_g, we only need to count how many monomial $x_1^{n_1} x_2^{n_2} \cdots x_7^{n_7}$ there are such that $x_1^{n_1} x_2^{n_2} \cdots x_7^{n_7}\omega$ has a pole somewhere along E. This is equivalent to find how many nonnegative integral vectors $(n_1, \ldots, n_7)$ are such that $n_1(x_1) + \cdots + n_7(x_7) + (\omega)$ is not effective. This can be done using Max Benson's computer program. We find that $p_g = 307$. On the other hand using (3.1) we get $K^2 = -608$. Hence $K^2 + 8p_g = 8(-76 + 307) = 8 \cdot 231 \not\equiv 0 (\text{mod } 16)$. In view of Theorem 2, the singularity $(V, 0)$ is not smoothable. Notice that the conditions of Laufer's conjecture (cf. [La3] or Wahl's version of laufer's conjecture (cf. [W]) on smoothable singularities are not satisfied for this example.

EXAMPLE 2. The weighted dual graph of the exceptional set is given as follows:



This is a Gorenstein graph and the determinant of the intersection matrix is odd. The canonical divisor K is given by

$$K = \begin{array}{cccccccc} & & & & -4 & & & \\ & & & & -92 & & & \\ -36 & -72 & -108 & -144 & -180 & -124 & -68 & -12 \end{array} \tag{3.4}$$

The manifold $\tilde{V}$ consists of eleven coordinate patches as shown in Example 1 above. The coordinate transformations are given also as in Example 1 except

$$u_7 = \frac{1}{v_6} \quad v_7 = u_6 v_6^{30}$$

A function $x = u_5^a v_5^b (1 - u_5)^c$ is holomorphic on $\tilde{V}$ if and only if

$$\begin{cases} a \geq 0, \quad b \geq 0, \quad c \geq 0 \\ 16a - 11b \geq 0 \\ 59c - 30b \geq 0 \\ 6b - 5a - 5c \geq 0 \end{cases} \tag{3.5}$$

The divisor of x is $(x) =$

$$\begin{array}{cccccccc} & & & & 2c-b & & & \\ & & & & c & & & \\ 5b-4a-4c & 4b-3a-3c & 3b-2a-2c & 2b-a-c & b & a & 2a-b & 3a-2b \end{array} \tag{3.6}$$

With the help of Max Benson's computer program, we find that the integral semigroup of (3.5) has seventeen generators. Therefore the minimal embedding dimension of V (the blown-down of $\tilde{V}$) is seventeen. By abusing the notation, we shall denote the divisor (x) of a holomorphic function x by (a, b, c). The actual formula for (x) is given by (3.6). The seventeen coordinate functions are as follows:

$$x_1 = (204, 296, 150)$$

$$x_2 = (649, 944, 480)$$

$$x_3 = (55, 80, 41)$$

$$x_4 = (31, 45, 23)$$

$$x_5 = (38, 55, 28)$$

$$x_6 = (66, 96, 49)$$

$$x_7 = (73, 106, 54)$$

$$x_8 = (77, 112, 57)$$

$$x_9 = (80, 116, 59)$$

$$x_{10} = (119, 173, 88)$$

$$x_{11} = (121, 175, 89)$$

$$x_{12} = (122, 177, 90)$$

$$x_{13} = (161, 234, 119)$$

$$x_{14} = (163, 236, 120)$$

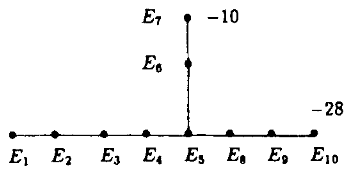
$$x_{15} = (203, 295, 150)$$

$$x_{16} = (242, 352, 179)$$

$$x_{17} = (284, 413, 210)$$

As in Example 1, p_g can be computed by Max Benson's computer program. We find that $p_g = 31$. On the other hand, using (3.4), we get $K^2 = -160$. Hence $K^2 + 8p_g = 8(-20 + 31) = 8 \cdot 11 \not\equiv 0 \pmod{16}$. In view of Theorem 2, the singularity $(V, 0)$ is not smoothable. Notice that the conditions of Laufer's conjecture (cf. (La3) or Wahl's version of Laufer's conjecture (cf. [W]) on non-smoothable singularities are not satisfied for this example.

EXAMPLE 3. The weighted dual graph of the exceptional set is given as follows:



This is a Gorenstein graph and the determinant of the intersection matrix is odd. The canonical divisor K is given by

$$\begin{array}{cccccccccc} & & & & & -14 & & & \\ & & & & & -132 & & & \\ K = & -50 & -100 & -150 & -200 & -250 & -168 & -86 & -4 \end{array} \tag{3.7}$$

The manifold $\tilde{V}$ consists of eleven coordinate patches as shown in Example 1 above. The coordinate transformations are given also as in Example 1 except

$$u_7 = \frac{1}{v_6} \qquad v_7 = u_6 v_6^{10}$$

$$u_{10} = \frac{1}{v_9} \qquad v_{10} = u_9 v_9^{28}$$

A function $x = u_5^a v_5^b (1 - u_5)^c$ is holomorphic on $\tilde{V}$ if and only if

$$\begin{cases} a \geq 0, \quad b \geq 0, \quad c \geq 0 \\ 82a - 55b \geq 0 \\ -5a + 6b - 5c \geq 0 \\ -10b + 19c \geq 0 \end{cases} \tag{3.8}$$

The divisor of x is $(x) =$

$$\begin{array}{cccccccc} & & & & 2c-b & & & \\ & & & & c & & & \\ 5b-4a-4c & 4b-3a-3c & 3b-2a-2c & 2b-a-c & b & a & 2a-b & 3a-2b \end{array} \tag{3.9}$$

With the help of Max Benson's computer program, we find that the integral semigroup of (3.8) has sixten generators. Therefore the minimal embedding dimension of V (the blown-down of $\tilde{V}$) is sixteen. By abusing the notation, we shall denote the divisor (x) of a holomorphic function x by (a, b, c). The actual formula for (x) is given by (3.9). The sixteen coordinate functions are as follows:

$$x_1 = (64, 95, 50)$$

$$x_2 = (1045, 1558, 820)$$

$$x_3 = (275, 410, 217)$$

$$x_4 = (37, 55, 29)$$

$$x_5 = (47, 70, 37)$$

$$x_6 = (51, 76, 40)$$

$$x_7 = (104, 155, 82)$$

$$x_8 = (108, 161, 85)$$

$$x_9 = (161, 240, 127)$$

$$x_{10} = (163, 243, 128)$$

$$x_{11} = (165, 246, 130)$$

$$x_{12} = (218, 325, 172)$$

$$x_{13} = (220, 328, 173)$$

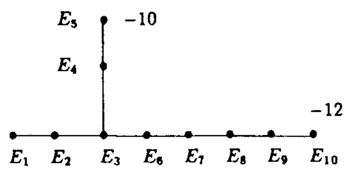
$$x_{14} = (275, 410, 216)$$

$$x_{15} = (330, 492, 259)$$

$$x_{16} = (548, 817, 430)$$

As in Example 1, we find that $p_g = 46$. On the other hand, using (3.7), we get $K^2 = -216$. Hence $K^2 + 8p_g = 8(-27 + 46) = 8.19 \not\equiv 0(\text{mod } 16)$. In view of Theorem 2, the singularity (V, O) is not smoothable. Notice that the conditions of Laufer's conjecture (cf. [La3]) or Wahl's version of Laufer's conjecture (cf. [W]) on nonsmoothable singularities are not satisfied for this example.

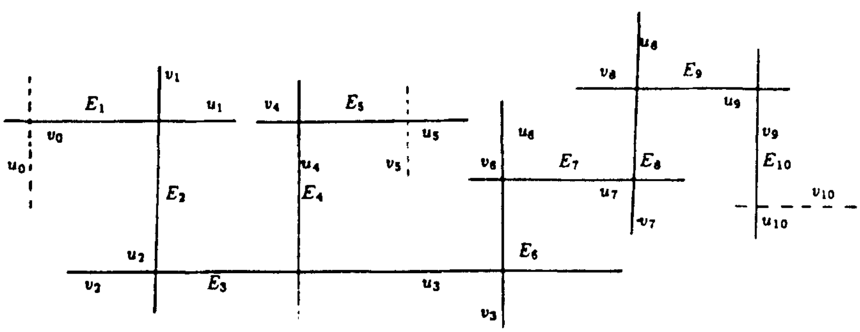
EXAMPLE 4. The weighted dual graph of the exceptional set is given as follows:



This is a Gorenstein graph and the determinant of the intersection matrix is odd. The canonical divisor K is given by

$$\begin{array}{lcccccccc} & & & -10 & & & & & \\ & & & -92 & & & & & \\ (K) = & -58 & -116 & -174 & -140 & -106 & -72 & -38 & -4 \end{array} \qquad (3.10)$$

The manifold $\tilde{V}$ consists of eleven coordinate patches as follows:



The coordinate transformations are given by

$$u_{10} = \frac{1}{v_9} \qquad v_{10} = u_9 v_9^{12}$$

$$u_9 = \frac{1}{v_8} \qquad v_9 = u_8 v_8^2$$

$$u_8 = \frac{1}{v_7} \qquad v_8 = u_7 v_7^2$$

$$u_7 = \frac{1}{v_6} \qquad v_7 = u_6 v_6^2$$

$$u_6 = \frac{1}{v_3(1-u_3)} \qquad v_6 = u_3 v_3^2(1-u_3)$$

$$u_5 = \frac{1}{v_4} \qquad v_5 = u_4 v_4^{10}$$

$$u_4 = \frac{1}{u_3 v_3} \qquad v_4 = u_3 v_3^2(1-u_3)$$

$$u_3 = \frac{1}{v_2} \qquad v_3 = u_2 v_2^2$$

$$u_2 = \frac{1}{v_1} \qquad u_2 = u_1 v_1^2$$

$$u_1 = \frac{1}{v_0} \qquad v_1 = u_0 v_0^2$$

A function

$$\begin{aligned} x &= u_3^a v_3^b (1-u_3)^c \\ &= u_6^{2a-b} v_6^a (1 + u_6^2 v_6)^{b-a-c} \\ &= u_7^{3a-2b} v_7^{2a-b} (1 + u_7^3 v_7^2)^{b-a-c} \\ &= u_8^{4a-3b} v_8^{3a-2b} (1 + u_8^4 v_8^3)^{b-a-c} \\ &= u_9^{5a-4b} v_9^{4a-3b} (1 + u_9^5 v_9^4)^{b-a-c} \\ &= u_{10}^{56a-45b} v_{10}^{5a-4b} (1 + u_{10}^{56} v_{10}^5)^{b-a-c} \\ &= u_4^{2c-b} v_4^c (1 + u_4^2 v_4)^{b-a-c} \\ &= u_5^{19c-10b} v_5^{2c-b} (1 + u_5^{19} v_5^2)^{b-a-c} \\ &= u_2^b v_2^{2b-a-c} (v_2 - 1)^c \\ &= u_1^{2b-a-c} v_1^{3b-2a-2c} (u_1 v_1^2 - 1)^c \\ &= u_0^{3b-2a-2c} v_0^{4b-3a-3c} (u_0^2 v_0^3 - 1)^c \end{aligned}$$

is holomorphic on $\tilde{V}$ if and only if

$$\begin{cases} a \geq 0, \quad b \geq 0, \quad c \geq 0 \\ 56a - 45b \geq 0 \\ -10b + 19c \geq 0 \\ -3a + 4b - 3c \geq 0 \end{cases} \tag{3.11}$$

The divisor of x is $(x) =$

$$\begin{array}{cccccccc} & & 2c-b & & & & & \\ & & c & & & & & \\ 3b-2a-2c & 2b-a-c & b & a & 2a-b & 3a-2b & 4a-3b & 5a-4b \end{array} \tag{3.12}$$

With the help of Max Benson's computer program, we find that the integral semigroup of (3.11) has ten generators. Therefore the minimal embedding dimension of V (the blowndown of $\tilde{V}$) is ten. By abusing the notation, we shall denote the divisor (x) of a holomorphic function x by (a, b, c). The actual formula for (x) is given by (3.13). The ten coordinate functions are as follows:

$$x_1 = (46, 57, 30)$$

$$x_2 = (855, 1064, 560)$$

$$x_3 = (135, 168, 89)$$

$$x_4 = (29, 36, 19)$$

$$x_5 = (41, 51, 27)$$

$$x_6 = (90, 112, 59)$$

$$x_7 = (107, 133, 70)$$

$$x_8 = (168, 209, 110)$$

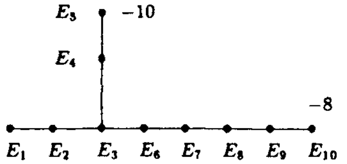
$$x_9 = (397, 494, 260)$$

$$x_{10} = (626, 779, 410)$$

As in Example 1, we find that $p_g = 30$. On the other hand, using (3.10), we get $K^2 = -120$. Hence $K^2 + 8p_g = 8(-15+30) = 8 \cdot 15 \not\equiv 0 (\mathrm{mod}\ 16)$. In view of Theorem 2.1, the singularity $(V, 0)$ is not smoothable. Notice that the conditions of Laufer's conjecture (cf. [La3]) or Wahl's version of Laufer's conjecture (cf. [W]) on nonsmoothable singularities are not satisfied for this example.

In the following two examples, we shall show that for smoothable singularities, the condition in Theorem 2 is satisfied.

EXAMPLE 5. The weighted dual graph of the exceptional set is given as follows



This is a Gorenstein graph and the determinant of the intersection matrix is odd. The canonical divisor K is given by

$$\begin{array}{lcccccccc} & & & -22 & & & & & \\ & & & -212 & & & & & \\ K = & -134 & -268 & -402 & -324 & -246 & -168 & -90 & -12 \end{array} \tag{3.13}$$

The manifold $\tilde{V}$ consists of eleven coordinate patches as in Example 4. The coordinate transformations are given as in Example 4 except for

$$u_{10} = \frac{1}{v_9} \quad v_{10} = u_9 v_9^8.$$

A function $x = u_3^a v_3^b (1-u_3)^c$ is holomorphic on $\tilde{V}$ if and only if

$$\begin{cases} a \geq 0, \quad b \geq 0, \quad c \geq 0 \\ 36a - 29b \geq 0 \\ -10b + 19c \geq 0 \\ -3a + 4b - 3c \geq 0 \end{cases} \tag{3.14}$$

The divisor of x is: $(x) =$

$$\begin{array}{ccccccccc} & & 2c-b & & & & & & \\ & & c & & & & & & \\ 3b-2a-2c & 2b-a-c & b & a & 2a-b & 3a-2b & 4a-3b & 5a-4b \end{array} \qquad (3.15)$$

With the help of Max Benson's computer program, we find that the integral semigroup of (3.14) has four generators. Therefore the minimal embedding dimension of V (the blown-down of $\tilde{V}$) is four. By abusing the notation, we shall denote the divisor (x) of a holomorphic function x by (a, b, c). The actual formula for (x) is given by (3.15). The four coordinate functions are as follows:

$$x_1 = (46, 57, 30)$$

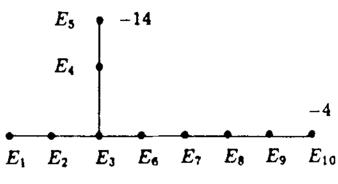
$$x_2 = (551, 684, 360)$$

$$x_3 = (29, 36, 19)$$

$$x_4 = (199, 247, 130)$$

As in Example 1, we find that $p_g = 59$. On the other hand, using (3.13), we get $K^2 = -248$. Hence $K^2 + 8p_g = 8(-31 + 59) = 16 \cdot 14 \equiv 0 (\text{mod } 16)$. $(V, 0)$ is a codimension 2 Gorenstein surface singularity in $\mathbf{C}^5$. By a result of Schaps [S] $(V, 0)$ is a determinantal scheme and is smoothable.

EXAMPLE 6. The weighted dual graph of the exceptional set is given as follows:



This is a Gorenstein graph and the determinant of the intersection matrix is odd. The canonical divisor K is given by

$$\begin{array}{ccccccccc} & & & -10 & & & & & \\ & & & -128 & & & & & \\ K = & -82 & -164 & -246 & -200 & -154 & -108 & -62 & -16 \end{array} \qquad (3.16)$$

The manifold $\tilde{V}$ consists of eleven coordinate patches as in Example 4. The coordinate transformations are given as in Example 4 except for

$$u_{10} = \frac{1}{v_9} \quad v_{10} = u_9 v_9^4$$

$$u_5 = \frac{1}{v_4} \quad v_5 = u_4 v_4^{14}$$

A function $x = u_3^a v_3^b (1 - u_3)^c$ is a holomorphic on $\tilde{V}$ if and only if

$$\begin{cases} a \geq 0, \quad b \geq 0, \quad c \geq 0 \\ 16a - 13b \geq 0 \\ -14b + 27c \geq 0 \\ -3a + 4b - 3c \geq 0 \end{cases} \tag{3.17}$$

The divisor of x is $(x) =$

$$\begin{matrix} & & 2c - b & & & & & \\ & & c & & & & & \\ 3b - 2a - 2c & 2b - a - c & b & a & 2a - b & 3a - 2b & 4a - 3b & 5a - 4b \end{matrix} \tag{3.18}$$

With the help of Max Benson's computer program, we find that the integral semigroup of (3.17) has four generators. Therefore the minimal embedding dimension of V (the blown-down of $\tilde{V}$) is four. By abusing the notation, we shall denote the divisor (x) of a holomorphic function x by (a, b, c). The actual formula for (x) is given by (3.18). The four coordinate functions are as follows:

$$x_1 = (22, 27, 14)$$

$$x_2 = (351, 432, 224)$$

$$x_3 = (39, 48, 25)$$

$$x_4 = (130, 160, 83)$$

As in Example 1, we find that $p_g = 37$. On the other hand, using (3.16), we get $K^2 = -152$. Hence $K^2 + 8p_g = 8(-19 + 37) = 16 \cdot 9 \equiv 0 (\text{mod } 16)$. By a result of Schaps [S], (V, O) is a determinantal scheme and is smoothable.

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