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On derivation Lie algebras of isolated complete intersection singularities

Naveed Hussain[1] · **Stephen S.-T. Yau**[2,3] · **Huaiqing Zuo**[2]

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Abstract
In this paper, we introduce a new invariant to isolated complete intersection singularities. We use this new invariant to obtain two characterization theorems for contact simple complete intersection singularities.

Keywords Derivations · Hessian algebra · Isolated singularity · Weighted homogeneous

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1 Introduction

In [4], we classify three-dimensional isolated weighted homogeneous rational complete intersection singularities, which define many new four-dimensional $\mathcal{N} = 2$ superconformal field theories. Many highly non-trivial physical questions such as the Coulomb branch spectrum and the Seiberg–Witten solution [16, 17] can be easily

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Naveed Hussain and Huaiqing Zuo have contributed equally to this work.

✉ Stephen S.-T. Yau
yau@uic.edu

Naveed Hussain
dr.nhussain@uaf.edu.pk

Huaiqing Zuo
hqzuo@mail.tsinghua.edu.cn

1 Department of Mathematics and Statistics, University of Agriculture, Faisalabad, Faisalabad, Punjab 38000, Pakistan

2 Department of Mathematical Sciences, Tsinghua University, Beijing 100084, People's Republic of China

3 Yanqi Lake Beijing Institute of Mathematical Sciences and Applications, Huairou 101400, People's Republic of China

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found by studying the mini-versal deformation of the singularity. In this article, we will introduce a new invariant to isolated complete intersection singularities. This new invariant is very useful in the classification theory of complete intersection singularities.

Finite-dimensional Lie algebras are the semi-direct product of the semi-simple Lie algebras and solvable Lie algebras. Brieskorn [2] gave a beautiful connection between simple Lie algebras and simple singularities. Simple Lie algebras have been well-understood, but not the solvable (nilpotent) Lie algebras. Thus, it is extremely important to establish a connection between singularities and solvable (nilpotent) Lie algebras. Recently, in [3, 8, 10, 11], the authors gave many new natural connections between the set of complex analytic isolated hypersurface singularities and the set of finite-dimensional solvable (nilpotent) Lie algebras. They introduced three different ways [12] to associate Lie algebras with isolated hypersurface singularities. These constructions are helpful to understand the solvable (nilpotent) Lie algebras from the geometric point of view [3]. Firstly, a new series of derivation Lie algebras $L_k(V), 0 \leq k \leq n$ associated to the isolated hypersurface singularity $(V, 0)$ defined by the holomorphic function $f(x_1, \ldots, x_n)$ are introduced in [10]. Let $\mathrm{Hess}(f)$ be the Hessian matrix (f_{ij}) of the second-order partial derivatives of f and $h(f)$ be the determinant of the matrix $\mathrm{Hess}(f)$. More generally, for each k satisfying $0 \leq k \leq n$ we denote by $h_k(f)$ the ideal in $\mathcal{O}_n$ generated by all $k \times k$-minors in the matrix $\mathrm{Hess}(f)$. In particular, $h_0(f) = 0$, the ideal $h_n(f) = (h(f))$ is a principal ideal. For each k as above, the graded k-th Hessian algebra of the polynomial f is defined by

$$H_k(V) = \mathcal{O}_n/(f + J(f) + h_k(f)).$$

The dimension of $H_k(V)$ as a $\mathbb{C}$-vector space is denoted as $h_k(V)$.

It is known that the isomorphism class of the local k-th Hessian algebra $H_k(f)$ is a contact invariant of f, i.e., depends only on the isomorphism class of the germ $(V, 0)$ [10]. In [10], we investigated the new Lie algebra $L_k(V)$ which is the Lie algebra of derivations of k-th Hessian algebra $H_k(f)$, i.e., $L_k(V) = \mathrm{Der}(H_k(V))$. The dimension of $L_k(V)$, denoted by $\lambda_k(V)$, is a new numerical analytic invariant of an isolated hypersurface singularity.

In particular, when $k = 0$, those are exactly the previous Yau algebra $L(V)$, and Yau number $\lambda(V)$ (cf. [5]), i.e., $L_0(V) = L(V), \lambda_0(V) = \lambda(V)$. Thus, the $L_k(V)$ is a generalization of Yau algebra $L(V)$. Moreover, $L_n(V)$ has been investigated intensively and many interesting results were obtained. In [3], it was shown that $L_n(V)$ completely distinguishes ADE singularities. Furthermore, the authors have proven Torelli-type theorems for some simple elliptic singularities. Therefore, this new Lie algebra $L_n(V)$ is a subtle invariant of isolated hypersurface singularities. It is a natural question whether we can distinguish singularities by only using part of the information of $L_n(V)$. In [9], we studied generalized Cartan matrices of the new Lie algebra $L_n(V)$ for simple hypersurface singularities and simple elliptic singularities. We introduced many other numerical invariants, namely, the dimension of the maximal nilpotent subalgebras (i.e., nilradical of nilpotent Lie algebra) $g(V)$ of $L_n(V)$; dimension of the maximal torus of $g(V)$, etc. We have proven that the generalized Cartan matrix of $L_n(V)$ can be used to characterize the ADE singularities except for the pair of A_6 and D_5 singularities [9].

Secondly, let $(V, 0)$ be an isolated hypersurface singularity defined by a holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. The multiplicity $\mathrm{mult}(f)$ of the singularity $(V, 0)$ is defined to be the order of the lowest non-vanishing term in the power series expansion of f at 0.

Definition 1 Let $(V, 0) = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \quad f(z_0, \dots, z_n) = 0\}$ be an isolated hypersurface singularity with $\mathrm{mult}(f) = m$. Let $J_k(f)$ be the ideal generated by all the k-th-order partial derivative of f, i.e., $J_k(f) = < \frac{\partial^k f}{\partial z_{i_1} \dots \partial z_{i_k}} \| 0 \le i_1, \dots, i_k \le n >$. For $1 \le k \le m$, we define the new k-th local algebra, $M_k(V) := \mathcal{O}_{n+1}/(f + J_1(f) + \dots + J_k(f))$. In particular, $M_m = 0$. The dimension of $M_k(V)$ as a $\mathbb{C}$-vector space is denoted as $d_k(V)$. In particular, $d_m(V) = 0$.

Recall that a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ is said to be weighted homogeneous if there exist positive rational numbers $w_1, \dots, w_n$ (weights of $x_1, \dots, x_n$) and d such that, $\sum a_i w_i = d$ for each monomial $\prod x_i^{a_i}$ appearing in f with nonzero coefficient. The number d is called weighted homogeneous degree (w-degree) of f with respect to weights w_j. The weight type of f is denoted as $(w_1, \dots, w_n; d)$. Without loss of generality, we can assume that w-$\deg f = 1$. An isolated hypersurface singularity $(V, 0)$ is called weighted homogeneous if it is defined by a weighted homogeneous polynomial f.

Remark 1 If f defines a weighted homogeneous isolated singularity at the origin, then $f \in J_1(f) \subset J_2(f) \subset \dots \subset J_k(f)$, thus $M_k(V) = \mathcal{O}_{n+1}/(f + J_1(f) + \dots + J_k(f)) = \mathcal{O}_{n+1}/(J_k(f))$.

The isomorphism class of the k-th local algebra $M_k(V)$ is a contact invariant of $(V, 0)$, i.e., depends only on the isomorphism class of the germ $(V, 0)$. The dimension of $M_k(V)$ is denoted by $d_k(V)$ which is a numerical analytic invariant of an isolated hypersurface singularity.

Theorem 1 [13] *Suppose* $(V, 0) = \{(x_1, \dots, x_n) \in \mathbb{C}^n \quad f(x_1, \dots, x_n) = 0\}$ *and* $(W, 0) = \{(x_1, \dots, x_n) \in \mathbb{C}^n \quad g(x_1, \dots, x_n) = 0\}$ *are isolated hypersurface singularities. If* $(V, 0)$ *is biholomorphically equivalent to* $(W, 0)$*, then* $M_k(V)$ *is isomorphic to* $M_k(W)$ *as a* $\mathbb{C}$*-algebra for all* $1 \le k \le m$*, where* $m = mult(f) = mult(g)$.

Based on Theorem 1, it is natural for us to introduce the new series of k-th derivation Lie algebras $\mathcal{L}_k(V)$ (or $\mathcal{L}_k((V, 0))$) which are defined to be the Lie algebra of derivations of the k-th local algebra $M_k(V)$, i.e., $\mathcal{L}_k(V) = \mathrm{Der}(M_k(V), M_k(V))$. Its dimension is denoted as $\delta_k(V)$ (or $\delta_k((V, 0))$). This number $\delta_k(V)$ is also a new numerical analytic invariant.

Finally, we recall that the well-known generalized Mather–Yau theorem as follows. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_n$

Theorem 2 *Let* $f, g \in \mathfrak{m} \subset \mathcal{O}_n$*. The following are equivalent:*

(1) $(V(f), 0) \cong (V(g), 0)$*;*

(2) *For all* $k \geq 0$, $\mathcal{O}_n/(f, \mathfrak{m}^k J(f)) \cong \mathcal{O}_n/(g, \mathfrak{m}^k J(g))$ *as* $\mathbb{C}$*-algebra;*
(3) *There is some* $k \geq 0$ *such that* $\mathcal{O}_n/(f, \mathfrak{m}^k J(f)) \cong \mathcal{O}_n/(g, \mathfrak{m}^k J(g))$ *as* $\mathbb{C}$*-algebra, where* $J(f) = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$.

In particular, if $k = 0$ and $k = 1$ above, then the claim of the equivalence of (1) and (3) is exactly the Mather–Yau theorem [14].

Motivated from Theorem 2, in [8, 11], we introduced the new series of k-th Yau algebras $L^k(V)$ (or $L^k((V, 0))$) which are defined to be the Lie algebra of derivations of the moduli algebra $T^k(V) = \mathcal{O}_n/(f, \mathfrak{m}^k J(f))$, $k \geq 0$, where $\mathfrak{m}$ is the maximal ideal, i.e., $L^k(V) := \mathrm{Der}(T^k(V), T^k(V))$. Its dimension is denoted as $\lambda^k(V)$ (or $\lambda^k((V, 0))$). This series of integers $\lambda^k(V)$ are new numerical analytic invariants of singularities. It is natural to call it k-th Yau number. In particular, when $k = 0$, those are exactly the previous Yau algebra and Yau number, i.e., $L^0(V) = L(V)$, $\lambda^0(V) = \lambda(V)$. In [20], Yau observed that the Yau algebra for the one-parameter family of simple elliptic singularities $\tilde{E}_6$ is constant. It turns out that the first Yau algebra $L^1(V)$ is also constant for the family of simple elliptic singularities $\tilde{E}_6$. However, the Torelli-type theorem for $L^k(V)$ for all $k > 1$ does hold on $\tilde{E}_6$ [7]. In general, the invariant $L^k(V)$, $k \geq 1$ is more subtle than the Yau algebra $L(V)$. In a word, we have reasons to believe that these three series of new Lie algebras and its numerical invariants will also play an important role in the study of singularities.

In this paper, we generalized the above construction to isolated complete intersection singularity. (This will be abbreviated in the sequel to ICIS.) Let X be an analytic space at the origin of $\mathbb{C}^n$ defined by an ideal $I_X = (f_1, \dots, f_p) \subset \mathfrak{m}^2$ as the fiber of the corresponding map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. It is well-known [18] that, in the case $n \geq p$, the map germ f is finitely contact determined if and only if X is an ICIS. Thus, the ICIS X is determined by the Artinian $\mathbb{C}$-algebra $\mathcal{O}_n/(I_X + \mathfrak{m}^{k+1})$ where k is the order of contact-determinacy of the map germ f. In this paper, we consider a different Artinian $\mathbb{C}$-algebra, more geometrically associated to X can play a similar role. More precisely, if X is an ICIS defined by an ideal I_X as above, then one can consider the singular subspace of X, which is the analytic space germ SX defined by the ideal $SI_X \subset \mathfrak{m}$ generated by the f_i and all the $p \times p$ minors in the Jacobian matrix $(\frac{\partial f_i}{\partial x_j})$, $i = 1, \dots, p$; $j = 1, \dots, n$. It is easy to see from the following example that $\mathcal{O}_n/SI_X$ is an interesting invariant of $(X, 0)$.

Example 1 If $f, g \in \mathbb{C}\{x, y, z\}$ are analytic functions defining an isolated curve singularity $(X, 0)$, then the Tjurina number of $(X, 0)$ (i.e., the dimension of the tangent space of the base space of the semiuniversal deformation of $(X, 0)$), $\tau(X, 0) = \dim_{\mathbb{C}} \mathbb{C}\{x, y, z\}/(f, g, M_1, M_2, M_3)$, where

$$M_1 = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix}; \; M_2 = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial z} \end{vmatrix}; \; M_3 = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{vmatrix};$$

are the 2-minors of the Jacobian matrix of f, g, i.e.,

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{pmatrix}.$$

For each ICIS X, it is natural for us to introduce X a new derivation Lie algebras $\mathcal{NL}(X)$ which is defined to be the Lie algebra of derivations of the local Artinian algebra $\mathcal{O}_n/SI_X$, i.e., $\mathcal{NL}(X) = \mathrm{Der}(\mathcal{O}_n/SI_X)$. Its dimension is denoted as $\upsilon(V)$. This number $\upsilon(V)$ is also a new numerical analytic invariant.

Recall that the classifications of contact simple and unimodal complete intersection singularities were done by Giusti [6] and Wall [19]. The classification of contact simple complete intersection curve singularities (i.e., with modality 0) is as follows [6].

Type S_μ $(x^2+y^2+z^{\mu-3},\ yz),\ \mu \geq 5$,

$$\text{Types}\begin{cases} T_7\ \left(x^2+y^3+z^3,\ yz\right), \\ T_8\ \left(x^2+y^3+z^4,\ yz\right), \\ T_9\ \left(x^2+y^3+z^5,\ yz\right), \end{cases}$$

$$\text{Types}\begin{cases} U_7\ \left(x^2+yz,\ xy+z^3\right), \\ U_8\ \left(x^2+yz+z^3,\ xy\right), \\ U_9\ \left(x^2+yz,\ xy+z^4\right), \end{cases}$$

$$\text{Types}\begin{cases} W_8\ \left(x^2+z^3,\ y^2+xz\right), \\ W_9\ \left(x^2+yz^2,\ y^2+xz\right), \end{cases}$$

$$\text{Types}\begin{cases} Z_9\ \ \left(x^2+z^3,\ y^2+z^3\right), \\ Z_{10}\ \left(x^2+yz^2,\ y^2+z^3\right). \end{cases}$$

In this paper, we obtain the following results.

Theorem A *The generalized Cartan matrix $C(X)$ arising from Lie algebra $\mathcal{NL}(X)$ completely distinguishes the contact simple complete intersection curve (CSCIC) singularities. Equivalently, if X and Y are two CSCIC singularities, then $C(X) = C(Y)$ if and only if X and Y are analytically isomorphic.*

Theorem B *If X and Y are two contact simple complete intersection curve singularities, then $\mathcal{NL}(X) \cong \mathcal{NL}(Y)$ as Lie algebras, if and only if X and Y are contact equivalent.*

2 Preliminaries

2.1 Isolated singularities

Let $\mathcal{O}_n$ be the algebra of germs of holomorphic functions at the origin of $\mathbb{C}^n$. Obviously, $\mathcal{O}_n$ can be naturally identified with the algebra of convergent power series in n indeterminates with complex coefficients. For $f \in \mathcal{O}_n$, we denote by $V = V(f)$ (or $(V, 0)$) the germ at the origin of $\mathbb{C}^n$ of hypersurface $\{f = 0\} \subset \mathbb{C}^n$. We say that V is a germ of isolated hypersurface singularity if the origin is an isolated zero of the gradient of f. The local (function) algebra of V is defined as the (commutative associative) algebra $F(V) \cong \mathcal{O}_n/(f)$, where (f) is the principal ideal generated by the germ of f at the origin. According to Hilbert's Nullstellensatz for an isolated singularity $V = V(f) = \{f = 0\}$ the factor algebra $A(V) = \mathcal{O}_n/(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$ is finite dimensional. This factor algebra is called the moduli algebra of V and its dimension $\tau(V)$ is called Tjurina number. The well-known Mather–Yau theorem states that

Theorem 3 [14] *The analytic isomorphism type of an isolated hypersurface singularity is determined by the isomorphism class of its moduli algebras, i.e.,*

$$(V_1, 0) \cong (V_2, 0) \Longleftrightarrow A(V_1) \cong A(V_2).$$

Definition 2 A n-dimensional singularity $(V, 0)$ is defined by:

$$(V, 0) = (\{f_1 = \cdots = f_m = 0\}, 0) \subset \left(\mathbb{C}^N, 0\right)$$

where $f_i : \left(\mathbb{C}^N, 0\right) \to (\mathbb{C}, 0)$ are germs of analytic functions with

$$r(p) = \text{rank} \left[\frac{\partial f_i}{\partial z_j}(p)\right]_{i=1,\ldots,m;\, j=1,\ldots,N} = N - n$$

for any generic (or smooth) point p of V. If $r(0) = N - n$ then $(V, 0)$ is a smooth germ, i.e., analytically isomorphic to $(\mathbb{C}^n, 0)$. If $r(0) < N - n$, but $r(p) = N - n$ for all $p \in V - \{0\}$ then we say that $(V, 0)$ has an isolated singularity at the origin. In general, $m \geq N - n$; if $m = N - n$, then it is called a complete intersection singularity.

2.2 Derivation Lie algebra

Let $\mathcal{O}_n$ denote the $\mathbb{C}$-algebra of germs of analytic functions defined at the origin of $\mathbb{C}^n$.

Definition 3 Let A and B be analytic algebras (i.e., $\mathcal{O}_n/I$) (or commutative associative algebras). A $\mathbb{C}$-linear map $\delta : A \to B$ satisfying the Leibniz rule, that is

$$\delta(f \cdot g) = \delta(f) \cdot g + f \cdot \delta(g),$$

is called a derivation of A with values in B. The set

$$\text{Der}(A, B) := \{\delta : A \to B \mid \delta \text{ is a derivation}\}$$

is via $(a \cdot \delta)(f) := a \cdot \delta(f)$ an A-module, the module of derivations of A with values in B. In case $A = B$ we define $\text{Der}(A) := \text{Der}(A, A)$.

Let A be an analytic algebra. Then $\text{Der}(A)$ is a vector space over $\mathbb{C}$, and it is also a Lie algebra, if we define the multiplication as follows:

$$[\delta, \sigma](f \cdot g) := (\delta \circ \sigma - \sigma \circ \delta)(f \cdot g)$$

with $\delta, \sigma \in \text{Der}(A)$, $f, g \in A$. A simple computation yields

$$[\delta, \sigma](f \cdot g) = [\delta, \sigma](f) \cdot g + f \cdot [\delta, \sigma](g)$$

hence the multiplication is closed. The other properties of a Lie algebra can also be verified by simple computations.

2.3 Kac–Moody Lie algebras of isolated hypersurface singularities

Let $(V, 0)$ be an isolated hypersurface singularity. Let $g(V)$ be the maximal ideal of $L(V)$ consisting of nilpotent elements. It is follows from [15] a generalized Cartan matrix $C(V)$, constructed from $g(V)$, is an invariant of $(V, 0)$ (cf. [21]).

Definition 4 An $l \times l$ matrix with entries in $\mathbb{Z}$, $C = (c_{ij})$ is a generalized Cartan matrix if
(a) $c_{ii} = 2 \quad \forall i = 1, \ldots, l,$
(b) $c_{ij} \leq 0 \quad \forall i, j = 1, \ldots, l, i \neq j,$
(c) $c_{ij} = 0$ if and only if $c_{ji} = 0\ \forall i, j = 1, \ldots, l, i \neq j.$

To each generalized Cartan matrix $C(V)$, one can associate a Lie algebra KM(C) (called a Kac–Moody Lie algebra) defined by generators:

$$\{f_1, \ldots, f_l, h_1, \ldots, h_l, e_1, \ldots, e_l\}$$

and relations:

$$\begin{aligned}
&[h_i, e_j] = c_{ij} e_j, \quad [h_i, f_j] = -c_{ij} f_j, \quad (\forall i, j = 1, \ldots, l),\\
&[h_i, h_j] = 0, \quad (\forall i, j = 1, \ldots, l), \quad [e_i, f_i] = h_i,\\
&[e_i, f_j] = 0, \quad (ade_i)^{-c_{ij+1}} e_j = 0 = (adf_i)^{-c_{ij+1}} f_j, \quad (\forall i \neq j).
\end{aligned}$$

Let $H = \mathbb{C}h_i + \cdots + \mathbb{C}h_l$; denote $\xi_+(C)$ (resp. $\xi_-(C)$) the subalgebra of KM(C) generated by $\{e_1, \ldots, e_l\}$ (resp. $(f_1, \ldots, f_l)$) one shows that:

$$\mathrm{KM}(C) = \xi_+(C) \oplus H \oplus \xi_-(C).$$

One can also define $\xi_+(C)$ by generators: $\{e_1, \ldots, e_l\}$ and relations:

$$(\mathrm{ad}e_i)^{-c_{ij}+1} e_j = 0 \quad \forall i, j = 1, \ldots, l, i \neq j.$$

We shall construct the generalized Cartan matrix from an isolated hypersurface singularity $(V, 0)$. Let $g(V)$ be the set of all nilpotent elements in $L(V)$, then $g(V)$ is the maximal nilpotent Lie subalgebra of $L(V)$ and Der$(g(V))$ be its derivation algebra.

Definition 5 A torus on $g(V)$ is a commutative subalgebra of Der$(g(V))$ whose elements are semisimple endomorphism. A maximal torus is a torus not contain in any other torus. The dimension of maximal torus is called generalized Mostow number (GMN). GMN is an invariant of isolated singularity $(V, 0)$.

Theorem 4 (Mostow's theorem [15]) *If T_1 and T_2 are maximal tori of $g(V)$, then there exist $\varphi \in Autg(V)$ (automorphism group of $g(V)$) such that $\varphi T_1 \varphi^{-1} = T_2$.*

Let T be a maximal torus and consider the root space decomposition of $g(V)$ relatively to T [15]:

$$g(V) = \sum_{\beta \in R(T)} g(V)^{\beta},$$

$$g(V)^{\beta} = \{x \in g(V) : tx = \beta(t)x, \forall t \in T\},$$

and

$$R(T) = \{\beta \in T^* : g(V)^{\beta} \neq (0)\} \text{ (root system)},$$

$$\begin{aligned} R^1(T) &= \{\beta \in R(T) : g(V)^{\beta} \not\subseteq [g(V), g(V)]\}, \\ l_{\beta} &= \dim\left(\frac{g(V)^{\beta}}{[g(V), g(V)] \cap g(V)^{\beta}}\right), \quad \forall \beta \in R^1(T), \\ d_{\beta} &= \dim(g(V)^{\beta}), \beta \in R^1(T). \end{aligned}$$

The map: $\beta \mapsto d_{\beta} \quad R^1(T) \to \mathbb{N}^*$ gives the partition:

$$\begin{aligned} R^1(T) = R^1(T)_{p_1} \cup \cdots \cup R^1(T)_{p_q}, \quad p_1 < \cdots < p_q, \quad R^1(T)_{p_i} \neq \emptyset, \\ R^1(T)_p = \{\beta \in R^1(T); d_{\beta} = p\}. \end{aligned}$$

Set $s_i = \sharp R^1(T)_{p_i}$ and $s = s_1 + \cdots + s_q$. We let $d_{\beta_i} = d_i$ and $l_{\beta_i} = l_i$.

Let $f : \{1, \ldots, l\} \longrightarrow \{1, \ldots, s\}$ be defined by:

$$f_i = \begin{cases} 1; & 1 \leq i \leq l_1, \\ 2; & l_1 \leq i \leq l_1 + l_2, \\ \vdots & \\ s; & l_1 + l_2 + \cdots + l_{s-1} \leq i \leq l. \end{cases}$$

Theorem 5 [15] *For $i, j \in \{1, \ldots, l\}$, $i \neq j$, let*

$$-c_{ij}(T) = \min\{-n \in N; (\mathrm{ad}v)^{-n+1}w = 0, \quad \forall v \in g^{\beta_{f(i)}}, \quad \forall w \in g^{\beta_{f(j)}}\},$$

with $(\mathrm{ad}0)^0 = 0$ *and let* $c_{ii}(T) = 2$ *for* $i = 1, \ldots, l$. *Then*

$$C(T) = (c_{ij}(T))_{1 \leq i, j \leq l}$$

is a generalized Cartan matrix.

3 Proof of theorems

Now we apply the above theory to study the $\mathcal{NL}(X)$ of contact simple complete intersection curve singularities. We use the following convention: $g^1 = [g, g], \ldots, g^{p+1} = [g, g^p]$. We use N to denote the set of positive integers.

Proposition 6 *Let* $X = \{x, y, z \in \mathbb{C}^3 : (x^2 + y^3 + z^3, yz)\}$ *be the* T_7 *contact simple complete intersection curve singularity and* $\mathcal{NL}(X)$ *be a derivation Lie algebra. Then*

$$C(T_7) = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ -2 & 0 & -1 & 2 \end{pmatrix}.$$

Proof It is easy to see that moduli algebra $\mathbb{C}\{x, y, z\}/(f, g, M_1, M_2, M_3) = <1, x, y, z, z^2, z^3, y^2>$. After simple calculation the Lie algebra $\mathcal{NL}(X)$ has the following basis:

$$\begin{aligned} &e_1 = 3x\partial_1 + 2y\partial_2 + 2z\partial_3, \quad e_2 = y^2\partial_2, \quad e_3 = z^2\partial_3, \\ &e_4 = -z^2\partial_2 + y^2\partial_3, \quad e_5 = z^3\partial_3, \\ &e_6 = 2z^2\partial_1 + x\partial_3, \quad e_7 = z^3\partial_2, \quad e_8 = 2y^2\partial_1 + x\partial_2, \quad e_9 = z^3\partial_1. \end{aligned}$$

The nilradical of Lie algebra $\mathcal{NL}(X)$ is given as

$$g(X) = <e_2, e_3, e_4, \ldots, e_9>.$$

The nilradical of Lie algebra $\mathcal{NL}(X)$ has the following multiplication table:

$$\begin{aligned} &[e_2, e_4] = -2e_5, \quad [e_2, e_8] = -4e_9, \quad [e_3, e_4] = 2e_7, \\ &[e_3, e_6] = -4e_9, \quad [e_6, e_8] = 2e_4, \\ &[e_6, e_9] = e_5, \quad [e_8, e_9] = e_7. \end{aligned}$$

The type of T_7 singularity = dim $g(X)/[g(X), g(X)] = 4$. The nilpotency of T_7 singularity $= \min\{p \in N \cup \{0\} : g(X)^{p+1} = 0\} = 2$. It is easy to see from [1] that the torus T of $g(X)$ is spanned by

$$\begin{array}{lll} t_1 : g(X) \longrightarrow g(X) & t_2 : g(X) \longrightarrow g(X) & t_3 : g(X) \longrightarrow g(X) \\ e_2 \longrightarrow e_2, & e_2 \longrightarrow 0, & e_2 \longrightarrow 0 \\ e_3 \longrightarrow 0, & e_3 \longrightarrow e_3, & e_3 \longrightarrow 0 \\ e_4 \longrightarrow 0, & e_4 \longrightarrow 0, & e_4 \longrightarrow e_4 \\ e_5 \longrightarrow e_5, & e_5 \longrightarrow 0, & e_5 \longrightarrow e_5 \\ e_6 \longrightarrow \dfrac{e_6}{2}, & e_6 \longrightarrow -\dfrac{e_6}{2}, & e_6 \longrightarrow \dfrac{e_6}{2} \\ e_7 \longrightarrow 0, & e_7 \longrightarrow e_7, & e_7 \longrightarrow e_7 \\ e_8 \longrightarrow -\dfrac{e_8}{2}, & e_8 \longrightarrow \dfrac{e_8}{2}, & e_8 \longrightarrow \dfrac{e_8}{2} \\ e_9 \longrightarrow \dfrac{e_9}{2}, & e_9 \longrightarrow \dfrac{e_9}{2}, & e_9 \longrightarrow \dfrac{e_9}{2}. \end{array}$$

Thus, $T = \mathbb{C}t_1 + \mathbb{C}t_2 + \mathbb{C}t_3$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2, 3$.

$$\begin{aligned} g(V) &= g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_3} \oplus g^{\beta_1+\beta_3} \oplus g^{\beta_1/2-\beta_2/2+\beta_3/2} \oplus g^{\beta_2+\beta_3} \oplus g^{-\beta_1/2+\beta_2/2+\beta_3/2} \\ &\quad \oplus g^{\beta_1/2+\beta_2/2+\beta_3/2} \\ &= \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_5 \oplus \mathbb{C}e_6 \oplus \mathbb{C}e_7 \oplus \mathbb{C}e_8 \oplus \mathbb{C}e_9. \end{aligned}$$

(e_2, e_3, e_6, e_8) is a T-minimal system of generators. The generalized Cartan matrix is

$$C(T_7) = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ -2 & 0 & -1 & 2 \end{pmatrix}.$$

□

Proposition 7 *Let $X = \{x, y, z \in \mathbb{C}^3 : (x^2 + y^3 + z^4, yz)\}$ be the T_8 contact simple complete intersection curve singularity and $\mathcal{NL}(X)$ be a derivation Lie algebra. Then*

$$C(T_8) = \begin{pmatrix} 2 & 0 & 0 & 0 & -1 \\ 0 & 2 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & 0 \\ 0 & -2 & 0 & 2 & -1 \\ -2 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Proof It is easy to see that moduli algebra $\mathbb{C}\{x, y, z\}/(f, g, M_1, M_2, M_3) = <1, x, y, z, z^2, z^3, z^4, y^2>$. After simple calculation the Lie algebra $\mathcal{NL}(X)$ has the following basis:

$$\begin{aligned} e_1 &= \frac{3x\partial_1}{4} + \frac{y\partial_2}{3} + \frac{3z\partial_3}{8}, \quad e_2 = \frac{3y^2\partial_2}{8}, \quad e_3 = \frac{z^2\partial_3}{3}, \quad e_4 = \frac{z^3\partial_3}{2}, \\ e_5 &= \frac{-4z^3\partial_2}{3} + y^2\partial_3, \quad e_6 = z^4\partial_3, \quad e_7 = \frac{7z^3\partial_1}{3} + x\partial_3, \quad e_8 = z^4\partial_2, \\ e_9 &= \frac{7y^2\partial_1}{4} + x\partial_2, \quad e_{10} = z^4\partial_1. \end{aligned}$$

The nilradical of Lie algebra $\mathcal{NL}(X)$ is given as

$$g(X) = <e_2, e_3, e_4, \ldots, e_{10}>.$$

The nilradical of Lie algebra $\mathcal{NL}(X)$ has the following multiplication table:

$$\begin{aligned} &[e_2, e_5] = -e_6, \quad [e_2, e_9] = \frac{-7e_{10}}{4}, \quad [e_3, e_4] = \frac{-e_6}{6}, \quad [e_3, e_5] = \frac{4e_8}{3}, \\ &[e_3, e_7] = \frac{-7e_{10}}{3}, [e_7, e_9] = \frac{7e_5}{4}, \quad [e_7, e_{10}] = e_6, \quad [e_9, e_{10}] = e_8. \end{aligned}$$

The type of T_8 singularity =dim $g(X)/[g(X), g(X)] = 5$. The nilpotency of T_8 singularity $= \min\{p \in N \cup \{0\} : g(X)^{p+1} = 0\} = 2$. It is easy to see from [1] that the torus T of $g(X)$ is spanned by

$$
\begin{array}{lll}
t_1 : g(X) \longrightarrow g(X) & t_2 : g(X) \longrightarrow g(X) & t_3 : g(X) \longrightarrow g(X) \\
e_2 \longrightarrow e_2, & e_2 \longrightarrow 0, & e_2 \longrightarrow 0 \\
e_3 \longrightarrow 0, & e_3 \longrightarrow e_3, & e_3 \longrightarrow 0 \\
e_4 \longrightarrow 0, & e_4 \longrightarrow 0, & e_4 \longrightarrow e_4 \\
e_5 \longrightarrow -e_5, & e_5 \longrightarrow e_5, & e_5 \longrightarrow e_5 \\
e_6 \longrightarrow 0, & e_6 \longrightarrow e_6, & e_6 \longrightarrow e_6 \\
e_7 \longrightarrow 0, & e_7 \longrightarrow 0, & e_7 \longrightarrow \frac{e_7}{2} \\
e_8 \longrightarrow -e_8, & e_8 \longrightarrow 2e_8, & e_8 \longrightarrow e_8 \\
e_9 \longrightarrow -e_9, & e_9 \longrightarrow e_9, & e_9 \longrightarrow \frac{e_9}{2} \\
e_{10} \longrightarrow 0, & e_{10} \longrightarrow e_{10}, & e_{10} \longrightarrow \frac{e_{10}}{2}.
\end{array}
$$

Thus, $T = \mathbb{C}t_1 + \mathbb{C}t_2 + \mathbb{C}t_3$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2, 3$.

$$
\begin{aligned}
g(X) = {} & g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_3} \oplus g^{-\beta_1+\beta_2+\beta_3} \oplus g^{\beta_2+\beta_2} \oplus g^{\beta_3/2} \oplus g^{-\beta_1+2\beta_2+\beta_3} \\
& \oplus g^{-\beta_1+\beta_2+\beta_3/2} \oplus g^{\beta_2+\beta_3/2} \\
= {} & \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_5 \oplus \mathbb{C}e_6 \oplus \mathbb{C}e_7 \oplus \mathbb{C}e_8 \oplus \mathbb{C}e_9 \oplus \mathbb{C}e_{10}.
\end{aligned}
$$

$(e_2, e_3, e_4, e_7, e_9)$ is a T-minimal system of generators. The generalized Cartan matrix is

$$
C(T_8) = \begin{pmatrix} 2 & 0 & 0 & 0 & -1 \\ 0 & 2 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & 0 \\ 0 & -2 & 0 & 2 & -1 \\ -2 & 0 & 0 & -1 & 2 \end{pmatrix}.
$$

□

Proposition 8 *Let $X = \{x, y, z \in \mathbb{C}^3 : (x^2 + y^3 + z^5, yz)\}$ be the T_9 contact simple complete intersection curve singularity and $\mathcal{NL}(X)$ be a derivation Lie algebra. Then*

$$
C(T_9) = \begin{pmatrix} 2 & 0 & 0 & 0 & -1 \\ 0 & 2 & -2 & -2 & -1 \\ 0 & -2 & 2 & -2 & -1 \\ 0 & -2 & -2 & 2 & -1 \\ -2 & -1 & -1 & -1 & 2 \end{pmatrix}.
$$

Proof It is easy to see that moduli algebra

$$\mathbb{C}\{x, y, z\}/(f, g, M_1, M_2, M_3) = <1, x, y, z, z^2, z^3, z^4, z^5, y^2>.$$

After simple calculation the Lie algebra $\mathcal{NL}(X)$ has the following basis:

$$\begin{aligned}
e_1 &= \frac{3x\partial_1}{4} + \frac{y\partial_2}{3} + \frac{3z\partial_3}{10}, \quad e_2 = \frac{3y^2\partial_2}{10},\\
e_3 &= \frac{z^2\partial_3}{4}, \quad e_4 = \frac{z^3\partial_3}{3}, \quad e_5 = \frac{z^4\partial_3}{2},\\
e_6 &= \frac{-5z^4\partial_2}{3} + y^2\partial_3, \quad e_7 = z^5\partial_3, \quad e_8 = \frac{8z^4\partial_1}{3} + x\partial_3, \quad e_9 = z^5\partial_2,\\
e_{10} &= \frac{8y^2\partial_1}{5} + x\partial_2, \quad e_{11} = z^5\partial_1.
\end{aligned}$$

The nilradical of Lie algebra $\mathcal{NL}(X)$ is given as

$$g(X) = <e_2, e_3, e_4, \ldots, e_{11}>.$$

The nilradical of Lie algebra $\mathcal{NL}(X)$ has the following multiplication table:

$$\begin{aligned}
&[e_2, e_6] = -e_7, \quad [e_2, e_{10}] = \frac{-8e_{11}}{5}, \quad [e_3, e_4] = \frac{-e_5}{6},\\
&[e_3, e_5] = \frac{-e_7}{4}, \quad [e_3, e_6] = \frac{5e_9}{3},\\
&[e_3, e_8] = \frac{-8e_{11}}{3}, \quad [e_8, e_{10}] = \frac{8e_6}{5}, \quad [e_8, e_{11}] = e_7 \quad [e_{10}, e_{11}] = e_9.
\end{aligned}$$

The type of T_9 singularity =dim $g(X)/[g(X), g(X)] = 5$. The nilpotency of T_9 singularity $= \min\{p \in N \cup \{0\} : g(X)^{p+1} = 0\} = 2$. It is easy to see from [1] that the torus T of $g(X)$ is spanned by

$$\begin{array}{ll}
t_1 : g(X) \longrightarrow g(X) & t_2 : g(X) \longrightarrow g(X))\\
e_2 \longrightarrow e_2, & e_2 \longrightarrow 0\\
e_3 \longrightarrow 0, & e_3 \longrightarrow e_3\\
e_4 \longrightarrow 0, & e_4 \longrightarrow e_4\\
e_5 \longrightarrow 0, & e_5 \longrightarrow 2e_5\\
e_6 \longrightarrow -e_6, & e_6 \longrightarrow 3e_6,\\
e_7 \longrightarrow 0, & e_7 \longrightarrow 3e_7\\
e_8 \longrightarrow 0, & e_8 \longrightarrow e_8\\
e_9 \longrightarrow -e_9, & e_9 \longrightarrow 4e_9\\
e_{10} \longrightarrow -e_{10}, & e_{10} \longrightarrow 2e_{10}\\
e_{11} \longrightarrow 0, & e_{11} \longrightarrow 2e_{11}.
\end{array}$$

Thus, $T = \mathbb{C}t_1 + \mathbb{C}t_2$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2$.

$$\begin{aligned} g(X) &= g^{\beta_1} \oplus g^{\beta_2} \oplus g^{2\beta_2} \oplus g^{-\beta_1+3\beta_2} \oplus g^{3\beta_2} \oplus g^{-\beta_1+4\beta_2} \oplus g^{-\beta_1+2\beta_2} \\ &= \mathbb{C}e_2 \oplus \mathbb{C}(e_3 \oplus e_4 \oplus e_8) \oplus \mathbb{C}(e_5 \oplus e_{11}) \oplus \mathbb{C}e_6 \oplus \mathbb{C}e_7 \oplus \mathbb{C}e_9 \oplus \mathbb{C}e_{10}. \end{aligned}$$

$(e_2, e_3, e_4, e_8, e_{10})$ is a T-minimal system of generators. The generalized Cartan matrix is

$$C(T_9) = \begin{pmatrix} 2 & 0 & 0 & 0 & -1 \\ 0 & 2 & -2 & -2 & -1 \\ 0 & -2 & 2 & -2 & -1 \\ 0 & -2 & -2 & 2 & -1 \\ -2 & -1 & -1 & -1 & 2. \end{pmatrix}.$$

□

Proposition 9 *Let $X = \{x, y, z \in \mathbb{C}^3 : (x^2 + yz, xy + z^3)\}$ be the U_7 contact simple complete intersection curve singularity and $\mathcal{NL}(X)$ be a derivation Lie algebra. Then*

$$C(U_7) = \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ -2 & 0 & 2 & -1 \\ 0 & -2 & -1 & 2 \end{pmatrix}.$$

Proof It is easy to see that moduli algebra $\mathbb{C}\{x, y, z\}/(f, g, M_1, M_2, M_3) = <1, x, y, z, z^2, xz, y^2>$. After simple calculation the Lie algebra $\mathcal{NL}(X)$ has the following basis:

$$e_1 = -x\partial_1 + \frac{y}{2}\partial_2 + z\partial_3, \quad e_2 = 6xz\partial_1, \quad e_3 = 2x\partial_1 - z\partial_3, \quad e_4 = 3xz\partial_3, \quad e_5 = y^2\partial_3,$$
$$e_6 = -xz\partial_1 + z^2\partial_3, \quad e_7 = -6xz\partial_2 + y\partial_3, \quad e_8 = y^2\partial_2, \quad e_9 = \frac{-y\partial_1}{6} + z^2\partial_2,$$
$$e_{10} = y^2\partial_1.$$

The nilradical of Lie algebra $\mathcal{NL}(X)$ is given as

$$g(V) = <e_2, e_4, e_5, e_6, \ldots, e_{10}>.$$

The nilradical of Lie algebra $\mathcal{NL}(X)$ has the following multiplication table:

$$\begin{aligned} &[e_2, e_4] = -3e_5, \quad [e_2, e_6] = e_{10}, \quad [e_2, e_7] = 6e_8, \\ &[e_4, e_6] = -e_5, \quad [e_4, e_9] = -e_8, \\ &[e_7, e_8] = e_5, \quad [e_7, e_9] = e_6, \quad [e_8, e_9] = \frac{e_{10}}{6}. \end{aligned}$$

The type of U_7 singularity =dim $g(X)/[g(X), g(X)] = 4$. The nilpotency of U_7 singularity $= \min\{p \in N \cup \{0\} : g(X)^{p+1} = 0\} = 2$. It is easy to see from [1] that the torus T of $g(X)$ is spanned by

$$
\begin{array}{lll}
t_1 : g(X) \longrightarrow g(X) & t_2 : g(X) \longrightarrow g(X) & t_3 : g(X) \longrightarrow g(X) \\
e_2 \longrightarrow e_2 + 3e_6, & e_2 \longrightarrow 3e_6, & e_2 \longrightarrow -3e_6 \\
e_4 \longrightarrow 0, & e_4 \longrightarrow e_4, & e_4 \longrightarrow 0 \\
e_5 \longrightarrow 0, & e_5 \longrightarrow 0, & e_5 \longrightarrow e_5 \\
e_6 \longrightarrow 0, & e_6 \longrightarrow -e_6, & e_6 \longrightarrow e_6 \\
e_7 \longrightarrow \dfrac{-e_7}{2}, & e_7 \longrightarrow 0, & e_7 \longrightarrow \dfrac{e_7}{2} \\
e_8 \longrightarrow \dfrac{e_8}{2}, & e_8 \longrightarrow 0, & e_8 \longrightarrow \dfrac{e_8}{2} \\
e_9 \longrightarrow \dfrac{e_9}{2}, & e_9 \longrightarrow -e_9, & e_9 \longrightarrow \dfrac{e_9}{2} \\
e_{10} \longrightarrow e_{10}, & e_{10} \longrightarrow -e_{10}, & e_{10} \longrightarrow e_{10}.
\end{array}
$$

Thus, $T = \mathbb{C}t_1 + \mathbb{C}t_2 + \mathbb{C}t_3$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2, 3$.

$$
\begin{aligned}
g(X) &= g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_3} \oplus g^{-\beta_2+\beta_3} \oplus g^{-\beta_1/2+\beta_3/2} \oplus g^{\beta_1/2+\beta_3/2} \oplus g^{\beta_1/2-\beta_2+\beta_3/2} \\
&\quad \oplus g^{\beta_1-\beta_2+\beta_3} \\
&= \mathbb{C}\left(\frac{e_2}{3} + e_6\right) \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_5 \oplus \mathbb{C}e_6 \oplus \mathbb{C}e_7 \oplus \mathbb{C}e_8 \oplus \mathbb{C}e_9 \oplus \mathbb{C}e_{10}.
\end{aligned}
$$

$(\frac{e_2}{3} + e_6, e_4, e_7, e_9)$ is a T-minimal system of generators. The generalized Cartan matrix is

$$
C(U_7) = \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ -2 & 0 & 2 & -1 \\ 0 & -2 & -1 & 2 \end{pmatrix}.
$$

□

Proposition 10 *Let $X = \{x, y, z \in \mathbb{C}^3 : (x^2 + xy + z^3, xy)\}$ be the U_8 contact simple complete intersection curve singularity and $\mathcal{NL}(X)$ be a derivation Lie algebra. Then*

$$
C(U_8) = \begin{pmatrix} 2 & -2 & -4 \\ -2 & 2 & -4 \\ -2 & -2 & 2 \end{pmatrix}.
$$

Proof It is easy to see that moduli algebra $\mathbb{C}\{x, y, z\}/(f, g, M_1, M_2, M_3) = <1, x, y, z, z^2, z^3, z^4, xz>$. After simple calculation the Lie algebra $\mathcal{NL}(X)$ has the following basis:

$$e_1 = \frac{3x\partial_1}{5} + \frac{4y\partial_2}{5} + \frac{2z\partial_3}{5}, \quad e_2 = z^3\partial_1, \quad e_3 = \frac{6y\partial_1}{4} + \frac{18z^2\partial_1}{7} + 3x\partial_3,$$
$$e_4 = \frac{xz\partial_1}{2} + \frac{z^3\partial_2}{3} - \frac{y\partial_3}{2}, \quad e_5 = \frac{z^3\partial_3}{2}, \quad e_6 = \frac{z^3\partial_1}{2} + xz\partial_3, \quad e_7 = z^4\partial_3,$$
$$e_8 = \frac{-7z^3\partial_2}{3} + \frac{3y\partial_3}{2} + z^2\partial_3, \quad e_9 = \frac{3y\partial_1}{14} + \frac{z^2\partial_1}{7} + xz\partial_2, \quad e_{10} = z^4\partial_2,$$
$$e_{11} = z^4\partial_1.$$

The nilradical of Lie algebra $\mathcal{NL}(X)$ is given as

$$g(X) = <e_2, e_3, e_4, \dots, e_{11}>.$$

The nilradical of Lie algebra $\mathcal{NL}(X)$ has the following multiplication table:

$$[e_2, e_3] = -6e_5, \quad [e_2, e_4] = \frac{e_{11}}{2}, \quad [e_2, e_6] = -e_7, \quad [e_2, e_9] = -e_{10},$$
$$[e_3, e_4] = \frac{3e_6}{2} + e_2, \quad [e_3, e_5] = \frac{18e_{11}}{7}, \quad [e_3, e_8] = -6e_6, \quad [e_3, e_9] = \frac{3e_8}{7},$$
$$[e_3, e_{10}] = \frac{6e_{11}}{7}, \quad [e_3, e_{11}] = 3e_7 \quad [e_4, e_5] = \frac{-e_7}{2}[e_4, e_6] = \frac{-e_{11}}{3},$$
$$[e_4, e_8] = \frac{7e_{10}}{3},$$
$$[e_4, e_9] = \frac{-e_6}{2}, \quad [e_4, e_{10}] = \frac{-e_7}{2}, \quad [e_5, e_8] = -e_7,$$
$$[e_5, e_9] = \frac{-e_{11}}{7}, \quad [e_8, e_9] = \frac{3e_6}{2},$$
$$[e_8, e_{10}] = \frac{3e_7}{2}, \quad [e_9, e_{10}] = \frac{3e_{11}}{14}.$$

The type of U_8 singularity =dim $g(X)/[g(X), g(X)] = 3$. The nilpotency of U_8 singularity $= \min\{p \in N \cup \{0\} : g(X)^{p+1} = 0\} = 4$. It is easy to see from [1] that the torus T of $g(X)$ is spanned by

$$t : g(X) \longrightarrow g(X)$$

$$\begin{array}{ll}
e_2 \longrightarrow e_2 & e_3 \longrightarrow \frac{e_3}{3} \\
e_4 \longrightarrow \frac{2e_4}{3} & e_5 \longrightarrow \frac{4e_5}{3} \\
e_6 \longrightarrow e_6 & e_7 \longrightarrow 2e_7 \\
e_8 \longrightarrow \frac{2e_8}{3} & e_9 \longrightarrow \frac{e_9}{3} \\
e_{10} \longrightarrow \frac{4e_{10}}{3} & e_{11} \longrightarrow \frac{5e_{11}}{3}.
\end{array}$$

Thus, $T = \mathbb{C}t$. Let $\beta : T \longrightarrow \mathbb{C}$ be a linear map with $\beta(t) = 1$.

$$\begin{aligned} g(X) &= g^{\beta} \oplus g^{\beta/3} \oplus g^{2\beta/3} \oplus g^{4\beta/3} \oplus g^{2\beta} \oplus g^{5\beta/3} \\ &= \mathbb{C}(e_2 \oplus e_6) \oplus \mathbb{C}(e_3 \oplus e_9) \oplus \mathbb{C}(e_4 \oplus e_8) \oplus \mathbb{C}(e_5 \oplus e_{10}) \oplus \mathbb{C}e_7 \oplus \mathbb{C}e_{11}. \end{aligned}$$

(e_3, e_9, e_4) is a T-minimal system of generators. The generalized Cartan matrix is

$$C(U_8) = \begin{pmatrix} 2 & -2 & -4 \\ -2 & 2 & -4 \\ -2 & -2 & 2 \end{pmatrix}.$$

□

Proposition 11 *Let $X = \{x, y, z \in \mathbb{C}^3 : (x^2 + yz, xy + z^4)\}$ be the U_9 contact simple complete intersection curve singularity and $\mathcal{NL}(V)$ be a derivation Lie algebra. Then*

$$C(U_9) = \begin{pmatrix} 2 & -2 & -2 & -1 & 0 \\ -2 & 2 & -2 & -1 & 0 \\ -1 & -1 & 2 & 0 & -1 \\ -2 & -2 & 0 & 2 & -1 \\ 0 & 0 & -2 & -1 & 2 \end{pmatrix}.$$

Proof It is easy to see that moduli algebra

$$\mathbb{C}\{x, y, z\}/(f, g, M_1, M_2, M_3) = <1, x, y, z, z^2, z^3, xz, xz^2, y^2>.$$

After simple calculation the Lie algebra $\mathcal{NL}(X)$ has the following basis:

$$\begin{aligned} &e_1 = -2x\partial_1 + \frac{y\partial_2}{2} + z\partial_3, \quad e_2 = -8xz\partial_1 + 8z^2\partial_3, \quad e_3 = 3x\partial_1 - z\partial_3, \\ &e_4 = 8xz^2\partial_1, \quad e_5 = 2xz\partial_1 - z^2\partial_3, \quad e_6 = \frac{8xz\partial_3}{3}, \quad e_7 = 4xz^2\partial_3, \quad e_8 = y^2\partial_3, \\ &e_9 = -xz^2\partial_1 + z^3\partial_3, \quad e_{10} = -8xz^2\partial_2 + y\partial_3, \quad e_{11} = y^2\partial_2, \quad e_{12} = \frac{-y\partial_1}{8} + z^3\partial_2, \\ &e_{13} = y^2\partial_1. \end{aligned}$$

The nilradical of Lie algebra $\mathcal{NL}(X)$ is given as

$$g(X) = <e_2, e_4, e_5, e_6, \ldots, e_{13}>.$$

The nilradical of Lie algebra $\mathcal{NL}(X)$ has the following multiplication table:

$$\begin{aligned} &[e_2, e_4] = -16e_{13}, \quad [e_2, e_5] = -e_4, \quad [e_2, e_6] = \frac{32e_7}{3}, \\ &[e_2, e_7] = 4e_8, \quad [e_2, e_9] = e_{13}, \end{aligned}$$

$$[e_2, e_{10}] = 8e_{11}, \quad [e_4, e_5] = -2e_{13}, \quad [e_4, e_6] = \frac{-8e_8}{3},$$
$$[e_5, e_6] = -2e_7, \quad [e_5, e_7] - e_8,$$
$$[e_6, e_9] = -e_8, \quad [e_6, e_{12}] = -e_{11}, \quad [e_{10}, e_{11}] = e_8,$$
$$[e_{10}, e_{12}] = e_9, \quad [e_{11}, e_{12}] = \frac{e_{13}}{8}.$$

The type of U_9 singularity =dim $g(X)/[g(X), g(X)] = 5$. The nilpotency of U_9 singularity $= \min\{p \in N \cup \{0\} : g(X)^{p+1} = 0\} = 2$. It is easy to see from [1] that the torus T of $g(V)$ is spanned by

$$\begin{array}{ll}
t_1 : g(X) \longrightarrow g(X) & t_2 : g(X) \longrightarrow g(X)) \\
e_2 \longrightarrow e_2, & e_2 \longrightarrow 0 \\
e_4 \longrightarrow 2e_4, & e_4 \longrightarrow 0 \\
e_5 \longrightarrow e_5, & e_5 \longrightarrow 0 \\
e_6 \longrightarrow 0, & e_6 \longrightarrow e_6 \\
e_7 \longrightarrow e_7, & e_7 \longrightarrow e_7 \\
e_8 \longrightarrow 2e_8, & e_8 \longrightarrow e_8 \\
e_9 \longrightarrow 2e_9, & e_9 \longrightarrow 0 \\
e_{10} \longrightarrow \frac{e_{10}}{2}, & e_{10} \longrightarrow \frac{e_{10}}{2} \\
e_{11} \longrightarrow \frac{3e_{11}}{2}, & e_{11} \longrightarrow \frac{e_{11}}{2} \\
e_{12} \longrightarrow \frac{3e_{12}}{2}, & e_{12} \longrightarrow \frac{-e_{12}}{2} \\
e_{13} \longrightarrow 3e_{13}, & e_{13} \longrightarrow 0.
\end{array}$$

Thus, $T = \mathbb{C}t_1 + \mathbb{C}t_2$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2$.

$$\begin{aligned}
g(X) &= g^{\beta_1} \oplus g^{2\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_1+\beta_2} \oplus g^{2\beta_1+\beta_2} \oplus g^{\beta_1/2+\beta_2/2} \oplus g^{3\beta_1/2+\beta_2/2} \\
&\quad \oplus g^{3\beta_1/2-\beta_2/2} \oplus g^{3\beta_1} \\
&= \mathbb{C}(e_2 \oplus e_5)\mathbb{C}(e_4 \oplus e_9) \oplus \mathbb{C}e_6 \oplus \mathbb{C}e_7 \oplus \mathbb{C}e_8 \oplus \mathbb{C}e_{10} \oplus \mathbb{C}e_{11} \oplus \mathbb{C}e_{12} \oplus \mathbb{C}e_{13}.
\end{aligned}$$

$(e_2, e_5, e_6, e_{10}, e_{12})$ is a T-minimal system of generators. The generalized Cartan matrix is

$$C(U_9) = \begin{pmatrix} 2 & -2 & -2 & -1 & 0 \\ -2 & 2 & -2 & -1 & 0 \\ -1 & -1 & 2 & 0 & -1 \\ -2 & -2 & 0 & 2 & -1 \\ 0 & 0 & -2 & -1 & 2 \end{pmatrix}.$$

□

Proposition 12 *Let* $X = \{x, y, z \in \mathbb{C}^3 : (x^2 + z^3, y^2 + xz)\}$ *be the* W_8 *contact simple complete intersection curve singularity and* $\mathcal{NL}(X)$ *be a derivation Lie algebra. Then*

$$C(W_8) = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}.$$

Proof It is easy to see that moduli algebra $\mathbb{C}\{x, y, z\}/(f, g, M_1, M_2, M_3) = {<}1, x, y, z, z^2, yz, y^2z, y^2{>}$. After simple calculation the Lie algebra $\mathcal{NL}(X)$ has the following basis:

$$\begin{aligned} &e_1 = -4x\partial_1 - y\partial_2 + 2z\partial_3, \quad e_2 = 2y^2\partial_1 - yz\partial_2, \\ &e_3 = x\partial_1 - z\partial_3, \quad e_4 = 2x\partial_2 + y\partial_3, \\ &e_5 = -y^2\partial_2, \quad e_6 = -y^2\partial_3, \quad e_7 = -y^2z\partial_3, \\ &e_8 = -y^2\partial_2 - yz\partial_3, \quad e_9 = -y^2\partial_1 - z^2\partial_3, \\ &e_{10} = -y^2z\partial_2, \quad e_{11} = -yz\partial_1 - z^2\partial_2, \quad e_{12} = -y^2z\partial_1. \end{aligned}$$

The nilradical of Lie algebra $\mathcal{NL}(X)$ is given as

$$g(X) = {<}e_2, e_4, e_5, e_6, \ldots, e_{12}{>}.$$

The nilradical of Lie algebra $\mathcal{NL}(X)$ has the following multiplication table:

$$\begin{aligned} &[e_2, e_4] = 2e_5 - e_8, \quad [e_2, e_5] = e_{10}, \quad [e_2, e_6] = 2e_7, \\ &[e_2, e_8] = -2e_{10}, \quad [e_2, e_9] = 2e_{12}, \\ &[e_4, e_5] = e_6, \quad [e_4, e_9] = -2e_8, \quad [e_4, e_{10}] = e_7, \\ &[e_4, e_{11}] = e_9, \quad [e_4, e_{12}] = 2e_{10}, \\ &[e_5, e_8] = e_7, \quad [e_5, e_{11}] = e_{12}, \quad [e_6, e_9] = 2e_7, \quad [e_6, e_{11}] = 2e_{10} \end{aligned}$$

The type of W_8 singularity =dim $g(X)/[g(X), g(X)] = 3$. The nilpotency of W_8 singularity $= \min\{p \in N \cup \{0\} : g(X)^{p+1} = 0\} = 4$. It is easy to see from [1] that the torus T of $g(X)$ is spanned by

$$\begin{array}{lll} t_1 : g(X) \longrightarrow g(X) & t_2 : g(X) \longrightarrow g(X) & t_3 : g(X) \longrightarrow g(X) \\ e_2 \longrightarrow e_2 + e_9, & e_2 \longrightarrow e_9, & e_2 \longrightarrow -e_9 \\ e_4 \longrightarrow 0, & e_4 \longrightarrow e_4, & e_4 \longrightarrow 0 \\ e_5 \longrightarrow e_5 + e_8, & e_5 \longrightarrow e_5 + e_8, & e_5 \longrightarrow \dfrac{-e_8}{2} \\ e_6 \longrightarrow e_6, & e_6 \longrightarrow 2e_6, & e_6 \longrightarrow 0 \\ e_7 \longrightarrow 0, & e_7 \longrightarrow 0, & e_7 \longrightarrow e_7 \\ e_8 \longrightarrow -e_8, & e_8 \longrightarrow -e_8, & e_8 \longrightarrow e_8 \\ e_9 \longrightarrow -e_9, & e_9 \longrightarrow -2e_9, & e_9 \longrightarrow e_9 \end{array}$$

$$\begin{array}{lll} e_{10} \longrightarrow 0, & e_{10} \longrightarrow -e_{10}, & e_{10} \longrightarrow e_{10} \\ e_{11} \longrightarrow -e_{11}, & e_{11} \longrightarrow -3e_{11}, & e_{11} \longrightarrow e_{11} \\ e_{12} \longrightarrow 0, & e_{12} \longrightarrow -2e_{12}, & e_{12} \longrightarrow e_{12}. \end{array}$$

Thus, $T = \mathbb{C}t_1 + \mathbb{C}t_2 + \mathbb{C}t_3$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2, 3$.

$$\begin{aligned} g(X) &= g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_1+2\beta_2} \oplus g^{\beta_3} \oplus g^{-\beta_1-\beta_2+\beta_3} \oplus g^{-\beta_1-2\beta_2+\beta_3} \oplus g^{-\beta_2+\beta_3} \\ &\quad \oplus g^{-\beta_1-3\beta_2+\beta_3} \oplus g^{-2\beta_2+\beta_3} \oplus g^{\beta_1+\beta_2} \\ &= \mathbb{C}(e_2 + e_9) \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_6 \oplus \mathbb{C}e_7 \oplus \mathbb{C}e_8 \oplus \mathbb{C}e_9 \\ &\quad \oplus \mathbb{C}e_{10} \oplus \mathbb{C}e_{11} \oplus \mathbb{C}e_{12} \oplus \mathbb{C}(2e_5 + e_8). \end{aligned}$$

$(e_2 + e_9, e_4, e_{11})$ is a T-minimal system of generators. The generalized Cartan matrix is

$$C(W_8) = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}.$$

□

Proposition 13 *Let $X = \{x, y, z \in \mathbb{C}^3 : (x^2+yz^2, y^2+xz)\}$ be the W_9 contact simple complete intersection curve singularity and $\mathcal{NL}(X)$ be a derivation Lie algebra. Then*

$$C(W_9) = \begin{pmatrix} 2 & -3 & -6 \\ -2 & 2 & -1 \\ -2 & -2 & 2 \end{pmatrix}.$$

Proof It is easy to see that moduli algebra $\mathbb{C}\{x, y, z\}/(f, g, M_1, M_2, M_3) = <1, x, y, z, z^2, z^3, z^4, yz, y^2>$. After simple calculation the Lie algebra $\mathcal{NL}(X)$ has the following basis:

$$e_1 = \frac{5x\partial_1}{8} + \frac{y\partial_2}{2} + \frac{3z\partial_3}{8}, \quad e_2 = -z^3\partial_1 - 2y^2\partial_2, \quad e_3 = -2z^2\partial_1 + 2x\partial_2 + 4y\partial_3,$$
$$e_4 = z^3\partial_2, \quad e_5 = 2z^2\partial_2 - 4x\partial_3, \quad e_6 = -y^2\partial_1 + \frac{z^2\partial_3}{3},$$
$$e_7 = \frac{z^3\partial_3}{2}, \quad e_8 = \frac{z^3\partial_2}{4} + y^2\partial_3,$$
$$e_9 = \frac{-z^3\partial_1}{4} + yz\partial_3, \quad e_{10} = z^4\partial_3, \quad e_{11} = y^2\partial_1 + yz\partial_2,$$
$$e_{12} = z^4\partial_2, \quad e_{13} = z^4\partial_1.$$

The nilradical of Lie algebra $\mathcal{NL}(X)$ is given as

$$g(X) = <e_2, e_4, e_5, e_6, \ldots, e_{13}>.$$

The nilradical of Lie algebra $\mathcal{NL}(X)$ has the following multiplication table:

$$[e_2,e_3]=-2e_4+8e_8,\quad [e_2,e_5]=-8e_7,\quad [e_2,e_8]=-e_{10},\quad [e_2,e_{11}]=-e_{13},$$
$$[e_3,e_4]=8e_7,\quad [e_3,e_5]=-16e_{11},\quad [e_3,e_6]=e_2-\frac{8e_9}{3},\quad [e_3,e_7]=-2e_{13},$$
$$[e_3,e_9]=-2e_8,\quad [e_3,e_{11}]=4e_9,\quad [e_3,e_{12}]=4e_{10},\quad [e_3,e_{13}]=2e_{12},\quad [e_4,e_6]=e_{12},$$
$$[e_4,e_9]=-e_{10},\quad [e_4,e_{11}]=-e_{12}\quad [e_5,e_6]=e_4+\frac{4e_8}{3},\quad [e_5,e_7]=2e_{12},$$
$$[e_5,e_{11}]=-4e_8,\quad [e_5,e_{13}]=-4e_{10},\quad [e_6,e_7]=\frac{-e_{10}}{6}\quad [e_6,e_8]=\frac{-e_{12}}{4},$$
$$[e_6,e_9]=\frac{e_{13}}{4}.$$

The type of W_9 singularity =dim $g(X)/[g(X),g(X)]=3$. The nilpotency of W_9 singularity $=\min\{p\in N\cup\{0\}: g(X)^{p+1}=0\}=6$. It is easy to see from [1] that the torus T of $g(X)$ is spanned by

$$t: g(X)\longrightarrow g(X)$$

$$\begin{array}{ll}
e_2\longrightarrow e_2 & e_3\longrightarrow \frac{e_3}{4}\\
e_4\longrightarrow \frac{5e_4}{4} & e_5\longrightarrow \frac{e_5}{2}\\
e_6\longrightarrow \frac{3e_6}{4} & e_7\longrightarrow \frac{3e_7}{2}\\
e_8\longrightarrow \frac{5e_8}{4} & e_9\longrightarrow e_9\\
e_{10}\longrightarrow \frac{9e_{10}}{4} & e_{11}\longrightarrow \frac{3e_{11}}{4}\\
e_{12}\longrightarrow 2e_{12} & e_{13}\longrightarrow \frac{7e_{13}}{4}.
\end{array}$$

Thus, $T=\mathbb{C}t$. Let $\beta_i: T\longrightarrow\mathbb{C}$ be a linear map with $\beta(t)=1$.

$$\begin{aligned}g(X)&=g^{\beta}\oplus g^{\beta/4}\oplus g^{5\beta/4}\oplus g^{\beta/2}\oplus g^{3\beta/4}\oplus g^{3\beta/2}\oplus g^{9\beta/4}\oplus g^{2\beta}\oplus g^{7\beta/4}\\
&=\mathbb{C}(e_2\oplus e_9)\oplus\mathbb{C}(e_4\oplus e_8)\oplus\mathbb{C}e_3\oplus\mathbb{C}e_5\oplus\mathbb{C}(e_6\oplus e_{11})\oplus\mathbb{C}e_7\\
&\quad\oplus\mathbb{C}e_{10}\oplus\mathbb{C}e_{12}\oplus\mathbb{C}e_{13}.\end{aligned}$$

(e_3,e_5,e_6) is a T-minimal system of generators. The generalized Cartan matrix is

$$C(W_9)=\begin{pmatrix}2&-3&-6\\-2&2&-1\\-2&-2&2\end{pmatrix}.$$

□

Proposition 14 *Let $X = \{x, y, z \in \mathbb{C}^3 : (x^2 + z^3, y^2 + z^3)\}$ be the Z_9 contact simple complete intersection curve singularity and $\mathcal{NL}(X)$ be a derivation Lie algebra. Then*

$$C(Z_9) = \begin{pmatrix} 2 & -2 & -4 \\ -2 & 2 & -4 \\ -2 & -2 & 2 \end{pmatrix}.$$

Proof It is easy to see that moduli algebra $\mathbb{C}\{x, y, z\}/(f, g, M_1, M_2, M_3) = <1, x, y, z, z^2, z^3, z^4, yz, xz>$. After simple calculation the Lie algebra $\mathcal{NL}(X)$ has the following basis:

$$e_1 = -3x\partial_1 - 3y\partial_2 - 2z\partial_3, \quad e_2 = -y\partial_1 + x\partial_2, \quad e_3 = -z^3\partial_1, \quad e_4 = -2z^2\partial_1 - x\partial_3,$$
$$e_5 = -z^3\partial_2, \quad e_6 = -2z^2\partial_2 - y\partial_3, \quad e_7 = -3xz\partial_1 - 3yz\partial_2 - 2z^2\partial_3, \quad e_8 = -z^3\partial_3,$$
$$e_9 = -z^3\partial_1 - xz\partial_3, \quad e_{10} = -z^3\partial_2 - yz\partial_3, \quad e_{11} = -z^4\partial_3, \quad e_{12} = yz\partial_1 - xz\partial_2,$$
$$e_{13} = -z^4\partial_2, \quad e_{14} = -z^4\partial_1.$$

The nilradical of Lie algebra $\mathcal{NL}(X)$ is given as

$$g(X) = <e_3, e_4, e_5, \ldots, e_{14}>.$$

The nilradical of Lie algebra $\mathcal{NL}(X)$ has the following multiplication table:

$$[e_3, e_4] = e_8, \quad [e_3, e_7] = -3e_{14}, \quad [e_3, e_9] = e_{11}, \quad [e_3, e_{12}] = e_{13}, \quad [e_4, e_6] = 4e_{12},$$
$$[e_4, e_7] = -6e_3 + e_9, \quad [e_4, e_8] = -4e_{14}, \quad [e_4, e_{12}] = e_{10}, \quad [e_4, e_{14}] = -e_{11},$$
$$[e_5, e_6] = e_8, \quad [e_5, e_7] = -3e_{13}, \quad [e_5, e_{10}] = e_{11},$$
$$[e_5, e_{12}] = -e_{14}, \quad [e_6, e_7] = e_{10} - 6e_5,$$
$$[e_6, e_8] = -4e_{13}, \quad [e_6, e_{12}] = -e_9, \quad [e_6, e_{13}] = -e_{11},$$
$$[e_7, e_8] = 2e_{11}, \quad [e_7, e_9] = 6e_{14},$$
$$[e_7, e_{10}] = 6e_{13}.$$

The type of Z_9 singularity =dim $g(X)/[g(X), g(X)] = 3$. The nilpotency of Z_9 singularity $= \min\{p \in N \cup \{0\} : g(X)^{p+1} = 0\} = 4$. It is easy to see from [1] that the torus T of $g(X)$ is spanned by

$$t : g(X) \longrightarrow g(X)$$

$$e_3 \longrightarrow e_3 \qquad e_4 \longrightarrow \frac{e_4}{3}$$
$$e_5 \longrightarrow e_5 \qquad e_6 \longrightarrow \frac{e_6}{3}$$
$$e_7 \longrightarrow \frac{2e_7}{3} \qquad e_8 \longrightarrow \frac{4e_8}{3}$$

$$e_9 \longrightarrow e_9 \qquad e_{10} \longrightarrow e_{10}$$
$$e_{11} \longrightarrow 2e_{11} \qquad e_{12} \longrightarrow \frac{2e_{12}}{3}$$
$$e_{13} \longrightarrow \frac{5e_{13}}{3} \qquad e_{14} \longrightarrow \frac{5e_{14}}{3}.$$

Thus, $T = \mathbb{C}t$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta(t) = 1$.

$$\begin{aligned} g(X) &= g^{\beta} \oplus g^{\beta/3} \oplus g^{2\beta/3} \oplus g^{4\beta/3} \oplus g^{2\beta} \oplus g^{5\beta/3} \\ &= \mathbb{C}(e_3 \oplus e_5 \oplus e_9 \oplus e_{10}) \oplus \mathbb{C}(e_4 \oplus e_6) \oplus \mathbb{C}(e_7 \oplus e_{12}) \oplus \mathbb{C}(e_{13} \oplus e_{14}) \\ &\quad \oplus \mathbb{C}e_8 \oplus \mathbb{C}e_{11}. \end{aligned}$$

(e_4, e_6, e_7) is a T-minimal system of generators. The generalized Cartan matrix is

$$C(Z_9) = \begin{pmatrix} 2 & -2 & -4 \\ -2 & 2 & -4 \\ -2 & -2 & 2 \end{pmatrix}.$$

□

Proposition 15 *Let $X = \{x, y, z \in \mathbb{C}^3 : (x^2+yz^2, y^2+z^3)\}$ be the Z_{10} contact simple complete intersection curve singularity and $\mathcal{NL}(X)$ be a derivation Lie algebra. Then*

$$C(Z_{10}) = \begin{pmatrix} 2 & -2 & -1 & 0 \\ -2 & 2 & -1 & -1 \\ -2 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2. \end{pmatrix}.$$

Proof It is easy to see that moduli algebra

$$\mathbb{C}\{x, y, z\}/(f, g, M_1, M_2, M_3) = <1, x, y, z, z^2, z^3, yz, yz^2, yz^3, xz>.$$

After simple calculation the Lie algebra $\mathcal{NL}(X)$ has the following basis:

$$\begin{aligned} &e_1 = -7x\partial_1 - 6y\partial_2 - 4z\partial_3, \quad e_2 = -yz^2\partial_1, \\ &e_3 = -2yz\partial_1 - x\partial_3, \quad e_4 = -z^2\partial_1 - x\partial_2, \\ &e_5 = -xz\partial_1 + 2yz\partial_2 - 2z^3\partial_3, \quad e_6 = -yz^2\partial_2, \quad e_7 = -2yz\partial_2 + z^2\partial_3, \\ &e_8 = 3z^3\partial_2 - 2yz\partial_3, \quad e_9 = -yz^2\partial_3, \quad e_{10} = -yz^2\partial_1 - xz\partial_3, \quad e_{11} = -yz^3\partial_3, \\ &e_{12} = yz^2\partial_2 - z^3\partial_3, \quad e_{13} = -z^3\partial_1 - xz\partial_2, \quad e_{14} = -yz^3\partial_2, \quad e_{15} = -yz^3\partial_1. \end{aligned}$$

The nilradical of Lie algebra $\mathcal{NL}(X)$ is given as

$$g(X) = <e_2, e_3, e_4, \ldots, e_{15}>.$$

The nilradical of Lie algebra $\mathcal{NL}(X)$ has the following multiplication table:

$[e_2, e_3] = e_9, \quad [e_2, e_4] = e_6, \quad [e_2, e_5] = -e_{15}, \quad [e_2, e_{10}] = e_{11}, \quad [e_2, e_{13}] = e_{14},$

$$
\begin{aligned}
&[e_3, e_4] = e_7, \quad [e_3, e_5] = 3e_{10} - 2e_2, \quad [e_3, e_6] = -2e_{15}, \quad [e_3, e_7] = -2e_{10},\\
&[e_3, e_{13}] = -e_{12}, \quad [e_3, e_{15}] = -e_{11}, \quad [e_4, e_5] = -3e_{13}, \quad [e_4, e_7] = 2e_{13},\\
&[e_4, e_8] = 2e_{10} - 6e_2, \quad [e_4, e_9] = -2e_{15}, \quad [e_4, e_{10}] = e_{12}, \quad [e_4, e_{15}] = -e_{14},\\
&[e_5, e_6] = 4e_{14}, \quad [e_5, e_7] = 2e_6, \quad [e_5, e_8] = -8e_9, \quad [e_5, e_9] = -2e_{11},\\
&[e_5, e_{10}] = 2e_{15}, \quad [e_5, e_{12}] = -2e_{14}, \quad [e_6, e_7] = 2e_{14},\\
&[e_6, e_8] = 2e_{11}, \quad [e_7, e_8] = 6e_9,\\
&[e_7, e_9] = 2e_{11}, \quad [e_8, e_{12}] = 6e_{11}, \quad [e_8, e_{13}] = 6e_{15}.
\end{aligned}
$$

The type of Z_{10} singularity =dim $g(X)/[g(X), g(X)] = 4$. The nilpotency of Z_{10} singularity $= \min\{p \in N \cup \{0\} : g(X)^{p+1} = 0\} = 4$. It is easy to see from [1] that the torus T of $g(X)$ is spanned by

$$
\begin{array}{ll}
t_1 : g(X) \longrightarrow g(X) & t_2 : g(X) \longrightarrow g(X))\\
e_2 \longrightarrow e_2, & e_2 \longrightarrow 0\\
e_3 \longrightarrow 0, & e_3 \longrightarrow e_3\\
e_4 \longrightarrow e_4, & e_4 \longrightarrow -2e_4\\
e_5 \longrightarrow e_5, & e_5 \longrightarrow -e_5\\
e_6 \longrightarrow 2e_6, & e_6 \longrightarrow -2e_6\\
e_7 \longrightarrow e_7, & e_7 \longrightarrow -e_7\\
e_8 \longrightarrow 0, & e_8 \longrightarrow 2e_8\\
e_9 \longrightarrow e_9, & e_9 \longrightarrow e_9\\
e_{10} \longrightarrow e_{10} & e_{10} \longrightarrow 0\\
e_{11} \longrightarrow 2e_{11}, & e_{11} \longrightarrow 0\\
e_{12} \longrightarrow 2e_{12} & e_{12} \longrightarrow -2e_{12}\\
e_{13} \longrightarrow 2e_{13}, & e_{13} \longrightarrow -3e_{13}\\
e_{14} \longrightarrow 3e_{14}, & e_{14} \longrightarrow -3e_{14}\\
e_{15} \longrightarrow 2e_{15}, & e_{15} \longrightarrow -e_{15}.
\end{array}
$$

Thus, $T = \mathbb{C}t_1 + \mathbb{C}t_2$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2$.

$$
\begin{aligned}
g(X) = &\, g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_1-2\beta_2} \oplus g^{\beta_1-\beta_2} \oplus g^{2\beta_1-2\beta_2} \oplus g^{2\beta_2} \oplus g^{\beta_1+\beta_2}\\
&\oplus g^{2\beta_1} \oplus g^{2\beta_1-3\beta_2} g^{3\beta_1-3\beta_2} \oplus g^{2\beta_1-\beta_2}\\
= &\, \mathbb{C}(e_2 \oplus e_{10}) \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4 \oplus \mathbb{C}(e_5 \oplus e_7) \oplus \mathbb{C}(e_6 \oplus e_{12}) \oplus \mathbb{C}e_8\\
&\oplus \mathbb{C}e_9 \oplus \mathbb{C}e_{11} \oplus \mathbb{C}e_{13} \oplus \mathbb{C}e_{14} \oplus \mathbb{C}e_{15}.
\end{aligned}
$$

(e_3, e_4, e_5, e_8) is a T-minimal system of generators. The generalized Cartan matrix is

$$C(Z_{10}) = \begin{pmatrix} 2 & -2 & -1 & 0 \\ -2 & 2 & -1 & -1 \\ -2 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2. \end{pmatrix}.$$

□

Proposition 16 *Let $X = \{x, y, z \in \mathbb{C}^3 : (x^2 + y^2 + z^{\mu-3},\ yz),\ \mu \geq 5\}$ be the S_μ contact simple complete intersection curve singularity and $\mathcal{NL}(X)$ be a derivation Lie algebra.*

Then

$$C(S_\mu) = \begin{cases} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}; & \mu=5, \\[2ex] \begin{pmatrix} 2 & -1 & -1 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}; & \mu=6, \\[2ex] \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{pmatrix}; & \mu=7, \\[2ex] \begin{pmatrix} 2 & -2 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{pmatrix}; & \mu=8, \\[2ex] \begin{pmatrix} 2 & -3 & -3 & -3 \\ -2 & 2 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{pmatrix}; & \mu=9, \\[2ex] \begin{pmatrix} 2 & -(\mu-6) & \frac{-(\mu-3)}{2} & \frac{-(\mu-3)}{2} \\ \frac{-(\mu-5)}{2} & 2 & -2 & -2 \\ -2 & -1 & 2 & 0 \\ -2 & -1 & 0 & 2 \end{pmatrix}; & \mu \text{ is odd and } \mu \geq 11, \\[2ex] \begin{pmatrix} 2 & -(\mu-6) & -1 & -1 \\ \frac{-(\mu-6)}{2} & 2 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{pmatrix}; & \mu \text{ is even and } \mu \geq 10. \end{cases}$$

Proof It is easy to see that the moduli algebra of S_μ series is given as:

$$\mathbb{C}\{x, y, z\}/(f, g, M_1, M_2, M_3) = <1, x, y, z, z^2, z^3, z^4, \ldots, z^{(\mu-3)}>.$$

The Lie algebra $\mathcal{NL}(X)$ arising from series S_μ has the following dimension:

$$\upsilon(X) = \mu + 2.$$

In case of μ is odd, then the Lie algebra $\mathcal{NL}(X)$ has the following basis:

$$\begin{aligned}
e_1 &= -\frac{(\mu-3)x\partial_1}{2} - \frac{(\mu-3)y\partial_2}{2} - z\partial_3, \quad e_2 = -z^2\partial_3, \quad e_3 = -z^3\partial_3,\\
&\quad \ldots, e_{\mu-3} = -z^{(\mu-3)}\partial_3, \quad e_{\mu-2} = \frac{(\mu-3)z^{(\mu-4)}\partial_2}{2} - y\partial_3,\\
e_{\mu-1} &= \frac{-(\mu-1)z^{\mu-4}\partial_1}{2} - x\partial_3, \quad e_\mu = -z^{(\mu-3)}\partial_2,\\
e_{\mu+1} &= \frac{-(\mu-1)y\partial_1}{2} - \frac{(\mu-3)x\partial_2}{2}, \quad e_{\mu+2} = -z^{(\mu-3)}\partial_1.
\end{aligned}$$

In case of μ is even, then the Lie algebra $\mathcal{NL}(X)$ has the following basis:

$$\begin{aligned}
e_1 &= \frac{x\partial_1}{2} + \frac{y\partial_2}{2} + \frac{z\partial_3}{\mu-3}, \quad e_2 = \frac{z^2\partial_3}{\mu-4}, \quad e_3 = \frac{z^3\partial_3}{\mu-5}, \quad e_4 = \frac{z^4\partial_3}{\mu-6},\\
&\quad \ldots, e_{\mu-3} = z^{(\mu-3)}\partial_3, \quad e_{\mu-2} = \frac{-(\mu-3)z^{(\mu-4)}\partial_2}{2} + y\partial_3,\\
e_{\mu-1} &= \frac{(\mu-1)z^{\mu-4}\partial_1}{2} + x\partial_3, \quad e_\mu = z^{(\mu-3)}\partial_2,\\
e_{\mu+1} &= \frac{(\mu-1)y\partial_1}{\mu-3} + x\partial_2, \quad e_{\mu+2} = z^{\mu-3}\partial_1.
\end{aligned}$$

The nilradical of Lie algebra $\mathcal{NL}(X)$ arising from S_5 is given as:

$$g(X) = <e_2, e_5, e_7>.$$

Set, $e_2 = x_1, \quad e_5 = x_2, \quad e_7 = x_3$. The nilradical of Lie algebra $\mathcal{NL}(X)$ has zero multiplication table.

The type of S_5 singularity =dim $g(X)/[g(X), g(X)] = 3$. The nilpotency of S_5 singularity $= \min\{p \in N \cup \{0\} : g(X)^{p+1} = 0\} = 0$. It is easy to see from [1] that the torus T of $g(X)$ is spanned by

$$\begin{array}{lll}
t_1 : g(X) \longrightarrow g(X) & t_2 : g(X) \longrightarrow g(X) & t_3 : g(X) \longrightarrow g(X)\\
x_1 \longrightarrow x_1, & x_1 \longrightarrow 0, & x_1 \longrightarrow 0\\
x_2 \longrightarrow 0, & x_2 \longrightarrow x_2, & x_2 \longrightarrow 0\\
x_3 \longrightarrow 0, & x_3 \longrightarrow 0, & x_3 \longrightarrow x_3.
\end{array}$$

Thus, $T = \mathbb{C}t_1 + \mathbb{C}t_2 + \mathbb{C}t_3$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2, 3$.

$$\begin{aligned} g(X) &= g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_3} \\ &= \mathbb{C}x_1 \oplus \mathbb{C}x_2 \oplus \mathbb{C}x_3. \end{aligned}$$

(x_1, x_2, x_3) is a T-minimal system of generators. The generalized Cartan matrix is

$$C(S_5) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

For S_6, the nilradical of Lie algebra $\mathcal{NL}(X)$ is given as

$$g(X) = <e_2, e_3, e_4, e_5, e_6, e_8>.$$

The nilradical of Lie algebra $\mathcal{NL}(X)$ has the following multiplication table:

$$[e_2, e_4] = \frac{3e_6}{2}, \quad [e_2, e_5] = \frac{-5e_8}{2}, \quad [e_4, e_6] = e_3, \quad [e_5, e_8] = e_3.$$

The type of S_6 singularity =dim $g(X)/[g(X), g(X)] = 3$. The nilpotency of S_6 singularity $= \min\{p \in N \cup \{0\} : g(X)^{p+1} = 0\} = 2$. It is easy to see from [1] that the torus T of $g(X)$ is spanned by

$$\begin{aligned} t_1 &: g(X) \longrightarrow g(X) & t_2 &: g(X) \longrightarrow g(X)) \\ e_2 &\longrightarrow e_2, & e_2 &\longrightarrow 0 \\ e_3 &\longrightarrow 0, & e_3 &\longrightarrow e_3 \\ e_4 &\longrightarrow \frac{-e_4}{2}, & e_4 &\longrightarrow \frac{e_4}{2} \\ e_5 &\longrightarrow \frac{-e_5}{2}, & e_5 &\longrightarrow \frac{e_5}{2} \\ e_6 &\longrightarrow \frac{e_6}{2}, & e_6 &\longrightarrow \frac{e_6}{2} \\ e_8 &\longrightarrow \frac{e_8}{2}, & e_8 &\longrightarrow \frac{e_8}{2}. \end{aligned}$$

Thus, $T = \mathbb{C}t_1 + \mathbb{C}t_2$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2$.

$$\begin{aligned} g(X) &= g^{\beta_1} \oplus g^{\beta_2} \oplus g^{-\beta_1/2+\beta_2/2} \oplus g^{2(\beta_1+\beta_2)} \\ &= \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathbb{C}(e_4 \oplus e_5) \oplus \mathbb{C}(e_6 \oplus e_8). \end{aligned}$$

(e_2, e_4, e_5) is a T-minimal system of generators. The generalized Cartan matrix is

$$C(S_6) = \begin{pmatrix} 2 & -1 & -1 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}.$$

For S_7, the nilradical of Lie algebra $\mathcal{NL}(X)$ is given as

$$g(X) = <e_2, e_3, e_4, e_5, e_6, e_7, e_9>.$$

The nilradical of Lie algebra $\mathcal{NL}(X)$ has the following multiplication table:

$$[e_2, e_3] = e_4, \quad [e_2, e_5]{=}{-}6e_7, \quad [e_2, e_6]{=}6e_9, \quad [e_5, e_7]{=}{-}e_4, \quad [e_6, e_9] = -e_4.$$

The type of S_7 singularity =dim $g(X)/[g(X), g(X)] = 4$. The nilpotency of S_7 singularity $= \min\{p \in N \cup \{0\} : g(X)^{p+1} = 0\} = 2$. It is easy to see from [1] that the torus T of $g(V)$ is spanned by

$$\begin{array}{ll}
t_1 : g(X) \longrightarrow g(X) & t_2 : g(X) \longrightarrow g(X)) \\
e_2 \longrightarrow e_2, & e_2 \longrightarrow 0 \\
e_3 \longrightarrow 0, & e_3 \longrightarrow e_3 \\
e_4 \longrightarrow e_4, & e_4 \longrightarrow e_4 \\
e_5 \longrightarrow 0, & e_5 \longrightarrow \dfrac{e_5}{2} \\
e_6 \longrightarrow 0, & e_6 \longrightarrow \dfrac{e_6}{2} \\
e_7 \longrightarrow e_7, & e_7 \longrightarrow \dfrac{e_7}{2} \\
e_9 \longrightarrow e_9, & e_8 \longrightarrow \dfrac{e_9}{2}.
\end{array}$$

Thus, $T = \mathbb{C}t_1 + \mathbb{C}t_2$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2$.

$$\begin{aligned} g(X) &= g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_1+\beta_2} \oplus g^{\beta_2/2} \oplus g^{\beta_1+\beta_2/2} \\ &= \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4 \oplus \mathbb{C}(e_5 \oplus e_6) \oplus \mathbb{C}(e_7 \oplus e_9). \end{aligned}$$

(e_2, e_3, e_5, e_6) is a T-minimal system of generators. The generalized Cartan matrix is

$$C(S_7) = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{pmatrix}.$$

For S_8, the nilradical of Lie algebra $\mathcal{NL}(X)$ is given as

$$g(X) = <e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_{10}>.$$

The nilradical of Lie algebra $\mathcal{NL}(X)$ has the following multiplication table:

$$[e_2, e_3] = \frac{-e_4}{6}, \quad [e_2, e_4] = \frac{-e_5}{4}, \quad [e_2, e_6] = \frac{5e_8}{2}, \quad [e_2, e_7] = \frac{-7e_{10}}{2},$$
$$[e_6, e_8] = e_5, [e_7, e_{10}] = e_5.$$

The type of S_8 singularity =dim $g(X)/[g(X), g(X)] = 4$. The nilpotency of S_7 singularity $= \min\{p \in N \cup \{0\} : g(X)^{p+1} = 0\} = 2$. It is easy to see from [1] that the torus T of $g(V)$ is spanned by

$$\begin{array}{ll}
t_1 : g(X) \longrightarrow g(X) & t_2 : g(X) \longrightarrow g(X)) \\
e_2 \longrightarrow e_2, & e_2 \longrightarrow 0 \\
e_3 \longrightarrow 0, & e_3 \longrightarrow e_3 \\
e_4 \longrightarrow e_4, & e_4 \longrightarrow e_4 \\
e_5 \longrightarrow 2e_5, & e_5 \longrightarrow e_5 \\
e_6 \longrightarrow \frac{e_6}{2}, & e_6 \longrightarrow \frac{e_6}{2} \\
e_7 \longrightarrow \frac{e_7}{2}, & e_7 \longrightarrow \frac{e_7}{2} \\
e_8 \longrightarrow \frac{3e_8}{2}, & e_8 \longrightarrow \frac{e_8}{2} \\
e_{10} \longrightarrow \frac{3e_{10}}{2}, & e_{10} \longrightarrow \frac{e_{10}}{2}.
\end{array}$$

Thus, $T = \mathbb{C}t_1 + \mathbb{C}t_2$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta_i(t_j) = \delta_{ij}$ for $i, j = 1, 2$.

$$\begin{aligned}
g(X) &= g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_1+\beta_2} \oplus g^{2\beta_1+\beta_2} \oplus g^{\beta_1/2+\beta_2/2} \oplus g^{3\beta_1/2+\beta_2/2} \\
&= \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_5 \oplus \mathbb{C}(e_6 \oplus e_7) \oplus \mathbb{C}(e_8 \oplus e_{10}).
\end{aligned}$$

(e_2, e_3, e_6, e_7) is a T-minimal system of generators. The generalized Cartan matrix is

$$C(S_8) = \begin{pmatrix} 2 & -2 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{pmatrix}.$$

For S_9, the nilradical of Lie algebra $\mathcal{NL}(V)$ is given as

$$g(X) = <e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{11}>.$$

The nilradical of Lie algebra $\mathcal{NL}(X)$ has the following multiplication table:

$$[e_2, e_3] = e_4, \quad [e_2, e_4] = 2e_5, \quad [e_2, e_5] = 3e_6, \quad [e_2, e_7] = -15e_9,$$
$$[e_2, e_8] = 20e_{11}, [e_3, e_4] = e_6, \quad [e_7, e_9] = -e_6, \quad [e_8, e_{11}] = -e_6,$$

The type of S_9 singularity =dim $g(V)/[g(V), g(V)] = 4$. The nilpotency of S_9 singularity $= \min\{p \in N \cup \{0\} : g(X)^{p+1} = 0\} = 3$. It is easy to see from [1] that the torus T of $g(X)$ is spanned by

$$t : g(X) \longrightarrow g(X)$$

$$\begin{aligned} e_2 &\longrightarrow e_2 & e_3 &\longrightarrow 2e_3 \\ e_4 &\longrightarrow 3e_4 & e_5 &\longrightarrow 4e_5 \\ e_6 &\longrightarrow 5e_6 & e_7 &\longrightarrow 2e_7 \\ e_8 &\longrightarrow 2e_8 & e_9 &\longrightarrow 3e_9 \\ e_{11} &\longrightarrow 3e_{11}. \end{aligned}$$

Thus, $T = \mathbb{C}t$. Let $\beta_i : T \longrightarrow \mathbb{C}$ be a linear map with $\beta(t) = 1$.

$$\begin{aligned} g(X) &= g^{\beta} \oplus g^{2\beta} \oplus g^{3\beta} \oplus g^{4\beta} \oplus g^{5\beta} \\ &= \mathbb{C}e_2 \oplus \mathbb{C}(e_3 \oplus e_7 \oplus e_8) \oplus \mathbb{C}(e_4 \oplus e_9 \oplus e_{11}) \oplus \mathbb{C}e_5 \oplus \mathbb{C}e_6. \end{aligned}$$

(e_2, e_3, e_7, e_8) is a T-minimal system of generators. The generalized Cartan matrix is

$$C(S_9) = \begin{pmatrix} 2 & -3 & -3 & -3 \\ -2 & 2 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{pmatrix}.$$

The nilradical of Lie algebra $\mathcal{NL}(X)$ arising from S_μ, $\mu \geq 10$ is given as:

$$g(X) = <e_2, e_3, e_4, \ldots, e_\mu, e_{\mu+2}>$$

Case 1: When μ is odd and $\mu \geq 11$, then nilradical $g(X)$ has the following multiplication table:

$$[e_2, e_3] = e_4, \quad [e_2, e_4] = 2e_5, \quad [e_2, e_5] = 3e_6, \ldots, [e_2, e_{\mu-4}] = (\mu - 6)e_{\mu-3},$$
$$[e_2, e_{\mu-2}] = -(\mu - 4)\frac{(\mu - 3)}{2}e_\mu, \quad [e_2, e_{\mu-1}] = \frac{(\mu - 4)(\mu - 1)e_{\mu+2}}{2}$$
$$[e_3, e_4] = e_5, \quad [e_3, e_5] = 2e_7, \quad [e_3, e_6] = 3e_8, \ldots, [e_3, e_{\mu-5}] = (\mu - 8)e_{\mu-3},$$
$$[e_4, e_5] = e_8, \quad [e_4, e_6] = 2e_9, \quad [e_4, e_7] = 3e_{10}, \ldots, [e_4, e_{\mu-6}] = (\mu - 10)e_{\mu-3},$$
$$[e_5, e_6] = e_{10}, \quad [e_5, e_7] = 2e_{11}, \quad [e_5, e_8] = 3e_{12}, \ldots, [e_5, e_{\mu-7}] = (\mu - 12)e_{\mu-3},$$
$$\vdots$$

$$\left[e_{\frac{\mu-3}{2}}, e_{\left(\frac{\mu-3}{2}+1\right)}\right] = e_{\mu-3}, \quad [e_{\mu-2}, e_{\mu}] = -e_{\mu-3}, \quad [e_{\mu-1}, e_{\mu+2}] = -e_{\mu-3}.$$

The type of S_μ singularity =dim $g(X)/[g(X), g(X)] = 4$. The nilpotency of S_μ singularity $= \min\{p \in N \cup \{0\} : g(X)^{p+1} = 0\} = \mu - 6$.

It is easy to see from [1] that the torus T of $g(X)$ is spanned by

$$\begin{aligned}
t &: g(X) \longrightarrow g(X)\\
e_2 &\longrightarrow e_2\\
e_3 &\longrightarrow 2e_3\\
e_4 &\longrightarrow 3e_4\\
&\vdots\\
e_{\mu-3} &\longrightarrow (\mu-4)e_{\mu-3}\\
e_{\mu-2} &\longrightarrow \frac{(\mu-5)e_{\mu-2}}{2}\\
e_{\mu-1} &\longrightarrow \frac{(\mu-5)e_{\mu-1}}{2}\\
e_{\mu} &\longrightarrow \frac{(\mu-3)e_{\mu}}{2}\\
e_{\mu+2} &\longrightarrow \frac{(\mu-3)e_{\mu+2}}{2}.
\end{aligned}$$

Thus, $T = \mathbb{C}t$ is a unique maximal torus of $g(V)$. Let $\beta : T \longrightarrow \mathbb{C}$ be a linear map with $\beta(t) = 1$.

$$\begin{aligned}
g(X) &= g^{\beta} \oplus g^{2\beta} \oplus g^{3\beta} \ldots \oplus g^{(\mu-4)\beta}\\
&= \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4 \ldots \oplus \mathbb{C}e_{\frac{\mu-5}{2}} \oplus \mathbb{C}\big(e_{\frac{\mu-3}{2}} \oplus e_{\mu-2} \oplus e_{\mu-1}\big) \oplus \mathbb{C}\big(e_{\frac{\mu-1}{2}} \oplus e_{\mu}\\
&\qquad \oplus e_{\mu+2}\big) \oplus \mathbb{C}e_{\frac{\mu+1}{2}} \oplus \mathbb{C}e_{\frac{\mu+3}{2}} \oplus \mathbb{C}e_{\frac{\mu+5}{2}} \oplus \cdots \oplus \mathbb{C}e_{\frac{2\mu-6}{2}}
\end{aligned}$$

$(e_2, e_3, e_{\mu-2}, e_{\mu-1})$ is a T-minimal system of generators. The generalized Cartan matrix is

$$C(S_\mu) = \begin{pmatrix} 2 & -(\mu-6) & \frac{-(\mu-3)}{2} & \frac{-(\mu-3)}{2}\\ \frac{-(\mu-5)}{2} & 2 & -2 & -2\\ -2 & -1 & 2 & 0\\ -2 & -1 & 0 & 2 \end{pmatrix}.$$

Case 2: When μ is even and $\mu \geq 10$, then nilradical $g(X)$ has the following multiplication table:

$$[e_2, e_3] = -e_4, \quad [e_2, e_4] = \frac{-e_5}{2}, \quad [e_2, e_5] = \frac{-e_6}{3}, \ldots, [e_2, e_{\mu-4}] = -\frac{e_{\mu-3}}{\mu-6},$$

$$[e_2, e_{\mu-2}] = \frac{e_\mu(\mu-3)}{2}, \quad [e_2, e_{\mu-1}] = \frac{-(\mu-1)e_{\mu+2}}{2}$$
$$[e_3, e_4] = -e_6, \quad [e_3, e_5] = \frac{-e_7}{2}, \quad [e_3, e_6] = \frac{-e_8}{3}, \ldots, [e_3, e_{\mu-5}] = \frac{-e_{\mu-3}}{(\mu-5)},$$
$$[e_4, e_5] = -e_8, \quad [e_4, e_6] = \frac{-e_9}{2}, \quad [e_4, e_7] = \frac{-e_{10}}{3}, \ldots, [e_4, e_{\mu-6}] = \frac{-e_{\mu-3}}{(\mu-5)},$$
$$[e_5, e_6] = -e_{10}, \quad [e_5, e_7] = \frac{-e_{11}}{2}, \quad [e_5, e_8] = \frac{-e_{12}}{3}, \ldots, [e_5, e_{\mu-7}] = \frac{-e_{\mu-3}}{(\mu-7)},$$
$$\vdots$$
$$[e_{\frac{\mu-4}{2}}, e_{\frac{\mu-2}{2}}] = \frac{-(\mu-8)e_{\mu-4}}{(\mu-5)(\mu-6)}, \quad [e_{\frac{\mu-4}{2}}, e_{\frac{\mu}{2}}] = \frac{-(\mu-8)e_{\mu-3}}{(\mu-5)(\mu-7)}, \quad [e_{\mu-2},$$
$$e_\mu] = e_{\mu-3}, [e_{\mu-1}, e_{\mu+2}] = e_{\mu-3}.$$

The type of S_μ singularity =dim $g(X)/[g(X), g(X)] = 4$. The nilpotency of S_μ singularity $= \min\{p \in N \cup \{0\} : g(X)^{p+1} = 0\} = \mu - 6$. It is easy to see from [1] that the torus T of $g(X)$ is spanned by

$$\begin{aligned}
t : g(X) &\longrightarrow g(X)\\
e_2 &\longrightarrow e_2\\
e_3 &\longrightarrow 2e_3\\
e_4 &\longrightarrow 3e_4\\
&\vdots\\
e_{\mu-3} &\longrightarrow (\mu-4)e_{\mu-3}\\
e_{\mu-2} &\longrightarrow \frac{(\mu-5)e_{\mu-2}}{2}\\
e_{\mu-1} &\longrightarrow \frac{(\mu-5)e_{\mu-1}}{2}\\
e_\mu &\longrightarrow \frac{(\mu-3)e_\mu}{2}\\
e_{\mu+2} &\longrightarrow \frac{(\mu-3)e_{\mu+2}}{2}.
\end{aligned}$$

Thus, $T = \mathbb{C}t$ is a unique maximal torus of $g(V)$. Let $\beta : T \longrightarrow \mathbb{C}$ be a linear map with $\beta(t) = 1$.

$$\begin{aligned}
g(X) &= g^\beta \oplus g^{2\beta} \oplus g^{3\beta} \ldots \oplus g^{(\mu-4)\beta} \oplus g^{\frac{(\mu-5)\beta}{2}} \oplus g^{\frac{(\mu-3)\beta}{2}}\\
&= \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4 \ldots \oplus \mathbb{C}e_{\mu-3} \oplus \mathbb{C}(e_{\mu-2} \oplus e_{\mu-1}) \oplus \mathbb{C}(e_\mu \oplus e_{\mu+2})
\end{aligned}$$

$(e_2, e_3, e_{\mu-2}, e_{\mu-1})$ is a T-minimal system of generators. The generalized Cartan matrix is

$$C(S_\mu) = \begin{pmatrix} 2 & -(\mu-6) & -1 & -1 \\ \frac{-(\mu-6)}{2} & 2 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{pmatrix}.$$

□

Proposition 17 *The following seven cases of Lie algebras $\mathcal{NL}(X)$ arising from contact simple complete intersection curve singularities are not isomorphic:*
(1) $\mathcal{NL}(T_7) \ncong \mathcal{NL}(S_7)$,
(2) $\mathcal{NL}(T_8) \ncong \mathcal{NL}(U_7), \mathcal{NL}(T_8) \ncong \mathcal{NL}(S_8), \mathcal{NL}(U_7) \ncong \mathcal{NL}(S_8)$,
(3) $\mathcal{NL}(T_9) \ncong \mathcal{NL}(U_8), \mathcal{NL}(T_9) \ncong \mathcal{NL}(S_9), \mathcal{NL}(U_8) \ncong \mathcal{NL}(S_9)$,
(4) $\mathcal{NL}(U_9) \ncong \mathcal{NL}(W_9), \mathcal{NL}(U_9) \ncong \mathcal{NL}(S_{11}), \mathcal{NL}(W_9) \ncong \mathcal{NL}(S_{11})$,
(5) $\mathcal{NL}(W_8) \ncong \mathcal{NL}(S_{10})$,
(6) $\mathcal{NL}(Z_9) \ncong \mathcal{NL}(S_{12})$,
(7) $\mathcal{NL}(Z_{10}) \ncong \mathcal{NL}(S_{13})$.

Proof Case (1) It is easy to see from Proposition 6, the nilradical of Lie algebra $\mathcal{NL}(T_7)$ Spanned by:

$$g(V) = <e_2, e_3, e_4, e_5, e_7, e_8, e_9>.$$

It is follow from Proposition 16, the nilradical of Lie algebra $\mathcal{NL}(S_7)$ Spanned by:

$$g(V) = <e_2, e_3, e_4, e_5, e_7, e_9>.$$

Therefore, the Lie algebras $\mathcal{NL}(T_7)$ and $\mathcal{NL}(S_7)$ have different dimensions of nilradical. Therefore, these two Lie algebras are pairwise non-isomorphic. Case(2) It is easy to see from Propositions 7, 9 and 16 the dimensions of nilradical of Lie algebras $\mathcal{NL}(T_8)$, $\mathcal{NL}(U_7)$ and $\mathcal{NL}(S_8)$ are 9, 8 and 8 respectively. From Proposition 9, the nilradical of $\mathcal{NL}(U_7)$ has [2, 4, 8] dimensions of upper central series. From Proposition 16, the nilradical of $\mathcal{NL}(S_8)$ has [1, 4, 8] dimensions of upper central series. Therefore the Lie algebras $\mathcal{NL}(T_8)$, $\mathcal{NL}(U_7)$ and $\mathcal{NL}(S_8)$ have different dimensions of nilradical and upper central series of nilradical. Therefore, these three Lie algebras are pairwise non-isomorphic. Similarly, we can prove cases (3)-(7). □

Proof of Theorem A The Theorem A is an immediate corollary from Propositions 6 to 16.

Proof of Theorem B From above Propositions 6 to 16 we have the following table:

Now we need to distinguish the pairs which have same dimensions of Lie algebra $\mathcal{NL}(X)$, and from the above table, we only need to treat the following seven cases:
(1) $\mathcal{NL}(T_7) \ncong \mathcal{NL}(S_7)$,
(2) $\mathcal{NL}(T_8) \ncong \mathcal{NL}(U_7), \mathcal{NL}(T_8) \ncong \mathcal{NL}(S_8), \mathcal{NL}(U_7) \ncong \mathcal{NL}(S_8)$,

type	equations	$\upsilon(V)$
S_μ	$\{(x^2+y^2+z^{\mu-3},\ yz),\ \mu\geq 5\}$	$\mu+2$
T_7	$(x^2+y^3+z^3,\ yz)$	9
T_8	$(x^2+y^3+z^4,\ yz)$	10
T_9	$(x^2+y^3+z^5,\ yz)$	11
U_7	$(x^2+yz,\ xy+z^3)$	10
U_8	$(x^2+yz+z^3,\ xy)$	11
U_9	$(x^2+yz,\ xy+z^4)$	13
W_8	$(x^2+z^3,\ y^2+xz)$	12
W_9	$(x^2+yz^2,\ y^2+xz)$	13
Z_9	$(x^2+z^3,\ y^2+z^3)$	14
Z_{10}	$(x^2+yz^2,\ y^2+z^3)$	15

(3) $\mathcal{NL}(T_9)\ncong\mathcal{NL}(U_8), \mathcal{NL}(T_9)\ncong\mathcal{NL}(S_9), \mathcal{NL}(U_8)\ncong\mathcal{NL}(S_9)$,
(4) $\mathcal{NL}(U_9)\ncong\mathcal{NL}(W_9), \mathcal{NL}(U_9)\ncong\mathcal{NL}(S_{11}), \mathcal{NL}(W_9)\ncong\mathcal{NL}(S_{11})$,
(5) $\mathcal{NL}(W_8)\ncong\mathcal{NL}(S_{10})$,
(6) $\mathcal{NL}(Z_9)\ncong\mathcal{NL}(S_{12})$,
(7) $\mathcal{NL}(Z_{10})\ncong\mathcal{NL}(S_{13})$.

It follows from Proposition 17 these seven cases are non-isomorphic. Therefore, we completely characterize the contact simple complete intersection curve singularities by using the derivation Lie algebra $\mathcal{NL}(X)$.

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Declarations

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