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Classification of weighted dual graphs consisting of −2-curves and exactly one −3-curve

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Abstract. Let (V,p) be a normal surface singularity. Let $\pi\colon (M,A)\to(V,p)$ be a minimal good resolution of V. The weighted dual graphs Γ associated with A completely describes the topology and differentiable structure of the embedding of A in M. In this paper, we classify all the weighted dual graphs of $A=\bigcup_{i=1}^n A_i$ such that one of the curves A_i is a -3-curve, and all the remaining ones are -2-curves. This is a natural generalization of Artin's classification of rational triple points. Moreover, we compute the fundamental cycles of maximal graphs (see § 5) which can be used to determine whether the singularities are rational, minimally elliptic or weakly elliptic. We also give formulas for computing arithmetic and geometric genera of star-shaped graphs.

Keywords: normal singularities, topological classification, weighted dual graph.

§ 1. Introduction

Let p be a normal singularity of the 2-dimensional Stein space V. Let $\pi\colon M\to V$ be a resolution of V such that the irreducible components A_i, $1\leqslant i\leqslant n$, of $A=\pi^{-1}(p)$ are non-singular and have only normal crossings. Associated with A is a weighted dual graph Γ (see, for example, [1] or [2]) which, along with the genera of the A_i, fully describes the topology and differentiable structure of A in M (see [3]). On a non-singular surface M, a $-k$-curve is a non-singular rational curve with self-intersection $-k$.

M. Artin has studied the rational singularities (those for which $R^1\pi_*(\mathcal{O})=0$). He has shown that all hypersurface rational singularities have multiplicity two and the graphs associated with those singularities belong to the graphs A_k, $k\geqslant 1$, D_k, $k\geqslant 4$, E_6, E_7, and E_8 arising from the classification of simple Lie groups (in the latter discussion we abuse the notation A_i for weighted dual graph as well as exceptional curves). He has also shown that the existence of a fundamental cycle (see Definiton 2.1) is equivalent to the negative definiteness of $(A_i\cdot A_j)$. Rational triple points are classified into nine classes according to the dual graphs in [4]. These nine classes of graphs consist of -2-curves and exactly one -3-curve. Here, simple

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means that only finite many isomorphism classes occur in the versal deformation. The rational double points and rational triple points are simple. Stevens [5] conjectured that the simple normal surface singularities are exactly those rational singularities whose resolution graphs can be obtained from the graph of a rational double or triple point by making some number of vertex weights more negative. He showed that no other rational singularities can be simple. He proved simpleness for some special classes of singularities, namely, rational quadruple points or sandwiched singularities in [5]. For the classification of certain classes of rational singularities, the interested readers can refer to the recent papers [6]–[9].

In [10], Laufer examined a class of elliptic singularities that satisfy a minimality condition. These minimally elliptic singularities have a theory much like that for rational singularities. Laufer [10] also listed all the dual graphs corresponding to the minimally elliptic hypersurface singularities. These singularities are exactly Gorenstein singularities with geometric genus 1. Such a list is extremely useful for researchers in this field. For a classification of Gorenstein singularities with geometric genus greater than 1, the interested readers can refer to [3], [11]–[18].

In [19] (respectively, [20]), the authors of the present paper generalized Laufer's list of dual graphs of minimally elliptic hypersurface singularities. They classified all weighted dual graphs of the simplest Gorenstein non-complete intersection (respectively, complete intersection) singularities of dimension two. These singularities are exactly those minimal elliptic singularities with fundamental cycle self-intersection number -5 (respectively, -4). In the present paper, the strategy of classification is different from that in [19]. We shall give a complete classification of singularities whose weighted dual graphs are $A=\bigcup_{i=1}^{n} A_i$ such that all A_i are -2-curves except one -3-curve A_j. Thus, we generalize Artin's list of dual graphs of rational triple points. The classification statements regarding topological types of normal surface singularities are important for some potential applications in our future work. In fact, these classes of singularities are the most interesting ones in the study of symmetries of surface singularities [21], since there is a natural homomorphism from the automorphism group of the singularities to the automorphism group of the central curve (that is, the -3-curve). Furthermore, we compute the fundamental cycles of maximal graphs. As an application, one can deduce whether these graphs are rational, minimally elliptic or weakly elliptic. Moreover, we discuss the arithmetic and geometric genera of the star-shaped graphs.

The main results of this paper are as follows.

Main results. We give a criteria to determine whether a weighted dual graph is negative-definite (cf. Corollary 3.7). Assuming that all the exceptional curves A_i are -2-curves except one -3-curve A_j, then the weighted dual graph Γ must be one of the three cases: a Tree graph, a Loop graph or a Multiple edges graph (cf. Theorem 4.13 for last two cases). The complete classifications are listed in § 4 (cf. Theorems 4.1, 4.3, 4.5, 4.10, and 4.13); the fundamental cycles of maximal graphs are computed and listed in § 5. Furthermore, when the graph is star-shaped, its arithmetic and geometric genus formulas are obtained in § 5 (cf. Corollary 6.8, Theorem 6.12).

Remark 1.1. Our new results also include Theorem 3.5 and Corollary 3.7, which can be of independent interest. These two results shed new light on the classification of more complicated weighted dual graphs of singularities.

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§ 2. Preliminaries

2.1. Riemann–Roch and fundamental cycle. Let $\pi\colon M \to V$ be a resolution of the normal two-dimensional Stein space V. We assume that p is the only singularity of V. Let $\pi^{-1}(p) = A = \bigcup A_i$, $1 \leqslant i \leqslant n$, be the decomposition of the exceptional set A into irreducible components.

A cycle $D = \sum d_i A_i$, $1 \leqslant i \leqslant n$, is an integral combination of the A_i, with d_i an integer. There is a natural partial ordering denoted by $\geqslant$, between cycles defined by comparing the coefficients: $\sum_i m_i A_i \geqslant \sum_i n_i A_i$ if $m_i \geqslant n_i$ for all i. If $D_1 \geqslant D_2$ but $D_1 \neq D_2$, then we write $D_1 > D_2$. We let $\operatorname{supp} D = \bigcup A_i$, $d_i \neq 0$, denote the support of D.

Let $\mathcal{O}$ be the sheaf of germs of holomorphic functions on M. Let $\mathcal{O}(-D)$ be the sheaf of germs of holomorphic functions on M which vanish to order d_i on A_i. Let $\mathcal{O}_D$ denote $\mathcal{O}/\mathcal{O}(-D)$. Define

$$\chi(D) := \dim H^0(M, \mathcal{O}_D) - \dim H^1(M, \mathcal{O}_D). \tag{2.1}$$

The Riemann–Roch theorem (see Proposition IV.4, p. 75 in [22]) says

$$\chi(D) = -\frac{1}{2}(D^2 + D \cdot K), \tag{2.2}$$

where K is the canonical divisor on M and $D \cdot K$ is the intersection number of D and K. In fact, let g_i be the geometric genus of A_i, that is, the genus of the desingularization of A_i. Then the adjunction formula (see Proposition IV, 5, p. 75 in [22]) says

$$A_i \cdot K = -A_i^2 + 2g_i - 2 + 2\delta_i, \tag{2.3}$$

where δ_i is the "number" of nodes and cusps on A_i. Each singular point on A_i other than a node or cusp counts as at least two nodes. It follows immediately from (2.2) that if B and C are cycles, then

$$\chi(B + C) = \chi(B) + \chi(C) - B \cdot C. \tag{2.4}$$

Definition 2.1. Associated to π is a unique fundamental cycle Z (see [4], pp. 131, 132) such that $Z > 0$, $A_i \cdot Z \leqslant 0$ for all A_i and such that Z is minimal with respect to those two properties.

The fundamental cycle Z may be computed from the intersection as follows via a computation sequence for Z in the sense of Laufer (see [23], Proposition 4.1, p. 607):

$$Z_0 = 0, \quad Z_1 = A_{i_1}, \quad Z_2 = Z_1 + A_{i_2}, \quad \dots, \quad Z_j = Z_{j-1} + A_{i_j}, \quad \dots, \\ Z_\ell = Z_{\ell-1} + A_{i_\ell} = Z,$$

where A_{i_1} is arbitrary and $A_{i_j} \cdot Z_{j-1} > 0$, $1 < j \leqslant \ell$.

Below, $\mathcal{O}(-Z_{j-1})/\mathcal{O}(-Z_j)$ represents the sheaf of germs of sections of a line bundle over A_{i_j} of Chern class $-A_{i_j}\cdot Z_{j-1}$. So

$$H^0(M,\mathcal{O}(-Z_{j-1})/\mathcal{O}(-Z_j))=0$$

for $j>1$.

Consider the exact sequence:

$$0\to\mathcal{O}(-Z_{j-1})/\mathcal{O}(-Z_j)\to\mathcal{O}_{Z_j}\to\mathcal{O}_{Z_{j-1}}\to 0. \tag{2.5}$$

From the long exact cohomology sequence for (2.5), it follows by induction that

$$H^0(M,\mathcal{O}_{Z_k})=\mathbb{C},\qquad 1\leqslant k\leqslant \ell, \tag{2.6}$$

$$\dim H^1(M,\mathcal{O}_{Z_k})=\sum_{1\leqslant j\leqslant k}\dim H^1\big(M,\mathcal{O}(-Z_{j-1}))/\mathcal{O}(-Z_j)\big). \tag{2.7}$$

Lemma 2.2 (see [10]). *Let Z_k be part of a computation sequence for Z such that $\chi(Z_k)=0$. Then $\dim H^1(M,\mathcal{O}_D)\leqslant 1$ for all cycles D such that $0\leqslant D\leqslant Z_k$. Also $\chi(D)\geqslant 0$.*

2.2. The canonical cycle.

Definition 2.3. A rational cycle Z_K is called a *canonical cycle* if $Z_K\cdot A_i=-KA_i$ for all i, that is,

$$Z_K\cdot A_i=A_i^2-2\delta_i-2g_i+2\quad\text{for all } i,$$

where δ_i is the "number" of nodes and cusps on A_i.

Definition 2.4. If the coefficients of Z_K are integers, then the singularity is called *numerical Gorenstein.*

2.3. Minimally elliptic and weakly elliptic singularities. The following definition of a minimally elliptic cycle was given by Laufer based on Lemma 2.2. We recall some properties of minimally elliptic singularities which we need for our classification problem.

Definition 2.5. A cycle $E>0$ is *minimally elliptic* if $\chi(E)=0$ and $\chi(D)>0$ for all cycles D such that $0<D<E$.

Wagreich [17] defined the singularity p to be elliptic if $\chi(D)\geqslant 0$ for all cycles $D\geqslant 0$ and $\chi(F)=0$ for some cycles $F>0$. He proved that this definition is independent of a resolution. It is easy to see that under this hypothesis, $\chi(Z)=0$. The converse is also true [10]. Henceforth, we shall adopt the following definition.

Definition 2.6. The singularity (V,p) is said to be *weakly elliptic* if $\chi(Z)=0$.

The following proposition holds for weakly elliptic singularities.

Proposition 2.7 (see [10]). *Suppose that $\chi(D)\geqslant 0$ for all cycles $D>0$. Let $B=\sum b_iA_i$ and $C=\sum c_iA_i$, $1\leqslant i\leqslant n$, be any cycles such that $B,C>0$ and $\chi(B)=\chi(C)=0$. Let $G=\sum\min(b_i,c_i)A_i$, $1\leqslant i\leqslant n$. Then $G>0$ and $\chi(G)=0$. In particular, there exists a unique minimally elliptic cycle E.*

Theorem 2.8 (see [10]). *Let $\pi\colon M \to V$ be the minimal resolution of the normal two dimensional variety V with one singular point p. Let Z be the fundamental cycle on the exceptional set $A = \pi^{-1}(p)$. Then the following statements are equivalent*:

(1) *Z is a minimally elliptic cycle*;

(2) $Z = Z_K$;

(3) $\chi(Z) = 0$ *and any connected proper subvariety of A is the exceptional set for a rational singularity.*

In [10], Laufer introduced the notion of minimally elliptic singularity.

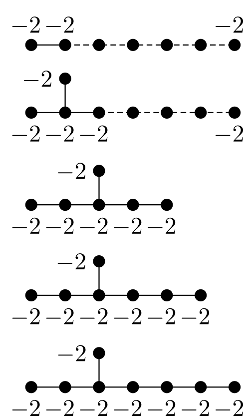
Definition 2.9. Let (V, p) be a normal two-dimensional singularity. (V, p) is said to be minimally elliptic if the minimal resolution $\pi\colon M \to V$ of a neighbourhood of p satisfies one of the conditions of Theorem 2.8.

2.4. Classification of weighted dual graphs. In this section, we recall two beautiful results given by Artin in [4]. Let (V, p) be a normal 2-dimensional singularity, $\pi\colon M \to V$ be the minimal resolution, and Z be the fundamental cycle.

Definition 2.10. The singularity (V, p) is said to be rational if $\chi(Z) = 1$.

If (V, p) is a rational singularity, then π is also a minimal good resolution, that is, exceptional set with non-singular A_i and normal crossings. Moreover, each A_i is a rational curve and $A_i^2 = -2$.

Theorem 2.11 (see [4]). *If (V, p) is a hypersurface rational singularity, then (V, p) is a rational double point. Moreover, the set of weighted dual graphs of hypersurface rational singularities consists of the following graphs*:



(1) A_n, $n \geqslant 1$ — $Z = 1\ 1 \ldots 1$;

(2) D_n, $n \geqslant 4$ — $Z = 1\ \overset{1}{2}\ 2 \ldots\ 2\ 1$;

(3) E_6 — $Z = 1\ 2\ \overset{2}{3}\ 2\ 1$;

(4) E_7 — $Z = 2\ 3\ \overset{2}{4}\ 3\ 2\ 1$;

(5) E_8 — $Z = 2\ 4\ \overset{3}{6}\ 5\ 4\ 3\ 2.$

To each such weighted dual graph there corresponds an intersection matrix whose (i, j) th entry is $A_i \cdot A_j$.

These graphs (1)–(5) in Theorem 2.11 are called ADE graphs in the literature. This theorem completely classifies the weighted dual graphs with all $A_i^2 = -2$. In general, according to [24] and [4], to classify the weighted dual graphs, we need to classify the corresponding negative-definite matrices.

Proposition 2.12 (see [4]). *Let* $\{A_i\}_{i=1,\dots,n}$ *be a connected bunch of complete curves on a regular two-dimensional scheme.*

(i) *Suppose that* $(A_i \cdot A_j)$ *is negative-definite, then there exist positive cycles* $Z = \sum r_i A_i$ *such that* $(Z \cdot A_i) \leqslant 0$ *for all* i.

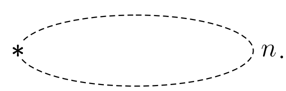
(ii) *Conversely, if there exists a positive cycle* $Z = \sum r_i A_i$ *such that* $(Z \cdot A_i) \leqslant 0$ *for all* i, *then* $(A_i \cdot A_j)$ *is negative semi-definite. If in addition* $(Z^2) < 0$, *then* $(A_i \cdot A_j)$ *is negative-definite.*

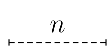
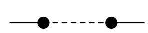
§ 3. A general determinant formula and a method for classification

In this section, we will give some key results, which will be helpful for classification of weighted dual graphs. The weighted dual graph consists of -2-curves and exactly one -3-curve, that is, all A_i's are non-singular rational curves with $A_j^2 = -3$ for some j and $A_i^2 = -2$ for all the other i's such that $i \neq j$. In a dual graph, the $*$ represents the -3-curve. We will call it the -3-point or the -3-cycle later. The others are the points corresponding to the -2-curves, denoted by $\bullet$, we will call them -2-points or -2-cycles later.

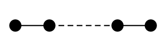
By Theorem 2.11, if all A_i have $A_i^2 = -2$, then the graph must be an ADE graph. Recall that a tree graph is a connected graph without loops. ADE graphs are all tree graphs. However, if there exists one $A_j^2 = -3$, then the following two cases are allowed:

and



Here,  denotes  with n vertices and $n+1$ edges.

We first begin with tree weighted dual graphs. We abuse the notation of weighted dual graph and the corresponding matrices in the following discussion when no confusion ensues. Henceforth, whether A_k is a weighted dual graph or a matrix should be clear from the context. For example, A_n could either denote the weighted dual graph:



or the matrix

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

We use Γ to denote the weighted dual tree graph of $A = \bigcup_{i=1}^{n} A_i$ such that all A_i are -2-curves except one -3-curve A_j. After removing the point corresponding to the -3-curve, the remaining connected graphs are denoted by $\Gamma_1, \dots, \Gamma_m$.

Lemma 3.1. *With the above notation,* $m \leqslant 5$, *and* Γ_i *must be ADE for any* $1 \leqslant i \leqslant m$.

Proof. Notice that the first column of the following matrix is $(-1/2)$ the sum of the other columns, hence the matrix

$$\begin{pmatrix} -3 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

has determinant 0, so the -3-curve can not be connected with six or more -2-curves, and so we have $m \leqslant 5$. As for Γ_i, note that if we require Γ to be negative-definite, then the fundamental cycle Z, when restricted to each Γ_i, satisfies $Z|_{\Gamma_i} \cdot A_j \leqslant 0$ $\forall\, A_j \in \Gamma_i$ (here it means that A_j is in the support of Γ_i). Denote by A_{j_0} the cycle in Γ_i connected with -3-point. Hence $Z|_{\Gamma_i} \cdot A_{j_0} < 0$. By Proposition 2.12, Γ_i is negative-definite. Hence it is an ADE. $\square$

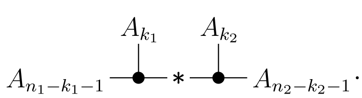
In the following, we illustrate a method for computing determinant of general tree graphs. Let us begin with the easiest case.

Example 3.2 (determinant formula for $m = 2$ in special case). Let $m = 2$ and Γ_i be $k_i - A_{n_i}$ (see the definition of $k - A_n$ in Theorem 4.1), $i = 1, 2$. Then

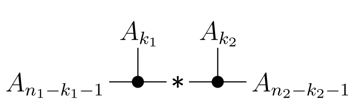
$$\begin{aligned} \det(\Gamma) = (-1)^{n_1+n_2}\big(&-3 \cdot |\det(A_{n_1})\det(A_{n_2})| \\ &+ |\det(A_{n_1})\det(A_{n_2-k_2-1})\det(A_{k_2})| \\ &+ |\det(A_{n_2})\det(A_{n_1-k_1-1})\det(A_{k_1})|\big), \end{aligned}$$

where $|\cdot|$ means absolute value.

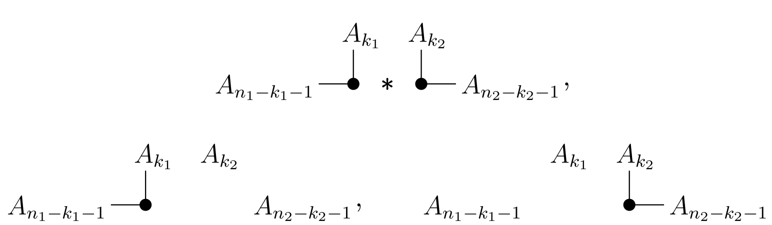
We explain what this formula means by using a weighted dual graph:



The formula tells us that the Laplacian expansion of $\det(\Gamma)$ is equivalent to "removing points and edges" on the graph Γ in some sense as follows:



is divided into



We take $k_1 = k_2 = 1$, $n_1 = n_2 = 3$ as an example to show how this happens. Note that the matrix Γ is

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

Using the Laplace expansion along the 4th row, we find that

$$\begin{aligned} \det(\Gamma) = (-3)\det(A_3)\det(A_3) \\ + 1 \cdot \det \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix} \\ + 1 \cdot \det \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}. \end{aligned}$$

The determinant of

$$\begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

is easy to compute, for the reason that the 1st row and 3rd row have only one non-zero element -2, that is,

$$\begin{aligned} \det \begin{pmatrix} (-2) & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & (1) & 0 & 0 & 0 \\ 0 & (-2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & \langle 1 \rangle & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix} \\ = (-1) \cdot (-2) \cdot (-2) \cdot (1) \cdot \det(A_3). \end{aligned}$$

It is similar for the 3rd term. Notice that $\det(A_1) = -2$, the formula is easily verified for this case. For general k_1, k_2, n_1 and n_2, the 2nd term of the matrix is:

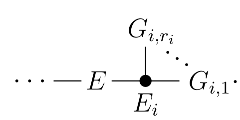
$$\begin{pmatrix}
-2 & \dots & 0 & & & & & & & \\
\vdots & \ddots & \vdots & & & & & & & \\
0 & \dots & -2 & & & & & & & \\
 & & 1 & 1 & 0 & \dots & (1) & & & \\
 & & & -2 & \dots & 0 & 0 & & & \\
 & & & \vdots & \ddots & \vdots & \vdots & & & \\
 & & & 0 & \dots & -2 & 0 & & & \\
 & & & & & & \vdots & -2 & \dots & 0 \\
 & & & & & & \langle 1 \rangle & \vdots & \ddots & \vdots \\
 & & & & & & & 0 & \dots & -2
\end{pmatrix},$$

where (1) lies on the $(n_1 - k_1 - 1)$th row and n_1th column, while $\langle 1 \rangle$ lies on the $(n_1 + k_2 + 1)$th row and n_1th column. The three -2 matrices are $A_{n_1-k_1-1}$, A_{k_1}, and A_{n_2}, respectively.

Remark 3.3. One may want to know how to determine the sign of each term. An easy way to see it is that the connection of -3 and Γ_i contributes to "positive" part in the determinant. Hence, besides the $(-3)|\det(A_{n_1})\det(A_{n_2})|$ term, the other terms are positive.

Now we try to generalize the formula of Γ_i to general graphs.

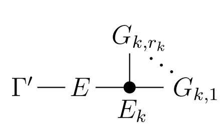
Notation 3.4. For a tree graph with central curve E, we denote the subgraphs connected to E as $\Gamma_1, \dots, \Gamma_s$, the points connected to E are denoted by $E_1, \dots, E_s$, and the subgraphs connected to E_i as are denoted by $G_{i,1}, \dots, G_{i,r_i}$:



Theorem 3.5 (general determinant formula). *Let the weighted dual graph* Γ *be as above. Then*

$$\det(\Gamma) = \left(\prod_{i=1}^{s} \det(\Gamma_i)\right)\left(E^2 + \sum_{j=1}^{s} \frac{(-1)^{n_j} \prod_{l=1}^{r_j} \det(G_{j,l})}{\det(\Gamma_j)}\right).$$

Proof. We argue by induction on s. Assume the formula holds for $s \leqslant k-1$, let us proceed to show it is true for k. Let the weighted dual graph be as in Notation 3.4 with k subgraphs connected to E, that is, the weighted dual graph is



Let n_i be the number of points of Γ_i. The intersection matrix can be written as

$$\begin{pmatrix} \Gamma' & 1 & 0 & 1 & \dots & 1 \\ 1 & E^2 & 1 & 0 & \dots & 0 \\ 0 & 1 & E_k^2 & 1 & \dots & 1 \\ 0 & 0 & 1 & G_{k,1} & \dots & 0 \\ 0 & 0 & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 1 & 0 & \dots & G_{k,r_k} \end{pmatrix}.$$

For simplicity, we use $\left(\begin{smallmatrix} \Gamma' & 1 \\ 1 & E^2 \end{smallmatrix}\right)$ to denote

$$\begin{pmatrix} E^2 & 1 & \dots & 1 \\ 1 & \Gamma_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & \Gamma_{k-1} \end{pmatrix}.$$

Using the Laplacian expansion on E, we obtain

$$\begin{aligned} \det(\Gamma) &= \det \begin{pmatrix} \Gamma' & 1 \\ 1 & E^2 \end{pmatrix} \det(\Gamma_k) + (-1)^{(n_k)} \det \begin{pmatrix} \Gamma' & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & G_{k,1} & \dots & 0 \\ 0 & 0 & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & G_{k,r_k} \end{pmatrix} \\ &= \det \begin{pmatrix} \Gamma' & 1 \\ 1 & E^2 \end{pmatrix} \det(\Gamma_k) + (-1)^{(n_k)} \det(\Gamma') \prod_{l=1}^{r_k} \det(G_{k,l}). \end{aligned}$$

By the induction hypothesis, we have

$$\det \begin{pmatrix} \Gamma' & 1 \\ 1 & E^2 \end{pmatrix} = \left(\prod_{i=1}^{k-1} \det(\Gamma_i) \right) \left((E^2) + \sum_{j=1}^{k-1} \frac{(-1)^{n_j} \prod_{l=1}^{r_j} \det(G_{j,l})}{\det(\Gamma_j)} \right).$$

Thus,

$$\begin{aligned} \det(\Gamma) &= \left(\prod_{i=1}^{k-1} \det(\Gamma_i) \right) \left((E^2) + \sum_{j=1}^{k-1} \frac{(-1)^{n_j} \prod_{l=1}^{r_j} \det(G_{j,l})}{\det(\Gamma_j)} \right) \det(\Gamma_k) \\ &\quad + (-1)^{(n_k)} \prod_{i=1}^{k-1} \det(\Gamma_i) \prod_{l=1}^{r_k} \det(G_{k,l}) \\ &= \left(\prod_{i=1}^{k} \det(\Gamma_i) \right) \left((E^2) + \sum_{j=1}^{s} \frac{(-1)^{n_j} \prod_{l=1}^{r_j} \det(G_{j,l})}{\det(\Gamma_j)} \right). \end{aligned}$$

□

Notice that the selection of E can be arbitrary in a weighted dual graph. Nevertheless, if we choose a suitable E (for example, an E such that Γ_i is negative-definite for $i = 1, \dots, s$), then testing the negative-definiteness of $\det(\Gamma)$ is reduced to checking the negative-definiteness of the determinants of the subgraphs containing E.

Remark 3.6. Theorem 3.5 tells us that the "removing points and edges" method in Example 3.2 also holds for arbitrary tree graphs.

Corollary 3.7 (criteria for negative definiteness). *With the assumptions in Theorem* 3.5, *let us further assume that each* Γ_i *is negative-definite, for* $i = 1, \ldots, s$. *Then the weighted dual graph is negative-definite if and only if*

$$E^2 + \sum_{j=1}^{s} \frac{\prod_{k=1}^{r_j} |\det(G_{j,k})|}{|\det(\Gamma_j)|} < 0.$$

Proof. Since Γ_i is negative-definite, we have

$$\det(\Gamma_i) = (-1)^{n_i} |\det(\Gamma_i)|, \qquad \prod_{k=1}^{r_j} \det(G_{j,k}) = (-1)^{n_k - 1} \prod_{k=1}^{r_j} |\det(G_{j,k})|.$$

Combining this fact with Theorem 3.5, we find that

$$\det(\Gamma) = (-1)^{\sum_{i=1}^{s} n_i} \left(\left| \prod_{i=1}^{s} \det(\Gamma_i) \right| \right) \left((E^2) + \sum_{j=1}^{s} \left| \frac{\prod_{k=1}^{r_j} \det(G_{j,k})}{\det(\Gamma_j)} \right| \right).$$

If Γ is negative-definite, then

$$(-1)^{(1 + \sum_{i=1}^{s} n_i)} \det(\Gamma) > 0.$$

Consequently,

$$E^2 + \sum_{j=1}^{s} \frac{\prod_{k=1}^{r_j} |\det(G_{j,k})|}{|\det(\Gamma_j)|} < 0.$$

Conversely, Sylvester's criterion tells us that Γ is negative-definite if and only if $(-1)^i \Delta_k > 0$, for $k = 1, \ldots, n$. Here, n denotes the number of points in Γ and Δ_k denotes the determinant of the upper-left $(k \times k)$-submatrix. We can have -3 as bottom right element. Then $\Delta_1, \ldots, \Delta_{n-1}$ only contains -2 and is negative-definite by assumption. Hence, $(-1)^k \Delta_k > 0$, for $k = 1, \ldots, n-1$. As for Δ_n, one computes directly that

$$\begin{aligned}
(-1)^n \Delta_n &= (-1)^n \det(\Gamma) \\
&= (-1)^n (-1)^{\sum_{i=1}^{s} n_i} \left(\left| \prod_{i=1}^{s} \det(\Gamma_i) \right| \right) \left((E^2) + \sum_{j=1}^{s} \left| \frac{\prod_{k=1}^{r_j} \det(G_{j,k})}{\det(\Gamma_j)} \right| \right) \\
&= (-1) \left(\left| \prod_{i=1}^{s} \det(\Gamma_i) \right| \right) \left((E^2) + \sum_{j=1}^{s} \left| \frac{\prod_{k=1}^{r_j} \det(G_{j,k})}{\det(\Gamma_j)} \right| \right) > 0.
\end{aligned}$$

Thus, Γ is negative-definite. □

Remark 3.8. The condition for Γ_i to be negative-definite is natural, because Proposition 2.12 and the proof of Lemma 3.1 imply that any subgraph of a negative-definite weighted dual graph is negative-definite. This criteria can also be used to classify other graphs. For example, the weighted dual graphs consist of exactly one $-k$-curve, for $k \geqslant 4$. The classification will be investigated in our subsequent paper.

§ 4. Classification of weighted dual graphs consists of −2-curves and exactly one −3-curve

In the following, we shall give the classification of the weighted dual graphs Γ for different values of m.

Case 1. $m = 0$.

The weighted dual graph has just one -3 point:

$$*.$$

Case 2. $m = 1$.

Theorem 4.1. *For* $m = 1$, Γ_1 *must be one of the following*:

(1) $k - A_n$: $k = 0, 1, 2$, *for arbitrary* $n \geqslant 2k+1$; *or* $k = 3$, $7 \leqslant n \leqslant 14$; *or* $k = 4$, $9 \leqslant n \leqslant 11$;

(2) $k - D_n$: $k = 0$ *for arbitrary* n; *or* $k = 1, n \geqslant 5$;

(3) D'_n: $5 \leqslant n \leqslant 11$;

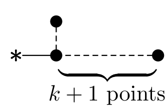
(4) E_6, E_7, E_8;

(5) E'_6;

(6) E'_7;

(7) D''_4.

Here, we use the notation $k - A_n$ to denote the following graph: $\Gamma_1 = A_n$ and Γ is

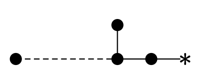


with $n \geqslant 2k + 1$. Here, $0 - A_n$ (or A_n) means that the -3-curve connects with A_n at the left or right end point.

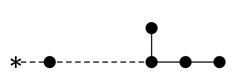
Similarly, the notation $1 - D_n$ means $\Gamma_1 = D_n$, with $n \geqslant 5$ and Γ is

$0 - D_n$ (or D_n) means that the -3-curve connects with the longest branch of D_n. In the later discussions, without the emphasis on the boundness of n, it means $n \geqslant 1$ can be arbitrary.

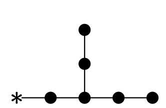
D'_n is



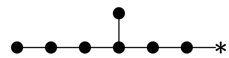
E_k, $k = 6, 7, 8$, are



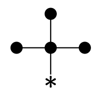
E'_6 is



E_7' is



and D_4'' is



Proof. This is clear by Corollary 3.7. □

Remark 4.2. Note that $k{-}A_n$ (also $k{-}D_n$) are terminologies to identify the different connection ways of Γ_1 and -3-point. D_n' is exactly $0{-}D_n$ when $n = 4$, so we require in (3) that $n \geqslant 5$. $1 - A_3$ can be regarded as a limiting case of D_n.

To classify all the tree graphs Γ, we need to compute the determinant of the corresponding matrices. One remembers that $\det(A_n) = (-1)^n(n+1)$, $\det(D_n) = 4 \cdot (-1)^n$, $\det(E_6) = 3$, $\det(E_7) = -2$, $\det(E_8) = 1$.
Case 3. $m = 2$.

Theorem 4.3. *For* $m = 2$, $\Gamma_1 + \Gamma_2$ *is one of the following*:
(1) $(k_1 - A_{n_1}) + (k_2 - A_{n_2})$:

$$\frac{(k_1+1)^2}{n_1+1} + \frac{(k_2+1)^2}{n_2+1} > k_1 + k_2 - 1;$$

(2) $(k_1 - D_{n_1}) + (k_2 - A_{n_2})$: $k_1 = 1$, $k_2 = 0$; *or* $k_1 = 0$, $k_2 = 0, 1$; *or* $k_1 = 0$, $k_2 = 2$, $n_2 \leqslant 7$;
(3) $D_{n_1} + D_{n_2}$;
(4) $D_{n_1}' + A_{n_2}$: $n_1 \leqslant 8$; *or* $n_1 = 9$, $n_2 = 1, 2$;
(5) $E_{n_1} + A_{n_2}$: $n_1 = 6, 7, 8$;
(6) $E_6 + (1 - A_{n_2})$: $3 \leqslant n_2 \leqslant 10$;
(7) $E_6' + A_{n_2}$;
(8) $E_7' + A_{n_2}$;
(9) $D_4'' + A_{n_2}$;
(10) $E_{n_1} + D_{n_2}$: $n_1 = 6, 7$;
(11) $D_{n_1}' + D_{n_2}$: $5 \leqslant n_1 \leqslant 7$;
(12) $E_{n_1} + E_6$: $n_1 = 6, 7$;
(13) $E_{n_1} + D_{n_2}'$: $n_1 = 6$, $n_2 = 5, 6$; *or* $n_1 = 7$, $n_2 = 5$.

Proof. We first prove (1). Plugging $|\det(A_n)| = n+1$ into the determinant formula in Example 3.2, we find that

$$\begin{aligned}\det(\Gamma) = (-1)^{n_1+n_2}\big(&-3 \cdot (n_1+1)(n_2+1)\\ &+ (n_1-k_1)(k_1+1)(n_2+1) + (n_2-k_2)(k_2+1)(n_1+1)\big).\end{aligned}$$

If we require Γ to be negative-definite, then $(-1)^{n_1+n_2}\det(\Gamma)$ must be negative, so this yields (1). The discussions for (2) to (13) are similar. Let us take the argument for (8) as an example here. For $E_7' + A_{n_2}$, we must require

$$-3 \cdot |\det(E_7)\det(A_{n_2})| + |\det(D_6)\det(A_{n_2})| + |\det(E_7)\det(A_{n_2-1})| < 0,$$

that is,

$$-3 \cdot 2(n_2+1) + 4(n_2+1) + 2n_2 < 0,$$

which is true for all $n_2 \geqslant 1$. □

Lemma 4.4. *The inequality*

$$\frac{(k_1+1)^2}{n_1+1} + \frac{(k_2+1)^2}{n_2+1} > k_1 + k_2 - 1$$

holds in the following cases:
(1) $k_1 = k_2 = 0$;
(2) $k_1 = 1, k_2 = 0$;
(3) $k_1 = 2, k_2 = 0$:
3.1. $5 \leqslant n_1 \leqslant 8$, $1 \leqslant n_2$;
3.2. $n_1 = 9$, $1 \leqslant n_2 \leqslant 8$;
3.3. $n_1 = 10$, $1 \leqslant n_2 \leqslant 4$;
3.4. $n_1 = 11, 12$, $1 \leqslant n_2 \leqslant 2$;
3.5. $n_1 = 13, 14, 15, 16$, $n_2 = 1$;
(4) $k_1 = 3, k_2 = 0$:
4.1. $n_1 = 7$, $1 \leqslant n_2$;
4.2. $n_1 = 8$, $1 \leqslant n_2 \leqslant 2$;
4.3. $n_1 = 9$, $n_2 = 1$;
(5) $k_1 = k_2 = 1$ (*we can assume* $n_1 \geqslant n_2$):
5.1. $n_1 \geqslant n_2 = 3$;
5.2. $18 \geqslant n_1 \geqslant n_2 = 4$;
5.3. $10 \geqslant n_1 \geqslant n_2 = 5$;
5.4. $8 \geqslant n_1 \geqslant n_2 = 6$;
(6) $k_1 = 2, k_2 = 1$:
6.1. $n_1 = 5$, $3 \leqslant n_2 \leqslant 6$;
6.2. $n_1 = 6$, $n_2 = 3, 4$;
6.3. $n_1 = 7$, $n_2 = 3$.

Proof. By Theorem 4.1, $n_1 \geqslant 2k_1 + 1$, $n_2 \geqslant 2k_2 + 1$. Thus,

$$\frac{(k_1+1)^2}{2k_1+2} + \frac{(k_2+1)^2}{2k_2+2} \geqslant \frac{(k_1+1)^2}{n_1+1} + \frac{(k_2+1)^2}{n_2+1} > k_1 + k_2 - 1.$$

This yields $k_1 + k_2 \leqslant 3$. Computations of (1)–(6) are similar. Let us take the argument for (5) as an example here. For $k_1 = k_2 = 1$, the inequality becomes

$$\frac{4}{n_1+1} + \frac{4}{n_2+1} > 1.$$

Thus, n_1 can be arbitrary, if $n_2 = 3$, and

$$n_1 < 3 + \frac{16}{n_2 - 3},$$

if $n_2 > 3$. Taking different n_2's, one gets 5.1–5.4.

Case 4. $m = 3$.

Theorem 4.5. *For* $m = 3$, $\Gamma_1 + \Gamma_2 + \Gamma_3$ *is one of the following*:

(1) $(k_1 - A_{n_1}) + (k_2 - A_{n_2}) + (k_3 - A_{n_3})$:

$$\sum_{i=1}^{3} \frac{(k_i+1)^2}{n_i+1} > \sum_{i=1}^{3} k_i;$$

(2) $D_{n_1} + (k_2 - A_{n_2}) + (k_3 - A_{n_3})$:

$$\sum_{i=2}^{3} \frac{(k_i+1)^2}{n_i+1} > \sum_{i=2}^{3} k_i;$$

(3) $E_6 + (1 - A_3) + A_1$;
(4) $E_6 + A_{n_2} + A_{n_3}$: $(n_2-2)(n_3-2) < 9$;
(5) $E_7 + A_{n_2} + A_{n_3}$: $(n_2-1)(n_3-1) < 4$;
(6) $D_{n_1} + D_{n_2} + A_{n_3}$;
(7) $D_{n_1} + E_6 + A_1$.

Proof. Computations of the above cases are similar. We take (4) as an example. Plugging $\det(E_6)$ and $\det(A_n)$ into the determinant formula in Theorem 3.5, and taking into account that Γ is negative-definite, we find that

$$\begin{aligned} 0 > \big(&-3 \cdot |\det(E_6)\det(A_{n_2})\det(A_{n_3})| + |\det(E_5)\det(A_{n_2})\det(A_{n_3})| \\ &+ |\det(E_6)\det(A_{n_2-1})\det(A_{n_3})| + |\det(E_6)\det(A_{n_2})\det(A_{n_3-1})|\big). \end{aligned}$$

A simple algebra shows that

$$\begin{aligned} 0 > &-3\cdot 3\cdot (n_2+1)(n_3+1) \\ &+ 4(n_2+1)(n_3+1) + 3n_2(n_3+1) + 3n_3(n_2+1), \end{aligned}$$

that is,

$$(n_2-2)(n_3-2) < 9.$$

□

Lemma 4.6. *The inequality*

$$\sum_{i=1}^{3} \frac{(k_i+1)^2}{n_i+1} > \sum_{i=1}^{3} k_i$$

holds in the following cases:

(1) $k_1 = k_2 = k_3 = 0$;
(2) $k_1 = 1$, $k_2 = k_3 = 0$ (*we assume* $n_2 \geqslant n_3$);
  2.1. $n_1 = 3$;
  2.2. $n_1 = 4$:
    2.2.1. $n_2 = 1, 2, \ldots, 8$, $1 \leqslant n_3 \leqslant n_2$;
    2.2.2. $n_2 = 9, 10$, $1 \leqslant n_3 \leqslant 8$;
    2.2.3. $n_2 = 11, 12$, $1 \leqslant n_3 \leqslant 7$;
    2.2.4. $n_2 = 13, 14, 15, 16$, $1 \leqslant n_3 \leqslant 6$;
    2.2.5. $n_2 = 17, \ldots, 28$, $1 \leqslant n_3 \leqslant 5$;

2.2.6. $n_2 \geqslant 29$, $1 \leqslant n_3 \leqslant 4$;
2.3. $n_1 = 5$:
2.3.1. $n_2 = 1, \dots, 4$, $1 \leqslant n_3 \leqslant n_2$;
2.3.2. $n_2 = 5, 6$, $1 \leqslant n_3 \leqslant 4$;
2.3.3. $n_2 = 7, \dots, 10$, $1 \leqslant n_3 \leqslant 3$;
2.3.4. $n_2 \geqslant 11$, $1 \leqslant n_3 \leqslant 2$;
2.4. $n_1 = 6$:
2.4.1. $n_2 = 1, 2, 3$, $n_3 \leqslant n_2$;
2.4.2. $n_2 = 4$, $1 \leqslant n_3 \leqslant 3$;
2.4.3. $n_2 = 5, \dots, 9$, $1 \leqslant n_3 \leqslant 2$;
2.4.4. $n_2 \geqslant 10$, $n_3 = 1$;
2.5. $n_1 = 7$:
2.5.1. $n_2 = 1, 2$, $n_3 \leqslant n_2$;
2.5.2. $n_2 = 3, 4$, $n_3 = 1, 2$;
2.5.3. $n_2 \geqslant 5$, $n_3 = 1$;
2.6. $n_1 = 8$:
2.6.1. $n_2 = 1, 2$, $n_3 \leqslant n_2$;
2.6.2. $n_2 = 3$, $n_3 = 1, 2$;
2.6.3. $4 \leqslant n_2 \leqslant 16$, $n_3 = 1$;
2.7. $n_1 = 9$:
2.7.1. $n_2 = 1, 2$, $n_3 \leqslant n_2$;
2.7.2. $3 \leqslant n_2 \leqslant 8$, $n_3 = 1$;
2.8. $n_1 = 10$:
2.8.1. $n_2 = 1, 2$, $n_3 \leqslant n_2$;
2.8.2. $3 \leqslant n_2 \leqslant 6$, $n_3 = 1$;
2.9. $n_1 = 11, 12$, $1 \leqslant n_2 \leqslant 4$, $n_3 = 1$;
2.10. $n_1 = 13, 14$, $1 \leqslant n_2 \leqslant 3$, $n_3 = 1$;
2.11. $n_1 = 15, 16, \dots, 22$, $1 \leqslant n_2 \leqslant 2$, $n_3 = 1$;
2.12. $n_1 \geqslant 23$, $n_2 = 1$, $n_3 = 1$;
(3) $k_1 = 2$, $k_2 = k_3 = 0$;
3.1. $n_1 = 5$:
3.1.1. $n_2 = 1$, $n_3 = 1$;
3.1.2. $n_2 = 2, 3, 4$, $n_3 = 1, 2$;
3.1.3. $n_2 \geqslant 5$, $n_3 = 1$;
3.2. $n_1 = 6$, $n_2 = 1, 2$, $n_3 = 1$;
3.3. $n_1 = 7$, $n_2 = n_3 = 1$;
(4) $k_1 = k_2 = 1, k_3 = 0$ (*we assume* $n_1 \geqslant n_2$):
4.1. $n_1 = n_2 = 3$;
4.2. $n_1 = 4$:
4.2.1. $n_2 = 3$, $n_3 = 1, 2, 3$;
4.2.2. $n_2 = 4$, $n_3 = 1$;
4.3. $n_1 = 5, 6$, $n_2 = 3$, $n_3 = 1$.

Lemma 4.7. *The inequality*

$$\sum_{i=2}^{3} \frac{(k_i+1)^2}{n_i+1} > \sum_{i=2}^{3} k_i$$

holds in the following cases:

(1) $k_2 = k_3 = 0$;

(2) $k_2 = 1$, $k_3 = 0$:

2.1. $n_2 = 3$;

2.2. $n_2 = 4$, $n_3 = 1, 2, 3$;

2.3. $n_2 = 5$, $n_3 = 1$;

2.4. $n_2 = 6$, $n_3 = 1$.

Lemma 4.8. *The inequality*

$$(n_2 - 2)(n_3 - 2) < 9$$

holds in the following cases (*we can assume* $n_2 \geqslant n_3$):

(1) $n_3 = 1, 2$;

(2) $n_3 = 3$, $3 \leqslant n_2 \leqslant 10$;

(3) $n_3 = 4$, $n_2 = 4, 5$.

Lemma 4.9. *The inequality*

$$(n_2 - 1)(n_3 - 1) < 4$$

holds in the following cases (*we can assume* $n_2 \geqslant n_3$):

(1) $n_3 = 1$;

(2) $n_3 = 2$, $2 \leqslant n_2 \leqslant 3$.

Case 5. $m = 4$.

Theorem 4.10. *For* $m = 4$, $\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$ *is one of the following*:

(1) $A_{n_1} + A_{n_2} + A_{n_3} + A_{n_4}$:

$$\sum_{i=1}^{4} \frac{1}{n_i + 1} > 1;$$

(2) $(1 - A_3) + A_{n_2} + A_{n_3} + A_1$:

$$\sum_{i=2}^{3} \frac{1}{n_i + 1} > \frac{1}{2};$$

(3) $D_{n_1} + A_{n_2} + A_{n_3} + A_1$:

$$\sum_{i=2}^{3} \frac{1}{n_i + 1} > \frac{1}{2}.$$

Proof. Let us only illustrate (3) in detail. Plugging $\det(A_n)$ and $\det(D_n)$ into the determinant formula in Theorem 3.5, we have, since Γ is negative-definite,

$$\sum_{i=2}^{4} \frac{1}{n_i + 1} > 1.$$

One can always assume that $n_2 \geqslant n_3 \geqslant n_4$, thus

$$\frac{1}{n_4 + 1} > \frac{1}{3},$$

that is, $n_4 = 1$. □

Lemma 4.11. *The inequality*

$$\sum_{i=1}^{4} \frac{1}{n_i+1} > 1$$

holds in the following cases (we assume $n_1 \geqslant n_2 \geqslant n_3 \geqslant n_4$*):*

(1) $n_4 = 1$:
- 1.1. $n_3 = 1$;
- 1.2. $n_3 = 2$:
  - 1.2.1. $2 \leqslant n_2 \leqslant 5$, $n_2 \leqslant n_1$;
  - 1.2.2. $n_2 = 6$, $6 \leqslant n_1 \leqslant 40$;
  - 1.2.3. $n_2 = 7$, $7 \leqslant n_1 \leqslant 22$;
  - 1.2.4. $n_2 = 8$, $8 \leqslant n_1 \leqslant 16$;
  - 1.2.5. $n_2 = 9$, $9 \leqslant n_1 \leqslant 13$;
  - 1.2.6. $n_2 = 10$, $10 \leqslant n_1 \leqslant 12$;
- 1.3. $n_3 = 3$:
  - 1.3.1. $n_2 = 3$, $3 \leqslant n_1$;
  - 1.3.2. $n_2 = 4$, $4 \leqslant n_1 \leqslant 18$;
  - 1.3.3. $n_2 = 5$, $5 \leqslant n_1 \leqslant 10$;
  - 1.3.4. $n_2 = 6$, $6 \leqslant n_1 \leqslant 8$;
- 1.4. $n_3 = 4$:
  - 1.4.1. $n_2 = 4$, $4 \leqslant n_1 \leqslant 8$;
  - 1.4.2. $n_2 = 5$, $n_1 = 5, 6$;

(2) $n_4 = 2$:
- 2.1. $n_3 = 2$:
  - 2.1.2. $n_2 = 2$;
  - 2.1.3. $n_2 = 3$, $3 \leqslant n_1 \leqslant 10$;
  - 2.1.4. $n_2 = 4$, $4 \leqslant n_1 \leqslant 6$;
- 2.2. $n_3 = n_2 = 3 \leqslant n_1 \leqslant 4$.

Lemma 4.12. *The inequality*

$$\sum_{i=2}^{3} \frac{1}{n_i+1} > \frac{1}{2}$$

holds in the following cases (we assume $n_2 \geqslant n_3$*):* $n_3 = 1$; *or* $n_3 = 2, n_2 = 2, 3, 4$.

Case 6. $m = 5$.

Theorem 4.13. *For* $m = 5$, $\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5$ *is one of the following*:

(1) $A_{n_1} + A_{n_2} + A_1 + A_1 + A_1$: $n_2 = 1$;

(2) $n_2 = 2$, $n_1 = 2, 3, 4$.

Proof. The determinant formula tells us that

$$\sum_{i=1}^{5} \frac{1}{n_i+1} > 2.$$

We can assume $n_1 \geqslant n_2 \geqslant n_3 \geqslant n_4 \geqslant n_5$, thus $n_5 = n_4 = n_3 = 1$. The inequality then becomes:

$$\sum_{i=1}^{2} \frac{1}{n_i + 1} > \frac{1}{2}.$$

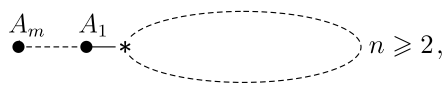
Consequently $n_2 = 1$ or $n_2 = 2$, $n_1 = 2, 3, 4$. □

Case 7. The weighted dual graph is not tree.

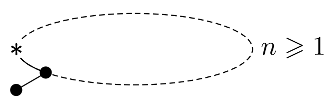
Now we turn to loop case and multiple edges case, these are stated below (intersection multiplicity > 1 means that $A_j \cdot A_i > 1$, here A_j is -3 cycle and A_i is a -2 cycle connected with A_j).

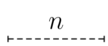
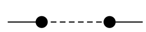
Theorem 4.14. *Any non-tree graph must has the following form.*

(1) *Loop cases*:

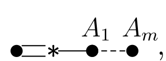


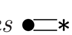
where $m \geqslant 0$ ($m = 0$ *means no points appear outside the loop*);

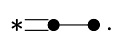


here,  *denotes*  *with* n *vertices and* $n+1$ *edges.*

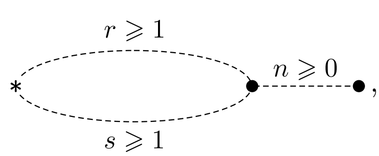
(2) *Multiple edges cases*:

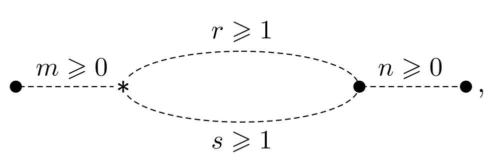


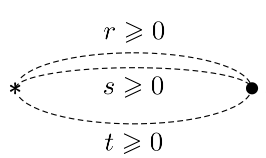
where $m \geqslant 0$ ($m = 0$ *denotes* ),

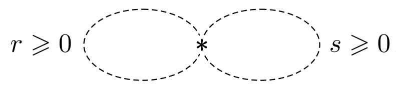


Proof. For (1), a direct evaluation of the determinant shows the following graphs are not allowed:



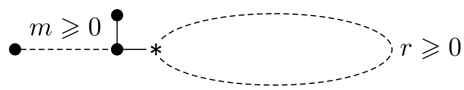


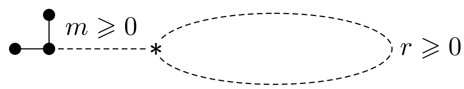




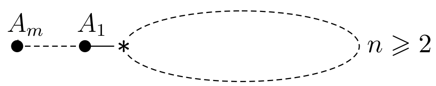
Here, we also used that each of these graphs is not negative-definite.

The following two graphs have determinant 0:



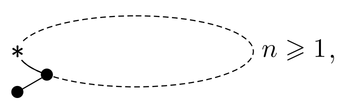


Next, for

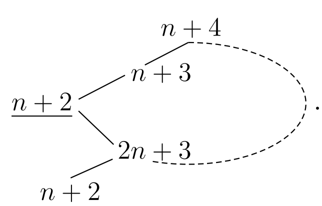


it can be shown that $\sum 1 \cdot A_i$ is the fundamental cycle, hence the graph is negative-definite.

For

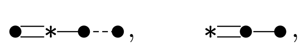


the fundamental cycle is (the underlined number represents the -3-point)



For example, for $n = 1$, the graph and the corresponding fundamental cycle are as follows:

For (2), it is easy to see that these two multiple edge cases are exactly the degeneration of the two loop cases by taking $n = 1$ and $n = 0$, respectively. Multiplicity greater than 3 is not allowed in our classification. The fundamental cycles of multiple edge cases are as follows:



$1 = \underline{1} — 1 \cdots 1$, $\underline{2} = 3 — 2$.

□

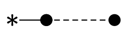
§ 5. The fundamental cycle of a weighted dual graph

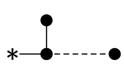
In § 4, we have obtained all weighted dual graphs consisting of −2-curves and exactly one −3-curve. It is interesting to know the classes to which the corresponding singularities belong to. A connected weighted dual graph Γ is called maximal if $\Gamma = \Gamma'$ for any other connected weighted dual graph $\Gamma' \supset \Gamma$. In this section, we compute the fundamental cycle of maximal graphs. For example, we only show the case of $n = 11$ for D'_n. By computing fundamental cycles, we show that some singularities are not weakly elliptic.

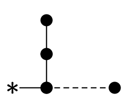
For $m = 0$, the weighted dual graph has just one point which is the −3-curve $*$; it is easy to see that $\chi(Z) = 1$, hence it is a rational singularity.

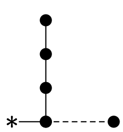
Theorem 5.1. *The fundamental cycles of* $m = 1$ *weighted dual graphs are as follows*:

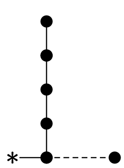
(1) $k - A_n$: $k = 0, 1, 2$, *for arbitrary* $n \geqslant 1$; *or* $k = 3$, $7 \leqslant n \leqslant 14$; *or* $k = 4$, $9 \leqslant n \leqslant 11$:

1.1. A_n:  $Z = \underline{1}\ 1\ \ldots\ 1$, $\chi(Z) = 1$;

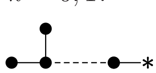
1.2. $1 - A_n$: 
1
$Z = \underline{1}\ 2\ 2\ \ldots\ 2\ 1$, $\chi(Z) = 1$;

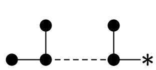
1.3. $2 - A_n$: 
1
2
$Z = \underline{1}\ 3\ 3\ \ldots\ 3\ 2\ 1$, $\chi(Z) = 1$;

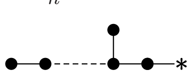
1.4. $3 - A_{14}$: 
3
6
9
$Z = \underline{4}\ 12\ 11\ \ldots\ 4\ 3\ 2$, $\chi(Z) = -1$;

1.5. $4 - A_{11}$: 
3
6
9
12
$Z = \underline{5}\ 15\ 13\ \ldots\ 9\ 7\ 5\ 3$, $\chi(Z) = -1$;

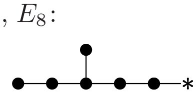
(2) $k - D_n$, $k = 0, 1$:

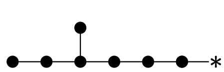
2.1. D_n: 
1
$Z = 1\ 2\ \ldots\ 2\ \underline{1}$, $\chi(Z) = 1$;

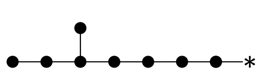
2.2. $1 - D_n$: 
$1 \quad 1$
$Z = 1\ 2\ \ldots 2\ \underline{1}$, $\chi(Z) = 0$;

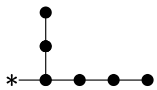
(3) D'_{11}: 
5
$Z = 2\ 3\ 4\ \ldots\ 10\ 6\ \underline{2}$, $\chi(Z) = 0$;

(4) E_6, E_7, E_8:

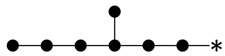
4.1. E_6: 
2
$Z = 2\ 3\ 4\ 3\ 2\ \underline{1}$, $\chi(Z) = 1$;

4.2. E_7: 
3
$Z = 2\ 4\ 6\ 5\ 4\ 3\ \underline{1}$, $\chi(Z) = 1$;

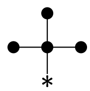
4.3. E_8: 
3
$Z = 2\ 4\ 6\ 5\ 4\ 3\ 2\ \underline{1}$, $\chi(Z) = 0$;

(5) E_6': 

$$\begin{matrix} & & 1 & & \\ & & 2 & & \\ Z = \underline{1} & 2 & 3 & 2 & 1 \end{matrix}, \ \chi(Z) = 0;$$

(6) E_7': 

$$\begin{matrix} & & & 2 & & & \\ Z = 1 & 2 & 3 & 4 & 3 & 2 & \underline{1} \end{matrix}, \ \chi(Z) = 0;$$

(7) D_4'': 

$$\begin{matrix} & 1 & \\ Z = 1 & 2 & 1 \\ & \underline{1} & \end{matrix}, \ \chi(Z) = 0.$$

Proof. The computation is by Definition 2.1. Here, $\chi(Z)$ can be evaluated using the Riemann–Roch theorem. □

In above cases, $3 - A_{14}$ and $4 - A_{11}$ are not rational or elliptic. Cases 2.2, (3), 4.3, (5), (6), and (7) are minimal elliptic. The others are rational.

Definition 5.2. Let (V, p) be a germ of weakly elliptic singularity. Let $\pi\colon M \to V$ be the minimal resolution with $\pi^{-1}(p) = A = \bigcup A_i$, $1 \leqslant i \leqslant n$, the irreducible decomposition of the exceptional set, and Z be the fundamental cycle. The set of effective cycles $\{A_{*1}, \dots, A_{*l}\}$ is the set $\{A_i\colon A_i \cdot Z < 0\}$.

Remark 5.3. A simple way to compute $\chi(Z)$ is by checking effective cycles. Let $c_1, \dots, c_l$ be the coefficient of effective cycles $A_{*1}, \dots, A_{*l}$ of Z. Let d be the coefficient of -3-cycle, where d may equal to some c_i. Let p_i be $-A_{*i} \cdot Z$, then

$$\chi(Z) = \frac{1}{2}\left(\sum_{i=1}^{l} p_i c_i - d\right).$$

If $l = 1$, $c_1 = d$, and $p_1 = 1$, then it is minimally elliptic.

For the sake of conciseness, we shall not draw all the dual graphs below, one remembers that a number on fundamental cycle represents a point in the dual graph. An underlined number represents the -3-cycle.

Theorem 5.4. *The fundamental cycles of $m = 2$ weighted dual graphs are as follows*:

(1) $k_1 - A_{n_1} + k_2 - A_{n_2}$:

1.1. $A_{n_1} + A_{n_2}$: $Z = 1 \ \dots \ \underline{1} \ \dots \ 1$, $\chi(Z) = 1$;

1.2. $1 - A_{n_1} + A_{n_2}$: $Z = \begin{matrix} & & & & 1 & & & \\ 1 & 2 & \dots & 2 & \underline{1} & 1 & \dots & 1 \end{matrix}$, $\chi(Z) = 1$;

1.3. $k_1 = 2, k_2 = 0$:

1.3.1. $2 - A_8 + A_{n_2}$: $Z = \begin{matrix} & & & & & & 2 & & & & & & \\ & & & & & & 4 & & & & & & \\ 1 & 2 & \dots & 4 & 5 & 6 & \underline{3} & 3 & \dots & 3 & 2 & 1 \end{matrix}$, $\chi(Z) = 0$;

1.3.2. $2 - A_9 + A_8$: $Z = \begin{matrix} & & & & & & 7 & & & & & & \\ & & & & & & 14 & & & & & & \\ 3 & 6 & \dots & 15 & 18 & 21 & \underline{10} & 9 & \dots & 4 & 3 & 2 \end{matrix}$, $\chi(Z) = -4$;

1.3.3. $2 - A_{10} + A_4$: $Z = \begin{matrix} & & & & & & 8 & & & & & \\ & & & & & & 16 & & & & & \\ 3 & 6 & \dots & 18 & 21 & 24 & \underline{11} & 9 & 7 & 5 & 3 \end{matrix}$, $\chi(Z) = -4$;

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 7 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 14 \\ 1.3.4.\ 2 - A_{12} + A_2\colon Z = 3\ 5\ \ldots\ 17\ 19\ 21\ \underline{9}\ 6\ 3,\ \chi(Z) = -3; \end{array}$$

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 5 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 10 \\ 1.3.5.\ 2 - A_{16} + A_1\colon Z = 2\ 3\ \ldots\ 13\ 14\ 15\ \underline{6}\ 3,\ \chi(Z) = -2; \end{array}$$

1.4. $k_1 = 3, k_2 = 0$:

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad\quad 1 \\ \qquad\qquad\qquad\qquad\qquad\quad 2 \\ \qquad\qquad\qquad\qquad\qquad\quad 3 \\ 1.4.1.\ 3 - A_7 + A_{n_2}\colon Z = 1\ 2\ 3\ 4\ \underline{2}\ 2\ \ldots\ 2\ 1,\ \chi(Z) = 0; \end{array}$$

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad\qquad 4 \\ \qquad\qquad\qquad\qquad\qquad\qquad 8 \\ \qquad\qquad\qquad\qquad\qquad\qquad 12 \\ 1.4.2.\ 3 - A_8 + A_2\colon Z = 4\ 7\ 10\ 13\ 16\ \underline{7}\ 5\ 3,\ \chi(Z) = 0; \end{array}$$

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad\qquad 3 \\ \qquad\qquad\qquad\qquad\qquad\qquad 6 \\ \qquad\qquad\qquad\qquad\qquad\qquad 9 \\ 1.4.3.\ 3 - A_9 + A_1\colon Z = 2\ 4\ 6\ 8\ 10\ 12\ \underline{5}\ 3,\ \chi(Z) = -1; \end{array}$$

1.5. $k_1 = k_2 = 1$:

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 11 \quad 10 \\ 1.5.1.\ 1 - A_8 + 1 - A_6\colon Z = 4\ 7\ \ldots\ 16\ 19\ 22\ \underline{14}\ 20\ 16\ 12\ 8\ 4,\ \chi(Z) = -5; \end{array}$$

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 5 \quad 4 \\ 1.5.2.\ 1 - A_{10} + 1 - A_5\colon Z = 2\ 3\ \ldots\ 8\ 9\ 10\ \underline{6}\ 8\ 6\ 4\ 2,\ \chi(Z) = -2; \end{array}$$

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 9 \quad 6 \\ 1.5.3.\ 1 - A_{18} + 1 - A_4\colon Z = 2\ 3\ \ldots\ 16\ 17\ 18\ \underline{10}\ 12\ 8\ 4,\ \chi(Z) = -4; \end{array}$$

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 2 \quad 1 \\ 1.5.4.\ 1 - A_{n_1} + 1 - A_3\colon Z = 1\ 2\ 3\ 4\ \ldots\ 4\ 4\ \underline{2}\ 2\ 1,\ \chi(Z) = 1; \end{array}$$

1.6. $k_1 = 2, k_2 = 1$:

$$\begin{array}{c} 2 \\ 4 \quad 3 \end{array}$$

1.6.1. $2 - A_5 + 1 - A_6$: $Z = 2\ 4\ 6\ \underline{4}\ 6\ 5\ 4\ 3\ 2$, $\chi(Z) = -1$;

$$\begin{array}{c} 3 \\ 6 \quad 3 \end{array}$$

1.6.2. $2 - A_6 + 1 - A_4$: $Z = 3\ 5\ 7\ 9\ \underline{5}\ 6\ 4\ 2$, $\chi(Z) = -1$;

$$\begin{array}{c} 4 \\ 8 \quad 3 \end{array}$$

1.6.3. $2 - A_7 + 1 - A_3$: $Z = 3\ 6\ 8\ 10\ 12\ \underline{6}\ 6\ 3$, $\chi(Z) = 0$;

(2) $k_1 - D_{n_1} + k_2 - A_{n_2}$:

1

2.1. $D_{n_1} + A_{n_2}$: $Z = 1\ 2 \ldots\ 2\ \underline{1}\ \ldots\ 1$, $\chi(Z) = 1$;

$1 \quad 1$

2.2. $1 - D_{n_1} + A_{n_2}$: $Z = 1\ 2 \ldots 2\ \underline{1}\ \ldots\ 1$, $\chi(Z) = 0$;

$1 \quad 2$

2.3. $D_{n_1} + 1 - A_{n_2}$: $Z = 1\ 2 \ldots\ 2\ \underline{2}\ 4\ 4\ \ldots\ 4\ 3\ 2\ 1$, $\chi(Z) = 1$;

4

$3 \quad 8$

2.4. $D_{n_1} + 2 - A_{n_7}$: $Z = 3\ 6 \ldots\ 6\ \underline{6}\ 12\ 10\ 8\ 6\ 3$, $\chi(Z) = 0$;

$1 \quad 1$

(3) $D_{n_1} + D_{n_2}$: $Z = 1\ 2 \ldots\ 2\ \underline{2}\ 2\ \ldots 2\ 1$, $\chi(Z) = 1$;

(4) $D'_{n_1} + A_{n_2}$:

3

4.1. $D'_8 + A_{n_2}$: $Z = 1\ 2\ \ldots\ 2\ \underline{2}\ 4\ 6\ 5\ 4\ 3\ 2\ 1$, $\chi(Z) = 0$;

7

4.2. $D'_9 + A_2$: $Z = 2\ 3\ \underline{4}\ 9\ 14\ 12\ \ldots\ 4\ 2$, $\chi(Z) = -1$;

(5)

2

5.1. $E_6 + A_{n_2}$: $Z = 2\ 3\ 4\ 3\ 2\ \underline{1}\ \ldots\ 1$, $\chi(Z) = 1$;

3

5.2. $E_7 + A_{n_2}$: $Z = 2\ 4\ 6\ 5\ 4\ 3\ \underline{2}\ 2\ \ldots\ 2\ 1$, $\chi(Z) = 1$;

3

5.3. $E_8 + A_{n_2}$: $Z = 2\ 4\ 6\ 5\ 4\ 3\ 2\ \underline{1}\ \ldots\ 1$, $\chi(Z) = 0$;

$6 \quad 5$

(6) $E_6 + 1 - A_{10}$: $Z = 4\ 8\ 12\ 10\ 8\ \underline{6}\ 10\ 9\ \ldots\ 4\ 3\ 2$, $\chi(Z) = -2$;

1

2

(7) $E'_6 + A_{n_2}$: $Z = 1\ 2\ 3\ 2\ \underline{1}\ \ldots\ 1$, $\chi(Z) = 0$;

2

(8) $E'_7 + A_{n_2}$: $Z = 1\ 2\ 3\ 4\ 3\ 2\ \underline{1}\ \ldots\ 1$, $\chi(Z) = 0$;

(9) $D_4'' + A_{n_2}$: $Z = 1\ \overset{1}{\underset{1}{2}}\ \underline{1}\ \ldots\ 1$, $\chi(Z) = 0$;

(10)

10.1. $E_6 + D_{n_2}$: $Z = 2\ 4\ \overset{3}{6}\ 5\ 4\ \underline{2}\ 2\ \ldots\ \overset{1}{2}\ 1$, $\chi(Z) = 1$;

10.2. $E_7 + D_{n_2}$: $Z = 2\ 4\ \overset{3}{6}\ 5\ 4\ 3\ \underline{2}\ 2\ \ldots\ \overset{1}{2}\ 1$, $\chi(Z) = 0$;

(11) $D_7' + D_{n_2}$: $Z = 2\ 3\ 4\ 5\ \overset{3}{6}\ 4\ \underline{2}\ 2\ \ldots\ \overset{1}{2}\ 1$, $\chi(Z) = 0$;

(12) $E_7 + E_6$: $Z = 4\ 8\ \overset{6}{12}\ 10\ 8\ 6\ \underline{4}\ 6\ 8\ \overset{5}{10}\ 7\ 4$, $\chi(Z) = 0$;

(13)

13.1. $E_6 + D_6'$: $Z = 4\ 7\ \overset{5}{10}\ 8\ 6\ \underline{4}\ 6\ \overset{4}{8}\ 6\ 4\ 2$, $\chi(Z) = 0$;

13.2. $E_7 + D_5'$: $Z = 2\ 4\ \overset{3}{6}\ 5\ 4\ 3\ \underline{2}\ 3\ \overset{2}{4}\ 3\ 2$, $\chi(Z) = 0$.

Theorem 5.5. *The fundamental cycles of $m = 3$ weighted dual graphs are as follows*:

(1) $k_1 - A_{n_1} + k_2 - A_{n_2} + k_3 - A_{n_3}$: $k_1 - A_{n_1} - \begin{array}{c} k_3 - A_{n_3} \\ | \\ * \end{array} - k_2 - A_{n_2}$:

1.1. $k_1 = k_2 = k_3 = 0$, *the coefficients of all cycles are* 1, *and* $\chi(Z) = 1$;

1.2. $k_1 = 1, k_2 = k_3 = 0$;

1.2.1. $n_1 = 3$: $Z = 1\ \overset{1}{2} - \begin{array}{c} 1 \\ 2 \\ \vdots \\ 2 \\ \underline{2} \end{array}\ 2\ \ldots\ 2\ 1$, $\chi(Z) = 1$;

1.2.2. $n_1 = 4$:

1.2.2.1. $1 - A_4 + A_{10} + A_8$: $Z = 18\ 36\ \overset{27}{54} - \begin{array}{c} 5 \\ 10 \\ \vdots \\ 40 \\ \underline{45} \end{array}\ 41\ \ldots\ 9\ 5$, $\chi(Z) = -20$;

1.2.2.2. $1 - A_4 + A_{12} + A_7$: $Z = 16\ 32\ \overset{24}{28} - \begin{array}{c} 5 \\ 10 \\ \vdots \\ 35 \\ \underline{40} \end{array}\ 37\ \ldots\ 7\ 4$, $\chi(Z) = -18$;

1.2.2.3. $1-A_4+A_{16}+A_6$: $Z = \begin{array}{ccccccccc} & & & & 5 & & & & \\ & & & & 10 & & & & \\ & & & & \vdots & & & & \\ & & 21 & & 30 & & & & \\ 14 & 28 & 42 & - & \underline{35} & 33 & \ldots & 5 & 3 \end{array}$, $\chi(Z) = -16$;

1.2.2.4. $1-A_4+A_{28}+A_5$: $Z = \begin{array}{ccccccccc} & & & & 5 & & & & \\ & & & & 10 & & & & \\ & & & & \vdots & & & & \\ & & 18 & & 25 & & & & \\ 12 & 24 & 36 & - & \underline{30} & 29 & \ldots & 3 & 2 \end{array}$, $\chi(Z) = -14$;

1.2.2.5. $1-A_4+A_{n_2}+A_4$: $Z = \begin{array}{ccccccccccc} & & & & 1 & & & & & & \\ & & & & 2 & & & & & & \\ & & & & 3 & & & & & & \\ & & 3 & & 4 & & & & & & \\ 2 & 4 & 6 & - & \underline{5} & 5 & \ldots & 5 & 4 & 3 & 2 \; 1 \end{array}$, $\chi(Z) = 0$;

1.2.3. $n_1 = 5$:

1.2.3.1. $1-A_5+A_6+A_4$: $Z = \begin{array}{cccccccccc} & & & & & 3 & & & & \\ & & & & & 6 & & & & \\ & & & & & 9 & & & & \\ & & & 10 & & 12 & & & & \\ 5 & 10 & 15 & 20 & - & \underline{15} & 13 & \ldots & 5 & 3 \end{array}$, $\chi(Z) = -6$;

1.2.3.2. $1-A_5+A_{10}+A_3$: $Z = \begin{array}{cccccccccc} & & & & & 3 & & & & \\ & & & & & 6 & & & & \\ & & & 8 & & 9 & & & & \\ 4 & 8 & 12 & 16 & - & \underline{12} & 11 & \ldots & 3 & 2 \end{array}$, $\chi(Z) = -5$;

1.2.3.3. $1-A_5+A_{n_2}+A_2$: $Z = \begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & 2 & & 2 & & & & & \\ 1 & 2 & 3 & 4 & - & \underline{3} & 3 & \ldots & 3 & 2 & 1 \end{array}$, $\chi(Z) = 0$;

1.2.4. $n_1 = 6$:

1.2.4.1. $1-A_6+A_4+A_3$: $Z = \begin{array}{ccccccccccc} & & & & & & 6 & & & & \\ & & & & & & 11 & & & & \\ & & & & 15 & & 16 & & & & \\ 6 & 12 & 18 & 24 & 30 & - & \underline{21} & 17 & 13 & 9 & 5 \end{array}$, $\chi(Z) = -8$;

1.2.4.2. $1-A_6+A_9+A_2$: $Z = \begin{array}{ccccccccccc} & & & & & & 7 & & & & \\ & & & & 15 & & 14 & & & & \\ 6 & 12 & 18 & 24 & 30 & - & \underline{21} & 19 & \ldots & 5 & 3 \end{array}$, $\chi(Z) = -9$;

1.2.4.3. $1-A_6+A_{n_2}+A_1$: $Z = \begin{array}{cccccccccccc} & & & & 3 & & 2 & & & & & \\ 2 & 3 & 4 & 5 & 6 & - & \underline{4} & 4 & \ldots & 4 & 3 & 2 \; 1 \end{array}$, $\chi(Z) = 1$;

1.2.5. $n_1 = 7$:

4

9 8

1.2.5.1. $1 - A_7 + A_4 + A_2$: $Z = 3\ 6\ 9\ 12\ 15\ 18 - \underline{12}\ 10\ 8\ 6\ 3$, $\chi(Z) = -3$;

3 2

1.2.5.2. $1 - A_7 + A_{n_2} + A_1$: $Z = 1\ 2\ 3\ 4\ 5\ 6 - \underline{4}\ 4\ \dots\ 4\ 3\ 2\ 1$, $\chi(Z) = 0$;

1.2.6. $n_1 = 8$:

3

7 6

1.2.6.1. $1 - A_8 + A_3 + A_2$: $Z = 2\ 4\ 6\ 8\ 10\ 12\ 14 - \underline{9}\ 7\ 5\ 3$, $\chi(Z) = -3$;

14 9

1.2.6.2. $1 - A_8 + A_{16} + A_1$: $Z = 4\ 8\ \dots\ 20\ 24\ 28 - \underline{18}\ 17\ \dots\ 4\ 3\ 2$, $\chi(Z) = -8$;

8 5

1.2.7. $1 - A_9 + A_8 + A_1$: $Z = 2\ 4\ \dots\ 14\ 16 - \underline{10}\ 9\ \dots\ 3\ 2$, $\chi(Z) = -4$;

1.2.8. $n_1 = 10$:

2

5 4

1.2.8.1. $1 - A_{10} + A_2 + A_2$: $Z = 2\ 3\ \dots\ 9\ 10 - \underline{6}\ 4\ 2$, $\chi(Z) = -2$;

18 11

1.2.8.2. $1 - A_{10} + A_6 + A_1$: $Z = 4\ 8\ \dots\ 32\ 36 - \underline{22}\ 19\ \dots\ 7\ 4$, $\chi(Z) = -9$;

17 10

1.2.9. $1 - A_{12} + A_4 + A_1$: $Z = 4\ 7\ \dots\ 31\ 34 - \underline{20}\ 16\ 12\ 8\ 4$, $\chi(Z) = -8$;

7 4

1.2.10. $1 - A_{14} + A_3 + A_1$: $Z = 2\ 3\ \dots\ 13\ 14 - \underline{8}\ 6\ 4\ 2$, $\chi(Z) = -3$;

11 6

1.2.11. $1 - A_{22} + A_2 + A_1$: $Z = 2\ 3\ \dots\ 21\ 22 - \underline{12}\ 8\ 4$, $\chi(Z) = -5$;

2 1

1.2.12. $1 - A_{n_1} + A_1 + A_1$: $Z = 1\ 2\ 3\ 4\ \dots\ 4\ 4 - \underline{2}\ 1$, $\chi(Z) = 1$;

1.3. $k_1 = 2$, $k_2 = k_3 = 0$:

1.3.1. $n_1 = 5$:

3 2

6 4

1.3.1.1. $2 - A_5 + A_4 + A_2$: $Z = 3\ 6\ 9 - \underline{6}\ 5\ 4\ 3\ 2$, $\chi(Z) = -2$;

1

2 1

1.3.1.2. $2 - A_5 + A_{n_2} + A_1$: $Z = 1\ 2\ 3 - \underline{2}\ 2\ \dots\ 2\ 1$, $\chi(Z) = 0$;

4

8 4

1.3.2. $2 - A_6 + A_2 + A_1$: $Z = 3\ 6\ 9\ 12 - \underline{7}\ 5\ 3$, $\chi(Z) = 0$;

4

8 3

1.3.3. $2 - A_7 + A_1 + A_1$: $Z = 3\ 6\ 8\ 10\ 12 - \underline{6}\ 3$, $\chi(Z) = 0$;

1.4. $k_1 = k_2 = 1$, $k_3 = 0$:

1.4.1. $1 - A_3 + 1 - A_3 + A_{n_3}$: $$Z = \begin{array}{ccccccc} & & & & 1 & & \\ & & & & 2 & & \\ & 1 & & & \vdots & & 1 \\ 1 & 2 & - & & \underline{2} & - & 2\ 1 \end{array},\ \chi(Z) = 0;$$

1.4.2. $n_1 = 4$:

1.4.2.1. $1 - A_4 + 1 - A_3 + A_3$: $$Z = \begin{array}{ccccccc} & & & & 2 & & \\ & & & & 4 & & \\ & & 5 & & 6 & & 4 \\ 4 & 7 & 10 & - & \underline{8} & - & 8\ 4 \end{array},\ \chi(Z) = -2;$$

1.4.2.2. $1 - A_4 + 1 - A_4 + A_1$: $$Z = \begin{array}{ccccccc} & & 3 & & 3 & & 3 \\ 2 & 4 & 6 & - & \underline{5} & - & 6\ 4\ 2 \end{array},\ \chi(Z) = -1;$$

1.4.3. $1 - A_6 + 1 - A_3 + A_1$: $$Z = \begin{array}{ccccccc} & & 3 & & 2 & & 2 \\ 2\ 3\ 4\ 5 & & 6 & - & \underline{4} & - & 4\ 2 \end{array},\ \chi(Z) = -1;$$

(2) $D_{n_1} + k_2 - A_{n_2} + k_3 - A_{n_3}$: $$D_{n_1} - \begin{array}{c} k_3 - A_{n_3} \\ | \\ * \end{array} - k_2 - A_{n_2}:$$

2.1. $D_{n_1} + A_{n_2} + A_{n_3}$: $$Z = \begin{array}{ccccccc} & & & & 1 & & \\ & & & & 2 & & \\ & 1 & & & \vdots & & \\ 1 & 2 \ldots & 2 & - & \underline{2} & - & 2\ \ldots\ 2\ 1 \end{array},\ \chi(Z) = 1;$$

2.2. $D_{n_1} + 1 - A_3 + A_{n_3}$: $$Z = \begin{array}{ccccccc} & & & & 1 & & \\ & & & & 2 & & \\ & 1 & & & \vdots & & 1 \\ 1 & 2 \ldots & 2 & - & \underline{2} & - & 2\ 1 \end{array},\ \chi(Z) = 0;$$

2.3. $D_{n_1} + 1 - A_4 + A_3$: $$Z = \begin{array}{ccccccc} & & & & 2 & & \\ & & & & 4 & & \\ & 4 & & & 6 & & 5 \\ 4 & 8 \ldots & 8 & - & \underline{8} & - & 10\ 7\ 4 \end{array},\ \chi(Z) = -2;$$

2.4. $D_{n_1} + 1 - A_6 + A_1$: $$Z = \begin{array}{ccccccc} & 2 & & & 2 & & 3 \\ 2 & 4 \ldots & 4 & - & \underline{4} & - & 6\ 5\ 4\ 3\ 2 \end{array},\ \chi(Z) = -1;$$

(3) $E_6 + 1 - A_3 + A_1$: $$Z = \begin{array}{ccccccc} & 6 & & & 3 & & 3 \\ 4\ 8 & 12 & 10\ 8 & - & \underline{6} & - & 6\ 3 \end{array},\ \chi(Z) = 0;$$

(4) $E_6 + A_{n_2} + A_{n_3}$: $$E_6 - \begin{array}{c} A_{n_3} \\ | \\ * \end{array} - A_{n_2}:$$

4.1. $E_6 + A_{n_2} + A_2$: $$Z = \begin{array}{ccccccc} & & & & 1 & & \\ & 3 & & & 2 & & \\ 2\ 4 & 6 & 5\ 4 & - & \underline{3} & - & 3\ \ldots\ 3\ 2\ 1 \end{array},\ \chi(Z) = 0;$$

4.2. $E_6 + A_{10} + A_3$: $Z = 8\ 16\ \overset{12}{24}\ 20\ 16 - \overset{\substack{3\\6\\9}}{\underline{12}} - 11\ \ldots\ 3\ 2$, $\chi(Z) = -5$;

4.3. $E_6 + A_5 + A_4$: $Z = 4\ 8\ \overset{6}{12}\ 10\ 8 - \overset{\substack{2\\3\\4\\5}}{\underline{6}} - 5\ 4\ 3\ 2\ 1$, $\chi(Z) = -2$;

(5) $E_7 + A_{n_2} + A_{n_3}$: $E_7 - \overset{\substack{A_{n_3}\\|}}{*} - A_{n_2}$:

5.1. $E_7 + A_{n_2} + A_1$: $Z = 2\ 4\ \overset{3}{6}\ 5\ 4\ 3 - \overset{1}{\underline{2}} - 2\ \ldots\ 2\ 1$, $\chi(Z) = 0$;

5.2. $E_7 + A_3 + A_2$: $Z = 4\ 8\ \overset{6}{12}\ 10\ 8\ 6 - \overset{\substack{2\\3}}{\underline{4}} - 3\ 2\ 1$, $\chi(Z) = -1$;

(6) $D_{n_1} + D_{n_2} + A_{n_3}$: $D_{n_1} - \overset{\substack{A_{n_3}\\|}}{*} - D_{n_2}$: $Z = 1\ \overset{1}{2}\ \ldots\ 2 - \overset{\substack{1\\2\\\vdots}}{\underline{2}} - 2\ \ldots\ \overset{1}{2}\ 1$, $\chi(Z) = 0$;

(7) $E_6 + D_{n_1} + A_1$: $E_6 - \overset{\substack{A_1\\|}}{*} - D_{n_1}$: $Z = 4\ 7\ \overset{5}{10}\ 8\ 6 - \overset{2}{\underline{4}} - 4\ \ldots \overset{2}{4}\ 2$, $\chi(Z) = 0$.

Remark 5.6. One can see that $1 - A_3$ is the limiting case of D_n, as their coefficients are the same. (For example, this is so in 1.2.1 and 2.1, in 1.4.1, 2.2 and (6).)

Theorem 5.7. *The fundamental cycles of $m = 4$ weighted dual graphs are as follows*:

(1) $A_{n_1} + A_{n_2} + A_{n_3} + A_{n_4}$: $A_{n_1} - \underset{\substack{|\\A_{n_4}}}{\overset{\substack{A_{n_3}\\|}}{*}} - A_{n_2}$:

1.1. $n_4 = 1$:

1.1.1. $A_{n_1} + A_{n_2} + A_1 + A_1$: $Z = 1\ 2\ \ldots\ 2 - \underset{1}{\overset{1}{\underline{2}}} - 2\ \ldots\ 2\ 1$, $\chi(Z) = 1$;

1.1.2. $A_{40} + A_6 + A_2 + A_1$: $Z = 2\ 3\ \ldots\ 40\ 41 - \underset{21}{\overset{\substack{14\\28}}{\underline{42}}} - 36\ \ldots\ 12\ 6$,

$\chi(Z) = -20$;

1.1.3. $A_{22} + A_7 + A_2 + A_1$: $Z = 2\ 3\ \ldots\ 22\ 23 - \begin{array}{c} 8 \\ 16 \\ \underline{24} \\ 12 \end{array} - 21\ \ldots\ 6\ 3,$

$\chi(Z) = -11;$

1.1.4. $A_{16} + A_8 + A_2 + A_1$: $Z = 2\ 3\ \ldots\ 17 - \begin{array}{c} 6 \\ 12 \\ \underline{18} \\ 9 \end{array} - 16\ \ldots\ 4\ 2,\ \chi(Z) = -8;$

1.1.5. $A_{13} + A_9 + A_2 + A_1$: $Z = 3\ 6\ 8\ 10\ \ldots\ 28 - \begin{array}{c} 10 \\ 20 \\ \underline{30} \\ 15 \end{array} - 27\ \ldots\ 6\ 3,$

$\chi(Z) = -12;$

1.1.6. $A_{12} + A_{10} + A_2 + A_1$: $Z = 6\ 11\ 16\ \ldots\ 61 - \begin{array}{c} 22 \\ 44 \\ \underline{66} \\ 33 \end{array} - 60\ \ldots\ 12\ 6,$

$\chi(Z) = -30;$

1.1.7. $A_{n_1} + A_3 + A_3 + A_1$: $Z = 1\ 2\ 3\ 4\ \ldots\ 4 - \begin{array}{c} 1 \\ 2 \\ 3 \\ \underline{4} \\ 2 \end{array} - 3\ 2\ 1,\ \chi(Z) = 0;$

1.1.8. $A_{18} + A_4 + A_3 + A_1$: $Z = 2\ 3\ \ldots\ 19 - \begin{array}{c} 5 \\ 10 \\ 15 \\ \underline{20} \\ 10 \end{array} - 16\ 12\ 8\ 4,\ \chi(Z) = -9;$

1.1.9. $A_{10} + A_5 + A_3 + A_1$: $Z = 2\ 3\ \ldots\ 11 - \begin{array}{c} 3 \\ 6 \\ 9 \\ \underline{12} \\ 6 \end{array} - 10\ \ldots\ 4\ 2,\ \chi(Z) = -5;$

1.1.10. $A_8 + A_6 + A_3 + A_1$: $Z = 4\ 7\ \ldots\ 25 - \begin{array}{c} 7 \\ 14 \\ 21 \\ \underline{28} \\ 14 \end{array} - 24\ \ldots\ 8\ 4,\ \chi(Z) = -12;$

1.1.11. $A_8 + A_4 + A_4 + A_1$: $Z = 2\ 3\ \ldots\ 9 - \begin{array}{c} 2 \\ 4 \\ 6 \\ 8 \\ \underline{10} \\ 5 \end{array} - 8\ 6\ 4\ 2$, $\chi(Z) = -4$;

1.1.12. $A_6 + A_5 + A_4 + A_1$: $Z = 5\ 10\ 14\ 18\ 22\ 26 - \begin{array}{c} 6 \\ \vdots \\ 24 \\ \underline{30} \\ 15 \end{array} - 25\ \ldots\ 10\ 5$, $\chi(Z) = -10$;

1.2. $n_4 = 2$:

1.2.1. $A_{n_1} + A_2 + A_2 + A_2$: $Z = 1\ 2\ 3\ \ldots\ 3 - \begin{array}{c} 1 \\ 2 \\ \underline{3} \\ 2 \\ 1 \end{array} - 2\ 1$, $\chi(Z) = 0$;

1.2.2. $A_{10} + A_3 + A_2 + A_2$: $Z = 2\ 3\ 4\ \ldots\ 11 - \begin{array}{c} 3 \\ 6 \\ 9 \\ \underline{12} \\ 8 \\ 4 \end{array} - 8\ 4$, $\chi(Z) = -5$;

1.2.3. $A_6 + A_4 + A_2 + A_2$: $Z = 3\ 5\ 7\ \ldots\ 13 - \begin{array}{c} 3 \\ 6 \\ 9 \\ 12 \\ \underline{15} \\ 10 \\ 5 \end{array} - 10\ 5$, $\chi(Z) = -6$;

1.2.4. $A_4 + A_3 + A_3 + A_2$: $Z = 3\ 6\ 8\ 10 - \begin{array}{c} 3 \\ 6 \\ 9 \\ \underline{12} \\ 8 \\ 4 \end{array} - 9\ 6\ 3$, $\chi(Z) = -3$;

(2) 2.1. $1 - A_3 + A_{n_1} + A_1 + A_1$: $Z = 1\ \begin{array}{c} 1 \\ 2 \end{array} - \begin{array}{c} 1 \\ \underline{2} \\ 1 \end{array} - 2\ \ldots\ 2\ 1$, $\chi(Z) = 0$;

$$2.2.\ A_{n_1}+A_4+A_2+A_1\colon\ Z=\begin{array}{cccccc} & & & 2 & & \\ & 3 & & 4 & & \\ 3 & 6 & - & \underline{6} & - & 5\ 4\ 3\ 2 \\ & & & 3 & & \end{array},\ \chi(Z)=-2;$$

$$(3)\ 3.1.\ D_{n_1}+A_{n_2}+A_1+A_1\colon\ Z=\begin{array}{cccccccc} & 1 & & & 1 & & \\ 1 & 2\ \ldots & 2 & - & \underline{2} & - & 2\ \ldots\ 2\ 1 \\ & & & & 1 & & \end{array},\ \chi(Z)=0;$$

$$3.2.\ D_{n_1}+A_4+A_2+A_1\colon\ Z=\begin{array}{ccccccc} & & & & 2 & & \\ & 3 & & & 4 & & \\ 3 & 6\ \ldots & 6 & - & \underline{6} & - & 5\ 4\ 3\ 2 \\ & & & & 3 & & \end{array},\ \chi(Z)=-2.$$

Theorem 5.8. *The fundamental cycles of $m = 5$ weighted dual graphs are as follows*:

$$(1)\ A_{n_1}+A_{n_2}+A_1+A_1+A_1\colon\quad \begin{array}{ccc} A_1 & A_{n_2} & \\ \diagdown & | & \\ & * & -A_{n_1}\colon \\ \diagup & | & \\ A_1 & A_1 & \end{array}$$

$$1.1.\ n_2=1\colon\ Z=\begin{array}{cc} 1 & 1 \\ & \underline{2}\ \ 2\ \ \ldots 2\ 1 \\ 1 & 1 \end{array},\ \chi(Z)=0;$$

$$1.2.\ n_2=2,\ n_1=4\colon\ Z=\begin{array}{cc} & 2 \\ 3 & 4 \\ & \underline{6}\ \ 5\ 4\ 3\ 2 \\ 3 & 3 \end{array},\ \chi(Z)=-2.$$

Remark 5.9. It is easy to see that if the coefficient of -3-cycle grows larger, then the singularity may always be not elliptic.

§ 6. The arithmetic and geometric genera of star-shaped graphs

In this section, we shall illustrate the methods for computing the arithmetic and geometric genera of star-shaped singularities contained in our classifications. With abuse of notations, E_i denotes the exceptional curve, and $E = \bigcup E_i$ is the exceptional divisor.

Definition 6.1. A weighted dual graph is called star-shaped if it is a connected tree in which at most one vertex is connected to more than two other vertices.

6.1. Splice quotient singularity. One can refer to [25] for the definition of splice quotient singularity. In this section, we recall the method for computing the geometric genus of splice quotient singularity.

Let $\delta_v = (E-E_v)\cdot E_v$ be the number of irreducible components of E intersecting with E_v. A curve E_v is called an end (respectively, a node) if $\delta_v = 1$ (respectively, $\delta_v \geqslant 3$). Let $\mathcal{E}$ (respectively, $\mathcal{N}$) denotes the set of indices of ends (respectively, nodes). A connected component of $E - E_v$ is called a branch of E_v.

Let $(a_{vw}) = -I^{-1}$, where I denotes the intersection matrix $(E_v \cdot E_w)$. Then every a_{vw} is a positive rational number and $E_v^* = \sum_{w\in\mathcal{V}} a_{vw}E_w$. We define positive integers e_v, ℓ_{vw} and m_{vw} as follows:

$$\ell_{vw} = |\det I|a_{vw}, \qquad e_v = |\det I|/\gcd\{\ell_{vw} \mid w \in \mathcal{V}\}, \qquad m_{vw} = e_v a_{vw}.$$

Definition 6.2. An element of the semigroup $\sum_{w\in\mathcal{E}} \mathbb{Z}_{\geqslant 0}E_w^*$, where $\mathbb{Z}_{\geqslant 0}$ is the set of nonnegative integers, is called a monomial cycle. Let $\mathbb{C}[z] := \mathbb{C}[z_w;\, w \in \mathcal{E}]$ be the polynomial ring in $\#\mathcal{E}$ variables. With a monomial cycle $D = \sum_{w\in\mathcal{E}} \alpha_w E_w^*$ we associate the monomial $z(D) := \prod_{w\in\mathcal{E}} z_w^{\alpha_w} \in \mathbb{C}[z]$.

Definition 6.3 (monomial condition). We say that E (or its weighted dual graph) satisfies the monomial condition if, for any branch C of any node E_v, there exists a monomial cycle D such that $D - E_v^*$ is an effective integral cycle supported on C. In this case, $z(D)$ is called an admissible monomial belonging to the branch C.

Notation 6.4. For a star-shaped graph with r branches, we assume that n_i is the number of points on ith branch and $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_r$. We denote the central curve by E_1, and $E_2, \ldots, E_{n_1+1}$ are the curves on the first branch, with E_{n_1+1} being the end curve:

$$\begin{array}{ccc} & E_{n_1+n_2+n_3+1} & \\ & \vdots & \\ & E_{n_1+n_2+2} & \\ & | & \\ E_{n_1+1} - E_{n_1} - \cdots - E_2 - & E_1 & - E_{n_1+2} - \cdots - E_{n_1+n_2+1}. \\ & | & \\ & E_{n_1+n_2+n_3+2} & \\ & \vdots & \\ & E_{n_1+n_2+n_3+n_4} & \end{array}$$

Furthermore, we require $E_1^2 = -3$, $E_i^2 = -2$, for $i \neq 1$. We denote this graph by $\Gamma(n_1, \ldots, n_r)$.

Proposition 6.5 (see [26], § 8). *Any star-shaped graph satisfies the monomial condition.*

Theorem 6.6 (see [27], Theorem 3.8). *Let (V, p) be a normal two-dimensional singularity and $\psi\colon (\widetilde{V}, A) \to (V, p)$ be the minimal good resolution. Assume that the dual graph of A is star-shaped with central curve E. Then the arithmetic genus $p_a(V, p)$ of (V, p) is*

$$p_a(V,p) = \max_{1\leqq r}\left\{ r(g-1) - \left(\sum_{k=0}^{r-1} \deg(D^{(k)})\right) + 1 \right\},$$

where g is the genus of E_1, and $D^{(k)}$ is defined by

$$D^{(k)} = kD - \sum_i \left\{ \frac{ke_i}{d_i} \right\} P_i,$$

where D is any divisor such that $O_{E_1}(D)$ is the conormal sheaf of E_1, P_i is the point at which E_1 intersects with the ith branch.

Proposition 6.7. *Let the weight dual graph be* $\Gamma(n_1, n_2, n_3, n_4)$, *then* $\deg(D^{(k)}) = \sum_{i=1}^{4} \lfloor k/(n_i+1) \rfloor - k$.

Proof. By definition, on the one hand, $D^{(k)} = kD - \sum_i \lceil e_i/d_i \rceil P_i$, where D is any divisor such that $O_{E_1}(D)$ is the conormal sheaf of E_1. Thus, $\deg(D) = -E_1 \cdot E_1 = 3$. On the other hand,

$$\frac{d_i}{e_i} = 2 - \cfrac{1}{2 - \cfrac{1}{\ddots - \cfrac{1}{2}}} = [2, \ldots, 2] = \frac{n_i + 1}{n_i},$$

and so

$$\deg(D^{(k)}) = 3k - \sum_{i=1}^{4} \left\lceil \frac{n_i}{n_i+1} \right\rceil = \sum_{i=1}^{4} \left\lfloor \frac{k}{n_i+1} \right\rfloor - k.$$

□

Corollary 6.8.

$$p_a = 1 + \max_{r \geqslant 1} \left\{ -r - \left(\sum_{k=0}^{r-1} \sum_{i=1}^{4} \left\lfloor \frac{k}{n_i+1} \right\rfloor - k \right) \right\}.$$

Table 1 indicates the p_a of maximal graphs of Theorem 4.10.

Table 1. Arithmetic genera of star-shaped graphs

n_1	n_2	n_3	n_4	p_a	n_1	n_2	n_3	n_4	p_a
40	6	2	1	216	22	7	2	1	69
16	8	2	1	39	13	9	2	1	26
12	10	2	1	108	18	4	3	1	48
10	5	3	1	17	8	6	3	1	32
8	4	4	1	12	6	5	4	1	13
10	3	2	2	17	6	4	2	2	13
4	3	3	2	4					

Next, we consider the formula for the geometric genus of splice-qoutient singularity with star-shaped graph in our special case.

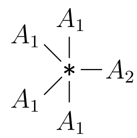
Lemma 6.9. *Let X be a splice-quotient singularity with E_1, which is the only node in the resolution graph. Let*

$$H(t) = \frac{1}{|\mathbb{H}|} \sum_{\lambda \in \Lambda} \prod_{w \in \mathcal{V}} \left(1 - \exp\left(2\pi\sqrt{-1}\, E_\lambda^* \cdot E_w^*\right) t^{m_{1w}}\right)^{\delta_w - 2}.$$

If we write $H(t) = p(t)/q(t) + r(t)$, *then* $p_g = r(1)$.

For a proof it suffices to use Proposition 3.8 in [25]. □

Example 6.10. Let the weighted dual graph be $\Gamma(2,1,1,1,1)$:



Then V is an elliptic singularity which is not minimally elliptic. Let I be the intersection matrix of exceptional curves, that is,

$$I = \begin{bmatrix} -3 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix},$$

and $|\det(I)| = 16 = |\mathbb{H}|$. Let us denote the central curve as E_1. The basis of $|\mathbb{H}|$ is given by E_1^*, E_4^*, E_5^*, E_6^*. By computing I^{-1}, we obtain

$$-I^{-1} = \begin{bmatrix} 3 & 2 & 1 & 3/2 & 3/2 & 3/2 & 3/2 \\ 2 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1/2 & 1/2 & 1/2 & 1/2 \\ 3/2 & 1 & 1/2 & 5/4 & 3/4 & 3/4 & 3/4 \\ 3/2 & 1 & 1/2 & 3/4 & 5/4 & 3/4 & 3/4 \\ 3/2 & 1 & 1/2 & 3/4 & 3/4 & 5/4 & 3/4 \\ 3/2 & 1 & 1/2 & 3/4 & 3/4 & 3/4 & 5/4 \end{bmatrix},$$

thus,

$$m_{1w} = (6,4,2,3,3,3,3).$$

Notice that $\Lambda = \{\lambda \in (\mathbb{Z}^4_{\geqslant 0}) | \lambda_i = 0 \text{ or } 1)\}$ and $E^*_\lambda = \lambda_1 E_1^* + \lambda_2 E_4^* + \lambda_3 E_5^* + \lambda_4 E_6^*$. The Hilbert polynomial with respect to E_1 is given by

$$H(t) = \frac{1}{|\mathbb{H}|} \sum_{\lambda \in \Lambda} \prod_{w=1}^{7} \left(1 - \exp\left(2\pi\sqrt{-1}\, E^*_\lambda \cdot E^*_w\right) t^{m_{1w}}\right)^{\delta_w - 2}.$$

We write $H(t) = p(t)/q(t) + r(t)$, thus $p_g = r(1) = 1$. However, Z_K is not an integral divisor, $Z_K \neq Z$, thus V is not minimally elliptic.

Remark 6.11. Nagy and Némethi [28] showed that, for a generic analytic structure on a given graph with a RHS (Rational Homology Sphere) link, the geometric genus equals to arithmetic genus.

The following theorem gives a computation formula for the graphs classified in Theorem 4.10.

Theorem 6.12. *Let X be a splice-quotient singularity with the resolution graph $\Gamma(n_1, n_2, n_3, n_4)$. All the exceptional curves are assumed to be rational and $E_1^2 = -3$, $E_i^2 = -2$ for all $i \neq 1$. Let*

$$S = \frac{1}{-1 + \sum_{i=1}^{4} \frac{1}{n_i+1}}.$$

Assume m is the smallest integer such that $mS/(n_j + 1)$ is integer, for all $j = 1, 2, 3, 4$. Let

$$\begin{aligned} H(t) &= \frac{1}{\prod_{i=1}^{4} m(n_i+1)} \\ &\times \sum_{k_i=0,\, i=1,2,3,4}^{mn_i} \frac{\left(1 - \exp\left(2\pi i\left(\sum_{i=1}^{4} \frac{k_i}{n_i+1}\right) \cdot S\right) t^{mS}\right)^2}{\prod_{j=1}^{4}\left(1 - \exp\left(2\pi i\left(\frac{S}{n_j+1} \sum_{i=1}^{4} \frac{k_i}{n_i+1} + \frac{n_j k_j}{n_j+1}\right) t^{mS/(n_j+1)}\right)\right)}. \end{aligned}$$

If we write $H(t) = p(t)/q(t) + r(t)$, then $p_g = r(1)$. Furthermore, X is numerical Gorenstein if and only if $S/(n_j + 1)$ is integral, for all $j = 1, 2, 3, 4$. (Notice that if X is numerical Gorenstein, then $m = 1$.)

Proof. We only detail the proof, when X is numerical Gorenstein. The general case is similar. Notice that $Z_K \cdot E_1 = -1$ and $Z_K \cdot E_i = 0$, for $i \neq 1$. Thus, $Z_K = E_1^*$. The first row of $-I^{-1}$ (that is, m_{1w}) is

$$\left(S, \frac{n_1 S}{n_1+1}, \ldots, \frac{S}{n_1+1}, \frac{n_1 S}{n_2+1}, \ldots, \frac{S}{n_2+1}, \frac{n_3 S}{n_3+1}, \ldots, \frac{S}{n_3+1}, \frac{n_4 S}{n_4+1}, \ldots, \frac{S}{n_4+1}\right).$$

Hence, numerical Gorenstein is equivalent to $S/(n_j+1)$ integral, for all $j = 1, 2, 3, 4$. The only node of star-shaped graph is central curve, thus by Lemma 6.9, it gives

$$H(t) = \frac{1}{|\mathbb{H}|} \sum_{\lambda \in \Lambda} \prod_{w \in \mathcal{V}} \left(1 - \exp\left(2\pi\sqrt{-1}\, E_\lambda^* \cdot E_w^*\right) t^{m_{1w}}\right)^{\delta_w - 2}.$$

To make $\delta_w - 2 \neq 0$, w must equal to 1, $n_1 + 1$, $n_1 + n_2 + 1$, $n_1 + n_2 + n_3 + 1$ or $n_1 + n_2 + n_3 + n_4 + 1$. We have

$$\begin{aligned} n_1 E_{n_1+1}^* &= E_1^*, & n_2 E_{n_1+n_2+1}^* &= E_1^*, \\ n_3 E_{n_1+n_2+n_3+1}^* &= E_1^*, & n_4 E_{n_1+n_2+n_3+n_4+1}^* &= E_1^*. \end{aligned}$$

Thus, we can take E_λ^* to be

$$\lambda_1 E_{n_1+1}^* + \lambda_2 E_{n_1+n_2+1}^* + \lambda_3 E_{n_1+n_2+n_3+1}^* + \lambda_4 E_{n_1+n_2+n_3+n_4+1}^*, \qquad 0 \leqslant \lambda_i \leqslant n_i.$$

And the average of $1/|\mathbb{H}|$ is replaced by $1/\prod(n_i+1)$. The rest part is to compute $E_\lambda^* \cdot E_w^*$, $w = 1$, $n_1 + 1$, $n_1 + n_2 + 1$, $n_1 + n_2 + n_3 + 1$, $n_1 + n_2 + n_3 + n_4 + 1$. This can be deduced from $-I^{-1}$ (for short, we write a, b, c, d in place of $n_1 + 1$, $n_2 + 1$, $n_3 + 1$, $n_4 + 1$, respectively):

$$\begin{bmatrix} S & \frac{n_1 S}{a} & \cdots & \frac{S}{a} & \frac{n_1 S}{b} & \cdots & \frac{S}{b} & \frac{n_3 S}{c} & \cdots & \frac{S}{c} & \frac{n_4 S}{d} & \cdots & \frac{S}{d} \\ \frac{n_1 S}{a} & \frac{\frac{n_1 S}{a}+1}{a} n_1 & \cdots & \frac{\frac{n_1 S}{a}+1}{a} & \frac{n_1 n_2 S}{ab} & \cdots & \frac{n_1 S}{ab} & \frac{n_1 n_3 S}{ac} & \cdots & \frac{n_1 S}{ac} & \frac{n_1 n_4 S}{ad} & \cdots & \frac{n_1 S}{ad} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{S}{a} & \frac{\frac{n_1 S}{a}+1}{a} & \cdots & \frac{\frac{n_1 S}{a}+n_1}{a} & \frac{n_2 S}{ab} & \cdots & \frac{S}{ab} & \frac{n_3 S}{ac} & \cdots & \frac{S}{ac} & \frac{n_4 S}{ad} & \cdots & \frac{S}{ad} \\ & & & & & \cdots & & & & & & & \end{bmatrix}.$$

We have

$$E^*_{n_1+1} \cdot E^*_1 = \frac{S}{a},$$
$$E^*_{n_1+1} \cdot E^*_{n_1+1} = \frac{n_1 S}{a^2} + \frac{n_1}{a},$$
$$E^*_{n_1+1} \cdot E^*_{n_1+n_2+1} = \frac{S}{ab},$$
$$E^*_{n_1+1} \cdot E^*_{n_1+n_2+n_3+1} = \frac{S}{ac},$$
$$E^*_{n_1+1} \cdot E^*_{n_1+n_2+n_3+n_4+1} = \frac{S}{ad}.$$

Exchanging the subscript n_1+1 by n_1+n_2+1, we find that

$$E^*_{n_1+n_2+1} \cdot E^*_1 = \frac{S}{b},$$
$$E^*_{n_1+n_2+1} \cdot E^*_{n_1+1} = \frac{S}{ba},$$
$$E^*_{n_1+n_2+1} \cdot E^*_{n_1+n_2+1} = \frac{n_2 S}{b^2} + \frac{n_2}{b},$$
$$E^*_{n_1+n_2+1} \cdot E^*_{n_1+n_2+n_3+1} = \frac{S}{bc},$$
$$E^*_{n_1+n_2+1} \cdot E^*_{n_1+n_2+n_3+n_4+1} = \frac{S}{bd}.$$

These computations can be done similarly for $n_1+n_2+n_3+1$ and $n_1+n_2+n_3+n_4+1$. The required conclusion now follows by taking above values in $H(t)$. ☐

Example 6.13. Let us consider X to be a splice-quotient singularity with the resolution graph $\Gamma(6,4,2,2)$. Then

$$S = \frac{1}{(-1+1/7+1/5+1/3+1/3) = 105}$$

and

$$H(t) = \frac{1}{3}\Bigg(2 \cdot \frac{(1-t^{105})^2}{(1-t^{15})(1-t^{21})(1-\exp(2\pi i \cdot 1/3)t^{35})(1-\exp(2\pi i \cdot 2/3)t^{35})} + \frac{(1-t^{105})^2}{(1-t^{15})(1-t^{21})(1-t^{35})^2}\Bigg).$$

Thus, $p_g = r(1) = \frac{1}{3}(2 \cdot 11 + 59) = 27$.

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