ON VARIATION OF COMPLEX STRUCTURES AND VARIATION OF NEW LIE ALGEBRAS ARISING FROM SINGULARITIES

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Abstract. Finite dimensional Lie algebras are semi-direct product of the semi-simple Lie algebras and solvable Lie algebras. Brieskorn gave the connection between simple Lie algebras and simple singularities. Simple Lie algebras have been well understood, but not the solvable (nilpotent) Lie algebras. It is extremely important to establish connections between singularities and solvable (nilpotent) Lie algebras. Recently, we have constructed a new natural connection between the set of complex analytic isolated hypersurface singularities and the set of finite dimensional solvable (nilpotent) Lie algebras, i.e., a new Lie algebra associated to an isolated hypersurface singularity has been constructed. The main purpose of this paper is to summarize the results that we have obtained on the new Lie algebras arising from singularities.

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1. Introduction

In this paper, we announce the recent results, obtained in [CHYZ], on the new Lie algebras arising from isolated hypersurface singularities.

Let G be a semi-simple Lie group acting on its Lie algebra $\mathcal{G}$ by the adjoint action and let $\mathcal{G}/G$ be the variety corresponding to the G-invariant polynomials on $\mathcal{G}$. The quotient morphism $\gamma : \mathcal{G} \to \mathcal{G}/G$ was intensively studied by Kostant ([Ko1], [Ko2]). Let $\mathcal{H} \subset \mathcal{G}$ be a Cartan subalgebra of $\mathcal{G}$ and W be the corresponding Weyl group.

(i) The space $\mathcal{G}/G$ may be identified with the set of semi-simple G classes in $\mathcal{G}$ such that γ maps an element $x \in \mathcal{G}$ to the class of its semi-simple part x_s. Thus $\gamma^{-1}(0) = N(\mathcal{G})$ is the nilpotent variety. An element $x \in N(\mathcal{G})$ is termed regular (resp., "subregular") if its centralizer has minimal dimension (resp., minimal dimension + 2).

(ii) By a theorem of Chevalley, the space $\mathcal{G}/G$ is isomorphic to $\mathcal{H}/W$, an affine space of dimension $r = \mathrm{rank}(\mathcal{G})$. The isomorphism is given by the map of a semi-simple class to its intersection with $\mathcal{H}$ (a W orbit).

The following beautiful theorem of Brieskorn [Br] conjectured by Grothendieck [Gr] establishes connections between the simple singularities and the simple Lie algebras.

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Theorem 1.1. ([Br]) *Let $\mathcal{G}$ be a simple Lie algebra over $\mathbb{C}$ of type A_r, D_r, E_r. Then*

(i) the intersection of the variety $N(\mathcal{G})$ of the nilpotent elements of $\mathcal{G}$ with a transverse slice S to the subregular orbit, which has codimension 2 in $N(\mathcal{G})$, is a surface $S \cap N(\mathcal{G})$ with an isolated rational double point of the type corresponding to the algebra $\mathcal{G}$.

(ii) the restriction of the quotient $\gamma : \mathcal{G} \to \mathcal{H}/W$ to the slice S is a realization of a semi-universal deformation of the singularity in $S \cap N(\mathcal{G})$.

The details of this Brieskorn's theory can be found in Slodowy's papers ([Sl1], [Sl2]).

Among many other things, Brieskorn's theory gives the way to construct rational double points from simple Lie algebras. It is known that finite dimensional Lie algebras are semi-direct product of the semi-simple Lie algebras and solvable Lie algebras. Brieskorn gave the connection between simple Lie algebras and simple singularities. Simple Lie algebras and semi-simple Lie algebras have been well understood, but not the solvable (nilpotent) Lie algebras. Historically, a marked difference is noted between the classification theory of semi-simple Lie algebras and the classification theories of solvable or nilpotent Lie algebras. The semi-simple theory can best be described as beautiful, while the others lack anything resembling elegance. For semi-simple Lie algebras over the complex numbers one has the Killing form, Dynkin diagrams, root space decompositions, the Serre presentation, the theory of highest weight representations, the Weyl character formula for finite-dimensional representations, and much more ([Hu], [Ja]). In the theory of solvable Lie algebras one has the theorems of Lie and Engel along with Malcev's reduction of the classification problem to the same problem for nilpotent algebras [Ma]. There does not seem to be any nice way to classify nilpotent Lie algebras (such as a graph or diagram for each algebra). Therefore, it is of great importance to establish connection between singularities and solvable (nilpotent) Lie algebras. In [CHYZ], a new natural connection between the set of complex analytic isolated hypersurface singularities and the set of finite dimensional solvable (nilpotent) Lie algebras has been constructed. These connections help people to understand the solvable (nilpotent) Lie algebras from the geometric point of view.

2. Yau algebra

For any isolated hypersurface singularity $(V, 0) \subset (\mathbb{C}^n, 0)$ where $V = V(f) = \{f = 0\}$, one can consider the moduli algebra $A(V) := \mathcal{O}_n/(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$, where $\mathcal{O}_n$ is the algebra of convergent power series in n indeterminates and $f \in \mathcal{O}_n$. In [MY], Mather and Yau proved that the complex structure of $(V, 0)$ determines and is determined by its moduli algebra.

Theorem 2.1. [MY] *The analytic isomorphism type of an isolated hypersurface singularity determine and is determined by the isomorphism class of its moduli algebra. i.e.,*

$$(V_1, 0) \cong (V_2, 0) \Longleftrightarrow A(V_1) \cong A(V_2).$$

Subsequently, motivated from Mather-Yau theorem, Yau [Ya2] introduced the Lie algebra to $(V, 0)$ as follows:

Let $V = \{f = 0\}$ be a germ of isolated hypersurface singularity at the origin of $\mathbb{C}^n$ defined by $f \in \mathbb{C}[x_1, x_2, ..., x_n]$ and $A(V)$ be the moduli algebra. We denote $L(V) := \mathrm{Der}_{\mathbb{C}}(A(V), A(V))$. Yu [Yu] call $L(V)$ the Yau algebra of V. Its dimension denoted as $\lambda(V)$ is called the Yau number by Elashvili and Khimshiashvili [EK].

He proved that $L(V)$ is solvable (cf. [Ya3]). Yau and his collaborators have systematically studied the Lie algebras of isolated hypersurface singularities since 1980s ([Ya1]-[Ya3], [BY], [SY], [YZ1, YZ2], [CYZ], [CCYZ], [HYZ1]-[HYZ3]). We shall denote as $\lambda(V)$ the dimension of $L(V)$. In [Yu], $L(V)$ is called Yau algebra while in [EK] $\lambda(V)$ is called Yau number.

3. New derivation Lie algebra

The following beautiful theorem of Dimca characterizes zero-dimensional isolated complete intersection singularities.

Theorem 3.1. *(Dimca* [Di]*) Two zero-dimensional isolated complete intersection singularities X and Y are isomorphic if and only if their singular subspaces Sing(X) and Sing(Y) are isomorphic.*

Remark 3.1. Let $V = V(f)$ be an isolated quasi-homogeneous hypersurface singularity. Assume that X defined by $(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$ is a zero-dimensional isolated complete intersection singularities. Then $\mathrm{Sing}(X)$ is defined by $(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}, \mathrm{Det}(\frac{\partial^2 f}{\partial x_i \partial x_j})_{i,j=1,\cdots,n})$.

Theorem 3.1 implies that in order to study analytic isomorphism type of zero dimensional isolated complete intersection singularity X, we only need to consider the Artinian local algebra $A^*(V)$ which is the coordinate ring of $\mathrm{Sing}(X)$. Thus $A^*(V)$ is defined as the quotient

$$\mathcal{O}_n/(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}, \mathrm{Det}(\frac{\partial^2 f}{\partial x_i \partial x_j})_{i,j=1,\cdots,n}).$$

Combining Theorem 3.1 with Mather-Yau theorem, we know that $A^*(V)$ is a complete invariant of quasi-homogeneous isolated hypersurface singularities (i.e., $A^*(V)$ determines and is determined by the analytic isomorphism type of the singularity). We call $A^*(V)$

the generalized moduli algebra of V. Based on this important observation, in [CHYZ], we introduce the following new invariants for isolated hypersurface singularities.

Definition 3.1. Let $V = \{f = 0\}$ be a germ of isolated hypersurface singularity at the origin of $\mathbb{C}^n$ defined by $f \in \mathbb{C}[x_1, x_2, ..., x_n]$. The new Lie algebra arising from the isolated hypersurface singularity V is defined as $L^*(V) := \mathrm{Der}(A^*(V), A^*(V))$ (or $\mathrm{Der}(A^*(V))$ for short). Its dimension is denoted as $\lambda^*(V)$.

It is natural to present the following question.

Question 3.1. What type of singularities such that $L^*(V)$ is a complete invariants. That is, if V_1, V_2 are two singularities of such type, then $L^*(V_1) \cong L^*(V_1)$ if and only if $V_1 \cong V_2$.

In [CHYZ], we have given an affirmative answer to Question 3.1 for simple singularities and simple elliptic singularities.

The following theorem by Saito will be used later.

Theorem 3.2. *([Sa1]) Let $f \in \mathcal{O}_n$ be a germ of a holomorphic function, defining an isolated quasi-homogeneous singularity at 0. Then*

$$Det(\frac{\partial^2 f}{\partial x_i \partial x_j})_{i,j=1,\cdots,n} \notin (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})\mathcal{O}_n$$

and

$$mDet(\frac{\partial^2 f}{\partial x_i \partial x_j})_{i,j=1,\cdots,n} \subseteq (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})\mathcal{O}_n$$

where m is the maximal ideal of $\mathcal{O}_n$.

We obtain the following result.

Theorem 3.3. *Let V be an isolated singularity defined by a quasi-homogeneous polynomial f. Then*

$$\mu^*(V) = \mu(V) - 1,$$

where $\mu^*(V)$ is the dimension of $A^*(V)$ and $\mu(V)$ is the Milnor number of V.

Proof. Since the Milnor algebra

$$\mathcal{O}_n/(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}),$$

is a Gorenstein local algebra and has a unique socle, it follows from Theorem 3.2 that $\mathrm{Det}(\frac{\partial^2 f}{\partial x_i \partial x_j})$ is the unique socle. Thus we have $\mu^*(V) = \mu(V) - 1$. □

Remark 3.2. It follows from Theorem 3.3 that $A^*(V) = 0$ when $\mu(V) = 1$. For this reason, our new Lie algebra $L^*(V)$ is defined only for singularities with Milnor number $\mu(V) \geq 2$.

Yau algebras are solvable. However, the new Lie algebra is not solvable in general. An example is: $x^3 + y^3$, and its new Lie algebra is spanned by $x\partial_x, y\partial_y, x\partial_y, y\partial_x$. Then it is easy to check that the derived series does not go down to zero. However, we prove that the new Lie algebra is solvable when the multiplicity of the singularity is at least 4. We first recall an important result obtained by Schulze.

Theorem 3.4. [Sc] *Let S be a zero-dimensional local $\mathbb{C}$-algebra of embedding dimension embdim(S) and order ord(S), and denote its first deviation by $\varepsilon_1(S)$. Then the Lie algebra $Der_{\mathbb{C}}(S, S)$ is solvable if $\varepsilon_1(S) + 1 <$ embdim$(S) +$ ord(S).*

Recall that, by definition, $\varepsilon_1(S) = \dim_{\mathbb{C}} H_1(S)$ where $H_\bullet(S)$ is the Koszul algebra of S. More explicitly, when $S = R/I$ in Theorem 3.4, where $R = \mathcal{O}_n$ and $I \subseteq R$ is a zero-dimensional ideal with $I \subseteq \mathsf{m}^m$, $\mathsf{m} = (x_1, \cdots, x_n)$ and $m \geq 2$ is chosen maximal. Then $n = \text{embdim}(S), m = \text{ord}(S)$, and $\varepsilon_1(S) = \dim_{\mathbb{C}}(I/\mathsf{m}I)$ is the minimal number of generators of I ([BH], Thm. 2.3.2(b)).

This result applies in particular to the generalized moduli algebra $A^*(V)$. If f is not quasi-homogeneous, then $\text{Der}(A^*(V), A^*(V))$ is the same as Yau algebra, thus it is solvable. Otherwise we have the following result.

Corollary 3.1. *If f is quasi-homogeneous and mult$(f) \geq 4$, then the new Lie algebra $Der(A^*(V), A^*(V))$ is solvable.*

Proof. Since f is quasi-homogeneous and $\text{mult}(f) \geq 4$, then $\text{embdim}(A^*(V)) = n, \varepsilon_1(S) = n + 1$, and $\text{ord}(A^*(V) \geq 3$. It follows from Theorem 3.4 that $\text{Der}(A^*(V), A^*(V))$ is solvable. $\square$

4. MAIN RESULTS

Given a family of complex projective hypersurfaces in $\mathbb{C}P^n$, the Torelli problem studied by Griffiths and his school asks whether the period map is injective on that family, i.e., whether the family of complex hypersurfaces can be distinguished by means of their Hodge structures. A complex projective hypersurface in $\mathbb{C}P^n$ can be viewed as a complex hypersurface with isolated singularity in $\mathbb{C}^{n+1}$. Let $V = \{z \in \mathbb{C}^{n+1} : f(z) = O\}$ be a complex hypersurface with isolated singularity at the origin. Seeley and Yau investigated the family of isolated complex hypersurface singularities using Yau algebras and obtained two deep Torelli-type theorems for simple elliptic singularities $\tilde{E}_7$ and $\tilde{E}_8$ [SY]. The natural question arises: whether the family of isolated complex hypersurface singularities can be distinguished by means of their new Lie algebras. The family of hypersurface singularities here is not arbitrary. First of all, as in projective case, we are actually studying the complex structures of an isolated hypersurface singularity. In view of the theorem of Lê and Ramanujan [LR], we require that the Milnor number μ is constant along this family. Recall that the dimension of the moduli algebra (denoted by τ) is a complex analytic invariant. So it suffices to consider only a (μ, τ)-constant family of isolated complex hypersurface singularities [SY]. The simple elliptic singularities are such families. We shall prove two Torelli-type theorems for simple elliptic singularities $\tilde{E}_7$ and $\tilde{E}_8$ respectively. However, there is no Torelli-type result for $\tilde{E}_6$, since $L^*(V_t)$ is a trivial family. Our method for $\tilde{E}_7$ is completely new and can be used to prove Torelli-type theorems for more general singularities. There are several advantages of our approach. First of all, it works for general complex hypersurface singularities without homogeneity assumption. Second, it allows us to construct a continuous invariant explicitly. Third, it gives a general method to produce a continuous family of nilpotent Lie algebras.

For recent progress on the new Lie algebras, please see [MYZ], we propose a new conjecture about the non-existence of negative weight derivations of the new moduli algebras of weighted homogeneous hypersurface singularities and verify this conjecture up to dimension three.

Due to the space limit, this paper is to summarize mainly the following results that we have obtained in [CHYZ]. The details and proofs can be found there.

Theorem A. *The Torelli-type theorem holds for simple elliptic singularities $\tilde{E}_8$. That is, $L^*(V_{t_1}) \cong L^*(V_{t_2})$ as Lie algebras, for $t_1 \neq t_2$ in $\mathbb{C} - \{t \in \mathbb{C} : 4t^3 + 27 = 0\}$, if and only if V_{t_1} and V_{t_2} are analytically isomorphic (i.e., $t_1^3 = t_2^3$). In particular, $\tilde{E}_8$ give rise to a non-trivial one-parameter family of solvable (resp. nilpotent) Lie algebras of dimension 12 (resp. 11).*

Theorem B. *The weak Torelli-type theorem holds for simple elliptic singularities $\tilde{E}_7$, i.e., $L^*(V_t)$ is a non-trivial one-parameter family. In particular, $\tilde{E}_7$ give rise to a non-trivial one-parameter family of solvable (resp. nilpotent) Lie algebras of dimension 11 (resp. 10).*

However the new Lie algebra can not distinguish $\widetilde{E_6}$.

$\widetilde{E_6}$ is a simple elliptic singularity defined by $\{(x, y, z) \in \mathbb{C}^3 \mid x^3 + y^3 + z^3 = 0\}$. Its (μ, τ)-constant family is given by

$$V_t = \{(x, y, z) \in \mathbb{C}^3 \mid f_t(x, y, z) = x^3 + y^3 + z^3 + txyz = 0\}$$

with $t^3+27\neq 0$ (cf. [Ya1]). The moduli algebra of V_t, denoted as $A(V_t)$, is given by

$$\begin{aligned} A(V_t) =& \mathbb{C}\{x,y,z\}/(\frac{\partial f_t}{\partial x},\frac{\partial f_t}{\partial y},\frac{\partial f_t}{\partial z}) \\ =& < 1,x,y,z,xy,yz,zx,xyz >, \end{aligned}$$

with multiplication rules

$$x^2=-\frac{t}{3}yz, y^2=-\frac{t}{3}zx, z^2=-\frac{t}{3}xy,$$

$$x^2y=xy^2=y^2z=yz^2=x^2z=0.$$

Let $\mathrm{Hess}(f_t)$ be the Hessian matrix of f_t. Then the generalized moduli algebra $A^*(V_t):=\mathcal{O}_n/(\frac{\partial f_t}{\partial x},\frac{\partial f_t}{\partial y},\frac{\partial f_t}{\partial z},\mathrm{Det}(\mathrm{Hess}(f_t)))=A(V_t)/(x^2y^2)=<1,x,y,z,xy,yz,zx>$ with multiplication rules

$$x^2=-\frac{t}{3}yz, y^2=-\frac{t}{3}zx, z^2=-\frac{t}{3}xy,$$

and

$$x^2y=xy^2=y^2z=yz^2=x^2z=xyz=0.$$

By calculation, a basis for the new Lie algebra $L^*(V_t)=\mathrm{Der}(A^*(V_t),A^*(V_t))$ denoted as L_t^* for short is:

$$x\partial_x+y\partial_y+z\partial_z, yz\partial_x, yz\partial_y, yz\partial_z, xz\partial_x, xz\partial_y, xz\partial_z, xy\partial_x, xy\partial_y, xy\partial_z,$$

for $t\neq 0$ and $216-\frac{t^6}{27}+7t^3\neq 0$. It is easy to see that in this case, L_t^* are isomorphic as Lie algebra. Thus L_t^* is a trivial family.

The classification of nilpotent Lie algebras in higher dimensions (>7) remains wide open. It is known that there are one-parameter families of non-isomorphic nilpotent Lie algebras (but no two-parameter families) in dimension seven. There are no such families in dimension less than seven. And the existence of such families is known in dimension greater than seven. However, such examples are hard to construct (cf. [Se]). As a corollary of Theorem A and Theorem B, we obtain non-trivial one-parameter families of 11-dimensional and 12-dimensional solvable (resp. 10-dimensional and 11-dimensional nilpotent) Lie algebras associated to $\tilde{E}_7$ and $\tilde{E}_8$ respectively.

Yau and Zuo [YZ2] formulated a sharp upper estimate conjecture for the Yau number of weighted homogeneous isolated hypersurface singularities and validated this conjecture for binomial isolated hypersurface singularities. A natural question is: what is the numerical relation between the new analytic invariant $\lambda^*(V)$ and the Yau number $\lambda(V)$? We propose the following conjecture:

Conjecture 4.1. Let $(V, 0)$ be an isolated hypersurface singularity defined by $f \in \mathcal{O}_n$, $n \geq 2$, and multiplicity greater than or equal to 3. Let $\lambda^*(V)$ be the dimension of $L^*(V) := \mathrm{Der}_{\mathbb{C}}(A^*(V), A^*(V))$, then $\lambda^*(V) = \lambda(V)$.

The above conjecture is obviously true when the isolated singularity $(V, 0)$ is not quasi-homogeneous. Recall the beautiful result of Saito ([Sa2], Corollary 3.8): let $f \in \mathcal{O}_n$ be a germ of a holomorphic function which defines a hypersurface with an isolated singularity at 0, then f is not quasi-homogeneous, precisely when

$$\mathrm{Det}(\frac{\partial^2 f}{\partial x_i \partial x_j})_{i,j=1,\cdots,n} \in (f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}).$$

Consequently, for non-quasi-homogeneous isolated hypersurface singularities, $A(V) = A^*(V)$. It follows that $L^*(V) = L(V)$ and $\lambda^*(V) = \lambda(V)$.

In this article, we shall also prove the following results:

Theorem C. *Let f be a weighted homogeneous polynomial in $\mathbb{C}[x_1, x_2, \ldots, x_n]$ $(n \geq 2)$ with respect to weight system $(w_1, w_2, \ldots, w_n; 1)$ and with mult$(f) \geq 3$. Suppose that f defines an isolated singularity $(V, 0)$, then*

$$\lambda^*(V) \leq \lambda(V).$$

Conjecture 4.1 is verified when $n \leq 4$.

Theorem D. *Let f be a weighted homogeneous polynomial in $\mathbb{C}[x_1, x_2, \ldots, x_n]$ $(2 \leq n \leq 4)$ with respect to weight system $(w_1, w_2, \ldots, w_n; 1)$ and with mult$(f) \geq 3$. Suppose that f defines an isolated singularity $(V, 0)$, then*

$$\lambda^*(V) = \lambda(V).$$

Elashvili and Khimshiashvili [EK] proved the following result: if X and Y are two simple singularities except the pair A_6 and D_5, then $L(X) \cong L(Y)$ as Lie algebras, if and only if X and Y are analytically isomorphic. Finally, we shall also show that the simple hypersurface singularities can be characterized completely by the new Lie algebra $L^*(V)$.

Theorem E. *If X and Y are two simple hypersurface singularities, then $L^*(X) \cong L^*(Y)$ as Lie algebras, if and only if X and Y are analytically isomorphic.*

The proof follows directly from the computation performed in section 6 by a straightforward analysis of the new Lie algebras.

5. Fewnomial singularities

In this subsection we recall the definition of fewnomial isolated singularities [Kh].

Definition 5.1. We say that a polynomial $f \in \mathbb{C}[z_1, z_2, \cdots, z_n]$ is fewnomial if the number of monomials in f does not exceed n.

Obviously, the number of monomials in f may depend on the system of coordinates. In order to obtain a rigorous concept we shall only allow linear transformations of coordinates and f (or rather its germ at the origin) is called a k-nomial if k is the smallest natural number such that f becomes a k-nomial after (possibly) a linear transformation of coordinates. An isolated hypersurface singularity V is called k-nomial if there exists an isolated hypersurface singularity Y analytically isomorphic to V which can be defined by a k-nomial and k is the smallest such number. It was shown in [CYZ] that a singularity defined by a fewnomial f is isolated only if f is a n-nomial in n variables when its multiplicity is at least 3 .

Definition 5.2. We say that an isolated hypersurface singularity V is fewnomial if it is defined by a fewnomial polynomial f. V is called a weighted homogenous fewnomial isolated singularity, if it is defined by a weighted homogenous fewnomial polynomial f. The 2-nomial (resp. 3-nomial) isolated hypersurface singularity is also called binomial (resp. trinomial) singularity.

The following proposition and corollary tell us that each simple singularity belongs to one of the following three types.

Proposition 5.1. [YZ2] *Let f be a weighted homogeneous fewnomial isolated hypersurface singularity with multiplicity at least 3. Then f is analytically equivalent to a linear combination of the following three series:*

Type A. $x_1^{a_1} + x_2^{a_2} + \cdots + x_{n-1}^{a_{n-1}} + x_n^{a_n}$, $n \geq 1$,

Type B. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}$, $n \geq 2$,

Type C. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1$, $n \geq 2$.

Corollary 5.1. [YZ2] *Each binomial isolated singularity is analytically equivalent to one of the three series: A) $x_1^{a_1} + x_2^{a_2}$, B) $x_1^{a_1}x_2 + x_2^{a_2}$, and C) $x_1^{a_1}x_2 + x_2^{a_2}x_1$.*

In many situations it is necessary to have an explicit basis of $A(V)$. It is well known that there always exist monomial bases. Recall that the monomial bases in moduli algebras of simple singularities $(A_k, D_k, E_6, E_7, E_8)$ are given in [AGV].

6. Computing the new Lie algebras

We compute the new Lie algebra for binomial singularities, which includes the simple singularities as special case. As an application, we prove that the simple hypersurface singularities can be characterized completely by the new Lie algebra.

Proposition 6.1. *Let $(V,0)$ be a weighted homogeneous fewnomial isolated singularity of type A, defined by $f = x_1^{a_1} + x_2^{a_2}$ $(a_1 \geq 2,\ a_2 \geq 3)$ with weight type $(\frac{1}{a_1}, \frac{1}{a_2}; 1)$. Then*

$$\lambda^*(V) = \begin{cases} 2a_1a_2 - 3(a_1+a_2) + 4, & a_1 \geq 3, a_2 \geq 3 \\ a_2 - 3, & a_1 = 2, a_2 \geq 3. \end{cases}$$

Remark 6.1. Since our new Lie algebra is not defined for the Milnor number $\mu(f) = 1$. The restriction $a_1 \geq 2,\ a_2 \geq 3$ in Proposition 6.1 follows from $\mu(f) \geq 2$. The similar restrictions also appear in Proposition 6.2 and Proposition 6.3 below.

Proposition 6.2. *Let $(V,0)$ be a binomial isolated singularity of type B defined by $f = x_1^{a_1}x_2 + x_2^{a_2}$ $(a_1 \geq 2,\ a_2 \geq 2)$ with weight type $(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}; 1)$. Then*

$$\lambda^*(V) = \begin{cases} 2a_1a_2 - 2a_1 - 3a_2 + 5, & a_1 \geq 2, a_2 \geq 3 \\ 2a_1 - 3, & a_1 \geq 2, a_2 = 2. \end{cases}$$

Proposition 6.3. *Let $(V,0)$ be a binomial isolated singularity of type C, defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_1$ $(a_1 \geq a_2 \geq 2)$ with weight type $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}; 1)$. Then*

$$\lambda^*(V) = \begin{cases} 2a_1a_2 - 2a_1 - 2a_2 + 6, & a_1 \geq a_2 \geq 3, \\ 2a_1, & a_1 \geq a_2 = 2 \end{cases}$$

In order to prove Theorem E, we need the following proposition.

Proposition 6.4. *The following three pairs of new Lie algebras arising from simple hypersurface singularities are not isomorphic:*

$$L^*(D_7) \ncong L^*(E_6), L^*(A_{10}) \ncong L^*(E_7), \textit{and } L^*(D_{10}) \ncong L^*(E_8).$$

It is easy to see that, from Propositions 6.1 and 6.2, we get $\dim L^*(A_k) = k-2$, $\dim L^*(D_k) = k$, $\dim L^*(E_6) = 7, \dim L^*(E_7) = 8$, and $\dim L^*(E_8) = 10$. Cartan subalgebras that from $L^*(A_k)$ and $L^*(D_k)$ are generated by $< x_2\partial_2 >$ and $< x_1\partial_1, x_2\partial_2 >$ respectively. It is then easy to verify that $\mathrm{rk} L^*(A_k) = \mathrm{rk} L^*(E_7) = 1$ while $\mathrm{rk} L^*(E_6) = \mathrm{rk} L^*(E_8) = \mathrm{rk} L^*(D_k) = 2$. When the dimensions or ranks of the new Lie algebras for all simple singularities are different, then they are certainly not isomorphic, so we only need to treat the three pairs of Lie algebras $(L^*(A_{10}), L^*(E_7)), (L^*(E_6), L^*(D_7)), (L^*(E_8), L^*(D_{10}))$ which have the same dimensions and ranks. It follows from the Proposition 6.4 that these three pairs are non-isomorphic. Therefore we have the following proposition.

Proposition 6.5 (i.e. Proposition E). *If X and Y are two simple hypersurface singularities then $L^*(X) \cong L^*(Y)$ as Lie algebras, if and only if X and Y are analytically isomorphic.*

In fact, we have obtained the following theorem which generalize Theorem E.

Theorem 6.1. [CHYZ]*(i.e., Conjecture 4.1) Conjecture 4.1 is true for binomial singularities.*

Proof. In order to prove Conjecture 4.1, i.e., $\lambda^*(V) = \lambda(V)$, we need the following propositions from [YZ2].

Proposition 6.6. [YZ2] *Let $(V, 0)$ be a weighted homogeneous fewnomial isolated singularity of type A defined by $f = x_1^{a_1} + x_2^{a_2} + \cdots + x_n^{a_n}$ with weight type $(\frac{1}{a_1}, \frac{1}{a_2}, \cdots, \frac{1}{a_n}; 1)$. Then the Yau number is*

$$\lambda(V) = n\prod_{i=1}^{n}(a_i - 1) - \sum_{i}^{n}(a_1 - 1)(a_2 - 1)\cdots\widehat{(a_i - 1)}\cdots(a_n - 1),$$

where $\widehat{(a_i - 1)}$ means that $a_i - 1$ is omitted.

Proposition 6.7. [YZ2] *Let $(V, 0)$ be a binomial isolated singularity of type B defined by $f = x_1^{a_1}x_2 + x_2^{a_2}$ with weight type $(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}; 1)$. Then the Yau number is*

$$\lambda(V) = 2a_1a_2 - 2a_1 - 3a_2 + 5.$$

Proposition 6.8. [YZ2] *Let $(V, 0)$ be a binomial isolated singularity of type C defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_1$ with weight type $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}; 1)$. If $mult(f) \geq 4$, i.e., $a_1, a_2 \geq 3$, then the Yau number is*

$$\lambda(V) = 2a_1a_2 - 2a_1 - 2a_2 + 6.$$

If $mult(f) = 3$, i.e., $f = x_1^2x_2 + x_2^{a_2}x_1$, then the Yau number is $\lambda(V) = 2a_2$.

Comparing the Yau number $\lambda(V)$ with the new analytic invariant $\lambda^*(V)$ in the case of binomial isolated singularities of type A, type B and type C (see Propositions 6.1-6.3), it is easy to see that the conjecture holds for binomial isolated singularities, i.e.

$$\lambda^*(V) = \lambda(V),$$

and hence Theorem 6.1 is proved.

□

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