HODGE MODULI ALGEBRAS AND COMPLETE INVARIANTS OF SINGULARITIES

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Abstract. We introduce the Hodge moduli algebras and Hodge moduli sequence associated with an isolated hypersurface singularity. These are new subtle invariants of singularities. We propose several characterization conjectures by using of these invariants. We investigate structural properties and numerical invariants of Hodge ideals naturally associated with isolated hypersurface singularities. In particular, we establish that the analytic isomorphisms class of an isolated two dimensional rational hypersurface singularities is determined by the Hodge moduli algebras and Hodge moduli sequence. As a result, we prove that Hodge moduli algebra together with the geometric genus give complete characterization of such singularities. In the proof, we concretely compute the Hodge ideals and the associated Hodge moduli algebras of these singularities.

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1. introdction

In [17] and [18], the authors extend the notion of Hodge ideals to the case when D is an arbitrary effective $\mathbb{Q}$-divisor on X. Hodge ideals $\{I_k(D)\}_{k\in\mathbb{N}}$ are defined in terms of the Hodge filtration $F_\bullet$ on some $\mathscr{D}_X$-module associated with D (cf. [17], §2 − §4 for more details). When D is an integral and reduced divisor, this recovers the definition of Hodge ideals $I_k(D)$ in [15].

Let X be a smooth complex variety, and $\mathscr{D}_X$ be the sheaf of differential operators on X. If H is an integral and reduced effective divisor on X, $D = \alpha H, \alpha \in \mathbb{Q} \cap (0, 1]$, let $\mathcal{O}_X(*D)$ be the sheaf of rational functions with poles along D. It is also a left $\mathscr{D}_X$-module underlying the mixed Hodge module $j_*\mathbb{Q}_U^H[n]$, where $U = X\backslash D$ and $j : U \hookrightarrow X$ is the inclusion map. Any $\mathscr{D}_X$-module associated with a mixed Hodge module has a good filtration $F_\bullet$, the Hodge filtration of the mixed Hodge module [20].

In order to study the Hodge filtration of $\mathcal{O}_X(*D)$, it seems easier to consider a series of ideal sheaves, defined by Mustaţă and Popa [15], which can be considered to be a generalization of multiplier ideals of divisors. The Hodge ideals $\{I_k(D)\}_{k\in\mathbb{N}}$ of the divisor D are defined by:

$$F_k\mathscr{O}_X(*D) = I_k(D) \otimes \mathscr{O}_X\big((k+1)D\big), \quad \text{for all } k \in \mathbb{N}.$$

These are coherent sheaves of ideals. See [15] for details and an extensive study of the ideals $I_k(D)$. Hodge ideals are indexed by the non-negative integers; at the 0-th step, they essentially

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coincide with multiplier ideals. It turns out that $I_0(D) = \mathscr{J}((1-\epsilon)D)$, the multiplier ideal of the divisor $(1-\epsilon)D,\ 0<\epsilon\ll 1$. The multiplier ideal sheaves are ubiquitous objects in birational geometry, encoding local numerical invariants of singularities, and satisfying Kodaira-type vanishing theorems in the global setting. The Hodge ideals are interesting invariants of the singularities, they have similar properties as multiplier ideals.

We summarize the properties and results (cf. [17] and [16]) of Hodge ideals as follows:

Given a reduced effective divisor H on a smooth complex variety X, $D=\alpha H, \alpha\in\mathbb{Q}\cap(0,1]$, we also denote by Z the support of D. The sequence of Hodge ideals $I_k(D)$, with $k\geq 0$, satisfies:

- $I_0(D)$ is the multiplier ideal $\mathcal{I}((1-\epsilon)D)$, so in particular $I_0(D)=\mathscr{O}_X$ if and only if the pair(X,D) is log canonical.
- When Z has simple normal crossings, then
$$I_k(D)=I_k(Z)\otimes\mathscr{O}_X(Z-\lceil D\rceil),$$
where $I_k(Z)$ can be computed explicitly as in [15], if Z is smooth, then $I_k(D)=\mathscr{O}_X(Z-\lceil D\rceil)$.
- The Hodge filtration is generated at level $n-1$, where $n=\dim X$, i.e.,
$$F_\ell\mathscr{D}_X\cdot\left(I_k(D)\otimes\mathscr{O}_X(kZ)h^{-\alpha}\right)=I_{k+\ell}(D)\otimes\mathscr{O}_X((k+\ell)Z)h^{-\alpha}$$
for all $k\geqslant n-1$ and $\ell\geqslant 0$.
- There are non-triviality criteria for $I_k(D)$ at a point $x\in D$ in terms of the multiplicity of D at x.
- If X is projective, $I_k(D)$ satisfy a vanishing theorem analogous to Nadel Vanishing for multiplier ideals.
- If Y is a smooth divisor in X such that $\left.Z\right|_Y$ is reduced, then $I_k(D)$ satisfy
$$I_k\left(\left.D\right|_Y\right)\subseteq I_k(D)\cdot\mathscr{O}_Y,$$
with equality when Y is general.
- If $X\to T$ is a smooth family with a section $s:T\to X$, and D is a relative divisor on X that satisfies a suitable condition then
$$\left\{t\in T\mid I_k\left(D_t\right)\nsubseteq\mathfrak{m}_{s(t)}^q\right\}$$
is an open subset of T, for each $q\geqslant 1$.
- If D_1 and D_2 are Q-divisors with supports Z_1 and Z_2, such that Z_1+Z_2 is also reduced, then the subadditivity property
$$I_k\left(D_1+D_2\right)\subseteq I_k\left(D_1\right)\cdot I_k\left(D_2\right)$$
holds.

For comparison, the list of properties of Hodge ideals in the case when D is reduced is summarized in [19]. The setting of Q-divisors is more intricate. For instance, the bounds for the generation level of the Hodge filtration can become worse. Moreover, it is not known whether the inclusions $I_k(D)\subseteq I_{k-1}(D)$ continue to hold for arbitrary Q-divisors. New phenomena appear as well: given two rational numbers $\alpha_1<\alpha_2$, usually the ideals $I_k\left(\alpha_1 Z\right)$ and $I_k\left(\alpha_2 Z\right)$ cannot be compared for $k\geqslant 1$, unlike in the case of multiplier ideals.

In classification theory of singularities, one always wants to find various invariants associated with singularities. Hopefully with enough invariants found, one can distinguish between different isolated singularities up to some certain equivalences. However, not many effective invariants are known. Moreover, most of known invariants. For example the geometric genus are hard to compute in general. In this article, we shall introduce a serious new numerical invariants to isolated hypersurface singularities. These invariants can be calculated easily compared with other invariants of isolated singularities. Moreover, it is a natural question whether some interesting singularities are distinguished by Hodge ideals. It is the first step to study the moduli space of singularities by using Hodge ideals. The most simple and important class of singularities are simple singularities. In this paper, we show that our newly introduced invariants (i.e., Hodge moduli algebras, see Definition 1.3) can be used to distinguish between different simple singularities.

From now on, throughout the paper we assume that X is $\mathbb{C}^n$, H is defined by f which has an isolated singularity at the origin. Set $D^\alpha = \alpha H, \alpha \in \mathbb{Q} \cap (0,1]$. Let $I_k(D^\alpha) \subset \mathbb{C}\{x_1, \cdots, x_n\}$ be the Hodge ideals for $k \in \mathbb{N}$. Since f has an isolated singularity at 0 and the $I_k(D^\alpha)$ are coherent, these are $\mathfrak{m}_{X,0}$-primary ideals, that is, $I_k(D^\alpha) \supset \mathfrak{m}_{X,0}^p$ for some $p \in \mathbb{Z}_{>0}$ (depending on k and α) with $\mathfrak{m}_{X,0} \subset \mathbb{C}\{x_1, \cdots, x_n\}$ the maximal ideal, and $\mathbb{C}\{x\}/I_k(D^\alpha)$ is finite-dimensional. I.e., the Hodge ideal $I_k(D^\alpha)$ is Artinian. The sequence of ideals $I_k(D^\alpha)$ are refined invariant of singularities than the multiplier ideal $I_0(D^\alpha)$ alone (cf. [15], Remark 17.13).

For recent progress in distinguishing between different simple singularities by using several derivation Lie algebras, the interested readers can refer to [3], [6], [8]-[10].

Definition 1.1. Let $f, g \in R = \mathbb{C}\{x_1, \ldots, x_n\}$ which is the convergent power series ring. We say f and g are contact equivalent if the local $\mathbb{C}$-algebras $R/(f)$ and $R/(g)$ are isomorphic.

Recall that we have the following well-known generalized Mather-Yau theorem ([7], Theorem 2.26).

Theorem 1.2. *Let $m = (x_1, \cdots, x_n)$ be the maximal ideal of $\mathcal{O}_n$. Let $f, g \in m \subset \mathcal{O}_n$. The following are equivalent:*

1) $(V(f), 0) \cong (V(g), 0)$*;*

2) For all $k \geq 0$, $\mathcal{O}_n/(f, m^k J(f)) \cong \mathcal{O}_n/(g, m^k J(g))$ *as* $\mathbb{C}$*-algebra;*

3) There is some $k \geq 0$ *such that* $\mathcal{O}_n/(f, m^k J(f)) \cong \mathcal{O}_n/(g, m^k J(g))$ *as* $\mathbb{C}$*-algebra,*

where $J(f) = (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$.

In particular, if $k = 0, 1$ above, then the claim of the equivalence of 1) and 3) is exactly the same as the Mather-Yau theorem [13].

Let $(V, 0)$ be a normal two-dimensional singularity, $\pi : M \to V$ be a resolution of $(V, 0)$. It is known that the geometric genus $p_g = \dim H^1(M, \mathcal{O}_M)$ is independent of resolution. One might classify singularities by p_g. Rational singularity is equivalent to $p_g = 0$. Minimally elliptic singularity is equivalent to saying that $p_g = 1$ and $\mathcal{O}_V$ is Gorenstein. It is well-known that isolated two-dimensional rational hypersurface singularities are exactly simple singularities.

Among Arnold's most famous results in the singularity theory is his classification of simple (or ADE) singularities [1]. Simple singularities can be considered in arbitrary dimensions. Simple surface singularities, consist of two series $A_k : \{x^{k+1} + y^2 + z^2 = 0\}, k \geq 1$, $D_k : \{x^{k-1} + xy^2 + z^2 = 0\}, k \geq 4$ and three exceptional singularities E_6, E_7, E_8 defined by polynomials $x^3 + y^4 + z^2, x^3 + xy^3 + z^2, x^3 + y^5 + z^2$ respectively. These singularities play an important role in algebraic geometry and singularity theory (cf. [2], [11]). Simple surface singularities (i.e., rational double points) can be characterized in many ways [5], all of which involve some form of finiteness. These characterizations build on the work of Artin, Brieskorn, Du Val, Arnold, Tjurina and many others. And they form a very interesting subject in singularity theory.

Notice that all these important invariants, Yau algebra [6], Milnor number, and geometric genus can not distinguish the simple singularities. It is a very interesting question to construct some subtle invariants which distinguish the rational isolated hypersurface singularities. In this article, we will introduce some new subtle invariants Hodge moduli algebras and Hodge moduli sequences. Motivated from the beautiful work in [6] and Mather-Yau theorem, one of our main goals is to show that singularities of certain types (i.e., rational isolated hypersurface singularities of dimension two) can be classified by their Hodge moduli algebras and Hodge moduli sequences. As a result, we prove that Hodge moduli algebra together with the geometric genus give a complete characterization of such singularities. This gives a new characterization for simple singularities, which can be viewed as an extension of the fifteen characterizations theorems summarized in [5].

We first introduce the following new definitions.

Definition 1.3. Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0), n \geq 2$, be an isolated hypersurface singularity. Let $H = \{f = 0\}$ be an integral and reduced effective divisor defined by $f, D^\alpha = \alpha H, \alpha \in \mathbb{Q} \cap (0, 1]$. We define the i-th Hodge moduli algebra of D^α to be $M_i(D^\alpha) := \mathbb{C}\{x_1, \ldots, x_n\}/I_i(D^\alpha)$ for $i \geq 0$ (or M_i for short), where $I_i(D^\alpha)$ be the i-th Hodge ideal (or I_i for short). The i-th Hodge moduli number of D^α is defined to be $m_i(D^\alpha) := dim(M_i(D^\alpha))$ for $i \geq 0$ (or m_i for short). We define the Hodge moduli sequence of D to be the sequence $\{m_i\} := \{m_0, m_1, m_2, \ldots\}$.

Definition 1.4. let $F_\Omega := \{f_\alpha : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0) \mid \alpha \in \Omega\}$ where f_α is an isolated hypersurface singularity, the Ω is contained in a finite-dimensional vector space $\mathbb{C}^d$, $d \in \mathbb{N}$. We call such F_Ω a bounded family of isolated hypersurface singularities.

Definition 1.5. Let $F_\Lambda := \{f_\alpha : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0) \mid \alpha \in \Lambda\}$, $n \geq 2$, be a bounded family of isolated hypersurface singularities. Let $H_\lambda = \{f_\lambda = 0\}$ be an integral and reduced effective divisor, $D^\alpha_\lambda = \alpha H_\lambda$, where $\alpha \in \mathbb{Q} \cap (0, 1]$. We call F_Λ has (α, N_α)-Hodge property where $N_\alpha \in \mathbb{N}$, if F_Λ satisfies the following two conditions.
(1) There exists $f_1, f_2 \in F_\Lambda$, such that f_1 is not contact equivalent to f_2, but $m_i(D^\alpha_1) = m_i(D^\alpha_2)$ for $0 \leq i \leq N_\alpha - 1$.
(2) For any $f_\lambda, f_\mu \in F_\Lambda$, then f_λ is contact equivalent to f_μ if and only if $m_i(D^\alpha_\lambda) = m_i(D^\alpha_\mu)$ for $0 \leq i \leq N_\alpha$.
If N_α above is finite, then we call F_Λ has finite Hodge property for the fixed α.

We propose the following new question and conjecture.

Question 1.6. *Let $F_\Omega := \{f_\alpha : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0) \mid \alpha \in \Omega\}$ be a bounded family of isolated hypersurface singularities. Which bounded family of isolated hypersurface singularities F_Ω will have finite Hodge property?*

Definition 1.7. Let $F_\Lambda := \{f_\alpha : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0) \mid \alpha \in \Lambda\}$, $n \geq 2$, be a bounded family of isolated hypersurface singularities. If $H_\lambda = \{f_\lambda = 0\}$ is an integral and reduced effective divisor defined by f_λ, and $D^\alpha_\lambda = \alpha H_\lambda$, where $\alpha \in \mathbb{Q} \cap (0, 1]$. If there exists $N_\alpha \in \mathbb{N}$ such that
(1) There exists $f_1, f_2 \in F_\Lambda$, such that f_1 is not contact equivalent to f_2, however $M_i(D^\alpha_1) \cong M_i(D^\alpha_2)$ for $0 \leq i \leq N_\alpha - 1$.
(2) For any $f_\lambda, f_\mu \in F_\Lambda$, then f_λ is contact equivalent to f_μ if and only if $M_{N_\alpha}(D^\alpha_\lambda) \cong M_{N_\alpha}(D^\alpha_\mu)$.
Such N_α is called the Hodge moduli threshold of F_Λ for the fixed α.

Conjecture 1.8. *Let $f, g : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0), n \geq 2$ define two isolated hypersurface singularities. Let $H_f = \{f = 0\}$ (resp. $H_g = \{g = 0\}$) be an integral and reduced effective divisor defined by f (resp. g). And $D^\alpha_f = \alpha H_f, D^\alpha_g = \alpha H_g$, where $\alpha \in \mathbb{Q} \cap (0, 1]$. Then there existss an $N_\alpha \in \mathbb{N}$,*

such that the following are equivalent.
(1) f is contact equivalent to g.
(2) For all $n \geq N_\alpha$, $M_n(D_f^\alpha) \cong M_n(D_g^\alpha)$.
(3) There is some $n \geq N_\alpha$ such that $M_n(D_f^\alpha) \cong M_n(D_g^\alpha)$.
Moreover, let $F_\Omega := \{f_\alpha : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0) \mid \alpha \in \Omega\}$ be a bounded family of isolated hypersurface singularities, then the Hodge moduli threshold of F_Ω is finite.

Remark 1.9. If F_Λ has (α, N_α)-Hodge property and its Hodge moduli threshold is equal to K_α, then $K_\alpha \leq N_\alpha$.

In this paper, we verify the above conjectures for simple singularities in three variables. We will prove the following proposition in section 3 which plays fundamental role in the proof of Main Theorem.

Proposition 1.10. *Then the first four Hodge moduli algebras and Hodge moduli numbers are given as follows.*
Let $f(x, y, z) = x^{k+1} + y^2 + z^2$ be A_k type singularity, where $k \geq 1$. Let $H = \{f = 0\}$ be an integral and reduced effective divisor defined by f. And $D^\alpha = \alpha H$ where $\alpha \in \mathbb{Q} \cap (0, 1]$. Then the first four Hodge moduli algebras and Hodge moduli numbers are given as follows.
(1) For $\alpha \in (0, \frac{1}{k+1}]$,

$$\begin{aligned}
M_0 =& M_1 = 0, m_0 = m_1 = 0.\\
M_2 =& \operatorname{span}\{1, x, \cdots, x^{k-1}\}, m_2 = k.\\
M_3 =& \operatorname{span}\{1, x, \cdots, x^k, y, xy, \cdots, x^{k-1}y, z, xz, \cdots, x^{k-1}z, z^2, xz^2, \cdots, x^{k-2}z^2\}, m_3 = 4k.
\end{aligned}$$

(2) For $\alpha \in (\frac{i}{k+1}, \frac{i+1}{k+1}]$ with $1 \leq i \leq k-1$,

$$\begin{aligned}
M_0 =& 0, m_0 = 0.\\
M_1 =& \operatorname{span}\{1, \cdots, x^{i-1}\}, m_1 = i.\\
M_2 =& \operatorname{span}\{1, x, \cdots, x^k, y, xy, \cdots, x^{i-1}y, z, xz, \cdots, x^{i-1}z, z^2, xz^2, \cdots, x^{i-2}z^2\}, m_2 = k + 3i.\\
M_3 =& \operatorname{span}\{1, x, \ldots, x^{2k+i-1}, y, xy, \ldots, x^{k+i-1}y, y^2, xy^2, \ldots, x^{i-1}y^2, z, xz, \ldots, x^{k+i-1}z,\\
& yz, xyz, \ldots, x^{i-1}yz, z^2, xz^2, \ldots, x^{i-1}z^2\}, m_3 = 4k + 6i.
\end{aligned}$$

(3) For $\alpha \in (\frac{k}{k+1}, 1]$,

$$\begin{aligned}
M_0 =& 0, m_0 = 0.\\
M_1 =& \operatorname{span}\{1, \cdots, x^{k-1}\}, m_1 = k.\\
M_2 =& \operatorname{span}\{1, x, \cdots, x^k, y, xy, \cdots, x^{k-1}y, z, xz, \cdots, x^{k-1}z, z^2, xz^2, \cdots, x^{k-2}z^2\}, m_2 = 4k.\\
M_3 =& \operatorname{span}\{1, x, \ldots, x^{3k-1}, y, xy, \ldots, x^{2k-1}y, y^2, xy^2, \ldots, x^{k-1}y^2, z, xz, \ldots, x^{2k-1}z,\\
& yz, xyz, \ldots, x^{k-1}yz, z^2, xz^2, \ldots, x^{k-1}z^2\}, m_3 = 10k.
\end{aligned}$$

Let $f(x, y) = x^{k-1} + xy^2 + z^2$ be D_k type singularity, where $k \geq 4$. Let $H = \{f = 0\}$ be an integral and reduced effective divisor defined by f. And $D^\alpha = \alpha H$ where $\alpha \in \mathbb{Q} \cap (0, 1]$. Then the first four Hodge moduli algebras and Hodge moduli numbers are given as follows. For D_k, when k is even, *(1) For* $\alpha \in (0, \frac{1}{2k-2}]$,

$$\begin{aligned}
M_0 =& M_1 = 0, m_0 = m_1 = 0.\\
M_2 =& \mathrm{span}\{1, x, \cdots, x^{k-2}, y\}, m_2 = k.\\
M_3 =& \mathrm{span}\{1, x, \cdots, x^{k-1}, y, xy, \cdots, x^{k-1}y, y^2, xy^2, \cdots, x^{k-2}y^2, y^3, z, xz, \cdots, x^{k-2}z, yz\},\\
& m_3 = 4k.
\end{aligned}$$

(2) For $\alpha \in (\frac{2i-3}{2k-2}, \frac{2i-1}{2k-2}]$, $2 \leq i \leq \frac{k}{2}$,

$$\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \mathrm{span}\{1, x \cdots, x^{i-2}\}, m_1 = i-1.\\
&M_2 = \mathrm{span}\{1, x, \cdots, x^{k+i-4}, y, xy, \cdots, x^{i-1}y, y^2, z, xz, \cdots, x^{i-2}z\}, m_2 = k+3i-3.\\
&M_3 = \mathrm{span}\{1, x, \cdots, x^{2k+i-6}, y, xy, \cdots, x^{k+i-3}y, y^2, xy^2, \cdots, x^iy^2, y^3, xy^3, y^4,\\
&\quad z, xz, \cdots, x^{k+i-4}z, yz, xyz, \cdots, x^{i-1}yz, y^2z, z^2, \cdots, x^{i-2}z^2\}, m_3 = 4k+6i-6.
\end{aligned}$$

(3) For $\alpha \in (\frac{2i-3}{2k-2}, \frac{2i-1}{2k-2}]$, $\frac{k}{2}+1 \leq i \leq k-1$,

$$\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \mathrm{span}\{1, x \cdots, x^{i-2}, y\}, m_1 = i.\\
&M_2 = \mathrm{span}\{1, x, \cdots, x^{k+i-4}, y, xy, \cdots, x^{i-1}y, y^2, xy^2, y^3z, xz, \cdots, x^{i-2}z, yz\},\\
&\quad m_2 = k+3i.\\
&M_3 = \mathrm{span}\{1, x, \cdots, x^{2k+i-6}, y, xy, \cdots, x^{k+i-3}y, y^2, xy^2, \cdots, x^iy^2, y^3, xy^3, x^2y^3,\\
&\quad y^4, xy^4, y^5, z, xz, \cdots, x^{k+i-4}z, yz, xyz, \cdots, x^{i-1}yz, y^2z, xy^2z, y^3z,\\
&\quad z^2, \cdots, x^{i-2}z^2, yz^2\}, m_3 = 4k+6i.
\end{aligned}$$

(4) For $\alpha \in (\frac{2k-3}{2k-2}, 1]$,

$$\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \mathrm{span}\{1, \cdots, x^{k-2}, y\}, m_1 = k.\\
&M_2 = \mathrm{span}\{1, x, \cdots, x^{2k-4}, y, xy, \cdots, x^{k-1}y, y^2, xy^2, y^3, z, xz, \cdots, x^{k-2}z, yz\}, m_2 = 4k.\\
&M_3 = \mathrm{span}\{1, x, \cdots, x^{3k-6}, y, xy, \cdots, x^{2k-3}y, y^2, xy^2, \cdots, x^ky^2, y^3, xy^3, x^2y^3,\\
&\quad y^4, xy^4, y^5, z, xz, \cdots, x^{2k-4}z, yz, xyz, \cdots, x^{k-1}yz, y^2z, xy^2z, y^3z,\\
&\quad z^2, \cdots, x^{k-2}z^2, yz^2\}, m_3 = 10k.
\end{aligned}$$

When k is odd,
(1) For $\alpha \in (0, \frac{1}{2k-2}]$,

$$\begin{aligned}
M_0 =& M_1 = 0, m_0 = m_1 = 0.\\
M_2 =& \mathrm{span}\{1, x, \cdots, x^{k-2}, y\}, m_2 = k.\\
M_3 =& \mathrm{span}\{1, x, \cdots, x^{k-1}, y, xy, \cdots, x^{k-1}y, y^2, xy^2, \cdots, x^{k-2}y^2, y^3, z, xz, \cdots, x^{k-2}z, yz\},\\
& m_3 = 4k.
\end{aligned}$$

(2) For $\alpha \in (\frac{2i-3}{2k-2}, \frac{2i-1}{2k-2}]$, $2 \le i \le \frac{k-1}{2}$,

$$\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \text{span}\{1, x \cdots, x^{i-2}\}, m_1 = i-1.\\
&M_2 = \text{span}\{1, x, \cdots, x^{k+i-4}, y, xy, \cdots, x^{i-1}y, y^2, z, xz, \cdots, x^{i-2}z\}, m_2 = k+3i-3.\\
&M_3 = \text{span}\{1, x, \cdots, x^{2k+i-6}, y, xy, \cdots, x^{k+i-3}y, y^2, xy^2, \cdots, x^i y^2, y^3, xy^3, y^4,\\
&\quad z, xz, \cdots, x^{k+i-4}z, yz, xyz, \cdots, x^{i-1}yz, y^2z, z^2, \cdots, x^{i-2}z^2\}, m_3 = 4k+6i-6.
\end{aligned}$$

(3) For $\alpha \in (\frac{k-2}{2k-2}, \frac{k-1}{2k-2}]$,

$$\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \text{span}\{1, x \cdots, x^{\frac{k-1}{2}}\}, m_1 = \frac{k+1}{2}.\\
&M_2 = \text{span}\{1, x, \cdots, x^{\frac{3k-5}{2}}, y, xy, \cdots, x^{\frac{k+1}{2}}y, y^2, z, xz, \cdots, x^{\frac{k-1}{2}}z\}, m_2 = \frac{5k+3}{2}.\\
&M_3 = \text{span}\{1, x, \cdots, x^{\frac{5k-9}{2}}, y, xy, \cdots, x^{\frac{3k-3}{2}}y, y^2, xy^2, \cdots, x^{\frac{k+3}{2}}y^2, y^3, xy^3, y^4,\\
&\quad z, xz, \cdots, x^{\frac{3k-5}{2}}z, yz, xyz, \cdots, x^{\frac{k+1}{2}}yz, y^2z, z^2, \cdots, x^{\frac{k-1}{2}}z^2\}, m_3 = 7k+3.
\end{aligned}$$

(4) For $\alpha \in (\frac{k-1}{2k-2}, \frac{k}{2k-2}]$,

$$\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \text{span}\{1, x \cdots, x^{i-2}, y\}, m_1 = \frac{k+3}{2}.\\
&M_2 = \text{span}\{1, x, \cdots, x^{\frac{3k-5}{2}}, y, xy, \cdots, x^{\frac{k+1}{2}}y, y^2, xy^2, y^3\\
&\quad z, xz, \cdots, x^{\frac{k-1}{2}}z, yz\}, m_2 = \frac{5k+9}{2}.\\
&M_3 = \text{span}\{1, x, \cdots, x^{\frac{5k-9}{2}}, y, xy, \cdots, x^{\frac{3k-3}{2}}y, y^2, xy^2, \cdots, x^{\frac{k+3}{2}}y^2,\\
&\quad y^3, xy^3, x^2y^3, y^4, xy^4, y^5, z, xz, \cdots, x^{\frac{3k-5}{2}}z, yz, xyz, \cdots, x^{\frac{k+1}{2}}yz,\\
&\quad y^2z, xy^2z, y^3z, z^2, \cdots, x^{\frac{k-1}{2}}z^2, yz^2\}, m_3 = 7k+9.
\end{aligned}$$

(5) For $\alpha \in (\frac{2i-3}{2k-2}, \frac{2i-1}{2k-2}]$, $\frac{k+3}{2} \le i \le k-1$,

$$\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \text{span}\{1, x \cdots, x^{i-2}, y\}, m_1 = i.\\
&M_2 = \text{span}\{1, x, \cdots, x^{k+i-4}, y, xy, \cdots, x^{i-1}y, y^2, xy^2, y^3z, xz, \cdots, x^{i-2}z, yz\},\\
&\quad m_2 = k+3i.\\
&M_3 = \text{span}\{1, x, \cdots, x^{2k+i-6}, y, xy, \cdots, x^{k+i-3}y, y^2, xy^2, \cdots, x^i y^2, y^3, xy^3, x^2y^3,\\
&\quad y^4, xy^4, y^5, z, xz, \cdots, x^{k+i-4}z, yz, xyz, \cdots, x^{i-1}yz, y^2z, xy^2z, y^3z,\\
&\quad z^2, \cdots, x^{i-2}z^2, yz^2\}, m_3 = 4k+6i.
\end{aligned}$$

(6) For $\alpha \in (\frac{2k-3}{2k-2}, 1]$,

$$\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \text{span}\{1, \cdots, x^{k-2}, y\}, m_1 = k.\\
&M_2 = \text{span}\{1, x, \cdots, x^{2k-4}, y, xy, \cdots, x^{k-1}y, y^2, xy^2, y^3z, xz, \cdots, x^{k-2}z, yz\},\\
&\quad m_2 = 4k.\\
&M_3 = \text{span}\{1, x, \cdots, x^{3k-6}, y, xy, \cdots, x^{2k-3}y, y^2, xy^2, \cdots, x^k y^2, y^3, xy^3, x^2y^3,\\
&\quad y^4, xy^4, y^5, z, xz, \cdots, x^{2k-4}z, yz, xyz, \cdots, x^{k-1}yz, y^2z, xy^2z, y^3z,\\
&\quad z^2, \cdots, x^{k-2}z^2, yz^2\}, m_3 = 10k.
\end{aligned}$$

Let $f(x,y,z) = x^3 + y^4 + z^2$ *be* E_6 *type singularity. Let* $H = \{f = 0\}$ *be an integral and reduced effective divisor defined by* f. *And* $D^\alpha = \alpha H$ *where* $\alpha \in \mathbb{Q} \cap (0,1]$. *Then the first four Hodge moduli algebras and Hodge moduli numbers are given as follows.*

(1) For $\alpha \in (0, \frac{1}{12}]$,

$$\begin{aligned}
&M_0 = M_1 = 0, m_0 = m_1 = 0.\\
&M_2 = \text{span}\{1, y, y^2, x, xy, xy^2\}, m_2 = 6.\\
&M_3 = \text{span}\{1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, y^3, xy^3,\\
&\quad y^4, xy^4, y^5, xy^5, y^6, z, xz, yz, xyz, y^2z, xy^2z\}, m_3 = 24.
\end{aligned}$$

(2) For $\alpha \in (\frac{1}{12}, \frac{4}{12}]$,

$$\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \text{span}\{1\}, m_1 = 1.\\
&M_2 = \text{span}\{1, x, x^2, y, xy, y^2, xy^2, y^3, z\}, m_2 = 9.\\
&M_3 = \text{span}\{1, x, x^2, x^3, x^4, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, x^3y^2, y^3, xy^3, x^2y^3,\\
&\quad y^4, xy^4, y^5, xy^5, y^6, z, xz, x^2z, yz, xyz, y^2z, xy^2z, y^3z, z^2\}, m_3 = 30.
\end{aligned}$$

(3) For $\alpha \in (\frac{4}{12}, \frac{5}{12}]$,

$$\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \text{span}\{1, y\}, m_1 = 2.\\
&M_2 = \text{span}\{1, x, x^2, y, xy, y^2, xy^2, y^3, z\}, m_2 = 12.\\
&M_3 = \text{span}\{1, x, x^2, x^3, x^4, y, xy, x^2y, x^3y, x^4y, y^2, xy^2, x^2y^2, x^3y^2,\\
&\quad y^3, xy^3, x^2y^3, y^4, xy^4, x^2y^4, y^5, xy^5, y^6, y^7, z, xz, x^2z, yz, xyz, x^2yz,\\
&\quad y^2z, xy^2z, y^3z, y^4z, z^2, yz^2\}, m_3 = 36.
\end{aligned}$$

(4) For $\alpha \in (\frac{5}{12}, \frac{7}{12}]$,

$$\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \text{span}\{1, x, y\}, m_1 = 3.\\
&M_2 = \text{span}\{1, x, x^2, x^3, y, xy, x^2y, y^2, xy^2, y^3, xy^3, y^4, z, xz, yz\}, m_2 = 15.\\
&M_3 = \text{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, y^2, xy^2, x^2y^2, x^3y^2,\\
&\quad y^3, xy^3, x^2y^3, x^3y^3, y^4, xy^4, x^2y^4, y^5, xy^5, y^6, xy^6, y^7, z, xz, x^2z, x^3z,\\
&\quad yz, xyz, x^2yz, y^2z, xy^2z, y^3z, xy^3z, y^4z, z^2, xz^2, yz^2\}, m_3 = 42.
\end{aligned}$$

(5) For $\alpha \in (\frac{7}{12}, \frac{8}{12}]$,

$$
\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \mathrm{span}\{1, x, y, y^2\}, m_1 = 4.\\
&M_2 = \mathrm{span}\{1, x, x^2, x^3, y, xy, x^2y, y^2, xy^2, x^2y^2, y^3, xy^3, y^4, y^5, z, xz, yz, y^2z\}, m_2 = 18.\\
&M_3 = \mathrm{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, y^2, xy^2, x^2y^2, x^3y^2, x^4y^2,\\
&\quad y^3, xy^3, x^2y^3, x^3y^3, y^4, xy^4, x^2y^4, y^5, xy^5, x^2y^5, y^6, xy^6, y^7, y^8, z, xz, x^2z, x^3z,\\
&\quad yz, xyz, x^2yz, y^2z, xy^2z, x^2y^2z, y^3z, xy^3z, y^4z, y^5z, z^2, xz^2, yz^2, y^2z^2\}, m_3 = 48.
\end{aligned}
$$

(6) For $\alpha \in (\frac{8}{12}, \frac{11}{12}]$,

$$
\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \mathrm{span}\{1, x, y, y^2, xy\}, m_1 = 5.\\
&M_2 = \mathrm{span}\{1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, y^3, xy^3,\\
&\quad y^4, xy^4, y^5, z, xz, yz, xyz, y^2z\}, m_2 = 21.\\
&M_3 = \mathrm{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y, y^2, xy^2, x^2y^2, x^3y^2, x^4y^2,\\
&\quad y^3, xy^3, x^2y^3, x^3y^3, y^4, xy^4, x^2y^4, x^3y^4, y^5, xy^5, x^2y^5, y^6, xy^6, y^7, xy^7, y^8,\\
&\quad z, xz, x^2z, x^3z, yz, xyz, x^2yz, x^3yz, y^2z, xy^2z, x^2y^2z,\\
&\quad y^3z, xy^3z, y^4z, xy^4z, y^5z, z^2, xz^2, yz^2, xyz^2, y^2z^2\}, m_3 = 54.
\end{aligned}
$$

(7) For $\alpha \in (\frac{11}{12}, 1]$,

$$
\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \mathrm{span}\{1, y, y^2, x, xy, xy^2\}, m_1 = 6.\\
&M_2 = \mathrm{span}\{1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, y^3, xy^3,\\
&\quad y^4, xy^4, y^5, xy^5, y^6, z, xz, yz, xyz, y^2z, xy^2z\}, m_2 = 24.\\
&M_3 = \mathrm{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y, y^2, xy^2, x^2y^2, x^3y^2, x^4y^2,\\
&\quad y^3, xy^3, x^2y^3, x^3y^3, y^4, xy^4, x^2y^4, x^3y^4, y^5, xy^5, x^2y^5, y^6, xy^6, x^2y^6,\\
&\quad y^7, xy^7, y^8, xy^8, y^9, z, xz, x^2z, x^3z, yz, xyz, x^2yz, x^3yz, y^2z, xy^2z, x^2y^2z,\\
&\quad y^3z, xy^3z, y^4z, xy^4z, y^5z, xy^5z, y^6z, z^2, xz^2, yz^2, xyz^2, y^2z^2, xy^2z^2\}, m_3 = 60.
\end{aligned}
$$

Let $f(x, y, z) = x^3 + xy^3 + z^2$ *be* E_7 *type singularity. Let* $H = \{f = 0\}$ *be an integral and reduced effective divisor defined by* f*. And* $D^\alpha = \alpha H$ *where* $\alpha \in \mathbb{Q} \cap (0, 1]$*. Then the first four Hodge moduli algebras and Hodge moduli numbers are given as follows.*

$$
\begin{aligned}
&M_0 = M_1 = 0, m_0 = m_1 = 0.\\
&M_2 = \mathrm{span}\{1, x, y, y^2, y^3, y^4, xy\}, m_2 = 7.\\
&M_3 = \mathrm{span}\{1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, y^3, xy^3, x^2y^3,\\
&\quad y^4, xy^4, y^5, xy^5, y^6, xy^6, y^7, z, xz, yz, xyz, y^2z, y^3z, y^4z\}, m_3 = 28.
\end{aligned}
$$

(2) For $\alpha \in (\frac{1}{18}, \frac{5}{18}]$,

$$\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \mathrm{span}\{1\}, m_1 = 1.\\
&M_2 = \mathrm{span}\{1, x, x^2, y, xy, y^2, xy^2, y^3, y^4, z\}, m_2 = 10.\\
&M_3 = \mathrm{span}\{1, x, x^2, x^3, x^4, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, x^3y^2, y^3, xy^3, x^2y^3,\\
&\quad y^4, xy^4, x^2y^4, y^5, xy^5, y^6, xy^6, y^7, z, xz, x^2z, yz, xyz, y^2z, xy^2z,\\
&\quad y^3z, y^4z, z^2\}, m_3 = 34.
\end{aligned}$$

(3) For $\alpha \in (\frac{5}{18}, \frac{7}{18}]$,

$$\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \mathrm{span}\{1, y\}, m_1 = 2.\\
&M_2 = \mathrm{span}\{1, x, x^2, y, xy, x^2y, y^2, xy^2, y^3, xy^3, y^4, z, yz\}, m_2 = 13.\\
&M_3 = \mathrm{span}\{1, x, x^2, x^3, x^4, y, xy, x^2y, x^3y, x^4y, y^2, xy^2, x^2y^2, x^3y^2,\\
&\quad y^3, xy^3, x^2y^3, x^3y^3, y^4, xy^4, x^2y^4, y^5, xy^5, y^6, xy^6, y^7, y^8,\\
&\quad z, xz, x^2z, yz, xyz, x^2yz, y^2z, xy^2z, y^3z, xy^3z, y^4z, z^2, yz^2\}, m_3 = 40.
\end{aligned}$$

(4) For $\alpha \in (\frac{7}{18}, \frac{9}{18}]$,

$$\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \mathrm{span}\{1, x, y\}, m_1 = 3.\\
&M_2 = \mathrm{span}\{1, x, x^2, x^3, y, xy, x^2y, y^2, xy^2, y^3, xy^3, y^4, y^5, z, xz, yz\}, m_2 = 16.\\
&M_3 = \mathrm{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, y^2, xy^2, x^2y^2, x^3y^2,\\
&\quad y^3, xy^3, x^2y^3, x^3y^3, y^4, xy^4, x^2y^4, y^5, xy^5, x^2y^5, y^6, xy^6, y^7, xy^7, y^8,\\
&\quad z, xz, x^2z, x^3z, yz, xyz, x^2yz, y^2z, xy^2z, y^3z, xy^3z, y^4z, y^5z, z^2, xz^2, yz^2\}, m_3 = 46.
\end{aligned}$$

(5) For $\alpha \in (\frac{9}{18}, \frac{11}{18}]$,

$$\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \mathrm{span}\{1, x, y, y^2\}, m_1 = 4.\\
&M_2 = \mathrm{span}\{1, x, x^2, x^3, y, xy, x^2y, y^2, xy^2, x^2y^2, y^3, xy^3, y^4, xy^4, y^5,\\
&\quad z, xz, yz, y^2z\}, m_2 = 19.\\
&M_3 = \mathrm{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, y^2, xy^2, x^2y^2, x^3y^2, x^4y^2,\\
&\quad y^3, xy^3, x^2y^3, x^3y^3, y^4, xy^4, x^2y^4, x^3y^4, y^5, xy^5, x^2y^5, y^6, xy^6, y^7, xy^7, y^8, y^9,\\
&\quad z, xz, x^2z, x^3z, yz, xyz, x^2yz, y^2z, xy^2z, x^2y^2z, y^3z, xy^3z,\\
&\quad y^4z, xy^4z, y^5z, z^2, xz^2, yz^2, y^2z^2\}, m_3 = 52.
\end{aligned}$$

(6) For $\alpha \in (\frac{11}{18}, \frac{13}{18}]$,

$$
\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \operatorname{span}\{1, x, y, y^2, xy\}, m_1 = 5.\\
&M_2 = \operatorname{span}\{1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, y^3, xy^3, y^4, xy^4, y^5, y^6,\\
&\quad z, xz, yz, xyz, y^2z\}, m_2 = 22.\\
&M_3 = \operatorname{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y, y^2, xy^2, x^2y^2, x^3y^2, x^4y^2,\\
&\quad y^3, xy^3, x^2y^3, x^3y^3, y^4, xy^4, x^2y^4, x^3y^4, y^5, xy^5, x^2y^5, y^6, xy^6, x^2y^6,\\
&\quad y^7, xy^7, y^8, xy^8, y^9, z, xz, x^2z, x^3z, yz, xyz, x^2yz, x^3yz, y^2z, xy^2z, x^2y^2z,\\
&\quad y^3z, xy^3z, y^4z, xy^4z, y^5z, y^6z, z^2, xz^2, yz^2, xyz^2, y^2z^2\}, m_3 = 58.
\end{aligned}
$$

(7) For $\alpha \in (\frac{13}{18}, \frac{17}{18}]$,

$$
\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \operatorname{span}\{1, x, y, y^2, y^3, xy\}, m_1 = 6.\\
&M_2 = \operatorname{span}\{1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, y^3, xy^3, x^2y^3,\\
&\quad y^4, xy^4, y^5, xy^5, y^6, z, xz, yz, xyz, y^2z, y^3z\}, m_2 = 25.\\
&M_3 = \operatorname{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y, y^2, xy^2, x^2y^2, x^3y^2, x^4y^2,\\
&\quad y^3, xy^3, x^2y^3, x^3y^3, x^4y^3, y^4, xy^4, x^2y^4, x^3y^4, y^5, xy^5, x^2y^5, x^3y^5, y^6, xy^6, x^2y^6,\\
&\quad y^7, xy^7, y^8, xy^8, y^9, y^{10}, z, xz, x^2z, x^3z, yz, xyz, x^2yz, x^3yz, y^2z, xy^2z, x^2y^2z,\\
&\quad y^3z, xy^3z, x^2y^3z, y^4z, xy^4z, y^5z, xy^5z, y^6z, z^2, xz^2, yz^2, xyz^2,\\
&\quad y^2z^2, y^3z^2\}, m_3 = 64.
\end{aligned}
$$

(8) For $\alpha \in (\frac{17}{18}, 1]$,

$$
\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \operatorname{span}\{1, x, y, y^2, y^3, y^4, xy\}, m_1 = 7.\\
&M_2 = \operatorname{span}\{1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, y^3, xy^3, x^2y^3, y^4, xy^4, y^5, xy^5,\\
&\quad y^6, xy^6, y^7, z, xz, yz, xyz, y^2z, y^3z, y^4z\}, m_2 = 28.\\
&M_3 = \operatorname{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y, y^2, xy^2, x^2y^2, x^3y^2, x^4y^2,\\
&\quad y^3, xy^3, x^2y^3, x^3y^3, x^4y^3, y^4, xy^4, x^2y^4, x^3y^4, y^5, xy^5, x^2y^5, x^3y^5, y^6, xy^6, x^2y^6,\\
&\quad y^7, xy^7, x^2y^7, y^8, xy^8, y^9, xy^9, y^{10}, y^{11}, z, xz, x^2z, x^3z, yz, xyz, x^2yz, x^3yz,\\
&\quad y^2z, xy^2z, x^2y^2z, y^3z, xy^3z, x^2y^3z, y^4z, xy^4z, y^5z, xy^5z, y^6z, xy^6z, y^7z,\\
&\quad z^2, xz^2, yz^2, xyz^2, y^2z^2, y^3z^2, y^4z^2\}, m_3 = 70.
\end{aligned}
$$

Let $f(x, y, z) = x^3 + y^5 + z^2$ *be* E_8 *type singularity. Let* $H = \{f = 0\}$ *be an integral and reduced effective divisor defined by* f*. And* $D^\alpha = \alpha H$ *where* $\alpha \in \mathbb{Q} \cap (0, 1]$*. Then the first four Hodge moduli algebras and Hodge moduli numbers are given as follows.*

(1) For $\alpha \in (0, \frac{1}{30}]$,

$$\begin{aligned}
&M_0 = M_1 = 0, m_0 = m_1 = 0.\\
&M_2 = \text{span}\{1, x, y, xy, y^2, xy^2, y^3, xy^3\}, m_2 = 8.\\
&M_3 = \text{span}\{1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, x^3y^2, y^3, xy^3, x^2y^3, y^4, xy^4,\\
&\quad y^5, xy^5, y^6, xy^6, y^7, xy^7, y^8, z, xz, yz, xyz, y^2z, xy^2z, y^3z, xy^3z\}, m_3 = 32.
\end{aligned}$$

(2) For $\alpha \in (\frac{1}{30}, \frac{7}{30}]$,

$$\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \text{span}\{1\}, m_1 = 1.\\
&M_2 = \text{span}\{1, x, x^2, y, xy, y^2, xy^2, y^3, xy^3, y^4, z\}, m_2 = 11.\\
&M_3 = \text{span}\{1, x, x^2, x^3, x^4, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, x^3y^2, y^3, xy^3, x^2y^3, x^3y^3,\\
&\quad y^4, xy^4, x^2y^4, y^5, xy^5, y^6, xy^6, y^7, xy^7, y^8, z, xz, x^2z, yz, xyz, y^2z, xy^2z,\\
&\quad y^3z, xy^3z, y^4z, z^2\}, m_3 = 38.
\end{aligned}$$

(3) For $\alpha \in (\frac{7}{30}, \frac{11}{30}]$,

$$\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \text{span}\{1, y\}, m_1 = 2.\\
&M_2 = \text{span}\{1, x, x^2, y, xy, x^2y, y^2, xy^2, y^3, xy^3, y^4, y^5, z, yz\}, m_2 = 14.\\
&M_3 = \text{span}\{1, x, x^2, x^3, x^4, y, xy, x^2y, x^3y, x^4y, y^2, xy^2, x^2y^2, x^3y^2, y^3, xy^3, x^2y^3, x^3y^3,\\
&\quad y^4, xy^4, x^2y^4, y^5, xy^5, x^2y^5, y^6, xy^6, y^7, xy^7, y^8, y^9, z, xz, x^2z, yz, xyz, x^2yz, y^2z, xy^2z,\\
&\quad y^3z, xy^3z, y^4z, y^5z, z^2, yz^2\}, m_3 = 44.
\end{aligned}$$

(4) For $\alpha \in (\frac{11}{30}, \frac{13}{30}]$,

$$\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \text{span}\{1, x, y, xy\}, m_1 = 4.\\
&M_2 = \text{span}\{1, x, x^2, x^3, y, xy, x^2y, y^2, xy^2, y^3, xy^3, y^4, xy^4, y^5, xy^5, y^6,\\
&\quad z, xz, yz, xyz\}, m_2 = 20.\\
&M_3 = \text{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, y^2, xy^2, x^2y^2, x^3y^2, y^3, xy^3, x^2y^3, x^3y^3,\\
&\quad y^4, xy^4, x^2y^4, x^3y^4, y^5, xy^5, x^2y^5, y^6, xy^6, x^2y^6, y^7, xy^7, y^8, xy^8, y^9, xy^9, y^{10}\\
&\quad z, xz, x^2z, x^3z, yz, xyz, x^2yz, y^2z, xy^2z, y^3z, xy^3z, y^4z, xy^4z, y^5z, xy^5z, y^6z,\\
&\quad z^2, yz^2, xz^2, xyz^2\}, m_3 = 56.
\end{aligned}$$

(5) For $\alpha \in (\frac{13}{30}, \frac{17}{30}]$,

$$\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \operatorname{span}\{1, x, y, y^2\}, m_1 = 4.\\
&M_2 = \operatorname{span}\{1, x, x^2, x^3, y, xy, x^2y, y^2, xy^2, x^2y^2, y^3, xy^3, y^4, xy^4, y^5, y^6,\\
&\quad z, xz, yz, y^2z\}, m_2 = 20.\\
&M_3 = \operatorname{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, y^2, xy^2, x^2y^2, x^3y^2, x^4y^2,\\
&\quad y^3, xy^3, x^2y^3, x^3y^3, y^4, xy^4, x^2y^4, x^3y^4, y^5, xy^5, x^2y^5, y^6, xy^6, x^2y^6, y^7,\\
&\quad xy^7, y^8, xy^8, y^9, y^{10}, z, xz, x^2z, x^3z, yz, xyz, x^2yz, y^2z, xy^2z, x^2y^2z,\\
&\quad y^3z, xy^3z, y^4z, xy^4z, y^5z, y^6z, z^2, xz^2, yz^2, y^2z^2\}, m_3 = 56.
\end{aligned}$$

(6) For $\alpha \in (\frac{17}{30}, \frac{19}{30}]$,

$$\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \operatorname{span}\{1, x, y, y^2, xy\}, m_1 = 5.\\
&M_2 = \operatorname{span}\{1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, y^3, xy^3, y^4, xy^4, y^5, xy^5, y^6,\\
&\quad z, xz, yz, xyz, y^2z\}, m_2 = 23.\\
&M_3 = \operatorname{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y, y^2, xy^2, x^2y^2, x^3y^2, x^4y^2,\\
&\quad y^3, xy^3, x^2y^3, x^3y^3, y^4, xy^4, x^2y^4, x^3y^4, y^5, xy^5, x^2y^5, x^3y^5, y^6, xy^6, x^2y^6, y^7,\\
&\quad xy^7, y^8, xy^8, y^9, xy^9, y^{10}, z, xz, x^2z, x^3z, yz, xyz, x^2yz, x^3yz, y^2z, xy^2z, x^2y^2z,\\
&\quad y^3z, xy^3z, y^4z, xy^4z, y^5z, xy^5z, y^6z, z^2, xz^2, yz^2, xyz^2, y^2z^2\}, m_3 = 62.
\end{aligned}$$

(7) For $\alpha \in (\frac{19}{30}, \frac{23}{30}]$,

$$\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \operatorname{span}\{1, x, y, xy, y^2, y^3\}, m_1 = 6.\\
&M_2 = \operatorname{span}\{1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, y^3, xy^3, x^2y^3, y^4, xy^4, y^5, xy^5, y^6, y^7,\\
&\quad z, xz, yz, xyz, y^2z, y^3z\}, m_2 = 26.\\
&M_3 = \operatorname{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y, y^2, xy^2, x^2y^2, x^3y^2, x^4y^2,\\
&\quad y^3, xy^3, x^2y^3, x^3y^3, x^4y^3, y^4, xy^4, x^2y^4, x^3y^4, y^5, xy^5, x^2y^5, x^3y^5, y^6, xy^6, x^2y^6, y^7,\\
&\quad xy^7, x^2y^7, y^8, xy^8, y^9, xy^9, y^{10}, y^{11}, z, xz, x^2z, x^3z, yz, xyz, x^2yz, x^3yz,\\
&\quad y^2z, xy^2z, x^2y^2z, y^3z, xy^3z, x^2y^3z, y^4z, xy^4z, y^5z, xy^5z, y^6z, y^7z,\\
&\quad z^2, xz^2, yz^2, xyz^2, y^2z^2, y^3z^2\}, m_3 = 68.
\end{aligned}$$

(8) For $\alpha \in (\frac{23}{30}, \frac{29}{30}]$,

$$\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \operatorname{span}\{1, x, y, xy, y^2, xy^2, y^3\}, m_1 = 7.\\
&M_2 = \operatorname{span}\{1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, x^3y^2, y^3, xy^3, x^2y^3, y^4, xy^4, y^5, xy^5,\\
&\quad y^6, xy^6, y^7, z, xz, yz, xyz, y^2z, xy^2z, y^3z\}, m_2 = 29.
\end{aligned}$$

$M_3 = \text{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y, y^2, xy^2, x^2y^2, x^3y^2, x^4y^2, x^5y^2,$
$y^3, xy^3, x^2y^3, x^3y^3, x^4y^3, y^4, xy^4, x^2y^4, x^3y^4, y^5, xy^5, x^2y^5, x^3y^5, y^6, xy^6, x^2y^6, x^3y^6, y^7,$
$xy^7, x^2y^7, y^8, xy^8, y^9, xy^9, y^{10}, xy^{10}, y^{11}, z, xz, x^2z, x^3z, yz, xyz, x^2yz, x^3yz,$
$y^2z, xy^2z, x^2y^2z, x^3y^2z, y^3z, xy^3z, x^2y^3z, y^4z, xy^4z, y^5z, xy^5z, y^6z, xy^6z, y^7z,$
$z^2, xz^2, yz^2, xyz^2, y^2z^2, xy^2z^2, y^3z^2\}, m_3 = 74.$

(9) For $\alpha \in (\frac{29}{30}, 1]$,

$M_0 = 0, m_0 = 0.$

$M_1 = \text{span}\{1, x, y, xy, y^2, xy^2, y^3, xy^3\}, m_1 = 8.$

$M_2 = \text{span}\{1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, x^3y^2, y^3, xy^3, x^2y^3, y^4, xy^4,$
$y^5, xy^5, y^6, xy^6, y^7, xy^7, y^8, z, xz, yz, xyz, y^2z, xy^2z, y^3z, xy^3z\}, m_2 = 32.$

$M_3 = \text{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y, y^2, xy^2, x^2y^2,$
$x^3y^2, x^4y^2, x^5y^2, y^3, xy^3, x^2y^3, x^3y^3, x^4y^3, y^4, xy^4, x^2y^4, x^3y^4,$
$y^5, xy^5, x^2y^5, x^3y^5, y^6, xy^6, x^2y^6, x^3y^6, y^7, xy^7, x^2y^7, y^8, xy^8, x^2y^8,$
$y^9, xy^9, y^{10}, xy^{10}, y^{11}, xy^{11}, y^{12}, z, xz, x^2z, x^3z, yz, xyz, x^2yz, x^3yz,$
$y^2z, xy^2z, x^2y^2z, x^3y^2z, y^3z, xy^3z, x^2y^3z, y^4z, xy^4z, y^5z, xy^5z, y^6z, xy^6z,$
$y^7z, xy^7z, y^8z, z^2, xz^2, yz^2, xyz^2, y^2z^2, xy^2z^2, y^3z^2, xy^3z^2\}, m_3 = 80.$

The following main result answers the Question 1.6 and verify Conjecture 1.8 for isolated two dimensional rational hypersurface singularities respectively.

For question 1.6, although the authors in [14] showed that one dimensional ADE singularities have finite Hodge properties. We obtain that in the case of two dimensional hypersurface singularities, $\forall \alpha \in (0, 1]$, ADE singularities don't have finite Hodge property. This follows from the following proposition 1.10. Since by tables 3.1, 3.2 and 3.3, different types of singularities have the same Hodge sequences. For example, if we observe A_6 singularity of $\alpha \in (\frac{6}{7}, 1]$, D_6 singularity of $\alpha \in (\frac{9}{10}, 1]$ and E_6 singularity of $\alpha \in (\frac{11}{12}, 1]$. Then we can see their m_i sequence are all the same: $m_0 = 0, m_1 = 6, m_2 = 24, m_3 = 60, \ldots$. Hence the two different families of singularities can not be distinguished by their Hodge sequences.

Main Theorem. *Let* $f, g : (\mathbb{C}^3, 0) \longrightarrow (\mathbb{C}, 0)$ *be two singularities with* $p_g = 0$*. Let* $H_f = \{f = 0\}$ *(resp.* $H_g = \{g = 0\}$*) be integral and reduced effective divisor defined by f (resp. g) and* $D_f^\alpha = \alpha H_f$ *(resp.* $D_g^\alpha = \alpha H_g$*), where* $\alpha \in \mathbb{Q} \cap (0, 1]$*. Then f is contact equivalent to g if and only if* $M_2(D_f^\alpha) \cong M_2(D_g^\alpha)$*. That is to say, let* $F_\Lambda := \{f_\alpha : (\mathbb{C}^3, 0) \longrightarrow (\mathbb{C}, 0) \mid \alpha \in \Lambda\}$ *be the family of isolated two dimensional rational hypersurface singularities, then the Hodge moduli threshold of* F_Λ *is two.*

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2. PRELIMINARY

We first recall the following definition.

Definition 2.1. A polynomial $f \in \mathbb{C}[x_1, \cdots, x_n]$ is called weighted homogeneous if there exists positive rational numbers $w_1, \cdots, w_n$ (i.e., weights of $x_1, \cdots, x_n$) and d such that, $\sum a_i w_i = d$ for each monomial $\prod x_i^{a_i}$ appearing in f with a non-zero coefficient. The number d is called the weighted homogeneous degree (w-deg) of f for weights $w_j, 1 \leq j \leq n$. These $w_j, 1 \leq j \leq n$ called the weight type of f.

The Hodge filtration $F_\bullet$ of $\mathcal{O}_X(*D)$ is usually hard to describe. However, it does have an explicit formula in the case when D is defined by a reduced weighted homogeneous polynomial f which has an isolated singularity at the origin, which is proved by M. Saito [21]. To state Saito's result, we first clarify the notations as follows. We denote

- $\mathcal{O} = \mathbb{C}\{x_1, \ldots, x_n\}$ the ring of germs of holomorphic function for local coordinates $x_1, \ldots, x_n$.
- $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ a germ of holomorphic function that is quasihomogeneous, i.e., $f \in \mathcal{J}(f) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$, and with an isolated singularity at the origin. Kyoji Saito [22] showed that after a biholomorphic coordinate change, we can assume f is a weighted homogeneous polynomial with an isolated singularity at the origin. We will keep this assumption for f unless otherwise stated.
- $w = w(f) = (w_1, \ldots, w_n)$ the weights of the weighted homogeneous polynomial f.
- $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ a germ of a holomorphic function, and we write
$$g = \sum_{A \in \mathbb{N}^n} g_A x^A,$$
where $A = (a_1, \ldots, a_n), g_A \in \mathbb{C}$ and $x^A = x_1^{a_1} \cdots x_n^{a_n}$.
- $\rho(g)$ the weight of an element $g \in \mathcal{O}$ defined by
$$\rho(g) = \left(\sum_{i=1}^{m} w_i\right) + \inf\{\langle w, A\rangle : g_A \neq 0\}.$$
The weight function ρ defines a filtration on $\mathcal{O}$ as
$$\mathcal{O}^{>k} = \{u \in \mathcal{O} : \rho(u) > k\},$$
$$\mathcal{O}^{\geq k} = \{u \in \mathcal{O} : \rho(u) \geq k\}.$$

Since we consider $\mathscr{D}_X$-modules locally around the isolated singularity, so we can assume $X = \mathbb{C}^n$ and identify the stalk at the singularity to be that of $\mathscr{D}_X$-modules on $\mathbb{C}^n$. For example, we replace $F_k\mathcal{O}_{X,0}(*D)$ with $F_k\mathcal{O}_X(*D)$. Now we can state the formula proved by M. Saito (see [21], Theorem 0.7]):

$$F_k\mathcal{O}_X(*D) = \sum_{i=0}^{k} F_{k-i}\mathscr{D}_X\left(\frac{\mathcal{O} \geq i+1}{f^{i+1}}\right), \forall k \in \mathbb{N}. \tag{1}$$

Since the Hodge filtration can be constructed on analogous $\mathscr{D}_X$-modules associated with any effective $\mathbb{Q}$-divisor D, so it satisfies a similar formula in the case when D is supported on a hypersurface defined by such a polynomial f.

Assume that the divisor is $D = \alpha Z$, where $0 < \alpha \leq 1$ and $Z = (f = 0)$ is an integral and reduced effective divisor defined by f, a weighted homogeneous polynomial with an isolated singularity at the origin. In this case, the associated $\mathscr{D}_X$-module is the well-known twisted localization $\mathscr{D}_X$-module $\mathcal{M}\left(f^{1-\alpha}\right) := \mathcal{O}_X(*Z)f^{1-\alpha}$ (see more details in [17] about how to construct the Hodge filtration $F_\bullet\mathcal{M}\left(f^{1-\alpha}\right)$). With new ingredients from Mustaţă and Popa's [MP18 b], where this Hodge filtration is compared to the V-filtration on $\mathcal{M}\left(f^{1-\alpha}\right)$, M. Zhang generalized Saito's formula and proved the following theorem:

Theorem 2.2. *(Zhang, [23]) If $D = \alpha Z$, where $0 < \alpha \leq 1$ and $Z = \{f = 0\}$ is an integral and reduced effective divisor defined by f, a weighted homogeneous polynomial with an isolated singularity at the origin, then we have*

$$F_k\mathcal{M}\left(f^{1-\alpha}\right) = \sum_{i=0}^{k} F_{k-i}\mathscr{D}_X\left(\frac{\mathcal{O}^{\geq\alpha+i}}{f^{i+1}}f^{1-\alpha}\right),$$

where the action $\cdot$ of $\mathscr{D}_X$ on the right hand side is the action on the left $\mathscr{D}_X$-module $\mathcal{M}\left(f^{1-\alpha}\right)$ defined by

$$D\cdot\left(wf^{1-\alpha}\right) := \left(D(w) + w\frac{(1-\alpha)D(f)}{f}\right)f^{1-\alpha}, \text{ for any } D \in \operatorname{Der}_{\mathbb{C}}\mathcal{O}_X.$$

Notice that if we set $\alpha = 1$, Theorem 2.2 recovers Saito's formula (1) mentioned above. For any polynomial f with an isolated singularity at the origin, it is well-known that the Milnor algebra

$$\mathcal{A}_f := \mathbb{C}\{x_1, \ldots, x_n\} / (\partial_1 f, \ldots, \partial_n f)$$

is a finite-dimensional $\mathbb{C}$-vector space. Fix a monomial basis $\{v_1, \ldots, v_\mu\}$ for this vector space, where μ is the dimension of $\mathcal{A}_f$ (i.e., Milnor number). The following theorem follows from Theorem 2.2.

Theorem 2.3. *(Zhang, [23]) If $D = \alpha Z$, where $0 < \alpha \leq 1$ and $Z = \{f = 0\}$ is an integral and reduced effective divisor defined by f, a weighted homogeneous polynomial with an isolated singularity at the origin, then we have*

$$F_0\mathcal{M}(f^{1-\alpha}) = f^{-1}\cdot\mathcal{O}^{\geq\alpha}f^{1-\alpha}$$

and

$$F_k\mathcal{M}(f^{1-\alpha}) = (f^{-1}\cdot\sum_{v_j\in\mathcal{O}^{\geq k+1+\alpha}}\mathcal{O}_X\cdot v_j)f^{1-\alpha} + F_1\mathfrak{D}_X\cdot F_{k-1}\mathcal{M}(f^{1-\alpha}).$$

Alternatively, in terms of Hodge ideals, these formulas say that

$$I_0(D) = \mathcal{O}^{\geq\alpha}$$

and

$$I_{k+1}(D) = \sum_{v_j\in\mathcal{O}^{\geq k+1+\alpha}}\mathcal{O}_X\cdot v_j + \sum_{1\leq i\leq n, a\in I_k(D)}\mathcal{O}_X(f\partial_i a - (\alpha+k)a\partial_i f).$$

3. Proof of the Proposition 1.10

The Proposition 1.10 is divided into the following Proposition 3.1-Proposition 3.5. And we only give proof for A_k type of $\alpha \in (0, \frac{1}{k+1}]$, D_k type of $\alpha \in (0, \frac{1}{2k-2}]$ and E type for simplicity, since technique and methods used in other cases are similar. The calculation can be summarized in two steps, that is, first we use formula in theorem 2.3 to calculate all the Hodge ideals I_0, I_1, I_2, then use our computer program singular to calculate the corresponding Hodge modular algebras M_0, M_1, M_2 and Hodge numbers m_0, m_1, m_2.

3.1. A Type.

Proposition 3.1. *Let $f(x,y,z) = x^{k+1} + y^2 + z^2$ be A_k type singularity, where $k \geq 1$. Let $H = \{f = 0\}$ be an integral and reduced effective divisor defined by f. And $D^\alpha = \alpha H$ where $\alpha \in \mathbb{Q} \cap (0,1]$. Then the first four Hodge moduli algebras and Hodge moduli numbers are given as follows.*
(1) For $\alpha \in (0, \frac{1}{k+1}]$,

$$\begin{aligned}
M_0 =& M_1 = 0, m_0 = m_1 = 0.\\
M_2 =& \operatorname{span}\{1, x, \cdots, x^{k-1}\}, m_2 = k.\\
M_3 =& \operatorname{span}\{1, x, \cdots, x^k, y, xy, \cdots, x^{k-1}y, z, xz, \cdots, x^{k-1}z, z^2, xz^2, \cdots, x^{k-2}z^2\}, m_3 = 4k.
\end{aligned}$$

(2) For $\alpha \in (\frac{i}{k+1}, \frac{i+1}{k+1}]$ with $1 \leq i \leq k-1$,

$$\begin{aligned}
M_0 =& 0, m_0 = 0.\\
M_1 =& \operatorname{span}\{1, \cdots, x^{i-1}\}, m_1 = i.\\
M_2 =& \operatorname{span}\{1, x, \cdots, x^k, y, xy, \cdots, x^{i-1}y, z, xz, \cdots, x^{i-1}z, z^2, xz^2, \cdots, x^{i-2}z^2\}, m_2 = k + 3i.\\
M_3 =& \operatorname{span}\{1, x, \ldots, x^{2k+i-1}, y, xy, \ldots, x^{k+i-1}y, y^2, xy^2, \ldots, x^{i-1}y^2, z, xz, \ldots, x^{k+i-1}z,\\
& yz, xyz, \ldots, x^{i-1}yz, z^2, xz^2, \ldots, x^{i-1}z^2\}, m_3 = 4k + 6i.
\end{aligned}$$

(3) For $\alpha \in (\frac{k}{k+1}, 1]$,

$$\begin{aligned}
M_0 =& 0, m_0 = 0.\\
M_1 =& \operatorname{span}\{1, \cdots, x^{k-1}\}, m_1 = k.\\
M_2 =& \operatorname{span}\{1, x, \cdots, x^k, y, xy, \cdots, x^{k-1}y, z, xz, \cdots, x^{k-1}z, z^2, xz^2, \cdots, x^{k-2}z^2\}, m_2 = 4k.\\
M_3 =& \operatorname{span}\{1, x, \ldots, x^{3k-1}, y, xy, \ldots, x^{2k-1}y, y^2, xy^2, \ldots, x^{k-1}y^2, z, xz, \ldots, x^{2k-1}z,\\
& yz, xyz, \ldots, x^{k-1}yz, z^2, xz^2, \ldots, x^{k-1}z^2\}, m_3 = 10k.
\end{aligned}$$

Proof. If $\alpha \in (0, \frac{1}{1+k}]$, we obtain

$$\begin{aligned}
I_0 =& \mathcal{O}_X,\\
I_1 =& \sum_{v_j \in \mathcal{O}^{\geq 1+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_0(D)} \mathcal{O}_X(f\partial_x a - \alpha a \partial_x f) + \mathcal{O}_X(f\partial_y a - \alpha a \partial_y f) + \mathcal{O}_X(f\partial_z a - \alpha a \partial_z f)\\
=& \mathcal{O}_X(1) + (x^k, y, z)\\
=& \mathcal{O}_X(1),\\
I_2 =& \sum_{v_j \in \mathcal{O}^{\geq 2+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_1(D)} \mathcal{O}_X(f\partial_x a - (\alpha+1) a \partial_x f)\\
&+ \mathcal{O}_X(f\partial_y a - (\alpha+1) a \partial_y f) + \mathcal{O}_X(f\partial_z a - (\alpha+1) a \partial_z f)\\
=& (x^k, y, z),
\end{aligned}$$

$$
\begin{aligned}
I_3 =& \sum_{v_j \in \mathcal{O}^{\geq 3+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_2(D)} \mathcal{O}_X(f\partial_x a - (\alpha+2)a\partial_x f) \\
&+ \mathcal{O}_X(f\partial_y a - (\alpha+2)a\partial_y f) + \mathcal{O}_X(f\partial_z a - (\alpha+2)a\partial_z f) \\
=&((-1-a-ak)x^{2k} + kx^{-1+k}y^2 + kx^{-1+k}z^2, -(2a+2)x^k y, -(2+2a)x^k z, \\
&(-1-a-k-ak)x^k y, x^{1+k} - (2a+1)y^2 + z^2, -(2+2a)yz, \\
&(-1-a-k-ak)x^k z, -(2+2a)yz, x^{1+k} + y^2 - (1+2a)z^2) \\
=&(-(1+a+ak)x^{2k} + kx^{k-1}y^2 + kx^{k-1}z^2, x^k y, x^k z, \\
&x^{1+k} - (2a+1)y^2 + z^2, yz, x^{1+k} + y^2 - (1+2a)z^2) \\
=&(-(1+a+ak)x^{2k} + kx^{k-1}y^2 + kx^{k-1}z^2, x^k y, x^k z, yz, x^{k+1}, y^2 - z^2) \\
=&(x^{k-1}y^2, x^{k-1}z^2, x^k y, x^k z, yz, x^{k+1}, y^2 - z^2).
\end{aligned}
$$

Hence, it is clear that M_0, M_1 have the forms on the above. For the other M_i, we just need to find a basis for them. For M_2, since x^k, y and $z \in I_2$, we obtain $1, x, \cdots, x^{k-1}$ form a basis. For M_3, we use computer program Singular to obtain this modular algebra from Hodge ideal I_3. since $x^{k+1} \in I_3$, we add $1, x, \cdots, x^k$ into a basis of M_3. Similarly since $x^k y, x^k z$ and $x^{k-1}z^2 \in I_3$, we can add $y, xy, \cdots, x^{k-1}y$, $z, xz, \cdots, x^{k-1}z$ and $z^2, xz^2, \cdots, x^{k-2}z^2$ into this basis. Finally, we note that $y^2 - z^2 \in I_2$, which means y^2 equals to z^2 in M_2. So those z^2 on the above can also be changed into y^2.

□

Here is a table with related to $\{m_0, m_1, m_2, m_3\}$ for some specific k.

Type	α	m_0	m_1	m_2	m_3
A_6	$(0, \frac{1}{7}]$	0	0	6	24
A_6	$(\frac{1}{7}, \frac{2}{7}]$	0	1	9	30
A_6	$(\frac{2}{7}, \frac{3}{7}]$	0	2	12	36
A_6	$(\frac{3}{7}, \frac{4}{7}]$	0	3	15	42
A_6	$(\frac{4}{7}, \frac{5}{7}]$	0	4	18	48
A_6	$(\frac{5}{7}, \frac{6}{7}]$	0	5	21	54
A_6	$(\frac{6}{7}, 1]$	0	6	24	60
A_7	$(0, \frac{1}{8}]$	0	0	7	28
A_7	$(\frac{1}{8}, \frac{2}{8}]$	0	1	10	34
A_7	$(\frac{2}{8}, \frac{3}{8}]$	0	2	13	40
A_7	$(\frac{3}{8}, \frac{4}{8}]$	0	3	16	46
A_7	$(\frac{4}{8}, \frac{5}{8}]$	0	4	19	52
A_7	$(\frac{5}{8}, \frac{6}{8}]$	0	5	22	58
A_7	$(\frac{6}{8}, \frac{7}{8}]$	0	6	25	64
A_7	$(\frac{7}{8}, 1]$	0	7	28	70
A_8	$(0, \frac{1}{9}]$	0	0	8	32
A_8	$(\frac{1}{9}, \frac{2}{9}]$	0	1	11	38
A_8	$(\frac{2}{9}, \frac{3}{9}]$	0	2	14	44
A_8	$(\frac{3}{9}, \frac{4}{9}]$	0	3	17	50

A_8	$(\frac{4}{9}, \frac{5}{9}]$	0	4	20	56
A_8	$(\frac{5}{9}, \frac{6}{9}]$	0	5	23	62
A_8	$(\frac{6}{9}, \frac{7}{9}]$	0	6	26	68
A_8	$(\frac{7}{9}, \frac{8}{9}]$	0	7	29	74
A_8	$(\frac{8}{9}, 1]$	0	8	32	80

3.2. D Type.

Proposition 3.2. *Let $f(x,y) = x^{k-1} + xy^2 + z^2$ be D_k type singularity, where $k \geq 4$. Let $H = \{f = 0\}$ be an integral and reduced effective divisor defined by f. And $D^\alpha = \alpha H$ where $\alpha \in \mathbb{Q} \cap (0, 1]$. Then the first four Hodge moduli algebras and Hodge moduli numbers are given as follows. For D_k, when k is even,*
(1) For $\alpha \in (0, \frac{1}{2k-2}]$,

$$M_0 = M_1 = 0, m_0 = m_1 = 0.$$
$$M_2 = \text{span}\{1, x, \cdots, x^{k-2}, y\}, m_2 = k.$$
$$M_3 = \text{span}\{1, x, \cdots, x^{k-1}, y, xy, \cdots, x^{k-1}y, y^2, xy^2, \cdots, x^{k-2}y^2, y^3, z, xz, \cdots, x^{k-2}z, yz\},$$
$$m_3 = 4k.$$

(2) For $\alpha \in (\frac{2i-3}{2k-2}, \frac{2i-1}{2k-2}]$, $2 \leq i \leq \frac{k}{2}$,

$$M_0 = 0, m_0 = 0.$$
$$M_1 = \text{span}\{1, x\cdots, x^{i-2}\}, m_1 = i - 1.$$
$$M_2 = \text{span}\{1, x, \cdots, x^{k+i-4}, y, xy, \cdots, x^{i-1}y, y^2, z, xz, \cdots, x^{i-2}z\}, m_2 = k + 3i - 3.$$
$$M_3 = \text{span}\{1, x, \cdots, x^{2k+i-6}, y, xy, \cdots, x^{k+i-3}y, y^2, xy^2, \cdots, x^iy^2, y^3, xy^3, y^4,$$
$$z, xz, \cdots, x^{k+i-4}z, yz, xyz, \cdots, x^{i-1}yz, y^2z, z^2, \cdots, x^{i-2}z^2\}, m_3 = 4k + 6i - 6.$$

(3) For $\alpha \in (\frac{2i-3}{2k-2}, \frac{2i-1}{2k-2}]$, $\frac{k}{2} + 1 \leq i \leq k - 1$,

$$M_0 = 0, m_0 = 0.$$
$$M_1 = \text{span}\{1, x\cdots, x^{i-2}, y\}, m_1 = i.$$
$$M_2 = \text{span}\{1, x, \cdots, x^{k+i-4}, y, xy, \cdots, x^{i-1}y, y^2, xy^2, y^3z, xz, \cdots, x^{i-2}z, yz\},$$
$$m_2 = k + 3i.$$
$$M_3 = \text{span}\{1, x, \cdots, x^{2k+i-6}, y, xy, \cdots, x^{k+i-3}y, y^2, xy^2, \cdots, x^iy^2, y^3, xy^3, x^2y^3,$$
$$y^4, xy^4, y^5, z, xz, \cdots, x^{k+i-4}z, yz, xyz, \cdots, x^{i-1}yz, y^2z, xy^2z, y^3z,$$
$$z^2, \cdots, x^{i-2}z^2, yz^2\}, m_3 = 4k + 6i.$$

(4) For $\alpha \in (\frac{2k-3}{2k-2}, 1]$,

$$M_0 = 0, m_0 = 0.$$
$$M_1 = \text{span}\{1, \cdots, x^{k-2}, y\}, m_1 = k.$$
$$M_2 = \text{span}\{1, x, \cdots, x^{2k-4}, y, xy, \cdots, x^{k-1}y, y^2, xy^2, y^3$$
$$z, xz, \cdots, x^{k-2}z, yz\}, m_2 = 4k.$$
$$M_3 = \text{span}\{1, x, \cdots, x^{3k-6}, y, xy, \cdots, x^{2k-3}y, y^2, xy^2, \cdots, x^ky^2, y^3, xy^3, x^2y^3,$$
$$y^4, xy^4, y^5, z, xz, \cdots, x^{2k-4}z, yz, xyz, \cdots, x^{k-1}yz, y^2z, xy^2z, y^3z,$$
$$z^2, \cdots, x^{k-2}z^2, yz^2\}, m_3 = 10k.$$

When k is odd,
(1) For $\alpha \in (0, \frac{1}{2k-2}]$,

$$\begin{aligned} &M_0 = M_1 = 0, m_0 = m_1 = 0.\\ &M_2 = \mathrm{span}\{1, x, \cdots, x^{k-2}, y\}, m_2 = k.\\ &M_3 = \mathrm{span}\{1, x, \cdots, x^{k-1}, y, xy, \cdots, x^{k-1}y, y^2, xy^2, \cdots, x^{k-2}y^2, y^3, z, xz, \cdots, x^{k-2}z, yz\},\\ &\quad m_3 = 4k. \end{aligned}$$

(2) For $\alpha \in (\frac{2i-3}{2k-2}, \frac{2i-1}{2k-2}]$, $2 \le i \le \frac{k-1}{2}$,

$$\begin{aligned} &M_0 = 0, m_0 = 0.\\ &M_1 = \mathrm{span}\{1, x \cdots, x^{i-2}\}, m_1 = i-1.\\ &M_2 = \mathrm{span}\{1, x, \cdots, x^{k+i-4}, y, xy, \cdots, x^{i-1}y, y^2, z, xz, \cdots, x^{i-2}z\}, m_2 = k+3i-3.\\ &M_3 = \mathrm{span}\{1, x, \cdots, x^{2k+i-6}, y, xy, \cdots, x^{k+i-3}y, y^2, xy^2, \cdots, x^{i}y^2, y^3, xy^3, y^4,\\ &\quad z, xz, \cdots, x^{k+i-4}z, yz, xyz, \cdots, x^{i-1}yz, y^2z, z^2, \cdots, x^{i-2}z^2\}, m_3 = 4k+6i-6. \end{aligned}$$

(3) For $\alpha \in (\frac{k-2}{2k-2}, \frac{k-1}{2k-2}]$,

$$\begin{aligned} &M_0 = 0, m_0 = 0.\\ &M_1 = \mathrm{span}\{1, x \cdots, x^{\frac{k-1}{2}}\}, m_1 = \frac{k+1}{2}.\\ &M_2 = \mathrm{span}\{1, x, \cdots, x^{\frac{3k-5}{2}}, y, xy, \cdots, x^{\frac{k+1}{2}}y, y^2, z, xz, \cdots, x^{\frac{k-1}{2}}z\}, m_2 = \frac{5k+3}{2}.\\ &M_3 = \mathrm{span}\{1, x, \cdots, x^{\frac{5k-9}{2}}, y, xy, \cdots, x^{\frac{3k-3}{2}}y, y^2, xy^2, \cdots, x^{\frac{k+3}{2}}y^2, y^3, xy^3, y^4,\\ &\quad z, xz, \cdots, x^{\frac{3k-5}{2}}z, yz, xyz, \cdots, x^{\frac{k+1}{2}}yz, y^2z, z^2, \cdots, x^{\frac{k-1}{2}}z^2\}, m_3 = 7k+3. \end{aligned}$$

(4) For $\alpha \in (\frac{k-1}{2k-2}, \frac{k}{2k-2}]$,

$$\begin{aligned} &M_0 = 0, m_0 = 0.\\ &M_1 = \mathrm{span}\{1, x \cdots, x^{i-2}, y\}, m_1 = \frac{k+3}{2}.\\ &M_2 = \mathrm{span}\{1, x, \cdots, x^{\frac{3k-5}{2}}, y, xy, \cdots, x^{\frac{k+1}{2}}y, y^2, xy^2, y^3\\ &\quad z, xz, \cdots, x^{\frac{k-1}{2}}z, yz\}, m_2 = \frac{5k+9}{2}.\\ &M_3 = \mathrm{span}\{1, x, \cdots, x^{\frac{5k-9}{2}}, y, xy, \cdots, x^{\frac{3k-3}{2}}y, y^2, xy^2, \cdots, x^{\frac{k+3}{2}}y^2,\\ &\quad y^3, xy^3, x^2y^3, y^4, xy^4, y^5, z, xz, \cdots, x^{\frac{3k-5}{2}}z, yz, xyz, \cdots, x^{\frac{k+1}{2}}yz,\\ &\quad y^2z, xy^2z, y^3z, z^2, \cdots, x^{\frac{k-1}{2}}z^2, yz^2\}, m_3 = 7k+9. \end{aligned}$$

(5) For $\alpha \in (\frac{2i-3}{2k-2}, \frac{2i-1}{2k-2}]$, $\frac{k+3}{2} \le i \le k-1$,

$$M_0 = 0, m_0 = 0.$$
$$M_1 = \text{span}\{1, x \cdots, x^{i-2}, y\}, m_1 = i.$$
$$M_2 = \text{span}\{1, x, \cdots, x^{k+i-4}, y, xy, \cdots, x^{i-1}y, y^2, xy^2, y^3z, xz, \cdots, x^{i-2}z, yz\},$$
$$m_2 = k + 3i.$$
$$M_3 = \text{span}\{1, x, \cdots, x^{2k+i-6}, y, xy, \cdots, x^{k+i-3}y, y^2, xy^2, \cdots, x^iy^2, y^3, xy^3, x^2y^3,$$
$$y^4, xy^4, y^5, z, xz, \cdots, x^{k+i-4}z, yz, xyz, \cdots, x^{i-1}yz, y^2z, xy^2z, y^3z,$$
$$z^2, \cdots, x^{i-2}z^2, yz^2\}, m_3 = 4k + 6i.$$

(6) For $\alpha \in (\frac{2k-3}{2k-2}, 1]$,

$$M_0 = 0, m_0 = 0.$$
$$M_1 = \text{span}\{1, \cdots, x^{k-2}, y\}, m_1 = k.$$
$$M_2 = \text{span}\{1, x, \cdots, x^{2k-4}, y, xy, \cdots, x^{k-1}y, y^2, xy^2, y^3z, xz, \cdots, x^{k-2}z, yz\},$$
$$m_2 = 4k.$$
$$M_3 = \text{span}\{1, x, \cdots, x^{3k-6}, y, xy, \cdots, x^{2k-3}y, y^2, xy^2, \cdots, x^ky^2, y^3, xy^3, x^2y^3,$$
$$y^4, xy^4, y^5, z, xz, \cdots, x^{2k-4}z, yz, xyz, \cdots, x^{k-1}yz, y^2z, xy^2z, y^3z,$$
$$z^2, \cdots, x^{k-2}z^2, yz^2\}, m_3 = 10k.$$

Proof. If $\alpha \in (0, \frac{1}{2k-2}]$, we obtain

$$\begin{aligned}
I_0 =& I_1 = \mathcal{O}_X,\\
I_2 =& \sum_{v_j \in \mathcal{O}^{\ge 2+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_1(D)} \mathcal{O}_X(f\partial_x a - (\alpha+1)a\partial_x f)\\
&+ \mathcal{O}_X(f\partial_y a - (\alpha+1)a\partial_y f) + \mathcal{O}_X(f\partial_z a - (\alpha+1)a\partial_z f)\\
=&((k-1)x^{k-2} + y^2, xy, z),\\
I_3 =& \sum_{v_j \in \mathcal{O}^{\ge 3+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_2(D)} \mathcal{O}_X(f\partial_x a - (\alpha+2)a\partial_x f)\\
&+ \mathcal{O}_X(f\partial_y a - (\alpha+2)a\partial_y f) + \mathcal{O}_X(f\partial_z a - (\alpha+2)a\partial_z f)\\
=&(((1+\alpha-\alpha k^2)x^{-4+2k} + (4+2\alpha-5k-2\alpha k+k^2)x^{-2+k}y^2 - (\alpha+1)y^4 + (2-3k+k^2)x^{-3+k})z^2,\\
&(4+2\alpha-2k-2\alpha k)x^{-1+k}y - 2\alpha xy^3 + 2yz^2, (2+2\alpha-2k-2\alpha k)x^{-2+k} - (2\alpha+2)y^2)z,\\
&(2+\alpha-k-\alpha k)x^{-1+k}y - \alpha xy^3 + yz^2, x^k - (2\alpha+1)x^2y^2 + xz^2, (-2-2\alpha)xyz,\\
&((1+\alpha-k-\alpha k)x^{-2+k} - (\alpha+1)y^2)z, (-2-2\alpha)xyz, x^{-1+k} + xy^2 - (1+2\alpha)z^2)\\
=&(((1+\alpha-\alpha k^2)x^{-4+2k} + (4+2\alpha-5k-2\alpha k+k^2)x^{-2+k}y^2 - (\alpha+1)y^4 + (2-3k+k^2)x^{-3+k})z^2,\\
&(2+\alpha-k-\alpha k)x^{-1+k}y - \alpha xy^3 + yz^2, x^k - (2\alpha+1)x^2y^2 + xz^2, (1-k)x^{k-2}z - y^2z,\\
&xyz, x^{-1+k} + xy^2 - (1+2\alpha)z^2).
\end{aligned}$$

Hence, it is clear that M_0, M_1 is the form on the above. For the other M_i, we just need to find a basis for them. For M_2, since $(k-1)x^{k-2} + y^2, xy$ and $z \in I_2$, we obtain that $1, x, \cdots, x^{k-2}, y$ form a basis. For M_3, we obtain the results by program SINGULAR.

□

Here is a table with related to $\{m_0, m_1, m_2, m_3\}$ for some specific k.

Type	α	m_0	m_1	m_2	m_3
D_6	$(0, \frac{1}{10}]$	0	0	6	24
D_6	$(\frac{1}{10}, \frac{3}{10}]$	0	1	9	30
D_6	$(\frac{3}{10}, \frac{5}{10}]$	0	2	12	36
D_6	$(\frac{5}{10}, \frac{7}{10}]$	0	4	18	48
D_6	$(\frac{7}{10}, \frac{9}{10}]$	0	5	21	54
D_6	$(\frac{9}{10}, 1]$	0	6	24	60
D_7	$(0, \frac{1}{12}]$	0	0	7	28
D_7	$(\frac{1}{12}, \frac{3}{12}]$	0	1	10	34
D_7	$(\frac{3}{12}, \frac{5}{12}]$	0	2	13	40
D_7	$(\frac{5}{12}, \frac{6}{12}]$	0	4	19	52
D_7	$(\frac{6}{12}, \frac{7}{12}]$	0	5	22	58
D_7	$(\frac{7}{12}, \frac{9}{12}]$	0	5	22	58
D_7	$(\frac{9}{12}, \frac{11}{12}]$	0	6	25	64
D_7	$(\frac{11}{12}, 1]$	0	7	28	70
D_8	$(0, \frac{1}{14}]$	0	0	8	32
D_8	$(\frac{1}{14}, \frac{3}{14}]$	0	1	11	38
D_8	$(\frac{3}{14}, \frac{5}{14}]$	0	2	14	44
D_8	$(\frac{5}{14}, \frac{7}{14}]$	0	4	17	50
D_8	$(\frac{7}{14}, \frac{9}{14}]$	0	5	23	62
D_8	$(\frac{9}{14}, \frac{11}{14}]$	0	6	26	68
D_8	$(\frac{11}{14}, \frac{13}{14}]$	0	7	29	74
D_8	$(\frac{13}{14}, 1]$	0	8	32	80

3.3. E Type.

Proposition 3.3. *Let $f(x,y,z) = x^3 + y^4 + z^2$ be E_6 type singularity. Let $H = \{f = 0\}$ be an integral and reduced effective divisor defined by f. And $D^\alpha = \alpha H$ where $\alpha \in \mathbb{Q} \cap (0,1]$. Then the first four Hodge moduli algebras and Hodge moduli numbers are given as follows.*
(1) For $\alpha \in (0, \frac{1}{12}]$,

$$
\begin{aligned}
&M_0 = M_1 = 0, m_0 = m_1 = 0.\\
&M_2 = \text{span}\{1, y, y^2, x, xy, xy^2\}, m_2 = 6.\\
&M_3 = \text{span}\{1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, y^3, xy^3,\\
&\quad y^4, xy^4, y^5, xy^5, y^6, z, xz, yz, xyz, y^2z, xy^2z\}, m_3 = 24.
\end{aligned}
$$

(2) For $\alpha \in (\frac{1}{12}, \frac{4}{12}]$,

$$
\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \text{span}\{1\}, m_1 = 1.\\
&M_2 = \text{span}\{1, x, x^2, y, xy, y^2, xy^2, y^3, z\}, m_2 = 9.\\
&M_3 = \text{span}\{1, x, x^2, x^3, x^4, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, x^3y^2, y^3, xy^3, x^2y^3,\\
&\quad y^4, xy^4, y^5, xy^5, y^6, z, xz, x^2z, yz, xyz, y^2z, xy^2z, y^3z, z^2\}, m_3 = 30.
\end{aligned}
$$

(3) For $\alpha \in (\frac{4}{12}, \frac{5}{12}]$,

$$
\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \text{span}\{1, y\}, m_1 = 2.\\
&M_2 = \text{span}\{1, x, x^2, y, xy, y^2, xy^2, y^3, z\}, m_2 = 12.\\
&M_3 = \text{span}\{1, x, x^2, x^3, x^4, y, xy, x^2y, x^3y, x^4y, y^2, xy^2, x^2y^2, x^3y^2,\\
&\quad y^3, xy^3, x^2y^3, y^4, xy^4, x^2y^4, y^5, xy^5, y^6, y^7, z, xz, x^2z, yz, xyz, x^2yz,\\
&\quad y^2z, xy^2z, y^3z, y^4z, z^2, yz^2\}, m_3 = 36.
\end{aligned}
$$

(4) For $\alpha \in (\frac{5}{12}, \frac{7}{12}]$,

$$
\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \text{span}\{1, x, y\}, m_1 = 3.\\
&M_2 = \text{span}\{1, x, x^2, x^3, y, xy, x^2y, y^2, xy^2, y^3, xy^3, y^4, z, xz, yz\}, m_2 = 15.\\
&M_3 = \text{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, y^2, xy^2, x^2y^2, x^3y^2,\\
&\quad y^3, xy^3, x^2y^3, x^3y^3, y^4, xy^4, x^2y^4, y^5, xy^5, y^6, xy^6, y^7, z, xz, x^2z, x^3z,\\
&\quad yz, xyz, x^2yz, y^2z, xy^2z, y^3z, xy^3z, y^4z, z^2, xz^2, yz^2\}, m_3 = 42.
\end{aligned}
$$

(5) For $\alpha \in (\frac{7}{12}, \frac{8}{12}]$,

$$
\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \text{span}\{1, x, y, y^2\}, m_1 = 4.\\
&M_2 = \text{span}\{1, x, x^2, x^3, y, xy, x^2y, y^2, xy^2, x^2y^2, y^3, xy^3, y^4, y^5, z, xz, yz, y^2z\}, m_2 = 18.\\
&M_3 = \text{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, y^2, xy^2, x^2y^2, x^3y^2, x^4y^2,\\
&\quad y^3, xy^3, x^2y^3, x^3y^3, y^4, xy^4, x^2y^4, y^5, xy^5, x^2y^5, y^6, xy^6, y^7, y^8, z, xz, x^2z, x^3z,\\
&\quad yz, xyz, x^2yz, y^2z, xy^2z, x^2y^2z, y^3z, xy^3z, y^4z, y^5z, z^2, xz^2, yz^2, y^2z^2\}, m_3 = 48.
\end{aligned}
$$

(6) For $\alpha \in (\frac{8}{12}, \frac{11}{12}]$,

$$
\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \text{span}\{1, x, y, y^2, xy\}, m_1 = 5.\\
&M_2 = \text{span}\{1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, y^3, xy^3,\\
&\quad y^4, xy^4, y^5, z, xz, yz, xyz, y^2z\}, m_2 = 21.\\
&M_3 = \text{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y, y^2, xy^2, x^2y^2, x^3y^2, x^4y^2,\\
&\quad y^3, xy^3, x^2y^3, x^3y^3, y^4, xy^4, x^2y^4, x^3y^4, y^5, xy^5, x^2y^5, y^6, xy^6, y^7, xy^7, y^8,\\
&\quad z, xz, x^2z, x^3z, yz, xyz, x^2yz, x^3yz, y^2z, xy^2z, x^2y^2z,\\
&\quad y^3z, xy^3z, y^4z, xy^4z, y^5z, z^2, xz^2, yz^2, xyz^2, y^2z^2\}, m_3 = 54.
\end{aligned}
$$

(7) For $\alpha \in (\frac{11}{12}, 1]$,

$$\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \text{span}\{1, y, y^2, x, xy, xy^2\}, m_1 = 6.\\
&M_2 = \text{span}\{1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, y^3, xy^3,\\
&\quad y^4, xy^4, y^5, xy^5, y^6, z, xz, yz, xyz, y^2z, xy^2z\}, m_2 = 24.\\
&M_3 = \text{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y, y^2, xy^2, x^2y^2, x^3y^2, x^4y^2,\\
&\quad y^3, xy^3, x^2y^3, x^3y^3, y^4, xy^4, x^2y^4, x^3y^4, y^5, xy^5, x^2y^5, y^6, xy^6, x^2y^6,\\
&\quad y^7, xy^7, y^8, xy^8, y^9, z, xz, x^2z, x^3z, yz, xyz, x^2yz, x^3yz, y^2z, xy^2z, x^2y^2z,\\
&\quad y^3z, xy^3z, y^4z, xy^4z, y^5z, xy^5z, y^6z, z^2, xz^2, yz^2, xyz^2, y^2z^2, xy^2z^2\}, m_3 = 60.
\end{aligned}$$

Proof. (1) If $\alpha \in (0, \frac{1}{12}]$, we obtain

$$\begin{aligned}
I_0 =& I_1 = \mathcal{O}_X,\\
I_2 =& \sum_{v_j \in \mathcal{O}^{\geq 2+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_1(D)} \mathcal{O}_X(f\partial_x a - (\alpha+1)a\partial_x f)\\
&+ \mathcal{O}_X(f\partial_y a - (\alpha+1)a\partial_y f) + \mathcal{O}_X(f\partial_z a - (\alpha+1)a\partial_z f)\\
=&((-3\alpha-3)x^2, -4\alpha y^3 - 4y^3, -2z - 2\alpha z),\\
I_3 =& \sum_{v_j \in \mathcal{O}^{\geq 3+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_2(D)} \mathcal{O}_X(f\partial_x a - (\alpha+2)a\partial_x f)\\
&+ \mathcal{O}_X(f\partial_y a - (\alpha+2)a\partial_y f) + \mathcal{O}_X(f\partial_z a - (\alpha+2)a\partial_z f)\\
=&((9\alpha^2 + 21\alpha + 12)x^4 + (-6\alpha y^4 - 6\alpha z^2 - 6y^4 - 6z^2)x,\\
&(12\alpha^2y^3 + 36\alpha y^3 + 24y^3)x^2,\\
&(-12\alpha y^2 - 12y^2)x^3 + 16\alpha^2y^6 + 36\alpha y^6 - 12\alpha y^2z^2 + 20y^6 - 12y^2z^2,\\
&(6z\alpha^2 + 18z\alpha + 12z)x^2,\\
&8z\alpha^2y^3 + 24z\alpha y^3 + 16zy^3,\\
&(-2\alpha-2)x^3 + 4\alpha^2z^2 - 2\alpha y^4 + 10\alpha z^2 - 2y^4 + 6z^2).
\end{aligned}$$

Hence, we can obtain the results by definition.
(2) If $\alpha \in (\frac{1}{12}, \frac{4}{12}]$, we obtain

$$\begin{aligned}
I_0 =& \mathcal{O}_X,\\
I_1 =& \sum_{v_j \in \mathcal{O}^{\geq 1+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_0(D)} \mathcal{O}_X(f\partial_x a - \alpha a\partial_x f) + \mathcal{O}_X(f\partial_y a - \alpha a\partial_y f) + \mathcal{O}_X(f\partial_z a - \alpha a\partial_z f)\\
=& \mathcal{O}_X(x, y) + \mathcal{O}_X(x^2, y^3, z)\\
=&(x, y, z),\\
I_2 =& \sum_{v_j \in \mathcal{O}^{\geq 2+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_1(D)} \mathcal{O}_X(f\partial_x a - (\alpha+1)a\partial_x f)
\end{aligned}$$

$$
\begin{aligned}
&+\mathcal{O}_X(f\partial_y a-(\alpha+1)a\partial_y f)+\mathcal{O}_X(f\partial_z a-(\alpha+1)a\partial_z f)\\
=&((-3\alpha-2)x^3+y^4+z^2,(-4\alpha y^3-4y^3)x,(-2z-2\alpha z)x,\\
&(-3y-3\alpha y)x^2,x^3+z^2-3y^4-4\alpha y^4,-2yz-2\alpha yz,\\
&(-3z-3\alpha z)x^2,-4y^3z-4\alpha y^3z,x^3+y^4-2\alpha z^2-z^2),\\
I_3=&\sum_{v_j\in\mathcal{O}^{\geq 3+\alpha}}\mathcal{O}_X v_j+\sum_{a\in I_2(D)}\mathcal{O}_X(f\partial_x a-(\alpha+2)a\partial_x f)\\
&+\mathcal{O}_X(f\partial_y a-(\alpha+2)a\partial_y f)+\mathcal{O}_X(f\partial_z a-(\alpha+2)a\partial_z f)\\
=&((9\alpha^2+15\alpha+6)x^5+(-12\alpha y^4-12\alpha z^2-12y^4-12z^2)x^2,\\
&(12\alpha^2y^3+32\alpha y^3+20y^3)x^3-4y^3z^2-4\alpha y^7-4y^7-4\alpha y^3z^2,\\
&(-12\alpha y^2-12y^2)x^4+(16\alpha^2y^6+36\alpha y^6-12\alpha y^2z^2+20y^6-12y^2z^2)x,\\
&(6z\alpha^2+16z\alpha+10z)x^3-2\alpha z^3-2y^4z-2z^3-2\alpha y^4z,\\
&(8z\alpha^2y^3+24z\alpha y^3+16zy^3)x,\\
&(-2\alpha-2)x^4+(4\alpha^2z^2-2\alpha y^4+10\alpha z^2-2y^4+6z^2)x,\\
&(9y\alpha^2+21y\alpha+12y)x^4+(-6\alpha y^5-6yz^2-6y^5-6\alpha yz^2)x,\\
&(-3\alpha-3)x^5+(12\alpha^2y^4+33\alpha y^4-3\alpha z^2+21y^4-3z^2)x^2,\\
&(-20\alpha y^3-20y^3)x^3+16\alpha^2y^7+28\alpha y^7-20\alpha y^3z^2+12y^7-20y^3z^2,\\
&(6yz\alpha^2+18yz\alpha+12yz)x^2,\\
&(-2z-2\alpha z)x^3+8\alpha^2y^4z+22\alpha y^4z-2\alpha z^3+14y^4z-2z^3,\\
&(-2y-2\alpha y)x^3+4\alpha^2yz^2-2\alpha y^5+10\alpha yz^2-2y^5+6yz^2,\\
&(9z\alpha^2+21z\alpha+12z)x^4+(-6\alpha z^3-6y^4z-6z^3-6\alpha y^4z)x,\\
&(-3\alpha-3)x^5+(6\alpha^2z^2-3\alpha y^4+15\alpha z^2-3y^4+9z^2)x^2,\\
&(12z\alpha^2y^3+36z\alpha y^3+24zy^3)x^2,\\
&(-12y^2z-12\alpha y^2z)x^3+16\alpha^2y^6z+36\alpha y^6z-12\alpha y^2z^3+20y^6z-12y^2z^3,\\
&(-4\alpha y^3-4y^3)x^3+8\alpha^2y^3z^2-4\alpha y^7+20\alpha y^3z^2-4y^7+12y^3z^2,\\
&(-6z-6\alpha z)x^3+4\alpha^2z^3-6\alpha y^4z+6\alpha z^3-6y^4z+2z^3).
\end{aligned}
$$

Hence, we can obtain the results by definition.
(3) If $\alpha\in(\frac{4}{12},\frac{5}{12}]$, we obtain

$$
\begin{aligned}
I_0=&\mathcal{O}_X,\\
I_1=&\sum_{v_j\in\mathcal{O}^{\geq 1+\alpha}}\mathcal{O}_X v_j+\sum_{a\in I_0(D)}\mathcal{O}_X(f\partial_x a-\alpha a\partial_x f)+\mathcal{O}_X(f\partial_y a-\alpha a\partial_y f)+\mathcal{O}_X(f\partial_z a-\alpha a\partial_z f)\\
=&\mathcal{O}_X(x,y^2)+\mathcal{O}_X(x^2,y^3,z)\\
=&(x,y^2,z),\\
I_2=&\sum_{v_j\in\mathcal{O}^{\geq 2+\alpha}}\mathcal{O}_X v_j+\sum_{a\in I_1(D)}\mathcal{O}_X(f\partial_x a-(\alpha+1)a\partial_x f)\\
&+\mathcal{O}_X(f\partial_y a-(\alpha+1)a\partial_y f)+\mathcal{O}_X(f\partial_z a-(\alpha+1)a\partial_z f)\\
=&((-3\alpha-2)x^3+y^4+z^2,(-4\alpha y^3-4y^3)x,(-2z-2\alpha z)x,
\end{aligned}
$$

$$
\begin{aligned}
&(-3\alpha y^2-3y^2)x^2, 2yx^3+2yz^2-4\alpha y^5-2y^5, -2y^2z-2\alpha y^2z,\\
&(-3z-3\alpha z)x^2, -4y^3z-4\alpha y^3z, x^3+y^4-2\alpha z^2-z^2),\\
I_3=&\sum_{v_j\in\mathcal{O}^{\geq 3+\alpha}}\mathcal{O}_Xv_j+\sum_{a\in I_2(D)}\mathcal{O}_X(f\partial_xa-(\alpha+2)a\partial_xf)\\
&+\mathcal{O}_X(f\partial_ya-(\alpha+2)a\partial_yf)+\mathcal{O}_X(f\partial_za-(\alpha+2)a\partial_zf)\\
=&((9\alpha^2+15\alpha+6)x^5+(-12\alpha y^4-12\alpha z^2-12y^4-12z^2)x^2,\\
&(12\alpha^2y^3+32\alpha y^3+20y^3)x^3-4y^3z^2-4\alpha y^7-4y^7-4\alpha y^3z^2,\\
&(-12\alpha y^2-12y^2)x^4+(16\alpha^2y^6+36\alpha y^6-12\alpha y^2z^2+20y^6-12y^2z^2)x,\\
&(6z\alpha^2+16z\alpha+10z)x^3-2\alpha z^3-2y^4z-2z^3-2\alpha y^4z,\\
&(8z\alpha^2y^3+24z\alpha y^3+16zy^3)x,\\
&(-2\alpha-2)x^4+(4\alpha^2z^2-2\alpha y^4+10\alpha z^2-2y^4+6z^2)x,\\
&(9\alpha^2y^2+21\alpha y^2+12y^2)x^4+(-6y^2z^2-6\alpha y^6-6y^6-6\alpha y^2z^2)x,\\
&(-6y-6\alpha y)x^5+(12\alpha^2y^5+30\alpha y^5-6\alpha yz^2+18y^5-6yz^2)x^2,\\
&2x^6+(4z^2-24y^4-28\alpha y^4)x^3+16\alpha^2y^8+20\alpha y^8-28\alpha y^4z^2+6y^8-24y^4z^2+2z^4,\\
&(6z\alpha^2y^2+18z\alpha y^2+12zy^2)x^2,\\
&(-4yz-4\alpha yz)x^3+8\alpha^2y^5z+20\alpha y^5z-4\alpha yz^3+12y^5z-4yz^3,\\
&(-2\alpha y^2-2y^2)x^3+4\alpha^2y^2z^2-2\alpha y^6+10\alpha y^2z^2-2y^6+6y^2z^2,\\
&(9z\alpha^2+21z\alpha+12z)x^4+(-6\alpha z^3-6y^4z-6z^3-6\alpha y^4z)x,\\
&(12z\alpha^2y^3+36z\alpha y^3+24zy^3)x^2,\\
&(-12y^2z-12\alpha y^2z)x^3+16\alpha^2y^6z+36\alpha y^6z-12\alpha y^2z^3+20y^6z-12y^2z^3,\\
&(-4\alpha y^3-4y^3)x^3+8\alpha^2y^3z^2-4\alpha y^7+20\alpha y^3z^2-4y^7+12y^3z^2,\\
&(-3\alpha-3)x^5+(6\alpha^2z^2-3\alpha y^4+15\alpha z^2-3y^4+9z^2)x^2,\\
&(-6z-6\alpha z)x^3+4\alpha^2z^3-6\alpha y^4z+6\alpha z^3-6y^4z+2z^3).
\end{aligned}
$$

Hence, we can obtain the results by definition.
(4) If $\alpha\in(\frac{5}{12},\frac{7}{12}]$, we obtain

$$
\begin{aligned}
I_0=&\mathcal{O}_X,\\
I_1=&\sum_{v_j\in\mathcal{O}^{\geq 1+\alpha}}\mathcal{O}_Xv_j+\sum_{a\in I_0(D)}\mathcal{O}_X(f\partial_xa-\alpha a\partial_xf)+\mathcal{O}_X(f\partial_ya-\alpha a\partial_yf)+\mathcal{O}_X(f\partial_za-\alpha a\partial_zf)\\
=&\mathcal{O}_X(xy,y^2)+\mathcal{O}_X(x^2,y^3,z)\\
=&(x^2,xy,y^2,z),\\
I_2=&\sum_{v_j\in\mathcal{O}^{\geq 2+\alpha}}\mathcal{O}_Xv_j+\sum_{a\in I_1(D)}\mathcal{O}_X(f\partial_xa-(\alpha+1)a\partial_xf)\\
&+\mathcal{O}_X(f\partial_ya-(\alpha+1)a\partial_yf)+\mathcal{O}_X(f\partial_za-(\alpha+1)a\partial_zf)\\
=&((-3\alpha-1)x^4+(2y^4+2z^2)x, (-4\alpha y^3-4y^3)x^2, (-2z-2\alpha z)x^2,\\
&(-2y-3\alpha y)x^3+y^5+yz^2, x^4+(z^2-3y^4-4\alpha y^4)x, (-2yz-2\alpha yz)x,\\
&(-3\alpha y^2-3y^2)x^2, 2yx^3+2yz^2-4\alpha y^5-2y^5, -2y^2z-2\alpha y^2z,
\end{aligned}
$$

$$
\begin{aligned}
&(-3z-3\alpha z)x^2, -4y^3z-4\alpha y^3z, x^3+y^4-2\alpha z^2-z^2),\\
I_3 =& \sum_{v_j\in\mathcal{O}^{\geq 3+\alpha}} \mathcal{O}_X v_j + \sum_{a\in I_2(D)} \mathcal{O}_X(f\partial_x a-(\alpha+2)a\partial_x f)\\
&+\mathcal{O}_X(f\partial_y a-(\alpha+2)a\partial_y f)+\mathcal{O}_X(f\partial_z a-(\alpha+2)a\partial_z f)\\
=&((9\alpha^2+9\alpha+2)x^6+(-18\alpha y^4-18\alpha z^2-14y^4-14z^2)x^3+2y^8+4y^4z^2+2z^4,\\
&(12\alpha^2y^3+28\alpha y^3+16y^3)x^4+(-8y^3z^2-8\alpha y^7-8y^7-8\alpha y^3z^2)x,\\
&(-12\alpha y^2-12y^2)x^5+(16\alpha^2y^6+36\alpha y^6-12\alpha y^2z^2+20y^6-12y^2z^2)x^2,\\
&(6z\alpha^2+14z\alpha+8z)x^4+(-4\alpha z^3-4y^4z-4z^3-4\alpha y^4z)x,\\
&(8z\alpha^2y^3+24z\alpha y^3+16zy^3)x^2,\\
&(-2\alpha-2)x^5+(4\alpha^2z^2-2\alpha y^4+10\alpha z^2-2y^4+6z^2)x^2,\\
&(9y\alpha^2+15y\alpha+6y)x^5+(-12\alpha y^5-12yz^2-12y^5-12\alpha yz^2)x^2,\\
&(-3\alpha-2)x^6+(12\alpha^2y^4+29\alpha y^4-3\alpha z^2+19y^4-z^2)x^3\\
&+z^4-4\alpha y^8-3y^8-2y^4z^2-4\alpha y^4z^2,\\
&(-20\alpha y^3-20y^3)x^4+(16\alpha^2y^7+28\alpha y^7-20\alpha y^3z^2+12y^7-20y^3z^2)x,\\
&(6yz\alpha^2+16yz\alpha+10yz)x^3-2yz^3-2y^5z-2\alpha yz^3-2\alpha y^5z,\\
&(-2z-2\alpha z)x^4+(8\alpha^2y^4z+22\alpha y^4z-2\alpha z^3+14y^4z-2z^3)x,\\
&(-2y-2\alpha y)x^4+(4\alpha^2yz^2-2\alpha y^5+10\alpha yz^2-2y^5+6yz^2)x,\\
&(9\alpha^2y^2+21\alpha y^2+12y^2)x^4+(-6y^2z^2-6\alpha y^6-6y^6-6\alpha y^2z^2)x,\\
&(-6y-6\alpha y)x^5+(12\alpha^2y^5+30\alpha y^5-6\alpha yz^2+18y^5-6yz^2)x^2,\\
&2x^6+(4z^2-24y^4-28\alpha y^4)x^3+16\alpha^2y^8+20\alpha y^8-28\alpha y^4z^2+6y^8-24y^4z^2+2z^4,\\
&(6z\alpha^2y^2+18z\alpha y^2+12zy^2)x^2,\\
&(-4yz-4\alpha yz)x^3+8\alpha^2y^5z+20\alpha y^5z-4\alpha yz^3+12y^5z-4yz^3,\\
&(-2\alpha y^2-2y^2)x^3+4\alpha^2y^2z^2-2\alpha y^6+10\alpha y^2z^2-2y^6+6y^2z^2,\\
&(9z\alpha^2+21z\alpha+12z)x^4+(-6\alpha z^3-6y^4z-6z^3-6\alpha y^4z)x,\\
&(12z\alpha^2y^3+36z\alpha y^3+24zy^3)x^2,\\
&(-12y^2z-12\alpha y^2z)x^3+16\alpha^2y^6z+36\alpha y^6z-12\alpha y^2z^3+20y^6z-12y^2z^3,\\
&(-3\alpha-3)x^5+(6\alpha^2z^2-3\alpha y^4+15\alpha z^2-3y^4+9z^2)x^2,\\
&(-4\alpha y^3-4y^3)x^3+8\alpha^2y^3z^2-4\alpha y^7+20\alpha y^3z^2-4y^7+12y^3z^2,\\
&(-6z-6\alpha z)x^3+4\alpha^2z^3-6\alpha y^4z+6\alpha z^3-6y^4z+2z^3).
\end{aligned}
$$

Hence, we can obtain the results by definition.
(5) If $\alpha\in(\frac{7}{12},\frac{8}{12}]$, we obtain

$$
\begin{aligned}
I_0 =&\mathcal{O}_X,\\
I_1 =& \sum_{v_j\in\mathcal{O}^{\geq 1+\alpha}} \mathcal{O}_X v_j + \sum_{a\in I_0(D)} \mathcal{O}_X(f\partial_x a-\alpha a\partial_x f)+\mathcal{O}_X(f\partial_y a-\alpha a\partial_y f)+\mathcal{O}_X(f\partial_z a-\alpha a\partial_z f)\\
=&\mathcal{O}_X(xy)+\mathcal{O}_X(x^2,y^3,z)\\
=&(x^2,xy,y^3,z),
\end{aligned}
$$

$$\begin{aligned}
I_2 = & \sum_{v_j \in \mathcal{O}^{\geq 2+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_1(D)} \mathcal{O}_X(f\partial_x a - (\alpha+1)a\partial_x f) \\
& + \mathcal{O}_X(f\partial_y a - (\alpha+1)a\partial_y f) + \mathcal{O}_X(f\partial_z a - (\alpha+1)a\partial_z f) \\
= & ((-3\alpha-1)x^4 + (2y^4+2z^2)x, (-4\alpha y^3 - 4y^3)x^2, (-2z-2\alpha z)x^2, \\
& (-2y-3\alpha y)x^3 + y^5 + yz^2, x^4 + (z^2 - 3y^4 - 4\alpha y^4)x, (-2yz - 2\alpha yz)x, \\
& (-3\alpha y^3 - 3y^3)x^2, 3y^2x^3 + 3y^2z^2 - 4\alpha y^6 - y^6, -2y^3z - 2\alpha y^3 z, \\
& (-3z - 3\alpha z)x^2, -4y^3z - 4\alpha y^3 z, x^3 + y^4 - 2\alpha z^2 - z^2),
\end{aligned}$$

$$\begin{aligned}
I_3 = & \sum_{v_j \in \mathcal{O}^{\geq 3+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_2(D)} \mathcal{O}_X(f\partial_x a - (\alpha+2)a\partial_x f) \\
& + \mathcal{O}_X(f\partial_y a - (\alpha+2)a\partial_y f) + \mathcal{O}_X(f\partial_z a - (\alpha+2)a\partial_z f) \\
= & ((9\alpha^2+9\alpha+2)x^6 + (-18\alpha y^4 - 18\alpha z^2 - 14y^4 - 14z^2)x^3 + 2y^8 + 4y^4z^2 + 2z^4, \\
& (6z\alpha^2 + 14z\alpha + 8z)x^4 + (-4\alpha z^3 - 4y^4 z - 4z^3 - 4\alpha y^4 z)x, \\
& (12\alpha^2 y^3 + 28\alpha y^3 + 16y^3)x^4 + (-8y^3z^2 - 8\alpha y^7 - 8y^7 - 8\alpha y^3 z^2)x, \\
& (-12\alpha y^2 - 12y^2)x^5 + (16\alpha^2 y^6 + 36\alpha y^6 - 12\alpha y^2 z^2 + 20y^6 - 12y^2z^2)x^2, \\
& (8z\alpha^2 y^3 + 24z\alpha y^3 + 16zy^3)x^2, (-2\alpha-2)x^5 + (4\alpha^2 z^2 - 2\alpha y^4 + 10\alpha z^2 - 2y^4 + 6z^2)x^2, \\
& (9y\alpha^2 + 15y\alpha + 6y)x^5 + (-12\alpha y^5 - 12yz^2 - 12y^5 - 12\alpha yz^2)x^2, \\
& (-3\alpha-2)x^6 + (12\alpha^2 y^4 + 29\alpha y^4 - 3\alpha z^2 + 19y^4 - z^2)x^3 \\
& + z^4 - 4\alpha y^8 - 3y^8 - 2y^4z^2 - 4\alpha y^4 z^2, \\
& (-20\alpha y^3 - 20y^3)x^4 + (16\alpha^2 y^7 + 28\alpha y^7 - 20\alpha y^3 z^2 + 12y^7 - 20y^3z^2)x, \\
& (6yz\alpha^2 + 16yz\alpha + 10yz)x^3 - 2yz^3 - 2y^5 z - 2\alpha yz^3 - 2\alpha y^5 z, \\
& (-2z - 2\alpha z)x^4 + (8\alpha^2 y^4 z + 22\alpha y^4 z - 2\alpha z^3 + 14y^4 z - 2z^3)x, \\
& (-2y - 2\alpha y)x^4 + (4\alpha^2 yz^2 - 2\alpha y^5 + 10\alpha yz^2 - 2y^5 + 6yz^2)x, \\
& (9\alpha^2 y^3 + 21\alpha y^3 + 12y^3)x^4 + (-6y^3z^2 - 6\alpha y^7 - 6y^7 - 6\alpha y^3 z^2)x, \\
& (6z\alpha^2 y^3 + 18z\alpha y^3 + 12zy^3)x^2, (-9\alpha y^2 - 9y^2)x^5 \\
& + (12\alpha^2 y^6 + 27\alpha y^6 - 9\alpha y^2 z^2 + 15y^6 - 9y^2z^2)x^2, 6yx^6 + (12yz^2 - 36\alpha y^5 - 24y^5)x^3 \\
& + 16\alpha^2 y^9 + 12\alpha y^9 - 36\alpha y^5 z^2 + 2y^9 - 24y^5z^2 + 6yz^4, \\
& (-6y^2z - 6\alpha y^2 z)x^3 + 8\alpha^2 y^6 z + 18\alpha y^6 z - 6\alpha y^2 z^3 + 10y^6 z - 6y^2z^3, \\
& (-2\alpha y^3 - 2y^3)x^3 + 4\alpha^2 y^3 z^2 - 2\alpha y^7 + 10\alpha y^3 z^2 - 2y^7 + 6y^3z^2, \\
& (9z\alpha^2 + 21z\alpha + 12z)x^4 + (-6\alpha z^3 - 6y^4 z - 6z^3 - 6\alpha y^4 z)x, \\
& (-3\alpha - 3)x^5 + (6\alpha^2 z^2 - 3\alpha y^4 + 15\alpha z^2 - 3y^4 + 9z^2)x^2, \\
& (12z\alpha^2 y^3 + 36z\alpha y^3 + 24zy^3)x^2, \\
& (-12y^2z - 12\alpha y^2 z)x^3 + 16\alpha^2 y^6 z + 36\alpha y^6 z - 12\alpha y^2 z^3 + 20y^6 z - 12y^2z^3, \\
& (-4\alpha y^3 - 4y^3)x^3 + 8\alpha^2 y^3 z^2 - 4\alpha y^7 + 20\alpha y^3 z^2 - 4y^7 + 12y^3z^2, \\
& (-6z - 6\alpha z)x^3 + 4\alpha^2 z^3 - 6\alpha y^4 z + 6\alpha z^3 - 6y^4 z + 2z^3).
\end{aligned}$$

Hence, we can obtain the results by definition.
(6) If $\alpha \in (\frac{8}{12}, \frac{11}{12}]$, we obtain

$$
\begin{aligned}
I_0 =&\mathcal{O}_X,\\
I_1 =& \sum_{v_j\in\mathcal{O}^{\geq 1+\alpha}} \mathcal{O}_X v_j + \sum_{a\in I_0(D)} \mathcal{O}_X(f\partial_x a - \alpha a\partial_x f) + \mathcal{O}_X(f\partial_y a - \alpha a\partial_y f) + \mathcal{O}_X(f\partial_z a - \alpha a\partial_z f)\\
=&\mathcal{O}_X(xy^2) + \mathcal{O}_X(x^2, y^3, z)\\
=&(x^2, xy^2, y^3, z),\\
I_2 =& \sum_{v_j\in\mathcal{O}^{\geq 2+\alpha}} \mathcal{O}_X v_j + \sum_{a\in I_1(D)} \mathcal{O}_X(f\partial_x a - (\alpha+1)a\partial_x f)\\
&+ \mathcal{O}_X(f\partial_y a - (\alpha+1)a\partial_y f) + \mathcal{O}_X(f\partial_z a - (\alpha+1)a\partial_z f)\\
=&((-3\alpha-1)x^4 + (2y^4+2z^2)x, (-4\alpha y^3 - 4y^3)x^2, (-2z-2\alpha z)x^2,\\
&(-3\alpha y^2 - 2y^2)x^3 + y^6 + y^2z^2, 2yx^4 + (2yz^2 - 4\alpha y^5 - 2y^5)x, (-2y^2z - 2\alpha y^2 z)x,\\
&(-3\alpha y^3 - 3y^3)x^2, 3y^2x^3 + 3y^2z^2 - 4\alpha y^6 - y^6, -2y^3z - 2\alpha y^3 z,\\
&(-3z - 3\alpha z)x^2, -4y^3z - 4\alpha y^3 z, x^3 + y^4 - 2\alpha z^2 - z^2),\\
I_3 =& \sum_{v_j\in\mathcal{O}^{\geq 3+\alpha}} \mathcal{O}_X v_j + \sum_{a\in I_2(D)} \mathcal{O}_X(f\partial_x a - (\alpha+2)a\partial_x f)\\
&+ \mathcal{O}_X(f\partial_y a - (\alpha+2)a\partial_y f) + \mathcal{O}_X(f\partial_z a - (\alpha+2)a\partial_z f)\\
=&((9\alpha^2+9\alpha+2)x^6 + (-18\alpha y^4 - 18\alpha z^2 - 14y^4 - 14z^2)x^3 + 2y^8 + 4y^4z^2 + 2z^4,\\
&(12\alpha^2y^3 + 28\alpha y^3 + 16y^3)x^4 + (-8y^3z^2 - 8\alpha y^7 - 8y^7 - 8\alpha y^3z^2)x,\\
&(-12\alpha y^2 - 12y^2)x^5 + (16\alpha^2y^6 + 36\alpha y^6 - 12\alpha y^2z^2 + 20y^6 - 12y^2z^2)x^2,\\
&(6z\alpha^2 + 14z\alpha + 8z)x^4 + (-4\alpha z^3 - 4y^4z - 4z^3 - 4\alpha y^4 z)x, (8z\alpha^2y^3 + 24z\alpha y^3 + 16zy^3)x^2,\\
&(-2\alpha - 2)x^5 + (4\alpha^2z^2 - 2\alpha y^4 + 10\alpha z^2 - 2y^4 + 6z^2)x^2,\\
&(9\alpha^2y^2 + 15\alpha y^2 + 6y^2)x^5 + (-12y^2z^2 - 12\alpha y^6 - 12y^6 - 12\alpha y^2z^2)x^2,\\
&(6z\alpha^2y^2 + 16z\alpha y^2 + 10zy^2)x^3 - 2y^2z^3 - 2y^6z - 2\alpha y^6 z - 2\alpha y^2z^3,\\
&(-4y - 6\alpha y)x^6 + (12\alpha^2y^5 + 26\alpha y^5 - 6\alpha yz^2 + 18y^5 - 2yz^2)x^3 + 2yz^4 - 4\alpha y^9 - 2y^9 - 4\alpha y^5z^2,\\
&2x^7 + (4z^2 - 24y^4 - 28\alpha y^4)x^4 + (16\alpha^2y^8 + 20\alpha y^8 - 28\alpha y^4z^2 + 6y^8 - 24y^4z^2 + 2z^4)x,\\
&(-4yz - 4\alpha yz)x^4 + (8\alpha^2y^5z + 20\alpha y^5z - 4\alpha yz^3 + 12y^5z - 4yz^3)x,\\
&(6z\alpha^2y^2 + 16z\alpha y^2 + 10zy^2)x^3 - 2y^2z^3 - 2y^6z - 2\alpha y^6z - 2\alpha y^2z^3,\\
&(-2\alpha y^2 - 2y^2)x^4 + (4\alpha^2y^2z^2 - 2\alpha y^6 + 10\alpha y^2z^2 - 2y^6 + 6y^2z^2)x,\\
&(9\alpha^2y^3 + 21\alpha y^3 + 12y^3)x^4 + (-6y^3z^2 - 6\alpha y^7 - 6y^7 - 6\alpha y^3z^2)x,\\
&(-9\alpha y^2 - 9y^2)x^5 + (12\alpha^2y^6 + 27\alpha y^6 - 9\alpha y^2z^2 + 15y^6 - 9y^2z^2)x^2,\\
&(6z\alpha^2y^3 + 18z\alpha y^3 + 12zy^3)x^2,\\
&6yx^6 + (12yz^2 - 36\alpha y^5 - 24y^5)x^3 + 16\alpha^2y^9 + 12\alpha y^9 - 36\alpha y^5z^2 + 2y^9 - 24y^5z^2 + 6yz^4,\\
&(-6y^2z - 6\alpha y^2z)x^3 + 8\alpha^2y^6z + 18\alpha y^6z - 6\alpha y^2z^3 + 10y^6z - 6y^2z^3,\\
&(-2\alpha y^3 - 2y^3)x^3 + 4\alpha^2y^3z^2 - 2\alpha y^7 + 10\alpha y^3z^2 - 2y^7 + 6y^3z^2,\\
&(9z\alpha^2 + 21z\alpha + 12z)x^4 + (-6\alpha z^3 - 6y^4z - 6z^3 - 6\alpha y^4z)x, (12z\alpha^2y^3 + 36z\alpha y^3 + 24zy^3)x^2,\\
&(-12y^2z - 12\alpha y^2z)x^3 + 16\alpha^2y^6z + 36\alpha y^6z - 12\alpha y^2z^3 + 20y^6z - 12y^2z^3,
\end{aligned}
$$

$$(-3\alpha-3)x^5+(6\alpha^2z^2-3\alpha y^4+15\alpha z^2-3y^4+9z^2)x^2,$$
$$(-4\alpha y^3-4y^3)x^3+8\alpha^2y^3z^2-4\alpha y^7+20\alpha y^3z^2-4y^7+12y^3z^2,$$
$$(-6z-6\alpha z)x^3+4\alpha^2z^3-6\alpha y^4z+6\alpha z^3-6y^4z+2z^3).$$

Hence, we can obtain the results by definition.

(7) If $\alpha\in(\frac{11}{12},1]$, we obtain

$$\begin{aligned}
I_0 =&\mathcal{O}_X,\\
I_1 =&\sum_{v_j\in\mathcal{O}^{\geq 1+\alpha}}\mathcal{O}_Xv_j+\sum_{a\in I_0(D)}\mathcal{O}_X(f\partial_xa-\alpha a\partial_xf)+\mathcal{O}_X(f\partial_ya-\alpha a\partial_yf)+\mathcal{O}_X(f\partial_za-\alpha a\partial_zf)\\
=&(x^2,y^3,z),\\
I_2 =&\sum_{v_j\in\mathcal{O}^{\geq 2+\alpha}}\mathcal{O}_Xv_j+\sum_{a\in I_1(D)}\mathcal{O}_X(f\partial_xa-(\alpha+1)a\partial_xf)\\
&+\mathcal{O}_X(f\partial_ya-(\alpha+1)a\partial_yf)+\mathcal{O}_X(f\partial_za-(\alpha+1)a\partial_zf)\\
=&((-3\alpha-1)x^4+(2y^4+2z^2)x,(-4\alpha y^3-4y^3)x^2,(-2z-2\alpha z)x^2,\\
&(-3\alpha y^3-3y^3)x^2,3y^2x^3+3y^2z^2-4\alpha y^6-y^6,-2y^3z-2\alpha y^3z,\\
&(-3z-3\alpha z)x^2,-4y^3z-4\alpha y^3z,x^3+y^4-2\alpha z^2-z^2),\\
I_3 =&\sum_{v_j\in\mathcal{O}^{\geq 3+\alpha}}\mathcal{O}_Xv_j+\sum_{a\in I_2(D)}\mathcal{O}_X(f\partial_xa-(\alpha+2)a\partial_xf)\\
&+\mathcal{O}_X(f\partial_ya-(\alpha+2)a\partial_yf)+\mathcal{O}_X(f\partial_za-(\alpha+2)a\partial_zf)\\
=&((9\alpha^2+9\alpha+2)x^6+(-18\alpha y^4-18\alpha z^2-14y^4-14z^2)x^3+2y^8+4y^4z^2+2z^4,\\
&(6z\alpha^2+14z\alpha+8z)x^4+(-4\alpha z^3-4y^4z-4z^3-4\alpha y^4z)x,\\
&(12\alpha^2y^3+28\alpha y^3+16y^3)x^4+(-8y^3z^2-8\alpha y^7-8y^7-8\alpha y^3z^2)x,\\
&(-12\alpha y^2-12y^2)x^5+(16\alpha^2y^6+36\alpha y^6-12\alpha y^2z^2+20y^6-12y^2z^2)x^2,\\
&(8z\alpha^2y^3+24z\alpha y^3+16zy^3)x^2,\\
&(-2\alpha-2)x^5+(4\alpha^2z^2-2\alpha y^4+10\alpha z^2-2y^4+6z^2)x^2,\\
&(9\alpha^2y^3+21\alpha y^3+12y^3)x^4+(-6y^3z^2-6\alpha y^7-6y^7-6\alpha y^3z^2)x,\\
&(-9\alpha y^2-9y^2)x^5+(12\alpha^2y^6+27\alpha y^6-9\alpha y^2z^2+15y^6-9y^2z^2)x^2,\\
&6yx^6+(12yz^2-36\alpha y^5-24y^5)x^3+16\alpha^2y^9+12\alpha y^9-36\alpha y^5z^2+2y^9-24y^5z^2+6yz^4,\\
&(6z\alpha^2y^3+18z\alpha y^3+12zy^3)x^2,\\
&(-6y^2z-6\alpha y^2z)x^3+8\alpha^2y^6z+18\alpha y^6z-6\alpha y^2z^3+10y^6z-6y^2z^3,\\
&(-2\alpha y^3-2y^3)x^3+4\alpha^2y^3z^2-2\alpha y^7+10\alpha y^3z^2-2y^7+6y^3z^2,\\
&(9z\alpha^2+21z\alpha+12z)x^4+(-6\alpha z^3-6y^4z-6z^3-6\alpha y^4z)x,\\
&(12z\alpha^2y^3+36z\alpha y^3+24zy^3)x^2,\\
&(-12y^2z-12\alpha y^2z)x^3+16\alpha^2y^6z+36\alpha y^6z-12\alpha y^2z^3+20y^6z-12y^2z^3,\\
&(-3\alpha-3)x^5+(6\alpha^2z^2-3\alpha y^4+15\alpha z^2-3y^4+9z^2)x^2,\\
&(-4\alpha y^3-4y^3)x^3+8\alpha^2y^3z^2-4\alpha y^7+20\alpha y^3z^2-4y^7+12y^3z^2,\\
&(-6z-6\alpha z)x^3+4\alpha^2z^3-6\alpha y^4z+6\alpha z^3-6y^4z+2z^3).
\end{aligned}$$

Hence, we can obtain the results by definition. □

Proposition 3.4. *Let $f(x,y,z) = x^3 + xy^3 + z^2$ be E_7 type singularity. Let $H = \{f = 0\}$ be an integral and reduced effective divisor defined by f. And $D^\alpha = \alpha H$ where $\alpha \in \mathbb{Q} \cap (0,1]$. Then the first four Hodge moduli algebras and Hodge moduli numbers are given as follows.*
(1) For $\alpha \in (0, \frac{1}{18}]$,

$$
\begin{aligned}
&M_0 = M_1 = 0, m_0 = m_1 = 0.\\
&M_2 = \mathrm{span}\{1, x, y, y^2, y^3, y^4, xy\}, m_2 = 7.\\
&M_3 = \mathrm{span}\{1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, y^3, xy^3, x^2y^3,\\
&\quad y^4, xy^4, y^5, xy^5, y^6, xy^6, y^7, z, xz, yz, xyz, y^2z, y^3z, y^4z\}, m_3 = 28.
\end{aligned}
$$

(2) For $\alpha \in (\frac{1}{18}, \frac{5}{18}]$,

$$
\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \mathrm{span}\{1\}, m_1 = 1.\\
&M_2 = \mathrm{span}\{1, x, x^2, y, xy, y^2, xy^2, y^3, y^4, z\}, m_2 = 10.\\
&M_3 = \mathrm{span}\{1, x, x^2, x^3, x^4, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, x^3y^2, y^3, xy^3, x^2y^3,\\
&\quad y^4, xy^4, x^2y^4, y^5, xy^5, y^6, xy^6, y^7, z, xz, x^2z, yz, xyz, y^2z, xy^2z,\\
&\quad y^3z, y^4z, z^2\}, m_3 = 34.
\end{aligned}
$$

(3) For $\alpha \in (\frac{5}{18}, \frac{7}{18}]$,

$$
\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \mathrm{span}\{1, y\}, m_1 = 2.\\
&M_2 = \mathrm{span}\{1, x, x^2, y, xy, x^2y, y^2, xy^2, y^3, xy^3, y^4, z, yz\}, m_2 = 13.\\
&M_3 = \mathrm{span}\{1, x, x^2, x^3, x^4, y, xy, x^2y, x^3y, x^4y, y^2, xy^2, x^2y^2, x^3y^2,\\
&\quad y^3, xy^3, x^2y^3, x^3y^3, y^4, xy^4, x^2y^4, y^5, xy^5, y^6, xy^6, y^7, y^8,\\
&\quad z, xz, x^2z, yz, xyz, x^2yz, y^2z, xy^2z, y^3z, xy^3z, y^4z, z^2, yz^2\}, m_3 = 40.
\end{aligned}
$$

(4) For $\alpha \in (\frac{7}{18}, \frac{9}{18}]$,

$$
\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \mathrm{span}\{1, x, y\}, m_1 = 3.\\
&M_2 = \mathrm{span}\{1, x, x^2, x^3, y, xy, x^2y, y^2, xy^2, y^3, xy^3, y^4, y^5, z, xz, yz\}, m_2 = 16.\\
&M_3 = \mathrm{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, y^2, xy^2, x^2y^2, x^3y^2,\\
&\quad y^3, xy^3, x^2y^3, x^3y^3, y^4, xy^4, x^2y^4, y^5, xy^5, x^2y^5, y^6, xy^6, y^7, xy^7, y^8,\\
&\quad z, xz, x^2z, x^3z, yz, xyz, x^2yz, y^2z, xy^2z, y^3z, xy^3z, y^4z, y^5z, z^2, xz^2, yz^2\}, m_3 = 46.
\end{aligned}
$$

(5) For $\alpha \in (\frac{9}{18}, \frac{11}{18}]$,

$$\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \mathrm{span}\{1, x, y, y^2\}, m_1 = 4.\\
&M_2 = \mathrm{span}\{1, x, x^2, x^3, y, xy, x^2y, y^2, xy^2, x^2y^2, y^3, xy^3, y^4, xy^4, y^5,\\
&\quad z, xz, yz, y^2z\}, m_2 = 19.\\
&M_3 = \mathrm{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, y^2, xy^2, x^2y^2, x^3y^2, x^4y^2,\\
&\quad y^3, xy^3, x^2y^3, x^3y^3, y^4, xy^4, x^2y^4, x^3y^4, y^5, xy^5, x^2y^5, y^6, xy^6, y^7, xy^7, y^8, y^9,\\
&\quad z, xz, x^2z, x^3z, yz, xyz, x^2yz, y^2z, xy^2z, x^2y^2z, y^3z, xy^3z,\\
&\quad y^4z, xy^4z, y^5z, z^2, xz^2, yz^2, y^2z^2\}, m_3 = 52.
\end{aligned}$$

(6) For $\alpha \in (\frac{11}{18}, \frac{13}{18}]$,

$$\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \mathrm{span}\{1, x, y, y^2, xy\}, m_1 = 5.\\
&M_2 = \mathrm{span}\{1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, y^3, xy^3, y^4, xy^4, y^5, y^6,\\
&\quad z, xz, yz, xyz, y^2z\}, m_2 = 22.\\
&M_3 = \mathrm{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y, y^2, xy^2, x^2y^2, x^3y^2, x^4y^2,\\
&\quad y^3, xy^3, x^2y^3, x^3y^3, y^4, xy^4, x^2y^4, x^3y^4, y^5, xy^5, x^2y^5, y^6, xy^6, x^2y^6,\\
&\quad y^7, xy^7, y^8, xy^8, y^9, z, xz, x^2z, x^3z, yz, xyz, x^2yz, x^3yz, y^2z, xy^2z, x^2y^2z,\\
&\quad y^3z, xy^3z, y^4z, xy^4z, y^5z, y^6z, z^2, xz^2, yz^2, xyz^2, y^2z^2\}, m_3 = 58.
\end{aligned}$$

(7) For $\alpha \in (\frac{13}{18}, \frac{17}{18}]$,

$$\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \mathrm{span}\{1, x, y, y^2, y^3, xy\}, m_1 = 6.\\
&M_2 = \mathrm{span}\{1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, y^3, xy^3, x^2y^3,\\
&\quad y^4, xy^4, y^5, xy^5, y^6, z, xz, yz, xyz, y^2z, y^3z\}, m_2 = 25.\\
&M_3 = \mathrm{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y, y^2, xy^2, x^2y^2, x^3y^2, x^4y^2,\\
&\quad y^3, xy^3, x^2y^3, x^3y^3, x^4y^3, y^4, xy^4, x^2y^4, x^3y^4, y^5, xy^5, x^2y^5, x^3y^5, y^6, xy^6, x^2y^6,\\
&\quad y^7, xy^7, y^8, xy^8, y^9, y^{10}, z, xz, x^2z, x^3z, yz, xyz, x^2yz, x^3yz, y^2z, xy^2z, x^2y^2z,\\
&\quad y^3z, xy^3z, x^2y^3z, y^4z, xy^4z, y^5z, xy^5z, y^6z, z^2, xz^2, yz^2, xyz^2,\\
&\quad y^2z^2, y^3z^2\}, m_3 = 64.
\end{aligned}$$

(8) For $\alpha \in (\frac{17}{18}, 1]$,

$$\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \mathrm{span}\{1, x, y, y^2, y^3, y^4, xy\}, m_1 = 7.\\
&M_2 = \mathrm{span}\{1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, y^3, xy^3, x^2y^3, y^4, xy^4, y^5, xy^5,\\
&\quad y^6, xy^6, y^7, z, xz, yz, xyz, y^2z, y^3z, y^4z\}, m_2 = 28.\\
&M_3 = \mathrm{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y, y^2, xy^2, x^2y^2, x^3y^2, x^4y^2,\\
&\quad y^3, xy^3, x^2y^3, x^3y^3, x^4y^3, y^4, xy^4, x^2y^4, x^3y^4, y^5, xy^5, x^2y^5, x^3y^5, y^6, xy^6, x^2y^6,\\
&\quad y^7, xy^7, x^2y^7, y^8, xy^8, y^9, xy^9, y^{10}, y^{11}, z, xz, x^2z, x^3z, yz, xyz, x^2yz, x^3yz,\\
&\quad y^2z, xy^2z, x^2y^2z, y^3z, xy^3z, x^2y^3z, y^4z, xy^4z, y^5z, xy^5z, y^6z, xy^6z, y^7z,\\
&\quad z^2, xz^2, yz^2, xyz^2, y^2z^2, y^3z^2, y^4z^2\}, m_3 = 70.
\end{aligned}$$

Proof. (1) If $\alpha \in (0, \frac{1}{18}]$, we have

$$\begin{aligned}
I_0 =& I_1 = \mathcal{O}_X,\\
I_2 =& \sum_{v_j \in \mathcal{O}^{\geq 2+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_1(D)} \mathcal{O}_X(f\partial_x a - (\alpha+1)a\partial_x f)\\
&+ \mathcal{O}_X(f\partial_y a - (\alpha+1)a\partial_y f) + \mathcal{O}_X(f\partial_z a - (\alpha+1)a\partial_z f)\\
=&((-3\alpha-3)x^2 - \alpha y^3 - y^3, (-3\alpha y^2 - 3y^2)x, -2z - 2\alpha z),\\
I_3 =& \sum_{v_j \in \mathcal{O}^{\geq 3+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_2(D)} \mathcal{O}_X(f\partial_x a - (\alpha+2)a\partial_x f)\\
&+ \mathcal{O}_X(f\partial_y a - (\alpha+2)a\partial_y f) + \mathcal{O}_X(f\partial_z a - (\alpha+2)a\partial_z f)\\
=&((9\alpha^2+21\alpha+12)x^4 + (6\alpha^2y^3+12\alpha y^3+6y^3)x^2 + (-6\alpha z^2-6z^2)x + \alpha^2y^6 + 3\alpha y^6 + 2y^6,\\
&(9\alpha^2y^2+24\alpha y^2+15y^2)x^3 + (3\alpha^2y^5+6\alpha y^5+3y^5)x - 3y^2z^2 - 3\alpha y^2z^2,\\
&(-6y-6\alpha y)x^4 + (9\alpha^2y^4+21\alpha y^4+12y^4)x^2 + (-6yz^2-6\alpha yz^2)x,\\
&(6z\alpha^2+18z\alpha+12z)x^2 + 2z\alpha^2y^3 + 6z\alpha y^3 + 4zy^3,\\
&(6z\alpha^2y^2+18z\alpha y^2+12zy^2)x,\\
&(-2\alpha-2)x^3 + (-2\alpha y^3-2y^3)x + 4\alpha^2z^2 + 10\alpha z^2 + 6z^2).
\end{aligned}$$

Hence, we can obtain the results by definition.
(2) If $\alpha \in (\frac{1}{18}, \frac{5}{18}]$, we obtain

$$\begin{aligned}
I_0 =& \mathcal{O}_X,\\
I_1 =& \sum_{v_j \in \mathcal{O}^{\geq 1+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_0(D)} \mathcal{O}_X(f\partial_x a - \alpha a\partial_x f) + \mathcal{O}_X(f\partial_y a - \alpha a\partial_y f) + \mathcal{O}_X(f\partial_z a - \alpha a\partial_z f)\\
=& \mathcal{O}_X(x, y) + \mathcal{O}_X(3x^2 + y^3, xy^2, z)\\
=& (x, y, z),
\end{aligned}$$

$$
\begin{aligned}
I_2 = & \sum_{v_j \in \mathcal{O}^{\geq 2+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_1(D)} \mathcal{O}_X(f\partial_x a - (\alpha+1)a\partial_x f) \\
& + \mathcal{O}_X(f\partial_y a - (\alpha+1)a\partial_y f) + \mathcal{O}_X(f\partial_z a - (\alpha+1)a\partial_z f) \\
= & ((-3\alpha-2)x^3 - \alpha x y^3 + z^2, (-3\alpha y^2 - 3y^2)x^2, (-2z - 2\alpha z)x, \\
& (-3y - 3\alpha y)x^2 - \alpha y^4 - y^4, x^3 + (-3\alpha y^3 - 2y^3)x + z^2, -2yz - 2\alpha yz, \\
& (-3z - 3\alpha z)x^2 - y^3 z - \alpha y^3 z, (-3y^2 z - 3\alpha y^2 z)x, x^3 + y^3 x - 2\alpha z^2 - z^2), \\
I_3 = & \sum_{v_j \in \mathcal{O}^{\geq 3+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_2(D)} \mathcal{O}_X(f\partial_x a - (\alpha+2)a\partial_x f) \\
& + \mathcal{O}_X(f\partial_y a - (\alpha+2)a\partial_y f) + \mathcal{O}_X(f\partial_z a - (\alpha+2)a\partial_z f) \\
= & ((9\alpha^2 + 15\alpha + 6)x^5 + (6\alpha^2 y^3 + 4\alpha y^3 - 2y^3)x^3 + \\
& (-12\alpha z^2 - 12z^2)x^2 + (\alpha^2 y^6 + \alpha y^6)x - 2y^3 z^2 - 2\alpha y^3 z^2, \\
& (9\alpha^2 y^2 + 21\alpha y^2 + 12y^2)x^4 + (3\alpha^2 y^5 + 3\alpha y^5)x^2 + (-6y^2 z^2 - 6\alpha y^2 z^2)x, \\
& (6z\alpha^2 + 16z\alpha + 10z)x^3 + (2z\alpha^2 y^3 + 4z\alpha y^3 + 2zy^3)x - 2\alpha z^3 - 2z^3, \\
& (-6y - 6\alpha y)x^5 + (9\alpha^2 y^4 + 21\alpha y^4 + 12y^4)x^3 + (-6yz^2 - 6\alpha yz^2)x^2, \\
& (6z\alpha^2 y^2 + 18z\alpha y^2 + 12zy^2)x^2, \\
& (-2\alpha - 2)x^4 + (-2\alpha y^3 - 2y^3)x^2 + (4\alpha^2 z^2 + 10\alpha z^2 + 6z^2)x, \\
& (9y\alpha^2 + 21y\alpha + 12y)x^4 + (6\alpha^2 y^4 + 12\alpha y^4 + 6y^4)x^2 \\
& + (-6yz^2 - 6\alpha yz^2)x + \alpha^2 y^7 + 3\alpha y^7 + 2y^7, \\
& (-3\alpha - 3)x^5 + (9\alpha^2 y^3 + 20\alpha y^3 + 11y^3)x^3 + (-3\alpha z^2 - 3z^2)x^2 \\
& + (3\alpha^2 y^6 + 5\alpha y^6 + 2y^6)x - 4y^3 z^2 - 4\alpha y^3 z^2, \\
& (6yz\alpha^2 + 18yz\alpha + 12yz)x^2 + 2z\alpha^2 y^4 + 6z\alpha y^4 + 4zy^4, \\
& (-12\alpha y^2 - 12y^2)x^4 + (9\alpha^2 y^5 + 15\alpha y^5 + 6y^5)x^2 + (-12y^2 z^2 - 12\alpha y^2 z^2)x, \\
& (-2z - 2\alpha z)x^3 + (6z\alpha^2 y^3 + 16z\alpha y^3 + 10zy^3)x - 2\alpha z^3 - 2z^3, \\
& (-2y - 2\alpha y)x^3 + (-2\alpha y^4 - 2y^4)x + 4y\alpha^2 z^2 + 10y\alpha z^2 + 6yz^2, \\
& (9z\alpha^2 + 21z\alpha + 12z)x^4 + (6z\alpha^2 y^3 + 12z\alpha y^3 + 6zy^3)x^2 \\
& + (-6\alpha z^3 - 6z^3)x + z\alpha^2 y^6 + 3z\alpha y^6 + 2zy^6, \\
& (9z\alpha^2 y^2 + 24z\alpha y^2 + 15zy^2)x^3 + (3z\alpha^2 y^5 + 6z\alpha y^5 + 3zy^5)x - 3y^2 z^3 - 3\alpha y^2 z^3, \\
& (-3\alpha - 3)x^5 + (-4\alpha y^3 - 4y^3)x^3 + (6\alpha^2 z^2 + 15\alpha z^2 + 9z^2)x^2 \\
& + (-\alpha y^6 - y^6)x + 2\alpha^2 y^3 z^2 + 5\alpha y^3 z^2 + 3y^3 z^2, \\
& (-6yz - 6\alpha yz)x^4 + (9z\alpha^2 y^4 + 21z\alpha y^4 + 12zy^4)x^2 + (-6yz^3 - 6\alpha yz^3)x, \\
& (-3\alpha y^2 - 3y^2)x^4 + (-3\alpha y^5 - 3y^5)x^2 + (6\alpha^2 y^2 z^2 + 15\alpha y^2 z^2 + 9y^2 z^2)x, \\
& (-6z - 6\alpha z)x^3 + (-6y^3 z - 6\alpha y^3 z)x + 4\alpha^2 z^3 + 6\alpha z^3 + 2z^3).
\end{aligned}
$$

Hence, we can obtain the results by definition.
(3) If $\alpha \in (\frac{5}{18}, \frac{7}{18}]$, we obtain $I_0 = \mathcal{O}_X$,

$$
\begin{aligned}
I_0 = & \mathcal{O}_X, \\
I_1 = & \sum_{v_j \in \mathcal{O}^{\geq 1+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_0(D)} \mathcal{O}_X(f\partial_x a - \alpha a\partial_x f) + \mathcal{O}_X(f\partial_y a - \alpha a\partial_y f) + \mathcal{O}_X(f\partial_z a - \alpha a\partial_z f)
\end{aligned}
$$

$$
\begin{aligned}
&=\mathcal{O}_X(x,y^2)+\mathcal{O}_X(3x^2+y^3,xy^2,z)\\
&=(x,y^2,z),\\
I_2=&\sum_{v_j\in\mathcal{O}^{\geq 2+\alpha}}\mathcal{O}_X v_j+\sum_{a\in I_1(D)}\mathcal{O}_X(f\partial_x a-(\alpha+1)a\partial_x f)\\
&+\mathcal{O}_X(f\partial_y a-(\alpha+1)a\partial_y f)+\mathcal{O}_X(f\partial_z a-(\alpha+1)a\partial_z f)\\
&=(-3\alpha-2)x^3-\alpha xy^3+z^2,(-3\alpha y^2-3y^2)x^2,(-2z-2\alpha z)x,\\
&(-3\alpha y^2-3y^2)x^2-\alpha y^5-y^5,2yx^3+(-3\alpha y^4-y^4)x+2yz^2,-2y^2z-2\alpha y^2z,\\
&(-3z-3\alpha z)x^2-y^3z-\alpha y^3z,(-3y^2z-3\alpha y^2z)x,x^3+y^3x-2\alpha z^2-z^2),\\
I_3=&\sum_{v_j\in\mathcal{O}^{\geq 3+\alpha}}\mathcal{O}_X v_j+\sum_{a\in I_2(D)}\mathcal{O}_X(f\partial_x a-(\alpha+2)a\partial_x f)\\
&+\mathcal{O}_X(f\partial_y a-(\alpha+2)a\partial_y f)+\mathcal{O}_X(f\partial_z a-(\alpha+2)a\partial_z f)\\
&=((9\alpha^2+15\alpha+6)x^5+(6\alpha^2y^3+4\alpha y^3-2y^3)x^3\\
&+(-12\alpha z^2-12z^2)x^2+(\alpha^2y^6+\alpha y^6)x-2y^3z^2-2\alpha y^3z^2,\\
&(9\alpha^2y^2+21\alpha y^2+12y^2)x^4+(3\alpha^2y^5+3\alpha y^5)x^2+(-6y^2z^2-6\alpha y^2z^2)x,\\
&(-6y-6\alpha y)x^5+(9\alpha^2y^4+21\alpha y^4+12y^4)x^3+(-6yz^2-6\alpha yz^2)x^2,\\
&(6z\alpha^2y^2+18z\alpha y^2+12zy^2)x^2,\\
&(6z\alpha^2+16z\alpha+10z)x^3+(2z\alpha^2y^3+4z\alpha y^3+2zy^3)x-2\alpha z^3-2z^3,\\
&(-2\alpha-2)x^4+(-2\alpha y^3-2y^3)x^2+(4\alpha^2z^2+10\alpha z^2+6z^2)x,\\
&(9\alpha^2y^2+21\alpha y^2+12y^2)x^4+(6\alpha^2y^5+12\alpha y^5+6y^5)x^2\\
&+(-6y^2z^2-6\alpha y^2z^2)x+\alpha^2y^8+3\alpha y^8+2y^8,\\
&(-6y-6\alpha y)x^5+(9\alpha^2y^4+16\alpha y^4+7y^4)x^3+(-6yz^2-6\alpha yz^2)x^2\\
&+(3\alpha^2y^7+4\alpha y^7+y^7)x-5y^4z^2-5\alpha y^4z^2,\\
&(6z\alpha^2y^2+18z\alpha y^2+12zy^2)x^2+2z\alpha^2y^5+6z\alpha y^5+4zy^5,\\
&2x^6+(-18\alpha y^3-14y^3)x^4+4z^2x^3+(9\alpha^2y^6+9\alpha y^6+2y^6)x^2\\
&+(-14y^3z^2-18\alpha y^3z^2)x+2z^4,\\
&(-4yz-4\alpha yz)x^3+(6z\alpha^2y^4+14z\alpha y^4+8zy^4)x-4yz^3-4\alpha yz^3,\\
&(-2\alpha y^2-2y^2)x^3+(-2\alpha y^5-2y^5)x+4\alpha^2y^2z^2+10\alpha y^2z^2+6y^2z^2,\\
&(9z\alpha^2+21z\alpha+12z)x^4+(6z\alpha^2y^3+12z\alpha y^3+6zy^3)x^2\\
&+(-6\alpha z^3-6z^3)x+z\alpha^2y^6+3z\alpha y^6+2zy^6,\\
&(9z\alpha^2y^2+24z\alpha y^2+15zy^2)x^3+(3z\alpha^2y^5+6z\alpha y^5+3zy^5)x-3y^2z^3-3\alpha y^2z^3,\\
&(-3\alpha-3)x^5+(-4\alpha y^3-4y^3)x^3+(6\alpha^2z^2+15\alpha z^2+9z^2)x^2\\
&+(-\alpha y^6-y^6)x+2\alpha^2y^3z^2+5\alpha y^3z^2+3y^3z^2,\\
&(-6yz-6\alpha yz)x^4+(9z\alpha^2y^4+21z\alpha y^4+12zy^4)x^2+(-6yz^3-6\alpha yz^3)x,\\
&(-3\alpha y^2-3y^2)x^4+(-3\alpha y^5-3y^5)x^2+(6\alpha^2y^2z^2+15\alpha y^2z^2+9y^2z^2)x,\\
&(-6z-6\alpha z)x^3+(-6y^3z-6\alpha y^3z)x+4\alpha^2z^3+6\alpha z^3+2z^3).
\end{aligned}
$$

Hence, we can obtain the results by definition.
(4) If $\alpha \in (\frac{7}{18}, \frac{9}{18}]$, we obtain

$$
\begin{aligned}
I_0 =& \mathcal{O}_X, \\
I_1 =& \sum_{v_j \in \mathcal{O}^{\geq 1+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_0(D)} \mathcal{O}_X(f\partial_x a - \alpha a \partial_x f) + \mathcal{O}_X(f\partial_y a - \alpha a \partial_y f) + \mathcal{O}_X(f\partial_z a - \alpha a \partial_z f) \\
=& \mathcal{O}_X(xy, y^2) + \mathcal{O}_X(3x^2 + y^3, xy^2, z) \\
=& (x^2, xy, y^2, z), \\
I_2 =& \sum_{v_j \in \mathcal{O}^{\geq 2+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_1(D)} \mathcal{O}_X(f\partial_x a - (\alpha+1) a \partial_x f) \\
& + \mathcal{O}_X(f\partial_y a - (\alpha+1) a \partial_y f) + \mathcal{O}_X(f\partial_z a - (\alpha+1) a \partial_z f) \\
=& ((-3\alpha - 1)x^4 + (y^3 - \alpha y^3)x^2 + 2z^2 x, (-3\alpha y^2 - 3y^2)x^3, (-2z - 2\alpha z)x^2, \\
& (-2y - 3\alpha y)x^3 + (-\alpha y^4)x + yz^2, x^4 + (-3\alpha y^3 - 2y^3)x^2 + xz^2, (-2yz - 2\alpha yz)x, \\
& (-3\alpha y^2 - 3y^2)x^2 - \alpha y^5 - y^5, 2yx^3 + (-3\alpha y^4 - y^4)x + 2yz^2, -2y^2 z - 2\alpha y^2 z, \\
& (-3z - 3\alpha z)x^2 - y^3 z - \alpha y^3 z, (-3y^2 z - 3\alpha y^2 z)x, x^3 + y^3 x - 2\alpha z^2 - z^2), \\
I_3 =& \sum_{v_j \in \mathcal{O}^{\geq 3+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_2(D)} \mathcal{O}_X(f\partial_x a - (\alpha+2) a \partial_x f) \\
& + \mathcal{O}_X(f\partial_y a - (\alpha+2) a \partial_y f) + \mathcal{O}_X(f\partial_z a - (\alpha+2) a \partial_z f) \\
=& ((9\alpha^2 + 9\alpha + 2)x^6 + (6\alpha^2 y^3 - 4\alpha y^3 - 6y^3)x^4 + (-18\alpha z^2 - 14z^2)x^3 \\
& + (\alpha^2 y^6 - \alpha y^6)x^2 + (-4\alpha y^3 z^2)x + 2z^4, \\
& (9\alpha^2 y^2 + 18\alpha y^2 + 9y^2)x^5 + (3\alpha^2 y^5 - 3y^5)x^3 + (-9y^2 z^2 - 9\alpha y^2 z^2)x^2, \\
& (6z\alpha^2 + 14z\alpha + 8z)x^4 + (2z\alpha^2 y^3 + 2z\alpha y^3)x^2 + (-4\alpha z^3 - 4z^3)x, \\
& (-6y - 6\alpha y)x^6 + (9\alpha^2 y^4 + 21\alpha y^4 + 12y^4)x^4 + (-6yz^2 - 6\alpha yz^2)x^3, \\
& (6z\alpha^2 y^2 + 18z\alpha y^2 + 12zy^2)x^3, \\
& (-2\alpha - 2)x^5 + (-2\alpha y^3 - 2y^3)x^3 + (4\alpha^2 z^2 + 10\alpha z^2 + 6z^2)x^2, \\
& (9y\alpha^2 + 15y\alpha + 6y)x^5 + (6\alpha^2 y^4 + 4\alpha y^4 - 2y^4)x^3 + (-12yz^2 - 12\alpha yz^2)x^2 \\
& + (\alpha^2 y^7 + \alpha y^7)x - 2y^4 z^2 - 2\alpha y^4 z^2, \\
& (-3\alpha - 2)x^6 + (9\alpha^2 y^3 + 17\alpha y^3 + 10y^3)x^4 + (-3\alpha z^2 - z^2)x^3 \\
& + (3\alpha^2 y^6 + 2\alpha y^6)x^2 + (-5y^3 z^2 - 7\alpha y^3 z^2)x + z^4, \\
& (6yz\alpha^2 + 16yz\alpha + 10yz)x^3 + (2z\alpha^2 y^4 + 4z\alpha y^4 + 2zy^4)x - 2yz^3 - 2\alpha yz^3, \\
& (-12\alpha y^2 - 12y^2)x^5 + (9\alpha^2 y^5 + 15\alpha y^5 + 6y^5)x^3 + (-12y^2 z^2 - 12\alpha y^2 z^2)x^2, \\
& (-2z - 2\alpha z)x^4 + (6z\alpha^2 y^3 + 16z\alpha y^3 + 10zy^3)x^2 + (-2\alpha z^3 - 2z^3)x, \\
& (-2y - 2\alpha y)x^4 + (-2\alpha y^4 - 2y^4)x^2 + (4y\alpha^2 z^2 + 10y\alpha z^2 + 6yz^2)x, \\
& (9\alpha^2 y^2 + 21\alpha y^2 + 12y^2)x^4 + (6\alpha^2 y^5 + 12\alpha y^5 + 6y^5)x^2 \\
& + (-6y^2 z^2 - 6\alpha y^2 z^2)x + \alpha^2 y^8 + 3\alpha y^8 + 2y^8, \\
& (-6y - 6\alpha y)x^5 + (9\alpha^2 y^4 + 16\alpha y^4 + 7y^4)x^3 + (-6yz^2 - 6\alpha yz^2)x^2 \\
& + (3\alpha^2 y^7 + 4\alpha y^7 + y^7)x - 5y^4 z^2 - 5\alpha y^4 z^2, \\
& (6z\alpha^2 y^2 + 18z\alpha y^2 + 12zy^2)x^2 + 2z\alpha^2 y^5 + 6z\alpha y^5 + 4zy^5,
\end{aligned}
$$

$$2x^6+(-18\alpha y^3-14y^3)x^4+4z^2x^3+(9\alpha^2y^6+9\alpha y^6+2y^6)x^2$$
$$+(-14y^3z^2-18\alpha y^3z^2)x+2z^4,$$
$$(-4yz-4\alpha yz)x^3+(6z\alpha^2y^4+14z\alpha y^4+8zy^4)x-4yz^3-4\alpha yz^3,$$
$$(-2\alpha y^2-2y^2)x^3+(-2\alpha y^5-2y^5)x+4\alpha^2y^2z^2+10\alpha y^2z^2+6y^2z^2,$$
$$(9z\alpha^2+21z\alpha+12z)x^4+(6z\alpha^2y^3+12z\alpha y^3+6zy^3)x^2$$
$$+(-6\alpha z^3-6z^3)x+z\alpha^2y^6+3z\alpha y^6+2zy^6,$$
$$(9z\alpha^2y^2+24z\alpha y^2+15zy^2)x^3+(3z\alpha^2y^5+6z\alpha y^5+3zy^5)x-3y^2z^3-3\alpha y^2z^3,$$
$$(-3\alpha-3)x^5+(-4\alpha y^3-4y^3)x^3+(6\alpha^2z^2+15\alpha z^2+9z^2)x^2$$
$$+(-\alpha y^6-y^6)x+2\alpha^2y^3z^2+5\alpha y^3z^2+3y^3z^2,$$
$$(-6yz-6\alpha yz)x^4+(9z\alpha^2y^4+21z\alpha y^4+12zy^4)x^2+(-6yz^3-6\alpha yz^3)x,$$
$$(-3\alpha y^2-3y^2)x^4+(-3\alpha y^5-3y^5)x^2+(6\alpha^2y^2z^2+15\alpha y^2z^2+9y^2z^2)x,$$
$$(-6z-6\alpha z)x^3+(-6y^3z-6\alpha y^3z)x+4\alpha^2z^3+6\alpha z^3+2z^3).$$

Hence, we can obtain the results by definition.
(5) If $\alpha\in(\frac{9}{18},\frac{11}{18}]$, we obtain

$$I_0=\mathcal{O}_X,$$
$$I_1=\sum_{v_j\in\mathcal{O}^{\geq 1+\alpha}}\mathcal{O}_Xv_j+\sum_{a\in I_0(D)}\mathcal{O}_X(f\partial_xa-\alpha a\partial_xf)+\mathcal{O}_X(f\partial_ya-\alpha a\partial_yf)+\mathcal{O}_X(f\partial_za-\alpha a\partial_zf)$$
$$=\mathcal{O}_X(xy,y^3)+\mathcal{O}_X(3x^2+y^3,xy^2,z)$$
$$=(x^2,xy,y^3,z),$$
$$I_2=\sum_{v_j\in\mathcal{O}^{\geq 2+\alpha}}\mathcal{O}_Xv_j+\sum_{a\in I_1(D)}\mathcal{O}_X(f\partial_xa-(\alpha+1)a\partial_xf)$$
$$+\mathcal{O}_X(f\partial_ya-(\alpha+1)a\partial_yf)+\mathcal{O}_X(f\partial_za-(\alpha+1)a\partial_zf)$$
$$=((-3\alpha-1)x^4+(y^3-\alpha y^3)x^2+2z^2x,(-3\alpha y^2-3y^2)x^3,(-2z-2\alpha z)x^2,$$
$$(-2y-3\alpha y)x^3+(-\alpha y^4)x+yz^2,x^4+(-3\alpha y^3-2y^3)x^2+xz^2,(-2yz-2\alpha yz)x,$$
$$(-3\alpha y^3-3y^3)x^2-\alpha y^6-y^6,3x^3y^2-3\alpha xy^5+3y^2z^2,-2y^3z-2\alpha y^3z,$$
$$(-3z-3\alpha z)x^2-y^3z-\alpha y^3z,(-3y^2z-3\alpha y^2z)x,x^3+y^3x-2\alpha z^2-z^2),$$
$$I_3=\sum_{v_j\in\mathcal{O}^{\geq 3+\alpha}}\mathcal{O}_Xv_j+\sum_{a\in I_2(D)}\mathcal{O}_X(f\partial_xa-(\alpha+2)a\partial_xf)$$
$$+\mathcal{O}_X(f\partial_ya-(\alpha+2)a\partial_yf)+\mathcal{O}_X(f\partial_za-(\alpha+2)a\partial_zf)$$
$$=((9\alpha^2+9\alpha+2)x^6+(6\alpha^2y^3-4\alpha y^3-6y^3)x^4+(-18\alpha z^2-14z^2)x^3$$
$$+(\alpha^2y^6-\alpha y^6)x^2+(-4\alpha y^3z^2)x+2z^4,$$
$$(9\alpha^2y^2+18\alpha y^2+9y^2)x^5+(3\alpha^2y^5-3y^5)x^3+(-9y^2z^2-9\alpha y^2z^2)x^2,$$
$$(6z\alpha^2+14z\alpha+8z)x^4+(2z\alpha^2y^3+2z\alpha y^3)x^2+(-4\alpha z^3-4z^3)x,$$
$$(-6y-6\alpha y)x^6+(9\alpha^2y^4+21\alpha y^4+12y^4)x^4+(-6yz^2-6\alpha yz^2)x^3,$$
$$(6z\alpha^2y^2+18z\alpha y^2+12zy^2)x^3,$$
$$(-2\alpha-2)x^5+(-2\alpha y^3-2y^3)x^3+(4\alpha^2z^2+10\alpha z^2+6z^2)x^2,$$
$$(9y\alpha^2+15y\alpha+6y)x^5+(6\alpha^2y^4+4\alpha y^4-2y^4)x^3+(-12yz^2-12\alpha yz^2)x^2$$

$$
\begin{aligned}
&+(\alpha^2y^7+\alpha y^7)x-2y^4z^2-2\alpha y^4z^2,\\
&(-3\alpha-2)x^6+(9\alpha^2y^3+17\alpha y^3+10y^3)x^4+(-3\alpha z^2-z^2)x^3\\
&+(3\alpha^2y^6+2\alpha y^6)x^2+(-5y^3z^2-7\alpha y^3z^2)x+z^4,\\
&(6yz\alpha^2+16yz\alpha+10yz)x^3+(2z\alpha^2y^4+4z\alpha y^4+2zy^4)x-2yz^3-2\alpha yz^3,\\
&(-12\alpha y^2-12y^2)x^5+(9\alpha^2y^5+15\alpha y^5+6y^5)x^3+(-12y^2z^2-12\alpha y^2z^2)x^2,\\
&(-2z-2\alpha z)x^4+(6z\alpha^2y^3+16z\alpha y^3+10zy^3)x^2+(-2\alpha z^3-2z^3)x,\\
&(-2y-2\alpha y)x^4+(-2\alpha y^4-2y^4)x^2+(4y\alpha^2z^2+10y\alpha z^2+6yz^2)x,\\
&(9\alpha^2y^3+21\alpha y^3+12y^3)x^4+(6\alpha^2y^6+12\alpha y^6+6y^6)x^2\\
&+(-6y^3z^2-6\alpha y^3z^2)x+\alpha^2y^9+3\alpha y^9+2y^9,\\
&(-9\alpha y^2-9y^2)x^5+(9\alpha^2y^5+12\alpha y^5+3y^5)x^3+(-9y^2z^2-9\alpha y^2z^2)x^2\\
&+(3\alpha^2y^8+3\alpha y^8)x-6y^5z^2-6\alpha y^5z^2,\\
&(6z\alpha^2y^3+18z\alpha y^3+12zy^3)x^2+2z\alpha^2y^6+6z\alpha y^6+4zy^6,\\
&6yx^6+(-24\alpha y^4-12y^4)x^4+12yz^2x^3+(9\alpha^2y^7+3\alpha y^7)x^2\\
&+(-12y^4z^2-24\alpha y^4z^2)x+6yz^4,\\
&(-6y^2z-6\alpha y^2z)x^3+(6z\alpha^2y^5+12z\alpha y^5+6zy^5)x-6y^2z^3-6\alpha y^2z^3,\\
&(-2\alpha y^3-2y^3)x^3+(-2\alpha y^6-2y^6)x+4\alpha^2y^3z^2+10\alpha y^3z^2+6y^3z^2,\\
&(9z\alpha^2+21z\alpha+12z)x^4+(6z\alpha^2y^3+12z\alpha y^3+6zy^3)x^2\\
&+(-6\alpha z^3-6z^3)x+z\alpha^2y^6+3z\alpha y^6+2zy^6,\\
&(9z\alpha^2y^2+24z\alpha y^2+15zy^2)x^3+(3z\alpha^2y^5+6z\alpha y^5+3zy^5)x-3y^2z^3-3\alpha y^2z^3,\\
&(-3\alpha-3)x^5+(-4\alpha y^3-4y^3)x^3+(6\alpha^2z^2+15\alpha z^2+9z^2)x^2\\
&+(-\alpha y^6-y^6)x+2\alpha^2y^3z^2+5\alpha y^3z^2+3y^3z^2,\\
&(-6yz-6\alpha yz)x^4+(9z\alpha^2y^4+21z\alpha y^4+12zy^4)x^2+(-6yz^3-6\alpha yz^3)x,\\
&(-3\alpha y^2-3y^2)x^4+(-3\alpha y^5-3y^5)x^2+(6\alpha^2y^2z^2+15\alpha y^2z^2+9y^2z^2)x,\\
&(-6z-6\alpha z)x^3+(-6y^3z-6\alpha y^3z)x+4\alpha^2z^3+6\alpha z^3+2z^3).
\end{aligned}
$$

Hence, we can obtain the results by definition.
(6) If $\alpha \in (\frac{11}{18}, \frac{13}{18}]$, we obtain

$$
\begin{aligned}
I_0 =&\mathcal{O}_X,\\
I_1 =& \sum_{v_j\in\mathcal{O}^{\geq 1+\alpha}}\mathcal{O}_X v_j+\sum_{a\in I_0(D)}\mathcal{O}_X(f\partial_x a-\alpha a\partial_x f)+\mathcal{O}_X(f\partial_y a-\alpha a\partial_y f)+\mathcal{O}_X(f\partial_z a-\alpha a\partial_z f)\\
=&\mathcal{O}_X(y^3)+\mathcal{O}_X(3x^2+y^3,xy^2,z)\\
=&(x^2,xy^2,y^3,z),\\
I_2 =& \sum_{v_j\in\mathcal{O}^{\geq 2+\alpha}}\mathcal{O}_X v_j+\sum_{a\in I_1(D)}\mathcal{O}_X(f\partial_x a-(\alpha+1)a\partial_x f)\\
&+\mathcal{O}_X(f\partial_y a-(\alpha+1)a\partial_y f)+\mathcal{O}_X(f\partial_z a-(\alpha+1)a\partial_z f)\\
=&((-3\alpha-1)x^4+(y^3-\alpha y^3)x^2+2z^2x,(-3\alpha y^2-3y^2)x^3,(-2z-2\alpha z)x^2,\\
&(-3\alpha y^2-2y^2)x^3+(-\alpha y^5)x+y^2z^2,2yx^4+(-3\alpha y^4-y^4)x^2+2yxz^2,(-2y^2z-2\alpha y^2z)x,
\end{aligned}
$$

$$
\begin{aligned}
&(-3\alpha y^3 - 3y^3)x^2 - \alpha y^6 - y^6, 3x^3y^2 - 3\alpha x y^5 + 3y^2z^2, -2y^3z - 2\alpha y^3 z,\\
&(-3z - 3\alpha z)x^2 - y^3 z - \alpha y^3 z, (-3y^2z - 3\alpha y^2 z)x, x^3 + y^3x - 2\alpha z^2 - z^2),\\
I_3 = &\sum_{v_j \in \mathcal{O}^{\geq 3+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_2(D)} \mathcal{O}_X(f\partial_x a - (\alpha+2)a\partial_x f)\\
&+ \mathcal{O}_X(f\partial_y a - (\alpha+2)a\partial_y f) + \mathcal{O}_X(f\partial_z a - (\alpha+2)a\partial_z f)\\
=&((9\alpha^2 + 9\alpha + 2)x^6 + (6\alpha^2y^3 - 4\alpha y^3 - 6y^3)x^4 + (-18\alpha z^2 - 14z^2)x^3\\
&+ (\alpha^2y^6 - \alpha y^6)x^2 + (-4\alpha y^3z^2)x + 2z^4,\\
&(9\alpha^2y^2 + 18\alpha y^2 + 9y^2)x^5 + (3\alpha^2y^5 - 3y^5)x^3 + (-9y^2z^2 - 9\alpha y^2z^2)x^2,\\
&(6z\alpha^2 + 14z\alpha + 8z)x^4 + (2z\alpha^2y^3 + 2z\alpha y^3)x^2 + (-4\alpha z^3 - 4z^3)x,\\
&(-6y - 6\alpha y)x^6 + (9\alpha^2y^4 + 21\alpha y^4 + 12y^4)x^4 + (-6yz^2 - 6\alpha y z^2)x^3,\\
&(6z\alpha^2y^2 + 18z\alpha y^2 + 12zy^2)x^3,\\
&(-2\alpha - 2)x^5 + (-2\alpha y^3 - 2y^3)x^3 + (4\alpha^2z^2 + 10\alpha z^2 + 6z^2)x^2,\\
&(9\alpha^2y^2 + 15\alpha y^2 + 6y^2)x^5 + (6\alpha^2y^5 + 4\alpha y^5 - 2y^5)x^3\\
&+ (-12y^2z^2 - 12\alpha y^2z^2)x^2 + (\alpha^2y^8 + \alpha y^8)x - 2y^5z^2 - 2\alpha y^5z^2,\\
&(-4y - 6\alpha y)x^6 + (9\alpha^2y^4 + 13\alpha y^4 + 8y^4)x^4 + (-2yz^2 - 6\alpha y z^2)x^3\\
&+ (3\alpha^2y^7 + \alpha y^7)x^2 + (-4y^4z^2 - 8\alpha y^4z^2)x + 2yz^4,\\
&(6z\alpha^2y^2 + 16z\alpha y^2 + 10zy^2)x^3 + (2z\alpha^2y^5 + 4z\alpha y^5 + 2zy^5)x - 2y^2z^3 - 2\alpha y^2z^3,\\
&2x^7 + (-18\alpha y^3 - 14y^3)x^5 + 4z^2x^4 + (9\alpha^2y^6 + 9\alpha y^6 + 2y^6)x^3\\
&+ (-14y^3z^2 - 18\alpha y^3z^2)x^2 + 2z^4x,\\
&(-4yz - 4\alpha yz)x^4 + (6z\alpha^2y^4 + 14z\alpha y^4 + 8zy^4)x^2 + (-4yz^3 - 4\alpha y z^3)x,\\
&(-2\alpha y^2 - 2y^2)x^4 + (-2\alpha y^5 - 2y^5)x^2 + (4\alpha^2y^2z^2 + 10\alpha y^2z^2 + 6y^2z^2)x,\\
&(9\alpha^2y^3 + 21\alpha y^3 + 12y^3)x^4 + (6\alpha^2y^6 + 12\alpha y^6 + 6y^6)x^2\\
&+ (-6y^3z^2 - 6\alpha y^3z^2)x + \alpha^2y^9 + 3\alpha y^9 + 2y^9,\\
&(-9\alpha y^2 - 9y^2)x^5 + (9\alpha^2y^5 + 12\alpha y^5 + 3y^5)x^3 + (-9y^2z^2 - 9\alpha y^2z^2)x^2\\
&+ (3\alpha^2y^8 + 3\alpha y^8)x - 6y^5z^2 - 6\alpha y^5z^2,\\
&(6z\alpha^2y^3 + 18z\alpha y^3 + 12zy^3)x^2 + 2z\alpha^2y^6 + 6z\alpha y^6 + 4zy^6,\\
&6yx^6 + (-24\alpha y^4 - 12y^4)x^4 + 12yz^2x^3 + (9\alpha^2y^7 + 3\alpha y^7)x^2\\
&+ (-12y^4z^2 - 24\alpha y^4z^2)x + 6yz^4,\\
&(-6y^2z - 6\alpha y^2z)x^3 + (6z\alpha^2y^5 + 12z\alpha y^5 + 6zy^5)x - 6y^2z^3 - 6\alpha y^2z^3,\\
&(-2\alpha y^3 - 2y^3)x^3 + (-2\alpha y^6 - 2y^6)x + 4\alpha^2y^3z^2 + 10\alpha y^3z^2 + 6y^3z^2,\\
&(9z\alpha^2 + 21z\alpha + 12z)x^4 + (6z\alpha^2y^3 + 12z\alpha y^3 + 6zy^3)x^2\\
&+ (-6\alpha z^3 - 6z^3)x + z\alpha^2y^6 + 3z\alpha y^6 + 2zy^6,\\
&(9z\alpha^2y^2 + 24z\alpha y^2 + 15zy^2)x^3 + (3z\alpha^2y^5 + 6z\alpha y^5 + 3zy^5)x - 3y^2z^3 - 3\alpha y^2z^3,\\
&(-3\alpha - 3)x^5 + (-4\alpha y^3 - 4y^3)x^3 + (6\alpha^2z^2 + 15\alpha z^2 + 9z^2)x^2\\
&+ (-\alpha y^6 - y^6)x + 2\alpha^2y^3z^2 + 5\alpha y^3z^2 + 3y^3z^2,\\
&(-6yz - 6\alpha yz)x^4 + (9z\alpha^2y^4 + 21z\alpha y^4 + 12zy^4)x^2 + (-6yz^3 - 6\alpha y z^3)x,
\end{aligned}
$$

$$(-3\alpha y^2-3y^2)x^4+(-3\alpha y^5-3y^5)x^2+(6\alpha^2y^2z^2+15\alpha y^2z^2+9y^2z^2)x,$$
$$(-6z-6\alpha z)x^3+(-6y^3z-6\alpha y^3z)x+4\alpha^2z^3+6\alpha z^3+2z^3).$$

Hence, we can obtain the results by definition.
(7) If $\alpha\in(\frac{13}{18},\frac{17}{18}]$, we obtain

$$\begin{aligned}
I_0 =&\mathcal{O}_X,\\
I_1 =&\sum_{v_j\in\mathcal{O}^{\geq 1+\alpha}}\mathcal{O}_Xv_j+\sum_{a\in I_0(D)}\mathcal{O}_X(f\partial_xa-\alpha a\partial_xf)\\
&+\mathcal{O}_X(f\partial_ya-\alpha a\partial_yf)+\mathcal{O}_X(f\partial_za-\alpha a\partial_zf)\\
=&\mathcal{O}_X(y^4)+\mathcal{O}_X(3x^2+y^3,xy^2,z)\\
=&(3x^2+y^3,xy^2,y^4,z),\\
I_2 =&\sum_{v_j\in\mathcal{O}^{\geq 2+\alpha}}\mathcal{O}_Xv_j+\sum_{a\in I_1(D)}\mathcal{O}_X(f\partial_xa-(\alpha+1)a\partial_xf)\\
&+\mathcal{O}_X(f\partial_ya-(\alpha+1)a\partial_yf)+\mathcal{O}_X(f\partial_za-(\alpha+1)a\partial_zf)\\
=&((-9\alpha-3)x^4+(-6\alpha y^3)x^2+6z^2x-\alpha y^6-y^6,(-9\alpha y^2-6y^2)x^3+(-3\alpha y^5)x+3y^2z^2,\\
&(-6z-6\alpha z)x^2-2y^3z-2\alpha y^3z,(-3\alpha y^2-2y^2)x^3+(-\alpha y^5)x+y^2z^2,\\
&2yx^4+(-3\alpha y^4-y^4)x^2+2yxz^2,(-2y^2z-2\alpha y^2z)x,\\
&(-3\alpha y^4-3y^4)x^2-\alpha y^7-y^7,4y^3x^3+(y^6-3\alpha y^6)x+4y^3z^2,-2y^4z-2\alpha y^4z,\\
&(-3z-3\alpha z)x^2-y^3z-\alpha y^3z,(-3y^2z-3\alpha y^2z)x,x^3+y^3x-2\alpha z^2-z^2),\\
I_3 =&\sum_{v_j\in\mathcal{O}^{\geq 3+\alpha}}\mathcal{O}_Xv_j+\sum_{a\in I_2(D)}\mathcal{O}_X(f\partial_xa-(\alpha+2)a\partial_xf)\\
&+\mathcal{O}_X(f\partial_ya-(\alpha+2)a\partial_yf)+\mathcal{O}_X(f\partial_za-(\alpha+2)a\partial_zf)\\
=&((27\alpha^2+27\alpha+6)x^6+(27\alpha^2y^3+9\alpha y^3-6y^3)x^4+(-54\alpha z^2-42z^2)x^3\\
&+(9\alpha^2y^6+9\alpha y^6+6y^6)x^2+(-6y^3z^2-18\alpha y^3z^2)x+\alpha^2y^9+3\alpha y^9+2y^9+6z^4,\\
&(27\alpha^2y^2+45\alpha y^2+18y^2)x^5+(18\alpha^2y^5+12\alpha y^5-6y^5)x^3\\
&+(-36y^2z^2-36\alpha y^2z^2)x^2+(3\alpha^2y^8+3\alpha y^8)x-6y^5z^2-6\alpha y^5z^2,\\
&(18z\alpha^2+42z\alpha+24z)x^4+(12z\alpha^2y^3+24z\alpha y^3+12zy^3)x^2\\
&+(-12\alpha z^3-12z^3)x+2z\alpha^2y^6+6z\alpha y^6+4zy^6,\\
&(-12y-18\alpha y)x^6+(27\alpha^2y^4+39\alpha y^4+24y^4)x^4+(-6yz^2-18\alpha yz^2)x^3\\
&+(9\alpha^2y^7+3\alpha y^7)x^2+(-12y^4z^2-24\alpha y^4z^2)x+6yz^4,\\
&(18z\alpha^2y^2+48z\alpha y^2+30zy^2)x^3+(6z\alpha^2y^5+12z\alpha y^5+6zy^5)x-6y^2z^3-6\alpha y^2z^3,\\
&(-6\alpha-6)x^5+(-8\alpha y^3-8y^3)x^3+(12\alpha^2z^2+30\alpha z^2+18z^2)x^2+(-2\alpha y^6-2y^6)x\\
&+4\alpha^2y^3z^2+10\alpha y^3z^2+6y^3z^2,\\
&(9\alpha^2y^2+15\alpha y^2+6y^2)x^5+(6\alpha^2y^5+4\alpha y^5-2y^5)x^3\\
&+(-12y^2z^2-12\alpha y^2z^2)x^2+(\alpha^2y^8+\alpha y^8)x-2y^5z^2-2\alpha y^5z^2,\\
&(-4y-6\alpha y)x^6+(9\alpha^2y^4+13\alpha y^4+8y^4)x^4+(-2yz^2-6\alpha yz^2)x^3\\
&+(3\alpha^2y^7+\alpha y^7)x^2+(-4y^4z^2-8\alpha y^4z^2)x+2yz^4,
\end{aligned}$$

$$
\begin{aligned}
&(6z\alpha^2y^2+16z\alpha y^2+10zy^2)x^3+(2z\alpha^2y^5+4z\alpha y^5+2zy^5)x-2y^2z^3-2\alpha y^2z^3,\\
&2x^7+(-18\alpha y^3-14y^3)x^5+4z^2x^4+(9\alpha^2y^6+9\alpha y^6+2y^6)x^3\\
&+(-14y^3z^2-18\alpha y^3z^2)x^2+2z^4x,\\
&(-4yz-4\alpha yz)x^4+(6z\alpha^2y^4+14z\alpha y^4+8zy^4)x^2+(-4yz^3-4\alpha yz^3)x,\\
&(-2\alpha y^2-2y^2)x^4+(-2\alpha y^5-2y^5)x^2+(4\alpha^2y^2z^2+10\alpha y^2z^2+6y^2z^2)x,\\
&(9\alpha^2y^4+21\alpha y^4+12y^4)x^4+(6\alpha^2y^7+12\alpha y^7+6y^7)x^2+(-6y^4z^2-6\alpha y^4z^2)x\\
&+\alpha^2y^{10}+3\alpha y^{10}+2y^{10},\\
&(-12\alpha y^3-12y^3)x^5+(9\alpha^2y^6+8\alpha y^6-y^6)x^3+(-12y^3z^2-12\alpha y^3z^2)x^2\\
&+(3\alpha^2y^9+2\alpha y^9-y^9)x-7y^6z^2-7\alpha y^6z^2,\\
&(6z\alpha^2y^4+18z\alpha y^4+12zy^4)x^2+2z\alpha^2y^7+6z\alpha y^7+4zy^7,\\
&12y^2x^6+(-30\alpha y^5-6y^5)x^4+24y^2z^2x^3+(9\alpha^2y^8-3\alpha y^8)x^2\\
&+(-6y^5z^2-30\alpha y^5z^2)x+12y^2z^4,\\
&(-8y^3z-8\alpha y^3z)x^3+(6z\alpha^2y^6+10z\alpha y^6+4zy^6)x-8y^3z^3-8\alpha y^3z^3,\\
&(-2\alpha y^4-2y^4)x^3+(-2\alpha y^7-2y^7)x+4\alpha^2y^4z^2+10\alpha y^4z^2+6y^4z^2,\\
&(9z\alpha^2+21z\alpha+12z)x^4+(6z\alpha^2y^3+12z\alpha y^3+6zy^3)x^2+(-6\alpha z^3-6z^3)x\\
&+z\alpha^2y^6+3z\alpha y^6+2zy^6,\\
&(9z\alpha^2y^2+24z\alpha y^2+15zy^2)x^3+(3z\alpha^2y^5+6z\alpha y^5+3zy^5)x-3y^2z^3-3\alpha y^2z^3,\\
&(-3\alpha-3)x^5+(-4\alpha y^3-4y^3)x^3+(6\alpha^2z^2+15\alpha z^2+9z^2)x^2\\
&+(-\alpha y^6-y^6)x+2\alpha^2y^3z^2+5\alpha y^3z^2+3y^3z^2,\\
&(-6yz-6\alpha yz)x^4+(9z\alpha^2y^4+21z\alpha y^4+12zy^4)x^2+(-6yz^3-6\alpha yz^3)x,\\
&(-3\alpha y^2-3y^2)x^4+(-3\alpha y^5-3y^5)x^2+(6\alpha^2y^2z^2+15\alpha y^2z^2+9y^2z^2)x,\\
&(-6z-6\alpha z)x^3+(-6y^3z-6\alpha y^3z)x+4\alpha^2z^3+6\alpha z^3+2z^3).
\end{aligned}
$$

Hence, we can obtain the results by definition.
(8) If $\alpha \in (\frac{17}{18}, 1]$, we obtain

$$
\begin{aligned}
I_0 =&\mathcal{O}_X,\\
I_1 =& \sum_{v_j\in\mathcal{O}^{\geq 1+\alpha}} \mathcal{O}_X v_j + \sum_{a\in I_0(D)} \mathcal{O}_X(f\partial_x a-\alpha a\partial_x f)+\mathcal{O}_X(f\partial_y a-\alpha a\partial_y f)+\mathcal{O}_X(f\partial_z a-\alpha a\partial_z f)\\
=&(3x^2+y^3, xy^2, z),\\
I_2 =& \sum_{v_j\in\mathcal{O}^{\geq 2+\alpha}} \mathcal{O}_X v_j + \sum_{a\in I_1(D)} \mathcal{O}_X(f\partial_x a-(\alpha+1)a\partial_x f)\\
&+\mathcal{O}_X(f\partial_y a-(\alpha+1)a\partial_y f)+\mathcal{O}_X(f\partial_z a-(\alpha+1)a\partial_z f)\\
=&((-9\alpha-3)x^4+(-6\alpha y^3)x^2+6z^2x-\alpha y^6-y^6, (-9\alpha y^2-6y^2)x^3+(-3\alpha y^5)x+3y^2z^2,\\
&(-6z-6\alpha z)x^2-2y^3z-2\alpha y^3z, (-3\alpha y^2-2y^2)x^3+(-\alpha y^5)x+y^2z^2,\\
&2yx^4+(-3\alpha y^4-y^4)x^2+2yxz^2, (-2y^2z-2\alpha y^2z)x,\\
&(-3z-3\alpha z)x^2-y^3z-\alpha y^3z, (-3y^2z-3\alpha y^2z)x, x^3+y^3x-2\alpha z^2-z^2),
\end{aligned}
$$

$$
\begin{aligned}
I_3 = & \sum_{v_j \in \mathcal{O}^{\geq 3+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_2(D)} \mathcal{O}_X (f\partial_x a - (\alpha+2) a \partial_x f) \\
& + \mathcal{O}_X (f\partial_y a - (\alpha+2) a \partial_y f) + \mathcal{O}_X (f\partial_z a - (\alpha+2) a \partial_z f) \\
=& ((27\alpha^2+27\alpha+6)x^6 + (27\alpha^2y^3+9\alpha y^3-6y^3)x^4 + (-54\alpha z^2-42z^2)x^3 + (9\alpha^2y^6 \\
& + 9\alpha y^6+6y^6)x^2 + (-6y^3z^2-18\alpha y^3z^2)x + \alpha^2y^9+3\alpha y^9+2y^9+6z^4, \\
& (27\alpha^2y^2+45\alpha y^2+18y^2)x^5 + (18\alpha^2y^5+12\alpha y^5-6y^5)x^3 + (-36y^2z^2-36\alpha y^2z^2)x^2 \\
& + (3\alpha^2y^8+3\alpha y^8)x - 6y^5z^2 - 6\alpha y^5z^2, \\
& (-12y-18\alpha y)x^6 + (27\alpha^2y^4+39\alpha y^4+24y^4)x^4 + (-6yz^2-18\alpha yz^2)x^3 \\
& + (9\alpha^2y^7+3\alpha y^7)x^2 + (-12y^4z^2-24\alpha y^4z^2)x + 6yz^4, \\
& (18z\alpha^2+42z\alpha+24z)x^4 + (12z\alpha^2y^3+24z\alpha y^3+12zy^3)x^2 + (-12\alpha z^3-12z^3)x \\
& + 2z\alpha^2y^6+6z\alpha y^6+4zy^6, \\
& (18z\alpha^2y^2+48z\alpha y^2+30zy^2)x^3 + (6z\alpha^2y^5+12z\alpha y^5+6zy^5)x - 6y^2z^3 - 6\alpha y^2z^3, \\
& (-6\alpha-6)x^5 + (-8\alpha y^3-8y^3)x^3 + (12\alpha^2z^2+30\alpha z^2+18z^2)x^2 \\
& + (-2\alpha y^6-2y^6)x + 4\alpha^2y^3z^2+10\alpha y^3z^2+6y^3z^2, \\
& (9\alpha^2y^2+15\alpha y^2+6y^2)x^5 + (6\alpha^2y^5+4\alpha y^5-2y^5)x^3 + (-12y^2z^2-12\alpha y^2z^2)x^2 \\
& + (\alpha^2y^8+\alpha y^8)x - 2y^5z^2 - 2\alpha y^5z^2, \\
& (-4y-6\alpha y)x^6 + (9\alpha^2y^4+13\alpha y^4+8y^4)x^4 + (-2yz^2-6\alpha yz^2)x^3 \\
& + (3\alpha^2y^7+\alpha y^7)x^2 + (-4y^4z^2-8\alpha y^4z^2)x + 2yz^4, \\
& (6z\alpha^2y^2+16z\alpha y^2+10zy^2)x^3 + (2z\alpha^2y^5+4z\alpha y^5+2zy^5)x - 2y^2z^3 - 2\alpha y^2z^3, \\
& 2x^7 + (-18\alpha y^3-14y^3)x^5 + 4z^2x^4 + (9\alpha^2y^6+9\alpha y^6+2y^6)x^3 + (-14y^3z^2-18\alpha y^3z^2)x^2 + 2z^4x, \\
& (-4yz-4\alpha yz)x^4 + (6z\alpha^2y^4+14z\alpha y^4+8zy^4)x^2 + (-4yz^3-4\alpha yz^3)x, \\
& (-2\alpha y^2-2y^2)x^4 + (-2\alpha y^5-2y^5)x^2 + (4\alpha^2y^2z^2+10\alpha y^2z^2+6y^2z^2)x, \\
& (9z\alpha^2+21z\alpha+12z)x^4 + (6z\alpha^2y^3+12z\alpha y^3+6zy^3)x^2 + (-6\alpha z^3-6z^3)x \\
& + z\alpha^2y^6+3z\alpha y^6+2zy^6, \\
& (9z\alpha^2y^2+24z\alpha y^2+15zy^2)x^3 + (3z\alpha^2y^5+6z\alpha y^5+3zy^5)x - 3y^2z^3 - 3\alpha y^2z^3, \\
& (-3\alpha-3)x^5 + (-4\alpha y^3-4y^3)x^3 + (6\alpha^2z^2+15\alpha z^2+9z^2)x^2 \\
& + (-\alpha y^6-y^6)x + 2\alpha^2y^3z^2+5\alpha y^3z^2+3y^3z^2, \\
& (-6yz-6\alpha yz)x^4 + (9z\alpha^2y^4+21z\alpha y^4+12zy^4)x^2 + (-6yz^3-6\alpha yz^3)x, \\
& (-3\alpha y^2-3y^2)x^4 + (-3\alpha y^5-3y^5)x^2 + (6\alpha^2y^2z^2+15\alpha y^2z^2+9y^2z^2)x, \\
& (-6z-6\alpha z)x^3 + (-6y^3z-6\alpha y^3z)x + 4\alpha^2z^3+6\alpha z^3+2z^3)
\end{aligned}
$$

Hence, we can obtain the results by definition. □

Proposition 3.5. *Let $f(x,y,z) = x^3+y^5+z^2$ be E_8 type singularity. Let $H = \{f=0\}$ be an integral and reduced effective divisor defined by f. And $D^\alpha = \alpha H$ where $\alpha \in \mathbb{Q} \cap (0,1]$. Then the first four Hodge moduli algebras and Hodge moduli numbers are given as follows.*

(1) For $\alpha \in (0, \frac{1}{30}]$,

$$
\begin{aligned}
&M_0 = M_1 = 0, m_0 = m_1 = 0.\\
&M_2 = \text{span}\{1, x, y, xy, y^2, xy^2, y^3, xy^3\}, m_2 = 8.\\
&M_3 = \text{span}\{1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, x^3y^2, y^3, xy^3, x^2y^3, y^4, xy^4,\\
&\quad y^5, xy^5, y^6, xy^6, y^7, xy^7, y^8, z, xz, yz, xyz, y^2z, xy^2z, y^3z, xy^3z\}, m_3 = 32.
\end{aligned}
$$

(2) For $\alpha \in (\frac{1}{30}, \frac{7}{30}]$,

$$
\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \text{span}\{1\}, m_1 = 1.\\
&M_2 = \text{span}\{1, x, x^2, y, xy, y^2, xy^2, y^3, xy^3, y^4, z\}, m_2 = 11.\\
&M_3 = \text{span}\{1, x, x^2, x^3, x^4, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, x^3y^2, y^3, xy^3, x^2y^3, x^3y^3,\\
&\quad y^4, xy^4, x^2y^4, y^5, xy^5, y^6, xy^6, y^7, xy^7, y^8, z, xz, x^2z, yz, xyz, y^2z, xy^2z,\\
&\quad y^3z, xy^3z, y^4z, z^2\}, m_3 = 38.
\end{aligned}
$$

(3) For $\alpha \in (\frac{7}{30}, \frac{11}{30}]$,

$$
\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \text{span}\{1, y\}, m_1 = 2.\\
&M_2 = \text{span}\{1, x, x^2, y, xy, x^2y, y^2, xy^2, y^3, xy^3, y^4, y^5, z, yz\}, m_2 = 14.\\
&M_3 = \text{span}\{1, x, x^2, x^3, x^4, y, xy, x^2y, x^3y, x^4y, y^2, xy^2, x^2y^2, x^3y^2, y^3, xy^3, x^2y^3, x^3y^3,\\
&\quad y^4, xy^4, x^2y^4, y^5, xy^5, x^2y^5, y^6, xy^6, y^7, xy^7, y^8, y^9, z, xz, x^2z, yz, xyz, x^2yz, y^2z, xy^2z,\\
&\quad y^3z, xy^3z, y^4z, y^5z, z^2, yz^2\}, m_3 = 44.
\end{aligned}
$$

(4) For $\alpha \in (\frac{11}{30}, \frac{13}{30}]$,

$$
\begin{aligned}
&M_0 = 0, m_0 = 0.\\
&M_1 = \text{span}\{1, x, y, xy\}, m_1 = 4.\\
&M_2 = \text{span}\{1, x, x^2, x^3, y, xy, x^2y, y^2, xy^2, y^3, xy^3, y^4, xy^4, y^5, xy^5, y^6,\\
&\quad z, xz, yz, xyz\}, m_2 = 20.\\
&M_3 = \text{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, y^2, xy^2, x^2y^2, x^3y^2, y^3, xy^3, x^2y^3, x^3y^3,\\
&\quad y^4, xy^4, x^2y^4, x^3y^4, y^5, xy^5, x^2y^5, y^6, xy^6, x^2y^6, y^7, xy^7, y^8, xy^8, y^9, xy^9, y^{10}\\
&\quad z, xz, x^2z, x^3z, yz, xyz, x^2yz, y^2z, xy^2z, y^3z, xy^3z, y^4z, xy^4z, y^5z, xy^5z, y^6z,\\
&\quad z^2, yz^2, xz^2, xyz^2\}, m_3 = 56.
\end{aligned}
$$

(5) For $\alpha \in (\frac{13}{30}, \frac{17}{30}]$,

$$M_0 = 0, m_0 = 0.$$
$$M_1 = \text{span}\{1, x, y, y^2\}, m_1 = 4.$$
$$M_2 = \text{span}\{1, x, x^2, x^3, y, xy, x^2y, y^2, xy^2, x^2y^2, y^3, xy^3, y^4, xy^4, y^5, y^6, z, xz, yz, y^2z\}, m_2 = 20.$$
$$M_3 = \text{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, y^2, xy^2, x^2y^2, x^3y^2, x^4y^2, y^3, xy^3, x^2y^3, x^3y^3, y^4, xy^4, x^2y^4, x^3y^4, y^5, xy^5, x^2y^5, y^6, xy^6, x^2y^6, y^7, xy^7, y^8, xy^8, y^9, y^{10}, z, xz, x^2z, x^3z, yz, xyz, x^2yz, y^2z, xy^2z, x^2y^2z, y^3z, xy^3z, y^4z, xy^4z, y^5z, y^6z, z^2, xz^2, yz^2, y^2z^2\}, m_3 = 56.$$

(6) For $\alpha \in (\frac{17}{30}, \frac{19}{30}]$,

$$M_0 = 0, m_0 = 0.$$
$$M_1 = \text{span}\{1, x, y, y^2, xy\}, m_1 = 5.$$
$$M_2 = \text{span}\{1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, y^3, xy^3, y^4, xy^4, y^5, xy^5, y^6, z, xz, yz, xyz, y^2z\}, m_2 = 23.$$
$$M_3 = \text{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y, y^2, xy^2, x^2y^2, x^3y^2, x^4y^2, y^3, xy^3, x^2y^3, x^3y^3, y^4, xy^4, x^2y^4, x^3y^4, y^5, xy^5, x^2y^5, x^3y^5, y^6, xy^6, x^2y^6, y^7, xy^7, y^8, xy^8, y^9, xy^9, y^{10}, z, xz, x^2z, x^3z, yz, xyz, x^2yz, x^3yz, y^2z, xy^2z, x^2y^2z, y^3z, xy^3z, y^4z, xy^4z, y^5z, xy^5z, y^6z, z^2, xz^2, yz^2, xyz^2, y^2z^2\}, m_3 = 62.$$

(7) For $\alpha \in (\frac{19}{30}, \frac{23}{30}]$,

$$M_0 = 0, m_0 = 0.$$
$$M_1 = \text{span}\{1, x, y, xy, y^2, y^3\}, m_1 = 6.$$
$$M_2 = \text{span}\{1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, y^3, xy^3, x^2y^3, y^4, xy^4, y^5, xy^5, y^6, y^7, z, xz, yz, xyz, y^2z, y^3z\}, m_2 = 26.$$
$$M_3 = \text{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y, y^2, xy^2, x^2y^2, x^3y^2, x^4y^2, y^3, xy^3, x^2y^3, x^3y^3, x^4y^3, y^4, xy^4, x^2y^4, x^3y^4, y^5, xy^5, x^2y^5, x^3y^5, y^6, xy^6, x^2y^6, y^7, xy^7, x^2y^7, y^8, xy^8, y^9, xy^9, y^{10}, y^{11}, z, xz, x^2z, x^3z, yz, xyz, x^2yz, x^3yz, y^2z, xy^2z, x^2y^2z, y^3z, xy^3z, x^2y^3z, y^4z, xy^4z, y^5z, xy^5z, y^6z, y^7z, z^2, xz^2, yz^2, xyz^2, y^2z^2, y^3z^2\}, m_3 = 68.$$

(8) For $\alpha \in (\frac{23}{30}, \frac{29}{30}]$,

$$M_0 = 0, m_0 = 0.$$
$$M_1 = \text{span}\{1, x, y, xy, y^2, xy^2, y^3\}, m_1 = 7.$$
$$M_2 = \text{span}\{1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, x^3y^2, y^3, xy^3, x^2y^3, y^4, xy^4, y^5, xy^5, y^6, xy^6, y^7, z, xz, yz, xyz, y^2z, xy^2z, y^3z\}, m_2 = 29.$$

$$M_3 = \text{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y, y^2, xy^2, x^2y^2, x^3y^2, x^4y^2, x^5y^2, y^3, xy^3, x^2y^3, x^3y^3, x^4y^3, y^4, xy^4, x^2y^4, x^3y^4, y^5, xy^5, x^2y^5, x^3y^5, y^6, xy^6, x^2y^6, x^3y^6, y^7, xy^7, x^2y^7, y^8, xy^8, y^9, xy^9, y^{10}, xy^{10}, y^{11}, z, xz, x^2z, x^3z, yz, xyz, x^2yz, x^3yz, y^2z, xy^2z, x^2y^2z, x^3y^2z, y^3z, xy^3z, x^2y^3z, y^4z, xy^4z, y^5z, xy^5z, y^6z, xy^6z, y^7z, z^2, xz^2, yz^2, xyz^2, y^2z^2, xy^2z^2, y^3z^2\}, m_3 = 74.$$

(9) For $\alpha \in (\frac{29}{30}, 1]$,

$$M_0 = 0, m_0 = 0.$$
$$M_1 = \text{span}\{1, x, y, xy, y^2, xy^2, y^3, xy^3\}, m_1 = 8.$$
$$M_2 = \text{span}\{1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, x^3y^2, y^3, xy^3, x^2y^3, y^4, xy^4, y^5, xy^5, y^6, xy^6, y^7, xy^7, y^8, z, xz, yz, xyz, y^2z, xy^2z, y^3z, xy^3z\}, m_2 = 32.$$
$$M_3 = \text{span}\{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y, y^2, xy^2, x^2y^2, x^3y^2, x^4y^2, x^5y^2, y^3, xy^3, x^2y^3, x^3y^3, x^4y^3, y^4, xy^4, x^2y^4, x^3y^4, y^5, xy^5, x^2y^5, x^3y^5, y^6, xy^6, x^2y^6, x^3y^6, y^7, xy^7, x^2y^7, y^8, xy^8, x^2y^8, y^9, xy^9, y^{10}, xy^{10}, y^{11}, xy^{11}, y^{12}, z, xz, x^2z, x^3z, yz, xyz, x^2yz, x^3yz, y^2z, xy^2z, x^2y^2z, x^3y^2z, y^3z, xy^3z, x^2y^3z, y^4z, xy^4z, y^5z, xy^5z, y^6z, xy^6z, y^7z, xy^7z, y^8z, z^2, xz^2, yz^2, xyz^2, y^2z^2, xy^2z^2, y^3z^2, xy^3z^2\}, m_3 = 80.$$

Proof. (1) If $\alpha \in (0, \frac{1}{30}]$, we obtain

$$\begin{aligned}
I_0 =& I_1 = \mathcal{O}_X, \\
I_2 =& \sum_{v_j \in \mathcal{O}^{\geq 2+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_1(D)} \mathcal{O}_X(f\partial_x a - (\alpha+1)a\partial_x f) \\
&+ \mathcal{O}_X(f\partial_y a - (\alpha+1)a\partial_y f) + \mathcal{O}_X(f\partial_z a - (\alpha+1)a\partial_z f) \\
=& ((-3\alpha-3)x^2, -5\alpha y^4 - 5y^4, -2z - 2\alpha z) \\
I_3 =& \sum_{v_j \in \mathcal{O}^{\geq 3+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_2(D)} \mathcal{O}_X(f\partial_x a - (\alpha+2)a\partial_x f) \\
&+ \mathcal{O}_X(f\partial_y a - (\alpha+2)a\partial_y f) + \mathcal{O}_X(f\partial_z a - (\alpha+2)a\partial_z f) \\
=& ((9\alpha^2 + 21\alpha + 12)x^4 + (-6\alpha y^5 - 6\alpha z^2 - 6y^5 - 6z^2)x, \\
& (15\alpha^2 y^4 + 45\alpha y^4 + 30y^4)x^2, (6z\alpha^2 + 18z\alpha + 12z)x^2, \\
& (-20\alpha y^3 - 20y^3)x^3 + 25\alpha^2 y^8 + 55\alpha y^8 - 20\alpha y^3 z^2 + 30y^8 - 20y^3z^2, \\
& 10z\alpha^2 y^4 + 30z\alpha y^4 + 20zy^4, (-2\alpha-2)x^3 + 4\alpha^2 z^2 - 2\alpha y^5 + 10\alpha z^2 - 2y^5 + 6z^2).
\end{aligned}$$

Hence, we can obtain the results by definition.
(2) If $\alpha \in (\frac{1}{30}, \frac{7}{30}]$, we obtain

$$\begin{aligned}
I_0 =& \mathcal{O}_X, \\
I_1 =& \sum_{v_j \in \mathcal{O}^{\geq 1+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_0(D)} \mathcal{O}_X(f\partial_x a - \alpha a\partial_x f) + \mathcal{O}_X(f\partial_y a - \alpha a\partial_y f) + \mathcal{O}_X(f\partial_z a - \alpha a\partial_z f)
\end{aligned}$$

$$
\begin{aligned}
&=\mathcal{O}_X(x,y)+\mathcal{O}_X(x^2,y^4,z)\\
&=\mathcal{O}_X(x,y,z),\\
I_2=&\sum_{v_j\in\mathcal{O}^{\geq 2+\alpha}}\mathcal{O}_Xv_j+\sum_{a\in I_1(D)}\mathcal{O}_X(f\partial_x a-(\alpha+1)a\partial_x f)\\
&+\mathcal{O}_X(f\partial_y a-(\alpha+1)a\partial_y f)+\mathcal{O}_X(f\partial_z a-(\alpha+1)a\partial_z f)\\
&=((-3\alpha-2)x^3+y^5+z^2,(-5\alpha y^4-5y^4)x,(-2z-2\alpha z)x,\\
&(-3y-3\alpha y)x^2,x^3+z^2-4y^5-5\alpha y^5,-2yz-2\alpha yz,\\
&(-3z-3\alpha z)x^2,-5y^4z-5\alpha y^4z,x^3+y^5-2\alpha z^2-z^2),\\
I_3=&\sum_{v_j\in\mathcal{O}^{\geq 3+\alpha}}\mathcal{O}_Xv_j+\sum_{a\in I_2(D)}\mathcal{O}_X(f\partial_x a-(\alpha+2)a\partial_x f)\\
&+\mathcal{O}_X(f\partial_y a-(\alpha+2)a\partial_y f)+\mathcal{O}_X(f\partial_z a-(\alpha+2)a\partial_z f)\\
&(15\alpha^2y^4+40\alpha y^4+25y^4)x^3-5y^4z^2-5\alpha y^9-5y^9-5\alpha y^4z^2,\\
&(-20\alpha y^3-20y^3)x^4+(25\alpha^2y^8+55\alpha y^8-20\alpha y^3z^2+30y^8-20y^3z^2)x,\\
&(6z\alpha^2+16z\alpha+10z)x^3-2\alpha z^3-2y^5z-2z^3-2\alpha y^5z,\\
&(10z\alpha^2y^4+30z\alpha y^4+20zy^4)x,\\
&(-2\alpha-2)x^4+(4\alpha^2z^2-2\alpha y^5+10\alpha z^2-2y^5+6z^2)x,\\
&(9y\alpha^2+21y\alpha+12y)x^4+(-6\alpha y^6-6yz^2-6y^6-6\alpha yz^2)x,\\
&(-3\alpha-3)x^5+(15\alpha^2y^5+42\alpha y^5-3\alpha z^2+27y^5-3z^2)x^2,\\
&(-30\alpha y^4-30y^4)x^3+25\alpha^2y^9+45\alpha y^9-30\alpha y^4z^2+20y^9-30y^4z^2,\\
&(6yz\alpha^2+18yz\alpha+12yz)x^2,\\
&(-2z-2\alpha z)x^3+10\alpha^2y^5z+28\alpha y^5z-2\alpha z^3+18y^5z-2z^3,\\
&(-2y-2\alpha y)x^3+4\alpha^2yz^2-2\alpha y^6+10\alpha yz^2-2y^6+6yz^2,\\
&(9z\alpha^2+21z\alpha+12z)x^4+(-6\alpha z^3-6y^5z-6z^3-6\alpha y^5z)x,\\
&(15z\alpha^2y^4+45z\alpha y^4+30zy^4)x^2,\\
&(-20y^3z-20\alpha y^3z)x^3+25\alpha^2y^8z+55\alpha y^8z-20\alpha y^3z^3+30y^8z-20y^3z^3,\\
&(-3\alpha-3)x^5+(6\alpha^2z^2-3\alpha y^5+15\alpha z^2-3y^5+9z^2)x^2,\\
&(-5\alpha y^4-5y^4)x^3+10\alpha^2y^4z^2-5\alpha y^9+25\alpha y^4z^2-5y^9+15y^4z^2,\\
&(-6z-6\alpha z)x^3+4\alpha^2z^3-6\alpha y^5z+6\alpha z^3-6y^5z+2z^3).
\end{aligned}
$$

Hence, we can obtain the results by definition.
(3) If $\alpha\in(\frac{7}{30},\frac{11}{30}]$, we obtain

$$
\begin{aligned}
I_0=&\mathcal{O}_X,\\
I_1=&\sum_{v_j\in\mathcal{O}^{\geq 1+\alpha}}\mathcal{O}_Xv_j+\sum_{a\in I_0(D)}\mathcal{O}_X(f\partial_x a-\alpha a\partial_x f)+\mathcal{O}_X(f\partial_y a-\alpha a\partial_y f)+\mathcal{O}_X(f\partial_z a-\alpha a\partial_z f)\\
&=\mathcal{O}_X(x,y^2)+\mathcal{O}_X(x^2,y^4,z)\\
&=\mathcal{O}_X(x,y^2,z),
\end{aligned}
$$

$$\begin{aligned}
I_2 =& \sum_{v_j\in\mathcal{O}^{\geq 2+\alpha}} \mathcal{O}_X v_j + \sum_{a\in I_1(D)} \mathcal{O}_X(f\partial_x a-(\alpha+1)a\partial_x f)\\
&+\mathcal{O}_X(f\partial_y a-(\alpha+1)a\partial_y f)+\mathcal{O}_X(f\partial_z a-(\alpha+1)a\partial_z f)\\
=&((-3\alpha-2)x^3+y^5+z^2,(-5\alpha y^4-5y^4)x,(-2z-2\alpha z)x,\\
&(-3\alpha y^2-3y^2)x^2,2yx^3+2yz^2-5\alpha y^6-3y^6,-2y^2z-2\alpha y^2z,\\
&(-3z-3\alpha z)x^2,-5y^4z-5\alpha y^4z,x^3+y^5-2\alpha z^2-z^2),\\
I_3 =& \sum_{v_j\in\mathcal{O}^{\geq 3+\alpha}} \mathcal{O}_X v_j + \sum_{a\in I_2(D)} \mathcal{O}_X(f\partial_x a-(\alpha+2)a\partial_x f)\\
&+\mathcal{O}_X(f\partial_y a-(\alpha+2)a\partial_y f)+\mathcal{O}_X(f\partial_z a-(\alpha+2)a\partial_z f)\\
=&((9\alpha^2+15\alpha+6)x^5+(-12\alpha y^5-12\alpha z^2-12y^5-12z^2)x^2,\\
&(15\alpha^2y^4+40\alpha y^4+25y^4)x^3-5y^4z^2-5\alpha y^9-5y^9-5\alpha y^4z^2,\\
&(-20\alpha y^3-20y^3)x^4+(25\alpha^2y^8+55\alpha y^8-20\alpha y^3z^2+30y^8-20y^3z^2)x,\\
&(6z\alpha^2+16z\alpha+10z)x^3-2\alpha z^3-2y^5z-2z^3-2\alpha y^5z,\\
&(10z\alpha^2y^4+30z\alpha y^4+20zy^4)x,\\
&(-2\alpha-2)x^4+(4\alpha^2z^2-2\alpha y^5+10\alpha z^2-2y^5+6z^2)x,\\
&(9\alpha^2y^2+21\alpha y^2+12y^2)x^4+(-6y^2z^2-6\alpha y^7-6y^7-6\alpha y^2z^2)x,\\
&(-6y-6\alpha y)x^5+(15\alpha^2y^6+39\alpha y^6-6\alpha yz^2+24y^6-6yz^2)x^2,\\
&2x^6+(4z^2-36y^5-40\alpha y^5)x^3+25\alpha^2y^{10}+35\alpha y^{10}-40\alpha y^5z^2+12y^{10}-36y^5z^2+2z^4,\\
&(6z\alpha^2y^2+18z\alpha y^2+12zy^2)x^2,\\
&(-4yz-4\alpha yz)x^3+10\alpha^2y^6z+26\alpha y^6z-4\alpha yz^3+16y^6z-4yz^3,\\
&(-2\alpha y^2-2y^2)x^3+4\alpha^2y^2z^2-2\alpha y^7+10\alpha y^2z^2-2y^7+6y^2z^2,\\
&(9z\alpha^2+21z\alpha+12z)x^4+(-6\alpha z^3-6y^5z-6z^3-6\alpha y^5z)x,\\
&(15z\alpha^2y^4+45z\alpha y^4+30zy^4)x^2,\\
&(-20y^3z-20\alpha y^3z)x^3+25\alpha^2y^8z+55\alpha y^8z-20\alpha y^3z^3+30y^8z-20y^3z^3,\\
&(-3\alpha-3)x^5+(6\alpha^2z^2-3\alpha y^5+15\alpha z^2-3y^5+9z^2)x^2,\\
&(-5\alpha y^4-5y^4)x^3+10\alpha^2y^4z^2-5\alpha y^9+25\alpha y^4z^2-5y^9+15y^4z^2,\\
&(-6z-6\alpha z)x^3+4\alpha^2z^3-6\alpha y^5z+6\alpha z^3-6y^5z+2z^3).
\end{aligned}$$

Hence, we can obtain the results by definition.
(4) If $\alpha \in (\frac{11}{30}, \frac{13}{30}]$, we obtain

$$\begin{aligned}
I_0 =&\mathcal{O}_X,\\
I_1 =& \sum_{v_j\in\mathcal{O}^{\geq 1+\alpha}} \mathcal{O}_X v_j + \sum_{a\in I_0(D)} \mathcal{O}_X(f\partial_x a-\alpha a\partial_x f)+\mathcal{O}_X(f\partial_y a-\alpha a\partial_y f)+\mathcal{O}_X(f\partial_z a-\alpha a\partial_z f)\\
=&\mathcal{O}_X(x^2,y^2)+\mathcal{O}_X(x^2,y^4,z)\\
=&\mathcal{O}_X(x^2,y^2,z),\\
I_2 =& \sum_{v_j\in\mathcal{O}^{\geq 2+\alpha}} \mathcal{O}_X v_j + \sum_{a\in I_1(D)} \mathcal{O}_X(f\partial_x a-(\alpha+1)a\partial_x f)\\
&+\mathcal{O}_X(f\partial_y a-(\alpha+1)a\partial_y f)+\mathcal{O}_X(f\partial_z a-(\alpha+1)a\partial_z f)
\end{aligned}$$

$$
\begin{aligned}
=&((-3\alpha-1)x^4+(2y^5+2z^2)x,(-5\alpha y^4-5y^4)x^2,(-2z-2\alpha z)x^2,\\
&(-3\alpha y^2-3y^2)x^2,2yx^3+2yz^2-5\alpha y^6-3y^6,-2y^2z-2\alpha y^2z,\\
&(-3z-3\alpha z)x^2,-5y^4z-5\alpha y^4z,x^3+y^5-2\alpha z^2-z^2),\\
I_3=&\sum_{v_j\in\mathcal{O}^{\geq 3+\alpha}}\mathcal{O}_Xv_j+\sum_{a\in I_2(D)}\mathcal{O}_X(f\partial_xa-(\alpha+2)a\partial_xf)\\
&+\mathcal{O}_X(f\partial_ya-(\alpha+2)a\partial_yf)+\mathcal{O}_X(f\partial_za-(\alpha+2)a\partial_zf)\\
=&((9\alpha^2+9\alpha+2)x^6+(-18\alpha y^5-18\alpha z^2-14y^5-14z^2)x^3+2y^{10}+4y^5z^2+2z^4,\\
&(15\alpha^2y^4+35\alpha y^4+20y^4)x^4+(-10y^4z^2-10\alpha y^9-10y^9-10\alpha y^4z^2)x,\\
&(-20\alpha y^3-20y^3)x^5+(25\alpha^2y^8+55\alpha y^8-20\alpha y^3z^2+30y^8-20y^3z^2)x^2,\\
&(10z\alpha^2y^4+30z\alpha y^4+20zy^4)x^2,\\
&(6z\alpha^2+14z\alpha+8z)x^4+(-4\alpha z^3-4y^5z-4z^3-4\alpha y^5z)x,\\
&(-2\alpha-2)x^5+(4\alpha^2z^2-2\alpha y^5+10\alpha z^2-2y^5+6z^2)x^2,\\
&(9\alpha^2y^2+21\alpha y^2+12y^2)x^4+(-6y^2z^2-6\alpha y^7-6y^7-6\alpha y^2z^2)x,\\
&(-6y-6\alpha y)x^5+(15\alpha^2y^6+39\alpha y^6-6\alpha yz^2+24y^6-6yz^2)x^2,\\
&2x^6+(4z^2-36y^5-40\alpha y^5)x^3+25\alpha^2y^{10}+35\alpha y^{10}-40\alpha y^5z^2+12y^{10}-36y^5z^2+2z^4,\\
&(6z\alpha^2y^2+18z\alpha y^2+12zy^2)x^2,\\
&(-4yz-4\alpha yz)x^3+10\alpha^2y^6z+26\alpha y^6z-4\alpha yz^3+16y^6z-4yz^3,\\
&(-2\alpha y^2-2y^2)x^3+4\alpha^2y^2z^2-2\alpha y^7+10\alpha y^2z^2-2y^7+6y^2z^2,\\
&(9z\alpha^2+21z\alpha+12z)x^4+(-6\alpha z^3-6y^5z-6z^3-6\alpha y^5z)x,\\
&(15z\alpha^2y^4+45z\alpha y^4+30zy^4)x^2,\\
&(-20y^3z-20\alpha y^3z)x^3+25\alpha^2y^8z+55\alpha y^8z-20\alpha y^3z^3+30y^8z-20y^3z^3,\\
&(-3\alpha-3)x^5+(6\alpha^2z^2-3\alpha y^5+15\alpha z^2-3y^5+9z^2)x^2,\\
&(-5\alpha y^4-5y^4)x^3+10\alpha^2y^4z^2-5\alpha y^9+25\alpha y^4z^2-5y^9+15y^4z^2,\\
&(-6z-6\alpha z)x^3+4\alpha^2z^3-6\alpha y^5z+6\alpha z^3-6y^5z+2z^3).
\end{aligned}
$$

Hence, we can obtain the results by definition.
(5) If $\alpha\in(\frac{13}{30},\frac{17}{30}]$, we obtain

$$
\begin{aligned}
I_0=&\mathcal{O}_X,\\
I_1=&\sum_{v_j\in\mathcal{O}^{\geq 1+\alpha}}\mathcal{O}_Xv_j+\sum_{a\in I_0(D)}\mathcal{O}_X(f\partial_xa-\alpha a\partial_xf)+\mathcal{O}_X(f\partial_ya-\alpha a\partial_yf)+\mathcal{O}_X(f\partial_za-\alpha a\partial_zf)\\
=&\mathcal{O}_X(x^2,xy,y^3)+\mathcal{O}_X(x^2,y^4,z)\\
=&\mathcal{O}_X(x^2,xy,y^3,z),\\
I_2=&\sum_{v_j\in\mathcal{O}^{\geq 2+\alpha}}\mathcal{O}_Xv_j+\sum_{a\in I_1(D)}\mathcal{O}_X(f\partial_xa-(\alpha+1)a\partial_xf)\\
&+\mathcal{O}_X(f\partial_ya-(\alpha+1)a\partial_yf)+\mathcal{O}_X(f\partial_za-(\alpha+1)a\partial_zf)\\
=&((-3\alpha-1)x^4+(2y^5+2z^2)x,(-5\alpha y^4-5y^4)x^2,(-2z-2\alpha z)x^2,\\
&(-2y-3\alpha y)x^3+y^6+yz^2,x^4+(z^2-4y^5-5\alpha y^5)x,(-2yz-2\alpha yz)x,\\
&(-3\alpha y^3-3y^3)x^2,3y^2x^3+3y^2z^2-5\alpha y^7-2y^7,-2y^3z-2\alpha y^3z,
\end{aligned}
$$

$$
\begin{aligned}
&(-3z-3\alpha z)x^2, -5y^4z-5\alpha y^4z, x^3+y^5-2\alpha z^2-z^2),\\
I_3=&\sum_{v_j\in\mathcal{O}^{\geq 3+\alpha}}\mathcal{O}_Xv_j+\sum_{a\in I_2(D)}\mathcal{O}_X(f\partial_xa-(\alpha+2)a\partial_xf)\\
&+\mathcal{O}_X(f\partial_ya-(\alpha+2)a\partial_yf)+\mathcal{O}_X(f\partial_za-(\alpha+2)a\partial_zf)\\
=&((9\alpha^2+9\alpha+2)x^6+(-18\alpha y^5-18\alpha z^2-14y^5-14z^2)x^3+2y^{10}+4y^5z^2+2z^4,\\
&(15\alpha^2y^4+35\alpha y^4+20y^4)x^4+(-10y^4z^2-10\alpha y^9-10y^9-10\alpha y^4z^2)x,\\
&(-20\alpha y^3-20y^3)x^5+(25\alpha^2y^8+55\alpha y^8-20\alpha y^3z^2+30y^8-20y^3z^2)x^2,\\
&(6z\alpha^2+14z\alpha+8z)x^4+(-4\alpha z^3-4y^5z-4z^3-4\alpha y^5z)x,\\
&(10z\alpha^2y^4+30z\alpha y^4+20zy^4)x^2,\\
&(-2\alpha-2)x^5+(4\alpha^2z^2-2\alpha y^5+10\alpha z^2-2y^5+6z^2)x^2,\\
&(9y\alpha^2+15y\alpha+6y)x^5+(-12\alpha y^6-12yz^2-12y^6-12\alpha yz^2)x^2,\\
&(6yz\alpha^2+16yz\alpha+10yz)x^3-2yz^3-2y^6z-2\alpha yz^3-2\alpha y^6z,\\
&(-3\alpha-2)x^6+(15\alpha^2y^5+37\alpha y^5-3\alpha z^2+24y^5-z^2)x^3+z^4-5\alpha y^{10}-4y^{10}-3y^5z^2-5\alpha y^5z^2,\\
&(-30\alpha y^4-30y^4)x^4+(25\alpha^2y^9+45\alpha y^9-30\alpha y^4z^2+20y^9-30y^4z^2)x,\\
&(-2z-2\alpha z)x^4+(10\alpha^2y^5z+28\alpha y^5z-2\alpha z^3+18y^5z-2z^3)x,\\
&(-2y-2\alpha y)x^4+(4\alpha^2yz^2-2\alpha y^6+10\alpha yz^2-2y^6+6yz^2)x,\\
&(9\alpha^2y^3+21\alpha y^3+12y^3)x^4+(-6y^3z^2-6\alpha y^8-6y^8-6\alpha y^3z^2)x,\\
&(-9\alpha y^2-9y^2)x^5+(15\alpha^2y^7+36\alpha y^7-9\alpha y^2z^2+21y^7-9y^2z^2)x^2,\\
&6yx^6+(12yz^2-50\alpha y^6-38y^6)x^3+25\alpha^2y^{11}+25\alpha y^{11}-50\alpha y^6z^2\\
&+6y^{11}-38y^6z^2+6yz^4, (6z\alpha^2y^3+18z\alpha y^3+12zy^3)x^2,\\
&(-6y^2z-6\alpha y^2z)x^3+10\alpha^2y^7z+24\alpha y^7z-6\alpha y^2z^3+14y^7z-6y^2z^3,\\
&(-2\alpha y^3-2y^3)x^3+4\alpha^2y^3z^2-2\alpha y^8+10\alpha y^3z^2-2y^8+6y^3z^2,\\
&(9z\alpha^2+21z\alpha+12z)x^4+(-6\alpha z^3-6y^5z-6z^3-6\alpha y^5z)x,\\
&(15z\alpha^2y^4+45z\alpha y^4+30zy^4)x^2,\\
&(-20y^3z-20\alpha y^3z)x^3+25\alpha^2y^8z+55\alpha y^8z-20\alpha y^3z^3+30y^8z-20y^3z^3,\\
&(-3\alpha-3)x^5+(6\alpha^2z^2-3\alpha y^5+15\alpha z^2-3y^5+9z^2)x^2,\\
&(-5\alpha y^4-5y^4)x^3+10\alpha^2y^4z^2-5\alpha y^9+25\alpha y^4z^2-5y^9+15y^4z^2,\\
&(-6z-6\alpha z)x^3+4\alpha^2z^3-6\alpha y^5z+6\alpha z^3-6y^5z+2z^3).
\end{aligned}
$$

Hence, we can obtain the results by definition.
(6) If $\alpha\in(\frac{17}{30},\frac{19}{30}]$, we obtain

$$
\begin{aligned}
I_0=&\mathcal{O}_X,\\
I_1=&\sum_{v_j\in\mathcal{O}^{\geq 1+\alpha}}\mathcal{O}_Xv_j+\sum_{a\in I_0(D)}\mathcal{O}_X(f\partial_xa-\alpha a\partial_xf)+\mathcal{O}_X(f\partial_ya-\alpha a\partial_yf)+\mathcal{O}_X(f\partial_za-\alpha a\partial_zf)\\
=&\mathcal{O}_X(x^2,xy^2,y^3)+\mathcal{O}_X(x^2,y^4,z)\\
=&\mathcal{O}_X(x^2,xy^2,y^3,z),
\end{aligned}
$$

$$
\begin{aligned}
I_2 = & \sum_{v_j \in \mathcal{O}^{\geq 2+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_1(D)} \mathcal{O}_X (f \partial_x a - (\alpha+1) a \partial_x f) \\
& + \mathcal{O}_X (f \partial_y a - (\alpha+1) a \partial_y f) + \mathcal{O}_X (f \partial_z a - (\alpha+1) a \partial_z f) \\
= & ((-3\alpha - 1)x^4 + (2y^5 + 2z^2)x, (-5\alpha y^4 - 5y^4)x^2, (-2z - 2\alpha z)x^2, \\
& (-3\alpha y^2 - 2y^2)x^3 + y^7 + y^2 z^2, 2yx^4 + (2yz^2 - 5\alpha y^6 - 3y^6)x, (-2y^2 z - 2\alpha y^2 z)x, \\
& (-3\alpha y^3 - 3y^3)x^2, 3y^2 x^3 + 3y^2 z^2 - 5\alpha y^7 - 2y^7, -2y^3 z - 2\alpha y^3 z, \\
& (-3z - 3\alpha z)x^2, -5y^4 z - 5\alpha y^4 z, x^3 + y^5 - 2\alpha z^2 - z^2),
\end{aligned}
$$

$$
\begin{aligned}
I_3 = & \sum_{v_j \in \mathcal{O}^{\geq 3+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_2(D)} \mathcal{O}_X (f \partial_x a - (\alpha+2) a \partial_x f) \\
& + \mathcal{O}_X (f \partial_y a - (\alpha+2) a \partial_y f) + \mathcal{O}_X (f \partial_z a - (\alpha+2) a \partial_z f) \\
= & ((9\alpha^2 + 9\alpha + 2)x^6 + (-18\alpha y^5 - 18\alpha z^2 - 14y^5 - 14z^2)x^3 + 2y^{10} + 4y^5 z^2 + 2z^4, \\
& (15\alpha^2 y^4 + 35\alpha y^4 + 20y^4)x^4 + (-10y^4 z^2 - 10\alpha y^9 - 10y^9 - 10\alpha y^4 z^2)x, \\
& (-20\alpha y^3 - 20y^3)x^5 + (25\alpha^2 y^8 + 55\alpha y^8 - 20\alpha y^3 z^2 + 30y^8 - 20y^3 z^2)x^2, \\
& (6z\alpha^2 + 14z\alpha + 8z)x^4 + (-4\alpha z^3 - 4y^5 z - 4z^3 - 4\alpha y^5 z)x, \\
& (10z\alpha^2 y^4 + 30z\alpha y^4 + 20zy^4)x^2, \\
& (-2\alpha - 2)x^5 + (4\alpha^2 z^2 - 2\alpha y^5 + 10\alpha z^2 - 2y^5 + 6z^2)x^2, \\
& (9\alpha^2 y^2 + 15\alpha y^2 + 6y^2)x^5 + (-12y^2 z^2 - 12\alpha y^7 - 12y^7 - 12\alpha y^2 z^2)x^2, \\
& (6z\alpha^2 y^2 + 16z\alpha y^2 + 10zy^2)x^3 - 2y^2 z^3 - 2y^7 z - 2\alpha y^7 z - 2\alpha y^2 z^3, \\
& (-4y - 6\alpha y)x^6 + (15\alpha^2 y^6 + 34\alpha y^6 - 6\alpha y z^2 + 23y^6 - 2yz^2)x^3 \\
& + 2yz^4 - 5\alpha y^{11} - y^6 z^2 - 3y^{11} - 5\alpha y^6 z^2, \\
& 2x^7 + (4z^2 - 36y^5 - 40\alpha y^5)x^4 + (25\alpha^2 y^{10} + 35\alpha y^{10} - 40\alpha y^5 z^2 + 12y^{10} - 36y^5 z^2 + 2z^4)x, \\
& (-4yz - 4\alpha yz)x^4 + (10\alpha^2 y^6 z + 26\alpha y^6 z - 4\alpha y z^3 + 16y^6 z - 4yz^3)x, \\
& (-2\alpha y^2 - 2y^2)x^4 + (4\alpha^2 y^2 z^2 - 2\alpha y^7 + 10\alpha y^2 z^2 - 2y^7 + 6y^2 z^2)x, \\
& (9\alpha^2 y^3 + 21\alpha y^3 + 12y^3)x^4 + (-6y^3 z^2 - 6\alpha y^8 - 6y^8 - 6\alpha y^3 z^2)x, \\
& (-9\alpha y^2 - 9y^2)x^5 + (15\alpha^2 y^7 + 36\alpha y^7 - 9\alpha y^2 z^2 + 21y^7 - 9y^2 z^2)x^2, \\
& 6yx^6 + (12yz^2 - 50\alpha y^6 - 38y^6)x^3 + 25\alpha^2 y^{11} + 25\alpha y^{11} - 50\alpha y^6 z^2 + 6y^{11} - 38y^6 z^2 + 6yz^4, \\
& (6z\alpha^2 y^3 + 18z\alpha y^3 + 12zy^3)x^2, \\
& (-6y^2 z - 6\alpha y^2 z)x^3 + 10\alpha^2 y^7 z + 24\alpha y^7 z - 6\alpha y^2 z^3 + 14y^7 z - 6y^2 z^3, \\
& (-2\alpha y^3 - 2y^3)x^3 + 4\alpha^2 y^3 z^2 - 2\alpha y^8 + 10\alpha y^3 z^2 - 2y^8 + 6y^3 z^2, \\
& (9z\alpha^2 + 21z\alpha + 12z)x^4 + (-6\alpha z^3 - 6y^5 z - 6z^3 - 6\alpha y^5 z)x, \\
& (-3\alpha - 3)x^5 + (6\alpha^2 z^2 - 3\alpha y^5 + 15\alpha z^2 - 3y^5 + 9z^2)x^2, \\
& (15z\alpha^2 y^4 + 45z\alpha y^4 + 30zy^4)x^2, \\
& (-20y^3 z - 20\alpha y^3 z)x^3 + 25\alpha^2 y^8 z + 55\alpha y^8 z - 20\alpha y^3 z^3 + 30y^8 z - 20y^3 z^3, \\
& (-5\alpha y^4 - 5y^4)x^3 + 10\alpha^2 y^4 z^2 - 5\alpha y^9 + 25\alpha y^4 z^2 - 5y^9 + 15y^4 z^2, \\
& (-6z - 6\alpha z)x^3 + 4\alpha^2 z^3 - 6\alpha y^5 z + 6\alpha z^3 - 6y^5 z + 2z^3).
\end{aligned}
$$

Hence, we can obtain the results by definition.
(7) If $\alpha \in (\frac{19}{30}, \frac{23}{30}]$, we obtain

$$
\begin{aligned}
I_0 =&\mathcal{O}_X,\\
I_1 =& \sum_{v_j \in \mathcal{O}^{\geq 1+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_0(D)} \mathcal{O}_X(f\partial_x a - \alpha a \partial_x f) + \mathcal{O}_X(f\partial_y a - \alpha a \partial_y f) + \mathcal{O}_X(f\partial_z a - \alpha a \partial_z f)\\
=&\mathcal{O}_X(x^2, xy^2, y^4) + \mathcal{O}_X(x^2, y^4, z)\\
=&\mathcal{O}_X(x^2, xy^2, y^4, z),\\
I_2 =& \sum_{v_j \in \mathcal{O}^{\geq 2+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_1(D)} \mathcal{O}_X(f\partial_x a - (\alpha+1) a \partial_x f)\\
&+ \mathcal{O}_X(f\partial_y a - (\alpha+1) a \partial_y f) + \mathcal{O}_X(f\partial_z a - (\alpha+1) a \partial_z f)\\
=&((-3\alpha - 1)x^4 + (2y^5 + 2z^2)x, (-5\alpha y^4 - 5y^4)x^2, (-2z - 2\alpha z)x^2,\\
&(-3\alpha y^2 - 2y^2)x^3 + y^7 + y^2z^2, 2yx^4 + (2yz^2 - 5\alpha y^6 - 3y^6)x, (-2y^2z - 2\alpha y^2 z)x,\\
&(-3\alpha y^4 - 3y^4)x^2, 4y^3x^3 + 4y^3z^2 - 5\alpha y^8 - y^8, -2y^4z - 2\alpha y^4 z,\\
&(-3z - 3\alpha z)x^2, -5y^4z - 5\alpha y^4 z, x^3 + y^5 - 2\alpha z^2 - z^2),\\
I_3 =& \sum_{v_j \in \mathcal{O}^{\geq 3+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_2(D)} \mathcal{O}_X(f\partial_x a - (\alpha+2) a \partial_x f)\\
&+ \mathcal{O}_X(f\partial_y a - (\alpha+2) a \partial_y f) + \mathcal{O}_X(f\partial_z a - (\alpha+2) a \partial_z f)\\
=&((9\alpha^2 + 9\alpha + 2)x^6 + (-18\alpha y^5 - 18\alpha z^2 - 14y^5 - 14z^2)x^3 + 2y^{10} + 4y^5z^2 + 2z^4,\\
&(15\alpha^2 y^4 + 35\alpha y^4 + 20y^4)x^4 + (-10y^4z^2 - 10\alpha y^9 - 10y^9 - 10\alpha y^4 z^2)x,\\
&(-20\alpha y^3 - 20y^3)x^5 + (25\alpha^2 y^8 + 55\alpha y^8 - 20\alpha y^3 z^2 + 30y^8 - 20y^3z^2)x^2,\\
&(6z\alpha^2 + 14z\alpha + 8z)x^4 + (-4\alpha z^3 - 4y^5 z - 4z^3 - 4\alpha y^5 z)x,\\
&(10z\alpha^2 y^4 + 30z\alpha y^4 + 20zy^4)x^2,\\
&(-2\alpha - 2)x^5 + (4\alpha^2 z^2 - 2\alpha y^5 + 10\alpha z^2 - 2y^5 + 6z^2)x^2,\\
&(9\alpha^2 y^2 + 15\alpha y^2 + 6y^2)x^5 + (-12y^2z^2 - 12\alpha y^7 - 12y^7 - 12\alpha y^2 z^2)x^2,\\
&(-4y - 6\alpha y)x^6 + (15\alpha^2 y^6 + 34\alpha y^6 - 6\alpha y z^2 + 23y^6 - 2yz^2)x^3\\
&+ 2yz^4 - 5\alpha y^{11} - y^6z^2 - 3y^{11} - 5\alpha y^6 z^2,\\
&2x^7 + (4z^2 - 36y^5 - 40\alpha y^5)x^4 + (25\alpha^2 y^{10} + 35\alpha y^{10} - 40\alpha y^5 z^2 + 12y^{10} - 36y^5z^2 + 2z^4)x,\\
&(6z\alpha^2 y^2 + 16z\alpha y^2 + 10zy^2)x^3 - 2y^2z^3 - 2y^7z - 2\alpha y^7 z - 2\alpha y^2 z^3,\\
&(-4yz - 4\alpha yz)x^4 + (10\alpha^2 y^6 z + 26\alpha y^6 z - 4\alpha y z^3 + 16y^6 z - 4yz^3)x,\\
&(-2\alpha y^2 - 2y^2)x^4 + (4\alpha^2 y^2 z^2 - 2\alpha y^7 + 10\alpha y^2 z^2 - 2y^7 + 6y^2z^2)x,\\
&(9\alpha^2 y^4 + 21\alpha y^4 + 12y^4)x^4 + (-6y^4z^2 - 6\alpha y^9 - 6y^9 - 6\alpha y^4 z^2)x,\\
&(-12\alpha y^3 - 12y^3)x^5 + (15\alpha^2 y^8 + 33\alpha y^8 - 12\alpha y^3 z^2 + 18y^8 - 12y^3z^2)x^2,\\
&12y^2x^6 + (24y^2z^2 - 60\alpha y^7 - 36y^7)x^3 + 25\alpha^2 y^{12} + 15\alpha y^{12}\\
&- 60\alpha y^7 z^2 + 2y^{12} - 36y^7z^2 + 12y^2z^4, (6z\alpha^2 y^4 + 18z\alpha y^4 + 12zy^4)x^2,\\
&(-8y^3z - 8\alpha y^3 z)x^3 + 10\alpha^2 y^8 z + 22\alpha y^8 z - 8\alpha y^3 z^3 + 12y^8 z - 8y^3z^3,\\
&(-2\alpha y^4 - 2y^4)x^3 + 4\alpha^2 y^4 z^2 - 2\alpha y^9 + 10\alpha y^4 z^2 - 2y^9 + 6y^4z^2,\\
&(9z\alpha^2 + 21z\alpha + 12z)x^4 + (-6\alpha z^3 - 6y^5 z - 6z^3 - 6\alpha y^5 z)x,
\end{aligned}
$$

$$(15z\alpha^2y^4 + 45z\alpha y^4 + 30zy^4)x^2,$$
$$(-20y^3z - 20\alpha y^3z)x^3 + 25\alpha^2y^8z + 55\alpha y^8z - 20\alpha y^3z^3 + 30y^8z - 20y^3z^3,$$
$$(-3\alpha - 3)x^5 + (6\alpha^2z^2 - 3\alpha y^5 + 15\alpha z^2 - 3y^5 + 9z^2)x^2,$$
$$(-5\alpha y^4 - 5y^4)x^3 + 10\alpha^2y^4z^2 - 5\alpha y^9 + 25\alpha y^4z^2 - 5y^9 + 15y^4z^2,$$
$$(-6z - 6\alpha z)x^3 + 4\alpha^2z^3 - 6\alpha y^5z + 6\alpha z^3 - 6y^5z + 2z^3).$$

Hence, we can obtain the results by definition.
(8) If $\alpha \in (\frac{23}{30}, \frac{29}{30}]$, we obtain

$$
\begin{aligned}
I_0 =& \mathcal{O}_X,\\
I_1 =& \sum_{v_j \in \mathcal{O}^{\geq 1+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_0(D)} \mathcal{O}_X(f\partial_x a - \alpha a \partial_x f) + \mathcal{O}_X(f\partial_y a - \alpha a \partial_y f) + \mathcal{O}_X(f\partial_z a - \alpha a \partial_z f)\\
=& \mathcal{O}_X(x^2, xy^3, y^4) + \mathcal{O}_X(x^2, y^4, z)\\
=& \mathcal{O}_X(x^2, xy^3, y^4, z),\\
I_2 =& \sum_{v_j \in \mathcal{O}^{\geq 2+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_1(D)} \mathcal{O}_X(f\partial_x a - (\alpha+1)a\partial_x f)\\
&+ \mathcal{O}_X(f\partial_y a - (\alpha+1)a\partial_y f) + \mathcal{O}_X(f\partial_z a - (\alpha+1)a\partial_z f)\\
=& ((-3\alpha - 1)x^4 + (2y^5 + 2z^2)x, (-5\alpha y^4 - 5y^4)x^2, (-2z - 2\alpha z)x^2,\\
& (-3\alpha y^3 - 2y^3)x^3 + y^8 + y^3z^2, 3y^2x^4 + (3y^2z^2 - 5\alpha y^7 - 2y^7)x, (-2y^3z - 2\alpha y^3z)x,\\
& (-3\alpha y^4 - 3y^4)x^2, 4y^3x^3 + 4y^3z^2 - 5\alpha y^8 - y^8, -2y^4z - 2\alpha y^4z,\\
& (-3z - 3\alpha z)x^2, -5y^4z - 5\alpha y^4z, x^3 + y^5 - 2\alpha z^2 - z^2),\\
I_3 =& \sum_{v_j \in \mathcal{O}^{\geq 3+\alpha}} \mathcal{O}_X v_j + \sum_{a \in I_2(D)} \mathcal{O}_X(f\partial_x a - (\alpha+2)a\partial_x f)\\
&+ \mathcal{O}_X(f\partial_y a - (\alpha+2)a\partial_y f) + \mathcal{O}_X(f\partial_z a - (\alpha+2)a\partial_z f)\\
=& ((9\alpha^2 + 9\alpha + 2)x^6 + (-18\alpha y^5 - 18\alpha z^2 - 14y^5 - 14z^2)x^3 + 2y^{10} + 4y^5z^2 + 2z^4,\\
& (6z\alpha^2 + 14z\alpha + 8z)x^4 + (-4\alpha z^3 - 4y^5z - 4z^3 - 4\alpha y^5z)x,\\
& (15\alpha^2y^4 + 35\alpha y^4 + 20y^4)x^4 + (-10y^4z^2 - 10\alpha y^9 - 10y^9 - 10\alpha y^4z^2)x,\\
& (-20\alpha y^3 - 20y^3)x^5 + (25\alpha^2y^8 + 55\alpha y^8 - 20\alpha y^3z^2 + 30y^8 - 20y^3z^2)x^2,\\
& (10z\alpha^2y^4 + 30z\alpha y^4 + 20zy^4)x^2,\\
& (-2\alpha - 2)x^5 + (4\alpha^2z^2 - 2\alpha y^5 + 10\alpha z^2 - 2y^5 + 6z^2)x^2,\\
& (9\alpha^2y^3 + 15\alpha y^3 + 6y^3)x^5 + (-12y^3z^2 - 12\alpha y^8 - 12y^8 - 12\alpha y^3z^2)x^2,\\
& (-9\alpha y^2 - 6y^2)x^6 + (15\alpha^2y^7 + 31\alpha y^7 - 9\alpha y^2z^2 + 22y^7 - 3y^2z^2)x^3\\
& + 3y^2z^4 + y^7z^2 - 5\alpha y^{12} - 2y^{12} - 5\alpha y^7z^2,\\
& 6yx^7 + (12yz^2 - 50\alpha y^6 - 38y^6)x^4 +\\
& (25\alpha^2y^{11} + 25\alpha y^{11} - 50\alpha y^6z^2 + 6y^{11} - 38y^6z^2 + 6yz^4)x,\\
& (6z\alpha^2y^3 + 16z\alpha y^3 + 10zy^3)x^3 - 2y^3z^3 - 2y^8z - 2\alpha y^8z - 2\alpha y^3z^3,\\
& (-6y^2z - 6\alpha y^2z)x^4 + (10\alpha^2y^7z + 24\alpha y^7z - 6\alpha y^2z^3 + 14y^7z - 6y^2z^3)x,\\
& (-2\alpha y^3 - 2y^3)x^4 + (4\alpha^2y^3z^2 - 2\alpha y^8 + 10\alpha y^3z^2 - 2y^8 + 6y^3z^2)x,
\end{aligned}
$$

$$(9\alpha^2y^4+21\alpha y^4+12y^4)x^4+(-6y^4z^2-6\alpha y^9-6y^9-6\alpha y^4z^2)x,$$
$$(-12\alpha y^3-12y^3)x^5+(15\alpha^2y^8+33\alpha y^8-12\alpha y^3z^2+18y^8-12y^3z^2)x^2,$$
$$12y^2x^6+(24y^2z^2-60\alpha y^7-36y^7)x^3+25\alpha^2y^{12}+15\alpha y^{12}$$
$$-60\alpha y^7z^2+2y^{12}-36y^7z^2+12y^2z^4,(6z\alpha^2y^4+18z\alpha y^4+12zy^4)x^2,$$
$$(-8y^3z-8\alpha y^3z)x^3+10\alpha^2y^8z+22\alpha y^8z-8\alpha y^3z^3+12y^8z-8y^3z^3,$$
$$(-2\alpha y^4-2y^4)x^3+4\alpha^2y^4z^2-2\alpha y^9+10\alpha y^4z^2-2y^9+6y^4z^2,$$
$$(9z\alpha^2+21z\alpha+12z)x^4+(-6\alpha z^3-6y^5z-6z^3-6\alpha y^5z)x,$$
$$(15z\alpha^2y^4+45z\alpha y^4+30zy^4)x^2,$$
$$(-20y^3z-20\alpha y^3z)x^3+25\alpha^2y^8z+55\alpha y^8z-20\alpha y^3z^3+30y^8z-20y^3z^3,$$
$$(-3\alpha-3)x^5+(6\alpha^2z^2-3\alpha y^5+15\alpha z^2-3y^5+9z^2)x^2,$$
$$(-5\alpha y^4-5y^4)x^3+10\alpha^2y^4z^2-5\alpha y^9+25\alpha y^4z^2-5y^9+15y^4z^2,$$
$$(-6z-6\alpha z)x^3+4\alpha^2z^3-6\alpha y^5z+6\alpha z^3-6y^5z+2z^3).$$

Hence, we can obtain the results by definition.
(9) If $\alpha\in(\frac{29}{30},1]$, we obtain

$$\begin{aligned}
I_0 =&\mathcal{O}_X,\\
I_1 =&\sum_{v_j\in\mathcal{O}^{\geq 1+\alpha}}\mathcal{O}_Xv_j+\sum_{a\in I_0(D)}\mathcal{O}_X(f\partial_xa-\alpha a\partial_xf)+\mathcal{O}_X(f\partial_ya-\alpha a\partial_yf)+\mathcal{O}_X(f\partial_za-\alpha a\partial_zf)\\
=&\mathcal{O}_X(x^2,xy^3,y^4)+\mathcal{O}_X(x^2,y^4,z)\\
=&\mathcal{O}_X(x^2,y^4,z),\\
I_2 =&\sum_{v_j\in\mathcal{O}^{\geq 2+\alpha}}\mathcal{O}_Xv_j+\sum_{a\in I_1(D)}\mathcal{O}_X(f\partial_xa-(\alpha+1)a\partial_xf)\\
&+\mathcal{O}_X(f\partial_ya-(\alpha+1)a\partial_yf)+\mathcal{O}_X(f\partial_za-(\alpha+1)a\partial_zf)\\
=&((-3\alpha-1)x^4+(2y^5+2z^2)x,(-5\alpha y^4-5y^4)x^2,(-2z-2\alpha z)x^2,\\
&(-3\alpha y^4-3y^4)x^2,4y^3x^3+4y^3z^2-5\alpha y^8-y^8,-2y^4z-2\alpha y^4z,\\
&(-3z-3\alpha z)x^2,-5y^4z-5\alpha y^4z,x^3+y^5-2\alpha z^2-z^2),\\
I_3 =&\sum_{v_j\in\mathcal{O}^{\geq 3+\alpha}}\mathcal{O}_Xv_j+\sum_{a\in I_2(D)}\mathcal{O}_X(f\partial_xa-(\alpha+2)a\partial_xf)\\
&+\mathcal{O}_X(f\partial_ya-(\alpha+2)a\partial_yf)+\mathcal{O}_X(f\partial_za-(\alpha+2)a\partial_zf)\\
=&((9\alpha^2+9\alpha+2)x^6+(-18\alpha y^5-18\alpha z^2-14y^5-14z^2)x^3+2y^{10}+4y^5z^2+2z^4,\\
&(15\alpha^2y^4+35\alpha y^4+20y^4)x^4+(-10y^4z^2-10\alpha y^9-10y^9-10\alpha y^4z^2)x,\\
&(6z\alpha^2+14z\alpha+8z)x^4+(-4\alpha z^3-4y^5z-4z^3-4\alpha y^5z)x,\\
&(-20\alpha y^3-20y^3)x^5+(25\alpha^2y^8+55\alpha y^8-20\alpha y^3z^2+30y^8-20y^3z^2)x^2,\\
&(10z\alpha^2y^4+30z\alpha y^4+20zy^4)x^2,\\
&(-2\alpha-2)x^5+(4\alpha^2z^2-2\alpha y^5+10\alpha z^2-2y^5+6z^2)x^2,\\
&(9\alpha^2y^4+21\alpha y^4+12y^4)x^4+(-6y^4z^2-6\alpha y^9-6y^9-6\alpha y^4z^2)x,\\
&(-12\alpha y^3-12y^3)x^5+(15\alpha^2y^8+33\alpha y^8-12\alpha y^3z^2+18y^8-12y^3z^2)x^2,
\end{aligned}$$

$(6z\alpha^2y^4 + 18z\alpha y^4 + 12zy^4)x^2,$
$12y^2x^6 + (24y^2z^2 - 60\alpha y^7 - 36y^7)x^3 + 25\alpha^2y^{12} + 15\alpha y^{12} - 60\alpha y^7z^2 + 2y^{12} - 36y^7z^2 + 12y^2z^4,$
$(-8y^3z - 8\alpha y^3z)x^3 + 10\alpha^2y^8z + 22\alpha y^8z - 8\alpha y^3z^3 + 12y^8z - 8y^3z^3,$
$(-2\alpha y^4 - 2y^4)x^3 + 4\alpha^2y^4z^2 - 2\alpha y^9 + 10\alpha y^4z^2 - 2y^9 + 6y^4z^2,$
$(9z\alpha^2 + 21z\alpha + 12z)x^4 + (-6\alpha z^3 - 6y^5z - 6z^3 - 6\alpha y^5z)x,$
$(15z\alpha^2y^4 + 45z\alpha y^4 + 30zy^4)x^2,$
$(-3\alpha - 3)x^5 + (6\alpha^2z^2 - 3\alpha y^5 + 15\alpha z^2 - 3y^5 + 9z^2)x^2,$
$(-20y^3z - 20\alpha y^3z)x^3 + 25\alpha^2y^8z + 55\alpha y^8z - 20\alpha y^3z^3 + 30y^8z - 20y^3z^3,$
$(-5\alpha y^4 - 5y^4)x^3 + 10\alpha^2y^4z^2 - 5\alpha y^9 + 25\alpha y^4z^2 - 5y^9 + 15y^4z^2,$
$(-6z - 6\alpha z)x^3 + 4\alpha^2z^3 - 6\alpha y^5z + 6\alpha z^3 - 6y^5z + 2z^3).$

Hence, we can obtain the results by definition. □

Here is a table with related to $\{m_0, m_1, m_2, m_3\}$.

Type	α	m_0	m_1	m_2	m_3
E_6	$(0, \frac{1}{12}]$	0	0	6	24
E_6	$(\frac{1}{12}, \frac{4}{12}]$	0	1	9	30
E_6	$(\frac{4}{12}, \frac{5}{12}]$	0	2	12	36
E_6	$(\frac{5}{12}, \frac{7}{12}]$	0	3	15	42
E_6	$(\frac{7}{12}, \frac{8}{12}]$	0	4	18	48
E_6	$(\frac{8}{12}, \frac{11}{12}]$	0	5	21	54
E_6	$(\frac{11}{12}, 1]$	0	6	24	60
E_7	$(0, \frac{1}{18}]$	0	0	7	28
E_7	$(\frac{1}{18}, \frac{5}{18}]$	0	1	10	34
E_7	$(\frac{5}{18}, \frac{7}{18}]$	0	2	13	40
E_7	$(\frac{7}{18}, \frac{9}{18}]$	0	3	16	46
E_7	$(\frac{9}{18}, \frac{11}{18}]$	0	4	19	52
E_7	$(\frac{11}{18}, \frac{13}{18}]$	0	5	22	58
E_7	$(\frac{13}{18}, \frac{17}{18}]$	0	6	25	64
E_7	$(\frac{17}{18}, 1]$	0	7	28	70
E_8	$(0, \frac{1}{30}]$	0	0	8	32
E_8	$(\frac{1}{30}, \frac{7}{30}]$	0	1	11	38
E_8	$(\frac{7}{30}, \frac{11}{30}]$	0	2	14	44
E_8	$(\frac{11}{30}, \frac{13}{30}]$	0	4	20	56
E_8	$(\frac{13}{30}, \frac{17}{30}]$	0	4	20	56
E_8	$(\frac{17}{30}, \frac{19}{30}]$	0	5	23	62
E_8	$(\frac{19}{30}, \frac{23}{30}]$	0	6	26	68
E_8	$(\frac{23}{30}, \frac{29}{30}]$	0	7	29	74
E_8	$(\frac{29}{30}, 1]$	0	8	32	80

4. Proof of the Main Theorem

Proof of the Main Theorem. In fact, we only give complete proof about A_k type v.s. D_k type and A_k type v.s. E_6 type, since other pairs are similar to these two.
(1) $f = x^{k+1} + y^2 + z^2$ be A_k type with $k \geq 1$ ($\alpha \in (0, \frac{1}{k+1}]$), $g = x^{k'-1} + xy^2 + z^2$ be $D_{k'}$ type

with $k' \geq 4$ ($\alpha \in (0, \frac{1}{2k'-2}]$). By Proposition 1.10, we obtain that

$$M_2(D_f^\alpha) = \text{span}\{1, x, \cdots, x^{k-1}\},$$
$$M_2(D_g^\alpha) = \text{span}\{1, x, \cdots, x^{k'-2}, y\}.$$

It is obvious that $M_2(D_f^\alpha) \ncong M_2(D_g^\alpha)$.
(2) $f = x^{k+1}+y^2+z^2$ be A_k type with $k \geq 1$ ($\alpha \in (\frac{i}{k+1}, \frac{i+1}{k+1}]$, $1 \leq i \leq k-1$), $g = x^{k'-1}+xy^2+z^2$ be $D_{k'}$ type with $k' \geq 4$ ($\alpha \in (\frac{2i'-3}{2k'-2}, \frac{2i'-1}{2k'-2}]$, $2 \leq i' \leq \frac{k'}{2}$ for k even or $2 \leq i' \leq \frac{k'-1}{2}$ for k odd). By Proposition 1.10, we obtain that

$$M_2(D_f^\alpha) = \text{span}\{1, x, \cdots, x^k, y, xy, \cdots, x^{i-1}y, z, xz, \cdots, x^{i-1}z, z^2, xz^2, \cdots, x^{i-2}z^2\},$$
$$M_2(D_g^\alpha) = \text{span}\{1, x, \cdots, x^{k'+i'-4}, y, xy, \cdots, x^{i'-1}y, y^2, z, xz, \cdots, x^{i'-2}z\}.$$

It is obvious that $M_2(D_f^\alpha) \ncong M_2(D_g^\alpha)$.
(3) $f = x^{k+1} + y^2 + z^2$ be A_k type with $k \geq 1$ ($\alpha \in (\frac{k}{k+1}, 1]$), $g = x^{k'-1} + xy^2 + z^2$ be $D_{k'}$ type with $k' \geq 4$ ($\alpha \in (\frac{2k'-3}{2k'-2}, 1]$. By Proposition 1.10, we obtain that

$$M_2(D_f^\alpha) = \text{span}\{1, x, \cdots, x^k, y, xy, \cdots, x^{k-1}y, z, xz, \cdots, x^{k-1}z, z^2, xz^2, \cdots, x^{k-2}z^2\},$$
$$M_2(D_g^\alpha) = \text{span}\{1, x, \cdots, x^{2k'-4}, y, xy, \cdots, x^{k'-1}y, y^2, xy^2, y^3, z, xz, \cdots, x^{k'-2}z, yz\}.$$

It is obvious that $M_2(D_f^\alpha) \ncong M_2(D_g^\alpha)$.
(4) $f = x^{k+1} + y^2 + z^2$ be A_k type with $k \geq 1$ ($\alpha \in (0, \frac{1}{k+1}]$), $g = x^3 + y^4 + z^2$ be E_6 type ($\alpha \in (0, \frac{1}{12}]$). By Proposition 1.10, we obtain that

$$M_2(D_f^\alpha) = \text{span}\{1, x, \cdots, x^{k-1}\},$$
$$M_2(D_g^\alpha) = \text{span}\{1, y, y^2, x, xy, xy^2\}.$$

It is obvious that $M_2(D_f^\alpha) \ncong M_2(D_g^\alpha)$.
(5) $f = x^{k+1} + y^2 + z^2$ be A_k type with $k \geq 1$ ($\alpha \in (\frac{i}{k+1}, \frac{i+1}{k+1}]$, $1 \leq i \leq k-1$), $g = x^3 + y^4 + z^2$ be E_6 type ($\alpha \in (\frac{1}{12}, \frac{4}{12}]$). By Proposition 1.10, we obtain that

$$M_2(D_f^\alpha) = \text{span}\{1, x, \cdots, x^k, y, xy, \cdots, x^{i-1}y, z, xz, \cdots, x^{i-1}z, z^2, xz^2, \cdots, x^{i-2}z^2\},$$
$$M_2(D_g^\alpha) = \text{span}\{1, x, x^2, y, xy, y^2, xy^2, y^3, z\}.$$

It is obvious that $M_2(D_f^\alpha) \ncong M_2(D_g^\alpha)$.
(6) $f = x^{k+1} + y^2 + z^2$ be A_k type with $k \geq 1$ ($\alpha \in (\frac{i}{k+1}, \frac{i+1}{k+1}]$, $1 \leq i \leq k-1$), $g = x^3 + y^4 + z^2$ be E_6 type ($\alpha \in (\frac{4}{12}, \frac{5}{12}]$). By Proposition 1.10, we obtain that

$$M_2(D_f^\alpha) = \text{span}\{1, x, \cdots, x^k, y, xy, \cdots, x^{i-1}y, z, xz, \cdots, x^{i-1}z, z^2, xz^2, \cdots, x^{i-2}z^2\},$$
$$M_2(D_g^\alpha) = \text{span}\{1, x, x^2, y, xy, y^2, xy^2, y^3, z\}.$$

It is obvious that $M_2(D_f^\alpha) \ncong M_2(D_g^\alpha)$.
(7) $f = x^{k+1} + y^2 + z^2$ be A_k type with $k \geq 1$ ($\alpha \in (\frac{i}{k+1}, \frac{i+1}{k+1}]$, $1 \leq i \leq k-1$), $g = x^3 + y^4 + z^2$ be E_6 type ($\alpha \in (\frac{5}{12}, \frac{7}{12}]$). By Proposition 1.10, we obtain that

$$M_2(D_f^\alpha) = \text{span}\{1, x, \cdots, x^k, y, xy, \cdots, x^{i-1}y, z, xz, \cdots, x^{i-1}z, z^2, xz^2, \cdots, x^{i-2}z^2\},$$
$$M_2(D_g^\alpha) = \text{span}\{1, x, x^2, x^3, y, xy, x^2y, y^2, xy^2, y^3, xy^3, y^4, z, xz, yz\}.$$

It is obvious that $M_2(D_f^\alpha) \ncong M_2(D_g^\alpha)$.
(8) $f = x^{k+1} + y^2 + z^2$ be A_k type with $k \geq 1$ ($\alpha \in (\frac{i}{k+1}, \frac{i+1}{k+1}]$, $1 \leq i \leq k-1$), $g = x^3 + y^4 + z^2$

be E_6 type ($\alpha \in (\frac{7}{12}, \frac{8}{12}]$). By Proposition 1.10, we obtain that

$$M_2(D_f^\alpha) = \text{span}\{1, x, \cdots, x^k, y, xy, \cdots, x^{i-1}y, z, xz, \cdots, x^{i-1}z, z^2, xz^2, \cdots, x^{i-2}z^2\},$$
$$M_2(D_g^\alpha) = \text{span}\{1, x, x^2, x^3, y, xy, x^2y, y^2, xy^2, x^2y^2, y^3, xy^3,$$
$$y^4, y^5, z, xz, yz, y^2z\}.$$

It is obvious that $M_2(D_f^\alpha) \ncong M_2(D_g^\alpha)$.

(9) $f = x^{k+1} + y^2 + z^2$ be A_k type with $k \geq 1$ ($\alpha \in (\frac{i}{k+1}, \frac{i+1}{k+1}]$, $1 \leq i \leq k-1$), $g = x^3 + y^4 + z^2$ be E_6 type ($\alpha \in (\frac{8}{12}, \frac{11}{12}]$). By Proposition 1.10, we obtain that

$$M_2(D_f^\alpha) = \text{span}\{1, x, \cdots, x^k, y, xy, \cdots, x^{i-1}y, z, xz, \cdots, x^{i-1}z, z^2, xz^2, \cdots, x^{i-2}z^2\},$$
$$M_2(D_g^\alpha) = \text{span}\{1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, y^3, xy^3, y^4, xy^4, y^5,$$
$$z, xz, yz, xyz, y^2z\}.$$

It is obvious that $M_2(D_f^\alpha) \ncong M_2(D_g^\alpha)$.

(10) $f = x^{k+1} + y^2 + z^2$ be A_k type with $k \geq 1$ ($\alpha \in (\frac{k}{k+1}, 1]$), $g = x^3 + y^4 + z^2$ be E_6 type ($\alpha \in (\frac{11}{12}, 1]$). By Proposition 1.10, we obtain that

$$M_2(D_f^\alpha) = \text{span}\{1, x, \cdots, x^k, y, xy, \cdots, x^{i-1}y, z, xz, \cdots, x^{i-1}z, z^2, xz^2, \cdots, x^{i-2}z^2\},$$
$$M_2(D_g^\alpha) = \text{span}\{1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, y^3, xy^3, y^4, xy^4, y^5, xy^5, y^6,$$
$$z, xz, yz, xyz, y^2z, xy^2z\}.$$

It is obvious that $M_2(D_f^\alpha) \ncong M_2(D_g^\alpha)$.

□

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