

# REGULARITY FOR THE HARVEY-LAWSON SOLUTIONS TO THE COMPLEX PLATEAU PROBLEM

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## 1. Introduction

It seems that one of the natural fundamental questions of complex geometry is the classical complex Plateau problem. Specifically, the problem asks which odd-dimensional, real submanifolds of  $\mathbf{C}^N$  are boundaries of complex submanifolds in  $\mathbf{C}^N$ .

Recall that a  $C^1$ -submanifold  $M$  of a complex manifold  $X$  is said to be maximally complex if

$$\text{codim}_{\mathbf{R}}(T_x M \cap J(T_x M)) = 1 \quad \text{for all } x \in M,$$

where  $J$  denotes the almost complex structure of  $X$ , and the codimension refers to  $M$ . It was a fundamental contribution to complex geometry by Harvey and Lawson [3] that if  $M$  is compact, oriented, and of dimension larger than 1, and if  $X$  is Stein, then maximal complexity implies that  $M$  forms the boundary of a holomorphic  $n$ -chain in  $X$ .

If  $M$  is a CR-manifold in the sense of Kohn [6], [2] (see Definition 2.1 below), then there is a natural filtration associated to the De Rham complex of  $M$  with complex coefficients [8], [9]. The  $E_1^{p,q}$  term of the spectral sequence associated to this filtration is called the Kohn-Rossi cohomology group  $H_{\text{KR}}^{p,q}(M)$  of  $M$  [7], [8], [9]. In [9], we gave smooth solutions to the classical complex Plateau problem in the following cases.

**Theorem 1.** *Let  $M$  be a compact, orientable, connected CR-manifold of real dimension  $2n - 1$ ,  $n \geq 3$ , in a Stein manifold  $X$  of complex dimension  $n + 1$ . Suppose that  $M$  is strongly pseudoconvex. Then  $M$  is a boundary of a complex submanifold  $V \subseteq X - M$  if and only if Kohn-Rossi's cohomology groups  $H_{\text{KR}}^{p,q}(M)$  are zero for  $1 \leq q \leq n - 2$ .*

However, for strongly pseudoconvex (see Definition 2.4 below) CR-manifolds of real dimension three in  $\mathbf{C}^3$ , the smoothness of Harvey-Lawson solutions to the classical complex plateau problem remains open.

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The purpose of this paper is to answer this question. In fact we shall introduce a numerical CR-invariant  $\tau(M)$  for any compact connected  $(2n - 1)$ -dimensional strongly pseudoconvex CR-manifold  $M$  in  $\mathbf{C}^{n+1}$ ,  $n \geq 2$ . The vanishing of  $\tau(M)$  will give a necessary and sufficient condition for the smoothness of Harvey-Lawson solutions to the classical complex Plateau problem for  $M$ . More precisely we have

**Definition.** Let  $M$  be a compact connected  $(2n - 1)$ -dimensional strongly pseudoconvex CR-manifold in  $\mathbf{C}^{n+1}$ ,  $n \geq 2$ . By a theorem of Harvey and Lawson (see [3]),  $M$  is the boundary of a complex variety  $V$  in the  $C^\infty$  sense.  $V$  is smooth except at finitely many isolated singular points  $\{p_1, \dots, p_k\}$ . Let  $\tau_i$  be the number of local moduli of  $V$  at  $p_i$  (see Definition 2.5 below). We define  $\tau(M)$  to be  $\tau_1 + \tau_2 + \dots + \tau_k$ .

**Theorem 2.** Let  $M$  be a compact connected  $(2n - 1)$ -dimensional strongly pseudoconvex CR-manifold in  $\mathbf{C}^{n+1}$ ,  $n \geq 2$ . The  $\tau(M)$  defined above is a CR-invariant in the sense that if  $M' \subseteq \mathbf{C}^{n+1}$  is another  $(2n - 1)$ -dimensional CR-manifold which is CR-diffeomorphic to  $M$ , then  $\tau(M) = \tau(M')$ . In fact for  $n \geq 3$ ,  $\tau(M) = \dim H_{\text{KR}}^{p,q}(M)$  for  $p + q = n - 1$ ,  $n$  and  $1 \leq q \leq n - 2$ . Moreover,  $M$  is a boundary of the complex submanifold  $V \subseteq \mathbf{C}^{n+1} - M$  if and only if  $\tau(M) = 0$ .

**Remark.** It will be of extreme interest to give an intrinsic interpretation to  $\tau(M)$  for  $n = 2$ . In fact, for  $n = 2$ , if  $M$  admits a transversal holomorphic  $S^1$ -action, Lawson and the present author [5] have proved that a necessary and sufficient condition for the regularity of the Harvey-Lawson solution to the complex Plateau problem is  $\pi_1 = 0$ .

## 2. Preliminaries and proof of Theorem 2

In this section, we shall recall some basic definitions which are needed in this paper.

**Definition 2.1.** Let  $M$  be a compact, connected, orientable real manifold of dimension  $2n - 1$ . A CR-structure on  $M$  is an  $(n - 1)$ -dimensional subbundle  $S$  of  $CTM$  such that the following are true:

- (1)  $S \cap \bar{S} = \{0\}$ .
- (2) If  $L, L'$  are local sections of  $S$ , then so is  $[L, L']$ .

**Definition 2.2.** Let  $L_1, \dots, L_{n-1}$  be the local basis for sections of  $S$  over an open subset  $U \subset M$  so that  $\bar{L}_1, \dots, \bar{L}_{n-1}$  form a local basis for sections of  $\bar{S}$ . Since  $S \oplus \bar{S}$  has complex codimension one in  $CTM$ , we may choose a local section  $N$  of  $CTM$  such that  $L_1, \dots, L_{n-1}, \bar{L}_1, \dots, \bar{L}_{n-1}, N$  span  $CTM$ . We may assume that  $N$  is purely imaginary. Then

the matrix  $(c_{ij})$ , defined by

$$[L_i, L_j] = \sum a_{ij}^k L_k + \sum b_{ij}^k \bar{L}_k + c_{ij} N,$$

is Hermitian and is called the Levi form.

The Levi form is noninvariant; however, its essential features are invariant.

**Proposition 2.3.** *The number of nonzero eigenvalues and the absolute value of the signature of  $(c_{ij})$  at each point are independent of the choice of  $L_1, \dots, L_{n-1}, N$ .*

**Definition 2.4.** Let  $M$  be a CR-manifold. Then  $M$  is strongly pseudoconvex if the Hermitian matrix  $(c_{ij})$  obtained in Definition 2.2 is always nonsingular and its eigenvalues are of the same sign.

**Definition 2.5.** Let  $f$  be a holomorphic function in  $\mathbf{C}^{n+1}$ . Suppose that  $V = \{z \in \mathbf{C}^{n+1} : f(z) = 0\}$  has an isolated singularity at the origin. Then the number of local moduli  $\tau$  of  $V$  at 0 is given by

$$\tau = \dim \mathbf{C}[[z_0, z_1, \dots, z_n]] / \left( f, \frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n} \right).$$

*Proof of Theorem 2.* By a theorem of Harvey and Lawson, there exist complex varieties  $V$  and  $V'$  in  $\mathbf{C}^{n+1}$  such that  $\partial V = M$  and  $\partial V' = M'$  in the  $C^\infty$  sense.  $V$  is smooth except at finitely many isolated singular points  $\{p_1, \dots, p_k\}$  while  $V'$  is smooth except at finitely many isolated singular points  $q_1, \dots, q_{k'}$ . Since  $M$  is strongly pseudoconvex, we can take a 1-convex exhaustion function  $\varphi$  on  $V$  such that  $\varphi \geq 0$  on  $V$  and  $\varphi(y) = 0$  if and only if  $y \in \{p_1, \dots, p_k\}$ . Put  $V_r = \{y \in V : \varphi(y) \leq r\}$ . Let  $\sigma = (\sigma_0, \sigma_1, \dots, \sigma_n)$  be the CR-diffeomorphism from  $M$  to  $M'$ . By the strong pseudoconvexity of  $M = \partial V$ , we see that the  $\sigma_i$ ,  $0 \leq i \leq n$ , extend to a holomorphic function defined in a neighborhood  $U$  of  $M$  in  $V$ . Since  $V - V_r \subseteq U$  for  $r$  large enough, the extension of  $\sigma_i$  is holomorphic on  $V - V_r$  for  $0 \leq i \leq n$ . On the other hand, by Andreotti and Grauert (Théorème 15 of [1]),  $H^0(V - \{p_1, \dots, p_k\}, \mathcal{O}) \rightarrow H^0(V - V_r, \mathcal{O})$  is an isomorphism. So the extension of  $\sigma_i$  is holomorphic on  $V - \{p_1, \dots, p_k\}$  for  $0 \leq i \leq n$ . Observe that  $V$  is a complex hypersurface. Therefore,  $p_1, \dots, p_k$  are hypersurface singularities, and in particular they are normal singularities. Hence  $\bar{\sigma}_i$ , the extension of  $\sigma_i$ , is actually holomorphic on  $V$  for all  $0 \leq i \leq n$ , and we have a holomorphic map  $\bar{\sigma} : V \rightarrow \mathbf{C}^{n+1}$  such that the restriction of  $\bar{\sigma}$  to  $M$  is  $\sigma$ . Clearly  $\bar{\sigma}(V)$  and  $V'$  are both complex varieties in  $\mathbf{C}^{n+1}$ , which have the same boundary  $M'$ . By the uniqueness of the solution to the complex

Plateau problem (see [3]), we see that  $\bar{\sigma}(V) = V'$ . Similarly let  $\sigma'$  be the inverse mapping of  $\sigma$ , which is a CR-diffeomorphism from  $M'$  to  $M$ . Since  $M'$  is also strongly pseudoconvex, the same argument as before shows that  $\sigma'$  extends to a holomorphic map  $\bar{\sigma}' : V' \rightarrow \mathbb{C}^{n+1}$  such that  $\bar{\sigma}'(V') = V$ .  $\bar{\sigma}' \circ \bar{\sigma} : V \rightarrow V$  is a holomorphic mapping which extends the identity map  $\text{Id} : \partial V \rightarrow \partial V$ . By the uniqueness of the extension, we conclude that  $\bar{\sigma}' \circ \bar{\sigma} : V \rightarrow V$  is an identity map. Similarly,  $\bar{\sigma} \circ \bar{\sigma}' : V' \rightarrow V'$  is an identity map. Hence,  $\bar{\sigma} : V \rightarrow V'$  is biholomorphic and  $k = k'$ . Without loss of generality, we may assume that  $\bar{\sigma}(p_i) = q_i$  and hence  $\tau_i = \tau'_i$  for all  $1 \leq i \leq k$ , where  $\tau_i$  and  $\tau'_i$  are local moduli of  $V$  and  $V'$  at  $p_i$  and  $q_i$ , respectively. It follows that

$$\tau(M) = \sum_{i=1}^k \tau_i = \sum_{i=1}^k \tau'_i = \tau(M').$$

For  $n \geq 3$ ,  $\tau(M) = \dim H_{\text{KR}}^{p,q}(M)$  for  $p+q = n-1$ ,  $n$  and  $1 \leq q \leq n-2$ . This was proved in our previous paper [9].

Finally, it is easy to see that  $\tau_i$  vanishes if and only if  $V$  is smooth at  $p_i$ . Hence  $\tau(M) = 0$  if and only if  $\tau_i = 0$  for all  $1 \leq i \leq k$  if and only if  $V$  is smooth. q.e.d.

The following well-known remark, which is included here for the convenience of the readers, implies that  $\tau_i = \tau'_i$  for all  $1 \leq i \leq k$  in the above proof of Theorem 2.

Let  $\mathcal{O}_{n+1}$  denote the ring of germs at the origin of holomorphic functions  $(\mathbb{C}^{n+1}, 0) \rightarrow \mathbb{C}$ . If  $(V, 0)$  is a germ at the origin of a hypersurface in  $\mathbb{C}^{n+1}$ , let  $I(V)$  be the ideal of functions in  $\mathcal{O}_{n+1}$  vanishing on  $V$ , and let  $f$  be a generator of  $I(V)$ . It is well known that  $V - \{0\}$  is nonsingular if and only if the  $\mathbb{C}$ -vector space

$$A(V) = \mathcal{O}_{n+1} / \left( f, \frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n} \right) \mathcal{O}_{n+1}$$

is finite dimensional. In this case,

$$A(V) \cong \mathbb{C}[[z_0, z_1, \dots, z_n]] / \left( f, \frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n} \right).$$

The germ at 0 of  $A(V)$  is the base space for the miniversal deformation of  $(V, 0)$ . So  $A(V)$ , provided with the obvious  $\mathbb{C}$ -algebra structure, is called the *moduli algebra* of  $V$ .

**Remark 2.6.** Suppose  $(V, 0)$  and  $(W, 0)$  are germs of hypersurfaces in  $\mathbb{C}^{n+1}$ , and  $V - \{0\}$  is nonsingular. If  $(V, 0)$  is biholomorphically equivalent to  $(W, 0)$ , then  $A(V)$  is isomorphic to  $A(W)$  as a  $\mathbb{C}$ -algebra.

*Proof.* Let  $f$  and  $g$  be generators of  $I(V)$  and  $I(W)$ , respectively. Let  $h: (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}^{n+1}, 0)$  be the germ at the origin of a biholomorphically mapping such that  $h(V) = W$ . Then there exists  $u \in \mathcal{O}_{n+1}$  such that  $f = u(g \circ h)$  and  $u(0) = 0$ . Write  $h = (h_0, h_1, \dots, h_n)$ , where  $h_i: (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ . Then

$$\frac{\partial f}{\partial z_i} = \frac{\partial u}{\partial z_i}(g \circ h) + u \sum_{j=0}^n \left( \frac{\partial g}{\partial z_j} \circ h \right) \frac{\partial h_j}{\partial z_i}.$$

Hence,  $\frac{\partial f}{\partial z_i}$  is in the ideal generated by  $g \circ h$ ,  $\frac{\partial g}{\partial z_0} \circ h$ ,  $\dots$ ,  $\frac{\partial g}{\partial z_n} \circ h$ . A similar argument shows that  $\frac{\partial g}{\partial z_i}$  is in the ideal generated by  $f \circ h^{-1}$ ,  $\frac{\partial f}{\partial z_0} \circ h^{-1}$ ,  $\dots$ ,  $\frac{\partial f}{\partial z_n} \circ h^{-1}$ . From this, it follows immediately that

$$h^* \left( g, \frac{\partial g}{\partial z_0}, \dots, \frac{\partial g}{\partial z_n} \right) \mathcal{O}_{n+1} = \left( f, \frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n} \right) \mathcal{O}_{n+1},$$

where  $h^*: \mathcal{O}_{n+1} \rightarrow \mathcal{O}_{n+1}$  is the  $\mathbb{C}$ -algebra isomorphism defined by  $h^*u = u \circ h$ . Hence  $h^*$  induces the  $\mathbb{C}$ -algebra isomorphism  $A(V) \cong A(W)$ .

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