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A sharp estimate of the number of integral points in a tetrahedron

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§ 1. Introduction

The general problem of counting the number Q of nonnegative integral points satisfying

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\leqq 1 \tag{1.1}$$

where a, b and c are positive integers, has been a challenge for many years. The difficulty lies from the fact that it is hard to estimate the number of nonnegative integral points satisfying the equality of (1.1). This problem has been discussed by Lehmer, Lochs and Ehrhart. There are extensive references in *Review in Number Theory* 1940–1972 *and* 1973–1983, especially in the sections H 40, Vol. 3 and P 28, Vol. 4. Perhaps the most interesting results in this direction are due to Ehrhart in his papers [Eh 1], [Eh 2] and [Ov]. In 1964, Ehrhart [Eh 1] claimed to have a formula for Q which is essentially the sum of the volume of the tetrahedron, the surface area of the boundary of the tetrahedron, a rational function in a, b and c and a correction term. However, the correction term in the formula is quite complicated and there is no estimate for it. So even if Ehrhart's formula is corrected, it will not be of much use from the point of view of getting the best upper bound of Q. Indeed, we do not understand Ehrhart's formula in [Eh 1] because one can take $a = 5$, $b = 4$ and $c = 3$ as counterexample. Ehrhart [Eh 2] in 1965 was able to get a clean estimate, but his estimate is far from being sharp. Overhagen [Ov] in 1975 was able to estimate the number of integral points lying inside an arbitrary convex body in 3-space. However, it is difficult to express Overhagen's estimate in terms of coordinates of the vertices of the tetrahedron. Indeed, we do not understand Overhagen's estimate also, as we can take the tetrahedron with vertices $(1,1,1)$, $(4,0,0)$, $(0,4,0)$ and $(0,0,4)$ as counterexample.

In this paper, we are interested in the problem of counting the number P of positive integral points satisfying (1.1), where a, b and c are positive real numbers. Of course one can

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deduce the estimate of Q once the estimate of P is known. The novelty in our problem is that we count the lattice points in a polytope whose vertices are not necessarily integer points (even rational points). Previous appearances of this sort of count, starting with Ramanujan and Hardy, and running through Spencer and Beukers, are entirely asymptotic. The purpose of this paper is to provide the best upper estimate for the number P.

Theorem. *Let $a \geqq b \geqq c \geqq 2$ be real numbers. Let P be the number of positive integral solutions of* (1.1) *i.e.* $P = \#\left\{(x, y, z) \in \mathbb{Z}_+^3 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leqq 1\right\}$ *where $\mathbb{Z}_+$ is the set of positive integers. Then*

$$6P \leqq (a-1)(b-1)(c-1) - c + 1 \tag{1.2}$$

and the equality is attained if and only if $a = b = c =$ integers.

Corollary. *Let $a \geqq b \geqq c \geqq 1$ be real numbers. Let Q be the number of nonnegative integral points satisfying $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leqq 1$. Then*

$$Q \leqq \frac{1}{6abc}\left(s^2 + (a+b)s\right) \text{ where } s = abc + ab + ac + bc$$

and the equality is attained if and only if $a = b = c =$ integers.

The original motivation of our work is to solve the Durfee conjecture. Let $f: (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be the germ of a complex analytic function with an isolated critical point at the origin. The Milnor number μ of the singularity is $\dim \mathbb{C}\{x, y, z\}/(f_x, f_y, f_z)$. Let $\pi: M \to V$ be a resolution of $V = \{(x, y, z): f(x, y, z) = 0\}$. The geometric genus p_g of the singularity $(V, 0)$ is the dimension of $H^1(M, \mathcal{O})$. In 1978, Durfee [Du] has made the following conjecture.

Durfee Conjecture. $6\,p_g \leqq \mu$ with equality only when $\mu = 0$.

The importance of the Durfee conjecture is that it gives a necessary condition for a singularity to be hypersurface. It also gives obstruction to embedding a strongly pseudo-convex 3-dimensional CR-manifold in $\mathbb{C}^3$ (cf. [Lu-Ya]).

If f is a weighted homogeneous polynomial with weights a, b and c (where a, b and c are rational numbers and we may assume without loss of generality that a, b, c are at least two), we consider the tetrahedron T with vertices $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$. In this case we have $\mu = (a-1)(b-1)(c-1)$ and $p_g = P$ where P is the number of positive integral points sitting in T. Thus the Durfee Conjecture can be written as

$$6P \leqq (a-1)(b-1)(c-1). \tag{1.3}$$

Actually many people like the estimate in (1.3) because of its symmetrical appearance in a, b and c. Unfortunately (or fortunately) it is a matter of fact that (1.2) is a sharper estimate than (1.3). Needless to say, for our purpose, (1.2) is more useful than (1.3). For instance, as a corollary of the Theorem, we can give a coordinate free characterization of homogeneous singularities. This result together with the proof of Durfee conjecture in the weighted homogeneous case are discussed in our paper [Xu-Ya].

The strategy of the proof of our Theorem is very simple. We first give an estimate of the number of positive integral solutions of $\frac{x}{r} + \frac{y}{s} \leqq 1$ where $r \geqq s > 0$ are real numbers (cf. Proposition 2.1). We then apply this result to estimate the number of positive integral solutions of (1.1) on planes parallel to the xy-planes and sum these estimates up. Our Theorem follows from a careful analysis on this sum. Due to the nature of the problem, we have to consider the case $\frac{a}{c} \geqq 2$ and the case $\frac{a}{c} < 2$ separetely. Each case will be split further into several subcases.

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§ 2. Sharp upper estimate of the number of integral points in a tetrahedron

Proposition 2.1. *Let N be the number of positive integral solutions of*

$$\frac{x}{r} + \frac{y}{s} \leqq 1 \tag{2.1}$$

where $r \geqq s > 0$ *are real numbers; i.e.* $N = \#\left\{(x, y) \in \mathbb{Z}_+^2 : \frac{x}{r} + \frac{y}{s} \leqq 1\right\}$. *Let* $s = [s] + \alpha$ *with* $0 \leqq \alpha < 1$, *where* $[s] \in \mathbb{Z}$. *If* $s < 1$, *then* $N = 0$. *If* $s > 1$, *then*

$$N \leqq \begin{cases} \dfrac{r(s-1)}{2} + \dfrac{r}{8s}, & (2.2) \\ \dfrac{r(s-1)}{2} + \dfrac{r-s}{2r} \quad \text{if } \alpha \geqq \dfrac{s}{r} \text{ and } \dfrac{s}{r} > \dfrac{1}{2}, & (2.3) \\ \dfrac{r(s-1)}{2} \quad \text{if } \alpha < \dfrac{s}{r}. & (2.4) \end{cases}$$

The equality of (2.2) *holds only if* $s = [s] + \frac{1}{2}$ *and* $\frac{r}{s} \geqq 2$, *while the equality of* (2.3) *holds only if* $s = [s] + \frac{s}{r}$ *with* $\frac{s}{r} < 1$.

Moreover if $r = s =$ *integer, then* $N = \dfrac{r(s-1)}{2}$.

Proof. We are going to sum the positive integral solutions of (2.1) horizontally. In view of (2.1), we have $x \leqq \dfrac{r(s-y)}{s}$. It follows that

(2.5) $$N \leqq \left(\sum_{k=1}^{[s-1]} \frac{r(s-k)}{s} \right) + \frac{r}{s}\alpha$$

$$= \frac{r}{s}\left\{ s[s-1] - \frac{([s-1]+1)[s-1]}{s} \right\} + \frac{r}{s}\alpha$$

$$= \frac{r}{s}[s-1]\left\{ s - \frac{[s]}{2} \right\} + \frac{r}{s}\alpha$$

$$= \frac{r}{s}(s-1-\alpha)\left(s - \frac{s-\alpha}{2} \right) + \frac{r}{s}\alpha$$

(2.6) $$= \frac{r(s-1)}{2} + \frac{r}{2s}(\alpha - \alpha^2)$$

$$\leqq \frac{r(s-1)}{2} + \frac{r}{8s}.$$

This yields (2.2).

The last inequality follows from the fact that $\alpha - \alpha^2 \leqq \frac{1}{4}$ while the equality holds only if $\alpha = \frac{1}{2}$. Notice that the term $\frac{r}{s}\alpha$ in (2.5) gives an upper estimate of number of integral points satisfying (2.1) on the horizontal line $y = [s]$. Therefore the term $\frac{r}{s}\alpha$ should stay in (2.5) only if $\frac{r}{s}\alpha \geqq 1$. Observe that the equality of (2.2) holds only if (i) $\frac{r}{s}\alpha$ stays in (2.5) and (ii) $\alpha = \frac{1}{2}$. The conditions (i) and (ii) are equivalent to $\frac{r}{s} \geqq 2$ and $\alpha = \frac{1}{2}$. Now we consider (2.4). If $\alpha < \frac{s}{r}$, the term $\frac{r}{s}\alpha$ should be omitted from (2.5) as indicated above. Hence we have the estimate:

$$N \leqq \sum_{k=1}^{[s-1]} \frac{r(s-k)}{s}$$

$$= \frac{r(s-1)}{2} + \frac{r}{2s}(\alpha - \alpha^2) - \frac{r}{s}\alpha$$

$$= \frac{r(s-1)}{2} - \frac{r}{2s}(\alpha + \alpha^2)$$

$$\leqq \frac{r(s-1)}{2}.$$

This yields (2.4).

For (2.3), we have $\alpha \geqq \frac{s}{r}$ i.e. $\frac{r}{s}\alpha \geqq 1$. Hence the term $\frac{r}{s}\alpha$ should stay in (2.5) and we should use (2.6) as the estimate of N. Notice that the function $\alpha - \alpha^2$ is a strictly decreasing function for $\alpha \geqq \frac{1}{2}$. If $\frac{s}{r} > \frac{1}{2}$, we have $\alpha \geqq \frac{s}{r} > \frac{1}{2}$ and, hence, $\alpha - \alpha^2 \leqq \left(\frac{s}{r}\right) - \left(\frac{s}{r}\right)^2$. Then from (2.6) we have

$$\begin{aligned} N &\leqq \frac{r(s-1)}{2} + \frac{r}{2s}\left(\frac{s}{r} - \left(\frac{s}{r}\right)^2\right) \\ &= \frac{r(s-1)}{2} + \frac{r-s}{2r}. \end{aligned}$$

This is (2.3). It is obvious that the equality holds only if $\alpha = \frac{s}{r}$ with $\frac{s}{r} < 1$.

The last statement of this proposition is well-known.

Corollary 2.2. *With the notation as in Proposition* 2.1

$$N \leqq \begin{cases} \dfrac{r(s-1)}{2} + \dfrac{r}{8s}, & (2.7) \\[2ex] \dfrac{r(s-1)}{2} + \dfrac{r-s}{2r} \quad \text{if } \dfrac{s}{r} > \dfrac{1}{2}. & (2.8) \end{cases}$$

The equality of (2.7) *holds only if* $s = [s] + \frac{1}{2}$ *and* $\frac{r}{s} \geqq 2$. *The equality of* (2.8) *holds only if* $s = [s] + \frac{s}{r}$ *and* $\frac{s}{r} < 1$.

Proof of the Theorem. We first remark that if $a < 3$, then $b < 3$, $c < 3$, and $P = 0$ and (1.2) holds trivially since $a \geqq 2$, $b \geqq 2$ and $c \geqq 2$ by assumption. It is clear that equality holds in (1.2) if and only if $a = b = c = 2$. From now on, we shall assume that $a \geqq 3$. There are two cases to be considered: (I) $\frac{a}{c} \geqq 2$ and (II) $\frac{a}{c} < 2$.

Case (I). $\frac{a}{c} \geqq 2$. Let N_k be the number of positive integral solutions of

$$\frac{x}{\frac{a}{c}(k+\beta)} + \frac{y}{\frac{b}{c}(k+\beta)} \leqq 1 \tag{2.9}$$

where $\beta = c - [c]$. By (2.7) of Corollary 2.2, we see that

$$N_k \leqq \frac{1}{2}\frac{a}{c}(k+\beta)\left[\frac{b}{c}(k+\beta)-1\right]+\frac{a}{8b}.$$

Notice that N_k is the number of positive integral solutions of (1.1) on plane $z=[c]-k$, we are going to sum these estimates for N_k's. There are two subcases: (i) $0 \leqq \beta < \frac{c}{b}$; (ii) $\frac{c}{b} \leqq \beta < 1$.

(i) $0 \leqq \beta < \frac{c}{b}$. By Proposition 2.1, we have

$$\begin{aligned}
P &\leqq \sum_{k=1}^{[c]-1}\left[\frac{1}{2}\frac{a}{c}(k+\beta)\left(\frac{b}{c}(k+\beta)-1\right)+\frac{a}{8b}\right]\\
&= \sum_{k=1}^{[c]-1}\left[\frac{1}{2}\frac{ab}{c^2}(k^2+2k\beta+\beta^2)-\frac{a(k+\beta)}{2c}+\frac{a}{8b}\right]\\
&= \frac{ab}{2c^2}\frac{([c]-1)([c])(2[c]-1)}{6}+\frac{ab\beta}{c^2}\left(\frac{[c]-1}{2}\right)[c]+\frac{ab\beta^2}{2c^2}([c])-1)\\
&\quad -\frac{a}{2c}\frac{([c]-1)[c]}{2}-\frac{a\beta}{2c}([c]-1)+\frac{a}{8b}([c]-1)\\
&= \frac{ab}{12c^2}(2[c]^3-3[c]^2+[c])+\frac{ab\beta[c]}{c^2}+\frac{a}{8b}\frac{([c]-1)}{2}\\
&= \frac{ab}{12c^2}(2[c]^3-3[c]^2+[c])+\frac{ab\beta[c]}{c^2}\frac{([c]-1}{2}+\frac{ab\beta^2}{2c^2}([c]-1)\\
&\quad -\frac{a}{4c}([c]^2-[c])-\frac{a\beta}{2c}([c]-1)+\frac{a}{8b}([c]-1)\\
&= \frac{ab}{6c^2}(c-\beta)^3-\frac{ab}{4c^2}(c-\beta)^2+\frac{ab}{12c^2}(c-\beta)+\frac{ab\beta}{c^2}(c-1-\beta)\frac{(c-\beta)}{2}\\
&\quad +\frac{ab\beta^2}{2c^2}(c-1-\beta)-\frac{a}{4c}(c-\beta)^2+\frac{a}{4c}(c-\beta)-\frac{a\beta}{2c}(c-1-\beta)\\
&\quad +\frac{a}{8b}(c-1-\beta)\\
&= \frac{1}{12}(c-1)\left(2ab-\frac{ab}{c}-3a+\frac{3}{2}\frac{a}{b}\right)+\left(-\frac{ab}{12c^2}+\frac{a}{4c}-\frac{a}{8b}\right)\beta\\
&\quad +\left(-\frac{ab}{4c^2}+\frac{a}{4c}\right)\beta^2-\frac{ab}{6c^2}\beta^3
\end{aligned}$$

(2.10) $$= \frac{1}{12}(c-1)\left(2ab - \frac{ab}{c} - 3a + \frac{a}{b}\right) + \frac{1}{12}R_1$$

where $R_1 = \frac{-2ab}{c^2}\beta^3 - \left(\frac{3ab}{c^2} - \frac{3a}{c}\right)\beta^2 + \left(\frac{3a}{c} - \frac{ab}{c^2} - \frac{3a}{2b}\right)\beta$.

We want to show

(2.11) $$\frac{1}{12}(c-1)\left(2ab - \frac{ab}{c} - 3a + \frac{3a}{2b}\right) + \frac{1}{12}R_1 < \frac{1}{6}(c-1)(ab-a-b).$$

Since $c \geqq 2$, and $\frac{ab}{c} + a - \frac{3a}{2b} > 0$ it is sufficient to show

(2.12) $$\frac{ab}{c} + a - \frac{3a}{2b} - R_1 > 2b.$$

Notice that $\frac{a}{c} \geqq 2$. Hence we only need to show

(2.13) $$a - \frac{3a}{2b} - R_1 > 0.$$

As $b \geqq c$ and $0 \leqq \beta < \frac{c}{b}$, we have the upper estimate of R_1

$$\begin{aligned} R_1 &= -\frac{2ab}{c^2}\beta^3 - \frac{3a(b-c)}{c^2}\beta^2 + \left(\frac{3a}{c} - \frac{ab}{c^2} - \frac{3a}{2b}\right)\beta \\ &\leqq \left(\frac{3a}{c} - \frac{ab}{c^2} - \frac{3a}{2b}\right)\beta. \end{aligned}$$

If $\frac{3a}{c} - \frac{ab}{c^2} - \frac{3a}{2b} \leqq 0$, then $R_1 \leqq 0$. In this case we only need to prove $a - \frac{3a}{2b} > 0$ in order to prove (2.13). But this follows from $b \geqq 2$. On the other hand if $\frac{3a}{c} - \frac{ab}{c^2} - \frac{3a}{2b} > 0$, then

$$\begin{aligned} R_1 &< \left(\frac{3a}{c} - \frac{ab}{c^2} - \frac{3a}{2b}\right)\frac{c}{b} \\ &= \frac{3a}{b} - \frac{a}{c} - \frac{3ac}{2b^2} \\ &\leqq \frac{a}{b}\left(2 - \frac{3c}{2b}\right). \end{aligned}$$

In order to prove (2.13), we need to show

$$a-\frac{3a}{2b}\geqq\frac{a}{b}\left(2-\frac{3c}{2b}\right),$$

which is equivalent to show

$$2b^2-7b+3c\geqq 0\,. \tag{2.14}$$

By completing the square, one sees that when $c>\frac{49}{24}$, (2.14) holds for arbitrary b. When $2\leqq c\leqq\frac{49}{24}$, we need

$$b\geqq\frac{7\pm\sqrt{49-24c}}{4} \tag{2.15}$$

in order (2.14) holds. But the right side of (2.15) is always less than or equal to 2 because by assumption $b\geqq c\geqq 2$. The proof of the strict inequality (1.2) is completed in this case.

(ii) $\frac{c}{b}\leqq\beta<1$. By Proposition 2.1, we have

$$\begin{aligned}P&\leqq\sum_{k=0}^{[c]-1}\left[\frac{1}{2}\frac{a}{c}(k+\beta)\left(\frac{b}{c}(k+\beta)-1\right)+\frac{a}{8b}\right]\\&=\frac{1}{12}(c-1)\left(2ab-\frac{ab}{c}-3a+\frac{3}{2}\frac{a}{b}\right)+\frac{1}{12}R_1+\frac{a\beta}{2c}\left(\frac{b\beta}{c}-1\right)+\frac{a}{8b}\\&=\frac{1}{12}(c-1)\left(2ab-\frac{ab}{c}-3a+\frac{3a}{2b}\right)+\frac{1}{12}R_2\end{aligned}$$

where

$$R_2=\frac{-2ab}{c^2}\beta^3+\left(\frac{3ab}{c^2}+\frac{3a}{c}\right)\beta^2+\left(-\frac{3a}{c}-\frac{ab}{c^2}-\frac{3a}{2b}\right)\beta+\frac{3a}{2b}.$$

We need to estimate R_2 from above when $\frac{c}{b}\leqq\beta<1$

$$\frac{dR_2}{d\beta}=-\frac{6ab}{c^2}\beta^2+\frac{6a}{c}\left(\frac{b+c}{c}\right)\beta-\left(\frac{3a}{c}+\frac{ab}{c^2}+\frac{3a}{2b}\right).$$

Letting $\frac{dR_2}{d\beta} = 0$, we have two critical points

$$\beta_1 = \frac{3(b+c)+\sqrt{3}b}{6b} \quad \text{and} \quad \beta_2 = \frac{3(b+c)-\sqrt{3}b}{6b}.$$

Since $\beta_1 > \beta_2$, we have $\sup_{\frac{c}{b} \leq \beta < 1} R_2(\beta) \leqq \max\{R_2(1), R_2\left(\frac{c}{b}\right), R_2(\beta_1)\}$ where $R_2(1) = 0$, $R_2\left(\frac{c}{b}\right) = \frac{-ac}{2b^2} - \frac{a}{b} + \frac{3a}{2b}$ and $R_2(\beta_1) = \frac{\sqrt{3}ab}{18c^2} - \frac{ac}{4b^2} + \frac{3a}{4b} - \frac{a}{2c}$. We want to show that

$$\frac{1}{12}(c-1)\left(2ab - \frac{ab}{c} - 3a + \frac{3a}{2b}\right) + \frac{R_2}{12} < \frac{1}{6}(c-1)(ab-a-b) \tag{2.16}$$

which is equivalent to

$$R_2 < (c-1)\left(a - 2b + \frac{ab}{c} - \frac{3a}{2b}\right). \tag{2.17}$$

As $c \geqq 2$, we only need to show

$$R_2 < a - 2b + \frac{ab}{c} - \frac{3a}{2b}. \tag{2.18}$$

There are three cases to be considered.

(α) $\max\left\{R_2(1), R_2\left(\frac{c}{b}\right), R_2(\beta_1)\right\} = R_2(1) = 0$. In order to prove (2.18), it suffices to prove $a - 2b + \frac{ab}{c} - \frac{3a}{2b} > 0$. However the latter inequality is obvious because $a - 2b + \frac{ab}{c} - \frac{3a}{2b} = \frac{a}{2b}(2b-3) + \mathrm{b}\left(\frac{a}{c} - 1\right)$, $b \geqq 2$ and $\frac{a}{c} \geqq 2$.

(β) $\max\left\{R_2(1), R_2\left(\frac{c}{b}\right), R_2(\beta_1)\right\} = R_2\left(\frac{c}{b}\right)$. In order to prove (2.18), we only need to prove $a - \frac{3a}{2b} > R_2$ because $\frac{a}{c} \geqq 2$. Observe that $R_2\left(\frac{c}{b}\right) = -\frac{ac}{2b^2} - \frac{a}{c} + \frac{3a}{2b} \leqq \frac{a}{2b}\left(1 - \frac{c}{b}\right)$. So it suffices to prove $a - \frac{3a}{2b} > \frac{a}{2b}\left(1 - \frac{c}{b}\right)$ which is equivalent to $1 - \frac{3}{2b} - \frac{1}{2b} + \frac{c}{2b^2} > 0$. The latter inequality is obvious true since $b \geqq 2$.

(γ) $\max\left\{R_2(1), R_2\left(\frac{c}{b}\right), R_2(\beta_1)\right\} = R_2(\beta_1)$. In this case $\beta_1 < 1$ i.e. $\frac{3+\sqrt{3}}{6} + \frac{1}{2}\frac{c}{b} < 1$, which implies $\frac{b}{c} > 2$. In order to prove (2.18), we need to prove

$$\frac{ab}{c} + a - 2b - \frac{3a}{2b} - \frac{\sqrt{3}ab}{18c^2} + \frac{ac}{4b^2} - \frac{3a}{4b} + \frac{a}{2c} > 0$$

i.e.

$$\frac{b}{c} + 1 - \frac{2b}{a} - \frac{3}{2b} - \frac{\sqrt{3}b}{18c^2} + \frac{c}{4b^2} - \frac{3}{4b} + \frac{1}{2c} > 0. \tag{2.19}$$

The left side of (2.19) is $\frac{b}{c}\left(1 - \frac{\sqrt{3}}{18c}\right) + \left(1 - \frac{2b}{a}\right) - \frac{3}{2b} + \frac{c}{4b^2} + \left(\frac{1}{2c} - \frac{3}{4b}\right)$. Since $\frac{a}{b} \geqq 1$ and $\frac{b}{c} > 2$, we have $1 - \frac{2b}{a} \geqq -1$ and $\frac{1}{2c} - \frac{3}{4b} > 0$. Hence the left side of (2.19) is bigger than $\frac{b}{c}\left(1 - \frac{\sqrt{3}}{18c}\right) - 1 - \frac{3}{2b}$. Observe that $\frac{b}{c} > 2$, $c \geqq 2$ imply $b > 4$. Thus $\frac{b}{c}\left(1 - \frac{\sqrt{3}}{18c}\right) - 1 - \frac{3}{2b} > 2\left(1 - \frac{\sqrt{3}}{36}\right) - 1 - \frac{3}{8} > \frac{17}{18} \times 2 - \frac{11}{8} = \frac{17}{9} - \frac{11}{8} > 0$. So (2.18) is proved and we have proved the strict inequality (1.2) in this case.

Case (II). $\frac{a}{c} > 2$. In this case $\frac{a}{b} < 2$ and $\frac{b}{c} < 2$. As in case (I), let N_k be the number of positive integral solutions of (2.9) where $\beta = c - [c]$. By (2.8) of Corollary 2.2, we see that

$$N_k \leqq \frac{1}{2}\frac{a}{c}(k+\beta)\left[\frac{b}{c}(k+\beta) - 1\right] + \frac{a-b}{2a}. \tag{2.20}$$

There are two subcases: (i) $0 \leqq \beta < \frac{c}{b}$ and (ii) $\frac{c}{b} \leqq \beta < 1$.

Case (i). $0 \leqq \beta < \frac{c}{b}$. By Proposition 2.1 and (2.20), we have

$$\begin{aligned} P &\leqq \sum_{k=1}^{[c]-1} \left[\frac{1}{2}\frac{a}{c}(k+\beta)\left(\frac{b}{c}(k+\beta) - 1\right) + \frac{a-b}{2a}\right] \\ &= \frac{1}{12}(c-1)\left(2ab - \frac{ab}{c} - 3a + \frac{3a}{2b}\right) + \frac{1}{12}R_1 - \frac{a}{8b}(c-1-\beta) \end{aligned} \tag{2.21}$$

$$
\begin{aligned}
&+\frac{a-b}{2a}(c-1-\beta)\\
&=\frac{1}{12}(c-1)\left(2ab-\frac{ab}{c}-3a+\frac{3a}{2b}\right)\\
&+\frac{1}{12}\left[\frac{-2ab}{c^2}\beta^3-\frac{3a(b-c)}{c^2}\beta^2+\left(\frac{3a}{c}-\frac{ab}{c^2}-\frac{3a}{2b}\right)\beta\right]\\
&-\frac{a}{8b}(c-1)+\frac{a}{8b}\beta+\frac{a-b}{2a}(c-1)-\frac{a-b}{2a}\beta\\
&=\frac{1}{12}(c-1)\left[2ab-\frac{ab}{c}-3a+\frac{6(a-b)}{a}\right]+\frac{1}{12}R_3
\end{aligned}
$$

where

$$
\begin{aligned}
(2.22)\qquad R_3&=\left[-\frac{2ab}{c^2}\beta^3-\frac{3a(b-c)}{c^2}\beta^2+\left(\frac{3a}{c}-\frac{ab}{c^2}-\frac{6(a-b)}{a}\right)\beta\right]\\
&\leqq-\frac{2ab}{c^2}\beta^3+\left(\frac{3a}{c}-\frac{ab}{c^2}-\frac{6(a-b)}{a}\right)\beta\,.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\frac{3a}{c}-\frac{ab}{c^2}-\frac{6(a-b)}{a}&=3\left(\frac{a}{c}\right)-\left(\frac{b}{a}\right)\left(\frac{a}{c}\right)^2-\frac{6(a-b)}{a}\\
&=-\left(\frac{b}{a}\right)\left[\frac{a}{c}-\frac{3a}{2b}\right]^2+\frac{9}{4}\left(\frac{a}{b}\right)^2\left(\frac{b}{a}\right)-\frac{6(a-b)}{a}\\
&\leqq\frac{9}{4}\frac{a}{b}-\frac{6(a-b)}{a}\\
&=\frac{9}{4}\left(\frac{a}{b}\right)-6+6\left(\frac{b}{a}\right)\\
&\leqq\frac{9}{4}\quad\text{since}\quad 1\leqq\frac{a}{b}<2\,.
\end{aligned}
$$

Thus

$$
\begin{aligned}
(2.23)\qquad R_3&\leqq-2\beta^3+\frac{9}{4}\beta\\
&\leqq-2\left(\frac{\sqrt{3}}{2\sqrt{2}}\right)^3+\frac{9}{4}\frac{\sqrt{3}}{2\sqrt{2}}<1\,.
\end{aligned}
$$

In order to prove (1.2), it suffices to show

$$\frac{1}{12}(c-1)\left[2ab-\frac{ab}{c}-3a+\frac{6(a-b)}{a}\right]+\frac{1}{12}R_3 \leqq \frac{1}{6}(c-1)(ab-a-b)$$

which is equivalent to

$$\text{(2.24)} \qquad \frac{1}{12}R_3 \leqq \frac{1}{12}(c-1)\left[a-2b+\frac{ab}{c}-\frac{6(a-b)}{a}\right].$$

Suppose that $\frac{b}{c} \geqq \frac{4}{3}$. Then we have

$$\begin{aligned} a-2b+\frac{ab}{c}-\frac{6(a-b)}{a} &\geqq a-2b+\frac{4}{3}a-\frac{6(a-b)}{a} \\ &= 2(a-b)-\frac{6(a-b)}{a}+\frac{a}{3} \\ &= \frac{2(a-b)(a-3)}{a}+\frac{a}{3} \\ &\geqq 1 \quad \text{since} \quad a \geqq 3 . \end{aligned}$$

In view of (2.23), (2.24) follows immediately. Thus we have $P < \frac{1}{6}(c-1)(ab-a-b)$ as claimed.

Hence we shall assume for the rest of case (i) that $\frac{b}{c} < \frac{4}{3}$. We need to consider two further subcases.

Subcase 1. $\beta < \frac{c}{a}+\frac{c}{b}-1$. Then

$$\frac{1}{\frac{a}{c}(1+\beta)}+\frac{1}{\frac{b}{c}(1+\beta)}=\frac{1}{(1+\beta)}\left(\frac{c}{a}+\frac{c}{b}\right)>1 .$$

Hence on the $z=c-1-\beta=[c]-1$ level, there is no positive integral solutin of (2.9), i.e. $N_{[c]-1}=0$. The estimate of P in (2.21) can be replaced by the following estimate:

$$\begin{aligned} P \leqq \frac{1}{12}(c-1)\left[2ab-\frac{ab}{c}-3a+\frac{6(a-b)}{a}\right]+\frac{1}{12}R_3 \\ -\frac{1}{2}\frac{a}{c}(1+\beta)\left[\frac{b}{c}(1+\beta)-1\right]-\frac{a-b}{2a}. \end{aligned}$$

Observe that

$$\begin{aligned}
&\frac{1}{12}R_3 - \frac{1}{2}\frac{a}{c}(1+\beta)\left[\frac{b}{c}(1+\beta)-1\right] - \frac{a-b}{2a} \\
&= \frac{1}{12}\left[-\frac{2ab}{c^2}\beta^3 - \frac{3a(b-c)}{c^2}\beta^2 + \left(\frac{3a}{c} - \frac{ab}{c^2} - \frac{6(a-b)}{a}\right)\beta\right] \\
&\quad - \frac{1}{2}\left[\frac{ab}{c^2}\beta^2 + \frac{2ab}{c^2}\beta + \frac{ab}{c^2} - \frac{a}{c}\beta - \frac{a}{c} + \frac{a-b}{a}\right] \\
&= \frac{1}{12}\left[-\frac{2ab}{c^2}\beta^3 - \frac{3a(3b-c)}{c^2}\beta^2 - \left(\frac{4ab}{c^2} + \frac{9a(b-c)}{c^2} + \frac{6(a-b)}{a}\right)\beta\right] \\
&\quad - \frac{a(b-c)}{2c^2} - \frac{a-b}{2a} \\
&\leqq 0\,.
\end{aligned}$$

Notice that the above equality holds if and only if $a = b = c =$ integers. Thus

$$P \leqq \frac{1}{12}(c-1)\left[2ab - \frac{ab}{c} - 3a + \frac{6(a-b)}{a}\right] \tag{2.25}$$

and the equality holds if and only if $a = b = c =$ integer because of the last statement of Proposition 2.1. We claim that

$$2ab - \frac{ab}{c} - 3a + \frac{6(a-b)}{a} \leqq 2ab - 2a - 2b\,. \tag{2.26}$$

This can be seen by showing

$$\frac{ab}{c} + a - 2b - 6 + \frac{6b}{a} = b\left(2 - \frac{a}{c}\right) + (a-3) + 6\frac{b}{a} - 3 \geqq 0\,.$$

The latter inequality follows from the fact that $a \geqq 3$, $\frac{a}{b} < 2$ and $\frac{a}{c} < 2$. Putting (2.25) and (2.26) together, we get

$$P \leqq \frac{1}{6}(c-1)(ab-a-b)$$

as desired.

Subcase 2. $\frac{c}{b} > \beta \geqq \frac{c}{a} + \frac{c}{b} - 1$. Recall that by (2.21), we have

$$\begin{aligned}
P &\leqq \frac{1}{12}(c-1)\left[2ab - \frac{ab}{c} - 3a + \frac{6(a-b)}{a}\right] + \frac{1}{12}R_3 \\
&= \frac{1}{12}(c-1)\left[2ab - 4a + \frac{6(a-b)}{a}\right] - \frac{1}{12}(c-1)\left(\frac{b}{c}-1\right)a + \frac{1}{12}R_3
\end{aligned} \tag{2.27}$$

$$\leqq \frac{1}{12}(c-1)\left[2ab-4a+\frac{6(a-b)}{a}\right]-\frac{1}{12}\left[\left(\frac{b}{c}-1\right)a-R_3\right]$$

$$\leqq \frac{1}{12}(c-1)[2ab-2a-2b]-\frac{1}{12}\left[\left(\frac{b}{c}-1\right)a-R_3\right]$$

because $a \geqq 3$ and $a \geqq b$. Thus if we can prove

$$\left(\frac{b}{c}-1\right)a-R_3>0 \tag{2.28}$$

then it follows from (2.27) that

$$P<\frac{1}{6}(c-1)(ab-a-b)$$

as desired.

Now it remains to prove (2.28). Consider

$$\frac{dR_3}{d\beta}=-\frac{6ab}{c^2}\beta^2-\frac{6a(b-c)}{c^2}\beta+\frac{3a}{c}-\frac{ab}{c^2}-\frac{6(a-b)}{a}.$$

Critical points of R_3 which is viewed as a function of β are given by

$$\beta_{\pm}=\frac{\dfrac{6a(b-c)}{c^2}\pm\sqrt{\dfrac{36a^2(b-c)^2}{c^2}+\dfrac{24ab}{c^2}\left(\dfrac{3a}{c}-\dfrac{ab}{c^2}-\dfrac{6(a-b)}{a}\right)}}{-\dfrac{12ab}{c^2}}$$

$$=\frac{\dfrac{6a(b-c)}{c^2}\pm\dfrac{1}{c^2}\sqrt{36a^2b^2-72a^2bc+36a^2c^2+72a^2bc-24a^2b^2-144abc^2+144b^2c^2}}{-\dfrac{12ab}{c^2}}$$

$$=\frac{6a(b-c)\pm\sqrt{12a^2b^2+36a^2c^2-144abc^2+144b^2c^2}}{-12ab}.$$

Clearly $\beta_->\beta_+$ and

$$\beta_-=\frac{-6a(b-c)+\sqrt{12a^2b^2+36a^2c^2-144abc^2+144b^2c^2}}{12ab}$$

$$=-\frac{1}{2}\left(1-\frac{c}{b}\right)+\sqrt{\frac{1}{12}+\frac{1}{4}\frac{c^2}{b^2}-\frac{c^2}{ab}+\frac{c^2}{a^2}}.$$

We claim that $\beta_- < \frac{c}{a} + \frac{c}{b} - 1$. This is equivalent to

$$\sqrt{\frac{1}{12} + \frac{1}{4}\frac{c^2}{b^2} - \frac{c^2}{ab} + \frac{c^2}{a^2}} < \frac{c}{a} + \frac{c}{b} - 1 + \frac{1}{2}\left(1 - \frac{c}{b}\right)$$
$$= \frac{c}{a} + \frac{c}{2b} - \frac{1}{2}$$

$$(2.29) \qquad \Leftrightarrow \frac{1}{12} + \frac{c^2}{4b^2} - \frac{c^2}{ab} + \frac{c^2}{a^2} < \frac{c^2}{a^2} + \frac{c^2}{4b^2} + \frac{1}{4} - \frac{c}{a} - \frac{c}{2b} + \frac{c^2}{ab}$$
$$\Leftrightarrow \frac{2c^2}{ab} + \frac{1}{6} > \frac{c}{a} + \frac{c}{2b}$$
$$\Leftrightarrow 12c^2 + ab > 6bc + 3ac\,.$$

Since $b < \frac{4}{3}c$, we have

$$6bc + 3ac < 8c^2 + 3ac\,.$$

In order to prove (2.29), we need

$$12c^2 + ab > 8c^2 + 3ac$$

i.e.

$$4c^2 + ab > 3ac\,.$$

However $a < 2c$ by hypothesis. Thus

$$4c^2 + ab > 2ac + ab \geqq 3ac\,.$$

So our claim $\beta_- < \frac{c}{a} + \frac{c}{b} - 1$ follows. Notice that $R_3(\beta)$ has local maximum at β_-. Therefore

$$\sup_{\frac{c}{a}+\frac{c}{b}-1<\beta<\frac{c}{b}} R_3 = \max\left\{R_3\left(\frac{c}{a} + \frac{c}{b} - 1\right),\ R_3\left(\frac{c}{b}\right)\right\},$$
$$R_3\left(\frac{c}{b}\right) = -\frac{2ab\,c^3}{c^2\,b^3} - \frac{3a(b-c)\,c^2}{c^2\,b^2} + \left(\frac{3a}{c} - \frac{ab}{c^2} - \frac{6(a-b)}{a}\right)\frac{c}{b}$$
$$= \frac{ac}{b^2} - \frac{a}{c} - \frac{6(a-b)c}{ab}$$

$$= \frac{a(c^2-b^2)}{b^2 c} - \frac{6(a-b)c}{ab} \leqq 0 .$$

The above inequality becomes an equality if and only if $a = b = c$. The latter condition is not satisfied because of the hypothesis $\frac{c}{b} > \beta \geqq \frac{c}{a} + \frac{c}{b} - 1$. Hence we have $R_3\left(\frac{c}{b}\right) < 0$.

If $\sup\limits_{\frac{c}{a}+\frac{c}{b}-1<\beta<\frac{c}{b}} R_3 = R_3\left(\frac{c}{b}\right)$, then (2.28) follows immediately. Therefore, it remains to consider the case $\sup\limits_{\frac{c}{a}+\frac{c}{b}-1<\beta<\frac{c}{b}} R_3 = R_3\left(\frac{c}{a}+\frac{c}{b}-1\right)$.

$$\begin{aligned}
R_3\left(\frac{c}{a}+\frac{c}{b}-1\right) &= -\frac{2ab}{c^2}\left(\frac{c}{a}+\frac{c}{b}-1\right)^3 - \frac{3a(b-c)}{c^2}\left(\frac{c}{a}+\frac{c}{b}-1\right)^2 \\
&\quad + \left(\frac{3a}{c} - \frac{ab}{c^2} - \frac{6(a-b)}{a}\right)\left(\frac{c}{a}+\frac{c}{b}-1\right) \\
&= -\frac{2ab}{c^2}\left(\frac{c^3}{a^3} + \frac{3c^3}{a^2 b} - \frac{3c^2}{a^2} + \frac{3c^3}{ab^2} - \frac{6c^2}{ab} + \frac{3c}{a} + \frac{c^3}{b^3} - 3\frac{c^2}{b^2} + 3\frac{c}{b} - 1\right) \\
&\quad - \frac{3(b-c)}{a} - \frac{3a(b-c)}{b^2} - \frac{3a(b-c)}{c^2} - \frac{6(b-c)}{b} + \frac{6(b-c)}{c} \\
&\quad + \frac{6a(b-c)}{bc} + 3 + \frac{3a}{b} - \frac{3a}{c} - \frac{b}{c} - \frac{a}{c} + \frac{ab}{c^2} - \frac{6c(a-b)}{a^2} \\
&\quad - \frac{6c(a-b)}{ab} + \frac{6(a-b)}{a} \\
&= 9 + \frac{4bc}{a^2} + \frac{ac}{b^2} - \frac{3c}{a} - \frac{3b}{a} - \frac{6c}{b} - \frac{b}{c} - \frac{a}{c}.
\end{aligned}$$

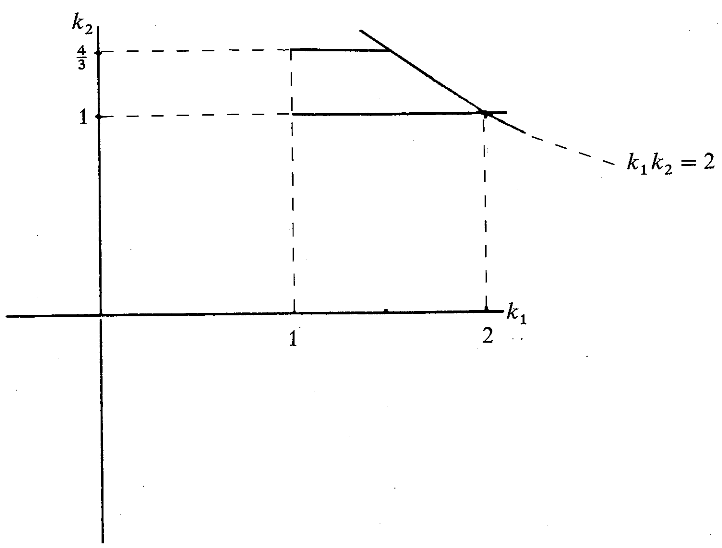
Set $\frac{a}{b} = k_1$ and $\frac{b}{c} = k_2$. Then $1 \leqq k_1 < 2$, $1 \leqq k_2 < \frac{4}{3}$, $k_1 k_2 < 2$ and

$$\begin{aligned}
R_3\left(\frac{c}{a}+\frac{c}{b}-1\right) &= 9 + \frac{4}{k_1^2 k_2} + \frac{k_1}{k_2} - \frac{3}{k_1 k_2} - \frac{3}{k_1} - \frac{6}{k_2} - k_2 - k_1 k_2 \\
&= \frac{1}{k_1^2 k_2}\left(9k_1^2 k_2 + 4 + k_1^3 - 3k_1 - 3k_1 k_2 - 6k_1^2 - k_1^2 k_2^2 - k_1^3 k_2^2\right).
\end{aligned}$$

Now we are going to estimate the left hand side of (2.28), which is denoted by I:

$$I = \left(\frac{b}{c} - 1\right)a - R_3\left(\frac{c}{a} + \frac{c}{b} - 1\right)$$
$$= (k_2 - 1)a - R_3\left(\frac{c}{a} + \frac{c}{b} - 1\right)$$
$$= \frac{1}{k_1^2 k_2}(ak_1^2k_2^2 - ak_1^2k_2 - 9k_1^2k_2 - 4 - k_1^3 + 3k_1 + 3k_1k_2 + 6k_1^2 + k_1^2k_2^2 + k_1^3k_2^2).$$

We want $I > 0$ when $(k_1, k_2) \in \Omega - \{1, 1)\}$. (Notice that $(k_1, k_2) = (1, 1)$ implies $a = b = c$. In this case the condition $\frac{c}{b} > \frac{c}{a} + \frac{c}{b} - 1$ is not satisfied.) Here Ω is the region in $\mathbb{R}^2$ defined by the inequalities $1 \leqq k_1 < 2$, $1 \leqq k_2 \leqq \frac{4}{3}$ and $k_1 k_2 < 2$.



Let

$$I_1 = ak_1^2k_2^2 - ak_1^2k_2 - 9k_1^2k_2 - 4 - k_1^3 + 3k_1 + 3k_1k_2 + 6k_1^2 + k_1^2k_2^2 + k_1^3k_2^2,$$

$$I_2 = 3k_1^2k_2^2 - 3k_1^2k_2 - 9k_1^2k_2 - 4 - k_1^3 + 3k_1 + 3k_1k_2 + 6k_1^2 + k_1^2k_2^2 + k_1^3k_2^2,$$
$$I_1 - I_2 = (a-3)k_1^2k_2^2 - (a-3)k_1^2k_2 = (a-3)k_1^2k_2(k_2 - 1) \geqq 0$$

since $a \geqq 3$ and $k_2 \geqq 1$. As $I = \frac{1}{k_1^2 k_2} I_1$, in order to prove $I > 0$, in $\Omega - \{(1, 1)\}$, it suffices to prove $I_2 > 0$ in $\Omega - \{(1, 1)\}$.

We first see that $\frac{\partial I_2}{\partial k_2}$ does not vanish in Ω. Suppose

$$\frac{\partial I_2}{\partial k_2} = 8k_1^2 k_2 - 12k_1^2 + 3k_1 + 2k_1^3 k_2 = 0$$

in Ω. Then

$$k_2 = \frac{3(4k_1 - 1)}{2(4k_1 + k_1^2)}.$$

In order that (k_1, k_2) is a point in Ω, we need

$$\frac{3(4k_1 - 1)}{2(4k_1 + k_1^2)} \geqq 1. \tag{2.30}$$

Notice that $k_1^2 + 4k_1 > 0$ for $k_1 \geqq 1$. Hence (2.30) is the same as

$$\begin{aligned} &3(4k_1 - 1) \geqq 2(4k_1 + k_1^2) \\ \Leftrightarrow\ &0 \geqq 3 - 4k_1 + 2k_1^2 \\ \Leftrightarrow\ &0 \geqq \frac{3}{2} - 2k_1 + k_1^2 = (k_1 - 1)^2 + \frac{1}{2}. \end{aligned}$$

The last inequality of course is absurd. So we have proved that $\frac{\partial I_2}{\partial k_2}$ has no zero point in Ω. Hence $\frac{\partial I_2}{\partial k_2}(k_1, k_2) > 0$ for all (k_1, k_2) in Ω because Ω is connected and $\frac{\partial I_2}{\partial k_2}(1, 1) > 0$. For each fixed k_1 with $1 \leqq k_1 < 2$, I_2 restricted on Ω is a strictly increasing function in k_2. To prove $I_2 > 0$ in $\Omega \setminus \{(1, 1)\}$, it suffices to prove $I_2(k_1, 1) > 0$ for $1 < k_1 < 2$. Observe that

$$\begin{aligned} I_2(k_1, 1) &= 4k_1^2 - 12k_1^2 - 4 - k_1^3 + 3k_1 + 3k_1 + 6k_1^2 + k_1^3 \\ &= -2k_1^2 + 6k_1 - 4 \\ &= -2\left(k_1 - \frac{3}{2}\right)^2 + \frac{1}{2} \end{aligned}$$

which is clearly bigger than zero whenever k_1 is strictly between one and two.

Case (ii). $\frac{c}{b} \leqq \beta < 1$. By Proposition 2.1 and (2.20), we have

$$P \leqq \sum_{k=0}^{[c]-1} \left[\frac{1}{2}\frac{a}{c}(k + \beta)\left(\frac{b}{c}(k + \beta) - 1\right) + \frac{a - b}{2a}\right] \tag{2.31}$$

$$= \frac{1}{12}(c-1)\left(2ab - \frac{ab}{c} - 3a + \frac{3a}{2b}\right) + \frac{1}{12}R_2 - \frac{a}{8b}(c-\beta)$$

$$+ \frac{a-b}{2a}(c-\beta)$$

$$= \frac{1}{12}(c-1)\left(2ab - \frac{ab}{c} - 3a + \frac{3a}{2b}\right)$$

$$+ \frac{1}{12}\left[-\frac{2ab}{c^2}\beta^3 + \left(\frac{3ab}{c^2} + \frac{3a}{c}\right)\beta^2 + \left(-\frac{3a}{c} - \frac{ab}{c^2} - \frac{3a}{2b}\right)\beta + \frac{3a}{2b}\right]$$

$$- \frac{ac}{8b} + \frac{a}{8b}\beta + \frac{(a-b)c}{2a} - \frac{a-b}{2a}\beta$$

$$= \frac{1}{12}(c-1)\left[2ab - \frac{ab}{c} - 3a + \frac{6(a-b)}{a}\right] + \frac{1}{12}R_4$$

where

$$R_4 = -\frac{2ab}{c^2}\beta^3 + \left(\frac{3ab}{c^2} + \frac{3a}{c}\right)\beta^2 + \left(-\frac{3a}{c} - \frac{ab}{c^2} - \frac{6(a-b)}{a}\right)\beta + \frac{6(a-b)}{a}.$$

Consider

$$\frac{dR_4}{d\beta} = -\frac{6ab}{c^2}\beta^2 + \frac{6a}{c^2}(b+c)\beta + \left(-\frac{3a}{c} - \frac{ab}{c^2} - \frac{6(a-b)}{a}\right).$$

Critical points of R_4 which is viewed as a function of β are given by

$$\beta^{\pm} = \frac{-\frac{6ab}{c^2}\left(1+\frac{c}{b}\right) \pm \sqrt{\frac{36a^2(b+c)^2}{c^4} - \frac{24b\left(3a^2c + a^2b + 6c^2(a-b)\right)}{c^4}}}{-\frac{12ab}{c^2}}$$

$$= \frac{1}{2} + \frac{c}{2b} \mp \frac{\sqrt{12a^2b^2 + 36a^2c^2 - 144abc^2 + 144b^2c^2}}{12ab}$$

$$= \frac{1}{2} + \frac{c}{2b} \mp \sqrt{\frac{1}{12} + \frac{c^2}{4b^2} - \frac{c^2}{ab} + \frac{c^2}{a^2}}.$$

Clearly $\beta^- > \beta^+$. Recall that $\frac{c}{b} > \frac{1}{2}$. It follows that

$$\beta^- = \frac{1}{2} + \frac{c}{2b} + \sqrt{\frac{1}{12} + \left(\frac{c}{2b} - \frac{c}{a}\right)^2} > 1.$$

Notice that $R_4(\beta)$ has local maximum at β^-.

Therefore

$$(2.32)\qquad \sup_{\frac{c}{b}\leqq\beta<1} R_4(\beta)=\max\left\{R_4\left(\frac{c}{b}\right), R_4(1)\right\}.$$

Clearly $R_4(1)=0$ and

$$(2.33)\quad R_4\left(\frac{c}{b}\right)=-\frac{2ab}{c^2}\frac{c^3}{b^3}+\left(\frac{3ab}{c^2}+\frac{3a}{c}\right)\frac{c^2}{b^2}+\left(-\frac{3a}{c}-\frac{ab}{c^2}-\frac{6(a-b)}{a}\right)\frac{c}{b}+\frac{6(a-b)}{a}$$

$$=\frac{ac}{b^2}-\frac{a}{c}+\frac{6(a-b)}{a}\left(1-\frac{c}{b}\right)$$

$$=\left(1-\frac{c}{b}\right)\left[\frac{6(a-b)}{a}-\frac{a}{c}\left(1+\frac{c}{b}\right)\right]$$

$$=\left(1-\frac{c}{b}\right)\left(6-\frac{6b}{a}-\frac{a}{c}-\frac{a}{b}\right).$$

Observe that the function $f(x)=6x^{-1}+2x$ defined on the interval $[1,2]$ has a minimum at $\sqrt{3}$. Hence

$$f(x)=6x^{-1}+2x\geqq f(\sqrt{3})=4\sqrt{3}>6 \quad\text{for}\quad 1\leqq x\leqq 2.$$

If follows immediately that

$$6\frac{b}{a}+\frac{a}{c}+\frac{a}{b}\geqq 6\frac{b}{a}+2\frac{a}{b}>6$$

because $1\leqq\frac{a}{b}<2$. Since $1-\frac{c}{b}>0$, the above inequality and (2.33) imply $R_4\left(\frac{c}{b}\right)<0$. We conclude from (2.32) that $R_4(\beta)<0$ for all $\beta\in\left[\frac{c}{b},1\right)$. By (2.31), we have

$$P<\frac{1}{12}(c-1)\left[2ab-\frac{ab}{c}-3a+\frac{6(a-b)}{a}\right]$$

$$\leqq\frac{1}{12}(c-1)(2ab-2a-2b)$$

by (2.26). Q.E.D.

Proof of the Corollary. By translation $x=x'-1$, $y=y'-1$ and $z=z'-1$, the Q in the Corollary is the same as P the number of positive integral points (x', y', z') which satisfy

$$\frac{x'-1}{a}+\frac{y'-1}{b}+\frac{z'-1}{c}\leqq 1$$

i.e.

$$\frac{x'}{a(1+u)}+\frac{y'}{b(1+u)}+\frac{z'}{c(1+u)}\leqq 1$$

where $u=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$. Observe that $c(1+u)=c+1+\frac{c}{a}+\frac{c}{b}\geqq 2$. So the result follows directly from the Theorem. Q.E.D.

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