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Gorenstein Quotient Singularities in Dimension Three

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ABSTRACT

Let G be a finite subgroup of $GL(3,\mathbb{C})$. Then G acts on $\mathbb{C}^3$. It is well known that $\mathbb{C}^3/G$ is Gorenstein if and only if $G \subseteq SL(3,\mathbb{C})$. In chapter 1, we sketch the classification of finite subgroups of $SL(3,\mathbb{C})$. We include two more types (J) and (K) which were usually missed in the work of many mathematicians. In chapter 2, we give general method to find invariant polynomials and their relations of finite subgroups of $GL(3,\mathbb{C})$. The method is in practice substantially better than the classical method due to Noether. In chapter 3, we recall some properties of quotient varieties and prove that $\mathbb{C}^3/G$ has isolated singularities if and only if G is abelian and 1 is not an eigenvalue of g in G. We also apply the method in chapter 2 to find minimal generators of ring of invariant polynomials as well as their relations.

Key words and phrases.

quotient singularities, isolated singularities, finite subgroups of $SL(3,\mathbb{C})$, invariant polynomials, minimal generators of invariants and their relations.

CHAPTER 0

INTRODUCTION

Let G be a finite subgroup of $GL(n, \mathbb{C})$. Then G acts on $\mathbb{C}^n$. The quotient variety $\mathbb{C}^n/G$ was studied by Chevalley [Ch] Shephard and Todd [Sh-To] in algebraic setting. However the first one who studied this was H. Cartan [Car]. He proved among other things that the singularities of $\mathbb{C}^n/G$ are normal. In particular, the singular set of $\mathbb{C}^n/G$ is at least condimension two in $\mathbb{C}^n/G$ or at most dimension $n-2$. Prill [Pr] later also made a substantial contribution in the subject. He showed that in order to study $\mathbb{C}^n/G$, it suffices to consider small subgroup G of $GL(n, \mathbb{C})$, i.e., G contains no element which has an eigenvalue 1 of multiplicity $n-1$. In this article, we are interested in the case that $\mathbb{C}^n/G$ is Gorenstein, i.e., the dualizing sheaf of $\mathbb{C}^n/G$ is trivial. We were told that physicists are interested in those three dimensional quotient singularities which admit a desingularization whose canonical bundle is trivial. As a first step, we have to understand three dimensional quotient singularities which are Gorenstein. By a theorem of Khinich [Kh] and Watanabe [Wa], we know that $\mathbb{C}^n/G$ is Gorenstein if and only if $G \subseteq SL(n, \mathbb{C})$. If G is a finite subgroup of $SL(2, \mathbb{C})$, then the quotient $\mathbb{C}^2/G$ has only isolated singularity and it must be a rational double point. Conversely every rational double point is analytically isomorphic to such an isolated quotient singularity. There are five types of these singularities. Four of them correspond to nonabelian subgroup of $SL(2, \mathbb{C})$. These were studied by Brieskorn [Br], Riemenschneider [Ri1] and others. For three dimensional quotient singularity, the singular set is of dimension either 0 or 1. Unlike the two dimensional case, our following result says that for three dimensional quotient singularity $\mathbb{C}^3/G$, we can get isolated singularity only if G is abelian.

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Theorem A. Let $G \subseteq SL(3,\mathbb{C})$ be a small subgroup. Then $\mathbb{C}^3/G$ has an isolated singularity if and only if G is abelian and 1 is not an eigenvalue of g for every nontrivial element g in G.

In view of Proposition 6 and Theorem 2 of [Pr], the classification of three dimensional Gorenstein quotient singularities corresponds to the classification of small finite subgroup of $SL(3,\mathbb{C})$ up to linear equivalence. The latter was done by Blichfeldt [Bl1] in 1917 and Miller-Blichfeldt-Dickson [Mi-Bl-Di] in 1916. 10 types of finite subgroups of $SL(3,\mathbb{C})$ (A)–(I) and (L) (see the following list) were obtained. Let

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad V = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},$$

$$U = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon\omega \end{pmatrix}, \quad P = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & \omega^2 \\ 1 & \omega & \omega \\ \omega & 1 & \omega \end{pmatrix}, \quad Q = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{pmatrix}.$$

where $abc = -1$, $\omega = e^{2\pi i/3}$, $\epsilon^3 = \omega^2$.

(A) Diagonal abelian groups.

(B) Group isomorphic to transitive finite subgroups of $GL(2,\mathbb{C})$.

(C) Group generated by (A) and T.

(D) Group generated by (C) and Q.

(E) Group of order 108 generated by S, T, V.

(F) Group of order 216 generated by (E) and $P = UVU^{-1}$.

(G) Hessian group of order 648 generated by (E) and U.

(H) Simple group of order 60 isomorphic to alternating group A_5.

(I) Simple group of order 168 isomorphic to permutation group generated by (1234567), (142)(356), (12)(35).

(L) Group G of order 1080 its quotient G/F isomorphic to alternating group A_6, where $F = \{I, W, W^2\}$ is the center of $SL(3,\mathbb{C})$, $I =$ identity and

$$W = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad \omega = e^{2\pi i/3}.$$

Although the classification of finite subgroups of $PGL(3,\mathbb{C})$ and their minimal realizations in $SL(3,\mathbb{C})$ was given in [Bl1] and [Mi-Bl-Di], except for a finite number of experts in finite group theory, people usually have a wrong impression that (A),...,(L) are all finite subgroups of $SL(3,\mathbb{C})$ because the works of [Bl1] and [Mi-Bl-Di] were not presented clearly enough or perhaps there is a generation gap. This can be seen, for example, in the work of Watanabe and Rotillon [Wa-Ro]. Their classification is incomplete because they missed two hypersurface singularities which correspond to the following tow groups (J) and (K). In order to obtain a complete classification up to conjugation, we actually need to distinguish two further types (see [Bl3, p.325]).

(J) Group of order 180 generated by (H) and F.

(K) Group of order 504 generated by (I) and F.

Because of this reason, we shall sketch the proof of classification of finite subgroups in $SL(3,\mathbb{C})$ in chapter 1.

Let $S=\mathbb{C}[x_1,\ldots,x_n]$. The subalgebra

$$S^G=\{f\in S, f(g(x))=f(x)\quad \text{for all } g\in G\}$$

is finitely generated and contains a minimal set of homogeneous polynomials $f_1,\ldots,f_k$ which generate it as a $\mathbb{C}$-algebra, i.e., any invariant is a polynomial in $f_1,\ldots,f_k$. f_i's are called the minimal generators of S^G. We have the following homomorphism of rings:

$$\phi : R=\mathbb{C}[y_1,\ldots,y_k]\to S=\mathbb{C}[x_1,\ldots,x_n]$$

where R and S are polynomial rings and $\phi(F)=F(f_1,\ldots,f_k)\in S$ for all $F(y_1,\ldots,y_k)\in\mathbb{C}[y_1,\ldots,y_k]$. Then $\mathrm{Im}\phi=S^G$. Let K be the kernel of ϕ, then K is an ideal of $\mathbb{C}[y_1,\ldots,y_k]$ and $S^G\cong\mathbb{C}[y_1,\ldots,y_k]/K$. The minimal generators of K are called relations of S^G. The main purpose of this article is to find a set of minimal generators of S^G and their relations for groups (A)–(L). Geometrically the number k of the minimal generators of S^G is the minimal embedding dimension of the quotient variety into complex Euclidean space. The relations of S^G are the equations which define the image of the quotient variety in $\mathbb{C}^k$ as affine algebraic variety.

It is classically known that the set of generators of S^G can be obtained by averaging over G all monomials

$$x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$$

of total degree $\sum_{i=1}^{n} b_i \leq g$ where g is the order of G (see Noether [No]). However this method in actual computation is practically useless when the order of the group is large even with the aid of computer. The following Theorem B is substantially better than Noether's theorem because in almost all the examples that we encounter, it reduces the computation drastically.

Theorem B. Let H be a subgroup of G and $\{f_1, \ldots, f_r\}$ be a set of minimal generators of S^H. Let $G = Ha_1 \cup Ha_2 \cup \cdots \cup Ha_s$, where $s = |G|/|H|$, $a_i \in G$, $i = 1, \ldots, s$. Then $(f_1^{d_1} \cdots f_r^{d_r})a_1 + \cdots + (f_1^{d_1} \cdots f_r^{d_r})a_s$, $\sum_{i=1}^{r} d_i(\deg f_i) \leq |G|$, form a set of generators of S^G.

In 1897, Molien [Mo] made an important progress in invariant theory. He showed that the number of linearly independent homogeneous invariants of G of degree d is the coefficient of λ^d in the expansion of

$$\phi(\lambda) = \frac{1}{g} \sum_{T \in G} \frac{1}{\det(I - \lambda T)}$$

where $g = |G|$ and I is the identity of G. We call $\phi((\lambda)$ the Molien series of G. By Noether normalization theorem, Quillen's solution to Serre conjecture and a result of Hochster and Eagon [Ho-Ea], one can easily see that S^G can be written as a direct sum

(1.1) $$S^G = \mathbb{C}[f_1, \ldots, f_n] \oplus \mathbb{C}[f_1, \ldots, f_n]g_1 \oplus \cdots \oplus \mathbb{C}[f_1, \ldots, f_n]g_k$$

where f_i, g_i are homogeneous invariants of G and f_i are algebraically independent. So Molien series of G is

(1.2) $$\phi(\lambda) = \frac{1 + \lambda^{b_1} + \cdots + \lambda^{b_k}}{(1 - \lambda^{d_1})(1 - \lambda^{d_2}) \cdots (1 - \lambda^{d_n})}$$

where d_i, b_j are the degrees of f_i, g_j respectively. This will play an important role in finding a set of minimal generators of invariants and their relations.

For any $a \in S^G$, a can be written as $p_0(f_1, \ldots, f_n) + p_1(f_1, \ldots, f_n)g_1 + \cdots + p_k(f_1, \ldots, f_n)g_k$ where $p_i(f_1, \ldots, f_n)$ are polynomials in $f_1, \ldots, f_n$. This is called the basic form of a. The invariants $f_1, \ldots, f_n$, $g_1, \ldots, g_k$ are called basic invariants of S^G. (1.1) is called a basic decomposition of S^G and (1.2) is called a basic series of G. Of course, we want

$$\{f_1, \ldots, f_n\} \subseteq \{\text{minimal generators}\} \subseteq \{\text{basic invariants}\}.$$

Now let $\{f_1, \ldots, f_n, g_1, \ldots, g_k\}$ be a set of basic invariants of S^G in which $\{f_1, \ldots, f_n, g_1, \ldots, g_t\}$ being a set of minimal generators of S^G, $t \leq k$ and $\{f_1, \ldots, f_n\}$ algebraically independent. Let $A = \{g_1 g_j : 1 \leq i \leq t, i \leq j \leq k\} - \{g_1, \ldots, g_k\}$ and $b(a)$ be a basic form of a, $a \in S^G$. Then $\{a - b(a) : a \in A\}$ generates K (see p.4) for g_m being a polynomial in $g_1, \ldots, g_t$, $t+1 \leq m \leq k$. Let $B = \{a \in A : a \text{ has no factor } a' \in A - \{a\}\}$. Then $\{a - b(a) : a \in B\}$ are relations of S^G.

We give a complete description of finding invariants and their relations for types (A)–(D) and we shall only consider the case that (B) is isomorphic to transitive small groups of $GL(2, \mathbb{C})$ (see section 1.2(B)). For types (E)–(L), we are able to write down all the invariants and their relations explicitly. In fact, for types (F)–(L), the number of the minimal generators of S^G is precisely 4. So the quotient varieties of these groups are hypersurface in $\mathbb{C}^4$. Therefore we know ápriori that there is only one relation of the minimal generators for these groups. Type (A) was treated also by MacWilliams-Mallows-Sloan [Mac-Mal-Sl, pp.800-801]. They only consider finding invariants, but our result is simpler than they got. The special case of type (C) for (A) being a diagonal cyclic group was treated by Maschke [Ma2] independently. Rotillon [Ro] treated the cases (E), (F) and (G) independently. However her result in case (E), second relation ([Ro], Theorem 1, p.346) is wrong. The invariants and relations of (H) and (I) were found by Klein (1884) [Kl, pp.236-243] and Weber (1899) [We, pp.518-529] (also see Gordan (1880) [Go]) respectively. Their methods are long and complicated and are difficult to comprehend. The cases (J), (K) and (L) are new except the fundamental invariants for (L) are given in [Mi-Bl-Di, §125].

(J) and (K) are needed in order to fill the gap of [Wa-Ro]. The most difficult one is type (L), its invariants take a few pages long to write down. We had a hard time to find their relation. It took us more than 3 months even with the aid of computer. However the final relation is quite simple. We summarize some of the results in the following table.

Theorem C. Let G be a finite subgroup of $SL(3,\mathbb{C})$. Then we have

Type of G	Minimal embedding dimension of $\mathbb{C}^3/G$	Equations of $\mathbb{C}^3/G$
(E)	5	$\begin{cases} 9y_4^2 - 12y_3^2 + y_1^2y_3 - y_1^2y_4 = 0 \\ 432y_5^2 - y_2^3 + 2y_1^3 - 36y_1y_4 \\ \qquad + 3y_1^2y_2 - 36y_2y_3 = 0 \end{cases}$
(F)	4	$4y_4^3 - 144y_2y_4^2 + 1728y_2y_4$ $- (y_1^3 - 432y_3^2 - 3y_1y_4 + 36y_1y_2)^2 = 0$
(G)	4	$4y_4^3 - 9y_3y_4^2 + 6y_3^2y_4 - 2592y_1^2y_3y_4 - y_3^3$ $+ 864y_1^2y_3^2 + 6912y_2^3y_3 - 186624y_1^4y_3 = 0$
(H)	4	$y_4^2 + 1728y_2^5 - y_3^3 - 720y_1y_2^3y_3$ $+ 80y_1^2y_2y_3^2 - 64y_1^3(5y_2^2 - y_1y_3)^2 = 0$
(I)	4	$y_4^2 - y_3^3 + 88y_1^2y_2y_3^2 - 1008y_1y_2^4y_3$ $- 1088y_1^4y_2^2y_3 + 256y_1^7y_3 - 1728y_2^7$ $+ 60032y_1^3y_2^5 - 22016y_1^6y_2^3 + 2048y_1^9y_2 = 0$
(J)	4	$y_4^3 - y_3[y_2^2 + 1728y_1^5 - 720y_1^3y_4$ $+ 80y_1y_4^2 - 64y_3(5y_1^2 - y_4)^2] = 0$
(K)	4	$y_4^3 - y_3(y_2^2 + 88y_3y_4 - 1008y_1^4y_4$ $- 1088y_1^2y_3y_4 + 256y_3^2y_4 - 1728y_1^7$ $+ 60032y_1^5y_3 - 22016y_1^3y_3^2 + 2048y_1y_3^3) = 0$
(L)	4	$459165024y_4^2 - 25509168y_3^3$ $- (236196 + 26244\sqrt{15}i)y_3^2y_1^5$ $+ 1889568(1 + \sqrt{15}i)y_3^2y_1^3y_2$

		$+ (8503056 - 2834352\sqrt{15}i)y_3^2y_1y_2^2$ $- (891 + 243\sqrt{15}i)y_3y_1^{10}$ $- (5346 - 8910\sqrt{15}i)y_3y_1^8y_2$ $+ (360612 - 51516\sqrt{15}i)y_3y_1^6y_2^2$ $+ (192456 + 21384\sqrt{15}i)y_3y_1^4y_2^3$ $- 3569184(1 + \sqrt{15}i)y_3y_1^2y_2^4$ $- (7558272 - 2519424\sqrt{15}i)y_3y_2^5$ $- 2426112(\sqrt{15}i)y_2^7y_1$ $+ (7978176 + 886464\sqrt{15}i)y_2^6y_1^3$ $- (3297168 - 471024\sqrt{15}i)y_2^5y_1^5$ $+ (78768 - 131280\sqrt{15}i)y_2^4y_1^7$ $+ (26928 + 7344\sqrt{15}i)y_2^3y_1^9$ $- (1560 - 40\sqrt{15}i)y_2^2y_1^{11}$ $+ (17 - 7\sqrt{15}i)y_2y_1^{13} = 0$

We have used the REDUCE program [He] to perform the following computations:

1. The relations for types (G), (H), (I), (L).
2. The Molien series for all types.
3. The invariants for types (H), (I), (L).

Since the groups of types (H), (I) and (L) are isomorphic to permutation groups, CAYLEY program [Can] helps us compute the conjugacy classes of these groups. With these results ready, we use REDUCE program to calculate the Molien series of these groups.

In Chapter 1, we sketch the classification of finite subgroups of $SL(3,\mathbb{C})$. We include two more types (J) and (K) which are not found in [Bl1] and [Mi-Bl-Di]. In Chapter 3, we recall some properties of quotient varieties and prove Theorem A. In Chapter 2, we give general method to find invariant polynomials and their relations of finite subgroups of $GL(n,\mathbb{C})$. In particular Theorem B is prove. In Chapter 3, we apply the method developed in Chapter 2 to find minimal generators of S^G for G being a finite subgroup of $SL(3,\mathbb{C})$. In particular Theorem C is proved.

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CHAPTER 1

CLASSIFICATION OF FINITE SUBGROUPS OF $SL(3, \mathbb{C})$

In this chapter we basically follow the work of Blichfeldt [Bl1] and Miller-Blichfeldt-Dickson [Mi-Bl-Di] to give a complete classification of $SL(3, \mathbb{C})$. He obtained 10 types of finite subgroups of $SL(3, \mathbb{C})$ (A)–(I) and (L); in view of [Bl3, p.325], in order to obtain a complete classification up to conjugation we distinguish two further types (J) and (K).

1.1. Definitions

1. Any finite subgroup G of $GL(n, \mathbb{C})$ is called a linear group in n variables. Any element of G is called a matrix or linear transformation or transformation of G.

2. If the n variables of a group G can be separated into two or more sets (either directly or after a suitable change of variables), such that the variables of any one set are transformed by all the transformations of G into linear functions of the variables of that set only, we say that G is intransitive. If such a division is not possible, the group is transitive. The different sets into which the variables of an intransitive group may be separated are called its sets of intransitivity.

3. A linear group in n variables is said to be reducible when, after a suitable choice of variables $x_1, \ldots, x_n$, a certain number of these (say $x_1, \ldots, x_m; m < n$) are transformed into linear functions of themselves by all the transformations of the group. We shall say that the m variables $x_1, \ldots, x_m$ constitute a reduced set of the group. A irreducible group is a group in which no such choice of variables is possible.

4. A transitive group G, in which the variables (either directly or after a suitable choice of new variables) can be separated into two or more sets $Y_1, \ldots, Y_k$, such that the variables of each set are transformed into linear functions of the variables of the same set or into linear functions of the variables of a different set, is said to be imprimitive. If such division is not possible, the group is primitive. The sets $Y_1, \ldots, Y_k$ are called sets of imprimitivity.

The determination of the linear groups in $SL(3,\mathbb{C})$ up to conjugation is based on the following classification:

$$\text{Linear groups of } SL(3,\mathbb{C}) \begin{cases} \text{intranditive} \\ \text{transitive} \begin{cases} \text{imprimitive} \\ \text{primitive} \end{cases} \end{cases}$$

$$\text{Primitive groups} \begin{cases} \text{having normal intransitive subgroups} \\ \text{having normal imprimitive subgroups} \\ \text{having normal primitive subgroups} \\ \text{which are simple} \end{cases}$$

1.2. Intransitive and imprimitive groups

We first discuss intransitive groups. There are two types of intransitive groups:

(A) Diagonal abelian groups. I.e., each element has the form of

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \quad \alpha\beta\gamma = 1.$$

(B) Groups isomorphic to transitive linear groups of $GL(2,\mathbb{C})$. I.e., each element has the form of

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}, \quad \alpha(ad-bc) = 1.$$

For this type, we shall only consider the case that

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = \alpha^{-1} \right\}$$

forms a small group[1] in $GL(2,\mathbb{C})$ in the following discussion (cf. Prill [Pr, Proposition 6, p.380]). According to the classification of small groups of $GL(2,\mathbb{C})$ which

[1] A linear group $G \subset GL(n,\mathbb{C})$ is small if no $T \in G$ has 1 as an eigenvalue of multiplicity precisely $(n-1)$.

were listed in the Riemenschneider's paper [Ri1, p.38], the type (B) now can be divided further in the following:

(B1) The dihedral groups $D_{n,q}$ where $1 < q < n$ and $(n, q) = 1$, generated by

(B1a) $\psi_{2q}, \tau, \phi_{2m}$, if $m = n - q \equiv 1 \bmod 2$,

(B1b) $\psi_{2q}, \tau \circ \phi_{4m}$, if $m \equiv 0 \bmod 2$,

where

$$\psi_k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_k & 0 \\ 0 & 0 & \zeta_k^{-1} \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \quad \phi_k = \begin{pmatrix} \zeta_k^{-2} & 0 & 0 \\ 0 & \zeta_k & 0 \\ 0 & 0 & \zeta_k \end{pmatrix},$$

and $\zeta_k = e^{(2\pi i)/k}$. Note that $|D_{n,q}| = 4mq$ and $(m, q) = 1$.

Let $D'_q = \langle \psi'_{2q}, \tau' \rangle$ where

$$\psi'_{2q} = \begin{pmatrix} \zeta_{2q} & 0 \\ 0 & \zeta_{2q}^{-1} \end{pmatrix}, \quad \tau' = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

then D'_q is a dihedral group of $SL(2, \mathbb{C})$. Note that $D'_q = D_{q+1,q}$ so $|D'_q| = 4q$.

(B2) The tetrahedral groups T_m, generated by

(B2a) $\psi_2, \tau, \eta, \phi_{2m}$, if $m \equiv 1, 5 \bmod 6$,

(B2b) $\psi_2, \tau, \eta \circ \phi_{6m}$, if $m \equiv 3 \bmod 6$, where τ, ϕ_k as in (B1) and

$$\psi_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}, \quad \eta = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \zeta_8 & \zeta_8^3 \\ 0 & \zeta_8 & \zeta_8^7 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1+i & -1+i \\ 0 & 1+i & 1-i \end{pmatrix},$$

and $\zeta_8 = e^{(2\pi i)/8}$. Note that $|T_m| = 24m$.

Let $T' = \langle \psi'_4, \tau', \eta' \rangle$ where τ' as in (B1) and

$$\psi'_4 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \eta' = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta_8 & \zeta_8^3 \\ \zeta_8 & \zeta_8^7 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}$$

then $T' = \langle \psi'_4, \eta' \rangle$ for $\tau' = \eta \psi'_4 \eta'^{-1}$ also $T' = \langle \psi'_4, S' \rangle$ where

$$S' = \frac{1}{2} \begin{pmatrix} \zeta_8^3 & \zeta_8^3 \\ \zeta_8 & \zeta_8^5 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1+i & -1+i \\ 1+i & -1-i \end{pmatrix}$$

for $S' = \eta'^2$ and $\eta' = (\psi_4')^2 S'^2$. T' is the tetrahedral group of $SL(2,\mathbb{C})$ and T'/F' is isomorphic to the alternating group A_4 where $F' = \{(1,1),(-1,-1)\}$ the center of $SL(2,\mathbb{C})$. As elements in $T'/F' \simeq A_4$, we have $\overline{S'} = (123)$, $\overline{\psi_4'} = (12)(34)$. Note that $T' = T_1$ so $|T'| = 24$.

(B3) The octahedral groups O_m, generated by ψ_8, τ, η, and ϕ_{2m} where $(m,6) = 1$ and τ, η, ϕ_{2m} as in (B2a) and

$$\psi_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_8 & 0 \\ 0 & 0 & \zeta_8^7 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 1-i \end{pmatrix},$$

and $\zeta_8 = e^{(2\pi i)/8}$. Note that $|O_m| = 48m$. Let $O' = \langle \psi_8', \tau', \eta' \rangle$ where τ', η' as in (B2) and

$$\psi_8' = \begin{pmatrix} \zeta_8 & 0 \\ 0 & \zeta_8^7 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}$$

then $O' = \langle \psi_8', S' \rangle$ where S' as in (B2) for $S' = \eta'^2$ and $\tau' = \eta'(\psi_8')^2 \eta'^{-1}$, $\eta' = (\psi_8')^4 S'^2$. O' is the octahedral group of $SL(2,\mathbb{C})$ and O'/F' is isomorphic to the symmetric group S_4 where $F' = \{(1,1),(-1,-1)\}$ the center of $SL(2,\mathbb{C})$. As elements in $O'/F' \simeq S_4$, we have $\overline{S'} = (123)$, $\overline{\psi_8'} = (1234)$. Note that $O' = O_1$ so $|O'| = 48$.

(B4) The icosahedral groups I_m, generated by σ, Ω, o and ϕ_{2m} where $(m,30) = 1$ and ϕ_{2m} as in (B1a) and

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \Omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_5^3 & 0 \\ 0 & 0 & \zeta_5^2 \end{pmatrix},$$

$$o = \frac{1}{\sqrt{5}} \begin{pmatrix} \sqrt{5} & 0 & 0 \\ 0 & \zeta_5^4 - \zeta_5 & \zeta_5^2 - \zeta_5^3 \\ 0 & \zeta_5^2 - \zeta_5^3 & \zeta_5 - \zeta_5^4 \end{pmatrix},$$

and $\zeta_5 = e^{(2\pi i)/5}$. Note that $|I_m| = 120m$.

Let $I' = \langle \sigma', \Omega', o' \rangle$ where

$$\sigma' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \Omega' = \begin{pmatrix} \zeta_5^3 & 0 \\ 0 & \zeta_5^2 \end{pmatrix}, \quad o' = \frac{1}{\sqrt{5}} \begin{pmatrix} \zeta_5^4 - \zeta_5 & \zeta_5^2 - \zeta_5^3 \\ \zeta_5^2 - \zeta_5^3 & \zeta_5 - \zeta_5^4 \end{pmatrix}$$

then $I' = \langle U', \Omega', o' \rangle$ where

$$U' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

for $U' = o'^2 \sigma'$. I' is the icosahedral group of $SL(2,\mathbb{C})$ and I'/F' is isomorphic to the alternating group A_5 where $F' = \{(1,1),(-1,-1)\}$ the center of $SL(2,\mathbb{C})$. As elements in $I'/F' \simeq A_5$, we have $\overline{U'} = (14)(23)$, $\overline{\Omega'} = (12345)$, $\overline{o'} = (12)(34)$. Note that $I' = I_1$ so $|I'| = 120$.

We next discuss imprimitive groups.

Theorem 1. Let G be an imprimitive linear group in n variables. These may be chosen in such a manner that they break up into a certain number of sets of imprimitivity $Y_1, \ldots, Y_k$ of m variables each ($n = km$), permuted according to a transitive permutation group K on k letters, homomorphic with G. That subgroup of G with corresponds to the subgroup of K leaving one letter unaltered, say Y_1, is primitive as far as the m variables of the set Y_1 are concerned.

If $m = 1$, $k = n$, then G is said to have the monomial form or to be a monomial group.

Proof. We refer to Theorem 9 of [Mi-Bl-Di, p.229].

The imprimitive groups of $SL(3,\mathbb{C})$ are all monomial by Theorem 1. There are two types:

(C) A group generated by (A) which is neither identity nor the center F of $SL(3,\mathbb{C})$ and a transformation Q which permutes the variables in the order (x_1, x_2, x_3). By suitable choice of variables, we may replace Q by T where

$$Q = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix}, \quad abc = 1, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

for $P^{-1}QP = T$, where

$$P = \frac{1}{\sqrt[3]{bc^2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & bc & 0 \\ 0 & 0 & c \end{pmatrix}$$

(D) A group generated by (A), T of (C) and the transformation

$$R = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{pmatrix}, \quad abc = -1.$$

In fact, the case (D) can be divided into two subcases:

(D1) $G = \langle (A) \neq \text{identity or center of } SL(3,\mathbb{C}), T, R \rangle$.

(D2) $G = \langle (A) = \text{identity or center of } SL(3,\mathbb{C}), T, R, (a,b,c) \neq (d,d,d) \rangle$.

For if $(a,b,c) = (d,d,d)$ then using new variable $y_1 = x_1 + x_2 + x_3$ and Theorem 5 below, (D2) will be intransitive.

Note. The center of $SL(3,\mathbb{C})$ is $F = \{I, W, W^2\}$ where I is the identity of $SL(3,\mathbb{C})$ and

$$W = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad \omega = e^{(2\pi i)/3}.$$

Diagonal entries of the diagonal matrix are called multipliers of the matrix, if they are equal, the matrix is called scalar matrix or scalar. Sometimes we write the diagonal matrix with multipliers $\alpha_1, \ldots, \alpha_n$ as $(\alpha_1, \ldots, \alpha_n)$.

1.3. Remarks on the invariants of the groups (C) and (D)

Interpreting x_1, x_2, x_3 as homogeneous coordinates of the projective plane, the triangle whose sides are $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ is transformed into itself by the operators of (C) and (D); in other words, $x_1x_2x_3$ is an invariant of these groups.

Assuming the existence of other invariant triangles, say

$$(1) \qquad (a_1x_1 + a_2x_2 + a_3x_3)(b_1x_1 + b_2x_2 + b_3x_3)(c_1x_1 + c_2x_2 + c_3x_3) = 0$$

We operate successively by the transformations of (A) and by T. Observing that α, β, γ cannot all be equal for every transformation of (A), as otherwise (C) would be intransitive, we find by examining the various possibilities that (1) could not be distinct from $x_1x_2x_3 = 0$ unless (A) is the particular group generated by the transformations

$$S_1 = (1, \omega, \omega^2), \quad S_2 = (\omega, \omega, \omega); \quad \omega = e^{(2\pi i)/3}.$$

There are then four invariant triangles for (C), namely:

$$x_1x_2x_3 = 0;$$

$$(2) \qquad (x_1 + x_2 + \theta x_3)(x_1 + \omega x_2 + \omega^2\theta x_3)(x_1 + \omega^2 x_2 + \omega\theta x_3) = 0$$

$$(\theta = 1, \omega, \text{ or } \omega^2).$$

In the case of (D), these four triangles will be invariant if the group is generated by S_1, S_2, T and R, the latter now having the form

$$R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

either directly or after multiplication by suitable powers of S_1 and S_2.

Lemma 2. A linear group G having a normal abelian subgroup H whose elements are not all scalar matrices is either intransitive or imprimitive.

Proof. We refer to Lemma of [Bl1, p.79].

Theorem 3. A linear group whose order is the power of a prime number can be written as a monomial group by a suitable choice of variables $x_1, \ldots, x_n$; that is, its transformations have the form:

$$x_s = a_{st} x'_t$$

where s and t run through the numbers $1, 2, \ldots, n$ though not necessarily in the same order.

Proof. We refer to Theorem 2 of [Bl1, p.80].

Corollary 4. A linear group in n variables whose order is the power of a prime greater than n is abelian.

Proof. We refer to Theorem 2 of [Bl1, p.81].

Theorem 5. A reducible group G is intransitive, and a reduced set becomes a set of intransitivity.

Proof. We refer to Theorem 6 of [Mi-Bl-Di, p.211].

1.4. Groups having normal intransitive subgroups

A group having a normal subgroup of type (A) whose elements are not all scalars is intransitive or imprimitive (by Lemma 2), and a group having a normal subgroup of type (B) is intransitive. For let V be any transformation of such group, and T any transformation of (B). Then $VTV^{-1} = T_1$ belong to (B), and if we put $(x_1)T_1 = \alpha x_1$, $(x_1)V = y$, we have $(y)T = (y)V^{-1}T_1V = \alpha(x_1)V = \alpha y$. This show that $y = 0$ is an invariant straight line for (B). But $x_1 = 0$ is the only such line, and therefore $y = (x_1)V = kx_1$, $k =$ constant. The group in question is therefore reducible and hence intransitive (by Theorem 5).

For groups having normal subgroup of type (A) whose elements are all scalars only, that is the center of $SL(3, \mathbb{C})$, we will discuss in section 1.7.

1.5. Primitive groups having normal imprimitive subgroups

We now consider a group G containing a normal subgroup of type (C) or (D). These types leave invariant triangle $x_1x_2x_3 = 0$, and if this is the only one, we could prove (see section 1.4) that G would also transform this triangle into itself. But then G would not be primitive. We therefore assume that there are four invariant triangles for (C) and (D), permuted among themselves by the transformations of G. Let us denote the triangles by t_1, t_2, t_3, t_4, in order as they are listed in (2).

We now associate with each transformation of G a permutation on the letters t_1, t_2, t_3, t_4, indicating the manner in which the transformation permutes the corresponding triangles. We thus obtain a permutation group K on four letters which is homomorphic to G, and the normal subgroup (C) or (D) corresponds to the identity of K. None of the four letters could be left unchanged by every permutation of K. For the corresponding triangle would be invariant under G; and bringing this triangle into form $x_1x_2x_3 = 0$ by a suitable choice of new variables, G would not be primitive. Moreover, no transformation can interchange two of the triangles and leave the other two fixed, as may be verified directly.

Under these conditions we find the following possible forms for K:

(E$'$) id, $(t_1t_2)(t_3t_4)$; (Other two types { id, $(t_1t_3)(t_2t_4)$}, { id $(t_1t_4)(t_2t_3)$} are conjugate to each other.)

(F$'$) id, $(t_1t_2)(t_3t_4), (t_1t_4)(t_2t_3), (t_1t_3)(t_2t_4)$;

(G$'$) the alternating group on four letters, generated by $(t_1t_2)(t_3t_4)$ and $(t_2t_3t_4)$.

We note that if a given transformation V permutes the triangles in a certain manner, then any transformation V' which permutes them in the same manner can be written in the form $V' = XV$, $X \in$ (D). For $V'V^{-1}$ must leave fixed each triangle.

By direct application we verify that the transformations U, V, UVU^{-1} permute the triangles in the following manner: $(t_2t_3t_4), (t_1t_2)(t_3t_4), (t_1t_4)(t_2t_3)$, where

$$
(3) \qquad \begin{aligned}
U &= \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon\omega \end{pmatrix}, \qquad \epsilon^3 = \omega^2, \\
V &= \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad UVU^{-1} = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & \omega^2 \\ 1 & \omega & \omega \\ \omega & 1 & \omega \end{pmatrix}.
\end{aligned}
$$

Accordingly, since all the groups required contain a transformation corresponding to $(t_1t_2)(t_3t_4)$, every such group must contain a transformation XV, X belonging to (D). Hence, if G contains (D) as a subgroup, it also contains V. If, however, (C) were a subgroup of G, but not (D), then either V is contained in G, or else XV, where X is a transformation contained in (D) but not in (C). In this event X may be written X_1R, where X_1 belong to (C). Hence, finally, either V or RV belongs to G. However, $V^2 = (RV)^2 = R$. Thus R, and therefore also V, are contained in G in any case.

Again, if G contains a transformation corresponding to $(t_2t_3t_4)$ or $(t_1t_4)(t_2t_3)$, such a transformation can be written XU or $XUVU^{-1}$, X belonging to (D). Hence, since G contains (D) as we have just seen, it will contain either U or UVU^{-1} in the cases considered. We therefore have the following types:

(E) Group of order 108 generated by S_1, T of (C) and V of (3).

(F) Group of order 216 generated by S_1, T, V and UVU^{-1} of (3).

(G) Group of order 648 generated by S_1, T, V and U of (3).

These groups are all primitive, and they all contain (D) as a normal subgroup (in fact (C) $\triangleleft$ (D) $\triangleleft$ (E) $\triangleleft$ (F) $\triangleleft$ (G)). The group (G) is called the Hessian group.

1.6. Primitive groups which are simple

Theorem 6. No prime $p > 7$ can divide the order of a primitive linear group G in $SL(3,\mathbb{C})$.

Proof. The process consists in showing that, if the order g of G contains a prime factor $p > 7$, then G is not primitive. We subdivide this process into four parts as follows: 1° proving the existence of an equation $F = 0$, where F is a certain sum of roots of unity; 2° giving a method for transforming such an equation into a congruence (mod p); 3° applying this method to the equation $F = 0$; 4° deriving an abelian normal subgroups P of order p^k.

1° G contains an element S of order of p. We choose such variables that S has the diagonal form

$$S = (\alpha_1, \alpha_2, \alpha_3); \quad \alpha_1^p = \alpha_2^p = \alpha_3^p = 1, \quad \alpha_1\alpha_2\alpha_3 = 1.$$

Two cases arise: two of them are equal, say $\alpha_1 = \alpha_2$, or they are all distinct. They cannot all be equal, since $\alpha_1^p = 1$ and $\alpha_1^3 = 1$ imply $\alpha_1 = 1$ whereas S is not the identity. Of the two cases we shall treat the latter only: the method would be the same in the former case (the congruence (10) would here be of the first degree in μ), and the result as stated in Theorem 6 would be the same.

Let V be any element of order p in G:

$$V = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}.$$

We form the products VS, VS^2, VS^μ. Their characteristics (i.e. sum of the eigenvalues) and that of V will be denoted by $[VS], [VS^2], [VS^\mu]$, and $[V]$ respectively and we have

$$\begin{aligned} &[V] = a_1 + b_2 + c_3, \\ &[VS] = a_1\alpha_1 + b_2\alpha_2 + c_3\alpha_3, \\ &[VS^2] = a_1\alpha_1^2 + b_2\alpha_2^2 + c_3\alpha_3^2, \\ &[VS^\mu] = a_1\alpha_1^\mu + b_2\alpha_2^\mu + c_3\alpha_3^\mu \end{aligned} \tag{4}$$

and

$$
\begin{vmatrix} [V] & 1 & 1 & 1 \\ [VS] & \alpha_1 & \alpha_2 & \alpha_3 \\ [VS^2] & \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \\ [VS^\mu] & \alpha_1^\mu & \alpha_2^\mu & \alpha_3^\mu \end{vmatrix} = 0. \tag{5}
$$

Expansion and division by $(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1)$ gives us

$$
[VS^\mu] + K[V] + L[VS] + M[VS^2] = 0, \tag{6}
$$

K, L, M being certain polynomials in $\alpha_1, \alpha_2, \alpha_3$, with the general term of the type $\alpha_1^1 \alpha_2^b \alpha_3^c$. Since $\alpha_1, \alpha_2, \alpha_3$ are power of a primitive pth root of unity α, the quantities K, L, M are certain sums of posers of α. Moreover, the characteristics $[V], [VS], [VS^2], [VS^\mu]$ are each the sum of three roots of unity. Therefore (6) is an equation of finite sums of roots of unity. By the Kronecker's Theorem (see [Mi-Bl-Di, p.240]), (6) can be written as the form

$$
\begin{aligned} a(1 + \alpha + \alpha^2 + \cdots + \alpha^{p-1}) + B(1 + \beta + \beta^2 + \cdots + \beta^{q-1}) \\ + C(1 + \gamma + \gamma^2 + \cdots + \gamma^{r-1}) + \cdots = 0, \end{aligned} \tag{7}
$$

$A, B, C, \ldots$ being certain sums of roots of unity; $\alpha, \beta, \gamma, \ldots$ primitive roots of unity of the equations $x^p = 1$, $x^q = 1$, $x^r = 1, \ldots$ respectively; and $p, q, r, \ldots$ different prime numbers.

The coefficients $A, B, C, \ldots$ may be put into certain standard forms. Thus, any root of unity $\epsilon \neq 1$ occurring in any of these sums will be assumed to be resolved into factors of prime-power orders: $\epsilon = \epsilon_p \epsilon_q \epsilon_r \cdots$, the root of unity ϵ_p being of order p^m, ϵ_1 of order q^n, etc.

To illustrate, let i be a root of unity of order 4 and τ a root of unity of order 9 (i.e., $\tau^3 = \omega$, ω has order 3), and let $p = 2$, $q = 3$. Then the standard form for the expression

$$
(\omega - 1)(1 - 1) + (r^2\omega - \omega^2 + i)(1 + \omega + \omega^2) \tag{8}
$$

would be

$$[\omega+(-1)][1+(-1)]+[r^5+(-1)\omega^2+i](1+\omega+\omega^2).$$

2° We shall now make certain changes in the values of the roots of unity in the equation (7). First we put 0 for every root of unity $\epsilon_p, \epsilon_q, \epsilon_r, \ldots$ whose order is divisible by the square of a prime, leaving undisturbed the roots of unity whose orders are not divisible by such a square, as $\alpha, \alpha^2, \ldots, \beta, \ldots$ the quantities $A, B, \ldots$ are thereby changed into certain sums $A', B', \ldots$. The equation (7) is still true, the vanishing sums $1+\alpha+\alpha^2+\cdots+\alpha^{p-1}$, etc., not having been affected.

Next we put 0 in place of $q-2$ of the roots of unity $\beta, \beta^2, \ldots, \beta^{q-1}$, and -1 for the remaining root of unity, thus changing $B'(1+\beta+\beta^2+\cdots+\beta^{q-1})$ into $B'(1+0+0+\cdots+(-1))$, for example, so that this product still remains equal to zero. Similarly, we put 0 in place of $r-2$ of the roots of unity $\gamma, \gamma^2, \ldots, \gamma^{r-1}$, and -1 for the remaining root of unity, and so on. Proceeding thus, we shall ultimately change (7) into an equation of the form

$$A''(1+\alpha+\alpha^2+\cdots+\alpha^{p-1})=0,$$

where A'' is a polynomial in α.

Finally, we put 1 in the place of every root of unity $\alpha, \alpha^2, \ldots, \alpha^{p-1}$. The left-hand member may then no longer vanish, but will in any event become a multiple of p.

The final value of the expression (8) would be $(\omega+1)(1+1)=2$ or 0, according as ω is replaced by 0 or -1.

Notation 1. Any expression N which is a sum of roots of unity, changed in the manner described above, shall denoted by N_p'.

3° We shall now study the effect of these changed upon the left-hand member of (6). Each of the characteristics $[VS], [VS^2], [VS^\mu]$, being the sum of three (unknown) roots of unity, will finally become one of the seven numbers $0, \pm1, \pm2, \pm3$, whereas $[V]$, being the sum of three roots of unity of order 1 or p (cf. 1°), will become 3. The left-hand member of (6) will thus take the form

$$(9) \qquad [VS^\mu]'_p + 3K'_p + L'_p[VS]'_p + M'_p[VS^2]'_p,$$

and this number is a multiple of p (by 2°).

By (5),

$$K = \frac{-1}{(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)} \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \\ \alpha_1^\mu & \alpha_2^\mu & \alpha_3^\mu \end{vmatrix},$$

$$L = \frac{1}{(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)} \begin{vmatrix} 1 & 1 & 1 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \\ \alpha_1^\mu & \alpha_2^\mu & \alpha_3^\mu \end{vmatrix},$$

$$M = \frac{-1}{(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)} \begin{vmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^\mu & \alpha_2^\mu & \alpha_3^\mu \end{vmatrix}.$$

The values K'_p, L'_p, M'_p may be obtained by treating them as indeterminates $\frac{0}{0}$. Thus, by l'Hôpital's Rule we find

$$K'_p = -\frac{1}{2}(\mu-1)(\mu-2), \quad L'_p = \mu(\mu-2), \quad M'_p = -\frac{1}{2}\mu(\mu-1),$$

and if we substitute in (9) and multiple by $p-1$ we obtain the congruence:

$$(10) \qquad [VS^\mu]'_p \equiv s\mu^2 + t\mu + v (\mathrm{mod} p),$$

s, t, v being certain integers, the same for all values of μ.

We finally substitute in succession $\mu = 0, 1, 2, \ldots, p-1$ in the right-hand member of (10). The remainders (mod p) should all lie between -3 and $+3$ inclusive, the interval of the value of $[VS^\mu]'_p$. Now, each of these seven remainders can correspond to at most two different values of μ less than p, if s and t are not both $\equiv 0$ (mod p), by the theory of such congruences. Hence, there will correspond to the seven remainders at most 14 different values of μ, so that p is not greater than 14 unless $s \equiv t \equiv 0$. Trying $p = 13$ and $p = 11$, choosing for s, t, v the different possible sets of number $< p$ (the problem can be simplified by special devices)[1], we find that not in no case the remainders all be contained in the sets $0, \pm 1, \pm 2, \pm 3$, unless $s \equiv t \equiv 0$. We have shown $s \equiv t \equiv 0 \bmod p$ for $p > 7$. Therefore, we get

$$[VS^{\mu}]'_p \equiv v(\text{mod} p).$$

In particular,

$$[VS]'_p \equiv v \equiv [V]'_p = 3(\text{mod} p),$$

from which it follows that $[VS]'_p = 3$. Again, from this equation we deduce that the roots of unity of $[VS]$ are of order 1 or p. For, if the order of one of these roots of unity were divisible by the square of a prime, or by a prime different from p, then the changes indicated in 2° could be made at the outset in such a way that 0 or -1 would take the place of this root of unity. But then $[VS]'_p$ would be one of the numbers $0, \pm 1, \pm 2, -3$.

4° Accordingly, the product VS of any two transformation both of order p is a transformation of order p or 1. The totality of such transformations in G, together with I (identity matrix), will therefore form a group P. The order of this group must be a power of p, since it contains no transformations whose order differs from p and 1. Moreover, P is normal in G, since the conjugate elements of order p has order p. Hence, G has a normal subgroup P of order p^k. But this subgroup is abelian (by Corollary 4), and therefore G is intransitive or imprimitive (by Lemma 2). Q.E.D.

Notation 2. A quantity N, which is the sum of a certain number of roots of unity, in which every root of unity ϵ_p is replaced by 1, but in which none of the other changes indicated in 2° of Theorem 6 are carried out, will denoted by N_p. If

---

[1] Since $a\mu + b$ runs through the p values $0, 1, 2, \ldots, p-1$ (mod p) when μ does, we may substitute this expression for μ in the right-hand member of (10) and select constants a, b so that this member takes a simple form. For instance, if $p = 11$, the right-hand member of (10) may be reduced by this substitution to one of the forms $\pm\mu^2 + c$; μ; c; according as $s \not\equiv 0$; $s \equiv 0$, $t \not\equiv 0$; $s \equiv t \equiv 0$ (mod p). When $p = 13$ we get the forms $\pm\mu^2 + c, \pm 2\mu^2 + c$; μ; c.

$N=0$, then $N_p \equiv 0 \pmod p$.

For example, let

$$N=[\omega+(-1)][1+(-1)]+[\tau^5+(-1)\omega^2+i](1+\omega+\omega^2)=0$$

in (8), then

$$N_p=(\omega+1)(1+1)+(\tau^5+\omega^2+1)(1+\omega+\omega^2)=2(\omega+1)\equiv 0\,(\text{mod}2).$$

Notation 3. Sometimes, we write the order of linear group G as $g\phi$ where $\phi=3$ or 1 according to G containing the center of $SL(3,\mathbb{C})$ or not respectively.

Theorem 7. If a group G contains a transformation S of order p^2 $(p \neq 3)$ or $p^2\phi(p=3)$, p being a prime >2, then there is a normal subgroup H_p in G (not excluding the possibility $G=H_p$) which contains S^p. Any transformation in H_p, say T, has the properly expressed by the following congruence:

$$(11) \qquad [V]_p \equiv [VT]_p\,(\text{mod}p),$$

V being any transformation of G.

In the case $p=2$ the group G contains a normal subgroup H_p if the order of S is p^3, and S^{p^2} will belong to H_p; also if $S=(-1,i,i)$, in which case S^2 will belong to H_p.

Proof. If $p>2$, we write S in diagonal form, and construct the products VS, VS^2, VR, where R denotes S^p. Assuming that the three entries in the diagonal of S are all distinct, we obtain an equation corresponding to (6):

$$[VR]+K[V]+L[VS]+M[VS^2]=0.$$

By 3° of Theorem 6, we have $L_p \equiv M_p \equiv 0 \pmod p$ and $K_p \equiv -1 \pmod p$. So

$$[VR]_p-[V]_p \equiv 0(\text{mod}p).$$

Now consider all the conjugates $R_1,\ldots,R_h$ to R within G. They generate a normal subgroup H_p, and $[VR_i]_p \equiv [V]_p \pmod p$ for $i=1,2,\ldots,h$, since $R_i=$

$A^{-1}RA = A^{-1}S^pA = (A^{-1}SA)^p$ and $A^{-1}SA$ and S have the same order, for some $A \in G$. Moreover, any transformation T in H_p satisfies the congruence (11), since such a transformation can be written as a product of powers of $R_1, \ldots, R_h$. For instance, let $T = R_1R_2$, and we have

$$[VR_1]_p \equiv [V]_p, \quad [(VR_1)R_2]_p \equiv [(VR_1)]_p \,(\mathrm{mod} p).$$

Hence,

$$[VT]_p = [VR_1R_2]_p \equiv [VR_1]_p \equiv [V]_p.$$

Finally, in the case the order of S is 8 $(= 2^3)$, we construct $[V], [VS], [VS^2]$ and $[VS^4]$. For $S = (-1, i, i)$, we construct

$$\begin{vmatrix} [V] & 1 & 1 \\ [VS] & -1 & i \\ [VS^2] & 1 & -1 \end{vmatrix}.$$

Q.E.D.

1.6.1. The normal group H_p

The order of this group is a power of p. For if its order contained a prime factor q, $q \neq p$, there would be a transformation of order q in H_p, say T. Then, by (11), we have

$$[T^j]_p = [IT^j]_p \equiv [I]_p = 3 \,(\mathrm{mod} p).$$

Hence, let α, β, γ be entries of diagonal of T (after choose a suitable variables making T a diagonal matrix), we have $\alpha^j + \beta^j + \gamma^j \equiv 3 \pmod p$, and therefore

$$\sum_{j=1}^{q} \alpha^{-j}(\alpha^j + \beta^j + \gamma^j) \equiv 3 \sum_{j=1}^{q} \alpha^{-1}. \tag{12}$$

We may assume α, β, γ are not all equal (otherwise $p \neq 3$ and $3\alpha \equiv 3 \pmod p$, these imply $\alpha = 1$ and $q = 1$). Hence, the left-hand sum is

$$\begin{cases} 2q, & \text{if } \alpha \in \{\beta, \gamma\}; \\ q, & \text{if } \alpha \notin \{\beta, \gamma\}. \end{cases}$$

for

$$\sum_{j=1}^{q} \alpha^{-j}\beta^{j} = \begin{cases} 0, & \text{if } \alpha \neq \beta; \\ q, & \text{if } \alpha = \beta. \end{cases}$$

The right-hand sum is

$$\begin{cases} 3q, & \text{if } \alpha = 1; \\ 0, & \text{if } \alpha \neq 1. \end{cases}$$

It follows that the congruence is impossible except when $p = 2$, and $\{\alpha \in \{\beta, \gamma\}$ and $\alpha \neq 1\}$ or $\{\alpha \notin \{\beta, \gamma\}$ and $\alpha = 1\}$. Substituting β^{-j} and γ^{-j} for α^{-j} in (12) we get similar results. Collecting these, we have $p = 2$ and $\{\alpha = \beta$ and $\gamma = 1$ and $\alpha \neq 1\}$. Since $\alpha\beta\gamma = 1$, we have $\alpha^2 = 1$ and $\alpha^q = 1$, then $\alpha = 1$, a contradiction.

The order of H_p is accordingly a power of p, and the group is monomial (by Theorem 3). The possibility $G = H_p$ is accordingly untenable if G is primitive.

Corollary 8. No primitive simple group can contain a transformation S of order p^2, if $p > 3$; or $p^2\phi$, if $p = 3$; or p^3, if $p = 2$; or p^2 if $S = (-1, i, i)$ and $p = 2$.

Proof. This follows from the Theorem 7 and above argument. Q.E.D.

Theorem 9. No primitive simple group can contain a transformation S of prime order p, $p > 3$, which has at most two distinct multipliers.

Proof. Let $S = (\alpha_1, \alpha_1, \alpha_2)$, $\alpha_1 \neq \alpha_2$ and assume first that $p = 7$ (Note $p > 7$ can not happen, by Theorem 6). This transformation leaves invariant a point ($x_1 = x_2 = 0$) and every straight line $L = \{(x_1, x_2, x_3) \,|\, ax_1 + bx_2 = 0\}$ through it. This will also be the case with any other transformation S' conjugate to S (for they have the same eigenvalues). Therefore, the line joining the two invariant points is invariant for both S and S'. If now the variables be changed so that the common invariants line is $y_1 = 0$, the group generated by S and S' will be reducible and therefore intransitive, by Theorem 5, breaking up into a group in one variable (y_1) and one in two variables (y_2, y_3). But, there being no primitive or imprimitive linear groups in two variables generated by two transformations of order 7, it follows that$S$ and S' are commutative.

Accordingly, all the conjugates to S are mutually commutative and generate an abelian group, which must be normal in G, and the latter cannot be primitive by Lemma 2.

Next, let $p = 5$. If S and S' are not commutative, they generate the icosahedral group, in the variables y_1, y_2. This contains a transformation of order 3 whose multipliers are ω, ω^2, and a scalar matrix whose multipliers are $-1, -1$. The product of these two transformations, as a transformation in the variables y_1, y_2, y_3, can be written in the canonical form $T = (1, -\omega, \omega^2)$. But such a transformation is excluded by the next theorem (put $S_1 = T^2 = (1, \omega^2, \omega)$ and $S_2 = T^3 = (1, -1, -1)$).

Q.E.D.

Theorem 10. No primitive simple group can contain a transformation S of order pq, where p and q are different prime numbers, and $S_1 = S^p$ has three distinct multipliers, while $S_2 = S^q$ has at least two.

Proof. Let S be written in diagonal form and assume $S_1 = (\alpha_1, \alpha_2, \alpha_3)$ and $S_2 = (\beta_1, \beta_2, \beta_3)$ (Note α_i are qth roots of unity while β_i are pth roots of unity). As in proof of Theorem 6, we get the equation

$$\begin{vmatrix} [V] & 1 & 1 & 1 \\ [VS_2] & \beta_1 & \beta_2 & \beta_3 \\ [VS_1] & \alpha_1 & \alpha_2 & \alpha_3 \\ [VS_1^2] & \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{vmatrix} = 0,$$

then

$$[V] \begin{vmatrix} \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{vmatrix} - [VS_2] \begin{vmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{vmatrix}$$
$$[VS_1] \begin{vmatrix} 1 & 1 & 1 \\ \beta_1 & \beta_2 & \beta_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{vmatrix} - [VS_1^2] \begin{vmatrix} 1 & 1 & 1 \\ \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix} = 0.$$

After putting unity for every root of unity of order p^k, the equation becomes the congruence:

$$\{[V]_p - [VS_2]_p\}(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1) \equiv 0 \,(\mathrm{mod} p)$$

which can be changed into the following:

$$\{[V]_p - [VS_2]_p\}q^3 \equiv 0\,(\mathrm{mod} p)$$

after multiplying by a suitable factor, since

$$(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)(\alpha_3-\alpha_1) = \alpha_1\alpha_2\alpha_3(1-\frac{\alpha_2}{\alpha_1})(1-\frac{\alpha_3}{\alpha_2})(1-\frac{\alpha_1}{\alpha_3}),$$

$\alpha_1\alpha_2\alpha_3 = 1$ and

$$(1-\epsilon)(1-\epsilon^2)\cdots(1-\epsilon^{q-1}) = \lim_{x\to 1}\frac{x^{q-1}}{x-1} = q,$$

where ϵ is a primitive qth root of unity.

Hence finally, $[VS_2]_p \equiv [V]_p$ (mod p), and the argument of Theorem 7 shows that G has an imprimitive normal subgroup H_p containing S_2. Q.E.D.

Corollary 11. No primitive simple group can contain a transformation of order $35, 15\phi, 21\phi, 10$ or 14 (If S_1, representing respectively $S^7, S^{3\phi}, S^{3\phi}, S^2$, or S^2, has not three distinct multipliers, Theorem 9 applies). In fact, if primitive group G contains a transformation of order $35, 15\phi, 21\phi, 10$ or 14 then G contains an imprimitive normal subgroup.

1.6.2. The Sylow subgroups

Theorem 12. If the group G contains a transformation S of order 5 and one T or order 7; i.e., if the order of G is divisible by 35, then G will contain a transformation of order 35.

Proof. We refer to Theorem 6 of [Bl2, p.565].

Theorem 13.

(a) A group of order $3^4\phi$ must contain a transformation of order $3^2\phi$.

(b) A group of order 2^4 must contain a transformation of order 2^3, or one of order 2^2 having the form $(-1, i, i)$.

Proof.

(a) We refer to Exercise 4 of [Mi-Bl-Di, p.233].

(b) We refer to Exercise 3 of [Mi-Bl-Di, p.232].

Theorem 14. If a linear group G has a transformation S of order 5 (or 7) and a transformation T whose multiplies are $\epsilon, \epsilon, \epsilon\omega^2$, where $\epsilon^3 = \omega$, then it has one of order 45 ($= 9 \cdot 5$) (or 63 ($= 9 \cdot 7$)).

Proof. We refer to Theorem 7 of [Bl2, p.566].

Consider now a primitive simple group G of order g. If G has a subgroup H of order 5^2 or 7^2, then H is abelian. If H is cyclic, then H has a transformation of order 5^2 or 7^2 violating Corollary 8. If H is not cyclic, then H has a transformation of order 5 or 7, which has at most two distinct multipliers violating Theorem 9. Again, if g is divisible by 35, we have a transformation of this order by Theorem 12 violating Corollary 11. Thus g is a factor of 5 or 7 but not both.

Next if G has a subgroup K of order $3^2\phi$ ($\phi = 1$), then K is abelian. If K is cyclic, then violating Corollary 8. If K is not cyclic then it must contain 8 elements of order 3. When we attempt to construct group K, we find that K will have a normal subgroup consisting the center of $SL(3, \mathbb{C})$.

Finally, G can not have a subgroup of order 2^4, otherwise violating Corollary 8 by Theorem 13(b).

Collecting our results, we find that g is a factor of one of the numbers

$$2^3 \cdot 3 \cdot 5\,(= 120), \quad 2^3 \cdot 3 \cdot 7\,(= 168).$$

Now, all simple groups whose orders do not exceed the largest of these numbers have been listed. There are two possibilities:

$$g = 60, \quad 168.$$

(H) Group of order 60 is isomorphic to the alternating group A_5. It generated by

$$S = (12345) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \epsilon^4 & 0 \\ 0 & 0 & \epsilon \end{pmatrix}, \quad U = (14)(23) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$T = (12)(34) = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 \\ 2 & s & t \\ 2 & t & s \end{pmatrix},$$

where $\epsilon = e^{(2\pi i)/5}$, $s = \epsilon^2 + \epsilon^3 = \frac{1}{2}(-1-\sqrt{5})$, $t = \epsilon + \epsilon^4 = \frac{1}{2}(-1+\sqrt{5})$.

(I) Group of order 168 is isomorphic to the permutation group generated by $(1234567), (142)(356)$, and $(12)(35)$. It generated by

$$S = (1234567) = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta^2 & 0 \\ 0 & 0 & \beta^4 \end{pmatrix}, \quad T = (142)(356) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$R = (12)(35) = \frac{1}{\sqrt{-7}} \begin{pmatrix} \beta^4 - \beta^3 & \beta^2 - \beta^5 & \beta - \beta^6 \\ \beta^2 - \beta^5 & \beta - \beta^6 & \beta^4 - \beta^3 \\ \beta - \beta^6 & \beta^4 - \beta^3 & \beta^2 - \beta^5 \end{pmatrix},$$

where $\beta = e^{(2\pi i)/7}$.

Blichfeldt [Bl2, p.571] incorrectly gave $T = (124)(365)$ instead of $(142)(356)$, for the stabilizer G_v of v in (I), where

$$v = (3\beta^5 - 2\beta^4 - 2\beta^3 + 3\beta^2 - 3, -3\beta^4 - 3\beta^3 - 5, -3\beta^5 + 5\beta^4 + 5\beta^3 - 3\beta^2 + 8) \in \mathbb{C}^3,$$

is $\{I, R, S^2TRS^2, S^5TRS^5\}$ which corresponding to:

$$G_v = \begin{cases} \{\mathrm{id}, (12)(35), (245)(376), (254)(367)\}, \text{if}\, T = (124)(365); \\ \{\mathrm{id}, (12)(35), (1523)(47), (1325)(47)\}, \text{if}\, T = (142)(356), \end{cases}$$

the former is not a subgroup.

1.7. Primitive groups having normal intransitive subgroups (continued)

Now we consider the primitive group G of order $g_1 = g\phi$ have the center $F = \{I, W, W^2\}$ of $SL(3,\mathbb{C})$ as a normal subgroup only, where $\phi = 3$ and g is the order of quotient group G/F. The quotient group G/F (existing as an abstract group) must be simple. As the same argument in section 1.6.2, we have the same restrictions for factors $5, 7$, and 2 of g. Next by Theorem 13(a), if G has a subgroup of order $3^4\phi$, then it must contain a transformation of order $3^2\phi$ violating Corollary 8. If now G has a subgroup of order $3^3\phi$ ($\phi = 3$), then it has a subgroup K of order $3^2\phi$ which is abelian. If K is cyclic, then violating Corollary 8 again. If K is not cyclic, then it must contain a transformation of type $T = (\epsilon, \epsilon, \epsilon\omega^2)$, where $\epsilon^3 = \omega$. Then g_1 is divisible by 5 or 7 at the same time, we would have a transformation of order 45 $(= 15 \cdot 3 = 15\phi)$ or 63 $(= 21 \cdot 3 = 2\phi)$ by Theorem 14 violating Corollary 11.

Collecting out results, we find that g is a factor of one of the numbers

$$2^3 \cdot 3^3\,(= 216), \quad 2^3 \cdot 3^2 \cdot 5\,(= 360), \quad 2^3 \cdot 3^2 \cdot 7\,(= 504).$$

There are four possibilities:

$$g = 60, \quad 168, \quad 360, \quad 504.$$

There can be no group in three variables isomorphic with the simple group of order 504. For, this has an abelian subgroup of order 8, formed of 7 distinct transformations of order 2 and the identity (see [Co1, p.312]). Attempting to write this subgroup in diagonal form, we find it impossible as a group in three variables.

(J) Group of order 180 $(= 3 \cdot 60)$ generated by F and the group (H).

(K) Group of order 504 $(= 3 \cdot 168)$ generated by F and the group (I).

(L) Group G of order 1080 $(= 3 \cdot 360)$ generated by S, U, T of (H) and

$$V = \frac{1}{\sqrt{5}} \begin{vmatrix} 1 & \lambda_1 & \lambda_1 \\ 2\lambda_2 & s & t \\ 2\lambda_2 & t & s \end{vmatrix},$$

where $\lambda_1 = \dfrac{1}{4}(-1 + \sqrt{15}i)$, $\lambda_2 = \dfrac{1}{4}(-1 - \sqrt{15}i)$, s and t are the same in the entries of T.

As elements in $G/F \simeq A{-}6$, we have $\overline{S} = (12345)$, $\overline{U} = (14)(23)$, $\overline{T} = (12)(34)$, $\overline{V} = (14)(56)$.

1.8. Primitive groups having normal primitive subgroups

A possible subgroup H_p is monomial (see section 1.6.1). We have already determined the primitive groups containing a normal subgroup of this type; it is therefore unnecessary to consider the groups containing the subgroup H_p. Hence, the orders of the groups still to be examined must be factor of the numbers

$$2^3 \cdot 3^3\phi, \quad 2^3 \cdot 3^2 \cdot 5\phi, \quad 2^3 \cdot 3^2 \cdot 7\phi \ (\text{all}\phi = 3),$$

as we have seen.

The group (L) has already a maximum order. If the primitive group has a normal subgroup (J) or (K), then it contains the center of $SL(3, \mathbb{C})$ as a normal intransitive subgroup only. This case has discussed in section 1.7. A primitive group G containing normal subgroup (H) must be of order $2^{2+a} \cdot 3^{1+b} \cdot 5$. Now, (H) has exactly 10 subgroups of order 3, say $\{H_1, \ldots, H_{10}\}$. Since (H) is a normal subgroup of G, G acts on $\{H_1, \ldots, H_{10}\}$ by conjugation. It is easy to check that (H) acts on $\{H_1, \ldots, H_{10}\}$ transitively. Let $G_1 = \{T \in G \big| THT_1T^{-1} = H_1\}$ then $|G/G_1| = 10$. Hence $|G_1| = |G|/10 = 2^{1+a} \cdot 3^{1+b}$. Choose $H_1 = \{\text{id}, (123), (132)\}$ as elements in (H) $\simeq A_5$. G_1 acts on H_1 by conjugation. Observe that $(23)(45) \in$ (H) is in G_1 which sends (123) to (132). Thus the orbit of G_1 containing (123) is precisely $\{(123), (132)\}$. The stabilizer of $V = (123)$ has order $|G_1|/2 = 2^a \cdot 3^{1+b}$. Therefore, if $a > 0$, G contains a transformation of order 2 commutative with V or order 3. But this would imply a group H_p by Theorem 10. Again (H) has exactly 6 subgroups of order 5, say $\{K_1, \ldots, K_6\}$. Since (H) is a normal subgroup of G, G acts on $\{K_1, \ldots, K_6\}$ by conjugation. It is easy to check that (H) acts on $\{K_1, \ldots, K_6\}$ transitively. Let $G_1 = \{T \in G \big| TK_1T^{-1} = K_1\}$ then $|G/G_1| = 6$. Hence $|G_1| = |G|/6 = 2^{1+a} \cdot 3^b \cdot 5$. G_1 acts on K_1 by conjugation. The cardinal number of the orbit of G_1 containing $V \neq I$ in K_1 is denoted by $|O_v|$. If $|O_v| \neq 3$, then the order of the stabilizer of V has factor 3^b. If $|O_v| = 3$ then there exists a $V' \neq I \in K_1$ such that $|O_{v'}| = 1$. Thus the order of the stabilizer of V' has factor 3^b too. Therefore, if $b > 0$, we would find in G a transformation S of order 3 commutative with a transformation in K_1 of order 5, which likewise would imply a group H_p by Theorem 10, if $S \notin F$, where F is the center of $SL(3, \mathbb{C})$. If $S \in F$, then $G =$ (J).

Similarly, a primitive group G having a normal subgroup (I) should be of order $2^3 \cdot 3^{1+b} \cdot 7$. Now, (I) has exactly 8 subgroups of order 7, say $\{H_1, \ldots, H_8\}$. Since (I) is a normal subgroup of G, G acts on $\{H_1, \ldots, H_8\}$ by conjugation. It is easy to check that (I) acts on $\{H_1, \ldots, H_8\}$ transitively. Let $G_1 = \{T \in G \big| TH_1T^{-1} = H_1\}$ then $|G/G_1| = 8$. Hence $|G_1| = |G|/8 = 3^{1+b} \cdot 7$. Choose $H_1 = \langle S = (1234567) \rangle$ as elements in (I) $\simeq \langle (1234567), (142)(256), (12)(35) \rangle$. G_1 acts on H_1 by conjugation. Observe that $(235)(476) = ((142)(356))^2 S^6 \in$ (I) is in G_1 which sends S to $S^2 = (1357246)$. Thus the orbit of G_1 containing S is precisely $\{S, S^2, S^4\}$. The stabilizer of S has order $|G_1|/3 = 3^b \cdot 7$. Therefore, if $b > 0$, we would find in G a transformation V of order 3 commutative with a transformation S of order 7. But this would imply a group H_p by Theorem 10, if $V \notin F$. If $V \in F$ then $G =$ (K).

There remain the groups (E), (F), (G). Now, any larger group would permutes among themselves the four triangles which form an invariant system for these groups. But this was just the condition under which the three given groups were determined. Hence no new types result.

CHAPTER 2

THE INVARIANT POLYNOMIALS AND THEIR RELATIONS OF LINEAR GROUPS OF $SL(3,\mathbb{C})$

Let G be a linear group (i.e. finite subgroup) of $GL(n,\mathbb{C})$ and $S = \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial ring. The action of G on S is defined to be

$$f(x_1, \ldots, x_n)T = f(\sum_{i=1}^{n} a_{1i}x_i, \ldots, \sum_{i=1}^{n} a_{ni}x_i),$$

where $f \in S$, $T = (a_{ij}) \in G$.

$f \in S$ is an invariant (polynomial) of G if

$$fT = f \quad \text{for all } T \in G.$$

Clearly, if f and g are invariants so are $f + g$ and fg; therefore, the invariants of G form a ring S^G (in fact S^G is a $\mathbb{C}$-algebra). Moreover, $\Sigma_{T \in G} fT \in S^G$. The first main problem of invariant theory [We, Ch. II, A] is to determine a set of polynomials $f_1, \ldots, f_k$ which generate S^G.

Let $\{f_1, \ldots, f_k\}$ be a set of minimal generators of S^G, i.e., a minimal set of invariants such that any invariant is a polynomial in $f_1, \ldots, f_k$, f_i are called the minimal generators of S^G, then we have substitution homomorphism of rings:

$$\phi : \mathbb{C}[y_1, \ldots, y_k] \rightarrow S$$

with $Im\,\phi = S^G$, where $\mathbb{C}[y_1, \ldots, y_k]$ is a polynomial ring and $\phi(F) = F(f_1, \ldots, f_k) \in S$ for all $F(y_1, \ldots, y_k) \in \mathbb{C}[y_1, \ldots, y_k]$. Let K be the kernel of ϕ, then K is an ideal of $\mathbb{C}[y_1, \ldots, y_k]$ and $S^G \simeq \mathbb{C}[y_1, \ldots, y_k]/K$. The minimal generators of K are called relations of S^G. The second main problem of invariant theory [We, Ch. II, C] is to determine generators for the ideal of relations on the $f's$.

2.1. Theorems

Theorem 15.(Noether) S^G has a set of generators consisting of not more than $\binom{g+n}{n}$ invariants, of degree not exceeding g, where g is the order of G. Furthermore, this set of generators can be obtained by taking the average over G of all monomials

$$x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$$

of total degree $\Sigma_{i=1}^{n} b_i \leq g$.

Proof. We refer to [No].

For a large g, we use the following theorem to find a set of generators of S^G.

Theorem 16. Let H be a subgroup of G and $\{f_1, \ldots, f_r\}$ be a set of minimal generators of S^H. Let $G = Ha_1 \cup Ha_2 \cup \cdots \cup Ha_s$, where $s = |G|/|H|$, $a_i \in G$, $i = 1, \ldots, s$. Then $(f_1^{d_1} \cdots f_r^{d_r})a_1 + \cdots + (f_1^{d_1} \cdot f_r^{d_r})a_s$, $\sum_{i=1}^{r} d_i(\deg f_i) \leq |G|$, form a set of generators of S^G.

Proof. Let $H = \{h_1, \ldots, h_m\}$ and given any monomial $g_1 = x_1^{b_1} \cdots x_n^{b_n}$ with deg $g_1 = \Sigma_{i=1}^{n} b_i \leq |G|$. Then $g_2 = \Sigma_{i=1}^{m} g_1 h_i \in S^H$ so $g_2 =$ polynomial in $f_1^{d_1} \cdots f_r^{d_r}$ with deg $g_2 = d_1(\deg f_1) + \cdots + d_r(\deg f_r) \leq |G|$ for deg $g_2 = \deg g_1 \leq |G|$. Now $\Sigma_{a \in G} g_1 a = \Sigma_{i=1}^{m} g_1 h_i a_1 + \cdots + \Sigma_{i=1}^{m} g_1 h_i a_s = g_2 a_1 + \cdots + g_2 a_s \in S^G$. Thus by Noether's theorem S^G is generated by $(f_1^{d_1} \cdots f_r^{d_r})a_1 + \cdots + (f_1^{d_1} \cdots f_r^{d_r})a_s$ with $\sum_{i=1}^{r} d_i(\deg f_i) \leq |G|$. Q.E.D.

Note. If H is a normal subgroup of G and $\{f_1, \ldots, f_r\}$ is a set of minimal generators of S^H, then $f_i a$ is a polynomial in $f_1, \ldots, f_r$ for any $a \in G$, $1 \leq i \leq r$. In fact, for any $h \in H$, $(f_i a)h = f_i(ah) = f_i(h'a) = f_i a$ for some $h' \in H$. For this case, the computations of S^G become simple.

In order to find the relations of S^G, we introduce the concept of basic invariants.

Theorem 17. There always exist n algebraically independent invariants of S^G, in other words, the transcendence degree trd S^G of S^G over $\mathbb{C}$ is n.

Proof. For any $f \in S = \mathbb{C}[x_1, \ldots, x_n]$ the polynomial $h(X) = \Pi_{T \in G}(X - fT) = X^g + a_{g-1}X^{g-1} + \cdots + a_0$ has coefficients a_i which are symmetric functions of

$\{fT \big| T \in G\}$ where $g = |G|$. Thus $a_i \in S^G$ by Cayley's theorem. Since $h(f) = 0$ this implies that S is an algebraic extension of S^G, so trd $S^G =$ trd $S = n$. Q.E.D.

Theorem 18. dim $S^G = n$.

Proof. S^G is an integral domain which is a finitely generated $\mathbb{C}$-algebra by Theorem 15. From the theorem of algebraic geometry (e.g., Hartshorne [Ha, Theorem 1.8A.(a), p.6]), the dimension (Krull dimension) of S^G is equal to trd S^G. Thus dim $S^G = n$ by Theorem 17. Q.E.D.

Theorem 19.(Molien) The number of linearly independent homogeneous invariants of G of degree d is the coefficient of λ^d in the expansion of

$$\phi(\lambda) = \frac{1}{g} \sum_{T \in G} \frac{1}{\det(I - \lambda T)},$$

where I is the identity of G. We call $\phi(\lambda)$ the Molien series of G.

Proof. We refer to [Mo].

Our aim is to find n algebraically independent homogeneous invariants $f_1, \ldots, f_n$ and $k \geq 0$ homogeneous invariants $g_1, \ldots, g_k$ such that S^G can be written as a direct sum

$$S^G = \mathbb{C}[f_1, \ldots, f_n] \oplus \mathbb{C}[f_1, \ldots, f_n]g_1 \oplus \cdots \oplus \mathbb{C}[f_1, \ldots, f_n]g_k. \tag{13}$$

Let d_i, b_i be the degrees of f_i, g_i, then (13) implies that the number of linearly independent homogeneous invariants of degree d is the number of solutions $(a_1, \ldots, a_n)$ of

$$\begin{aligned} a_1 d_1 + \cdots + a_n d_n + b_0 &= d, \\ a_1 d_1 + \cdots + a_n d_n + b_1 &= d, \\ &\vdots \\ a_1 d_1 + \cdots + a_n d_n + b_k &= d \end{aligned}$$

for which $a_i \geq 0$, $b_0 = 0$, i.e., the coefficient of λ^d in the expansion of

$$\frac{1 + \lambda^{b_1} + \cdots + \lambda^{b_k}}{(1 - \lambda^{d_1})(1 - \lambda^{d_2}) \cdots (1 - \lambda^{d_n})} \tag{14}$$

which must therefore be Molien series $\phi(\lambda)$.

For any $a \in S^G$, a can be written as $p_0(f_1, \ldots, f_n) + p_1(f_1, \ldots, f_n)g_1 + \cdots + p_k(f_1, \ldots, f_n)g_k$ where $p_i(f_1, \ldots, f_n)$ are polynomials in $f_1, \ldots, f_n$. This is called the basic form of a. The invariants $f_1, \ldots, f_n, g_1, \ldots, g_k$ are called basic invariants of S^G. (13) is called a basic decomposition of S^G and (14) is called a basic series of G. Of course, we want

$$\{f_1, \ldots, f_n\} \subseteq \{\text{minimal generators}\} \subseteq \{\text{basic invariants}\}.$$

The following theorem 20 is classical and well-known. See for example, [Ho-Ea, p.1039]. We shall present a simple proof by using Quillen's solution to Serre conjecture.

Theorem 20. For any linear group G of $GL(n, \mathbb{C})$, S^G has a basic decomposition.

Proof. Suppose that S^G is generated by homogeneous polynomials $f_1(x), \ldots, f_{n+k}(x)$ as a $\mathbb{C}$-algebra. Define

$$\phi : \mathbb{C}[y_1, \ldots, y_{n+k}] \to S = \mathbb{C}[x_1, \ldots, x_n]$$

by $\phi(F) = F(f_1, \ldots, f_{n+k}) \in S$ for all $F(y_1, \ldots, y_{n+k}) \in \mathbb{C}[y_1, \ldots, y_{n+k}]$. Then $Im\phi = S^G$. Let K be the kernel of ϕ. Then K is an ideal of $\mathbb{C}[y_1, \ldots, y_{n+k}]$ and $R = \mathbb{C}[y_1, \ldots, y_{n+k}]/K \cong S^G$. Since dim $S^G = n$, by the Noether normalization theorem (cf. [At-Mac]), we may assume without loss of generality that R is integral over $A = \mathbb{C}[y_1, \ldots, y_n]$. Moreover, by a result of [Ho-Ea], S^G is Cohen-Macaulay. We may assume further that $f_1, \ldots, f_n$ is a regular sequence. So R is a finite module flat over $\mathbb{C}[y_1, \ldots, y_n] = A$ (cf. p.154 of [Fi]). Therefore R_m is flat over A_m for all maximal ideal m in A (cf. p.116 of [Bo]). But then R_m is projective over A_m for all maximal ideal m in A (cf. p.107 of [Bo]). In view of the local characterization of projective module (cf. p.138 of [Bo]), we know that R is projective over A. By Quillen's solution to Serre conjecture, R is free over A. Q.E.D.

Note.

1. Basic decomposition (13) means that S^G is a free $\mathbb{C}[f_1, \ldots, f_n]$-module with basis 1, $g_1, \ldots, g_k$.

2. We know that a basic series of G can be obtained from a basic decomposition of S^G. However, it is not always true that a basic decomposition can be obtained from a basic series of G. This is shown by the following example:

Let G be a cyclic group of order 7 with a generator:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta^3 & 0 \\ 0 & 0 & \zeta^4 \end{pmatrix}, \quad \zeta = e^{(2\pi i)/7}.$$

The basic series is

$$\begin{aligned} \phi(\lambda) &= \frac{1+\lambda^2+\lambda^4+\lambda^6+\lambda^8+\lambda^{10}+\lambda^{12}}{(1-\lambda)(1-\lambda^7)^2} \\ &= \frac{1+\lambda^7}{(1-\lambda)(1-\lambda^2)(1-\lambda^7)}. \end{aligned}$$

A basic decomposition exists corresponding to the first equation, namely,

$$S^G = \mathbb{C}[x, y^7, z^7] \bigoplus_{i=1}^{6} (yz)^i \mathbb{C}[x, y^7, z^7],$$

but there is no basic decomposition corresponding to the second equation.

3. If we know the conjugacy classes of G, then the computation of Molien series of G becomes simple, for $\det(I - \lambda A) = \det(I - \lambda B)$ if A and B being in the same conjugacy class of G.

Now let $\{f_1, \ldots, f_n, g_1, \ldots, g_k\}$ be a set of basic invariants of S^G in which $\{f_1, \ldots, f_n, g_1, \ldots, g_t\}$ being a set of minimal generators S^G, $t \le k$ and $\{f_1, \ldots, f_n\}$ algebraically independent. Let $A = \{g_i g_j \big| 1 \le i \le t, i \le j \le k\} - \{g_1, \ldots, g_k\}$ and $b(a)$ be a basic form of a, $a \in S^G$. Then $\{a - b(a) \big| a \in A\}$ generates K (see p.32) for g_m being a polynomial in $g_1, \ldots, g_t$, $t+1 \le m \le k$. Let $B = \{a \in A \big| a$ has no a factor $a' \in A - \{a\}\}$, then $\{a - b(a) \big| a \in B\}$ are relations of S^G.

2.2. The invariants of group type (A)

Let G be a group of type (A), then G is the product of cyclic groups:

$$D = G_1 \times G_2 \times \cdots \times G_m$$

by the fundamental theorem for finite abelian groups. G_i is generated by a diagonal matrix $(\zeta_i^{a_{i1}}, \zeta_i^{a_{i2}}, \zeta_i^{a_{i3}})$ of degree g_i, for $i = 1, \ldots, m$ and $a_{i1} + a_{i2} + a_{i3} \equiv 0$ (mod g_i) and ζ_i being a primitive g_ith root of unity. Clearly, $g_1 g_2 \cdots g_m = g$ the order of G.

Since G is diagonal, the average over G of a monomial $x^b y^c z^d$ is again a monomial. Furthermore, a monomial $x^b y^c z^d$ is invariant if and only if

$$(15) \qquad a_{i1} b + a_{i2} c + a_{i3} d \equiv 0 \,(\mathrm{mod} g_i), \quad i = 1, \ldots, m.$$

Let b_1, b_2, b_3 be smallest positive integers such that $x^{b_1}, y^{b_2}, z^{b_3}$ are invariants, i.e., $a_{i1} b_1 \equiv a_{i2} b_2 \equiv a_{i3} b_3 \equiv 0$ (mod g_i), $i = 1, \ldots, m$. Of course, $1 \leq b_1, b_2, b_3 \leq g$. Then any solution (c_1, c_2, c_3) of (15) can be written as

$$(c_1, c_2, c_3) = (d_1, d_2, d_3) + (b_1 \ell_1, b_2 \ell_2, b_3 \ell_3),$$

where $0 \leq d_i < b_i$, ℓ_i nonnegative integer, $i = 1, 2, 3$. To every solution $(d_1, d_2, d_3) \neq (0, 0, 0)$ with $0 \leq d_i \leq b_i$ there corresponds the invariant $x^{d_1} y^{d_2} z^{d_3}$. Let these monomial be called $g_1, \ldots, g_k$, of degrees $r_1, \ldots, r_k$. Then a set of basic invariants consists of the 3 algebraically independent monomials $x^{b_1}, y^{b_2}, z^{b_3}$ together with $g_1, \ldots, g_k$. So the Molien series (basic series) of G is

$$\phi(\lambda) = \frac{1 + \lambda^{r_1} + \cdots + \lambda^{r_k}}{(1 - \lambda^{b_1})(1 - \lambda^{b_2})(1 - \lambda^{b_3})}.$$

The set of minimal generators is contained in the set of basic invariants which necessarily have degrees $\leq g$ according to Noether's theorem. After finding a set of minimal generators, the relations can be found by the method described in the last paragraph of section 2.1.

Note. The above argument are true for every finite abelian subgroup of $GL(n, \mathbb{C})$. This was treated also by MacWilliams-Mallows-Sloan [Mac-Ma-Sl, pp.800-801]. Then only consider finding invariants, but they gave x^g, y^g, z^g instead of $x^{b_1}, y^{b_2}, z^{b_3}$.

Example. Let

$$G = \left\langle \begin{pmatrix} \zeta_4^2 & 0 & 0 \\ 0 & \zeta_4 & 0 \\ 0 & 0 & \zeta_4 \end{pmatrix}, \begin{pmatrix} \zeta_6 & 0 & 0 \\ 0 & \zeta_6^3 & 0 \\ 0 & 0 & \zeta_6^2 \end{pmatrix} \right\rangle, \qquad \zeta_n = e^{(2\pi i)/n}$$

of order 24. Then $G = G_1 \times G_2$, where G_1 being a cyclic group generated by the first generator of G, while G_2 generated by the second one. Now $6, 4, 12$ are smallest positive integers such that x^6, y^4, z^{12} in S^G. Next we want to find i, j, k with $0 \le i < 6$, $0 \le j < 4$, $o \le k < 12$ such that

$$\begin{cases} 2i + j + k \equiv 0 \pmod 4, \\ i + 3j + 2k \equiv 0 \pmod 6). \end{cases}$$

This implies that

$$\begin{cases} j = 3i + 4\ell + 6m, \\ k = -5i - 6m; \end{cases}$$

where ℓ and m are integers.

So we have

$$\begin{aligned}
& f_1 = x^6, \quad f_2 = y^4, \quad f_3 = z^{12}; \\
& g_1 = xyz, \quad g_2 = y^2z^6, \quad g_3 = x^4z^4, \quad g_4 = x^2z^8; \\
& g_5 = x^2y^2z^2 = g_1^2, \\
& g_6 = x^3y^3z^3 = g_1^3, \\
& g_7 = xy^3z^7 = g_1g_2, \\
& g_8 = x^5yz^5 = g_1g_3, \\
& g_9 = x^3yz^9 = g_1g_4, \\
& g_{10} = x^4y^2z^{10} = g_1^2g_4, \\
& g_{11} = x^5y^3z^{11} = g_1^3g_4;
\end{aligned}$$

in which f_1, f_2, f_3 are algebraically independent and $f_1, f_2, f_3, g_1, g_2, g_3, g_4$ minimal generators and $f_1, f_2, f_3, g_1, \ldots, g_{11}$ basic invariants. Thus the Molien series is

$$\phi(\lambda) = \frac{1 + \lambda^3 + \lambda^6 + 2\lambda^8 + \lambda^9 + \lambda^{10} + 2\lambda^{11} + \lambda^{13} + \lambda^{16} + \lambda^{19}}{(1-\lambda^4)(1-\lambda^6)(1-\lambda^{12}}.$$

The relations are

$$g_1^4 = f_2 g_3,$$
$$g_1^2 g_2 = f_2 g_4,$$
$$g_1^2 g_3 = f_1 g_2,$$
$$g_2^2 = f_2 f_3,$$
$$g_2 g_3 = g_1^2 g_4,$$
$$g_2 g_4 = f_3 g_1^2,$$
$$g_3^2 = f_1 g_4,$$
$$g_3 g_4 = f_1 f_3,$$
$$g_4^2 = f_3 g_3.$$

Conclusion: $S^G = \mathbb{C}[f_1, f_2, f_3, g_1, g_2, g_3, g_4]$ which is a free $\mathbb{C}[f_1, f_2, f_3]$-module with basis $1, g_1, g_2, g_3, g_4, g_1^2, g_1^3, g_1g_2, g_1g_3, g_1g_4, g_1^2g_4, g_1^3g_4$. Also $S^G \simeq \mathbb{C}[y_1, y_2, y_3, y_4, y_5, y_6, y_7]/K$ where y_i are indeterminates and K is an ideal generated by $y_4^4 - y_2y_6$, $y_4^2y_5 - y_2y_7$, $y_4^2y_6 - y_1y_5$, $y_5^2 - y_2y_3$, $y_5y_6 - y_4^2y_7$, $y_5y_7 - y_3y_4^2$, $y_6^2 - y_1y_7$, $y_6y_7 - y_1y_3$, and $y_7^2 - y_3y_6$.

2.3. The invariants of group type (B)

Let G be a group of type (B), we shall only consider the case that G is isomorphic to a transitive small group of $GL(2, \mathbb{C})$ (see section 1.2 (B)).

2.3.1 The invariants of dihedral groups $D_{n,q}$

(a) Let $G = D_{n,q} = \langle \psi_{2q}, \tau, \phi_{2m} \rangle$ where $m = n - q \equiv 1 \pmod 2$, $1 < q < n$, $(n, q) = 1$ and

$$\psi_{2q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_{2q} & 0 \\ 0 & 0 & \zeta_{2q}^{-1} \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \quad \phi_{2m} = \begin{pmatrix} \zeta_{2m}^{-2} & 0 & 0 \\ 0 & \zeta_{2m} & 0 \\ 0 & 0 & \zeta_{2m} \end{pmatrix},$$

and $\zeta_k = e^{(2\pi i)/k}$.

Let $G_2 = \langle \psi_{2q} \rangle$ the $1, 2q$ are smallest positive integers such that x, y^{2q}, z^{2q} in S^{G_2}. Now we want to find ℓ, k with $0 \le \ell,\ k < 2q$ such that $\ell - k \equiv 0$ (mod

$2q$). This implies $\ell \equiv k \pmod{2q}$. Therefore, S^{G_2} has a set of minimal generators: x, y^{2q}, z^{2q}, yz. Let $G_1 = \langle \psi_{2q}, \tau \rangle$ then $G_1 = G_2 \cup G_2\tau$ for $|G_1| = 4q$. Now $x\tau = x$, $y^{2q}\tau = (-1)^q z^{2q}$, $z^{2q}\tau = (-1)^q y^{2q}$, $(yz)\tau = -yz$. Thus S^{G_1} is generated by x and $(y^{2q})^j(z^{2q})^k(yz)^\ell + [(y^{2q})^j(z^{2q})^k(yz)^\ell]\tau$ where $0 \le (2q)j + (2q)k + 2\ell \le 4q$ and $0 \le \ell \le 2q-1$ by Theorem 16 and since $(yz)^{2q} = (y^{2q})(z^{2q})$. The results are

$$
\begin{aligned}
f_1 &= x, \\
f_2 &= y^{2q} + (-1)^q z^{2q}, \\
f_3 &= y^2 z^2, \\
f_4 &= yz(y^{2q} + (-1)^{q-1} z^{2q}), \\
f_5 &= y^{4q} + z^{4q} = f_2^2 - 2(-1)^q f_3^q, \\
f_6 &= y^{2q} z^{2q} = f_3^q, \\
f_7 &= y^\ell z^\ell = f_3^{\frac{\ell}{2}}, \quad \ell \,\text{even and}\, 2 < \ell \le 2q-2, \\
f_8 &= (yz)^\ell [y^{2q} + (-1)^{q+\ell} z^{2q}], \quad 2 \le \ell \le q \\
&= \begin{cases} f_2 f_3^{\frac{\ell}{2}}, & \ell \text{ even}, \\ f_3^{\frac{\ell-1}{2}} f_4, & \ell \text{ odd}. \end{cases}
\end{aligned}
$$

So S^{G_1} has a set of minimal generators: f_1, f_2, f_3, f_4 and we have

$$f_4^2 = f_2^2 f_3 + 4(-1)^{q+1} f_3^{q+1}. \tag{16}$$

Now $G = G_1 \cup G_1\phi_{2m} \cup G_1\phi_{2m}^2 \cup \cdots \cup G_1\phi_{2m}^{m-1}$ for $|G| = 4mq$. Therefore, S^G is generated by $\Sigma_{s=0}^{m-1}(f_1^j f_2^k f_3^\ell f_4^t)\phi_{2m}^s$ where $0 \le j + (2q)k + 4\ell + (2q+2)t \le 4mq$ and $t = 0, 1$ (by Theorem 16 and (16)).

Example. Let $m = 3$, $q = 2$ then $G = D_{5,2} = \langle \psi_4, \tau, \phi_6 \rangle$ and $|G| = 24$ where

$$\psi_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \quad \phi_6 = \begin{pmatrix} \zeta_6^{-2} & 0 & 0 \\ 0 & \zeta_6 & 0 \\ 0 & 0 & \zeta_6 \end{pmatrix}$$

and $\zeta_6 = e^{(2\pi i)/6}$.

Let $G_1 = \langle \psi_4, \tau \rangle$ then form above S^{G_1} has a set of minimal generators

$$f_1 = x,$$
$$f_2 = y^4 + z^4,$$
$$f_3 = y^2 z^2,$$
$$f_4 = yz(y^4 - z^4)$$

and we have $f_4^2 = f_2^2 f_3 - 4f_3^3$. Now $G = G_1 \cup G_1\phi_6 \cup G_1\phi_6^2$ and

$$f_1\phi_6 = \omega^2 f_1, \quad f_1\phi_6^2 = \omega f_1;$$
$$f_2\phi_6 = \omega^2 f_2, \quad f_2\phi_6^2 = \omega f_2;$$
$$f_3\phi_6 = \omega^2 f_3, \quad f_3\phi_6^2 = \omega f_3;$$
$$f_4\phi_6 = f_4, \qquad f_4\phi_6^2 = f_4;$$

where $\omega = e^{(2\pi i)/3}$. So S^G is generated by f_4 and $f_1^j f_2^k f_3^\ell + (f_1^j f_2^k f_3^\ell)\phi_6 + (f_1^j f_2^k f_3^\ell)\phi_6^2$ where $0 \leq j + 4k + 4\ell \leq 24$. The minimal generators of S^G are

$$h_1 = f_1^3,$$
$$h_2 = f_2^3,$$
$$h_3 = f_3^3,$$
$$g_1 = f_4,$$
$$g_2 = f_1^2 f_2,$$
$$g_3 = f_1^2 f_3,$$
$$g_4 = f_1 f_2^2,$$
$$g_5 = f_1 f_3^2,$$
$$g_6 = f_1 f_2 f_3,$$
$$g_7 = f_2 f_3^2.$$

Therefore, $S^G = \mathbb{C}[h_1, h_2, h_3, g_1, g_2, g_3, g_4, g_5, g_6, g_7]$. The Molien series of G is

$$\phi(\lambda) = \frac{1 + 3\lambda^6 + 3\lambda^9 + 4\lambda^{12} + 3\lambda^{15} + 3\lambda^{18} + \lambda^{24}}{(1 - \lambda^3)(1 - \lambda^{12})^2}.$$

Let $g_8 = g_1^2$, $g_9 = g_1g_2$, $g_{10} = g_1g_3$, $g_{11} = g_1g_4$, $g_{12} = g_1g_5$, $g_{13} = g_1g_6$, $g_{14} = g_1^3$, $g_{15} = g_1g_7$, $g_{16} = g_6^2$, $g_{17} = g_1g_6^2$, then $h_1, h_2, h_3, g_1, \ldots, g_{17}$ are basic invariants of S^G.

The relations are

$$\begin{aligned}
g_1^2g_2 &= h_2g_3 - 4h_3g_2, \quad g_1^2g_3 = g_6^2 - 4h_3g_3,\\
g_1^2g_4 &= h_2g_6 - 4h_3g_4, \quad g_1^2g_5 = h_3g_4 - 4h_3g_5,\\
g_1^2g_6 &= h_2g_5 - 4h_3g_6, \quad g_1^4 = h_2g_7 - 8h_3g_1^2 - 16h_3^2,\\
g_1^2g_7 &= h_2h_3 - 4h_3g_7, \quad g_2^2 = h_1g_4,\\
g_2g_3 &= h_1g_6, \quad g_2g_4 = h_1h_2,\\
g_2g_5 &= h_1g_7, \quad g_2g_6 = h_1g_1^2 + 4h_1h_3,\\
g_2g_7 &= g_6^2, \quad g_3^2 = h_1g_5,\\
g_3g_4 &= h_1g_1^2 + 4h_1h_3, \quad g_3g_5 = h_1h_3,\\
g_3g_6 &= h_1g_7, \quad g_3g_7 = h_3g_2,\\
g_4^2 &= h_2g_2, \quad g_4g_5 = g_6^2,\\
g_4g_6 &= h_2g_3, \quad g_4g_7 = h_2g_5,\\
g_5^2 &= h_3g_3, \quad g_5g_6 = h_3g_2,\\
g_5g_7 &= h_3g_6, \quad g_6g_7 = h_3g_4,\\
g_6^3 &= h_1h_2h_3, \quad g_7^2 = h_3g_1^2 + 4h_3^2.
\end{aligned}$$

S^G is a free $\mathbb{C}[h_1, h_2, h_3]$-module with basis $1, g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_1^2, g_1g_2, g_1g_3, g_1g_4, g_1g_5, g_1g_6, g_1^3, g_1g_7, g_6^2, g_1g_6^2$ and $S^G \simeq \mathbb{C}[y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}]/K$ where y_i are indeterminates and K is an ideal generated by $y_4^2y_5 - y_2y_6 + 4y_3y_5$, $y_4^2y_6 - y_9^2 + 4y_3y_6$, $y_4^2y_7 - y_2y_9 + 4y_3y_7$, $y_4^2y_8 - y_3y_7 + 4y_3y_8$, $y_4^2y_9 - y_2y_8 + 4y_3y_9$, $y_4^4 - y_2y_{10} + 8y_3y_4^2 + 16y_3^2$, $y_4^2y_{10} - y_2y_3 + 4y_3y_{10}$, $y_5^2 - y_1y_7$, $y_5y_6 - y_1y_9$, $y_5y_7 - y_1y_2$, $y_5y_8 - y_1y_{10}$, $y_5y_9 - y_1y_4^2 - 4y_1y_3$, $y_5y_{10} - y_9^2$, $y_6^2 - y_1y_8$, $y_6y_7 - y_1y_4^2 - 4y_1y_3$, $y_6y_8 - y_1y_3$, $y_6y_9 - y_1y_{10}$, $y_6y_{10} - y_3y_5$, $y_7^2 - y_2y_5$, $y_7y_8 - y_9^2$, $y_7y_9 - y_2y_6$, $y_7y_{10} - y_2y_8$, $y_8^2 - y_3y_6$, $y_8y_9 - y_3y_5$, $y_8y_{10} - y_3y_9$, $y_9y_{10} - y_3y_7$, $y_9^3 - y_1y_2y_3$ and $y_{10}^2 - y_3y_4^2 - 4y_3^2$.

(b) Let $G = D_{n,q} = \langle \psi_{2q}, \tau \circ \phi_{4m} \rangle$ where $m = n - q \equiv 0 \pmod 2$, $1 < q < n$, $(n, q) = 1$ and

$$\tau \circ \phi_{4m} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} \zeta_{4m}^{-2} & 0 & 0 \\ 0 & \zeta_{4m} & 0 \\ 0 & 0 & \zeta_{4m} \end{pmatrix} = \begin{pmatrix} \zeta_{4m}^{-2} & 0 & 0 \\ 0 & 0 & \zeta_{4m}^{m+1} \\ 0 & \zeta_{4m}^{m+1} & 0 \end{pmatrix},$$

$$\psi_{2q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_{2q} & 0 \\ 0 & 0 & \zeta_{2q}^{-1} \end{pmatrix}, \quad \zeta_k = e^{(2\pi i)/k}.$$

Let $G_1 = \langle \psi_{2q} \rangle$ then by section 2.3.1 (a), S^{G_1} has a set of minimal generators

$$f_1 = x,$$
$$f_2 = y^{2q},$$
$$f_3 = z^{2q},$$
$$f_4 = yz.$$

Now let $A = \tau \circ \phi_{4m}$ then $G = G_1 \cup G_1 A \cup \cdots \cup G_1 A^{2m-1}$ for $|G| = 4mq$. Thus S^G is generated by $\Sigma_{s=0}^{2m-1} (f_1^j f_2^k f_3^\ell f_4^t) A^s$ where $0 \leq j + (2q)k + (2q)\ell + 2t \leq 4mq$ and $0 \leq t \leq 2q - 1$ (by Theorem 16 and since $f_4^{2q} = f_2 f_3$).

Example. Let $m = 2$, $q = 3$ then $G = D_{5,3} = \langle \psi_6, \tau \circ \phi_8 \rangle$ and $|G| = 24$ where

$$\tau \circ \phi_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} \zeta_8^{-2} & 0 & 0 \\ 0 & \zeta_8 & 0 \\ 0 & 0 & \zeta_8 \end{pmatrix} = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & \zeta_8^3 \\ 0 & \zeta_8^3 & 0 \end{pmatrix},$$

$$\psi_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_6 & 0 \\ 0 & 0 & \zeta_6^{-1} \end{pmatrix}, \quad \zeta_k = e^{(2\pi i)/k}.$$

Let $G_2 = \langle \psi_6 \rangle$ then from above S^{G_2} has a set of minimal generators: $f_1 = x$, $f_2 = y^6$, $f_3 = z^6$, $f_4 = yz$. Let $A = \tau \circ \phi_8$ and $G_1 = \langle \psi_6, A^2 = (-1, -i, -i) \rangle$ then $G_1 = G_2 \cup G_2 A^2$ for $|G_1| = 12$. Now $f_1 A^2 = -f_1$, $f_2 A^2 = -f_2$, $f_3 A^2 = -f_3$, $f_4 A^2 = -f_4$. Thus S^{G_1} is generated by $f_1^j f_2^k f_3^\ell f_4^t + (f_1^j f_2^k f_3^\ell f_4^t) A^2$ where

$0 \leq j + 6k + 6\ell + 2t \leq 12$ and $0 \leq t \leq 5$ by Theorem 16 and since $f_4^6 = f_2 f_3$. The minimal generators of S^{G_1} are

$$F_1 = f_1^2 = x^2,$$

$$F_2 = f_2^2 = y^{12},$$

$$F_3 = f_3^2 = z^{12},$$

$$F_4 = f_4^2 = y^2 z^2,$$

$$F_5 = f_1 f_2 = xy^6,$$

$$F_6 = f_1 f_3 = xz^6,$$

$$F_7 = f_1 f_4 = xyz,$$

$$F_8 = f_2 f_4 = y^7 z,$$

$$F_9 = f_3 f_4 = yz^7.$$

Now $G = G_1 \cup G_1 A$ for $|G| = 24$ and $F_1 A = -F_1$, $F_2 A = -F_3$, $F_3 A = -F_2$, $F_4 A = -F_4$, $F_5 A = F_6$, $F_6 A = F_5$, $F_7 A = -F_7$, $F_8 A = F_9$, $F_9 A = F_8$. Thus S^G is generated by $\Sigma_{j=1}^{9} F_j^{a_j} + (\Sigma_{j=1}^{9} F_j^{a_j})A$ where $0 \leq 2a_1 + 12a_2 + 12a_3 + 4a_4 + 7a_5 + 7a_6 + 3a_7 + 8a_8 + 8a_9 \leq 24$ and $0 \leq a_4 \leq 5$ by Theorem 16 and since $F_4^6 = F_2 F_3$. The minimal generator of S^G are

$$h_1 = F_1^2 = x^4,$$

$$h_2 = F_4^2 = y^4 z^4,$$

$$h_3 = F_2 - F_3 = y^{12} - z^{12},$$

$$g_1 = F_1 F_7 = x^3 yz,$$

$$g_2 = F_7^2 = x^2 y^2 z^2,$$

$$g_3 = F_5 + F_6 = x(y^6 + z^6),$$

$$g_4 = F_4 F_7 = xy^3 z^3,$$

$$g_5 = F_8 + F_9 = yz(y^6 + z^6),$$

$$g_6 = F_1 F_5 - F_1 F_6 = x^3(y^6 - z^6),$$

$$g_7 = F_1 F_8 - F_1 F_9 = x^2 yz(y^6 - z^6),$$

$$g_8 = F_4 F_5 - F_4 F_6 = xy^2 z^2(y^6 - z^6),$$

$$g_9 = F_4 F_8 - F_4 F_9 = y^3 z^3(y^6 - z^6).$$

Therefore, $S^G = \mathbb{C}[h_1, h_2, h_3, g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, g_9]$. The Molien series of G is

$$\phi(\lambda) = \frac{1 + \lambda^5 + \lambda^6 + 2\lambda^7 + \lambda^8 + \lambda^9 + \lambda^{10} + \lambda^{11} + \lambda^{12} + \lambda^{13} + 2\lambda^{14} + \lambda^{15} + \lambda^{16} + \lambda^{21}}{(1-\lambda^4)(1-\lambda^8)(1-\lambda^{12})}.$$

Let $g_{10} = g_1g_5$, $g_{11} = g_2g_5$, $g_{12} = g_3^2$, $g_{13} = g_3g_5$, $g_{14} = g_5^2$, $g_{15} = g_1g_5^2$, then $h_1, h_2, h_3, g_1, g_2, \ldots, g_{15}$ are basic invariants of S^G.

The relations are

$$g_1^2 = h_1g_2, \quad g_1g_2 = h_1g_4,$$

$$g_1g_3 = h_1g_5, \quad g_1g_4 = h_1h_2,$$

$$g_1g_6 = h_1g_7, \quad g_1g_7 = h_1g_8,$$

$$g_1g_8 = h_1g_9, \quad g_1g_9 = h_2g_6,$$

$$g_2^2 = h_1h_2,$$

$$g_2g_3 = g_1g_5, \quad g_2g_4 = h_2g_1,$$

$$g_2g_6 = h_1g_8, \quad g_2g_7 = h_1g_9,$$

$$g_2g_8 = h_2g_6, \quad g_2g_9 = h_2g_7,$$

$$g_2g_5^2 = h_2g_3^2, \quad g_3g_4 = g_2g_5,$$

$$g_3g_6 = h_1h_3, \quad g_3g_7 = h_3g_1,$$

$$g_3g_8 = h_3g_2, \quad g_3g_9 = h_3g_4,$$

$$g_3^3 = h_3g_6 + 4h_2g_1g_5, \quad g_3^2g_5 = h_3g_7 + 4h_2g_2g_5,$$

$$g_3g_5^2 = h_3g_8 + 4h_2^2g_3, \quad g_4^2 = h_2g_2,$$

$$g_4g_5 = h_2g_3, \quad g_4g_6 = h_1g_9,$$

$$g_4g_7 = h_2g_6, \quad g_4g_8 = h_2g_7,$$

$$g_4g_9 = h_2g_9, \quad g_5g_6 = h_3g_1,$$

$$g_5g_7 = h_3g_2, \quad g_5g_8 = h_3g_4,$$

$$g_5g_9 = h_2g_3, \quad g_5^3 = h_3g_9 + 4h_2^2g_5,$$

$$g_6^2 = h_1g_3^2 - 4h_1h_2g_2, \quad g_6g_7 = h_1g_3g_5 - 4h_1h_2g_4,$$

$$g_6g_8 = h_1g_5^2 - 4h_1h_2^2, \quad g_6g_9 = g_1g_5^2 - 4h_2^2g_1,$$

$$g_7^2 = h_1g_5^2 - 4h_1h_2^2, \quad g_7g_8 = g_1g_5^2 - 4h_2^2g_1,$$

$$g_7g_9 = h_2g_3^2 - 4h_2^2g_2, \quad g_8^2 = h_2g_3^2 - 4h_2^2g_2,$$

$$g_8g_9 = h_2g_3g_5 - 4h_2^2g_4, \quad g_9^2 = h_2g_5^2 - 4h_2^3.$$

S^G is a free $\mathbb{C}[h_1, h_2, h_3]$-module with basis $1, g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, g_9, g_1g_5, g_2g_5, g_3^2, g_3g_5, g_5^2, g_1g_5^2$ and $S^G \simeq \mathbb{C}[y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12}]/K$ where y_i are indeterminates and K is an ideal generated by $y_4^2 - y_1y_5$, $y_4y_5 - y_1y_7$, $y_4y_6 - y_1y_8$, $y_4y_7 - y_1y_2$, $y_4y_9 - y_1y_{10}$, $y_4y_{10} - y_1y_{11}$, $y_4y_{11} - y_1y_{12}$, $y_4y_{12} - y_2y_9$, $y_5^2 - y_1y_2$, $y_5y_6 - y_4y_8$, $y_5y_7 - y_2y_4$, $y_5y_9 - y_1y_{11}$, $y_5y_{10} - y_1y_{12}$, $y_5y_{11} - y_2y_9$, $y_5y_{12} - y_2y_{10}$, $y_5y_8^2 - y_2y_6^2$, $y_6y_7 - y_5y_8$, $y_6y_9 - y_1y_3$, $y_6y_{10} - y_3y_4$, $y_6y_{11} - y_3y_5$, $y_6y_{12} - y_3y_7$, $y_6^3 - y_3y_9 - 4y_2y_4y_8$, $y_6^2y_8 - y_3y_{10} - 4y_2y_5y_8$, $y_6y_8^2 - y_3y_{11} - 4y_2^2y_6$, $y_7^2 - y_2y_5$, $y_7y_8 - y_2y_6$, $y_7y_9 - y_1y_{12}$, $y_7y_{10} - y_2y_9$, $y_7y_{11} - y_2y_{10}$, $y_7y_{12} - y_2y_{11}$, $y_8y_9 - y_3y_4$, $y_8y_{10} - y_3y_5$, $y_8y_{11} - y_3y_7$, $y_8y_{12} - y_2y_3$, $y_8^3 - y_3y_{12} - 4y_2^2y_8$, $y_9^2 - y_1y_6^2 + 4y_1y_2y_5$, $y_9y_{10} - y_1y_6y_8 + 4y_1y_2y_7$, $y_9y_{11} - y_1y_8^2 + 4y_1y_2^2$, $y_9y_{12} - y_4y_8^2 + 4y_2^2y_4$, $y_{10}^2 - y_1y_8^2 + 4y_1y_2^2$, $y_{10}y_{11} - y_4y_8^2 + 4y_2^2y_4$, $y_{10}y_{12} - y_2y_6^2 + 4y_2^2y_5$, $y_{11}^2 - y_2y_6^2 + 4y_2^2y_5$, $y_{11}y_{12} - y_2y_6y_8 + 4y_2^2y_7$ and $y_{12}^2 - y_2y_8^2 + 4y_2^3$.

The rest of this section, for each element

$$T = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}, \quad \alpha(ab - cd) = 1$$

in G $(= T_m, O_m$ or $I_m)$, we let

$$T' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and $G' = \{T' \big| T \in G\}$. Clearly, $T' \in GL(2, \mathbb{C})$ and G' is a transitive small group of $GL(2, \mathbb{C})$ isomorphic to Gf. Also, let $F' = \{J = (1, 1), (-1, -1)\}$ be the center of $SL(2, \mathbb{C})$.

Definitions.

1. Let $f \in \mathbb{C}[y, z]$, the Hessian of f is defined to be

$$H(f) = \begin{vmatrix} \dfrac{\partial^2 f}{\partial y^2} & \dfrac{\partial^2 f}{\partial y \partial z} \\ \dfrac{\partial^2 f}{\partial z \partial y} & \dfrac{\partial^2 f}{\partial z^2} \end{vmatrix}.$$

2. Let $f, g \in \mathbb{C}[y,z]$, the Jacobian of f, g is defined to be

$$J(f,g) = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{vmatrix}.$$

Facts.

1. If f is an invariant then so is the Hessian of f and deg $H(f) = 2(\deg f - 2)$.
2. If f, g are invariants then so is the Jacobian of f, g and deg $J(f,g) = \deg f + \deg g - 2$.

2.3.2. The invariants of tetrahedral group T_m

(a) Let $G = T_m = \langle \psi_4, \tau, \eta, \phi_{2m} \rangle$ where $m \equiv 1, 5 \pmod 6$ and

$$\psi_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix}, \quad \eta = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \zeta_8 & \zeta_8^3 \\ 0 & \zeta_8 & \zeta_8^7 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1+i & -1+i \\ 0 & 1+i & 1-i \end{pmatrix},$$

$$\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \quad \phi_{2m} = \begin{pmatrix} \zeta_{2m}^{-2} & 0 & 0 \\ 0 & \zeta_{2m} & 0 \\ 0 & 0 & \zeta_{2m} \end{pmatrix}, \quad \zeta_k = e^{(2\pi i)/k}.$$

Let $G_2 = \langle \psi_4, \tau \rangle$ then $G_2 = D_{3,2}$. From section 2.3.1 (a), S^{G_2} has a set of minimal generators: $x, y^4+z^4, y^2z^2, yz(y^4-z^4)$. Let $G_1 = \langle \psi_4, \tau, \eta \rangle$ then S^{G_1} has a set of minimal generators: x and a set of minimal generators of $S^{G_1'} \in \mathbb{C}[y,z]$. Also, $S^{G_2'} = \mathbb{C}[y^4+z^4, y^2z^2, yz(y^4-z^4)]$. As elements in $G_1'/F' \simeq A_4$, we have $\overline{S'} = (123)$, $\overline{\psi_4'} = (12)(34)$ where $S' = \eta'^2$ and $G_1' = \langle \psi_4', \tau', \eta' \rangle = \langle \psi_4', S' \rangle$ for $\tau' = \eta' \psi_4' \eta'^{-1}$ and $\eta' = (\psi_4')^2 S'^2$. The Molien series of G_1' is

$$\phi(\lambda) = \frac{1}{2}(\phi_1(\lambda) + \phi_1(-\lambda)) = \frac{1+\lambda^{12}}{(1-\lambda^6)(1-\lambda^8)}$$

where

$$\phi_1(\lambda) = \frac{1}{12}\Big[\frac{1}{\det(J-\lambda J)} + \frac{4}{\mathrm{Det}(J-\lambda S')} + \frac{4}{\mathrm{Det}(J-\lambda S'^2)} + \frac{3}{\mathrm{Det}(J-\lambda \psi_4')}\Big].$$

Let

$$f_1 = yz(y^4 - z^4).$$

Since $f_1\eta' = f_1$, f_1 is an invariant of degree 6 of G_1'. The Hessian of f_1 is $H(f_1)$ of degree 8. Let $f_2 = -\frac{1}{25}H(f_1)$ then

$$f_2 = y^8 + z^8 + 14y^4z^4.$$

The Jacobian of f_1, f_2 is $J(f_1, f_2)$ of degree 12. Let $f_3 = \frac{1}{8}J(f_1, f_2)$ then

$$f_3 = 33y^4z^4(y^4 + z^4) - (y^{12} + z^{12}).$$

So $S^{G_1'}$ has a set of minimal generators: f_1, f_2, f_3 and we have $f_3^2 = f_2^3 - 108f_1^4$. Thus S^{G_1} has a set of minimal generator: x, f_1, f_2, f_3.

Now $G = G_1 \cup G_1\phi_{2m} \cup \cdots \cup G_1\phi_{2m}^{m-1}$ for $|G_1| = 24$ and $|G| = 24m$. Therefore, S^G is generated by $\Sigma_{s=0}^{m-1}(x^j f_1^k f_2^\ell f_3^t)\phi_{2m}^s$ where $0 \leq j + 6k + 8\ell + 12t \leq 24m$ and $t = 0, 1$ (by Theorem 16 and since $f_3^2 = f_2^3 - 108f_1^4$).

Example. Let $m = 1$ then $G = T_1 = \langle \psi_4, \tau, \eta, \phi_2 \rangle = \langle \psi_4, \tau, \eta \rangle$ for $\phi_2 = \psi_4^2$ and $|G| = 24$. The Molien series of G is

$$\phi(\lambda) = \frac{1 + \lambda^{12}}{(-\lambda)(1 - \lambda^6)(1 - \lambda^8)}.$$

From above, $S^G = \mathbb{C}[x, f_1, f_2, f_3]$ and the relation is

$$f_3^2 = f_2^3 - 108f_1^4.$$

S^G is a free $\mathbb{C}[x, f_1, f_2]$-module with basis 1, f_3 and $S^G \simeq \mathbb{C}[y_1, y_2, y_3, y_4]/(L)$ where y_i are indeterminates and $L = y_4^2 - y_3^3 + 108y_2^4$.

(b) Let $G = T_m = \langle \psi_4, \tau, \eta \circ \phi_{6m} \rangle$ where $m \equiv 3 \pmod 6$ and

$$\eta \circ \phi_{6m} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \zeta_8 & \zeta_8^3 \\ 0 & \zeta_8 & \zeta_8^7 \end{pmatrix} \begin{pmatrix} \zeta_{6m}^{-2} & 0 & 0 \\ 0 & \zeta_{6m} & 0 \\ 0 & 0 & \zeta_{6m} \end{pmatrix}$$
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\zeta_{6m}^{-2} & 0 & 0 \\ 0 & \zeta_{24m}^{3m+4} & \zeta_{24m}^{9m+4} \\ 0 & \zeta_{24m}^{3m+4} & \zeta_{24m}^{21m+4} \end{pmatrix},$$
$$\psi_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \quad \zeta_k = e^{(2\pi i)/k}.$$

Let $G_1 = \langle\psi_4, \tau\rangle$ then $G_1 = D_{3,2}$. From section 2.3.1 (a), S^{G_1} has a set of minimal generators: $f_1 = x$, $f_2 = y^4 + z^4$, $f_3 = y^2z^2$, $f_4 = yz(y^4 - z^4)$. Now let $A = \eta \circ \phi_{6m}$ the $G = G_1 \cup G_1A \cup \cdots \cup G_1A^{3m-1}$ for $|G_1| = 8$ and $|G| = 24m$. Thus S^G is generated by $\Sigma_{s=0}^{3m-1}(f_1^j f_2^k f_3^\ell f_4^t)A^s$ where $0 \leq j + 4k + 4\ell + 6t \leq 24m$ and $t = 0, 1$ (by Theorem 16 and since $f_4^2 = f_3^2 f_3 - 4f_3^3$).

2.3.3. The invariants of octahedral groups O_m

Let $G = O_m = \langle\psi_8, \tau, \eta, \phi_{2m}\rangle$ where $(m, 6) = 1$ and

$$\psi_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_8 & 0 \\ 0 & 0 & \zeta_8^7 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 1-i \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix},$$

$$\eta = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \zeta_8 & \zeta_8^3 \\ 0 & \zeta_8 & \zeta_8^7 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1+i & -1+i \\ 0 & 1+i & 1-i \end{pmatrix},$$

$$\phi_{2m} = \begin{pmatrix} \zeta_{2m}^{-2} & 0 & 0 \\ 0 & \zeta_{2m} & 0 \\ 0 & 0 & \zeta_{2m} \end{pmatrix}$$

and $\zeta_k = e^{(2\pi i)/k}$.

Let $G_1 = \langle\psi_8, \tau, \eta\rangle$ then as elements in $G_1'/F' \simeq S_4$, we have $\overline{S'} = (123)$, $\overline{\psi_8'} = (1324)$, where $S' = \eta'^2$ and $G_1' = \langle\psi_8', \tau', \eta'\rangle = \langle\psi_8', S'\rangle$ for $\tau' = \eta'(\psi_8')^2\eta'^{-1}$ and $\eta' = (\psi_8')^4 S'^2$. The Molien series of G_1' is

$$\phi(\lambda) = \frac{1}{2}(\phi_1(\lambda) + \phi_1(-\lambda)) \frac{1 + \lambda^{18}}{(1 - \lambda^8)(1 - \lambda^{12})}$$

where

$$\phi_1(\lambda) = \frac{1}{24}\Big[\frac{1}{\det(J - \lambda J)} + \frac{8}{\det(J - \lambda S')} + \frac{6}{\det(J - \lambda(\psi_8')^3 S'^2)} + \frac{6}{\det(J - \lambda\psi_8' S'^2)} + \frac{3}{\det(J - \lambda S'^2(\psi_8')^2 S')}\Big].$$

Let $G_2 = \langle\psi_4, \tau, \eta\rangle$ where $\psi_4 = \psi_8^2$ then $G_1' = G_2' \cup G_2'\psi_8'$ for $|G_2'| = 24$ and $|G_1'| = 48$. Let

$$g_1 = y^8 + z^8 + 14y^4z^4$$

then g_1 is an invariant of G_2' by section 2.3.2 (a). Since $g_1\psi_8' = g_1$, g_1 is an invariant of degree 8 of G_1'. The Hessian of g_1 is $H(g_1)$ of degree 12. Let $g_2 = \dfrac{1}{9408}H(g_1)$ then

$$g_2 = y^2z^2(y^8 + z^8 - 2y^4z^4).$$

The Jacobian of g_1, g_2 is $J(g_1, g_2)$ of degree 18. Let $g_3 = \dfrac{1}{16}J(g_1, g_2)$ then

$$g_3 = yz(y^{16} - z^{16}) - 34y^5z^5(y^8 - z^8).$$

So $S^{G_1'}$ has a set of minimal generators: g_1, g_2, g_3 and we have $g_3^2 = g_2g_1^3 - 108g_2^3$. Thus S^{G_1} has a set of minimal generators: x, g_1, g_2, g_3.

Now $G = G_1 \cup G_1\phi_{2m} \cup \cdots \cup G_1\phi_{2m}^{m-1}$ for $|G_1| = 48$ and $|G| = 48m$. Therefore, S^G is generated by $\Sigma_{s=0}^{m-1}(x^jg_1^kg_2^\ell g_3^t)\phi_{2m}^s$ where $0 \leq j + 8k + 12\ell + 18t \leq 48m$ and $t = 0, 1$ (by Theorem 16 and since $g_3^2 = g_2g_1^3 - 108g_2^3$).

Example. Let $m = 1$ then $G = O_1 = \langle\psi_8, \tau, \eta, \phi_2\rangle = \langle\psi_8, \tau, \eta\rangle$ for $\phi_2 = \psi_8^4$ and $|G| = 48$. The Molien series of G is

$$\phi(\lambda) = \frac{1 + \lambda^{18}}{(1 - \lambda)(1 - \lambda^8)(1 - \lambda^{12})}.$$

From above, $S^G = \mathbb{C}[x, g_1, g_2, g_3]$ and the relation is

$$g_3^2 = g_2g_1^3 - 108g_2^3.$$

S^G is a free $\mathbb{C}[x, g_1, g_2]$-module with basis 1, g_3 and $S^G \simeq \mathbb{C}[y_1, y_2, y_3, y_4]/(L)$ where y_i are indeterminates and $L = y_4^2 - y_3y_2^3 + 108y_3^3$.

2.3.4. The invariants of icosahedral groups I_m

Let $G = I_m = \langle\sigma, \Omega, o, \phi_{2m}\rangle$ where $(m, 30) = 1$ and

$$0 = \frac{1}{\sqrt{5}}\begin{pmatrix} \sqrt{5} & 0 & 0 \\ 0 & \zeta_5^4 - \zeta_5 & \zeta_5^2 - \zeta_5^3 \\ 0 & \zeta_5^2 - \zeta_5^3 & \zeta_5 - \zeta_5^4 \end{pmatrix}, \quad \phi_{2m} = \begin{pmatrix} \zeta_{2m}^{-2} & 0 & 0 \\ 0 & \zeta_{2m} & 0 \\ 0 & 0 & \zeta_{2m} \end{pmatrix},$$

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_5^3 & 0 \\ 0 & 0 & \zeta_5^2 \end{pmatrix}, \quad \zeta_k = e^{(2\pi i)/k}.$$

Let $G_4 = \langle \Omega \rangle$. Then $1, 5$ are smallest positive integers such that x, y^5, z^5 in S^{G_4}. Now we want to find j, k with $0 \leq j, k < 5$ such that $3j + 2k \equiv 0 \pmod 5$. This implies $k \equiv j \pmod 5$. Therefore, S^{G_4} has a set of minimal generators: $f_1 = x$, $f_2 = y^5$, $f_3 = z^5$, $f_4 = yz$ and $f_4^5 = f_2 f_3$. Let $G_3 = \langle \Omega, \sigma^2 \rangle$ then $G_3 = G_4 \cup G_4\sigma^2$ for $|G_3| = 10$. Note that $\sigma^2 = (1, -1, -1)$. Now $f_1\sigma^2 = f_1$, $f_2\sigma^2 = -f_2$, $f_3\sigma^2 = -f_3$, $f_4\sigma^2 = f_4$. Thus S^{G_3} is generated by f_1, f_4 and $f_2^a f_2^b + (f_2^a f_2^b)\sigma^2$ where $0 \leq 5a + 5b \leq 10$ by Theorem 16. The results are

$$g_1 = f_2^2,$$

$$g_2 = f_3^2,$$

$$g_3 = f_2 f_3 = f_4^5.$$

So S^{G_3} has a set of minimal generators: f_1, f_4, g_1, g_2 and $f_4^{10} = g_1 g_2$. Let $G_2 = \langle \Omega, \sigma \rangle$ then $G_2 = G_3 \cup G_3\sigma$ for $|G_2| = 20$. Now $f_1\sigma = f_1$, $f_4\sigma = -f_4$, $g_1\sigma = g_2$, $g_2\sigma = g_1$. Thus S^{G_2} is a generated by f_1 and $f_4^a g_1^b g_2^c + (f_4^a g_1^b g_2^c)\sigma$ where $0 \leq 2a + 10b + 10c \leq 20$ and $0 \leq a \leq 9$ by Theorem 16 and since $f_4^{10} = g_1 g_2$. The minimal generators of S^{G_2} are f_1 and

$$h_1 = f_4^2,$$

$$h_2 = g_1 + g_2,$$

$$h_3 = f_4(g_1 - g_2).$$

Thus $S^{G_2'} = \mathbb{C}[h_1, h_2, h_3]$. Let $G_1 = \langle \Omega, \sigma, o \rangle$ then as elements in $G_1'/F' \simeq A_5$, we have $\overline{U'} = (14)(23)$, $\overline{\Omega'} = (12345)$, $\overline{o'} = (12)(34)$ where $U' = o'^2\sigma'$ and $G_1' = \langle \Omega', \sigma', o' \rangle = \langle U', \Omega', o' \rangle$. The Molien series of G_1' is

$$\phi(\lambda) = \frac{1}{2}(\phi_1(\lambda) + \phi_1(-\lambda)) = \frac{1 + \lambda^{30}}{(1 - \lambda^{12})(1 - \lambda^{20})}.$$

where

$$\phi_1(\lambda) = \frac{1}{60}\Big[\frac{1}{\det(J - \lambda J)} + \frac{12}{\det(J - \lambda \Omega')} + \frac{12}{\det(J - \lambda \Omega'^2)} + \frac{15}{\det(J - \lambda U'\Omega'^2)} + \frac{205}{\det(J - \lambda \Omega'^3 o' \Omega'^3)}\Big].$$

Now $G_1' = G_2' \cup G_2' o' \cup G_2'(o'\Omega') \cup G_2'(o'\Omega'^2) \cup G_2'(o'\Omega'^3) \cup G_2'(o'\Omega'^4)$. So an invariant of degree 12 of G_1' is

$$A = yz(y^{10} + 11y^5z^5 - z^{10})$$

for $h_1^3 + h_1^3 o' + h_1^3(o'\Omega') + h_1^3(o'\Omega'^2) + h_1^2(o'\Omega'^3) + h_1^3(o'\Omega'^4) = \frac{6}{25} yz(y^{10} + 11y^5z^5 - z^{10})$. The Hessian of A is $H(A)$ of degree 20. Let $B = \frac{1}{121} H(A)$ then

$$B = y^{20} + z^{20} - 228y^5z^5(y^{10} - z^{10}) + 494y^{10}z^{10}.$$

The Jacobina of A, B is $J(A, B)$ of degree 30. Let $C = -\frac{1}{20} J(A, B)$ then

$$C = y^{30} + z^{30} + 522y^5z^5(y^{20} - z^{20}) + 10005y^{10}z^{10}(y^{10} + z^{10}).$$

So $S^{G_1'}$ has a set of minimal generators: A, B, C and we have $C^2 = B^3 + 1728A^5$. Thus S^G has a set of minimal generators: x, A, B, C.

Now $G = G_1 \cup G_1\phi_{2m} \cup \cdots \cup G_1\phi_{2m}^{m-1}$ for $|G_1| = 120$ and $|G| = 120m$. Therefore, S^G is generated by $\Sigma_{s=0}^{m-1}(x^j A^k B^\ell C^t)\phi_{2m}^s$ where $0 \leq j + 12k + 20\ell + 30t \leq 120m$ and $t = 0, 1$ (by Theorem 16 and since $C^2 = B^3 + 1728A^5$).

Example. Let $m = 1$ then $G = I_1 = \langle \sigma, \Omega, o, \phi_2 \rangle = \langle \sigma, \Omega, o \rangle$ for $\phi_2 = o^2$ and $|G| = 120$. The Molien series of G is

$$\phi(\lambda) = \frac{1 + \lambda^{30}}{(1 - \lambda)(1 - \lambda^{12})(1 - \lambda^{20})}.$$

From above, $S^G = \mathbb{C}[x, A, B, C]$ and the relation is

$$C^2 = B^3 + 1728A^5.$$

S^G is a free $\mathbb{C}[x, A, B]$-module with basis 1, C and $S^G \simeq \mathbb{C}[y_1, y_2, y_3, y_4]/(L)$ where y_i are indeterminates and $L = y_4^2 - y_3^3 + 1728y_2^5$.

2.4. The invariants of group type (C)

Let G be a group of type (C), i.e., generated by a nontrivial diagonal abelian group H and

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

In this case, we can find a normal diagonal abelian subgroup H' of G such that $G = H' \cup H'T \cup H'T^2$. To prove this, we may assume

$$H = \left\langle \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \right\rangle, \quad a_1 a_2 a_3 = 1.$$

for H is a direct product of cyclic groups.

Claim. If $G = \langle H, T \rangle$ then $G = H' \cup H'T \cup H'T^2$ and $H' \triangleleft G$, where

$$H' = \left\langle \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \begin{pmatrix} a_2 & 0 & 0 \\ 0 & a_3 & 0 \\ 0 & 0 & a_1 \end{pmatrix} \right\rangle.$$

Proof. Let

$$A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \quad B = TAT^{-1} = \begin{pmatrix} a_2 & 0 & 0 \\ 0 & a_3 & 0 \\ 0 & 0 & a_1 \end{pmatrix},$$
$$C = T^{-1}AT = \begin{pmatrix} a_3 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_2 \end{pmatrix}, \quad H' = \langle A, B \rangle.$$

Since $a_1 a_2 a_3 = 1$, $C = (AB)^{-1}$, so $C \in H'$. Therefore, $TH'T^{-1} = H'$, i.e., $H' \triangleleft G$ and $G = H' \cup H'T \cup H'T^2$. Q.E.D.

To find invariants of this group, we find the invariants of H' first (see section 2.2), then Theorem 16 applied.

Note. The special case of (C) for H being a diagonal cyclic group was treated by Maschke [Ma2] independently.

Example. Let $G = \langle S, T \rangle$, where

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \omega = e^{(2\pi i)/3}.$$

Let $H' = \langle S, TST^{-1} = (\omega, \omega^2, 1) \rangle$, then $G = H' \cup H'T \cup H'T^2$ and $H' \triangleleft G$. Moreover, $G = \{ W^a S^b T^c \,|\, 0 \leq a, b, c \leq 2 \}$ for $TS = WST$, so $|G| = 27$, where $W = (\omega, \omega, \omega)$.

First we find $S^{H'}$. Now 3 is the smallest positive integer such that x^3, y^3, z^3 in $S^{H'}$. Next we want to find i, j, k, with $0 \le i, j, k < 3$ such that

$$\begin{cases} j + 2k \equiv 0 \pmod 3, \\ i + 2j \equiv 0 \pmod 3. \end{cases}$$

This implies that $i \equiv j \equiv k \pmod 3$. Thus $S^{H'} = \mathbb{C}[x^3, y^3, z^3, xyz]$.

Next, since $x^3T = y^3$, $y^3T = z^3$, $z^3T = x^3$, $(xyz)T = xyz$, S^G is generated by xyz and $(x^3)^i(y^3)^j(z^3)^k + ((x^3)^i(y^3)^j(z^3)^k)T + ((x^3)^i(y^3)^j(z^3)^k)T^2$ where $0 \le 3i + 3j + 3k \le 27$.

So the minimal generator of S^G are

$$\begin{aligned} f_1 &= xyz, \\ f_2 &= x^3 + y^3 + z^3, \\ f_3 &= x^3y^3 + y^3z^3 + z^3x^3, \\ \Delta &= (x^3 - y^3)(y^3 - z^3)(z^3 - x^3). \end{aligned}$$

Therefore, $S^G = \mathbb{C}[f_1, f_2, f_3, \Delta]$. The Molien series of G is

$$\phi(\lambda) = \frac{1 + \lambda^9}{(1 - \lambda^3)^2(1 - \lambda^6)}.$$

So the relation is

$$\Delta^2 = 18f_1^3f_2f_3 - 4f_1^3f_2^3 - 4f_3^3 + f_2^2f_3^2 - 27f_1^6.$$

Thus S^G is a free $\mathbb{C}[f_1, f_2, f_3]$-module with basis, $1, \Delta$ and $S^G \simeq \mathbb{C}[y_1, y_2, y_3, y_4]/(L)$ where y_i are indeterminates and $L = y_4^2 - 18y_1^3y_2y_3 + 4y_1^3y_2^3 + 4y_3^3 - y_2^2y_3^2 + 27y_1^6$.

2.5. The invariants of group type (D)

Let G be a group of type (D), i.e., generated by H of type (A), T of type (C) and

$$R = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{pmatrix}, \quad abc = -1.$$

Since

$$R^{2s} = \begin{pmatrix} a^{2s} & 0 & 0 \\ 0 & (bc)^s & 0 \\ 0 & 0 & (bc)^s \end{pmatrix} \quad \text{and} \quad R^{2s-1} = \begin{pmatrix} a^{2s-1} & 0 & 0 \\ 0 & 0 & b^s c^{s-1} \\ 0 & b^{s-1}c^s & 0 \end{pmatrix}$$

the order of R should be even. Let $|R| = 2s$, then $a^{2s} = (bc)^s = 1$.

To find S^G, first let $G_1 = \langle H, T \rangle$ and find S^{G_1} (see section 2.4). Next construct the cosets of G over G_1, then apply Theorem 16.

Example. Let $G = \langle S, T, R \rangle$, where

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$
$$R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \omega = e^{(2\pi i)/3}.$$

Let $G_1 = \langle S, T \rangle$ then $G = G_1 \cup G_1 R$ and $G_1 \triangleleft G$. In fact, $G = \{W^a S^b T^c R^d \,\big|\, 0 \leq a, b, c \leq 2, 0 \leq d \leq 1\}$ for $RS = S^2 R$, $RT = T^2 R$, so $|G| = 54$, where $W = (\omega, \omega, \omega)$.

From example of section 2.4, S^{G_1} has a set of minimal generators

$$f_1 = xyz,$$
$$f_2 = x^3 + y^3 + z^3,$$
$$f_3 = x^3y^3 + y^3z^3 + z^3x^3,$$
$$\Delta = (x^3 - y^3)(y^3 - z^3)(z^3 - x^3).$$

Now $f_1 R = -f_1$, $f_2 R = -f_2$, $f_3 R = f_3$, $\Delta R = \Delta$. Thus S^G is generated by f_3, Δ and $f_1^j f_2^k + (f_1^j f_2^k)R$ where $0 \leq 3j + 3k \leq 54$ and $0 \leq j \leq 5$ by Theorem 16 and since $\Delta^2 = 18 f_1^3 f_2 f_3 - 4 f_1^3 f_2^3 - 4 f_3^3 + f_2^2 f_3^2 - 27 f_1^6$. The minimal generators of S^G are f_3, Δ and

$$g_1 = f_1^2,$$
$$g_2 = f_1 f_2,$$
$$g_3 = f_2^2.$$

So $S^G = \mathbb{C}[f_3, g_1, g_3, g_2, \Delta]$. The Molien series of G is

$$\phi(\lambda) = \frac{1 + \lambda^6 + \lambda^9 + \lambda^{15}}{(1-\lambda^6)^3}.$$

Thus the relations are

$$g_2^2 = g_1 g_3,$$
$$\Delta^2 = 18 g_1 g_2 f_3 - 4 g_1 g_2 g_3 - 4 f_3^3 + g_3 f_3^2 - 27 g_1^3.$$

S^G is a free $\mathbb{C}[f_3, g_1, g_3]$-module with basis $1, g_2, \Delta, g_2\Delta$ and $S^G \simeq \mathbb{C}[y_1, y_2, y_3, y_4, y_5]/K$ where y_i are indeterminates and K is an ideal generated by $y_4^2 - y_2 y_3$ and $y_5^2 - 18 y_1 y_2 y_4 + 4 y_2 y_3 y_4 + 4 y_1^3 - y_1^2 y_3 + 27 y_2^3$.

2.6. The invariants of group type (E)

Let G be the group (E), i.e., generated by S, T, and V where

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$
$$V = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad \omega = e^{(2\pi i)/3}.$$

Let $G_1 = \langle S, T, R \rangle$ where $R = V^2$ as in the example of section 2.5, then $G = G_1 \cup G_1 V$ and $G_1 \triangleleft G$. In fact, $G = \{W^a S^b T^c V^d \,|\, 0 \le a, b, c \le 2, 0 \le d \le 3\}$ for $VS = TV$, $VT = S^2 V$, so $|G| = 108$, where $W = (\omega, \omega, \omega)$.

From example of section 2.5, S^{G_1} has a set of minimal generators

$$g_1 = f_1^2,$$
$$g_2 = f_1 f_2,$$
$$g_3 = f_2^2,$$
$$f_3 = x^3 y^3 + y^3 z^3 + z^3 x^3,$$
$$\Delta = (x^3 - y^3)(y^3 - z^3)(z^3 - x^3),$$

where $f_1 = xyz$, $f_2 = x^3 + y^3 + z^3$.

Now

$$
\begin{aligned}
f_1V &= \frac{i}{\sqrt{27}}(f_2 - 3f_1),\\
f_2V &= \frac{i}{\sqrt{3}}(f_2 + 6f_1),\\
f_3V &= -\frac{1}{9}(-9f_3 + f_2^2 + 3f_1f_2 + 9f_1^2) = -\frac{1}{9}(-9g_1 + 3g_2 + g_3 - 9f_3),\\
\Delta V &= \Delta.
\end{aligned}
$$

So

$$
\begin{aligned}
g_1V &= (f_1^2)V = (f_1V)^2 = -\frac{1}{27}(9g_1 - 6g_2 + g_3),\\
g_2V &= (f_1f_2)V = (f_1V)(f_2V) = -\frac{1}{9}(-18g_1 + 3g_2 + g_3),\\
g_3V &= (f_2^2)V = (f_2V)^2 = -\frac{1}{3}(36g_1 + 12g_2 + g_3).
\end{aligned}
$$

Thus S^G is generated by Δ and $g_1^jg_2^kg_3^\ell f_3^m + (f_1^jg_2^kg_3^\ell f_3^m)V$ where $0 \le 6j+6k+6\ell+6m \le 108$ and $k = 0, 1$ (by Theorem 16 and since $g_2^2 = g_1g_3$).
The minimal generators of S^G are Δ and

$$
\begin{aligned}
h_1 &= 18g_1 + 6g_2 - g_3,\\
h_2 &= g_3 - 12f_3,\\
h_3 &= 27g_1^2 - g_2g_3,\\
h_4 &= 18g_1g_2 - 3g_1g_3 - g_2g_3.
\end{aligned}
$$

Thus $S^G = \mathbb{C}[h_1, h_2, h_3.h_4, \Delta]$. The Molien series of G is

$$
\phi(\lambda) = \frac{1 + \lambda^9 + \lambda^{12} + \lambda^{21}}{(1-\lambda^6)^2(1-\lambda^{12})}.
$$

So the relations are

$$
\begin{aligned}
9h_4^2 &= 12h_3^2 - h_1^2h_3 + h_1^2h_4,\\
432\Delta^2 &= h_2^3 - 2h_1^3 + 36h_1h_4 - 3h_1^2h_2 + 36h_2h_3.
\end{aligned}
\tag{17}
$$

S^G is a free $\mathbb{C}[h_1, h_2, h_3]$-module with basis $1, h_4, \Delta, h_4\Delta$ and $S^G \simeq \mathbb{C}[y_1, y_2, y_3, y_4, y_5]/K$ where y_i are indeterminates and K is an ideal generated by $9y_4^2 - 12y_3^2 + y_1^2y_3 - y_1^2y_4$ and $432y_5^2 - y_2^3 + 2y_1^3 - 36y_1y_4 + 3y_1^2y_2 - 36y_2y_3$.

2.7. The invariants of group (F)

Let G be the group (F), i.e., generated by S, T, V, and $P = UVU^{-1}$ where

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad V = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},$$

$$U = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon\omega \end{pmatrix}, \quad P = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & \omega^2 \\ 1 & \omega & \omega \\ \omega & 1 & \omega \end{pmatrix}, \quad \omega = e^{(2\pi i)/3},\ \epsilon^3 = \omega^2.$$

Let $G_1 = \langle S, T, V \rangle$ then $G = G_1 \cup G_1 P$ for $P^2 = S^2 V^2$ and $|G| = 216$, thus $G_1 \triangleleft G$. So $G = \{W^a S^b T^c V^d P^e \big| \, 0 \le a, b, c \le 2,\ 0 \le d \le 3,\ 0 \le e \le 1\}$ where $W = (\omega, \omega, \omega)$.

From section 2.6, S^{G_1} has a set of minimal generators

$$\Delta = (x^3 - y^3)(y^3 - z^3)(z^3 - x^3),$$

$$h_1 = 18g_1 + 6g_2 - g_3,$$

$$h_2 = g_3 - 12f_3,$$

$$h_3 = 27g_1^2 - g_2 g_3,$$

$$h_4 = 18g_1 g_2 - 3g_1 g_3 - g_2 g_3.$$

where $g_1 = f_1^2$, $g_2 = f_1 f_2$, $g_3 = f_2^2$, $f_1 = xyz$, $f_2 = x^3 + y^3 + z^3$, $f_3 = x^3y^3 + y^3z^3 + z^3x^3$.

Now

$$\Delta U = \Delta, \quad \Delta U^{-1} = \Delta;$$

$$f_1 U = f_1, \quad f_1 U^{-1} = f_1;$$

$$f_2 U = \omega^2 f_2, \quad f_2 U^{-1} = \omega f_2;$$

$$f_3 U = \omega f_3, \quad f_3 U^{-1} = \omega^2 f_3;$$

$$g_1 U = g_1, \quad g_1 U^{-1} = g_1;$$

$$g_2 U = \omega^2 g_2, \quad g_2 U^{-1} = \omega g_2;$$

$$g_3 U = \omega g_3, \quad g_3 U^{-1} = \omega^2 g_3;$$

together with the equalities of g_1V, g_2V, g_3V, f_3V and ΔV in section 2.6, we have

$$\begin{aligned} g_1P &= -\frac{1}{27}(9g_1 - 6\omega g_2 + \omega^2 g_3), \\ g_2P &= \frac{\omega^2}{9}(-18g_1 + 3\omega g_2 + \omega^2 g_3), \\ g_3P &= -\frac{\omega}{3}(36g_1 + 12\omega g_2 + \omega^2 g_3), \\ f_3P &= -\frac{\omega}{9}(9g_1 + 3\omega g_2 + \omega^2 g_3 - 9\omega^2 f_3), \\ \Delta P &= \Delta. \end{aligned}$$

Thus

$$\begin{aligned} h_1P &= -h_1, \\ h_2P &= h_2, \\ h_3P &= h_3, \\ h_4P &= \frac{1}{9}(h_1^2 - 9h_4). \end{aligned}$$

So S^G is generated by Δ, h_2, h_3 and $h_1^j h_4^k + (h_1^j h_4^k)P$ where $0 \leq 6j + 12k \leq 216$ and $k = 0, 1$ (by Theorem 16 and (17)). The minimal generators of S^G are Δ, h_2, h_3 and

$$d = h_1^2.$$

Therefore $S^G = \mathbb{C}[h_2, h_3, \Delta, d]$. The Molien series of G is

$$\phi(\lambda) = \frac{1 + \lambda^{12} + \lambda^{24}}{(1 - \lambda^6)(1 - \lambda^9)(1 - \lambda^{12})}.$$

So the relation is

$$(18) \qquad 4d^3 - 144h_3d^2 + 1728h_3^2 d = (h_2^3 - 432\Delta^2 - 3h_2d + 36h_2h_3)^2.$$

S^G is a free $\mathbb{C}[h_2, h_3, \Delta]$-module with basis $1, d, d^2$ and $S^G \simeq \mathbb{C}[y_1, y_2, y_3, y_4]/(L)$ where y_i are indeterminates and $L = 4y_4^3 - 144y_2y_4^2 + 1728y_2y_4 - (y_1^3 - 432y_3^2 - 3y_1y_4 + 36y_1y_2)^2$.

2.8. The invariants of group (G)

Let G be the group (G), i.e., generated by S, T, V, U where

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad V = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},$$
$$U = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon\omega \end{pmatrix}, \quad \omega = e^{(2\pi i)/3},\ \epsilon^3 = \omega^2.$$

Let $G_1 = \langle S, T, V, P \rangle$ where $P = UVU^{-1}$ as in the section 2.7, then $G = G_1 \cup G_1 U \cup g_1 U^2$ for $U^3 = W^2$ and $|G| = 648$. So $G = \{W^a S^b T^c V^d P^e U^f \,|\, 0 \leq a, b, c \leq 2,\ 0 \leq d \leq 3,\ 0 \leq e \leq 1,\ 0 \leq f \leq 2\}$ and $G_1 \lhd G$ for $UPU^{-1} = W^2 S V^3 P$, $U^2 P U^{-2} = V$, $UVU^{-1} = P$, $U^2 V U^{-2} = W^2 S V^3 P$, $UTU^{-1} = S^2 T$, $U^2 T U^{-2} = ST$, where $W = (\omega, \omega, \omega)$.

From section 2.7, S^{G_1} has a set of minimal generators

$$\Delta = (x^3 - y^3)(y^3 - z^3)(z^3 - x^3),$$
$$h_2 = g_3 - 12 f_3,$$
$$h_3 = 27 g_1^2 - g_2 g_3,$$
$$d = h_1^2 = (18 g_1 + 6 g_2 - g_3)^2.$$

where $g_1 = f_1^2$, $g_2 = f_1 f_2$, $g_3 = f_2^2$, $f_1 = xyz$, $f_2 = x^3 + y^3 + z^3$, $f_3 = x^3 y^3 + y^3 z^3 + z^3 x^3$.

Now

$$g_1 U = g_1, \quad g_1 U^2 = g_1;$$
$$g_2 U = \omega^2 g_2, \quad g_2 U^2 = \omega g_2;$$
$$g_3 U = \omega g_3, \quad g_3 U^2 = \omega^2 g_3;$$
$$f_3 U = \omega f_3, \quad f_3 U^2 = \omega^2 f_3;$$
$$\Delta U = \Delta, \quad \Delta U^2 = \Delta;$$
$$h_2 U = \omega h_2, \quad h_2 U^2 = \omega^2 h_2;$$
$$h_3 U = h_3, \quad h_3 U^2 = h_3;$$
$$dU = \omega^2 d + (12 - 12\omega^2) h_3, \quad dU^2 = \omega d + (12 - 12\omega) h_3.$$

Thus S^G is generated by Δ, h_3 and $h_2^j d^k + (h_2^j d^k)U + (h_2^j d^k)U^2$ where $0 \le 6j+12k \le 648$ and $0 \le k \le 2$ (by Theorem 16 and (18)). The minimal generators of S^G are Δ, h_3, and $b_1 = h_2^3$, $b_2 = h_2 d - 12h_2h_3$.
So $S^G = \mathbb{C}[\Delta, h_3, b_1, b_2]$. The Molien series of G is

$$\phi(\lambda) = \frac{1+\lambda^{18}+\lambda^{36}}{(1-\lambda^9)(1-\lambda^{12})(1-\lambda^{18})}.$$

Thus the relation is

$$4b_2^2 = 9b_2^2b_1 - 6b_2b_1^2 + 2592b_2b_1\Delta^2 + b_1^3 - 864b_1^2\Delta^2 - 6912b_1h_3^3 + 186624b_1\Delta^4.$$

S^G is a free $\mathbb{C}[\Delta, h_3b_1]$-module with basis $1, b_2, b_2^2$ and $S^G \simeq \mathbb{C}[y_1, y_2, y_3, y_4]/(L)$ where y_i are indeterminates and $L = 4y_4^3 - 9y_3y_4^2 + 6y_3^2y_4 - 2592y_1^2y_3y_4 - y_3^3 + 864y_1^2y_3^2 + 6912y_2^3y_3 - 186624y_1^4y_3$.

Note. Rotillon [Ro] also treated the cases (E), (F), (G) independently, but her result in case (E), second relation ([Ro, Theorem 1, p.346]) is wrong.

The groups (H)–(L) are isomorphic to permutation groups, so the conjugacy classes of those groups can be found easily. Thus we compute the Molien series first, then use them to find a set of minimal generators of invariants.

Definitions.

1. Let $f \in \mathbb{C}[x,y,z]$, the Hessian of f is defined to be

$$H(f) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{vmatrix}.$$

2. Let $f, g \in \mathbb{C}[x,y,z]$, the bordered Hessian of f, g is defined to be

$$BH(f,g) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial g}{\partial x} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial g}{\partial y} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} & \frac{\partial g}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} & 0 \end{vmatrix}.$$

3. Let $f, g, h \in \mathbb{C}[x, y, z]$, the Jacobian of f, g, h is defined to be

$$J(f,g,h) = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial g}{\partial x} & \frac{\partial h}{\partial x} \\ \frac{\partial f}{\partial y} & \frac{\partial g}{\partial y} & \frac{\partial h}{\partial y} \\ \frac{\partial f}{\partial z} & \frac{\partial g}{\partial z} & \frac{\partial h}{\partial z} \end{vmatrix}.$$

Facts.

1. If f is an invariant then so is the Hessian of f and $\deg H(f) = 3(\deg f - 2)$.
2. If f, g are invariants then so is the bordered Hessian of f, g and $\deg BH(f,g) = 2(\deg f - 2) + 2(\deg g - 1)$.
3. If f, g, h are invariants then so is the Jacobian of f, g, h and $\deg J(f,g,h) = \deg f + \deg g + \deg h - 3$.

The invariants and relations of (H) and (I) were found by Klein (1884) [Kl, pp.236-243] and Weber (1899) [We, pp.518-529] (also see Gordan (1880) [Go]) respectively. Their methods are long and complicated and are difficult to comprehend.

2.9. The invariants of group (H)

Let G be the group (H), i.e., isomorphic to the alternating group A_5. It generated by

$$S = (12345) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \epsilon^4 & 0 \\ 0 & 0 & \epsilon \end{pmatrix}, \quad U = (14)(23) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$T = (12)(34) = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 \\ 2 & s & t \\ 2 & t & s \end{pmatrix},$$

where $\epsilon = e^{(2\pi i)/5}$, $s = \epsilon^2 + \epsilon^3 = \frac{1}{2}(-1-\sqrt{5})$, $t = \epsilon + \epsilon^4 = \frac{1}{2}(-1+\sqrt{5})$.

We have $|G| = 60$ and $G = \{U^a S^b, S^b T U^a S^{b_1} \mid 0 \le a \le 1, 0 \le b, b_1 \le 4\}$. The Molien series of G is

$$\begin{aligned} \phi(\lambda) &= \frac{1}{60}\Big[\frac{1}{\det(I-\lambda I)} + \frac{12}{\det(I-\lambda S)} + \frac{12}{\det(I-\lambda S^2)} \\ &\quad + \frac{15}{\det(I-\lambda U S^2)} + \frac{20}{\det(I-\lambda S^3 T S^3)}\Big] \\ &= \frac{1+\lambda^{15}}{(1-\lambda^2)(1-\lambda^6)(1-\lambda^{10})}. \end{aligned}$$

Let $G_2 = \langle S \rangle$. Then $1, 5$ are smallest positive integers such that x, y^5, z^5 in S^{G_2}. Now we want to find ℓ, k with $0 \leq \ell,\ k < 5$ such that $4\ell + k \equiv 0 \pmod 5$. This implies $\ell \equiv k \pmod 5$. Therefore, S^{G_2} has a set of minimal generators: $f_1 = x$, $f_2 = y^5$, $f_3 = z^5$, $f_4 = yz$ and $f_4^5 = f_2 f_3$. Let $G_1 = \langle S, U \rangle$ then $G_1 = G_2 \cup G_2 U$ for $|G_1| = 10$. Now $f_1 U = -f_1$, $f_2 U = -f_3$, $f_3 U = -f_2$, $f_4 U = f_4$. Thus S^{G_2} is generated by f_4 and $f_1^a f_2^b f_3^c + (f_1^a f_2^b f_3^c)U$ where $o \leq a + 5b + 5c \leq 10$ (by Theorem 16). The minimal generators of S^{G_1} are f_4 and

$$g_1 = f_1^2,$$

$$g_2 = f_2 - f_3,$$

$$g_3 = f_1(f_2 + f_3).$$

Now $G = G_1 \cup G_1 T \cup G_1(TS) \cup G_1(TS^2) \cup G_1(TS^3) \cup G_1(TS^4)$. So an invariant of degree 2 of G is

$$h_1 = x^2 + yz$$

for $g_1 + g_1 T + g_1(TS) + g_1(TS^2) + g_1(TS^3) + g_1(TS^4) = 2(x^2 + yz)$ and an invariant of degree 6 of G is

$$h_2 = 8x^4yz - 2x^2y^2z^2 - x(y^5 + z^5) + y^3z^3$$

for $h_2' = g_3 + g_3 T + g_3(TS) + g_3(TS^2) + g_3(TS^3) + g_3(TS^4) = \frac{4}{25}(16x^6 - 120x^4yz + 90x^2y^2z^2 + 21xy^5 + 21xz^5 - 5y^3z^3)$ and $16h_1^3 - \frac{25}{4}h_2' = 21(8x^4yz - 2x^2y^2z^2 - xy^5 - xz^5 + y^3z^3)$. The bordered Hessian of h_2, h_1 is $BH(h_2, h_1)$ of degree 10. Let $h_3 = -\frac{1}{25}(256h_1^5 - BH(h_2, h_1) - 480h_1^2 h_2)$ then

$$\begin{aligned} h_3 = {} & 320x^6y^2z^2 - 160x^4y^3z^3 + 20x^2y^4z^4 + 6y^5z^5 \\ & - 4x(y^5 + z^5)(32x^4 - 20x^2yz + 5y^2z^2) + y^{10} + z^{10}. \end{aligned}$$

The Jacobian of h_2, h_1, h_3 is $J(h_2, h_1, h_3)$ of degree 15. Let $h_4 = \frac{1}{10}J(h_2, h_1, h_3)$ then

$$\begin{aligned} h_4 = {} & (y^5 - z^5)[-1024x^{10} + 3840x^8yz - 3840x^6y^2z^2 + 1200x^4y^3z^3 - 100x^2y^4z^4 \\ & + y^{10} + z^{10} + 2y^5z^5 + x(y^5 + z^5)(352x^4 - 160x^2yz + 10y^2z^2)]. \end{aligned}$$

So $S^G = \mathbb{C}[h_1, h_2, h_3, h_4]$.

The relation is

$$(19) \qquad h_4^2 = -1728h_2^5 + h_3^3 + 720h_1h_2^3h_3 - 80h_1^2h_2h_3^2 + 64h_1^3(5h_2^2 - h_1h_3)^2.$$

S^G is a free $\mathbb{C}[h_1, h_2, h_3]$-module with basis $1, h_4$ and $S^G \simeq \mathbb{C}[y_1, y_2, y_3, y_4]/(L)$ where y_i are indeterminates and $L = y_4^2 + 1728y_2^5 - y_3^3 - 720y_1y_2^3y_3 + 80y_1^2y_2y_3^2 - 64y_1^3(5y_2^2 - y_1y_3)$.

2.10. The invariants of group (I)

Let G be the group (I), i.e., isomorphic to the permutation group generated by $(1234567), (142)(356), (12)(35)$. It generated by

$$S = (1234567) = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta^2 & 0 \\ 0 & 0 & \beta^4 \end{pmatrix}, \quad T = (142)(356) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$
$$R = (12)(35) = \frac{-1}{\sqrt{-7}} \begin{pmatrix} \beta^4 - \beta^3 & \beta^2 - \beta^5 & \beta - \beta^6 \\ \beta^2 - \beta^5 & \beta - \beta^6 & \beta^4 - \beta^3 \\ \beta - \beta^6 & \beta^4 - \beta^3 & \beta^2 - \beta^5 \end{pmatrix},$$

where $\beta = e^{(2\pi i)/7}$.

We have $|G| = 168$ and $G = \{T^aS^b, S^bT^aRS^{b_1} \mid 0 \le a \le 2,\ 0 \le b,\ b_1 \le 6\}$. The Molien series of G is

$$\begin{aligned} \phi(\lambda) &= \frac{1}{168}\Big[\frac{1}{\det(I - \lambda I)} + \frac{24}{\det(I - \lambda S)} + \frac{24}{\det(I - \lambda S^3)} + \frac{56}{\det(I - \lambda TS^4)} \\ &\quad + \frac{42}{\det(I - \lambda RS^6)} + \frac{21}{\det(I - \lambda S^4T^2RS^3)}\Big] \\ &= \frac{1 + \lambda^{21}}{(1 - \lambda^4)(1 - \lambda^6)(1 - \lambda^{14})}. \end{aligned}$$

Let $G_2 = \langle S \rangle$. Then 7 is the smallest positive integer such that x^7, y^7, z^7 in S^{G_2}. Now we want to find j, k, ℓ with $0 \le j, k, \ell < 7$ such that $j + 2k + 4\ell \equiv 0 \pmod 7$. Therefore, S^{G_2} has a set of minimal generators: $xyz, xy^3, yz^3, zx^3, x^3y^2, y^3z^2, z^3x^2,$

$x^5y, y^5z, z^5x, x^7, y^7, z^7$. Let $G_1 = \langle S, T\rangle$ then $G_1 = G_2 \cup G_2T \cup G_2T^2$ for $|G_1| = 21$. The Molien series of G_1 is

$$\begin{aligned}\phi_{G_1}(\lambda) &= \frac{1}{21}\Big[\frac{1}{\det(I-\lambda I)} + \frac{3}{\det(I-\lambda S)} + \frac{3}{\det(I-\lambda S^3)} \\ &\quad + \frac{7}{\det(I-\lambda TS^4)} + \frac{7}{\det(I-\lambda T^2S^6)}\Big] \\ &= \frac{1+\lambda^5+\lambda^6+\lambda^{11}}{(1-\lambda^3)(1-\lambda^4)(1-\lambda^7)}.\end{aligned}$$

Now

$$\begin{aligned}&xyz + (xyz)T + (xyz)T^2 = 3xyz,\\ &xy^3 + (xy^3)T + (xy^3)T^2 = xy^3 + yz^3 + zx^3,\\ &x^3y^2 + (x^3y^2)T + (x^3y^2)T^2 = x^3y^2 + y^3z^2 + z^3x^2,\\ &x^5y + (x^5y)T + (x^5y)T^2 = x^5y + y^5z + z^5x,\\ &x^7 + x^7T + x^7T^2 = x^7 + y^7 + z^7.\end{aligned}$$

Thus S^{G_1} has a set of minimal generators: $xyz, xy^3+yz^3+zx^3$, $x^3y^2+y^3z^2+z^3x^2$, $x^5y+y^5z+z^5x$, $x^7+y^7+z^7$.

Let

$$a = xy^3 + yz^3 + zx^3.$$

Since $aR = a$, a is invariant of degree 4 of G. The Hessian of a is $H(a)$ of degree 6. Let $b = \frac{1}{54}H(a)$ then

$$b = 5x^2y^2z^2 - x^5y - y^5z - z^5x.$$

The bordered Hessian of a, b is $BH(a,b)$ of degree 14. Let $c = \frac{1}{9}BH(a,b)$ then

$$\begin{aligned}c &= x^{14}+y^{14}+z^{14} - 34\Sigma x^{11}yz^2 - 250\Sigma x^9y^4z + 375\Sigma x^8y^2z^4\\ &\quad + 18(x^7y^7+y^7z^7+z^7x^7) - 126\Sigma x^6y^5z^3\end{aligned}$$

where $\Sigma x^jy^kz^\ell = x^jy^kz^\ell + y^jz^kx^\ell + z^jx^ky^\ell$. The Jacobian of a, b, c is $J(a,b,c)$ of degree 21. Let $d = \frac{1}{14}J(a,b,c)$ then

$$\begin{aligned}d &= x^{21}+y^{21}+z^{21} - 7\Sigma x^{18}yz^2 + 217\Sigma x^{16}y^4z - 308\Sigma x^{15}y^2z^4\\ &\quad - 57(x^7y^{14}+y^7z^{14}+z^7x^{14}) - 2890(x^{14}y^7+y^{14}z^7+z^{14}x^7)\\ &\quad + 4018\Sigma x^{13}y^5z^3 + 637\Sigma x^{12}y^3z^6 + 1638\Sigma x^{11}yz^9 - 6279\Sigma x^{11}y^8z^2\\ &\quad + 7007\Sigma x^{10}y^6z^5 - 10010\Sigma x^9y^4z^8 + 10296x^7y^7z^7.\end{aligned}$$

(Gordan [Go, p.372] incorrectly gave the coefficient of $x^7y^7z^7$ in d as 3432 instead of 10296). So $S^G = \mathbb{C}[a,b,c,d]$.

The relation is

$$d^2 = c^3 - 88a^2bc^2 + 1008ab^4z + 1088a^4b^2c - 256a^7c + 1728b^7 - 60032a^3b^5 + 22016a^6b^3 - 2048a^9b. \tag{20}$$

S^G is a free $\mathbb{C}[a,b,c]$-module with basis $1, d$ and $S^G \simeq \mathbb{C}[y_1,y_2,y_3,y_4]/(L)$ where y_i are indeterminates and $L = y_4^2 - y_3^3 + 88y_1^2y_2y_3^2 - 1008y_1y_2^4y_3 - 1088y_1^4y_2^2y_3 + 256y_1^7y_3 - 1728y_2^7 + 60032y_1^3y_2^5 - 22016y_1^6y_2^3 + 2048y_1^9y_2$.

2.11. The invariants of group (J)

Let G be the group (J) then $|G| = 180$ and $G = G_1 \times G_2$ where G_1 is the group (H) and G_2 is the center of $SL(3,\mathbb{C})$, i.e., $G_2 = \{I, W, W^2\}$ where $W = (\omega,\omega,\omega)$ and $\omega = e^{(2\pi i)/3}$. Clearly, $G_1 \lhd G$ and $G = G_1 \cup G_1W \cup G_1W^2$. From section 2.9, S^{G_1} has a set of minimal generators h_i, $i = 1,2,3,4$. Now

$$h_1W = \omega^2 h_1, \quad h_1W^2 = \omega h_1;$$

$$h_2W = h_2, \quad h_2W^2 = h_2;$$

$$h_3W = \omega h_3, \quad h_3W^2 = \omega^2 h_3;$$

$$h_4W = h_4, \quad h_4W^2 = h_4.$$

Thus S^G is generated by h_1, h_4 and $h_1^jh_3^k + (h_1^jh_3^k)W + (h_1^jh_3^k)W^2$ where $0 \le 2j + 10k \le 108$ and $0 \le k \le 2$ (by Theorem 16 and (19)). The minimal generators of S^G are h_2, h_4 and $a_1 = h_1^3$, $a_2 = h_1h_3$.
So $S^G = \mathbb{C}[h_2,h_4,a_1,a_2]$. The Molien series of G is

$$\phi(\lambda) = \frac{1}{3}(\phi_{G_1}(\lambda) + \phi_{G_1}(\omega\lambda) + \phi_{G_1}(\omega^2\lambda)) + \frac{1+\lambda^{12}+\lambda^{24}}{(1-\lambda^6)^2(1-\lambda^{15})}$$

where $\phi_{G_1}(\lambda)$ is the Molien series of G_1.

Thus the relations

$$a_2^3 = a_1(h_4^2 + 1728h_2^5 - 720h_2^3a_2 + 80h_2a_2^2 - 64a_1(5h_2^2 - a_2)^2).$$

S^G is a free $\mathbb{C}[h_2,h_4,a_1]$-module with basis $1, a_2, a_2^2$ and $S^G \simeq \mathbb{C}[y_1,y_2,y_3,y_4]/(L)$ where y_i are indeterminates and $L = y_4^3 - y_3(y_2^2 + 1728y_1^5 - 720y_1^3y_4 + 80y_1y_4^2 - 64y_3(5y_1^2 - y_4)^2)$.

2.12. The invariants of group (K)

Let G be the group (K) then $|G| = 504$ and $G = G_1 \times G_2$ where G_1 is the group (I) and G_2 is the center of $SL(3,\mathbb{C})$, i.e., $G_2 = \{I, W, W^2\}$ where $W = (\omega,\omega,\omega)$ and $\omega = e^{(2\pi i)/3}$. Clearly, $G_1 \triangleleft G$ and $G = G_1 \cup G_1W \cup G_1W^2$. From section 2.10, S^{G_1} has a set of minimal generators a, b, c, d. Now

$$aW = \omega a, \quad aW^2 = \omega^2 a;$$

$$bW = b, \quad bW^2 = b;$$

$$cW = \omega^2 c, \quad cW^2 = \omega c;$$

$$dW = d, \quad dW^2 = d.$$

Thus S^G is generated by b, d and $a^jc^k + (a^jc^k)W + (a^jc^k)W^2$ where $0 \le 4j + 14k \le 504$ and $0 \le k \le 2$ (by Theorem 16 and (20)). The minimal generators of S^G are b, d and $b_1 = a_1^3$, $b_2 = ac$.
So $S^G = \mathbb{C}[b, d, b_1, b_2]$. The Molien series of G is

$$\phi(\lambda) = \frac{1}{3}(\phi_{G_1}(\lambda) + \phi_{G_1}(\omega\lambda) + \phi_{G_1}(\omega^2\lambda)) + \frac{1 + \lambda^{18} + \lambda^{36}}{(1-\lambda^6)(1-\lambda^{12})(1-\lambda^{21})}$$

where $\phi_{G_1}(\lambda)$ is the Molien series of G_1.

Thus the relations is

$$b_2^3 = b_1(d^2 + 88bb_2^2 - 1008b^4b_2 - 1088b^2b_1b_2 + 256b_1^2b_2 - 1728b^7$$
$$+ 60032b^5b_1 - 22016b^3b_1^2 + 2048bb_1^3).$$

S^G is a free $\mathbb{C}[b, d, b_1]$-module with basis $1, b_2, b_2^2$ and $S^G \simeq \mathbb{C}[y_1, y_2, y_3, y_4]/(L)$ where y_i are indeterminates and $L = y_4^3 - y_3(y_2^2 + 88y_1y_4^2 - 1008y_1^4y_4 - 1088y_1^2y_3y_4 + 256y_3^2y_4 - 1728y_1^7 + 60032y_1^5y_3 - 22016y_1^3y_3^2 + 2048y_1y_3^3)$.

2.13. The invariants of group (L)

Let G be the group (L) then $|G| = 1080$ and G/F is isomorphic to the alternating group A_6 where F is the center of $SL(3,\mathbb{C})$. It generated by

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \epsilon^4 & 0 \\ 0 & 0 & \epsilon \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$T = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 \\ 2 & s & t \\ 2 & t & s \end{pmatrix}, \quad V = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \lambda_1 & \lambda_1 \\ 2\lambda_2 & s & t \\ 2\lambda_2 & t & s \end{pmatrix},$$

where $\epsilon = e^{(2\pi i)/5}$, $s = \epsilon^2 + \epsilon^3 = \frac{1}{2}(-1 - \sqrt{5})$, $t = \epsilon + \epsilon^4 = \frac{1}{2}(-1 - \sqrt{5})$, $\lambda_1 = \frac{1}{4}(-1+\sqrt{15}i)$, $\lambda_2 = \frac{1}{4}(-1-\sqrt{15}i)$. As elements in $G/F \simeq A_6$, we have $\overline{S} = (12345)$, $\overline{U} = (14)(23)$, $\overline{T} = (12)(34)$, $\overline{V} = (14)(56)$.

We have $G = \{U^a S^b W^c, S^b T U^a S^{b_1} W^c, S^b V U^a S^{b_1} W^c, S^b T U S^{b_1} V U^a S^{b_2} W^c \big| 0 \leq a \leq 1,\ 0 \leq b,\ b_1, b_2 \leq 4,\ 0 \leq c \leq 2\}$ where $W = (\omega, \omega, \omega)$ and $\omega = e^{(2\pi i)/3}$. In fact, $W = (VS^4)(S^2TUSVS^3TUS^3)^{-1}$. The Molien series of G is

$$\phi(\lambda) = \frac{1}{3}(\phi_1(\lambda) + \phi_1(\omega\lambda) + \phi_1(\omega^2\lambda)) + \frac{1+\lambda^{45}}{(1-\lambda^6)(1-\lambda^{12})(1-\lambda^{30})}$$

where

$$\phi_1(\lambda) = \frac{1}{360}\Big[\frac{1}{\det(I-\lambda I)} + \frac{72}{\det(I-\lambda S)} + \frac{72}{\det(I-\lambda S^2)} + \frac{40}{\det(I-\lambda S^4 V)} + \frac{45}{\det(I-\lambda US^2)} + \frac{40}{\det(I-\lambda S^3TS^3)} + \frac{90}{\det(I-\lambda STUS^4VUS)}\Big].$$

Let $G_1 = \langle S, U, T\rangle$ then $G = G_1 \cup G_1 V \cup G_1(VS) \cup G_1(VS^2) \cup G_1(VS^3) \cup G_1(VS^4)$. Note that G_1 is the group (J) of order 180. From section 2.11, $h_1^3 = (x^2+yz)^3$, $h_2 = 8x^4yz - 2x^2y^2z^2 - x(y^5+z^5) + y^3z^3$ are invariants of G_1. Let $c_1 = 10(h_2 + h_2V + h_2(VS) + h_2(VS^2) + h_2(VS^3) + h_2(VS^4)) = (5+13\sqrt{15}i)x_6 + (375+15\sqrt{15}i)x^4yz - (75-45\sqrt{15}i)x^2y^2z^2 - (45-3\sqrt{15}i)xy^5 - (45-3\sqrt{15}i)xz^5 + (50+10\sqrt{15}i)y^3z^3$ and $c_2 = \frac{160}{3}(h_1^3 + h_1^3V + h_1^3(VS) + h_1^3(VS^2) + h_1^3(VS^3) + h_1^3(VS^4)) = (80+16\sqrt{15}i)x^6 + (600-120\sqrt{15}i)x^4yz + (150+90\sqrt{15}i)x^2y^2z^2 - (45-21\sqrt{15}i)xy^5 - (45-21\sqrt{15}i)xz^5 + (125-5\sqrt{15}i)y^3z^3$. Let $A = \frac{1}{15}(7c_1 - c_2)$ then

$$A = (-3+5\sqrt{15}i)x^6 + (135+15\sqrt{15}i)x^4yz - (45-15\sqrt{15}i)x^2y^2z^2 - 18xy^5 - 18xz^5 + (15+5\sqrt{15}i)y^3z^3$$

where is an invariant of degree of 6 of G. The Hessian of A is $H(A)$ of degree 12. Let

$$B = \frac{1}{81000 H(A)}.$$

The bordered Hessian of A, B is $BH(A,B)$ of degree 30. Let

$$C = \frac{1}{145800} BH(A,B).$$

The Jacobian of A, B, C is $J(A,B,C)$ of degree 45. Let

$$D = \frac{1}{9720} J(A,B,C).$$

Since the invariants B, C, D take a few pages long to write down, we don't want to give them explicitly here. So $S^G = \mathbb{C}[A,B,C,D]$.

The relation is

$$\begin{aligned} 459165024D^2 = {} & 25509168C^3 + (236196 + 26244\sqrt{15}i)C^2A^5 \\ & - 1889568(1+\sqrt{15}i)C^2A^3B - (8503056 - 2834352\sqrt{15}i)C^2AB^2 \\ & + (891 + 243\sqrt{15}i)CA^{10} + (5346 - 8910\sqrt{15}i)CA^8B \\ & - (360612 - 51516\sqrt{15}i)CA^6B^2 - (192456 + 21384\sqrt{15}i)CA^4B^3 \\ & + 3569184(1+\sqrt{15}i)CA^2B^4 + (7558272 - 2519424\sqrt{15}i)CB^5 \\ & + 2426112(1+\sqrt{15}i)B^7A - (7978176 + 886464\sqrt{15}i)B^6A^3 \\ & + (3297168 - 471024\sqrt{15}i)B^5A^5 - (78768 - 131280\sqrt{15}i)B^4A^7 \\ & - (26928 + 7344\sqrt{15}i)B^3A^9 + (1560 - 40\sqrt{15}i)B^2A^{11} \\ & - (17 - 7\sqrt{15}i)BA^{13}. \end{aligned}$$

S^G is a free $\mathbb{C}[A,B,C]$-module with basis 1, D and $S^G \simeq \mathbb{C}[y_1, y_2, y_3, y_4]/(L)$ where y_i are indeterminates and

$$\begin{aligned} L = {} & 459165204y_4^2 - 25509168y_3^3 - (236196 + 26244\sqrt{15}i)y_3^2y_1^5 \\ & + 1889568(1+\sqrt{15}i)y_3^2y_1^3y_2 + (8503056 - 2834352\sqrt{15}i)y_3^2y_1y_2^2 \\ & - (891 + 243\sqrt{15}i)y_3y_1^{10} - (5346 - 8910\sqrt{15}i)y_3y_1^8y_2 \\ & + (360612 - 51516\sqrt{15}i)y_3y_1^6y_2^2 + (192456 + 21384\sqrt{15}i)y_3y_1^4y_2^3 \\ & - 3569184(1+\sqrt{15}i)y_3y_1^2y_2^4 - (7558272 - 2519424\sqrt{15}i)y_3y_2^5 \\ & - 2426112(1+\sqrt{15}i)y_2^7y_1 + (7978176 + 886464\sqrt{15}i)y_2^6y_1^3 \\ & - (3297168 - 471024\sqrt{15}i)y_2^5y_1^5 + (78768 - 131280\sqrt{15}i)y_2^4y_1^7 \\ & + (26928 + 7344\sqrt{15}i)y_2^3y_1^9 - (1560 - 40\sqrt{15}i)y_2^2y_1^{11} + (17 - 7\sqrt{15}i)y_2y_1^{13}. \end{aligned}$$

We have used the REDUCE program [He] to perform the following computations.

1. The relations for types (G), (H), (I), (L).
2. The Molien series for all types.
3. The invariants for types (H), (I), (L).

Since the groups of types (H), (I), (L) are isomorphic to permutation groups, CAYLEY program [Can] helps us compute the conjugacy classes of these groups. With these results ready, we use REDUCE program to calculate the Molien series of these groups.

CHAPTER 3

GORENSTEIN QUOTIENT SINGULARITIES IN DIMENSION THREE

We first recall some results of H. Cartan [Car]. Given a finite subgroup G of the complex linear group $GL(n,\mathbb{C})$. The analytic space $\mathbb{C}^n/G$ is defined as follows:

(i) A point of $\mathbb{C}^n/G$ is an orbit of G acting on the complex n-dimensional vector space $\mathbb{C}^n$. The space $\mathbb{C}^n/G$ has the quotient topology.

(ii) Let $\tau : \mathbb{C}^n \to \mathbb{C}^n/G$ assign to each point its orbit. A continuous function f on an open set $U \subset \mathbb{C}^n/G$ is "analytic" if $f \circ \tau$ is analytic on $\tau^{-1}(U)$.

In [Car], Cartan proved that the space $\mathbb{C}^n/G$ is a normal analytic variety, with the analytic functions given by (ii).

Let $S = \mathbb{C}[x_1, \ldots, x_n]$. By Theorem 15, the subalgebra S^G is finitely generated. Let $f_1, \ldots, f_k$ be a minimal set of homogeneous polynomials in S^G which generate S^G as a $\mathbb{C}$-algebra. The vector $(f_1, \ldots, f_k)$ is a $\mathbb{C}$-generic point of an affine algebraic subvariety V_G of $\mathbb{C}^k$. The map $\psi : \mathbb{C}^n \to V_G$ defined by

$$\psi(z) = (f_1(z), \ldots, f_k(z)), \quad z \in \mathbb{C}^n$$

is constant on orbits of G. Define $\phi : \mathbb{C}^n/G \to V_G$ by requiring that

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{\psi} & V_G \subseteq \mathbb{C}^k \\ {\scriptstyle\tau}\big\downarrow & \phi \nearrow & \\ \mathbb{C}^n/G & & \end{array}$$

be commutative. Then Cartan [Car] showed that ϕ is a biholomorphic map. The relations of S^G defined in chapter 2 are precisely the defining equations of this affine algebraic subvariety V_G of $\mathbb{C}^k$.

$g \in G$ is called a quasi-reflection if it has 1 as an eigenvalue of multiplicity precisely $n-1$. A group $G \subset GL(n,\mathbb{C})$ is small if it contains no quasi-reflection elements. A classical theorem of Shephard and Todd [Sh-To] and Chevalley [Ch] says that V_G is smooth if and only if G is generated by quasi-reflections. An important observation of Prill [Pr] says that: for every finite group $G \subset GL(n,\mathbb{C})$, there exists a small subgroup $G' \subset GL(n,\mathbb{C})$ such that V_G is biholomorphically equivalent to

V'_G. Therefore, in order to study $\mathbb{C}^n/G$, it suffices to consider small subgroup G of $GL(n,\mathbb{C})$. The following lemmas are quite standard (cf. [Kh],[Wa] for a proof).

Lemma 21. Let $G \subset GL(n,\mathbb{C})$ be a small subgroup, and let $S = \{z \in \mathbb{C}^n : g(z) = z \text{ for some } g \neq 1\}$. Then the singular locus of V_G is S/G.

Lemma 22.(Khinich and Watanabe) V_G is Gorenstein if and only if $G \subset SL(n,\mathbb{C})$.

The following Theorem is a surprising result of the classification theorem in chapter 1.

Theorem 23. Three dimensional Gorenstein quotient singularity V_G is isolated if and only if G is an abelian subgroup of $SL(n,\mathbb{C})$ and 1 is not an eigenvalue of g for every nontrivial element g in G.

In view of the classification theorem of finite subgroups in $SL(3,\mathbb{C})$ and Lemma 22 that the finite subgroups (B)–(L) listed in chapter 1 give nonisolated quotient singularities. For finite subgroups (C)–(L) this is clear because all these groups contain the element

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

which has 1 as eigenvalue. By Lemma 21, finite subgroups (C)–(L) listed in chapter 1 give nonisolated quotient singularities.

Group (B) is isomorphic to transitive linear groups of $GL(2,\mathbb{C})$, i.e., each element has the form of

$$g = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}, \quad \alpha(ad - bc) = 1.$$

Clearly eigenvalue of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is also an eigenvalue of g. Observe that if the subgroup $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = \alpha^{-1} \right\} \subseteq GL(2,\mathbb{C})$ is not small, then of course 1 as an eigenvalue of a nontrivial element of G. On the other hand if $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = \alpha^{-1} \right\}$ is small, then we have the complete list (B_1)–(B_4) as shown in chapter 1. In both cases (B_1)–(B_4) contain a nontrivial element which has 1 as eigenvalue. In view of Lemma 21, our proof is completed. Q.E.D.

Finally, by the previous discussion, we can summarize some of the results of chapter 2 in the following Theorem.

Theorem 24. Let G be finite subgroup of $SL(3,\mathbb{C})$. Then we have

Type of G	Minimal embedding dimension of $\mathbb{C}^3/G$	Equations of $\mathbb{C}^3/G$
(E)	5	$\begin{cases} 9y_4^2 - 12y_3^2 + y_1^2y_3 - y_1^2y_4 = 0 \\ 432y_5^2 - y_2^3 + 2y_1^3 - 36y_1y_4 \\ \qquad + 3y_1^2y_2 - 36y_2y_3 = 0 \end{cases}$
(F)	4	$4y_4^3 - 144y_2y_4^2 + 1728y_2y_4$ $- (y_1^3 - 432y_3^2 - 3y_1y_4 + 36y_1y_2)^2 = 0$
(G)	4	$4y_4^3 - 9y_3y_4^2 + 6y_3^2y_4 - 2592y_1^2y_3y_4 - y_3^3$ $+ 864y_1^2y_3^2 + 6912y_2^3y_3 - 186624y_1^4y_3 = 0$
(H)	4	$y_4^2 + 1728y_2^5 - y_3^3 - 720y_1y_2^3y_3$ $+ 80y_1^2y_2y_3^2 - 64y_1^3(5y_2^2 - y_1y_3)^2 = 0$
(I)	4	$y_4^2 - y_3^3 + 88y_1^2y_2y_3^2 - 1008y_1y_2^4y_3$ $- 1088y_1^4y_2^2y_3 + 256y_1^7y_3 - 1728y_2^7$ $+ 60032y_1^3y_2^5 - 22016y_1^6y_2^3 + 2048y_1^9y_2 = 0$
(J)	4	$y_4^3 - y_3[y_2^2 + 1728y_1^5 - 720y_1^3y_4$ $+ 80y_1y_4^2 - 64y_3(5y_1^2 - y_4)^2] = 0$
(K)	4	$y_4^3 - y_3(y_2^2 + 88y_3y_4 - 1008y_1^4y_4$ $- 1088y_1^2y_3y_4 + 256y_3^2y_4 - 1728y_1^7$ $+ 60032y_1^5y_3 - 22016y_1^3y_3^2 + 2048y_1y_3^3) = 0$
(L)	4	$459165024y_4^2 - 25509168y_3^3 - (236196$ $+ 26244\sqrt{15}i)y_3^2y_1^5 + 1889568(1 + \sqrt{15}i)y_3^2y_1^3y_2$ $+ (8503056 - 2834352\sqrt{15}i)y_3^2y_1y_2^2$

			$-(891+243\sqrt{15}i)y_3y_1^{10}$
			$-(5346-8910\sqrt{15}i)y_3y_1^8y_2$
			$+(360612-51516\sqrt{15}i)y_3y_1^6y_2^2$
			$+(192456+21384\sqrt{15}i)y_3y_1^4y_2^3$
			$-3569184(1+\sqrt{15}i)y_3y_1^2y_2^4$
			$-(7558272-2519424\sqrt{15}i)y_3y_2^5$
			$-2426112(1+\sqrt{15}i)y_2^7y_1$
			$+(7978176+886464\sqrt{15}i)y_2^6y_1^3$
			$-(3297168-471024\sqrt{15}i)y_2^5y_1^5$
			$+(78768-131280\sqrt{15}i)y_2^4y_1^7$
			$+(26928+7344\sqrt{15}i)y_2^3y_1^9$
			$-(1560-40\sqrt{15}i)y_2^2y_1^{11}$
			$+(17-7\sqrt{15}i)y_2y_1^{13}=0$

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