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# Cohomology and Splitting Criterion for Holomorphic Vector Bundles on $\mathbb{CP}^n$

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## 1. Introduction

A holomorphic vector bundle $E$ on $\mathbb{CP}^n$ splits if it is isomorphic to the direct sum of line bundles. One of the most important problems in Algebraic Geometry is to decide whether $E$ splits if the rank of $E$ is relatively small. In fact, the famous Hartshorne conjecture says that any rank two holomorphic vector bundle on $\mathbb{CP}^n (n \geq 6)$ splits.

Let $\tilde{E}$ be the pull back of $E$ to $\mathbb{C}^{n+1} - \{0\}$. Then the famous Horrocks splitting criterion asserts that $E$ splits if and only if $H^i(\mathbb{C}^{n+1} - \{0\}, \tilde{E}) = 0$ for $1 \leq i \leq n-1$. The purpose of this paper is to introduce a method to prove the vanishing of $H^i(\mathbb{C}^{n+1} - \{0\}, \tilde{E})$.

**Theorem A.** *Let be a holomorphic vector bundle of rank k over* $\mathbb{C}^{n+1} - \{0\}$, $n \geq 2$. *Suppose that E admits a* (1, 0)*-connection which is holomorphic in direction* $z_0$ (cf. Definition 2.6 below). *Then* $H^i(\mathbb{C}^{n+1} - \{0\}, E) = 0$ *for* $1 \leq i \leq n-1$.

As a consequence of this result, we can prove:

**Theorem B.** *Let E be a holomorphic vector bundle on* $\mathbb{CP}^n$, $n \geq 2$. *Then E is a direct sum of holomorphic line bundles if and only if* $H^1(\mathbb{CP}^n, (E \otimes E^*)(k)) = 0$ *for all* $k \in \mathbb{Z}$.

KEMPF [4] proved a result similar to Theorem B. Although he needs $H^1(\mathbb{CP}^n, (E \otimes E^*)(k)) = 0$ for $k$ negative only, he does need to make an additional condition that $E$ is a restriction of holomorphic vector bundle on $\mathbb{CP}^{n+2}$. In view of this and the fact that our method is not familiar to many algebraic geometers, perhaps it will be of some use to publish the results.

If $E$ is a rank 2 holomorphic vector bundle, then we have even a stronger theorem.

**Theorem C.** *Let* $\tilde{E}$ *be the pull back of a holomorphic rank* 2 *bundle E from* $\mathbb{CP}^n$ *to*

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$\mathbb{C}^{n+1} - \{0\}$. *Suppose that* $\tilde{E}$ *admits a* (1, 0)-*connection which is weakly holomorphic in direction* $z_0$ (cf. Definition 3.1 below). *Then* $H^i(\mathbb{C}^{n+1} - \{0\}, \tilde{E}) = 0$ *for* $1 \le i \le n-1$ *and* $E$ *is a direct sum of line bundles.*

## 2. Holomorphic vector bundles over $\mathbb{CP}^n$

We start with the following three well known lemmas.

**Lemma 2.1.** *Let* $L$ *be a holomorphic line bundle on* $\mathbb{C}^{n+1} - \{0\}, n \ge 2$. *Then* $L$ *is holomorphically trivial and* $H^i(\mathbb{C}^{n+1} - \{0\}, L) = 0$ *for* $1 \le i \le n-1$.

Proof. The proof follows from the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 1$$

and the fact that $H^i(\mathbb{C}^{n+1} - \{0\}, \mathcal{O}) = 0$, $1 \le i \le n-1$, and $H^2(\mathbb{C}^{n+1} - \{0\}, \mathbb{Z}) = 0$ for $n \ge 2$. □

**Lemma 2.2.** *Let* $\pi: L \to X$ *be a holomorphic line bundle over a complex manifold* $X$. *Let* $E$ *vector bundle over* $X$. *Then* $H^i(L - X, \pi^* E) = \bigoplus_{n=-\infty}^{\infty} H^i(X, E \otimes L^n)$.

Proof. If we represent $H^i(L - X, \pi^* E)$ as Čech-cohomology, then the result follows by taking the power series of the Čech-cocycle along the fiber of $L$. □

**Lemma 2.3. (Splitting criterion of Horrocks).** *Let* $E$ *be a holomorphic vector bundle over* $\mathbb{CP}^n$ *and* $\pi: \mathbb{C}^{n+1} - \{0\} \to \mathbb{CP}^n$ *be the natural projection. Then* $E$ *is a direct sum of holomorphic line bundles, if and only if* $H^i(\mathbb{C}^{n+1} - \{0\}, \pi^* E) = 0$ *for all* $1 \le i \le n-1$.

**Definition 2.4.** Let $E$ be a holomorphic vector bundle. We say that a connection $D$ on $E$ is *compatible with the complex structure* if $D'' = \bar{\partial}$ where $D = D' + D''$ with $D': A^{(0,0)}(E) \to A^{(0,1)}(E)$ and $D'': A^{(0,0)}(E) \to A^{(0,1)}(E)$, $A^{(p,q)}(E)$ being the space of $C^\infty(p, q)$-forms with coefficients in $E$.

**Definition 2.5.** Let $e = \{e_1, \dots, e_k\}$ be a holomorphic frame for $E$ over an open set $U$. Given a connection $D$ on $E$ which is compatible with the complex structure, we can decompose $D' e_i$ into its components, writing

$$D' e_i = \sum \theta_{ij} e_j.$$

The matrix $\theta = (\theta_{ij})$ of (1, 0)-forms is called the *connection matrix of D with respect to e.*

**Definition 2.6.** Let $E$ be a holomorphic vector bundle of rank $k$ over $\mathbb{C}^{n+1} - \{0\}$. We say that the connection $D'$ is *holomorphic in direction* $z_0$ if $\theta_{ij}^0$ is holomorphic for all $i, j$ where $\theta_{ij} = \theta_{ij}^0 dz_0 + \theta_{ij}^1 dz_1 + \dots + \theta_{ij}^n dz_n$.

**Remark 2.7.** Definition 2.6 above is well-defined: if $e' = \{e'_1, \dots, e'_k\}$ is another holomorphic frame with

$$e'_i(z) = \sum g_{ij}(z)\, e_j(z) \quad \text{where} \quad g_{ij}(z) \quad \text{are holomorphic},$$

then

(2.1) $$D'e'_i = \sum \partial g_{ij} e_j + \sum g_{ik}\theta_{kj} e_j$$

so that

(2.2) $$\theta_{e'} = \partial g \cdot g^{-1} + g \cdot \theta_e \cdot g^{-1} \quad \text{where} \quad g = (g_{ij}).$$

Therefore if $\theta_e$ is holomorphic in direction $z_0$, then so is $\theta_{e'}$.

**Theorem 2.8.** *Let $E$ be a holomorphic vector bundle of rank $k$ over $\mathbb{C}^{n+1} - \{0\}$, $n \geq 2$. Suppose that $E$ admits a connection $D'$ which is holomorphic in direction $z_0$. Then $H^i(\mathbb{C}^{n+1} - \{0\}, E) = 0$ for $1 \leq i \leq n-1$.*

Proof. If we view $E$ as a locally free sheaf over $\mathbb{C}^{n+1} - \{0\}$, then $E$ can be extended to $\mathbb{C}^{n+1}$ as coherent analytic sheaf. By the local cohomology exact sequence

$$H^q(\mathbb{C}^{n+1}, E) \to H^q(\mathbb{C}^{n+1} - \{0\}, E) \to H^{q+1}_{\{0\}}(\mathbb{C}^{n+1}, E) \to H^{q+1}(\mathbb{C}^{n+1}, E),$$

we conclude that $H^q(\mathbb{C}^{n+1} - \{0\}, E) \cong H^{q+1}_{\{0\}}(\mathbb{C}^{n+1}, E)$ for $q \geq 1$. On the other hand, local duality asserts that $H^{q+1}_{\{0\}}(\mathbb{C}^{n+1}, E)$ is dual to $\mathrm{Ext}^{n-q}_{\mathcal{O}_0}(E_0, \mathcal{O}_0)$. Since $\mathrm{Ext}^{n-q}_{\mathcal{O}}(E, \mathcal{O})$ is a coherent analytic sheaf and is supported at 0 because $E$ is locally free outside 0, we conclude that $H^q(\mathbb{C}^{n+1} - \{0\}, E)$ is finite dimensional for $1 \leq q \leq n-1$.

Let $\sigma \in H^q(\mathbb{C}^{n+1} - \{0\}, E)$. Then $\sigma$ can be represented as a Dolbeault cohomology class $\sigma = \sum \sigma_i e_i \in A^{(0,q)}(E)$ such that $\bar\partial\sigma = 0$. Here $(e_1, \ldots, e_k\}$ is a holomorphic frame for $E$ over an open set $U$. Let $\lambda = \dfrac{\partial}{\partial z_0}$. Clearly

$$\lambda D'\sigma = \sum_{j=1}^{k} \lambda\left(\partial\sigma_j + \sum_{i=1}^{k} \sigma_i\theta_{ij}\right)e_j$$

is an element in $A^{(0,q)}(E)$ again. We claim that $\bar\partial(\lambda D'\sigma) = 0$ so that $\lambda D'\sigma$ is again a cohomology class in $H^q(\mathbb{C}^{n+1} - \{0\}, E)$. To see this, observe that $\bar\partial$ commutes with $\lambda$. Hence

$$\bar\partial(\lambda D'\sigma) = \bar\partial\lambda\left(\sum_{j=1}^{k} \partial\sigma_j + \sum_{j=1}^{k}\sum_{i=1}^{k} \sigma_i\theta_{ij}\right)e_j = \sum_{j=1}^{k} \lambda\bar\partial(\partial\sigma_j)e_j + \sum_{j=1}^{k}\sum_{i=1}^{k} \bar\partial(\sigma_i\theta^0_{ij})e_j$$
$$= 0 \quad \text{since} \quad \bar\partial\sigma_i = 0 \quad \text{and} \quad \bar\partial\theta^0_{ij} = 0.$$

Let $f$ be any holomorphic function on $\mathbb{C}^{n+1}$. Then

(2.3) $$(\lambda D'f - f\lambda D')(\sigma) = \lambda D'(f\sigma) - f\,\lambda D'(\sigma)$$
$$= \lambda\sum_{j=1}^{k}\left(\partial(f\sigma_j) + \sum_{i=1}^{k} f\sigma_i\theta_{ij}\right)e_j - f\lambda\sum_{j=1}^{k}\left(\partial\sigma_j + \sum_{i=1}^{k}\sigma_i\theta_{ij}\right)e_j$$
$$= \lambda\sum_{j=1}^{k} f\left(\partial\sigma_j + \sum_{i=1}^{k}\sigma_i\theta_{ij}\right)e_j + \lambda\sum_{j=1}^{k}(\partial f)\sigma_j e_j - f\lambda\sum_{j=1}^{k}\left(\partial\sigma_j + \sum_{i=1}^{k}\sigma_i\theta_{ij}\right)e_j$$
$$= (\lambda\partial f)\sigma = \frac{\partial f}{\partial z_0}\sigma.$$

Observe that $z$ acts on $H^q(\mathbb{C}^{n+1} - \{0\}, E)$. Since for $1 \leq q \leq n-1$, $H^q(\mathbb{C}^{n+1} - \{0\}, E)$ is finite dimensional, so it has a minimal polynomial $p(z_0)$. Of course $p(z_0)$ acts on $H^q(\mathbb{C}^{n+1} - \{0\}, E)$ trivially. By taking $f = p(z_0)$ in (2.3), we see that $\dfrac{\partial p}{\partial z_0}(z_0)$ also acts on $H^q(\mathbb{C}^{n+1}-\{0\},E)$ trivially. We conclude that a nonzero constant acts on $H^q(\mathbb{C}^{n+1}-\{0\},E)$ trivially by repeating the above argument. This simply means that $H^q(\mathbb{C}^{n+1}-\{0\},E)=0$.

**Theorem 2.9.** *Let $E$ be a holomorphic vector bundle on $\mathbb{CP}^n$, $n\ge 2$. Then $E$ is a direct sum of holomorphic line bundles if and only if $H^1(\mathbb{CP}^n,(E\otimes E^*)(k))=0$ for all $k\in\mathbb{Z}$.*

Proof. Suppose that $E$ is a direct sum of holomorphic line bundles on $\mathbb{CP}^n$. Then $\tilde{E}=\pi^*(E)$, the pull back of $E$ to $\mathbb{C}^{n+1}-\{0\}$, is also a direct sum of holomorphic line bundles on $\mathbb{C}^{n+1}-\{0\}$. So we have $H^1(\mathbb{C}^{n+1}-\{0\},\tilde{E}\otimes\tilde{E}^*)=0$ by Lemma 2.1. In view of Lemma 2.2, we have $H^1(\mathbb{CP}^n,(E\otimes E^*)(k))=0$ for all $k\in\mathbb{Z}$.

Conversely, suppose that $H^1(\mathbb{CP}^n,(E\otimes E^*)(k))=0$ for all $k\in\mathbb{Z}$, i.e. $H^1(\mathbb{C}^{n+1}-\{0\},\tilde{E}\otimes\tilde{E}^*)=0$. In view of Lemma 2.3 and Theorem 2.8, it suffices to prove that $\tilde{E}$ admits a connection $D'$ which is holomorphic in direction $z_0$. Applying $\bar\partial$ to (2.2), we have

(2.4) $$\bar\partial\theta_{e'}=g\cdot\bar\partial\theta_e\cdot g^{-1}.$$

Write $\theta_{e'}=\left(\sum_{\alpha=0}^{n}\theta'^{\alpha}_{ij}dz_\alpha\right)_{1\le i,j\le k}$. Then (2.4) implies

(2.5) $$(\bar\partial\theta'_{ij})_{1\le i,j\le k}=g\cdot(\bar\partial\theta^0_{ij})_{1\le i,j\le k}\cdot g^{-1}.$$

Therefore $(\bar\partial\theta'^0_{ij})_{1\le i,j\le k}$ represents a $C^\infty$ section of $A^{(0,1)}(\tilde{E}\otimes\tilde{E}^*)$. Clearly it is $\bar\partial$ closed. By the hypothesis $H^1(\mathbb{C}^{n+1}-\{0\},\tilde{E}\otimes\tilde{E}^*)=0$, we conclude that there exists $\gamma\in A^{(0,0)}(\mathbb{C}^{n+1}-\{0\},\tilde{E}\otimes\tilde{E}^*)$ such that

$$\bar\partial(\gamma'^0_{ij})_{1\le i,j\le k}=(\bar\partial\theta'^0_{ij})_{1\le i,j\le n}.$$

Consider $\tilde\theta_{e'}=\theta_{e'}-\gamma_{e'}dz_0$. Then clearly $\bar\partial\tilde\theta'^0_{ij}=\bar\partial\theta'^0_{ij}-\bar\partial\gamma'^0_{ij}=0$. So $\tilde{D}$ is a connection of $\tilde{E}$ which is holomorphic in direction $z_0$. □

## 3. Rank 2 holomorphic bundles

In this section, we shall assume that $E$ is a holomorphic rank 2 bundle over $\mathbb{C}^{n+1}-\{0\}$, $n\ge 2$. The purpose is to show that the vanishing result is still true even under a weaker condition. We shall use the same notations as in § 2.

Let $e^*=\{e_1^*,e_2^*\}$ be a dual basis for $E^*$ over $U$, i.e. $\langle e_i,e_j^*\rangle=\delta_{ij}$. Observe that

$$\delta_{ij}=\langle e'_i,e'^*_j\rangle=\left\langle\sum_{m=1}^{2}g_{im}e_m,\sum_{l=1}^{2}g^*_{jl}e^*_l\right\rangle=\sum_{m,l=1}^{2}g_{im}g^*_{jl}\delta_{ml}=\sum_{m=1}^{2}g_{im}g^*_{jm}.$$

Therefore $g^*=(g^{-1})^T$. Let $\theta^*_{e'}=-\theta^T_{e'}$. Then we claim that

$$\theta^*_{e'}=\partial g^*\cdot {g^*}^{-1}+g^*\cdot\theta^*_e\cdot {g^*}^{-1}.$$

This follows immediately from (2.2). Alternately, $D^*$ on $E^*$ can be defined by the requirement

$$d\langle\sigma,\tau\rangle=\langle D\sigma,\tau\rangle+\langle\sigma,D^*\tau\rangle$$

for $\sigma\in A^0(E)$ and $\tau\in A^0(E^*)$.

Recall that $D_{E\otimes E^*} = D_E \otimes I + I \otimes D_{E^*}$ (cf. [1]). Therefore we have

$$\begin{aligned} D_{E\otimes E^*}(e_1 \otimes e_1^*) &= De_1 \otimes e_1^* + e_1 \otimes De_1^* \\ &= \theta_{11} e_1 \otimes e_1^* + \theta_{12} e_2 \otimes e_1^* + e_1 \otimes \theta_{11}^* e_1^* + e_1 \otimes \theta_{12}^* e_2^* \\ &= (\theta_{11} + \theta_{11}^*) e_1 \otimes e_1^* + \theta_{12}^* e_1 \otimes e_2^* + \theta_{12} e_2 \otimes e_1^* \\ &= -\theta_{21} e_1 \otimes e_2^* + \theta_{12} e_2 \otimes e_1^* . \end{aligned}$$

Similarly one can show that

$$\begin{aligned} D_{E\otimes E^*}(e_1 \otimes e_2^*) &= -\theta_{12} e_1 \otimes e_1^* + (\theta_{11} - \theta_{22}) e_1 \otimes e_2^* + \theta_{12} e_2 \otimes e_2^* \\ D_{E\otimes E^*}(e_2 \otimes e_1^*) &= \theta_{21} e_1 \otimes e_1^* + (\theta_{22} - \theta_{11}) e_2 \otimes e_1^* - \theta_{21} e_2 \otimes e_2^* \\ D_{E\otimes E^*}(e_2 \otimes e_2^*) &= \theta_{21} e_1 \otimes e_2^* - \theta_{12} e_2 \otimes e_1^* . \end{aligned}$$

The connection matrix of $D_{E\otimes E^*}$ is of the form

$$\theta_{E\otimes E^*} = \begin{bmatrix} 0 & -\theta_{12} & \theta_{21} & 0 \\ -\theta_{21} & \theta_{11} - \theta_{22} & 0 & \theta_{21} \\ \theta_{12} & 0 & \theta_{22} - \theta_{11} & -\theta_{12} \\ 0 & \theta_{12} & -\theta_{21} & 0 \end{bmatrix} . \tag{3.1}$$

**Definition 3.1.** Let $E$ be a holomorphic vector bundle of rank 2 over $\mathbb{C}^{n+1} - \{0\}$. The connection $D$ is said to be *weakly holomorphic in direction* $z_0$ if $\theta_{12}^0$, $\theta_{21}^0$ and $\theta_{11}^0 - \theta_{22}^0$ are holomorphic.

**Lemma 3.2.** *Definition* 3.1 *above is well-defined.*

Proof. It follows easliy from (2.2) and the fact that

$$g \cdot \theta_e \cdot g^{-1} = \frac{1}{\det g} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix}$$

$$= \frac{1}{\det g} \begin{bmatrix} \begin{array}{l} g_{22}g_{11}\theta_{11} + g_{22}g_{12}\theta_{21} \\ -g_{21}g_{11}\theta_{12} - g_{21}g_{12}\theta_{22} \end{array} & \begin{array}{l} -g_{12}g_{11}(\theta_{11} - \theta_{12}) \\ -g_{12}^2\theta_{21} + g_{12}^2\theta_{12} \end{array} \\ \begin{array}{l} g_{22}g_{21}(\theta_{11} - \theta_{22}) \\ +g_{22}^2\theta_{21} - g_{21}^2\theta_{12} \end{array} & \begin{array}{l} -g_{12}g_{21}\theta_{11} - g_{12}g_{22}\theta_{21} \\ +g_{11}g_{21}\theta_{12} + g_{11}g_{22}\theta_{22} \end{array} \end{bmatrix} . \quad \square$$

We are now ready to prove the following vanishing theorem.

**Theorem 3.3.** *Let $E$ be a holomorphic vector bundle of rank 2 over $\mathbb{CP}^n$ and $\tilde{E}$ be its pull back to $\mathbb{C}^{n+1} - \{0\}$. Suppose that $\tilde{E}$ admits a (1, 0)-connection which is weakly holomorphic in direction $z_0$. Then $H^i(\mathbb{C}^{n+1} - \{0\}, \tilde{E}) = 0$ for $1 \le i \le n-1$, and $E$ is a direct sum of line bundles.*

Proof. Since $\tilde{E}$ admits a weakly holomorphic connection in direction $z_0$, in view of (3.1) $\tilde{E} \otimes \tilde{E}^*$ admits a holomorphic connection in direction $z_0$. By Theorem 2.8, $H^1(\mathbb{C}^{n+1} - \{0\}, \tilde{E} \otimes \tilde{E}^*) = 0$ i.e. $H^1(\mathbb{CP}^n, (E \otimes E^*)(k)) = 0$ for all $k \in \mathbb{Z}$. Because of Theorem 2.9, we conclude that $E$ is a direct sum of line bundles. The rest of the theorem follows from Lemma 2.1. $\square$

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