# GORENSTEIN SINGULARITIES WITH GEOMETRIC GENUS EQUAL TO TWO

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**Introduction.** Let $p$ be a singularity of a normal two-dimensional analytic space $V$. In [1], M. Artin introduced a definition for $p$ to be rational. Rational singularities have also been studied by, for instance, DuVal [5], Tyurina [31], Laufer [17], and Lipman [22]. In [33], Wagreich introduced a definition for $p$ to be weakly elliptic. Weakly elliptic singularities have occurred naturally in papers by Grauert [6], Hirzebruch [10], Laufer [19], Orlik and Wagreich [24], [25], and Wagreich [34]. Karras [12, 13] and Saito [27] have studied some of these particular elliptic singularities. Recently, Laufer [20] developed a theory for a general class of weakly elliptic singularities which satisfy a minimality condition. These are so-called the minimally elliptic singularities. In [36], we develop a theory for a general class of weakly elliptic singularities which satisfy a maximality condition. Maximally elliptic singularities include minimally elliptic singularities in the sense of Laufer as a particular case.

Let $\pi: M \to V$ be a resolution of $V$. It is known that $h = \dim H^1(M, \mathcal{O})$ is independent of resolution. One might classify singularities by $h$. Rational singularity is equivalent to $h=0$. Minimally elliptic singularity is equivalent to saying that $h=1$ and ${}_V\mathcal{O}_p$ is Gorenstein. Maximally elliptic singularities may have $h = \dim H^1(M, \mathcal{O})$ arbitrarily large. It is a natural question to ask for a theory for $h=2$ and ${}_V\mathcal{O}_p$ Gorenstein. Our main interest is to build up a theory for those singularities which has $h=2$ and ${}_V\mathcal{O}_p$ is Gorenstein, although we sometimes refer to almost minimally elliptic singularities.

All undefined terms and notation are standard and are described in [20] and [36]. Throughout this paper, $E$ will denote the minimally elliptic cycle and $Z$ will denote the fundamental cycle.

Our main results are the following. Recall that $h=2$ and ${}_V\mathcal{O}_p$ Gorenstein implies that $p$ is weakly elliptic, i.e., that $\chi(Z)=0$.

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THEOREM A. *Suppose $p$ is an almost minimally elliptic singularity (see Definition 1.1) and ${}_V\mathcal{O}_p$ is Gorenstein. Then $H^1(M,\mathcal{O})=\mathbb{C}^2$.*

THEOREM B. *Let $\pi: M\to V$ be the minimal good resolution of a normal two-dimensional Stein space $V$ with $p$ as its only singularity. Suppose $H^1(M,\mathcal{O})=\mathbb{C}^2$ and ${}_V\mathcal{O}_p$ is Gorenstein. Then $p$ is an almost minimally elliptic singularity if and only if $H^0(M,\mathcal{O}(-Z)/\mathcal{O}(-Z-E))=\mathbb{C}$.*

*Definition* 0.1. Let $D=\sum_{i\in\Lambda}d_iA_i$ be a positive cycle. Let $B=\bigcup_{i\in\Lambda_1}A_i\subseteq|D|$ where $\Lambda_1\subseteq\Lambda$. Then $D/B=\sum_{i\in\Lambda}f_iA_i$ is a positive cycle such that $f_i=d_i$ if $A_i\subseteq B$ and $f_i=0$ if $A_i\not\subseteq B$.

*Definition* 0.2. Let $A$ be the exceptional set of the minimal good resolution $\pi: M\to V$ where $V$ is a normal two-dimensional Stein space with $p$ as its only weakly elliptic singularity. If $E\cdot Z<0$, we say that this *elliptic sequence* is $\{Z\}$ and the *length of the elliptic sequence* is equal to one. Suppose $E\cdot Z=0$. Let $B_1$ be the maximal connected subvariety of $A$ such that $B_1\supseteq\operatorname{supp}E$ and $A_i\cdot Z=0$ $\forall A_i\subseteq B_1$. Since $A$ is an exceptional set, $Z\cdot Z<0$. So $B_1$ is properly contained in $A$. Suppose $Z_{B_1}\cdot E=0$. Let $B_2$ be the maximal connected subvariety of $B_1$ such that $B_2\supseteq|E|$ and $A_i\cdot Z_{B_1}=0$ $\forall A_i\subseteq B_2$. By the same argument as above, $B_2$ is properly contained in $B_1$. Continuing this process, we finally obtain $B_m$ with $Z_{B_m}\cdot E<0$. We call $\{Z_{B_0}=Z, Z_{B_1},\dots,Z_{B_m}\}$ *the elliptic sequence*, and *the length of the elliptic sequence* is $m+1$.

THEOREM C. *Let $\pi: M\to V$ be the minimal good resolution of normal two-dimensional Stein space $V$ with $p$ as its only weakly elliptic singularity. Suppose $H^1(M,\mathcal{O})=\mathbb{C}^2$, $H^1(|E|,\mathbb{Z})=0$, and ${}_V\mathcal{O}_p$ is Gorenstein. Let $Z_{B_0}, Z_{B_1},\dots,Z_{B_l}Z_E$ be the elliptic sequence. Let $D$ be the subvariety of $B_l$ consisting of those irreducible components $A_i\subseteq B_l$ such that $A_i\cap|E|\neq\varnothing$. If $Z/D=Z_{B_l}/D$, then $l=0$, i.e., $p$ is an almost minimally elliptic singularity.*

THEOREM D. *Let $\pi: M\to V$ be the minimal good resolution of normal two-dimensional Stein space with $p$ as its only singularity. Suppose $H^1(M,\mathcal{O})=\mathbb{C}^2$ and ${}_V\mathcal{O}_p$ is Gorenstein. Let $Z_{B_0}=Z$, $Z_{B_1},\dots,Z_{B_l}, Z_E$ be the elliptic sequence. Then $m\mathcal{O}\subseteq\mathcal{O}(-\sum_{i=0}^{l}Z_{B_i})$. If $Z_E\cdot Z_E\leqslant-2$, then $m\mathcal{O}=\mathcal{O}(-\sum_{i=0}^{l}Z_{B_i})$. If $Z_E\cdot Z_E\leqslant-3$, then $m^n\cong H^0(A,\mathcal{O}(-n(\sum_{i=0}^{l}Z_{B_i})))$ and the Hilbert function $H_{V,p}^{(0)}(n)=\dim m^n/m^{n+1}=-n(\sum_{i=0}^{l}Z_{B_i}^2)$ for $n\geqslant1$.*

THEOREM E. *Let $\pi: M\to V$ be the minimal good resolution of normal two-dimensional Stein space with $p$ as its only almost minimally elliptic singularity. If $Z_E\cdot Z_E\leqslant-3$ and ${}_V\mathcal{O}_p$ is Gorenstein, then $p$ is absolutely isolated. Moreover, the blowup $p$ at its maximal ideal yields exactly those curves $A_i$ such* *that* $A_i \cdot Z > 0$. *The singularities remaining after the blowup are the rational double points and a minimally elliptic singularity corresponding to deleting the* $A_i$ *with* $A_i \cdot Z < 0$ *from the exceptional set. The self-intersection number of the fundamental cycle of the minimally elliptic singularity is less than or equal to* $-3$.

Theorem A explains why almost minimally elliptic singularities are interesting. The converse of Theorem A is false. (See [36], Chapter III, §2, Example 3.) However, partial converses are shown for hypersurface singularities. Theorem B gives a necessary and sufficient condition for $p$ to be an almost minimally elliptic singularity. Theorem C provides us a comprehensible condition for $p$ to be an almost minimally elliptic singularity. This condition is readable from the intersection matrix. In Theorem D, we are able to identify the maximal ideal. Therefore the important invariants of the singularities (such as the multiplicity, the Hilbert fenction) are extracted from the topological information. Using Theorem D, we can list all possible hypersurface weighted dual graphs with $h=2$. (There are 250 types of them.) Since the topology of the singularity is determined by the weighted dual graph, this will give a topological classification of hypersurface singularities with $h=2$. By virtue of this classification and Theorem C, we have the following theorem, which is a partial converse of Theorem A.

THEOREM. *Let* $\pi: M \to V$ *be a resolution of normal two-dimensional Stein space with* $p$ *as its only singular point. Suppose* $H^1(M, \mathcal{O}) \cong \mathbb{C}^2$ *and* $p$ *is a hypersurface singularity. Let* $A$ *be the exceptional set. If* $H^1(A, \mathbb{Z}) = 0$, *then* $p$ *is an almost minimally elliptic singularity.*

Since room does not allow a proof of the theorem or of the topological classification, these will be included in the accompanying paper "Hypersurface weighted dual graphs of normal singularities of surfaces." It is not true that every two-dimensional isolated singularity can be resolved by means of a sequence of $\sigma$-processes with centers at points. Theorem E tells us, however, that under a certain condition almost minimally elliptic singularities do have this property. This is one of the many reasons that the almost minimally elliptic singularities are very interesting. Our presentation goes as follows:

The necessarily basic knowledge to read this paper can be found in [20] and [36].

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## 1. General Theory for Almost Minimally Elliptic Singularities.

*Definition* 1.1. Let $\pi: M \to V$ be the minimal good resolution of a normal two-dimensional Stein space with $p$ as its only weakly elliptic singular point. Suppose $p$ is not a minimally elliptic singularity, i.e., $|E| \subsetneqq \pi^{-1}(p)$. If for all $A_i \nsubseteq |E|$ and $A_i \cap |E| \neq \varnothing$, then $A_i \cdot Z < 0$. We call $p$ an *almost minimally elliptic singularity*.

THEOREM 1.2. *Let $\pi: M \to V$ be the minimal good resolution of a normal two-dimensional Stein space with $p$ as its only almost minimally elliptic singularity. Suppose ${}_V\mathcal{O}_p$ is Gorenstein. Then $H^1(M, \mathcal{O}) = \mathbb{C}^2$.*

*Proof.* If $\dim H^1(M, \mathcal{O}) = 0$, then $p$ is a rational singularity, which implies $\chi(Z) = 1$. This is a contradiction. If $\dim H^1(M, \mathcal{O}) = 1$, then $p$ is a minimally elliptic singularity by Theorem 3.10 of [20]. This contradicts our definition of almost minimal elliptic singularity. Therefore $\dim H^1(M, \mathcal{O}) \geqslant 2$. On the other hand $\dim H^1(M, \mathcal{O}) \leqslant 2$ by Theorem 3.9 of [36]. We conclude that $\dim H^1(M, \mathcal{O}) = 2$. Q.E.D.

Example 3 in Chapter III, §2 of [36] shows that $H^1(M, \mathcal{O}) = \mathbb{C}^2$ and ${}_V\mathcal{O}_p$ Gorenstein do not imply that $p$ is an almost minimal elliptic singularity. However, a partial converse of Theorem 1.2 will be shown later.

LEMMA 1.3. *Let $\pi: M \to V$ be the minimal good resolution of a normal two-dimensional Stein space with $p$ as its only weakly elliptic singularity. If $\dim H^1(M, \mathcal{O}) \neq 1$, then one of the following cases holds:*

(1) $H^0(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-E)) \cong \mathbb{C} \cong H^1(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-E))$,

(2) $H^0(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-E)) \cong 0 \cong H^1(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-E))$.

*Proof.* Since $H^1(M, \mathcal{O}) \neq 1$, we have $E \cdot Z = 0$ by Theorem 4.1 of [20]. Choose a computation sequence for $Z$ as follows: $Z_0 = 0, \dots, Z_k = E, \dots,$. Consider the following sheaf exact sequences:

$$\begin{aligned}&0\to\mathcal{O}(-Z-Z_1)/\mathcal{O}(-Z-E)\to\mathcal{O}(-Z)/\mathcal{O}(-Z-E)\\&\quad\to\mathcal{O}(-Z)/\mathcal{O}(-Z-Z_1)\to 0,\\&0\to\mathcal{O}(-Z-Z_2)/\mathcal{O}(-Z-E)\to\mathcal{O}(-Z-Z_1)/\mathcal{O}(-Z-E)\\&\quad\to\mathcal{O}(-Z-Z_1)/\mathcal{O}(-Z-Z_2)\to 0,\\&\qquad\vdots\\&0\to\mathcal{O}(-Z-Z_{k-1})/\mathcal{O}(-Z-E)\to\mathcal{O}(-Z-Z_{k-2})/\mathcal{O}(-Z-E)\\&\quad\to\mathcal{O}(-Z-Z_{k-2})/\mathcal{O}(-Z-Z_{k-1})\to 0.\end{aligned}$$

By the Riemann-Roch theorem, the usual long-cohomology-sequence argument will show that either (1) or (2) holds. Q.E.D.

THEOREM 1.4. *Let $\pi: M\to V$ be the minimal good resolution of normal two-dimensional Stein space with $p$ as its only weakly elliptic singularity. Suppose $H^1(M,\mathcal{O})=\mathbb{C}^2$ and ${}_V\mathcal{O}_p$ is Gorenstein. Then $p$ is an almost minimally elliptic singularity if and only if $H^0(M,\mathcal{O}(-Z)/\mathcal{O}(-Z-E))=\mathbb{C}$.*

*Proof.* $\Rightarrow$: By (1.1) and (1.6) of [36], $H^0(M,\mathcal{O}_Z)\cong\mathbb{C}\cong H^1(M,\mathcal{O}_Z)$. The long exact cohomology sequence

$$\begin{aligned}&0\to H^0(M,\mathcal{O}(-Z))\to H^0(M,\mathcal{O})\to H^0(M,\mathcal{O}_Z)\\&\quad\to H^1(M,\mathcal{O}(-Z))\to H^1(M,\mathcal{O})\to H^1(M,\mathcal{O}_Z)\to 0\end{aligned}$$

will show that $H^1(M,\mathcal{O}(-Z))=\mathbb{C}$. Since $p$ is an almost minimally elliptic singularity, $-K'=Z+E$. By (1.2), $H^1(M,\mathcal{O}(-Z-E))=0$. Now the exact sequence

$$H^1(M,\mathcal{O}(-Z-E))\to H^1(M,\mathcal{O}(-Z))\to H^1(M,\mathcal{O}(-Z)/\mathcal{O}(-Z-E))\to 0$$

will show that $H^1(M,\mathcal{O}(-Z)/\mathcal{O}(-Z-E))\cong\mathbb{C}$.

$\Leftarrow$: Conversely, suppose $H^1(M,\mathcal{O}(-Z)/\mathcal{O}(-Z-E))=\mathbb{C}$. Let $Z_{B_0},Z_{B_1},\dots,Z_{B_l},Z_E$ be the elliptic sequence. Then $-K'=B+E$, where $B=\sum_{i=0}^{l}Z_{B_i}$. Choose a computation sequence for $Z$ as follows: $Z_0=0$, $Z_1,\dots,Z_k=E,\dots,Z_{r_1}=Z_{B_l},\dots,Z_{r_l}=Z_{B_1},\dots,Z_{r_{l+1}}=Z_{B_0}=Z$. Suppose $p$ is not an almost minimal elliptic singularity; then $l\geqslant 1$. Look at the following exact cohomology sequence

$$0 \to \underset{\cong \mathbb{C}}{H^0(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-E))} \to H^0(M, \mathcal{O}_{Z+E}) \to \underset{\cong \mathbb{C}}{H^0(M, \mathcal{O}_Z)}$$
$$\to \underset{\cong \mathbb{C}}{H^1(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-E))} \to H^1(M, \mathcal{O}_{Z+E}) \to \underset{\cong \mathbb{C}}{H^1(M, \mathcal{O}_Z)} \to 0.$$

It is easy to see that $H^0(M, \mathcal{O}_{Z+E}) \to H^0(M, \mathcal{O}_Z)$ is surjective. Therefore $H^1(M, \mathcal{O}_{Z+E}) \cong \mathbb{C}^2 \cong H^0(M, \mathcal{O}_{Z+E})$. Since the two sequences

$$H^1(M, \mathcal{O}_B) \to H^1(M, \mathcal{O}_{Z+E}) \to 0,$$
$$H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}_B) \to 0$$

are exact, $H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}_B)$ is an isomorphism by dimensional considerations. It follows that $H^1(M, \mathcal{O}(-B)) \to H(M, \mathcal{O})$ is a zero map. As ${}_V\mathcal{O}_p$ is Gorenstein, there exists $\omega \in H^0(M - A, \Omega)$ having no zeros near $A$. Let $(\omega)$ be the divisor of $\omega$. Then $(\omega) = -B - E$. Let $w_1$ be the order of the pole of $\omega$ on $A_1 \subseteq |E|$.

Consider a cover as in Lemma 3.8 of [20]. On $P_1$,

$$\omega = \frac{\omega_1(x_1, y_1)}{y_1^{w_1}}\, dx_1 \wedge dy_1,$$

where $\omega_1(x_1, y_1)$ is a holomorphic function, $\omega_1(x_1, 0) \not\equiv 0$. There is a holomorphic function $f(x_1)$, $r \leqslant x_1 \leqslant R$, such that

$$\int_{\substack{|x_1| = R \\ |y_1| = R}} y_1^{w_1 - 1} f(x_1) \frac{\omega_1(x_1, y_1)}{y_1^{w_1}}\, dx_1 \wedge dy_1 \neq 0.$$

Let $\lambda_{01} = y_1^{w_1 - 1} f(x_1)$ and $\lambda_{0j} = 0$ for $j \neq 1$. Then by Lemma 3.8 of [20], $\operatorname{cls}[\lambda] \neq 0$ in $H^1(M, \mathcal{O})$. However, $w_1 - 1 \geqslant \sum_{i=0}^{l} z_{1, B_i}$ where $Z_{B_i} = \sum_j z_{j, B_i} A_j$. Hence $\lambda$ may be thought of as also a cocycle in $H^1(N(u), \mathcal{O}(-B))$. Consequently, $\operatorname{cls}[\lambda] = 0$ in $H^1(M, \mathcal{O})$ because $H^1(M, \mathcal{O}(-B)) \to H^1(M, \mathcal{O})$ is a zero map. This leads to a contradiction. Q.E.D.

THEOREM 1.5. *Let $\pi: M \to V$ be the minimal good resolution of normal two-dimensional Stein space $V$ with $p$ as its only weakly elliptic singularity. If $\mathcal{O}(-Z)/\mathcal{O}(-Z-E)$ corresponds to a trivial line bundle $L$ over $(|E|, \mathcal{O}_E)$, then $H^0(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-E)) \cong \mathbb{C}$. Conversely, suppose $H^1(M, \mathcal{O}) \cong \mathbb{C}^2$ and ${}_V\mathcal{O}_p$ is Gorenstein. If $H^0(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-E)) = \mathbb{C}$, then $\mathcal{O}(-Z)/\mathcal{O}(-Z-E)$ corresponds to a trivial line bundle $L$ over $(|E|, \mathcal{O}_E)$.*

*Proof.* Suppose $\mathcal{O}(-Z)/\mathcal{O}(-Z-E)$ corresponds to a trivial line bundle $L$ over $(|E|,\mathcal{O}_E)$. Let $U$ be an holomorphically convex neighborhood of $|E|$ such that $\Phi: U \to V_1$ represents $|E|$ as an exceptional set where $V_1$ is a normal two-dimensional Stein space with $\Phi(|E|)$ as its only minimally elliptic singularity. The group of sections of $L$ is isomorphic to $H^0(M,\mathcal{O}(-Z)/\mathcal{O}(-Z-E))$. However, $L$ is a trivial bundle over $(|E|,\mathcal{O}_E)$. So the group of sections of $L$ is isomorphic to $H^0(U,\mathcal{O}_E)\cong\mathbb{C}$. Therefore $H^0(M,\mathcal{O}(-Z)/\mathcal{O}(-Z-E))=\mathbb{C}$.

Conversely, suppose $H^1(M,\mathcal{O})=\mathbb{C}^2$ and ${}_V\mathcal{O}_p$ is Gorenstein. Then $H^0(M,\mathcal{O}(-Z)/\mathcal{O}(-Z-E))\cong\mathbb{C}$ implies that $p$ is an almost minimally elliptic singularity by Theorem 1.4. There exists $f\in H^0(M,\mathcal{O}(-Z))$ such that the image of $f$ in $H^0(M,\mathcal{O}(-Z)/\mathcal{O}(-Z-E))$ viewed as section of the line bundle $L$ is nowhere zero, by Proposition 3.13 of [36]. Hence $L$ is a trivial bundle over $(|E|,\mathcal{O}_E)$. Q.E.D.

With notation as above, let $\phi:\mathcal{O}\to\mathcal{O}_E=\mathcal{O}/\mathcal{O}(-E)$ be the quotient map. Define $\mathcal{O}_E^*=\phi(\mathcal{O}^*)\subseteq\mathcal{O}_E$. Let $\alpha:\mathbb{Z}\to\mathcal{O}_E$ be $\phi\circ i$, where $i:\mathbb{Z}\to\mathcal{O}$ is the obvious inclusion map. $\beta:\mathcal{O}_E\to\mathcal{O}_E^*$ is defined as follows. For a germ $f$ in a stalk of $\mathcal{O}_E$, let $F$ be a germ in $\mathcal{O}$ such that $\phi(F)=f$. Then we set $\beta(f)=\phi(\exp 2\pi iF)$. We claim that $\beta$ is well defined. Let $F_1$ be another germ in $\mathcal{O}$ such that $\phi(F_1)=f$. Then $F_1=F+g$, where $g$ can be considered as germ in $\mathcal{O}(-E)$. Hence

$$\begin{aligned}\phi(\exp(2\pi iF_1)) &= \phi(\exp(2\pi iF+2\pi ig))\\ &= \phi\left(1+\frac{2\pi iF+2\pi ig}{1!}+\frac{(2\pi iF+2\pi ig)^2}{2!}+\cdots+\frac{(2\pi iF+2\pi ig)^n}{n!}+\cdots\right)\\ &= \phi\left[\left(1+\frac{2\pi iF}{1}+\frac{(2\pi iF)^2}{2!}+\cdots+\frac{(2\pi iF)^n}{n!}+\cdots\right)+gh\right]\\ &= \phi(\exp 2\pi iF).\end{aligned}$$

LEMMA 1.6. $0\to\mathbb{Z}\xrightarrow{\alpha}\mathcal{O}_E\xrightarrow{\beta}\mathcal{O}_E^*\to 0$ *is an exact sheaf sequence.*

PROPOSITION 1.7. *Let $\pi: M\to V$ be the minimal good resolution of normal two-dimensional Stein space $V$ with $p$ as its only weakly elliptic singularity. Let $\mathcal{O}(-Z)/\mathcal{O}(-Z-E)$ correspond to a line bundle $L$ over $(|E|,\mathcal{O}_E)$. Suppose $H^1(M,\mathcal{O}\cong\mathbb{C}^2$ and ${}_V\mathcal{O}_p$ is Gorenstein. Let $Z_{B_0}=Z,\dots,Z_{B_l},Z_{B_{l+1}}=Z_E$ be the elliptic sequence. Let $D$ be the subvariety of $B_l$ consisting of those irreducible components $A_i\subseteq B_l$ such that $A_i\cap|E|\neq\varnothing$. Suppose $Z/D$, the restriction of $Z$ to $D$, is equal to $Z_{B_l}/D$, the restriction of $Z_{B_l}$ to $D$. Then $L^{l+1}$ is a trivial line bundle over $(|E|,\mathcal{O}_E)$.*

Note. Let $A=\bigcup_{i=1}^n A_i\supseteq D=\bigcup_{i=1}^t A_i$. If $Z=\sum_{i=1}^n z_iA_i$, then $Z/D=\sum_{i=1}^t z_iA_i$.

*Proof.* Let $Z_0=0,\dots, Z_k=E,\dots$ be a computation sequence for $Z$. Look at the following sheaf exact sequences (recall that $B=\Sigma_{i=0}^{l}Z_{B_i}$):

$$\begin{aligned}
0\to \mathcal{O}(-B-Z_1)/\mathcal{O}(&-B-E)\to \mathcal{O}(-B)/\mathcal{O}(-B-E)\\
&\to \mathcal{O}(-B)/\mathcal{O}(-B-Z_1)\to 0,\\
0\to \mathcal{O}(-B-Z_2)/\mathcal{O}(&-B-E)\to \mathcal{O}(-B-Z_1)/\mathcal{O}(-B-E)\\
&\to \mathcal{O}(-B-Z_1)/\mathcal{O}(-B-Z_2)\to 0,\\
&\vdots\\
0\to \mathcal{O}(-B-Z_{k-1})/\mathcal{O}(&-B-E)\to \mathcal{O}(-B-Z_{k-2})/\mathcal{O}(-B-E)\\
&\to \mathcal{O}(-B-Z_{k-2})/\mathcal{O}(-B-Z_{k-1})\to 0.
\end{aligned}$$

By the Riemann-Roch theorem, the usual long exact cohomology sequence will show that either $H^0(M,\mathcal{O}(-B)/\mathcal{O}(-B-E))=\mathbf{C}=H^1(M,\mathcal{O}(-B)/\mathcal{O}(-B-E))$ or $H^0(M,\mathcal{O}(-B)/\mathcal{O}(-B-E))=0=H^1(M,\mathcal{O}(-B)/\mathcal{O}(-B-E))$. We claim that the latter case cannot occur. Otherwise $H^0(M,\mathcal{O}(-B-E))\to H^0(M,\mathcal{O}(-B))$ will be an isomorphism. However, by Theorem 2.1, which will be proved later, we have $m\mathcal{O}\equiv\mathcal{O}(-B)$. It follows that the maximal ideal cycle $Y\geqslant B+E=-K'$. This is absurd, and our claim is proved. Hence we have the following exact sequence:

$$\begin{aligned}
0\to H^0(M,\mathcal{O}(-B-E))&\to H^0(M,\mathcal{O}(-B))\\
&\to H^0(M,\mathcal{O}(-B)/\mathcal{O}(-B-E))\cong \mathbf{C}\to 0.
\end{aligned}$$

Let $f\in H^0(M,\mathcal{O}(-B))$ be such that the image of $f$ in $H^0(M,\mathcal{O}(-B)/\mathcal{O}(-B-E))$ is not zero. Then $f\notin H^0(M,\mathcal{O}(-B-E))$. We are going to prove that actually $f\notin H^0(M,\mathcal{O}(-B-A_1))$ for any $A_1\subseteq|E|$. Choose a computation sequence of the following form: $Z_0=0,\ Z_1=A_1,\dots,\ Z_k=E,\dots$. Consider the following sheaf exact sequences:

$$\begin{aligned}
&0\to \mathcal{O}(-B-Z_2)\to \mathcal{O}(-B-Z_1)\to \mathcal{O}(-B-Z_1)/\mathcal{O}(-B-Z_2)\to 0,\\
&0\to \mathcal{O}(-B-Z_3)\to \mathcal{O}(-B-Z_2)\to \mathcal{O}(-B-Z_2)/\mathcal{O}(-B-Z_3)\to 0,\\
&\qquad\vdots\\
&0\to \mathcal{O}(-B-E)\to \mathcal{O}(-B-Z_{k-1})\to \mathcal{O}(-B-Z_{k-1})/\mathcal{O}(-B-E)\to 0.
\end{aligned}$$

By the Riemann-Roch theorem, the usual long cohomology exact sequence will show that $H^0(M,\mathcal{O}(-B-Z_j))\to H^0(M,\mathcal{O}(-B-Z_{j-1}))$, $2\leqslant j\leqslant k$, are isomorphisms. By composing the maps, we get that $H^0(M,\mathcal{O}(-B-E))\to$

$H^0(M, \mathcal{O}(-B-A_1))$ is an isomorphism. Hence $f \notin H^0(M, \mathcal{O}(-B-A_1))$. The image of $f$ in $H^0(M, \mathcal{O}(-B)/\mathcal{O}(-B-E))$ viewed as section of the line bundle $N$ over $(|E|, \mathcal{O}_E)$ corresponding to the sheaf $\mathcal{O}(-B)/\mathcal{O}(-B-E)$ is nowhere zero. Hence $N$ is a trivial bundle over $(|E|, \mathcal{O}_E)$.

Let us prove that for any $A_i \not\subseteq B_l$, $A_i \cap |E| = \varnothing$. First observe that $Z/D = Z_{B_l}/D$ implies $Z_{B_i}/D = Z_{B_l}/D$ for all $0 \leqslant i \leqslant l$. Suppose first that $A_i \subseteq B_{l-1}$ and $A_i \not\subseteq B_l$. If $A_i \cap |E| \neq \varnothing$, then there exists $A_j \subseteq |E|$ such that $A_i \cap A_j \neq \varnothing$. Since $Z_{B_{l-1}}/D = Z_{B_l}/D$ and $A_j \cdot Z_{B_l} = 0$, we have $A_j \cdot Z_{B_{l-1}} \geqslant A_j \cdot (Z_{B_{l-1}}/D + A_i) = A_j \cdot (Z_{B_l}/D + A_i) = A_j(Z_{B_l} + A_i) = 1 > 0$. This is a contradiction. Suppose that if $A_i \subseteq B_h$ and $A_i \not\subseteq B_l$, then $A_i \cap |E| = \varnothing$. We want to prove that this is also true for $B_{h-1}$. Then the decreasing induction argument will complete the proof. Let $A_i \subseteq B_{h-1}$ and $A_i \not\subseteq B_l$. If $A_i \cap |E| \neq \varnothing$, then there exists $A_j \subseteq |E|$ such that $A_i \cap A_j \neq \varnothing$. By the induction hypothesis, $0 = A_j \cdot Z_{B_h} = A_j \cdot Z_{B_h}/D$. Hence $A_j \cdot Z_{B_{h-1}} \geqslant A_j \cdot (Z_{B_{h-1}}/D + A_i) = A_j \cdot (Z_{B_h}/D + A_i) = 1 > 0$. This is absurd. Our claim is proved.

Hence

$$\begin{aligned} L^{l+1} &= \underbrace{\mathcal{O}(-Z)/\mathcal{O}(-Z-E) \otimes_{\mathcal{O}_E} \cdots \otimes_{\mathcal{O}_E} \mathcal{O}(-Z)/\mathcal{O}(-Z-E)}_{l+1} \\ &\cong \mathcal{O}(-(l+1)Z)/\mathcal{O}(-(l+1)Z-E) \\ &\cong \mathcal{O}(-B)/\mathcal{O}(-B-E). \end{aligned}$$

It follows that $L^{l+1} \cong N$ is a trivial bundle over $(|E|, \mathcal{O}_E)$. Q.E.D.

THEOREM 1.8. *Let $\pi: M \to V$ be the minimal good resolution of normal two-dimensional Stein space $V$ with $p$ as its only weakly elliptic singularity. Suppose $H^1(M, \mathcal{O}) = \mathbb{C}^2$, $H^1(|E|, \mathbb{Z}) = 0$, and ${}_V\mathcal{O}_p$ is Gorenstein. Let $Z_{B_0}, Z_{B_1}, \ldots, Z_{B_l}, Z_E = Z_{B_{l+1}}$ be the elliptic sequence. Let $D$ be the subvariety of $B_l$ consisting of those irreducible components $A_i \subseteq B_l$ such that $A_i \cap |E| \neq \varnothing$. If $Z/D = Z_{B_l}/D$, then $l = 0$, i.e., $p$ is an almost minimally elliptic singularity.*

*Proof.* Let $L$ be a line bundle over $(|E|, \mathcal{O}_E)$ corresponding to $\mathcal{O}(-Z)/\mathcal{O}(-Z-E)$. Consider the following commutative diagram:

$$\begin{array}{ccc} H^1(|E|, \mathcal{O}_E^*) & \xrightarrow{c^*} & H^2(|E|, \mathbb{Z}) \\ \downarrow \phi_1 & & \downarrow \wr \\ H^1(|E|, \mathcal{O}_{|E|}^*) & \rightarrow & H^2(|E|, \mathbb{Z}) \\ \downarrow \phi_2 & & \downarrow \wr \\ \bigoplus_{A_i \subseteq |E|} H^1(A_i, \mathcal{O}_{A_i}^*) & \xrightarrow{c} & H^2(|E|, \mathbb{Z}) \cong \bigoplus_{A_i \subseteq |E|} H^2(A_i, \mathbb{Z}) \end{array}$$

Since $A_i \cdot Z = 0$, $c(\phi_2 \circ \phi_1(L)) = 0$. Therefore $c^*(L) = 0$. Look at the following exact sequence:

$$0 \cong H^1(|E|, \mathbb{Z}) \to H^1(|E|, \mathcal{O}_E) \to H^1(|E|, \mathcal{O}_E^*) \xrightarrow{c^*} H^2(|E|, \mathbb{Z}).$$

From $H^1(|E|, \mathcal{O}_E) = \mathbb{C}$, $c^*(L) = 0$, and the fact that $L^{l+1}$ is a trivial bundle by Proposition 1.7, it follows that $L$ is a trivial bundle itself. By Theorem 1.4 and Theorem 1.5, $p$ is an almost minimally elliptic singularity, i.e., $l = 0$. Q.E.D.

**2. Calculation of Multiplicities.** Suppose $H^1(M, \mathcal{O}) \cong \mathbb{C}^2$ and ${}_V\mathcal{O}_p$ is Gorenstein. In this section we identify the maximal ideal, and in particular, we get a formula for the multiplicity of a singularity.

THEOREM 2.1. *Let $\pi: M \to V$ be the minimal good resolution of normal two-dimensional Stein space with $p$ as its only weakly elliptic singularity. Suppose $H^1(M, \mathcal{O}) = \mathbb{C}^2$ and ${}_V\mathcal{O}_p$ is Gorenstein. Let $Z_{B_0} = Z, \ldots, Z_{B_1}, \ldots, Z_{B_l}$, $Z_{B_{l+1}} = Z_E$ be the elliptic sequence. Let $B = \sum_{i=0}^{l} Z_{B_i}$. Then $m\mathcal{O} \subseteq \mathcal{O}(-B)$. If $Z_E \cdot Z_E \leqslant -2$, then $m\mathcal{O} = \mathcal{O}(-B)$.*

*Proof.* Since $\chi(B) = 0$ by (1.4) of [36], $\dim H^0(M, \mathcal{O}_B) = \dim H^1(M, \mathcal{O}_B)$.The two exact sequences

$$H^1(M, \mathcal{O}_B) \to H^1(M, \mathcal{O}_Z) \cong \mathbb{C} \to 0,$$
$$H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}_B) \to 0$$

say that $\dim H^1(M, \mathcal{O}_B)$ is either two or one. If $H^1(M, \mathcal{O}_B) = \mathbb{C}^2$, then $H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}_B)$ is an isomorphism by dimensional considerations. It follows that $H^1(M, \mathcal{O}(-B)) \to H^1(M, \mathcal{O})$is a zero map. As ${}_V\mathcal{O}_p$ is Gorenstein, by the proof of Theorem 1.4, we will get a contradiction. We conclude that $H^1(M, \mathcal{O}_B) = \mathbb{C}$.

Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 \to & H^0(M, \mathcal{O}(-B)) & \to & H^0(M, \mathcal{O}) & \to & H^0(M, \mathcal{O}_B) \cong \mathbb{C} & \to 0 \\
& \downarrow & & \downarrow \wr & & \wr \downarrow & \\
0 \to & H^0(M, \mathcal{O}(-Z)) & \to & H^0(M, \mathcal{O}) & \to & H^0(M, \mathcal{O}_Z) \cong \mathbb{C} & \to 0
\end{array}$$

By the five lemma, $H^0(M,(-B)) \to H^0(M, \mathcal{O}(-Z))$ is an isomorphism.

Since $m\mathcal{O} \subseteq \mathcal{O}(-Z)$, it follows easily that $m\mathcal{O} \subseteq \mathcal{O}(-B)$.

Suppose $Z_E \cdot Z_E \leqslant -2$. We want to prove $m\mathcal{O} = \mathcal{O}(-B)$. It suffices to prove $\mathcal{O}(-B) \subseteq m\mathcal{O}$. Let us first show that

$$\rho: H^0(M, \mathcal{O}(-B)) \to H^0(M, \mathcal{O}(-B)/\mathcal{O}(-B-A_1)) \qquad (2.1)$$

is surjective for all $A_1 \subseteq |E|$. If $E = A_1$ is a nonsingular elliptic curve, then $-K' = B + A_1$. Since $H^1(M, \mathcal{O}(-B-A_1)) = 0$ by Theorem 3.2 of [17], $\rho$ is surjective by the usual long-cohomology-exact-sequence argument. If $|E|$ has at least two irreducible components, then $H^1(M, \mathcal{O}(-B)/\mathcal{O}(-B-A_1)) = 0$ by the Riemann-Roch theorem. We are going to show $H^1(M, \mathcal{O}(-B-A_1)) \cong \mathbb{C} \cong H^1(M, \mathcal{O}(-B))$. The exact sequence

$$0 \to H^1(M, \mathcal{O}(-B)) \to \overset{\cong \mathbb{C}^2}{H^1(M, \mathcal{O})} \to H^1(M, \mathcal{O}_B) \cong \mathbb{C} \to 0$$

shows that we indeed have $H^1(M, \mathcal{O}(-B)) = \mathbb{C}$. Choose a computation sequence for $Z$ of the following form: $Z_0 = 0$, $Z_1 = A_{i_1} = A_1, \ldots, Z_{k-1}, Z_k = E, \cdots$. The long exact cohomology sequence

$$\begin{aligned}
&0 \to H^0\big(M, \mathcal{O}(-B)/\mathcal{O}(-B-A_1)\cong \mathbb{C} \to H^0\big(M, \mathcal{O}_{B+A_1}\big) \\
&\to H^0(M, \mathcal{O}_B) \cong \mathbb{C} \to H^1(M, \mathcal{O}(-B)/\mathcal{O}(-A_1)) = 0 \\
&\to H^1\big(M, \mathcal{O}_{B+A_1}\big) \to H^1(M, \mathcal{O}_B) \to 0
\end{aligned}$$

will show that $H^0(M, \mathcal{O}_{B+A_1}) = \mathbb{C}^2$ and $H^1(M, \mathcal{O}_B) = \mathbb{C}$.

Consider the following long exact cohomology sequence:

$$\begin{aligned}
0 &\to H^0(M, \mathcal{O}(-B-A_1)) \to H^0(M, \mathcal{O}) \to H^0\big(M, \mathcal{O}_{B+A_1}\big) \cong \mathbb{C}^2 \\
&\to H^1(M, \mathcal{O}(-B-A_1)) \to H^1(M, \mathcal{O}) \to H^1\big(M, \mathcal{O}_{B+A_1}\big) = \mathbb{C} \to 0.
\end{aligned}$$

We claim that $H^0(M, \mathcal{O}) \to H^0(M, \mathcal{O}_{B+A_1})$ is surjective. Otherwise the image $R$ of $H^0(M, \mathcal{O}) \to H^0(M, \mathcal{O}_{B+A_1})$ will be isomorphic to $\mathbb{C}$. The five lemma together with the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \to & H^0(M, \mathcal{O}(-B-A_1)) & \to & H^0(M, \mathcal{O}) & \to & R & \cong \mathbb{C} \to 0 \\
& \downarrow & & \downarrow \wr & & \downarrow \wr & \\
0 \to & H^0(M, \mathcal{O}(-Z)) & \to & H^0(M, \mathcal{O}) & \to & H^0(M, \mathcal{O}_Z) & \cong \mathbb{C} \to 0
\end{array}$$

will show that $H^0(M, \mathcal{O}(-B-A_1)) \to H^0(M, \mathcal{O}(-Z))$ is an isomorphism. The following exact sequences of sheaves

$$\begin{aligned}
&0 \to \mathcal{O}(-B-Z_2) \to \mathcal{O}(-B-Z_1) \to \mathcal{O}(-B-Z_1)/\mathcal{O}(-B-Z_2) \to 0, \\
&0 \to \mathcal{O}(-B-Z_3) \to \mathcal{O}(-B-Z_2) \to \mathcal{O}(-B-Z_2)/\mathcal{O}(-B-Z_3) \to 0, \\
&\qquad\qquad\vdots \\
&0 \to \mathcal{O}(-B-E) \to \mathcal{O}(-B-Z_{k-1}) \to \mathcal{O}(-B-Z_{k-1})/\mathcal{O}(-B-Z_k) \to 0
\end{aligned}$$

will show that $H^0(M, \mathcal{O}(-B-Z_j)) \to H^0(M, \mathcal{O}(-B-Z_{j-1}))$ are isomorphisms for $2 \leqslant j \leqslant k$. By composing the maps, we get that $H^0(M, \mathcal{O}(-B-E)) \to H^0(M, \mathcal{O}(-Z))$ is an isomorphism. Since $m\mathcal{O} \subseteq \mathcal{O}(-Z)$, the maximal ideal cycle $Y \geqslant B+E=-K'$. This contradicts Theorem 2.20 of [36]. We conclude that $H^0(M, \mathcal{O}) \to H^0(M, \mathcal{O}_{B+A_1}$ is surjective. It follows that $H^1(M, \mathcal{O}(-B-A_1)) = \mathbb{C}$.

Look at the following exact cohomology sequence:

$$\begin{aligned} 0 &\to H^0(M, \mathcal{O}(-B-A_1)) \to H^0(M, \mathcal{O}(-B)) \\ &\to H^0(M, \mathcal{O}(-B)/\mathcal{O}(-B-A_1)) \to H^1(M, \mathcal{O}(-B-A_1)) \\ &\to H^1(M, \mathcal{O}(-B)) \to H^1(M, \mathcal{O}(-B)/\mathcal{O}(-B-A_1)) \to 0. \end{aligned}$$

Since $H^1(M, \mathcal{O}(-B)/\mathcal{O}(-B-A_1)) = 0$, $H^1(M, \mathcal{O}(-B-A_1)) \to H^1(M, \mathcal{O}(-B))$ is an isomorphism by dimensional considerations. Therefore $\rho$ in (2.1) is surjective. Given a point $a_1 \in A_1$, let $F \in H^0(M, \mathcal{O}(-B)/\mathcal{O}(-B-A_1))$ be nonzero near $a_1$ as a section of the line bundle associated to $\mathcal{O}(-B)/\mathcal{O}(-B-A_1)$. Then $f \in H^0(M, \mathcal{O}(-B))$ projecting onto $F$ will generate $\mathcal{O}(-B)$ near $a_1$, since it must vanish to the prescribed orders on the $A_1$ near $a_1$ and will have no other zeros near $a_1$.

In order to prove $\mathcal{O}(-B) \subseteq m\mathcal{O}$, it remains to prove $\mathcal{O}(-B) \subseteq m\mathcal{O}$ near $A$-supp$E$. There are two subcases.

*Case* (1). *There exists* $A_i \subseteq |E|$ *such that* $E \cdot Z_E + 1 \leqslant A_i \cdot Z_E \leqslant -1$ *or* $E = A_i$ *is a nonsingular elliptic curve.* For any $A_1 \not\subseteq |E|$, choose a computation sequence for $Z$ of the following form: $Z_0 = 0$, $Z_1 = A_{i_1} = A_1, \ldots, Z_r, Z_{r+1}, \ldots, Z_{r+k} = Z_r + E, \ldots, Z_{r_{l+1}} = Z$, where $\operatorname{supp} Z_r \subseteq \overline{A - |E|}$ and $Z_{r+1} - Z_r, \ldots, Z_{r+k} - Z_r = E$ is part of a computation sequence for $Z$. Our hypothesis guarantees that the computation sequence can be so chosen such that $A_{i_{r+k}} \cdot Z_E < 0$ by Corollary 2.3 of [36]. Consider the following exact sheaf sequence for $n \geqslant 0$:

$$\begin{aligned} 0 &\to \mathcal{O}(-B-Z_E-nZ-Z_1) \to \mathcal{O}(-B-Z_E-nZ) \\ &\quad \to \mathcal{O}(-B-Z_E-nZ)/\mathcal{O}(-B-Z_E-nZ-Z_1) \to 0, \\ 0 &\to \mathcal{O}(-B-Z_E-nZ-Z_j) \to \mathcal{O}(-B-Z_E-nZ-Z_{j-1}) \\ &\quad \to \mathcal{O}(-B-Z_E-nZ-Z_{j-1})/\mathcal{O}(-B-Z_E-nZ-Z_j) \to 0, \\ 0 &\to \mathcal{O}(-B-Z_E-nZ-Z_{r_{l+1}}) \to \mathcal{O}(-B-Z_E-nZ-Z_{r_{l+1}-1}) \\ &\quad \to \mathcal{O}(-B-Z_E-nZ-Z_{r_{l+1}-1})/\mathcal{O}(-B-Z_E-nZ-Z). \end{aligned} \tag{2.2}$$

We recall that $(B+Z_E) \cdot A_i \leqslant 0$ for all $A_i \subseteq A$. Then $\mathcal{O}(-B-Z_E-nZ-Z_{j-1})/\mathcal{O}(-B-Z_E-nZ-Z_j)$ is the sheaf of germs of sections of a line bundle over $A_{i_j}$ with the Chern class $-A_{i_j}(B+Z_E+nZ+Z_{j-1})$. If $|E|$ has at least two irreducible components, then from Proposition 2.5 of [36], $A_{i_{r+k}}\cdot Z_{r+k-1}=2$ and $A_{i_j}\cdot Z_{j-1}=1$ for $j\neq r+k$. So $A_{i_j}\cdot(B+Z_E+nZ+Z_{j-1})\leqslant 1$ for all $j$ and all $n$. Thus $H^1(M,\mathcal{O}(-B-Z_E-nZ-Z_{j-1})/\mathcal{O}(-B-Z_E-nZ-Z_j))=0$, and the maps $H^1(M,\mathcal{O}(-B-Z_E-nZ-Z_j))\to H^1(M,\mathcal{O}(-B-Z_E-nZ-Z_{j-1}))$ in (2.2) are surjective. Composing the maps, we see that

$$\phi:H^1\big(M,\mathcal{O}(-B-Z_E-nZ-Z_j)\big)\to H^1\big(M,\mathcal{O}(-B-Z_E-Z_j)\big)$$

is surjective for all $n\geqslant 0$. For sufficiently large $n$, $\phi$ is the 0 map by [6, §4, Satz 1, p. 355]. Hence $H^1(M,\mathcal{O}(-B-Z_E-Z_j))=0$. If $|E|=A_i$ is a nonsingular elliptic curve, then $A_i\cdot A_i\leqslant -2$. By Corollary 2.6 of [36], we know that $e_i=z_i=1$. Since $A_{i_j}\cdot Z_{j-1}=1$ for all $j$ by Proposition 2.5 of [36], $A_{i_j}\cdot(B+Z_E+nZ+Z_{j-1})\leqslant 1$ for all $j\neq r+1$ and $A_{i_{r+1}}(B+Z_E+nZ+Z_r)\leqslant -1$. Thus $H^1(M,\mathcal{O}(-B-Z_E-nZ-Z_{j-1})/\mathcal{O}(-B-Z_E-nZ-Z_j))=0$ for all $j$ and $n$. A similar argument to the one above will show that $H^1(M,\mathcal{O}(-B-Z_E-Z_j))=0$. In particular $H^1(M,\mathcal{O}(-B-Z_E-A_1))=0$. Therefore, $H^0(M,\mathcal{O}(-B-Z_E))\to H^0(M,\mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-A_1))$ is surjective. We remark that the above argument is also applicable to the following situation: With notation as above, there exists $A_j\subseteq \operatorname{supp} E$ such that $A_j\neq A_{i_{r+1}}$ and $A_j\cdot Z_E<0$.

*Case* (2). *$|E|$ has at least two irreducible components and there exists $A_i\subseteq|E|$ such that $e_i=1$, $A_i\cdot Z_E<0$, and $A_j\cdot Z_E=0$ for all $A_j\subseteq|E|$ where $A_j\neq A_i$.* The proof of case (1) fails only because $A_{i_{r+k}}\neq A_i$, i.e., $A_{i_{r+1}}\cdot Z_E<0$. Suppose first that $A_1\cap A_{i_{r+1}}=A_1\cap A_i\neq\varnothing, A_1\nsubseteq|E|$. Choose a computation sequence for $Z$ with $E=Z_k$, $A_{i_k}=A_i$, and $A_{i_{k+1}}=A_1$. By Proposition 2.7 of [36], $H^1(M,\mathcal{O}(-B-Z_E-Z_j))=0$ for all $j$. Therefore,

$$H^0(M,\mathcal{O}(-B-Z_E))\to H^0\big(M,\mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-Z_{k+1})\big)$$

is surjective. It follows that $H^0(M,\mathcal{O}(-B-Z_E))$ and $H^0(M,\mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-Z_{k+1}))$ have the same image $R$ in $H^0(M,\mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-A_1))$.

$$\begin{aligned}0&\to H^0\big(M,\mathcal{O}(-B-Z_E-Z_k)/\mathcal{O}(-B-Z_E-Z_{k+1})\big)\\&\to H^0\big(M,\mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-Z_{k+1})\big)\\&\to H^0\big(M,\mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-Z_k)\big)\to 0\end{aligned}$$

is an exact sequence. Thus the image of $H^0(M,\mathcal{O}(-B-Z_E-Z_k)/\mathcal{O}(-B-Z_E-Z_{k+1}))$ which is injected into $H^0(M,\mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-A_1))$ via the natural map is contained in $R$. If $H^0(M,\mathcal{O}(-B-Z_E-Z_k)/\mathcal{O}(-B-Z_E-Z_{k+1}))\neq 0$, then the elements of $R$ have no common zeros on $A_1-A_1\cap A_i$ as section of the line bundle $L$ on $A_1$ associated to $\mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-A_1)$. If $H^0(M,\mathcal{O}(-B-Z_E-Z_k)/\mathcal{O}(-B-Z_E-Z_{k+1}))=0$, then $A_1\cdot(B+Z_E)=0$. Hence $H^0(M,\mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-A_1))=\mathbb{C}$. It suffices to prove that $H^0(M,\mathcal{O}(-B-Z_E))\to H^0(M,\mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-A_1))$ is not a zero map. Since $A_1\not\subseteq|E|$, $A_1\cdot Z_E=1$ and $A_1\cap A_i\neq\varnothing$, the coefficient of $A_i$ in $Z_E$ is equal to 1. Hence $A_i\cdot Z_E=Z_E\cdot Z_E\leqslant -2$. It follows that $A_i(B+Z_E)\leqslant -2$ and $\dim H^0(M,\mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-A_i))\geqslant 3$. The image of $\rho: H^0(M,\mathcal{O}(-B-Z_E))\to H^0(M,\mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-A_i))$ is a subspace of codimension 1 in $H^0(M,\mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-A_i))$, and the elements of $S$ have no common zeros as sections of the line bundle $L_i$ on $A_i$ associated to $\mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-A_i)$ by Proposition 2.8 of [36]. It follows that $H^0(M,\mathcal{O}(-B-Z_E))\to H^0(M,\mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-A_1))$ is not a zero map.

In order to finish the proof of case (2), it remains to consider those $A_1\not\subseteq|E|$ such that $A_1\cap A_i=\varnothing$ and the computation sequence for $Z$ starting from $A_1$ must first reach $A_i$ in order to reach $|E|$. Choose a computation sequence for $Z$ of the following form $E=Z_k$, $A_{i_k}=A_i$ $(A_{i_{k+1}}\cap A_i\neq\varnothing)$, $A_{i_{k+t}}=A_1$, where $A_{i_j}$ $(k+1\leqslant j\leqslant k+t)$ are distinct from each other and not contained in $|E|$. Since $H^1(M,\mathcal{O}(-B-Z_E-Z_j))=0$ for all $j$ by Proposition 2.7 of [36], $H^0(M,\mathcal{O}(-B-Z_E))\to H^0(M,\mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-Z_{k+t}))$ is surjective. It follows that $H^0(M,\mathcal{O}(-B-Z_E))$ and $H^0(M,\mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-Z_{k+t}))$ have the same image $R$ in $H^0(M,\mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-A_1))$.

$$\begin{aligned}0&\to H^0(M,\mathcal{O}(-B-Z_E-Z_{k+t-1})/\mathcal{O}(-B-Z_E-Z_{k+t}))\\&\to H^0(M,\mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-Z_{k+t}))\\&\to H^0(M,\mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-Z_{k+t}-1))\to 0\end{aligned}$$

is an exact sequence. Thus the image of $H^0(M,\mathcal{O}(-B-Z_E-Z_{k+t-1})/\mathcal{O}(-B-Z_E-Z_{k+t}))$ which is injected into $H^0(M,\mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-A_1))$ via the natural map is contained in $R$. If $H^0(M,\mathcal{O}(-B-Z_E-Z_{k+t-1})/\mathcal{O}(-B-Z_E-Z_{k+t}))\neq 0$, then the elements of $R$ have no common zeros on $A_1-(A_1\cap A_{i_{k+t-1}})$ as sections of the line bundle $L_1$ on $A_1$ associated to $\mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-A_1)$. If $H^0(M,\mathcal{O}(-B-Z_E-Z_{k+t-1})/\mathcal{O}(-B-Z_E-Z_{k+t}))=0$, then $A_1\cdot(B+Z_E)=0$. Hence $H^0(M,\mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-A_1))=\mathbb{C}$. But by induction, we know that the elements of image of

$$H^0(M,\mathcal{O}(-B-Z_E))\to H^0\big(M,\mathcal{O}(-B-Z_E)/\mathcal{O}\big(-B-Z_E-A_{i_{k+t-1}}\big)\big)$$

have no common zeros on $A_{i_{k+t-1}}-(A_{i_{k+t-1}}\cap A_{i_{k+t-2}})$ as sections of the line bundle $L_{i_{k+t-1}}$ on $A_{i_{k+t-1}}$ associated to $\mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-A_{i_{k+t-1}})$. It follows that $H^0(M, \mathcal{O}(-B-Z_E)) \to H^0(M, \mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-A_1))$ is surjective. Q.E.D.

COROLLARY 2.2. *Let $\pi: M \to V$ be the minimal good resolution of normal two-dimensional Stein space with $p$ as its only weakly elliptic singularity. Suppose $H^1(M, \mathcal{O}) = \mathbb{C}^2$ and ${}_V\mathcal{O}_p$ is Gorenstein. Let $Z_{B_0} = Z, Z_{B_1}, \dots, Z_{B_l}, Z_{B_{l+1}} = Z_E$ be the elliptic sequence. Suppose $Z_E \cdot Z_E = -1$. Let $A_i \subseteq |E|$ be such that $A_i \cdot Z_E = -1$. Let $S$ be the image of $\rho: H^0(M, \mathcal{O}(-B-Z_E)) \to H^0(M, \mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-A_i))$. Then $m\mathcal{O} = \mathcal{O}(-B)$ provided that the following condition holds: Let $A_1 \not\subseteq |E|$ and $A_1 \cap A_i \neq \varnothing$; then either $A_1 \cdot (B+Z_E) < 0$ or the elements of $S$ have no common zeros at $A_1 \cap A_i$ as sections of the line bundle $L_i$ on $A_i$ associated to $\mathcal{O}(-B-Z_E)\mathcal{O}(-B-Z_E-A_i)$.*

*Proof.* By the proof of Theorem 2.1.

COROLLARY 2.3. *Let $\pi: M \to V$ be the minimal good resolution of normal two-dimensional Stein space with $p$ as its only weakly elliptic singularity. Suppose $H^1(M\mathcal{O}) = \mathbb{C}^2$ and ${}_V\mathcal{O}_p$ is Gorenstein. Let $Z_{B_0} = Z, Z_{B_1}, \dots, Z_{B_l}, Z_E$ be the elliptic sequence. Then the multiplicity $({}_V\mathcal{O}_p) \geq -\sum_{i=0}^{l} Z_{B_i}^2$. If $Z_E \cdot Z_E \leq -2$, then* multiplicity$({}_V\mathcal{O}_p) = -\sum_{i=0}^{l} Z_{B_i}^2$.

*Proof.* Theorem 2.1 says that $m\mathcal{O} \subseteq \mathcal{O}(-\sum_{i=0}^{l} Z_{B_i})$. Hence the maximal ideal cycle $Y$ relative to $\pi$ is greater than or equal to $B = \sum_{i=0}^{l} Z_{B_i}$. By Theorem 2.17 of [36] multiplicity $({}_V\mathcal{O}_p) \geq -Y \cdot Y$. But $-Y \cdot Y \geq -(B)\cdot(B) = -\sum_{i=0}^{l} Z_{B_i}^2$ by Lemma 2.15 of [36]. Hence multiplicity $({}_V\mathcal{O}_p) \geq -\sum_{i=0}^{l} Z_{B_i}^2$. The rest of the corollary is easy.

**3. Calculation of Hilbert Functions.** Suppose $H^1(M, \mathcal{O}) = \mathbb{C}^2$ and ${}_V\mathcal{O}_p$ is Gorenstein. In this section we calculate the Hilbert function of ${}_V\mathcal{O}_p$. In particular, the dimension of the Zariski tangent space is computed. Hence we know the lowest possible embedding dimension of the singularity.

THEOREM 3.1. *Let $V$ be a normal two-dimensionsal Stein space with $p$ as its only weakly elliptic singularity. Let $\pi: M \to V$ be the minimal good resolution. Suppose ${}_V\mathcal{O}_p$ is Gorenstein and $H^1(M, \mathcal{O}) = \mathbb{C}^2$. Let $Z_{B_0} = Z, Z_{B_1}, \dots, Z_{B_l}, Z_E$ be the elliptic sequence. If $Z_E \cdot Z_E \leq -3$, then $m^n \cong H^0(A, \mathcal{O}(-n(B)))$, $n \geq 0$, where $B = \sum_{i=0}^{l} Z_{B_i}$.*

*Proof.* It is true that $H^0(A, \mathcal{O}(-B)) = \varinjlim H^0(U, \mathcal{O}(-B))$, $U$ a neighborhood of $A$. Since $m\mathcal{O} = \mathcal{O}(-B)$ by Theorem 2.1, $H^0(A, \mathcal{O}(-B)) = m$.

*Step* 1. We are going to show

$$H^0(M,\mathcal{O}(-B-Z_E))\otimes_{\mathbb{C}}H^0(M,\mathcal{O}(-nB-Z_E))$$
$$\to H^0(M,\mathcal{O}(-(n+1)B-2Z_E))$$

is surjective. It suffices to show

$$\begin{aligned}\tau: &H^0(M,\mathcal{O}(-B-Z_E)/\mathcal{O}(-2B-Z_E))\\ &\otimes_{\mathbb{C}}H^0(M,\mathcal{O}(-nB-Z_E)/\mathcal{O}(-(n+1)B-Z_E))\\ &\to H^0(M,\mathcal{O}(-(n+1)B-2Z_E)/\mathcal{O}(-(n+2)B-2Z_E))\end{aligned} \tag{3.1}$$

is surjective.

Let us first demonstrate this fact. We first show that the image of $H^0(M,\mathcal{O}(-B-Z_E))\otimes_{\mathbb{C}}H^0(M,\mathcal{O}(-nB-Z_E))$ contains $H^0(M,\mathcal{O}(-mB-2Z_E))$ for some $m$. Let $f_1,\dots,f_s\in H^0(M,\mathcal{O}(-nB-Z_E))$ generate $\mathcal{O}(-nB-Z_E)$ as an $\mathcal{O}$-module. The proof of Theorem 2.1 and Proposition 2.8 of [36] show that such $f_i$'s do exist. The $\mathcal{O}$-module map

$$\rho:\oplus_s\mathcal{O}(-B-Z_E)\to\mathcal{O}(-(n+1)B-2Z_E)$$

given by $(g_1,\dots,g_s)\to\Sigma f_ig_i$ is then surjective. Let $K=\ker\rho$.

$$0\to K\to\oplus_s\mathcal{O}(-B-Z_E)$$
$$\xrightarrow{\rho}\mathcal{O}(-(n+1)B-2Z_E)\to 0$$

is exact. Multiplying by $\mathcal{O}(-kB)$, we get

$$\begin{tikzcd}
0 \arrow[r] & K\mathcal{O}(-kB) \arrow[r] \arrow[d, "\sigma"] & \oplus_s\mathcal{O}(-(k+1)B-Z_E) \arrow[r] \arrow[d] & \mathcal{O}(-(n+k+1)B-2Z_E) \arrow[r] \arrow[d, "\lambda"] & 0\\
0 \arrow[r] & K \arrow[r] & \oplus_s\mathcal{O}(-B-Z_E) \arrow[r] & \mathcal{O}(-(n+1)B-2Z_E) \arrow[r] & 0
\end{tikzcd}$$

with the vertical maps the inclusion maps, is commutative. The verification that the first line is exact is the same as the verification that (5.5) of [16] was exact.

$$\begin{array}{ccccc} H^0(M, \oplus_s \mathcal{O}(-(k+1)B - Z_E)) & \to & H^0(M, \mathcal{O}(-(n+k+1)B - 2Z_E)) & \to & H^1(M, K\mathcal{O}(-kB)) \\ \downarrow & & \downarrow \lambda_* & & \downarrow \sigma_* \\ H^0(M, \oplus_s \mathcal{O}(-B - Z_E)) & \to & H^0(M, \mathcal{O}(-(n+1)B - 2Z_E)) & \to & H^1(M, K) \end{array}$$

is commutative with exact rows. By Theorem 5.4 of [16], $\sigma*$ is the zero map for sufficiently large $k$. Then given $h \in H^0(M, \mathcal{O}(-(n+k+1)B - 2Z_E))$, we have $\lambda*(h) = \rho*(g)$ for some $g$, by exactness. Letting $m = n+1+k$, we have that the image of $H^0(M, \mathcal{O}(-B - Z_E)) \otimes_{\mathbb{C}} H^0(M, \mathcal{O}(-nB - Z_E))$ contains $H^0(M, \mathcal{O}(-mB - 2Z_E))$ as required.

If $m > n+1 \geq 2$, we shall show that the image of $H^0(M, \mathcal{O}(-B - Z_E)) \otimes_{\mathbb{C}} H^0(M, \mathcal{O}(-nB - Z_E))$ contains $H^0(M, \mathcal{O}(-(m-1)B - 2Z_E))$. By induction argument, we will be done. Look at the following diagram:

$$\begin{array}{ccccc} & & & & 0 \\ & & & & \downarrow \\ & & & & H^0(M, \mathcal{O}(-mB - 2Z_E)) \\ & & & & \downarrow \\ & & & & H^0(M, \mathcal{O}(-(m-1)B - 2Z_E)) \\ & & & & \downarrow \\ H^0\left(M, \dfrac{\mathcal{O}(-B - Z_E)}{\mathcal{O}(-2B - Z_E)}\right) & \otimes_{\mathbb{C}} & H^0\left(M, \dfrac{\mathcal{O}(-(m-2)B - Z_E)}{\mathcal{O}(-(m-1)B - Z_E)}\right) & \to & H^0\left(M, \dfrac{\mathcal{O}(-(m-1)B - 2Z_E)}{\mathcal{O}(-mB - 2Z_E)}\right) \\ & & & & \downarrow \\ & & & & 0 \end{array}$$

with the vertical sequence exact because $H^1(M, \mathcal{O}(-mB - 2Z_E)) = 0$ and the horizontal map surjective. Since $H^1(M, \mathcal{O}(-2B - Z_E)) = 0 = H^1(M, \mathcal{O}(-(m-1)B - Z_E))$, it follows that the image of $H^0(M, \mathcal{O}(-B - Z_E)) \otimes_{\mathbb{C}} H^0(M, \mathcal{O}(-nB - Z_E))$ contains $H^0(M, \mathcal{O}(-(m-1)B - 2Z_E))$.

It remains to prove (3.1) is surjective for all $n \geq 1$. The proof breaks up into three subcases:

(i) There is an $A_i$ (call it $A_1$) such that $Z_E \cdot Z_E + 1 \leq A_1 \cdot Z_E \leq -2$.
(ii) There is an $A_i$ (call it $A_1$) such that $A_1 \cdot Z_E = Z_E \cdot Z_E$.
(iii) $A_i \cdot Z_E = -1$ or 0, all $A_i \subseteq |E|$. Take $A_1 \cdot Z_E = -1$.

In case (i), all irreducible components are nonsingular rational curves. Choose a computation sequence for $Z$ as follows: $Z_0 = 0$, $Z_1, \ldots,$ $Z_k = E = Z_{k-1} + A_{i_k}, \ldots,$ $Z_{r_0} = Z_E, \ldots,$ $Z_{r_1} = Z_{B_l}, \ldots,$ $Z_{r_2} = Z_{B_{l-1}}, \ldots,$ $Z_{r_l} = Z_{B_1}, \ldots,$ $Z_{r_{l+1}} = Z_{B_0} = Z,$

where $A_{i_k} = A_1$. Consider

$$\tau(B_0,\dots,B_h,j): H^0\left(M, \frac{\mathcal{O}(-B-Z_E)}{\mathcal{O}(-B-Z_E-A_{i_j})}\right) \otimes_{\mathbb{C}} H^0\left(M, \frac{\mathcal{O}(-nB-Z_E-G_h-Z_{j-1})}{\mathcal{O}(-nB-Z_E-G_h-Z_j)}\right)$$
$$\to H^0\left(M, \frac{\mathcal{O}(-(n+1)B-2Z_E-G_h-Z_{j-1})}{\mathcal{O}(-(n+1)B-2Z_E-G_h-Z_j)}\right)$$
$$\forall \quad -1 \leqslant h \leqslant l-1, \quad 1 \leqslant j \leqslant r_{l-h} \quad \text{where } G_h = \sum_{i=0}^{h} Z_{B_i}. \tag{3.2}$$

To show that (3.1) is surjective, it will suffice to show that $\tau(B_0,\dots,B_h,j)$ is surjective $\forall\ -1 \leqslant h \leqslant l-1$, $1 \leqslant j \leqslant r_{l-h}$. Indeed, since$(G_h)\cdot A_j \leqslant 0$ for all $A_j \subseteq A$, all of the first cohomology groups

$$H^1(M, \mathcal{O}(-nB-G_h-z_j)) = 0 \quad \text{and} \quad H^1(M, \mathcal{O}(-(n+1)B-2Z_E-G_h-Z_j)) = 0$$

by Proposition 2.7 of [36]. Hence $H^0(M, \mathcal{O}(-nB-Z_E)/\mathcal{O}(-(n+1)B-Z_E)$ can be written via successive quotients:

$$\begin{aligned} 0 &\to H^0(M, \mathcal{O}(-nB-Z_E-G_h-Z_j)/\mathcal{O}(-(n+1)B-Z_E)) \\ &\to H^0(M, \mathcal{O}(-nB-Z_E-G_h-Z_{j-1})/\mathcal{O}(-(n+1)B-Z_E)) \\ &\to H^0(M, \mathcal{O}(-nB-Z_E-G_h-Z_{j-1})/\mathcal{O}(-nB-Z_E-G_h-Z_j)) \to 0, \end{aligned}$$
$$-1 \leqslant h \leqslant l-1, \quad 1 \leqslant j \leqslant r_{l-h},$$

where we denote $\Sigma_{i=0}^{-1} Z_{B_i} = 0$, $Z_0 = 0$, $B = \Sigma_{i=0}^{l} Z_{B_i}$ and $G_h = \Sigma_{i=0}^{h} Z_{B_i}$.

$$H^0\left(M, \frac{\mathcal{O}(-(n+1)B-2Z_E)}{\mathcal{O}(-(n+2)B-2Z_E)}\right)$$

also can be written via similar successive quotients. Moreover, by Proposition 2.7 of [36] and the proof of Theorem 2.1, we have $H^1(M, \mathcal{O}(-B-Z_E-A_{i_j})) = 0$. Hence

$$H^0\left(M, \frac{\mathcal{O}(-B-Z_E)}{\mathcal{O}(-2B-Z_E)}\right) \to H^0\left(M, \frac{\mathcal{O}(-B-Z_E)}{\mathcal{O}(-B-Z_E-A_{i_j})}\right)$$

is surjective. Look at the following commutative diagrams:

$$\begin{array}{ccccc}
H^0\left(M, \dfrac{\mathcal{O}(-B-Z_E)}{\mathcal{O}(-2B-Z_E)}\right) & \otimes_{\mathbf{C}} & H^0\left(M, \dfrac{\mathcal{O}(-nB-Z_E-G_h-Z_j)}{\mathcal{O}(-(n+1)B-Z_E)}\right) & \to & H^0\left(M, \dfrac{\mathcal{O}(-(n+1)B-2Z_E-G_h-Z_j)}{\mathcal{O}(-(n+2)B-2Z_E)}\right) \\
\downarrow & & \downarrow & & \downarrow \\
H^0\left(M, \dfrac{\mathcal{O}(-B-Z_E)}{\mathcal{O}(-2B-Z_E)}\right) & \otimes_{\mathbf{C}} & H^0\left(M, \dfrac{\mathcal{O}(-nB-Z_E-G_h-Z_{j-1})}{\mathcal{O}(-(n+1)B-Z_E)}\right) & \to & H^0\left(M, \dfrac{\mathcal{O}(-(n+1)B-2Z_E-G_h-Z_{j-1})}{\mathcal{O}(-(n+1)G-2Z_E)}\right) \\
\downarrow & & \downarrow & & \downarrow \\
H^0\left(M, \dfrac{\mathcal{O}(-B-Z_E)}{\mathcal{O}(-2B-Z_E-A_{i_j})}\right) & \otimes_{\mathbf{C}} & H^0\left(M, \dfrac{\mathcal{O}(-nB-Z_E-G_h-Z_{j-1})}{\mathcal{O}(-nB-Z_E-G_h-Z_j)}\right) & \to & H^0\left(M, \dfrac{\mathcal{O}(-(n+1)B-2Z_E-G_h-Z_{j-1})}{\mathcal{O}(-(n+1)B-2Z_E-G_h-Z_j)}\right)
\end{array}$$

$$-1 \leqslant h \leqslant l-1, \quad 1 \leqslant j \leqslant r_{l-h}$$

Thus, if (3.2) is surjective for all $1 \leqslant j \leqslant r_{l-h}$, $-1 \leqslant h \leqslant l-1$, then (3.1) is also surjective.

Suppose that the target space in (3.2) is nonzero, i.e., $-A_{i_j} \cdot ((n+1)B + 2Z_E + G_h + Z_{j-1}) \geqslant 0$. We need $-A_{i_j} \cdot (B + Z_E) \geqslant 0$ and $-A_{i_j} \cdot (nB + Z_E + G_h + Z_{j-1}) \geqslant 0$. For $j \neq k$, $A_{i_j} \cdot Z_{j-1} = 1$. If $-A_{i_j}(B + Z_E) > 0$, then $-A_{i_j}(nB + Z_E) > 0$. Hence $-A_{i_j} \cdot (nB + Z_E + G_h + Z_{j-1}) \geqslant 0$. If $-A_{i_j} \cdot (B + Z_E) = 0$, then $-A_{i_j} \cdot (nB + Z_E + G_h + Z_{j-1}) = -A_{i_j} \cdot ((n+1)B + 2Z_E + G_h + Z_{j-1}) \geqslant 0$. For $j = k$, $A_{i_k} \cdot Z_{k-1} = 2$. But by construction $A_{i_k} \cdot Z_E \leqslant -2$, and so (3.2) is surjective for all $0 \leqslant j \leqslant r_{l-h} - 1 \leqslant h \leqslant l - 1$.

Let us do case (ii). Suppose $|E|$ has more than one irreducible component. The proof of case (i) fails only because the maps

$$H^0(M, \mathcal{O}(-B - Z_E)) \to H^0\left(M, \frac{\mathcal{O}(-B - Z_E)}{\mathcal{O}(-B - Z_E - A_{i_k})}\right)$$

and

$$H^0(M, \mathcal{O}(-B - Z_E)) \to H^0\left(M, \frac{\mathcal{O}(-B - Z_E)}{\mathcal{O}(-B - Z_E - A_{i_{t+k}})}\right)$$

need not be surjective where $A_{i_{t+k}} \not\subseteq |E|$ and the computation sequence starting from $A_{i_{t+k}}$ in order to reach $|E|$ must first reach $A_1$. In (3.2)

$$H^0\left(M, \frac{\mathcal{O}(-B - Z_E)}{\mathcal{O}(-B - Z_E - A_1)}\right)$$

must be replaced by the subspace S of Proposition 2.8 of [36]. $\dim S = -A_1(B + Z_E) = -A_1 \cdot Z_E = -Z_E \cdot Z_E \geqslant 2$. Also

$$\begin{aligned}\dim H^0&\left(M, \frac{\mathcal{O}(-nB - Z_E - G_h - Z_{k-1})}{\mathcal{O}(-nB - Z_E - G_h - Z_k)}\right)\\ &= -A_1 \cdot (nB + Z_E + G_h + Z_{k-1}) + 1\\ &= -nA_1 \cdot Z_E + 1 \geqslant 2.\end{aligned}$$

Under these conditions

$$\begin{aligned}\tau(B_0, \ldots, B_h, k): S \otimes_{\mathbb{C}} H^0&\left(M, \frac{\mathcal{O}(-nB - Z_E - G_h - Z_{k-1})}{\mathcal{O}(-nB - Z_E - G_h - Z_k)}\right)\\ &\to H^0\left(M, \frac{\mathcal{O}(-(n+1)B - 2Z_E - G_h - Z_{k-1})}{\mathcal{O}(-(n+1)B - 2Z_E - G_h - Z_k)}\right)\end{aligned}$$

is still surjective $\forall\ -1 \leqslant h \leqslant l-1$. Namely, consider the subspace $T$ of $S$ of sections which vanish at some given point, say $a \in A_1$. $T$ has codimension 1 in $S$. If all the elements of $T$ have a common zero at some point $b \neq a \in A_1$ or if all have a second order zero at $a$, then $T$, having codimension 2 in

$$H^0\left(M, \frac{\mathcal{O}(-B-Z_E)}{\mathcal{O}(-B-Z_E-A_{i_k})}\right),$$

represents all sections of a suitable line bundle over $A_1$. Then $\tau(B_0, \ldots, B_h; k)$ is readily seen to be surjective as in proof of [16, Lemma 7.9, p. 144–146], but more easily. If the elements of $T$ have no common zeros, then think of $T$ as a codimension-1 subspace of the sections of a line bundle, and replace S by $T$ in the previous case. Eventually we see that $\tau(B_0, \ldots, B_h; k)$ is surjective when dim $T=1$.

Also in (3.2),

$$H^0\left(M, \frac{\mathcal{O}(-B-Z_E)}{\mathcal{O}(-B-Z_E-A_{i_{t+k}})}\right)$$

must be replaced by the subspace $R_{t+k}$ which is the image of

$$\varphi_{t+k}: H^0(M, \mathcal{O}(-B-Z_E)) \to H^0\left(M, \frac{\mathcal{O}(-B-Z_E)}{\mathcal{O}(-B-Z_E-A_{i_{t+k}})}\right)$$

if $\varphi_{t+k}$ is not surjective. By the proof of Theorem 2.1, case (ii), we know that $R_{t+k}$ has at most codimension one in

$$H^0\left(M, \frac{\mathcal{O}(-B-Z_E)}{\mathcal{O}(-B-Z_E-A_{i_{t+k}})}\right).$$

Moreover the elements of $R_{t+k}$ have no common zeros as sections of the line bundle on $A_{i_{t+k}}$ associated to $\mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-A_{i_{t+k}})$, and

$$\dim H^0\left(M, \frac{\mathcal{O}(-nB-Z_E-Z_{t+k}-G_h)}{\mathcal{O}(-nB-Z_E-G_h-Z_{t+k})}\right) \geqslant 2.$$

In fact, since we assume that

$$\varphi_{t+k}: H^0(M, \mathcal{O}(-B-Z_E)) \to H^0\left(M, \frac{\mathcal{O}(-B-Z_E)}{\mathcal{O}(-B-Z_E-A_{i_{t+k}})}\right)$$

is not surjective, it follows from the proof of case (ii) of Theorem 2.1 that $-A_{i_{k+t}}(B+Z_E) \geqslant 1$. We claim that $-A_{i_{k+t}} \cdot (B+Z_E) \neq 1$. Otherwise

$$H^0\left(M, \frac{\mathcal{O}(-B-Z_E)}{\mathcal{O}(-B-Z_E-A_{i_{k+t}})}\right) = \mathbb{C}^2.$$

Inductive argument as in the proof of case (ii) of Theorem 2.1 will show that there exists $f \in H^0(M, \mathcal{O}(-B-Z_E))$ such that the image of $f$ in

$$H^0\left(M, \frac{\mathcal{O}(-B-Z_E)}{\mathcal{O}(-B-Z_E-A_{i_{k+t}})}\right)$$

as section of the line bundle associated to $\mathcal{O}(-B-Z_E)/\mathcal{O}(-B-Z_E-A_{i_{k+t}})$ has no zero on $A_{i_{k+t}} \cap A_{i_{k+t-1}}$. Hence the image of $f$ cannot be in the image of

$$H^0\left(M, \frac{\mathcal{O}(-B-Z_E-Z_{k+t-1})}{\mathcal{O}(-B-Z_E-Z_{k+t})}\right) = \mathbb{C}$$

which is injected into

$$H^0\left(M, \frac{\mathcal{O}(-B-Z_E)}{\mathcal{O}(-B-Z_E-A_{i_{k+t}})}\right)$$

via the natural map and which is contained in $R_{t+k}$. Hence $\varphi_{t+k}$ is surjective. This contradicts our assumption. We conclude that $-A_{i_{k+t}} \cdot (B+Z_E) > 1$ and hence

$$\dim H^0\left(M, \frac{\mathcal{O}(-nB-Z_E-Z_{t+k-1}-G_h)}{\mathcal{O}(-nB-Z_E-Z_{t+k}-G_h)}\right) \geqslant 2.$$

Now repeating the argument above, we get that

$$\tau(B_0, \ldots, B_h; k): R_{t+k} \otimes_{\mathbb{C}} H^0\left(M, \frac{\mathcal{O}(-nB-Z_E-G_h-Z_{t+k-1})}{\mathcal{O}(-nB-Z_E-G_h-Z_{t+k})}\right)$$
$$\to H^0\left(M, \frac{\mathcal{O}(-(n+1)B-2Z_E-G_h-Z_{t+k-1})}{\mathcal{O}(-(n+1)B-2Z_E-G_h-Z_{t+k})}\right)$$

is surjective. When $|E| = A_1$ is an elliptic curve, we know that

$$H^0\left(M, \frac{\mathcal{O}(-B-Z_E)}{\mathcal{O}(-B-Z_E-A_1)}\right) \otimes_{\mathbb{C}} H^0\left(M, \frac{\mathcal{O}(-nB-Z_E-G_h)}{\mathcal{O}(-nB-Z_E-G_h-A_1)}\right)$$
$$\to H^0\left(M, \frac{\mathcal{O}(-(n+1)B-2Z_E-G_h)}{\mathcal{O}(-(n+1)B-2Z_E-G_h-A_1)}\right)$$

is surjective. This is shown in [27]. The result follows from the above and the proof of case (i).

Let us now do case (iii). The proof of case (i) fails only because

$$H^0\left(M, \frac{\mathcal{O}(-nB-Z_E-G_h-Z_{k-1})}{\mathcal{O}(-nB-Z_E-G_h-Z_k)}\right) = 0.$$

We can still get

$$H^0\left(M, \frac{\mathcal{O}(-(n+1)B-2Z_E-G_h-Z_{k-1})}{\mathcal{O}(-(n+2)B-2Z_E)}\right)$$

as an image as follows. There are two subcases. First, suppose that $A_1$ can be chosen so that $A_1 \cdot Z_E < 0$ and $e_1 > 1$, where $E = \sum e_i A_i$, $1 \leqslant i \leqslant t$. In this subcase $Z_E = E$. Then choose a computation sequence for $Z_E$ with $A_{i_1} \cdot Z_E < 0$, $E = Z_k = Z_E$, $A_1 = A_{i_k}$ and with a $Z_q$, $q < k$, such that $A_{i_q} = A_1$, $A_1 \not\subset \operatorname{supp}(E - A_1 - Z_q)$ and $A_i \cdot Z_{q-1} \leqslant 0$, $i \neq 1$, $A_i \subseteq \operatorname{supp} E$. Such a computation sequence can be formed by letting $A_{i_j} = A_1$ only when $A_{i_j} \subseteq |E|$ cannot be chosen otherwise. Then also $0, Z_q - Z_{q-1}, Z_{q+1} - Z_{q-1}, \ldots, Z_k - Z_{q-1}$ is part of a computation sequence for $Z_E = Z_k$ which, by Corollary 2.3 of [36] can be continued to terminate a $A_{i_1}$. Recall that $A_{i_1} \cdot Z_E < 0$ by construction. So by Proposition 2.7 of [36], $H^1(M, \mathcal{O}(-B - Z_E - Z_q)) = 0$ and also $H^1(M, \mathcal{O}(-nB - Z_E - G_h - (Z_k - Z_{q-1})))$ $= 0$. In place of (3.2) we use

$$H^0\left(M, \frac{\mathcal{O}(-B-Z_E-Z_{q-1})}{\mathcal{O}(-B-Z_E-Z_q)}\right) \otimes_{\mathbb{C}} H^0\left(M, \frac{\mathcal{O}(-nB-Z_E-G_h-(Z_{k-1}-Z_{q-1}))}{\mathcal{O}(-nB-Z_E-G_h-(Z_k-Z_{q-1}))}\right)$$
$$\to H^0\left(M, \frac{\mathcal{O}(-(n+1)B-2Z_E-G_h-Z_{k-1})}{\mathcal{O}(-(n+1)B-2Z_E-G_h-Z_k)}\right)$$

Look at the following diagram:

$$
\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \downarrow \\
H^0\left(M, \dfrac{\mathcal{O}(-B-Z_E)}{\mathcal{O}(-2B-Z_E)}\right) & \otimes_{\mathbf{C}} & H^0\left(M, \dfrac{\mathcal{O}(-nB-Z_E-G_h-Z_k)}{\mathcal{O}(-(n+1)B-Z_E)}\right) & \to & H^0\left(M, \dfrac{\mathcal{O}(-(n+1)B-2Z_E-G_h-Z_k)}{\mathcal{O}(-n+2)B-2Z_E}\right) \\
 & \downarrow & \downarrow & & \downarrow \\
H^0\left(M, \dfrac{\mathcal{O}(-B-Z_E)}{\mathcal{O}(-2B-Z_E)}\right) & \otimes_{\mathbf{C}} & H^0\left(M, \dfrac{\mathcal{O}(-nB-Z_E-G_h-Z_{k-1})}{\mathcal{O}(-(n+1)B-Z_E)}\right) & \to & H^0\left(M, \dfrac{\mathcal{O}(-(n+1)B-2Z_E-G_h-Z_{k-1})}{\mathcal{O}(-(n+2)B-2Z_E)}\right) \\
 & & & & \downarrow \\
H^0\left(M, \dfrac{\mathcal{O}(-B-Z_E-Z_{q-1})}{\mathcal{O}(-B-Z_E-Z_q)}\right) & \otimes_{\mathbf{C}} & H^0\left(M, \dfrac{\mathcal{O}(-nB-Z_E-G_h-(Z_{k-1}-Z_{q-1}))}{\mathcal{O}(-nB-Z_E-G_h-(Z_k-Z_{q-1}))}\right) & \to & H^0\left(M, \dfrac{\mathcal{O}(-(n+1)B-2Z_E-G_h-Z_{k-1})}{\mathcal{O}(-(n+1)B-2Z_E-G_h-Z_k)}\right) \\
 & & & & \downarrow \\
 & & & & 0
\end{array}
$$

---

$$
\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \downarrow \\
H^0\left(M, \dfrac{\mathcal{O}(-B-Z_E)}{\mathcal{O}(-2B-Z_E)}\right) & \otimes & H^0\left(M, \dfrac{\mathcal{O}(-nB-Z_E-G_h-Z_k)}{\mathcal{O}(-(n+1)B-Z_E)}\right) & \to & H^0\left(M, \dfrac{\mathcal{O}(-(n+1)B-2Z_E-G_h-Z_k)}{\mathcal{O}(-(n+2)B-2Z_E)}\right) \\
 & \downarrow & \downarrow & & \downarrow \\
H^0\left(M, \dfrac{\mathcal{O}(-B-Z_E)}{\mathcal{O}(-2B-Z_E)}\right) & \otimes & H^0\left(M, \dfrac{\mathcal{O}(-nB-Z_E-G_h-Z_{k-1})}{\mathcal{O}(-(n+1)B-Z_E)}\right) & \to & H^0\left(M, \dfrac{\mathcal{O}(-(n+1)B-2Z_E-G_h-Z_{k-1})}{\mathcal{O}(-(n+2)B-2Z_E)}\right) \\
 & & & & \downarrow \\
H^0\left(M, \dfrac{\mathcal{O}(-B-Z_E-Z_{q-1})}{\mathcal{O}(-B-Z_E-Z_{q-1}-A_1)}\right) & \otimes & H^0\left(M, \dfrac{\mathcal{O}(-nB-Z_E-G_h-(Z_{k-1}-Z_{q-1}))}{\mathcal{O}(-nB-Z_E-G_h-(Z_k-Z_{q-1}))}\right) & \to & H^0\left(M, \dfrac{\mathcal{O}(-(n+1)B-2Z_E-G_h-Z_{k-1})}{\mathcal{O}(-(n+1)B-2Z_E-G_h-Z_k)}\right) \\
 & & & & \downarrow \\
 & & & & 0
\end{array}
$$

with the column on the right exact. Our result follows easily.

In the other subcase, there must be $A_1$, $A_2$, and $A_3$ all distinct, such that $A_i \cdot Z_E < 0$, $1 \leqslant i \leqslant 3$, and $e_i = 1$, $1 \leqslant i \leqslant 3$. Choose a computation sequence for $Z_E$ with $E = Z_k$ such that $A_{i_1} \cdot A_1 > 0$, $A_{i_k} = A_1$, and such that when $Z_q$ with $q < k$, $A_{i_q} = A_2$ is reached, $A_i \cdot Z_{q-1} \leqslant 0$ for $i \neq 1,2$. We may suppose $A_3 \subset \operatorname{supp} Z_{q-1}$, for otherwise we may reverse the roles of $A_2$ and $A_3$, since $A_{i_1} \cdot A_1 > 0$ and $e_1 = 1$, $Z_{q-1} + A_1$, is part of a computation sequence for $Z_E$. $0, Z_q - Z_{q-1}, \ldots, Z_k - Z_{q-1}$ is also part of a computation sequence for $Z_E$. Therefore

$$H^1(M, \mathcal{O}(-B - Z_E - Z_{q-1} - A_1)) = 0$$

and

$$H^1(M, \mathcal{O}(-nB - Z_E - G_h - (Z_k - Z_{q-1})) = 0$$

by Proposition 2.7 of [36]. In place of (3.2) we use

$$\begin{aligned} H^0\left(M, \frac{\mathcal{O}(-B - Z_E - Z_{q-1})}{\mathcal{O}(-B - Z_E - Z_{q-1} - A_1)}\right) & \\ \otimes H^0\left(M, \frac{\mathcal{O}(-nB - Z_E - G_h - (Z_{k-1} - Z_{q-1}))}{\mathcal{O}(-nB - Z_E - G_h - (Z_k - Z_{q-1}))}\right) & \\ \to H^0\left(M, \frac{\mathcal{O}(-(n+1)B - 2Z_E - G_h - Z_{k-1})}{\mathcal{O}(-(n+1)B - 2Z_E - G_h - Z_k)}\right) & \end{aligned}$$

Look at the commutative diagram at the bottom of the opposite page:

with the column sequence on the right exact. The result follows easily.

*Step* 2. We are going to show

$$\begin{aligned} \gamma : H^0\left(M, \frac{\mathcal{O}(-B)}{\mathcal{O}(-B - 2Z_E)}\right) \otimes_{\mathbf{C}} H^0\left(M, \frac{\mathcal{O}(-nB)}{\mathcal{O}(-nB - 2Z_E)}\right) & \\ \to H^0\left(M, \frac{\mathcal{O}(-(n+1)B)}{\mathcal{O}(-(n+1)B - 2Z_E)}\right) & \qquad (3.3) \end{aligned}$$

is surjective. The proof breaks up into two subcases.

(i) $\operatorname{supp} E$ has more than one irreducible component;
(ii) $\operatorname{supp} E = A_1$ is a nonsingular elliptic curve.

In case (i), all irreducible components are nonsingular rational curves. Choose a computation sequence for $Z$ as follows: $Z_0=0$, $Z_1,\ldots,$ $Z_k=Z_{k-1}+A_{i_k},\ldots,Z_{r_0}=Z_E,\ldots,$ $Z_{r_1}=Z_{B_l},\ldots,$ $Z_{r_l}=Z_{B_1},\ldots,$ $Z_{r_{l+1}}=Z_{B_0}=Z$, where $A_{i_k}\cdot Z_E<0$, $A_{i_k}=A_1$. By Proposition 2.7 of [36],

$$H^1(M,\mathcal{O}(-nB-Z_E-Z_j))=0 \qquad \text{for} \quad n\geqslant 1,\ j\geqslant 0.$$

Consider

$$\begin{aligned}
\gamma_j : H^0\left(M,\frac{\mathcal{O}(-B)}{\mathcal{O}(-B-A_{i_j})}\right)\otimes_{\mathbb{C}} H^0\left(M,\frac{\mathcal{O}(-nB-Z_{j-1})}{\mathcal{O}(-nB-Z_j)}\right)&\\
\to H^0\left(M,\frac{\mathcal{O}(-(n+1)B-Z_{j-1})}{\mathcal{O}(-(n+1)B-Z_j)}\right)&,\\
1\leqslant j\leqslant r_0,&\\
\gamma_j' : H^0\left(M,\frac{\mathcal{O}(-B)}{\mathcal{O}(-B-A_{i_j})}\right)\otimes_{\mathbb{C}} H^0\left(M,\frac{\mathcal{O}(-nB-Z_E-Z_{j-1})}{\mathcal{O}(-nB-Z_E-Z_j)}\right)&\\
\to H^0\left(M,\frac{\mathcal{O}(-(n+1)B-Z_E-Z_{j-1})}{\mathcal{O}(-(n+1)B-Z_E-Z_j)}\right)&,\\
1\leqslant j\leqslant r_0.&
\end{aligned}\tag{3.4}$$

To show that (3.3) is surjective, it will suffice to show that $\gamma_j,\gamma_j'$ are surjective for all $1\leqslant j\leqslant r_0$. Consider the following exact sheaf sequence:

$$\begin{aligned}
&0\to\frac{\mathcal{O}(-nB-Z_j)}{\mathcal{O}(-nB-2Z_E)}\to\frac{\mathcal{O}(-nB-Z_{j-1})}{\mathcal{O}(-nB-2Z_E)}\\
&\quad\to\frac{\mathcal{O}(-nB-Z_{j-1})}{\mathcal{O}(-nB-Z_j)}\to 0,\\
&0\to\frac{\mathcal{O}(-nB-Z_E-Z_j)}{\mathcal{O}(-nB-2Z_E)}\to\frac{\mathcal{O}(-nB-Z_E-Z_{j-1})}{\mathcal{O}(-nB-2Z_E)}\\
&\quad\to\frac{\mathcal{O}(-nB-Z_E-Z_{j-1})}{\mathcal{O}(-nB-Z_E-Z_j)}\to 0,\\
&\qquad 1\leqslant j\leqslant r_0,
\end{aligned}$$

where $B=\sum_{i=0}^{l} Z_{Bi}$ and $Z_0=0$.We claim that

$$H^0\left(M, \frac{\mathcal{O}(-nB-Z_j)}{\mathcal{O}(-nB-2Z_E)}\right) \to H^0\left(M, \frac{\mathcal{O}(-nB-Z_j)}{\mathcal{O}(-nB-Z_{j+1})}\right)$$

is surjective for all $0 \leqslant j \leqslant r_0-1$. The Chern class of the line bundle associated to $\mathcal{O}(-nB-Z_j)/\mathcal{O}(-nB-Z_{j+1})$ is $-A_{i_{j+1}}\cdot(nB+Z_j)=-A_{i_{j+1}}\cdot Z_j$, which is less than 0 for $j>1$ and 0 for $j=1$. Therefore for $j>1$, the claim is trivially true because

$$H^0\left(M, \frac{\mathcal{O}(-nB-Z_j)}{\mathcal{O}(-nB-Z_{j+1})}\right) = 0.$$

For $j=0$,

$$H^0\left(M, \frac{\mathcal{O}(-B)}{\mathcal{O}(-nB-A_{i_1})}\right) = \mathbb{C}.$$

It suffices to produce a function $f \in H^0(M, \mathcal{O}(-nB))$ such that the image of $f$ in

$$H^0\left(M, \frac{\mathcal{O}(-nB)}{\mathcal{O}(-nB-A_{i_1})}\right)$$

is nonzero. By proof of Theorem 2.1 we know that

$$\rho: H^0(M, \mathcal{O}(-B)) \to H^0\left(M, \frac{\mathcal{O}(-B)}{\mathcal{O}(-B-A_{i_1})}\right) = \mathbb{C}$$

is surjective. There exists $g \in H^0(M, \mathcal{O}(-B))$ such that the image of $g$ in

$$H^0\left(M, \frac{\mathcal{O}(-B)}{\mathcal{O}(-B-A_{i_1})}\right)$$

is nonzero. Let $f=g^n$. Then $f \in H^0(M, \mathcal{O}(-nB))$ and $f \in H^0(M, \mathcal{O}(-nB-Z_1))$, i.e., the image of $f$ in

$$H^0\left(M, \frac{\mathcal{O}(-nB)}{\mathcal{O}(-nB-A_{i_1})}\right)$$

is nonzero. We next prove that

$$H^1\left(M, \frac{\mathcal{O}(-nB-Z_E-Z_j)}{\mathcal{O}(-nB-Z_E-Z_{j+1})}\right)=0, \qquad 0 \leqslant j \leqslant r_0-1.$$

$\mathcal{O}(-nB-Z_E-Z_j)/\mathcal{O}(-nB-Z_E-Z_{j+1})$ is the sheaf of germs of sections of a line bundle over $A_{i_{j+1}}$ of Chern class $-A_{i_{j+1}}(nB+Z_E+Z_j)$. Recall that by construction $A_{i_k}\cdot Z_E \leqslant -1$. Hence $-A_{i_{j+1}}\cdot(nB+Z_E+Z_j)\geqslant -1$. By the Serre duality theorem and Riemann-Roch theorem, we have

$$H^1\left(M, \frac{\mathcal{O}(-nB-Z_E-Z_j)}{\mathcal{O}(-nB-Z_E-Z_{j+1})}\right)=0.$$

Now the long cohomology exact sequence argument will show that

$$H^0\left(M, \frac{\mathcal{O}(-nB-Z_E-Z_j)}{\mathcal{O}(-nB-2Z_E)}\right)\to H^0\left(M, \frac{\mathcal{O}(-nB-Z_E-Z_j)}{\mathcal{O}(-nB-Z_E-Z_{j+1})}\right)$$

is surjective for all $0\leqslant j\leqslant r_0-2$. So far we have proved

$$H^0\left(M, \frac{\mathcal{O}(-nB)}{\mathcal{O}(-nB-2Z_E)}\right)$$

can be written via successive quotients:

$$\begin{aligned}
0\to H^0\left(M, \frac{\mathcal{O}(-nB-Z_j)}{\mathcal{O}(-nB-2Z_E)}\right)&\to H^0\left(M, \frac{\mathcal{O}(-nB-Z_{j-1})}{\mathcal{O}(-nB-2Z_E)}\right)\\
\to H^0\left(M, \frac{\mathcal{O}(-nB-Z_{j-1})}{\mathcal{O}(-nB-Z_j)}\right)&\to 0,\\
0\to H^0\left(M, \frac{\mathcal{O}(-nB-Z_E-Z_j)}{\mathcal{O}(-nB-2Z_E)}\right)&\to H^0\left(M, \frac{\mathcal{O}(-nB-Z_E-Z_{j-1})}{\mathcal{O}(-nB-2Z_E)}\right)\\
\to H^0\left(M, \frac{\mathcal{O}(-nB-Z_E-Z_{j-1})}{\mathcal{O}(-nB-Z_E-Z_j)}\right)&\to 0,\\
1\leqslant j\leqslant r_0,&
\end{aligned}$$

where $B=\Sigma_{i=0}^{-1}Z_{B_i}$ and $Z_0=0$.

By the proof of Theorem 2.1, we know that

$$H^0(M, \mathcal{O}(-B)) \to H^0\left(M, \frac{\mathcal{O}(-B)}{\mathcal{O}(-B-A_{i_j})}\right)$$

is surjective for all $A_{i_j} \subseteq |E|$. Hence

$$H^0\left(M, \frac{\mathcal{O}(-B)}{\mathcal{O}(-B-2Z_E)}\right) \to H^0\left(M, \frac{\mathcal{O}(-B)}{\mathcal{O}(-B-A_{i_j})}\right)$$

is surjective for all $A_{i_j} \subseteq |E|$. Look at the following commutative diagrams:

$$\begin{array}{ccc}
H^0\left(M, \frac{\mathcal{O}(-B)}{\mathcal{O}(-B-2Z_E)}\right) \otimes_{\mathbf{C}} H^0\left(M, \frac{\mathcal{O}(-nB-Z_j)}{\mathcal{O}(-nB-2Z_E)}\right) & \to H^0\left(M, \frac{\mathcal{O}(-(n+1)B-Z_j)}{\mathcal{O}(-(n+1)B-2Z_E)}\right), \\
\downarrow \qquad\qquad \downarrow & \downarrow \\
H^0\left(M, \frac{\mathcal{O}(-B)}{\mathcal{O}(-B-2Z_E)}\right) \otimes_{\mathbf{C}} H^0\left(M, \frac{\mathcal{O}(-nB-Z_{j-1})}{\mathcal{O}(-nB-2Z_E)}\right) & \to H^0\left(M, \frac{\mathcal{O}(-(n+1)B-Z_{j-1})}{\mathcal{O}(-(n+1)B-2Z_E)}\right), \\
\downarrow \qquad\qquad \downarrow & \downarrow \\
H^0\left(M, \frac{\mathcal{O}(-B)}{\mathcal{O}(-B-A_{i_j})}\right) \otimes_{\mathbf{C}} H^0\left(M, \frac{\mathcal{O}(-nB-Z_{j-1})}{\mathcal{O}(-nB-Z_j)}\right) & \to H^0\left(M, \frac{\mathcal{O}(-(n+1)B-Z_{j-1})}{\mathcal{O}(-(n+1)B-Z_j)}\right), \\
H^0\left(M, \frac{\mathcal{O}(-B)}{\mathcal{O}(-B-2Z_E)}\right) \otimes_{\mathbf{C}} H^0\left(M, \frac{\mathcal{O}(-nB-Z_E-Z_j)}{\mathcal{O}(-nB-2Z_E)}\right) & \to H^0\left(M, \frac{\mathcal{O}(-(n+1)B-Z_E-Z_j)}{\mathcal{O}(-(n+1)B-2Z_E)}\right), \\
\downarrow \qquad\qquad \downarrow & \downarrow \\
H^0\left(M, \frac{\mathcal{O}(-B)}{\mathcal{O}(-B-2Z_E)}\right) \otimes_{\mathbf{C}} H^0\left(M, \frac{\mathcal{O}(-nB-Z_E-Z_{j-1})}{\mathcal{O}(-nB-Z_E)}\right) & \to H^0\left(M, \frac{\mathcal{O}(-(n+1)B-Z_E-Z_{j-1})}{\mathcal{O}(-(n+1)B-2Z_E)}\right), \\
\downarrow \qquad\qquad \downarrow & \downarrow \\
H^0\left(M, \frac{\mathcal{O}(-B)}{\mathcal{O}(-B-A_{i_j})}\right) \otimes_{\mathbf{C}} H^0\left(M, \frac{\mathcal{O}(-nB-Z_E-Z_{j-1})}{\mathcal{O}(-nB-Z_E-Z_j)}\right) & \to H^0\left(M, \frac{\mathcal{O}(-(n+1)B-Z_E-Z_{j-1})}{\mathcal{O}(-(n+1)B-Z_E-Z_j)}\right),
\end{array}$$

$$1 \leqslant j \leqslant r_0.$$

Thus if $\gamma_j$ and $\gamma_j'$ are surjective for all $j$, (3.3) is also surjective. By the Riemann-Roch theorem, the target space of $\gamma_j$ is nonzero only if $j=1$. In that case, $-A_{i_1}\cdot(B)=0$ and $-A_{i_1}\cdot(nB)=0$. Hence $\gamma_j$ is surjective for all $j$. It remains to prove $\gamma_j'$ is surjective. Suppose that the target space $\gamma_j'$ is nonzero, i.e., $-A_{i_j}\cdot((n+1)B+Z_E+Z_{j-1})\geqslant 0$. We need $-A_{i_j}\cdot(B)\geqslant 0$ and $-A_{i_j}\cdot(nB+Z_E+Z_{j-1})\geqslant 0$. But this is obviously true because $A_{i_j}\cdot(B)=0$ for $A_{i_j}\subseteq|E|$.

In case (ii), $Z_E=E=A_1$ is an elliptic curve. By Proposition 2.7 of [36], $H^1(M, \mathcal{O}(-nB-E))=0$ for $n\geqslant 1$. Hence $H^0(M, \mathcal{O}(-nB))\to H^0(M, \mathcal{O}(-nB)/\mathcal{O}(-nB-E))$ is surjective for all $n\geqslant 1$. We have the following commutative diagram:

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
H^0\left(M, \frac{\mathcal{O}(-B-E)}{\mathcal{O}(-B-2E)}\right) & \otimes_{\mathbb{C}} & H^0\left(M, \frac{\mathcal{O}(-nB-E)}{\mathcal{O}(-nB-2E)}\right) & \to & H^0\left(M, \frac{\mathcal{O}(-(n+1)B-E)}{\mathcal{O}(-(n+1)B-2E)}\right) \\
\downarrow & & \downarrow & & \downarrow \\
\gamma: H^0\left(M, \frac{\mathcal{O}(-B)}{\mathcal{O}(-B-2E)}\right) & \otimes_{\mathbb{C}} & H^0\left(M, \frac{\mathcal{O}(-nB)}{\mathcal{O}(-nB-2E)}\right) & \to & H^0\left(M, \frac{\mathcal{O}(-(n+1)B)}{\mathcal{O}(-(n+1)B-2E)}\right) \\
\downarrow & & \downarrow & & \downarrow \\
\varphi: H^0\left(M, \frac{\mathcal{O}(-B)}{\mathcal{O}(-B-E)}\right) & \otimes_{\mathbb{C}} & H^0\left(M, \frac{\mathcal{O}(-nB)}{\mathcal{O}(-nB-E)}\right) & \to & H^0\left(M, \frac{\mathcal{O}(-(n+1)B)}{\mathcal{O}(-(n+1)B-E)}\right) \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
\end{array}$$

with the column sequences on the right exact. Let $N$ be the line bundle over $A_1 = E$ whose sheaf of germs of sections is $\mathcal{O}(-B)/\mathcal{O}(-B-E)$. By the proof of Proposition 1.7, $N$ is a trivial line bundle over $(A_1, \mathcal{O}_{A_1})$ and $H^0(M, \mathcal{O}(-B)/\mathcal{O}(-B-E)) \cong \mathbb{C}$. Since

$$\mathcal{O}(-nB)/\mathcal{O}(-nB-E) \cong \underbrace{\mathcal{O}(-B)/\mathcal{O}(-B-E) \otimes_{\mathcal{O}_E} \cdots \otimes_{\mathcal{O}_E} \mathcal{O}(-B)/\mathcal{O}(-B-E)}_{n},$$

$\mathcal{O}(-nB)/\mathcal{O}(-nB-E)$ corresponds to a trivial line bundle $N^n$ over $A_1 = |E|$. Therefore $H^0(M, \mathcal{O}(-nB)/\mathcal{O}(-nB-E)) = \mathbb{C}$ by the same argument as in the proof of Theorem 1.5. It follows that the map $\varphi$ is surjective. The map $\psi$ is also surjective. This is shown in [27]. It follows that $\gamma$ is surjective. This completes the proof of step 2.

*Step* 3. To show that $m^n \cong H^0(A, \mathcal{O}(-nB))$, we shall show that $H^0(M, \mathcal{O}(-B)) \otimes_{\mathbb{C}} H^0(M, \mathcal{O}(-nB)) \to H^0(M, \mathcal{O}(-(n+1)B))$ is surjective. Consider the following commutative diagram:

$$\begin{array}{cccccc}
 & & & & 0 & \\
 & & & & \downarrow & \\
H^0(M, \mathcal{O}(-B-Z_E)) & \otimes_{\mathbb{C}} & H^0(M, \mathcal{O}(-nB-Z_E)) & \to & H^0(M, \mathcal{O}(-(n+1)B-2Z_E)) & \to 0 \\
\downarrow & & \downarrow & & \downarrow & \\
H^0(M, \mathcal{O}(-B)) & \otimes_{\mathbb{C}} & H^0(M, \mathcal{O}(-nB)) & \to & H^0(M, \mathcal{O}(-(n+1)B)) & \to 0 \\
\downarrow & & \downarrow & & \downarrow & \\
H^0\left(M, \frac{\mathcal{O}(-B)}{\mathcal{O}(-B-2Z_E)}\right) & \otimes_{\mathbb{C}} & H^0\left(M, \frac{\mathcal{O}(-nB)}{\mathcal{O}(-nB-2Z_E)}\right) & \to & H^0\left(M, \frac{\mathcal{O}(-(n+1)B)}{\mathcal{O}(-(n+1)B-2Z_E)}\right) & \to 0 \\
 & & & & \downarrow & \\
 & & & & 0 &
\end{array}$$

with the column sequence on the right exact. The first row and third row are exact by step 1 and step 2 respectively. Since $H^1(M, \mathcal{O}(-nB-2Z_E)) = 0$ for

$n \geqslant 1$ by Proposition 2.7 of [36], it follows that $H^0(M, \mathcal{O}(-nB)) \to H^0(M, \mathcal{O}(-nB)/\mathcal{O}(-nB-2Z_E))$ is onto for $n \geqslant 1$. Consequently, the second row is exact.

Q.E.D.

THEOREM 3.2. *Let $V$ be normal two-dimensional Stein space with $p$ as its only weakly elliptic singularity. Let $\pi: M \to V$ be the minimal good resolution. Suppose ${}_V\mathcal{O}_p$ is Gorenstein and $H^1(M, \mathcal{O}) = \mathbb{C}^2$. Let $Z_{B_0} = Z, Z_{B_1}, \ldots, Z_{B_l}, Z_E$ be the elliptic sequence. If $Z_E \cdot Z_E \leqslant -3$, then $\dim m^n/m^{n+1} = -n(\sum_{i=0}^{l} Z_{B_i}^2)$, $n \geqslant 1$.*

*Proof.* The long cohomology exact sequence

$$0 \to H^0\left(M, \frac{\mathcal{O}(-nB)}{\mathcal{O}(-(n+1)B)}\right) \to H^0(M, \mathcal{O}_{(n+1)B})$$
$$\to H^0(M, \mathcal{O}_{nB}) \to H^1\left(M, \frac{\mathcal{O}(-nB)}{\mathcal{O}(-(n+1)B)}\right)$$
$$\to H^1(M, \mathcal{O}_{(n+1)B}) \to H^1(M, \mathcal{O}_{nB}) \to 0$$

says that

$$\dim H^0\big(M, \mathcal{O}(-nB)/\mathcal{O}(-(n+1)B)\big) - \dim H^1\big(M, \mathcal{O}(-nB)/\mathcal{O}(-(n+1)B)\big)$$

$$= \dim H^0(M, \mathcal{O}_{(n+1)B}) - \dim H^1(M, \mathcal{O}_{(n+1)B})$$

$$- \dim H^0(M, \mathcal{O}_{nB}) + \dim H^1(M, \mathcal{O}_{nB})$$

$$= \chi((n+1)B) - \chi(nB)$$

$$= \chi(B) + \chi(nB) - n(B)\cdot(B) - \chi(nB)$$

$$= -n\left(\sum_{i=0}^{l} Z_{B_i}^2\right)$$

Consider the following cohomology exact sequence:

$$0 \to H^0(M, \mathcal{O}(-(n+1)B)) \to H^0(M, \mathcal{O}(-nB))$$
$$\to H^0\left(M, \frac{\mathcal{O}(-nB)}{\mathcal{O}(-(n+1)B)}\right) \to H^1(M, \mathcal{O}(-(n+1)B))$$
$$\to H^1(M, \mathcal{O}(-nB)) \to H^1\left(M, \frac{\mathcal{O}(-nB)}{\mathcal{O}(-(n+1)B)}\right) \to 0.$$

By Theorem 3.1,

$$
\begin{aligned}
&\dim m^n/m^{n+1}\\
&\quad= \dim H^0(M,\mathcal{O}(-nB))/H^0(M,\mathcal{O}(-(n+1)B))\\
&\quad= \dim H^0\left(M,\frac{\mathcal{O}(-nB)}{\mathcal{O}(-(n+1)B)}\right) - \dim H^1(M,\mathcal{O}(-(n+1)B))\\
&\qquad+\dim H^1(M,\mathcal{O}(-nB)) - \dim H^1\left(M,\frac{\mathcal{O}(-nB)}{\mathcal{O}(-(n+1)B)}\right)\\
&\quad= -n\left(\sum_{i=0}^{l} Z_{B_i}^2\right) + \dim H^1(M,\mathcal{O}(-nB)) - \dim H^1(M,\mathcal{O}(-(n+1)B)).
\end{aligned}
$$

We claim that $H^1(M,\mathcal{O}(-nB))\cong\mathbb{C}$ for all $n\geqslant 1$. Choose a computation sequence for $Z$ of the following form: $Z_0=0,\dots,\ Z_k=E,\dots,\ Z_{r_0}=Z_E,\dots,\ Z_{r_1}=Z_{B_l},\dots,\ Z_{r_l}=Z_{B_1},\dots,\ Z_{r_{l+1}}=Z_{B_0}=Z$. Consider the following sheaf exact sequence:

$$
\begin{aligned}
&0\to\frac{\mathcal{O}(-nB-Z_1)}{\mathcal{O}(-nB-E)}\to\frac{\mathcal{O}(-nB)}{\mathcal{O}(-nB-E)}\to\frac{\mathcal{O}(-nB)}{\mathcal{O}(-nB-Z_1)}\to 0,\\
&0\to\frac{\mathcal{O}(-nB-Z_2)}{\mathcal{O}(-nB-E)}\to\frac{\mathcal{O}(-nB-Z_1)}{\mathcal{O}(-nB-E)}\to\frac{\mathcal{O}(-nB-Z_1)}{\mathcal{O}(-nB-Z_2)}\to 0,\\
&\qquad\vdots\\
&0\to\frac{\mathcal{O}(-nB-Z_{k-1})}{\mathcal{O}(-nB-E)}\to\frac{\mathcal{O}(-nB-Z_{k-2})}{\mathcal{O}(-nB-E)}\to\frac{\mathcal{O}(-nB-Z_{k-2})}{\mathcal{O}(-nB-Z_{k-1})}\to 0.
\end{aligned}
$$

By the proof of Theorem 2.1, we know that there exists $f\in H^0(M,\mathcal{O}(-nB))$ such that the image of $f$ in $H^0(M,\mathcal{O}(-B)/\mathcal{O}(-B-Z_1))$ is nonzero. The usual long-cohomology-exact-sequence argument will show that

$$H^1(M,\mathcal{O}(-nB)/\mathcal{O}(-nB-E))\cong\mathbb{C}.$$

Since $H^1(M,\mathcal{O}(-nB-E))=0$, the exact sequence

$$
\begin{aligned}
&H^1(M,\mathcal{O}(-nB-E))\to H^1(M,\mathcal{O}(-nB))\\
&\qquad\to H^1\left(M,\frac{\mathcal{O}(-nB)}{\mathcal{O}(-nB-E)}\right)\to 0
\end{aligned}
$$

will show that $H^1(M,\mathcal{O}(-nB))\cong\mathbb{C}$. Hence $\dim m^n/m^{n+1}=-n\sum_{i=0}^{l}Z_{B_i}^2$.

Q.E.D.

**4. Absolute Isolatedness of Almost Minimally Elliptic Singularities.** The name *absolutely isolated singularity* is given in [3] and [14, 15] to a two-dimensional normal singularity, realized in $\mathbb{C}^3$, which can be resolved by means of a sequence of $\sigma$-processes with centers at points. It is proved in [3] and [32] that double rational points are always absolutely isolated and, conversely, an arbitrary double absolutely singularity in $\mathbb{C}^3$ is rational. In this paper we shall say that a two-dimensional isolated singularity is absolutely isolated if it can be resolved by means of a sequence of $\sigma$-processes with centers at points, without requiring, in what follows, that it should be realized in $\mathbb{C}^3$. It is in this sense that Laufer proved that minimally elliptic singularities which are not double points are absolutely isolated. In this section, we will prove the following theorem.

THEOREM 4.1. *Let $\pi: M \to V$ be the minimal good resolution of normal two-dimensional Stein space with $p$ as its only almost minimally elliptic singularity. If $Z_E \cdot Z_E \leqslant -3$ and ${}_V\mathcal{O}_p$ is Gorenstein, then $p$ is absolutely isolated. Moreover, blowing up $p$ at its maximal ideal yields exactly those curves $A_i$ such that $A_i \cdot Z < 0$. The singularities remaining after the blowup are the rational double points and a minimally elliptic singularity corresponding to deleting the $A_i$ with $A_i \cdot Z < 0$ from the exceptional set. The self-intersection number of the fundamental cycle of the minimally elliptic singularity is less than or equal to $-3$.*

*Proof.* Since $p$ is an almost minimally elliptic singularity, the elliptic sequence is of the form $Z, Z_E$. Let $\sigma: V' \to V$ be the blowup of $V$ at the maximal ideal $m$ at $p$. Let $A_{i_1}, \dots, A_{i_k}$ be those irreducible components of $A = \pi^{-1}(p) = \cup_{i=1}^{N} A_i$ for which $Z \cdot A_i < 0$. We consider the curve $\overline{A_1 | (A_{i_1} \cup \cdots \cup A_{i_k})}$. Generally speaking, this is a reducible curve. Let $|E|, C_2, \dots, C_s$ be its connected components. For any $A_i \subseteq C_j$, $2 \leqslant j \leqslant s$, we have $A_i \cap |E| = \varnothing$, so $-A_i \cdot A_i + 2g_i - 2 = A_i \cdot K' = -A_i \cdot (Z + E) = -A_1 \cdot Z = 0$, i.e., $A_i \cdot A_i = -2$. Therefore, $C_2, \dots, C_s$ are exceptional sets of rational double points. We shall contract to a point each of the curves $|E|, C_2, \dots, C_s$ on the surface $M$. We obtain a surface $M'$ with $s-1$ rational double points and one minimally elliptic singularity. We denote by $A'$ the image of the curve $A$ on the surface $M'$. In order to prove the theorem, we need the following proposition.

PROPOSITION 4.2. *The surface $M'$ is biholomorphically equivalent to the surface $V'$.*

*Proof.* Use the notation of Theorem 4.1. Let $\pi': M \to V'$ be the induced map. $\pi'$ is holomorphic, since $m\mathcal{O} = \mathcal{O}(-Z)$ by Theorem 2.1. Choose $f_1, \dots, f_d \in$

$H^0(M, \mathcal{O}(-Z))$ project to a basis of $m/m^2$. Then $\sigma^{-1}(p)$ is the image of $A$ in $\mathbb{P}^{d-1}$ of the map given by the (well-defined) homogeneous coordinate $[f_1(q),\ldots,f_d(q)]$, $q \in A$. Suppose that $f_i$ generate $\mathcal{O}(-Z)$ near $q$. Then functions near $\pi'(q) \in V'$ include quotients $g/f_i^r$ where $g \in m^r$ and $r$ is a nonnegative integer. $\pi'^*(g/f^r)$ is holomorphic near $q$. More precisely, let a neighborhood $U$ of the singular point $p \in V$ be realized in the space $\mathbb{C}^d$, and let $z_1,\ldots,z_d$ be restrictions of coordinates in $\mathbb{C}^n$ to $U$. We consider the functions $f_i = \pi^*(z_i)$ in a neighborhood of the curve $A$ on the surface $M$ and define by means of them the mapping $\pi' : M \to V'$. Let $a$ be a point on the curve $A$. It follows from the proof of Theorem 2.1 that there exists a neighborhood $U_a$ of the point $a$ and a number $1 \leqslant i \leqslant d$ such that the divisor of the function $f_i$ in the neighborhood $U_a$ is precisely the cycle $Z$ and $f_j = g_j f_i$, $i \neq j$, where $g_j$ are holomorphic functions in $U_a$. Then by definition the mapping $\pi'$ transfers $U_a$ into the neighborhood $A_i$ of the set $\sigma(\mathbb{C}^d)$, the blowup of $\mathbb{C}^d$ at the maximal ideal at origin $p$, with coordinates $(t_1,\ldots,t_{i-1},z_i,t_{i+1},\ldots,t_d)$ in accordance with the formula

$$\pi'(q) = (g_1(q),\ldots,g_{i-1}(q),f_i(q),g_{i+1}(q),\ldots,g_d(q)), \tag{4.1}$$

where $q \in U_a$. Since the mapping $\sigma$ on the neighborhood $A_i$ is given by the formula $\sigma(t_1,\ldots,t_{i-1},z_i,t_{i+1},\ldots,t_d) = (t_1 z_i,\ldots,t_{i-1}z_i,z_i,t_{i+1}z_i,\ldots,t_d z_i)$, the mapping $\sigma \circ \pi'$ transfers the point $q \in U_a$ to the point $(f_1(q),\ldots,f_d(q))$, i.e., $\pi = \sigma \circ \pi'$. It is easily verified that the mapping defined in this way is concordant on the intersection of the neighborhoods $U_a$ and $U_b$, $a,b \in A$. If we put $\pi' = \sigma^{-1} \circ \pi$ on $M \backslash A$, the mapping will be concordant on the intersection of $U_a$ and $M \backslash A$. Thus we have defined a holomorphic mapping $\pi'$ of surface $M$ into $V'$ which is biholomorphic on $M \backslash A$. As we know, the manifold $\sigma(\mathbb{C}^d)$ is a line bundle with fiber $\mathbb{C}$ and base $\mathbb{C}P^{d-1}$, and $\pi^{-1}(A) = \sigma^{-1}(p) \subseteq \mathbb{C}P^{d-1}$. The mapping $\pi'/A$ can be given as whole by the formula

$$\pi'/A(q) = (f_1(q),\ldots,f_d(q)), \qquad q \in A.$$

To prove the proposition, it is sufficient to show that the mapping $\pi'$ contracts the curves $|E|, C_2,\ldots,C_s$ to points, and is biholomorphic on $M - (\cup_{i=2}^s C_i \cup |E|)$. Also we must show that $V'$ is normal.

If $A_i \subseteq |E|$, by almost minimally ellipticity, we have $A_i \cdot Z = 0$ and $H^0(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-A_i)) \cong \mathbb{C}$. If $A_i \subseteq C_j$, $2 \leqslant j \leqslant s$, then $A_i$ is a nonsingular rational curve and $H^0(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-A_i)) \cong \mathbb{C}$. Since $m\mathcal{O} = \mathcal{O}(-Z)$ by Theorem 2.1, $H^0(M, \mathcal{O}(-Z)) \to H^0(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-A_i))$ is surjective for all $A_i \subseteq \cup_{i=2}^s C_i \cup |E|$, and the mapping $\pi'$ transfers the component $A_i$ to a point. If $A_1 \cdot Z < 0$, then $A_1 \nsubseteq |E|$. We shall first show that $\pi'$ is biholomorphic near the regular points $R_1$ of $A$ within $A_1$. There are two subcases. First suppose that $A_1\cdot(Z+Z_E)\leqslant -1$. If there exists $A_i\subseteq|E|$ such that $Z_E\cdot E+1\leqslant A_i\cdot Z_E\leqslant -1$, then as in the proof of Theorem 2.1, case (i), $H^1(M,\mathcal{O}(-Z-Z_E-A_1))=0$. Hence $H^0(M,\mathcal{O}(-Z-Z_E))\to H^0(M,\mathcal{O}(-Z-Z_E-A_1))$ is surjective. Elements of $H^0(M,\mathcal{O}(-Z-Z_E))$ suffice to show that $\pi'$ is biholomorphic on $A_1\backslash(A_1\cap|E|)$. We claim that actually $\pi'$ isomorphically embeds $A_1$ in $\mathbb{C}P^{d-1}$. For $A_1\cap|E|=\varnothing$, this is clear. Suppose $A_1\cap|E|\neq\varnothing$. Let $A_2\subseteq|E|$ such that $A_1\cdot A_2=1$. Since $H^0(M,\mathcal{O}(-Z))\to H^0(M,\mathcal{O}(-Z)/\mathcal{O}(-Z-A_2))\cong\mathbb{C}$ is surjective by the proof of Theorem 2.1, there is an $f\in H^0(M,\mathcal{O}(-Z))$ whose image in $H^0(M,\mathcal{O}(-Z)/\mathcal{O}(-Z-A_1))$ as a section of line bundle on $A_1$ associated to $\mathcal{O}(-Z)/\mathcal{O}(-Z-A_1)$ is nonzero at $A_1\cap A_2=A_1\cap|E|$. Hence $\pi'$ isomorphically embeds $A_1$ in $\mathbb{C}P^{d-1}$. Suppose that $A_1$ has the following property: Any computation sequence for $Z$ starting from $A_1$ must first reach $A_i$ in order to reach $|E|$, where $A_i\cdot Z_E=Z_E\cdot Z_E$. If $H^0(M,\mathcal{O}(-Z-Z_E))\to H^0(M,\mathcal{O}(-Z-Z_E)/\mathcal{O}(-Z-Z_E-A_1))$ is surjective, then the previous argument shows that $\pi'$ isomorphically embeds $A_1$. Suppose $H^0(M,\mathcal{O}(-Z-Z_E))\to H^0(M,\mathcal{O}(-Z-Z_E)/\mathcal{O}(-Z-Z_E-A_1))$ is not surjective. By the proof of Theorem 3.1, we have $-A_i\cdot(Z+Z_E)\geqslant 2$. So $\dim H^0(M,\mathcal{O}(-Z-Z_E-Z_{k+t-1})/\mathcal{O}(-Z-Z_E-Z_{k+t}))\geqslant 2$, where $\{Z_i\}$ is the computation sequence chosen in Theorem 2.1, last part of case (2). As proved there, the image of $H^0(M,\mathcal{O}(-Z-Z_E-Z_{k+t-1})/\mathcal{O}(-Z-Z_E-Z_{k+t}))$ which is injected into $H^0(M,\mathcal{O}(-Z-Z_E)/\mathcal{O}(-Z-Z_E-A_1))$ via natural map is contained in $R$, the image of $H^0(M,\mathcal{O}(-Z-Z_E))\to H^0(M,\mathcal{O}(-Z-Z_E)/\mathcal{O}(-Z-Z_E-A_1))$. Therefore elements of $H^0(M,\mathcal{O}(-Z-Z_E))$ still suffice to show that $\pi'$ is biholomorphic on $A_1-(A_1\cap|E|)$. Consequently $\pi'$ isomorphically embeds $A_1$ in $\mathbb{C}P^{d-1}$. The other subcase is $A_1\cdot(Z+Z_E)=0$. Then $A_1\cap|E|\neq\varnothing$. Let $A_j\subseteq|E|$, $A_j\cdot A_1=1$. Choose a computation sequence for $Z$ as follows: $Z_0=0$, $Z_1=A_{i_1}=A_1$, $Z_2=Z_1+A_j,\ldots,$ $Z_{1+k}=Z_1+E,\ldots,$ $Z_{r_0+1}=Z_1+Z_E,\ldots$ . Consider the following sheaf exact sequence:

$$0\to\frac{\mathcal{O}(-Z-Z_2)}{\mathcal{O}(-Z-Z_E-A_1)}\to\frac{\mathcal{O}(-Z-A_1)}{\mathcal{O}(-Z-Z_E-A_1)}\to\frac{\mathcal{O}(-Z-A_1)}{\mathcal{O}(-Z-Z_2)}\to 0,$$

$$\vdots$$

$$0\to\frac{\mathcal{O}(-Z-Z_k)}{\mathcal{O}(-Z-Z_E-A_1)}\to\frac{\mathcal{O}(-Z-Z_{k-1})}{\mathcal{O}(-Z-Z_E-A_1)}\to\frac{\mathcal{O}(-Z-Z_{k-1})}{\mathcal{O}(-Z-Z_k)}\to 0,$$

$$\vdots$$

$$0\to\frac{\mathcal{O}(-Z-Z_{r_0})}{\mathcal{O}(-Z-Z_{r_0+1})}\to\frac{\mathcal{O}(-Z-Z_{r_0-1})}{\mathcal{O}(-Z-Z_E-A_1)}\to\frac{\mathcal{O}(-Z-Z_{r_0-1})}{\mathcal{O}(-Z-Z_{r_0})}\to 0.$$

By Proposition 2.5 of [36], we know that $A_{i_j}\cdot Z_{j-1}=1$ for $j\neq k+1$ and $A_{i_{k+1}}\cdot Z_k=2$. Hence the Riemann-Roch theorem and the usual long-cohomology-exact-sequence argument will show that $H^1(M,\mathcal{O}(-Z-A_1)/\mathcal{O}(-Z-Z_E-A_1))\cong\mathbb{C}$. Consider the following exact cohomology sequence:

$$H^1(M,\mathcal{O}(-Z-Z_E-A_1))\to H^1(M,\mathcal{O}(-Z-A_1))$$
$$\to H^1(M,\mathcal{O}(-Z-A_1)/\mathcal{O}(-Z-Z_E-A_1))\to 0.$$

If there exists $A_i\neq A_j$ such that $A_i\cdot Z_E<0$, then by the proof of case (i) of Theorem 2.1, $H^1(M,\mathcal{O}(-Z-E-A_1))=0$. Hence $H^1(M,\mathcal{O}(-Z-A_1))\cong\mathbb{C}$. If $A_j\cdot Z_E=Z_E\cdot Z_E$, the elements of S which constitute the image of $H^0(M,\mathcal{O}(-Z-Z_E))\to H^0(M,\mathcal{O}(-Z-Z_E)/\mathcal{O}(-Z-Z_E-A_j))$ as section of the line bundle associated to $\mathcal{O}(-Z-Z_E)/\mathcal{O}(-Z-Z_E-A_j)$ have no common zeros on $A_j$. Since $H^0(M,\mathcal{O}(-Z-Z_E)/\mathcal{O}(-Z-Z_E-A_1))\cong\mathbb{C}$, we conclude that $H^0(M,\mathcal{O}(-Z-Z_E))\to H^0(M,\mathcal{O}(-Z-Z_E)/\mathcal{O}(-Z-Z_E-A_1))$ is surjective. The following cohomology exact sequence

$$0\to H^0(M,\mathcal{O}(-Z-Z_E-A_1))\to H^0(M,\mathcal{O}(-Z-Z_E))$$
$$\to H^0(M,\mathcal{O}(-Z-Z_E)/\mathcal{O}(-Z-Z_E-A_1))\cong\mathbb{C}$$
$$\to H^1(M,\mathcal{O}(-Z-Z_E-A_1))\to H^1(M,\mathcal{O}(-Z-Z_E))\cong 0$$
$$\to H^1(M,\mathcal{O}(-Z-Z_E)/\mathcal{O}(-Z-Z_E-A_1))\to 0$$

shows that $H^1(M,\mathcal{O}(-Z-Z_E-A_1))=0$. Therefore we still have $H^1(M,\mathcal{O}(-Z-Z_E-A_1))\cong\mathbb{C}$. Now consider the following exact cohomology sequence:

$$0\to H^0(M,\mathcal{O}(-Z-A_1))\to H^0(M,\mathcal{O}(-Z))$$
$$\to H^0(M,\mathcal{O}(-Z)/\mathcal{O}(-Z-A_1))\to H^1(M,\mathcal{O}(-Z-A_1))$$
$$\to H^1(M,\mathcal{O}(-Z))\to H^1(M,\mathcal{O}(-Z)/\mathcal{O}(-Z-A_1))\to 0.$$

By Theorem 1.2 and (1.6) of [20], $H^1(M,\mathcal{O}(-Z))\cong\mathbb{C}$. The Riemann-Roch theorem and Serre duality will show that $H^1(M,\mathcal{O}(-Z)/\mathcal{O}(-Z-A_1))=0$. Therefore $H^1(M,\mathcal{O}(-Z-A_1))\to H^1(M,\mathcal{O}(-Z))$ is an isomorphism by dimensional considerations. It follows that $H^0(M,\mathcal{O}(-Z))\to H^0(M,\mathcal{O}(-Z)/\mathcal{O}(-Z-A_1))$ is surjective. Hence $\pi'$ isomorphically embeds $A_1$ into $\mathbb{C}P^{d-1}$. Let $q$ be a point on the curve $A_1$, $A_1\cdot Z<0$, which is not a point of intersection of components. The mapping $\pi'$ acts in accordance with the formula (4.1), and it may be assumed that $f_1=\tau^{z_1}$, where $\tau=0$ is the local equation of the curve $A_1$, and $z_1$ is the coefficient of $A_1$ in the cycle $Z$. If $z_1=1$, then the formula (4.1) gives $\pi'(z)=(\tau(z),g_2(z),\ldots,g_d(z))$, where the point $z$ lies in some neighborhood of the point $q$. Clearly the rank of the differential $d\pi'$ is 2, since we know by the previous proof that the mapping $\pi'$ on the curve $A_1$ is a biholomorphic embedding.

Let $z_1 \geqslant 2$. To complete the proof that $\pi'$ is biholomorphic near $A_1$ for each $q \in R_1$, we need a function $g \in H^0(M, \mathcal{O})$ vanishing on $A_1$ to exactly order $z_1+1$ near $q$. Let $Y$ be the least cycle such that $Y \geqslant Z+Z_E+A_1$ and $A_i \cdot Y \leqslant 0$ for all $A_i$. With $Y=\sum y_i A_i$, $1 \leqslant i \leqslant n$, we claim that $y_1 = z_1+1$. Suppose first that $(Z+Z_E)\cdot A_1 < 0$. $Y$ is formed via a computation sequence as for fundamental cycles: $Y_0 = Z+Z_E$, $Y_1 = Z+Z_E+A_1$, $Y_2,\ldots,$ $Y_r = Y$. Then $0, A_1, Y_2-Y_1,\ldots, Y_r - Y_0$ is part of a computation sequence for $Z$. Then $A_{i_j}=A_1$, $1<j\leqslant r$, is impossible, because if $A_1\cdot(Y_{j-1}-Y_0)=1$, then $A_1\cdot Y_{j-1}\leqslant 0$ and $Y_j$ was not part of the computation sequence for $Y$. While if $A_1\cdot(Y_{j-1}-Y_0)=2$, the other possibility, then $Y_j - Y_0$ contains a minimally elliptic subgraph and $A_{i_j}=A_1 \subseteq |E|$. This contradicts our assumption that $A_1 \nsubseteq |E|$. Notice that the above reasoning also shows that $A_2 \not\subset \operatorname{supp}(Y-Y_0)$ if $A_2\cdot Y_0<0$. If $A_1\cdot Y_0=0$, then $A_1 \cap |E| \neq \varnothing$, since $A_1 \cdot Z < 0$. In this case, $Y$ is actually the least cycle such that $Y \geqslant Z + A_1$ and $Y\cdot A_i \leqslant 0$ for all $A_i \subseteq A$, because $E\cdot Z = 0$ and $|E|$ is connected. $Y$ can also be found via a computation sequence as for fundamental cycle as follows: $X_0 = Z$, $X_1 = Z + A_1,\ldots,$ $X_r = Y$. Then $0, A_1, X_2 - X_0,\ldots, X_r - X_0$ is part of a computation sequence for $Z$. $A_{i_j}=A_1$, $1<j\leqslant r$ is impossible, because if $A_1\cdot(X_{j-1}-X_0)=1$, then $A_1\cdot X_{j-1}\leqslant 0$ and $X_j$ was not part of the computation sequence for $Y$. If $A_1\cdot(X_{j-1}-X_0)=2$, the other possibility, then $X_j - X_0$ contains a minimally elliptic subgraph and $A_{i_j}=A_1\subseteq|E|$. This contradicts the fact that $A_1 \nsubseteq |E|$. To prove that there exists $g \in H^0(M,\mathcal{O})$ vanishing on $A_1$ to exactly order $z_1+1$ near $q$, we have to examine $Y$ more closely. Suppose firstly that $A_1 \cap |E| \neq \varnothing$. $Y = Z+Z_E+D_1+A_1+D_2$ where $D_1, D_2$ are positive cycles such that $|D_1| \subseteq |E|$ and $|D_2| \cap |E| = \varnothing$. Hence $Z_E\cdot Y = Z_E\cdot Z_E + Z_E\cdot D_1 + Z_E \cdot A_1 = Z_E\cdot Z_E + Z_E\cdot D_1 + 1 \leqslant -2$. Let $A_i \subseteq |E|$ such that $A_i\cdot A_1 = 1$. If there exists $A_j \subseteq |E|$ such that $A_j \cdot Y < 0$, $A_j \neq A_i$, then by the proof of Theorem 2.1, case (i), $H^1(M,\mathcal{O}(-Y-A_1))=0$. Hence $H^0(M,\mathcal{O}(-Y)) \to H^0(M,\mathcal{O}(-Y)/\mathcal{O}(-Y-A_1))$ is surjective. Suppose $A_j\cdot Y=0$ for all $A_j \subseteq |E|$, $A_j \neq A_i$. Choose a computation sequence for $Z$ with $E = Z_k$, $A_{i_k}=A_i$, $A_{i_{k+1}}=A_1$. By Proposition 2.7 of [36], $H^1(M,\mathcal{O}(-Y-Z_j))=0$ for all $j$. So $H^0(M,\mathcal{O}(-Y)) \to H^0(M,\mathcal{O}(-Y)/\mathcal{O}(-Y-Z_{k+1}))$ is surjective. It follows that $H^0(M,\mathcal{O}(-Y))$ and $H^0(M,\mathcal{O}(-Y)/\mathcal{O}(-Y-Z_{k+1}))$ have the same image $R$ in $H^0(M,\mathcal{O}(-Y)/\mathcal{O}(-Y-A_1))$, and $0 \to H^0(M,\mathcal{O}(-Y-Z_k)/\mathcal{O}(-Y-Z_{k+1})) \to H^0(M,\mathcal{O}(-Y)/\mathcal{O}(-Y-Z_{k+1})) \to H^0(M,\mathcal{O}(-Y)/\mathcal{O}(-Y-Z_k)) \to 0$ is an exact sequence. Thus the image of $H^0(M,\mathcal{O}(-Y-Z_k)/\mathcal{O}(-Y-Z_{k+1}))$ which is injected into

$H^0(M,\mathcal{O}(-Y)/\mathcal{O}(-Y-A_1))$ via the natural map is contained in $R$. If $H^0(M,\mathcal{O}(-Y-Z_k)/\mathcal{O}(-Y-Z_{k+1}))\neq 0$, then the elements of $R$ have no common zeros on $A_1-(A_1\cap A_i)$ as sections of the line bundle $L$ on $A_1$ associated to $\mathcal{O}(-Y)/\mathcal{O}(-Y-A_1)$. If $H^0(M,\mathcal{O}(-Y-Z_k)/\mathcal{O}(-Y-Z_{k+1}))=0$, then $A_1\cdot Y=0$. So $H^0(M,\mathcal{O}(-Y)/\mathcal{O}(-Y-A_1))\cong\mathbb{C}$. We claim that $H^0(M,\mathcal{O}(-Y))\to H^0(M,\mathcal{O}(-Y)/\mathcal{O}(-Y-A_1))$ is surjective. It suffices to prove that the map is not zero. Since the coefficient of $A_i$ in $Z_E$ is one, $A_i\cdot Y\leqslant -2$ and $\dim H^0(M,\mathcal{O}(-Y)/\mathcal{O}(-Y-A_i))\geqslant 3$. The image of $\rho: H^0(M,\mathcal{O}(-Y))\to H^0(M,\mathcal{O}(-Y)/\mathcal{O}(-Y-A_i))$ is a subspace $S$ of codimension 1 in $H^0(M,\mathcal{O}(-Y)/\mathcal{O}(-Y-A_i))$. Hence elements of $S$ have no common zeros as sections of the line bundle $L_i$ on $A_i$ associated to $\mathcal{O}(-Y)/\mathcal{O}(-Y-A_i)$ by Proposition 2.8 of [36]. Suppose secondly that $A_1\cap|E|=\varnothing$. Let $A_i\subseteq|E|$ such that the computation sequence starting from $A_1$ in order to reach $|E|$ must first reach $A_i$. Let $C$ be the union of $A_1$ and connected components $C_n$ of those $A_j$ such that $A_j\cdot(Z+Z_E)=0$ and $A_1\cap C_n\neq\varnothing$. We claim that $Y=Z+Z_E+Z_C$, where $Z_C$ is the fundamental cycle on $C$. Obviously $Y\geqslant Z+Z_E+Z_C$. For any $A_j\not\subseteq C$ and $A_j\cap C\neq\varnothing$, we have $A_j\cdot(Z+Z_E)<0$. By previous argument, we know that $A_j\subseteq\operatorname{supp}(Y-Z-Z_E)$. Hence $Y\leqslant Z+Z_E+Z_C$. If $A_t\not\subseteq C$ and $A_t\cdot Z_C=2$, then $A_t\subseteq|E|$ and $\operatorname{supp}(E-A_t)\subseteq C$. If $\pi$ is the minimal resolution, then $Z_E=E$ and the coefficient ${}_Ez_t$ of $A_t$ in $Z_E$ is equal to one. If $\pi$ is not the minimal resolution, we still get ${}_Ez_t=1$ by Proposition 2.2 of [36] and case-by-case checking. Hence $A_t\cdot Z_E=Z_E\cdot Z_E\leqslant -3$ and $A_t\cdot Y\leqslant -1$. The proof of case (i) of Theorem 2.1 shows that $H^1(M,\mathcal{O}(-Y-A_1))=0$. Therefore $H^1(M,\mathcal{O}(-Y))\to H^1(M,\mathcal{O}(-Y)/\mathcal{O}(-Y-A_1))$ is surjective. Suppose for all $A_t\not\subseteq C$, $A_t\cdot Z_C\leqslant 1$. If $Z_E=E$, then the connected components $C_n$ of $\overline{C-A_1}$ are exceptional sets of rational double points. $Z_E\cdot Z_E\leqslant -3$ will imply that either $H^1(M,\mathcal{O}(-Y-A_1))=0$ or $Z_E\cdot Y\leqslant -2$. If $Z_E\neq E$, $Z_E\cdot Z_E\leqslant -3$ still implies that either $H^1(M,\mathcal{O}(-Y-A_1))=0$ or $Z_E\cdot Y\leqslant -2$ by Proposition 2.2 of [36] and case-by-case checking. If there exists $A_s\subseteq|E|$, $A_s\neq A_i$, such that $A_s\cdot Y<0$, then the proof of case (i) of Theorem 2.1 shows that $H^1(M,\mathcal{O}(-Y-A_1))=0$. Therefore we may suppose that $A_i\cdot Y\leqslant -2$. Choose a computation sequence for $Z$ with $E=Z_k$, $A_{i_k}=A_i, A_{i_{k+1}}\cap A_i\neq\varnothing$, $A_{i_{k+t}}=A_1$, $A_1\not\subseteq|Z_{k+t-1}|$, and such that $A_{i_j}$, $k+1\leqslant j\leqslant k+t$, are distinct from each other and not contained in $|E|$. By Proposition 2.7 of [36], $H^1(M,\ \mathcal{O}(-Y-Z_j))=0$ for all $j$, so $H^0(M,\mathcal{O}(-Y))\to H^0(M,\mathcal{O}(-Y)/\mathcal{O}(-Y-Z_{k+t}))$ is surjective. It follows that $H^0(M,\mathcal{O}(-Y))$ and $H^0(M,\mathcal{O}(-Y)/\mathcal{O}(-Y-Z_{k+t}))$ have the same image $R$ in $H^0(M,\mathcal{O}(-Y)/\mathcal{O}(-Y-A_1))$, and $0\to H^0(M,\mathcal{O}(-Y-Z_{k+t-1})/\mathcal{O}(-Y-Z_{k+t}))\to H^0(M,\mathcal{O}(-Y)/\mathcal{O}(-Y-Z_{k+t}))\to H^0(M,\mathcal{O}(-Y)/\mathcal{O}(-Y-Z_{k+t-1}))\to 0$ is an exact sequence. Thus the image of $H^0(M,\mathcal{O}(-Y-Z_{k+t-1})/\mathcal{O}(-Y-Z_{k+t}))$

which is injected into $H^0(M, \mathcal{O}(-Y)/\mathcal{O}(-Y-A_1))$ via the natural map is contained in $R$. If $H^0(M, \mathcal{O}(-Y-Z_{k+t-1})/\mathcal{O}(-Y-Z_{k+t})) \neq 0$, then the elements of $R$ have no common zeros on $A_1-(A_1 \cap A_{i_{k+t-1}})$ as sections of the line bundle $L_1$ on $A_1$ associated to $\mathcal{O}(-Y)/\mathcal{O}(-Y-A_1)$. If $H^0(M, \mathcal{O}(-Y-Z_{k+t-1})/\mathcal{O}(-Y-Z_{k+t}))=0$, then $A_1 \cdot Y=0$. Hence $H^0(M, \mathcal{O}(-Y)/\mathcal{O}(-Y-A_1)) \cong \mathbb{C}$. But by induction, we know that the elements of the image of $H^0(M, \mathcal{O}(-Y)) \to H^0(M, \mathcal{O}(-Y)/\mathcal{O}(-Y-A_{i_{k+t-1}}))$ have no common zeros on $A_{i_{k+t-1}}-(A_{i_{k+t-1}} \cap A_{i_{k+t-2}})$ as sections of the line bundle $L_{i_{k+t-1}}$ on $A_{i_{k+t-1}}$ associated to $\mathcal{O}(-Y)/\mathcal{O}(-Y-A_{i_{k+t-1}})$. It follows that $H^0(M, \mathcal{O}(-Y)) \to H^0(M, \mathcal{O}(-Y)/\mathcal{O}(-Y-A_1))$ is surjective. So far we have proved that there exists $g \in H^0(M, \mathcal{O})$ vanishing on $A_1$ to exactly order $z_1+1$ near $q$.

Let us now show that $\pi'$ is as one-to-one as possible on $A$, i.e., $\pi'$ should map the connected components $|E|, C_2, \ldots, C_s$ to distinct points and otherwise be one-to-one on $A$. We showed above that $\pi'$ is one-to-one on each $A_i$ with $A_i \cdot Z<0$. So suppose that $A_1 \cdot Z<0$ and $A_2 \cdot Z<0$. Form $Y$, the least cycle $Y \geqslant Z+Z_E+A_1$ such that $A_i \cdot Y \leqslant 0$ for all $A_i$. If both $A_1$ and $A_2$ are disjoint from $|E|$, then $A_1 \cdot (Z+Z_E)<0$, $A_2 \cdot (Z+Z_E)<0$. As shown in the previous paragraph, $A_2 \not\subset \operatorname{supp}(Y-Z-Z_E)$. In the other case, interchange the role of $A_1$ and $A_2$. If necessary, we may assume that $A_1 \cap |E| \neq \varnothing$. Then $Y$ is the least cycle $\geqslant Z+A_1$ such that $A_i \cdot Y \leqslant 0$ for all $A_i$. As shown in the previous paragraph, we still have $A_2 \not\subset \operatorname{supp}(Y-Z-Z_E)$. Since $m\mathcal{O}=\mathcal{O}(-Z)$, by the proof of the previous paragraph, we know that there are functions which separate $A_1$ from any given point in $A_2-B_j$, where $B_j=|E|$ or $C_i$, $2 \leqslant i \leqslant s$, and $B_j \cap A_1 \neq \varnothing$.

Finally, we must examine $V'$ at the singular points of $\pi'(A)$. Suppose that $q \in A_1 \cap A_2$, $A_1 \cdot Z<0$, and $A_2 \cdot Z<0$. Let $f_1 \in H^0(M, \mathcal{O})$ generate $\mathcal{O}(-Z)$ at $q$. The function $f_1$ has in a neighborhood of $q$ the form $f_1=\tau_1^{z_1}\tau_2^{z_2}$ where $\tau_1=0$, $\tau_2=0$ are the local equations of the curves $A_1$ and $A_2$. Since $A_1$ and $A_2$ intersect at the point $q$ transversely, the functions $\tau_1$ and $\tau_2$ may be taken as the local coordinate system on the surface $M$. If both $A_1$ and $A_2$ are distinct from $|E|$, then there exist $f_2, f_3 \in H^0(M, \mathcal{O})$ such that near $q$

$$f_2 = \psi_2 \tau_1^{z_1+1}\tau_2^{z_2}, \qquad \psi_2(q) \neq 0,$$
$$f_3 = \psi_3 \tau_1^{z_1}\tau_2^{z_2+1}, \qquad \psi_3(q) \neq 0.$$

In this case the formula for the mapping $\pi'$ looks like

$$\pi' = \left(\tau_1^{z_1}\tau_2^{z_2}, \psi_2\tau_1, \psi_3\tau_2, g_4, \ldots, g_d\right)$$

and its differential is of rank 2 at $q$. Suppose $A_1 \cap |E| \neq \varnothing$. Then $A_2 \cap |E| = \varnothing$.

The previous proof shows that there exists $f_2 \in H^0(M, \mathcal{O})$ such that near $q$, $f_2 = \psi_2 \tau_1^{z_1+1} \tau_2^{z_2}$, where $\psi_2(q) \neq 0$. If $A_1 \cdot Z \leqslant -2$, there is also a function $f_3 \in H^0(M, \mathcal{O})$ such that near $q$, $f_3 = \psi_3 \tau_1^{z_1+1} \tau_2^{z_2}$. The same reasoning as before will show that the differential of $\pi'$ is of rank 2 at $q$. Suppose $A_1 \cdot Z = -1$. Then $H^0(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-A_1)) \cong \mathbb{C}^2$. We need $H^0(M, \mathcal{O}(-Z)) \to H^0(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-A_1))$ to be surjective. But this has been shown on pp. 847–848. Choose instead $f_3 \in H^0(M, \mathcal{O}(-Z))$ to have a first-order zero at $q$ in $H^0(M, \mathcal{O}(-Z)/\mathcal{O}(-Z-A_1))$. Then near $q$, $f_3$ looks like $\tau_1^{z_1} \tau_2^{z_2}(\alpha \tau_2 + \dots)$, where $\alpha \neq 0$. In this case the formula for the mapping $\pi'$ looks like

$$\pi' = (\tau_1^{z_1} \tau_2^{z_2}, \psi_2 \tau_1, \alpha \tau_2 + \dots, g_4, \dots, g_d), \qquad \alpha \neq 0,$$

and its differential is of rank 2 at $q$. Lastly, let $C$ be a connected component of $\cup_{i=2}^{s} C_i \cup |E|$. We need that $V'$ is normal at $\pi'(C)$. Take $A_1 \subset A'$. Let $Y$ be the least cycle such that $Y \geqslant Z + A_1$ and $A_i \cdot Y \leqslant 0$ for all $A_i$. Then, arguing as before, one sees that $A_i \not\subset \operatorname{supp}(Y - Z)$ if $A_i \cdot Z < 0$. Since $Y - Z$ is part of a computation sequence for $Z$, $\operatorname{supp}(Y - Z)$ is connected. Then $Y - Z = Z'$ is the fundamental cycle for $C$. There is an $f \in H^0(M, \mathcal{O}(-Z))$ which generates $\mathcal{O}(-Z)$ in a neighborhood of $C$. Functions on $V'$ near $\pi'(C)$ thus include $g/f^2$ for $g \in H^0(M, \mathcal{O}(-2Z - Z')) \subset m^2$. Division by $f^2$ gives an isomorphism $H^0(M, \mathcal{O}(-2Z - Z')/\mathcal{O}(-2Z - 2Z')) \cong H^0(M, \mathcal{O}(-Z')/\mathcal{O}(-2Z'))$. The proposition is proved.

It follows from the above proposition that after applying a $\sigma$-process at $p$, we obtain a surface which has only rational double points and minimally elliptic singular points. Moreover, the self-intersection number of the fundamental cycle of the minimally elliptic singularity is less than or equal to $-3$. By Theorem 3.15 of [20] and Theorem 1 of [31], our theorem follows. Q.E.D.

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