COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 92, nº 2 (1994), p. 133-155.

<http://www.numdam.org/item?id=CM_1994__92_2_133_0>

Compositio Mathematica **92**: 133–155, 1994.

Diffeomorphic types of the complements of arrangements of hyperplanes

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Received 29 October 1992; accepted in revised form 31 May 1993

1. Introduction

An *arrangement of hyperplanes* is a finite collection of **C**-linear subspace of dimension $(l-1)$ in $\mathbf{C}^l$. For such an arrangement $\mathscr{A}$, there is a natural projective arrangement $\mathscr{A}^*$ of hyperplanes in $\mathbf{CP}^{l-1}$ associated to it. Let $M(\mathscr{A}) = \mathbf{C}^l - \bigcup\{H: H \in \mathscr{A}\}$ and $M(\mathscr{A}^*) = \mathbf{CP}^{l-1} - \bigcup\{H^*: H^* \in \mathscr{A}^*\}$. Then it is clear that $M(\mathscr{A}) = M(\mathscr{A}^*) \times \mathbf{C}^*$. The central problem in the theory of arrangements is to find connections between the topology or differentiable structure of $M(\mathscr{A})$ (or $M(\mathscr{A}^*)$) and the combinatorial geometry of $\mathscr{A}$.

The study of the topology of $M(\mathscr{A})$ is important both in the theory of hypergeometric functions. (See the work of Gelfand [Ge] and his subsequent papers, the work of Deligne and Mostow [De-Mo] and subsequent papers by Mostow) and in the singularity theory ([Ar], [Br], [De] and also [Ca].). Moreover, it plays a role in some interesting problems in algebraic geometry (see especially the works of Hirzebruch [Hi] and Moishezon [Mo].).

Let M_l denote the braid space with l strands i.e., M_l is the complement of complexified braid arrangement A_l defined by $Q = \Pi_{1 \leqslant i<j \leqslant l}(z_i - z_j)$. In 1969, Arnold [Ar] was able to calculate the Poincaré polynomial of the pure braid space M_l and the cohomology ring structure of $H^*(M_l)$. In general for an arbitrary arrangement $\mathscr{A}$, define holomorphic differential forms $\omega_H = (1/(2\pi i))(d\alpha_H/\alpha_H)$ where α_H is the linear form defining the hyperplane H for $H \in \mathscr{A}$ and let $[\omega_H]$ denote the corresponding cohomology class. Let $R(\mathscr{A}) = \bigoplus_{p=0}^{l} R_p$ be the graded **C**-algebra of holomorphic differential forms on $M(\mathscr{A})$ generated by the ω_H and 1. Arnold conjectured that the natural map $\eta \to [\eta]$ of $R(\mathscr{A}) \to H^*(M(\mathscr{A}), \mathbf{C})$ is an isomorphism of graded algebras. This was proved by Brieskorn [Br] in 1971 who showed in fact that the **Z**-subalgebra of $R(\mathscr{A})$ generated by the forms ω_H and 1 is isomorphic to the singular cohomology $H^*(M(\mathscr{A}), \mathbf{Z})$. Although Brieskorn proved the Arnold conjecture, it was not known whether the algebra $R(\mathscr{A})$ is determined by the combinatorial data of $\mathscr{A}$, since the linear forms enter the definition of $R(\mathscr{A})$. In 1980, Orlik and Solomon [Or-Sol] showed that for an arbitrary arrange-

This research is partially supported by the NSF.

ment $\mathscr{A}$ the Poincarté polynomial of $M(\mathscr{A})$ equals the Poincaré polynomial of $\mathscr{A}$. Hence the betti number of $M(\mathscr{A})$ is combinatorially determined. They also introduced a graded algebra $A(\mathscr{A})$ in [Or-So1]. It is a combinatorial invariant of $\mathscr{A}$. The main result of [Or-So1] asserts that there is an isomorphism of algebras $A(\mathscr{A}) \simeq R(\mathscr{A})$. This, together with the Brieskorn solution to Arnold's conjecture, imply that the cohomology ring $H^*(M(\mathscr{A}), \mathbf{C})$ is a combinatorial invariant of $\mathscr{A}$.

The next difficult unsolved problems involve the homotopy groups of $M(\mathscr{A})$. In a Bourbaki Seminar talk, Brieskorn [Br] generalized Arnold's results. He replaced the symmetric group and the braid arrangement by a Coxeter group W acting in $\mathbf{R}^l$. Then A acts as a reflection group in $\mathbf{C}^l$. Let $\mathscr{A} = \mathscr{A}(W)$ be its reflection arrangement. Brieskorn conjectured that $\mathscr{A}(W)$ is a $K(\pi, 1)$ arrangement for all Coxeter groups W. He proved this for some of the groups by representing M as the total space of a sequence of fibrations. Deligne [De] settled the question by proving that complement of complexification of real simplicial arrangement is $K(\pi, 1)$. This result proves Brieskorn's conjecture because the arrangement of a Coxeter group is simplicial. Shepherd and Todd [Sh-To] classified finite irreducible complex reflection groups. Recall that real reflection groups are also called Coxeter groups because finite irreducible real reflection groups were classified by Coxeter [Co]. Every real reflection group may be viewed as a complex reflection group. There are examples of complex reflection groups which are not Coxeter groups. Orlik and Solomon [Or-So2] conjectured that all the complex reflection arrangements are $K(\pi, 1)$. For a subclass of irreducible complex reflection groups called Shepherd groups, this was proved by Orlik and Solomon [Or-So3]. The conjecture is still open for the remaining irreducible complex reflection groups. In [Sa1], Salvetti made a fundamental contribution to the understanding of the higher homotopy in the complement of an arrangement. He considered a union of real affine hyperplanes in $\mathbf{C}^l$ with complement M and constructed explicitly a CW-complex $X \subset M$ of dimension l with the homotopy type of M. Recently, he introduced a class of cellular complexes by which Deligne's result is re-proved and generalized [Sa3]. Orlik [Or] has constructed for all arrangements a finite simplicial complex of the homotopy type of M. In [Ar], Arvola exhibits a simplicial homotopy equivalence between Salvetti complex and Orlik complex in the case of real arrangement. Recently, Björner and Ziegler presented a general method for constructing regular complexes with the homotopy type of $M(\mathscr{A})$. They generalized the construction of Salvetti [Sa1] to complex arrangements, gave a new proof of Brieskorn-Orlik-Solomon theorem and investigated a class of topological deformed complex arrangements [Bj-Zi].

The difficult and still unsolved problem is whether the topological or diffeomorphic type of complement $M(\mathscr{A})$ of an arrangement is combinatorial in nature. The purpose of this paper is to give a partial solution to this problem. Let $\mathscr{A}$ be a central arrangement in $\mathbf{C}^3$ and $\mathscr{A}^*$ be the corresponding projective arrangement in $\mathbf{CP}^2$. We can define a graph $G(\mathscr{A}^*)$ which depends only on the combinatorial data of the arrangement. An arrangement $\mathscr{A}^*$ is called a nice arrangement if after removing pairwise disjoint star shaped subgraphs of G, the graph becomes a forest.

MAIN THEOREM. *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be two central nice arrangements in $\mathbf{C}^3$ and $\mathscr{A}_1^*$, $\mathscr{A}_2^*$ be the corresponding projective arrangements in $\mathbf{CP}^2$. If the lattices of $\mathscr{A}_1$ and $\mathscr{A}_2$ are isomorphic, then the complements of the projective arrangements A_1^*, $\mathscr{A}_2^*$ in $\mathbf{CP}^2$ are diffeomorphic to each other.*

The proof of our Main Theorem above actually works for much general arrangements. We shall give an example in Section 4 to demonstrate this. Any arrangement such that the proof of our Main Theorem works is called a good arrangement. Given a projective arrangement $\mathscr{A}^*$ in $\mathbf{CP}^2$, it is important to find a presentation of the fundamental group of the complement $M(\mathscr{A}^*)$ and determine whether $\pi_1(M(\mathscr{A}^*))$ depends only on the lattice $L(\mathscr{A})$. If $\mathscr{A}^*$ is the complexification of a real arrangement, then this problem was solved by Randall [Ra]. The following corollary is an immediate application of our Main Theorem above.

COROLLARY. *A presentation of the fundamental group of the complement $M(\mathscr{A}^*)$ of a nice arrangement in $\mathbf{CP}^2$ can be explicitly written and it depends only on the lattice of $\mathscr{A}$.*

The second author would like to thank M. Falk for his invitation to participate in the AMS-CBMS on Arrangement of Hyperplanes at Northern Arizona University in 1988. The Main Theorem was announced in the survey paper [Ji-Ya]. The Main Theorem is based on the observation that the diffeomorphic types of the complements of arrangements are the same in a one parameter family of arrangements with isomorphic lattices. This follows immediately from a Teissier's numerical characterization of Whitney condition [Te] and Thom's first isotopy theorem. It was also observed independently during the AMS-CBMS on Arrangement of Hyperplanes by Randell [Ra]. Recently Arvola [Ar] has determined $\pi_1(M(\mathscr{A}^*))$ from a certain planar graph. In Section 2, we shall recall some terminology in abstract lattice theory. In Section 3 we shall prove the Main Theorem. For each projective arrangement in $\mathbf{CP}^2$, we associate a variety in $(\mathbf{CP}^1)^p$ where p is the number of lines in the graph $G(\mathscr{A}^*)$. This variety plays an important role in studying the diffeomorphic type of arrangement. In Section 4 we shall study two examples of these varieties.

2. Arrangement $\mathscr{A}$ and its lattice $L(\mathscr{A})$

We begin by recalling some terminology in lattice theory.

DEFINITION 2.1. Let P be a poset. An *upper bound* of a subset X of P is an element $a \in P$ such that $x \leqslant a$ for every $x \in X$. The least upper bound is an upper bound less than or equal to every other upper bound; it is denoted by $\sup X$. The notion of lower bound of X and greatest lower bound ($\inf X$) of X are defined dually.

DEFINITION 2.2. A *lattice* is a poset P any two of whose elements have a greatest lower bound or "meet" denoted by $x \wedge y$, and a lowest upper bound or "join" denoted by $x \vee y$.

DEFINITION 2.3. An element y covers an element x in a lattice L if and only if $x < y$, but $x < z < y$ for no element z in L.

DEFINITION 2.4. A chain in a lattice L is any linearly ordered subset of L.

DEFINITION 2.5. A lattice having no infinite chains is said to be semimodular whenever it has the covering property: for all lattice elements x, y, if x and y cover $x \wedge y$, then $x \vee y$ covers x and y.

DEFINITION 2.6. Let L be a lattice with finite elements. The *length* of a chain C of L is defined as $|C| - 1$. The rank of $a \in L$, denoted by $r(a)$, is the length of the longest chain in L below a. Let $\hat{0} = \inf L$ and $\hat{1} = \sup L$. Then $r(\hat{0}) = 0$. The rank of L (rank L) is defined to be $r(\hat{1})$. If a in L has rank 1, then a is called a point or an atom of the lattice.

DEFINITION 2.7. A *point lattice* (or atomic lattice) is a lattice in which every element is a join of points. A *geometric lattice* is a semimodular point lattice with no infinite chains.

In this paper an arrangement $\mathscr{A}$ is a finite collection of hyperplanes $\{H_1, \ldots, H_n\}$ through the origin in $\mathbf{C}^l$. The lattice $L(\mathscr{A})$ is the set of all intersections of subsets of $\mathscr{A}$, partially ordered by reverse inclusion i.e. $X \leqslant Y \leftrightarrow Y \subseteq X$. The rank function r on $L(\mathscr{A})$ is $r(X) = \operatorname{codim} X = l - \dim_{\mathbf{C}} X$ for $X \in L(\mathscr{A})$, each H_i is an *atom* of $L(\mathscr{A})$, the *join* is by $X \vee Y = X \cap Y$ and the *meet* is by $X \wedge Y = \bigcap\{Z : Z \in L(\mathscr{A}),\ X \cup Y \subset Z\}$.

LEMMA 2.8. *Let $\mathscr{A}$ be an arrangement. Then*

(i) *for every $X \in L(\mathscr{A})$ all chains from X to $\mathbf{C}^l$ have the same cardinality,*
(ii) *every element of $L(\mathscr{A}) - \{\mathbf{C}^l\}$ is join of atoms,*
(iii) *for all X, $Y \in L(\mathscr{A})$ the rank function satisfies*

$$r(X \wedge Y) + r(X \vee Y) \leqslant r(X) + r(Y).$$

Thus $L(\mathscr{A})$ is a geometric lattice.

DEFINITION 2.9. Let $L_p = L_p(\mathscr{A}) := \{X \in L(\mathscr{A}) : r(X) = p\}$. The Hasse diagram of $L(\mathscr{A})$ has vertices labelled by the elements of $L(\mathscr{A})$ and arranged on levels L_p, $p \geqslant 0$. Suppose $X \in L_p$ and $Y \in L_{p+1}$. An edge connects X with Y if $X < Y$.

EXAMPLE 2.10. Let $\mathscr{A}$ be an arrangement of hyperplanes in $\mathbf{C}^3$ consisting of the elements

$\{(x, y, z) \in \mathbf{C}^3 : x = y\}$, $\{(x, y, z) \in \mathbf{C}^3 : x = -y\}$. $\{(x, y, z) \in \mathbf{C}^3 : y = z\}$,
$\{(x, y, z) \in \mathbf{C}^3 : y = -z\}$, $\{(x, y, z) \in \mathbf{C}^3 : x = z\}$, $\{(x, y, z) \in \mathbf{C}^3 : x = -z\}$.
Figure 2.1 shows the Hasse diagram of $L(\mathscr{A})$.

3. Proof of the Main Theorem

In this section, we denote $\mathscr{A}$ the (central) arrangement of hyperplanes in $\mathbf{C}^3$ and $\mathscr{A}^*$ its associated projective arrangement of hyperplanes in $\mathbf{CP}^2$. Let $L(\mathscr{A})$ be the lattice associated with $\mathscr{A}$.

DEFINITION 3.1. A point p in $\mathbf{CP}^2$ is of multiplicity k in $\mathscr{A}^*$ if p is the intersection of exactly k lines in $\mathscr{A}^*$. Let $t_k(\mathscr{A}^*)$ be the number of k-tuple points in the arrangement $\mathscr{A}^*$. Then the complexity of $\mathscr{A}^*$ is defined to be $\Sigma_{k \geqslant 3}(k-2)t_k(\mathscr{A}^*)$.

Let us define the graph G from an arrangement $\mathscr{A}^*$ in $\mathbf{CP}^2$. Let VG be the set of vertices of G consisting of all points of $\mathscr{A}^*$ with multiplicity greater than 2. Let EG be the set of edges of G. Each edge in EG is a pair of distinct vertices (v_1, v_2) of VG which span a line $\langle v_1, v_2 \rangle$ of $\mathscr{A}^*$. A *reduced path* of G is denoted by a $(n+1)$-tuple $(v_0, \ldots, v_n)$ such that $(v_{i-1}, v_i) \in EG$ and

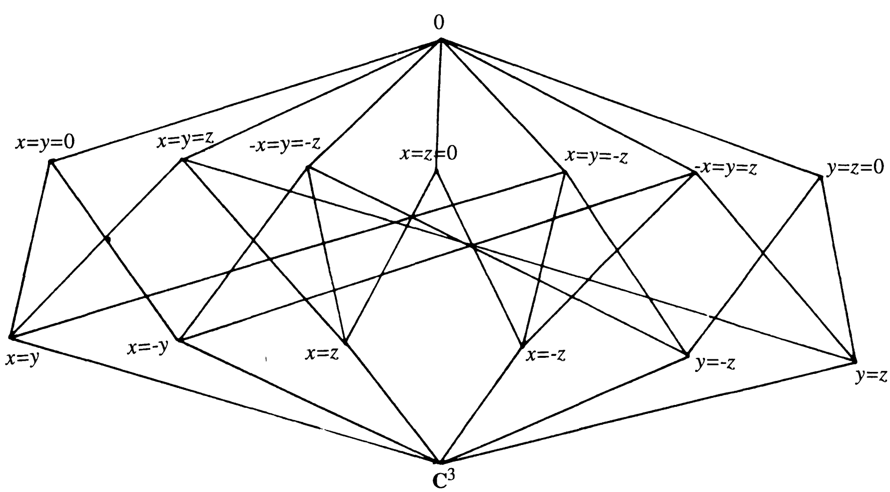


Fig. 2.1. Hasse diagram of Example 2.10.

$\langle v_{i-1}, v_i \rangle \neq \langle v_i, v_{i+1} \rangle$ for $i = 1, \ldots, n-1$. Furthermore, it is a *loop* when $v_0 = v_n$, $n \geqslant 3$. G is called a *forest* if it does not contain such a loop.

For a $v_0 \in VG$, define a subgraph $\mathrm{St}(v_0)$ of G by setting $V\mathrm{St}(v_0) = \{v_0\} \cup \{v \in VG : \langle v_0, v \rangle \in \mathscr{A}^*\}$ and $E\mathrm{St}(v_0) = \{(v, w) \in EG : v = v_0$ or $w = 0$, otherwise $\langle v, w \rangle = \langle v_0, v \rangle\}$.

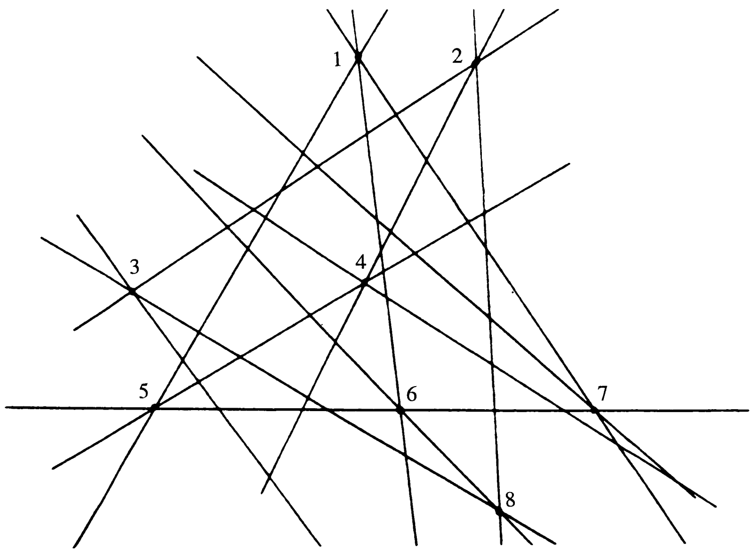


Fig. 3.1. A nice arrangement $\mathscr{A}^*$ including the line in infinite.

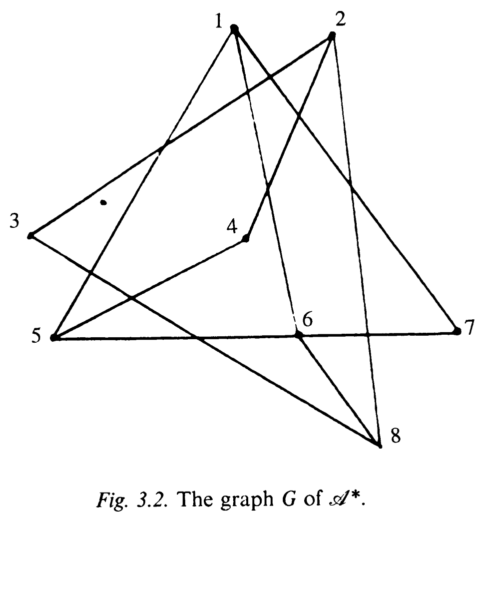


Fig. 3.2. The graph G of $\mathscr{A}^*$.

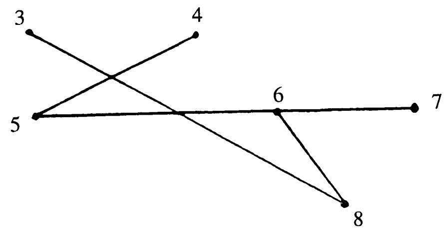


Fig. 3.3. The graph $G' = G - (E\,\mathrm{St}(v_1) \cup \{v_1\} \cup E\,\mathrm{St}(v_2) \cup \{v_2\})$.

DEFINITION 3.2. An arrangement $\mathscr{A}^*$ in $\mathbf{CP}^2$ is said to be *nice* if the graph G from $\mathscr{A}^*$ has the following property: There are $v_1, \ldots, v_m \in VG$ such that $\mathrm{St}(v_1), \ldots, \mathrm{St}(v_m)$ are pairwise disjoint in G and $G' = G - \bigcup_{i=1}^{m} (E\,\mathrm{St}(v_i) \cup \{v_i\})$ is a forest.

THEOREM 3.3. *Let $\mathscr{A}_0^*$ and $\mathscr{A}_1^*$ be two nice projective arrangements in $\mathbf{CP}^2$. If the lattices of $\mathscr{A}_0$ and $\mathscr{A}_1$ are isomorphic, then the complements $M(\mathscr{A}_0^*)$ and $M(\mathscr{A}_1^*)$ of the projective arrangements $\mathscr{A}_0^*$ and $\mathscr{A}_1^*$ in $\mathbf{CP}^2$ are diffeomorphic to each other.*

Proof. We represent the two arrangements as $\mathscr{A}_1^* = \{H_1, H_2, \ldots, H_n\}$ and $\mathscr{A}_0^* = \{G_1, G_2, \ldots, G_n\}$ where $H_i = (h_{i1}, h_{i2}, h_{i3})$ and $G_i = (g_{i1}, g_{i2}, g_{i3})$ are in $\mathbf{CP}^2$. We shall construct a one-parameter family of arrangements $\mathscr{A}^*(t)$ such that $\mathscr{A}^*(0) = \mathscr{A}_0^*$, $\mathscr{A}^*(1) = \mathscr{A}_1^*$ and $L(\mathscr{A}(t)) \equiv L(\mathscr{A}_0)$ for all $t \in [0, 1]$.

Let $\mathscr{A}^* = \{F_1, F_2, \ldots, F_n\}$ where $F_i = x_i G_i + y_i H_i$ for some $x_i, y_i \in \mathbf{C}$ such that F_i is in $\mathbf{CP}^2$, $i = 1, 2, \ldots, n$. Let $I = \{(i, j, k): 1 \leqslant i < j < k \leqslant n\}$. So $|I| = \binom{n}{3}$. Consider any triple (F_i, F_j, F_k), $(i, j, k) \in I$. Denote the matrix

$$\begin{bmatrix} x_i g_{i1} + y_i h_{i1} & x_i g_{i2} + y_i h_{i2} & x_i g_{i3} + y_i h_{i3} \\ x_j g_{j1} + y_j h_{j1} & x_j g_{j2} + y_j h_{j2} & x_j g_{j3} + y_j h_{j3} \\ x_k g_{k1} + y_k h h_{k1} & x_k g_{k2} + y_k h_{k2} & x_k g_{k3} + y_k h_{k3} \end{bmatrix}$$

by $[F_i F_j F_k]$ and its determinant by $|F_i F_j F_k|$. We now can write

$$\begin{aligned} |F_i F_j F_k| = {} & |G_i G_j G_k| x_i x_j x_k + |H_i G_j G_k| y_i x_j x_k \\ & + |G_i H_j G_k| x_i y_j x_k + |G_i G_j H_k| x_i x_j y_k \\ & + |G_i H_j H_k| x_i y_j y_k + |H_i G_j H_k| y_i x_j y_k \\ & + |H_i H_j G_k| y_i y_j x_k + |H_i H_j H_k| y_i y_j y_k. \end{aligned} \tag{3.1}$$

Since each two lines in $\mathbf{CP}^2$ meet exactly at one point, to get $L(\mathscr{A}) \equiv L(\mathscr{A}_0)$, it is sufficient to have the following

For any $(i, j, k) \in I$, $|F_i F_j F_k| = 0$ if and only if $|G_i G_j G_k| = 0$ $\quad$ (3.2)

Let $l = \Sigma_{j \geq 3} \binom{j}{3} t_j(\mathscr{A}_1^*)$. By (3.2), we need to have l equations and $\binom{n}{3} - l$ inequalities

$$P_1 = 0, \ldots, P_l = 0 \tag{3.3}$$

$$Q_1 \neq 0, \ldots, Q_{\binom{n}{3}-l} \neq 0. \tag{3.4}$$

Both P_i and Q_j have the forms like (3.1). But for P_i, the first term and the last term are zero since $|G_i G_j G_k| = |H_i H_j H_k| = 0$ by (3.2). Among $P_1, \ldots, P_l$ at most $c(\mathscr{A}_1) = \Sigma_{j \geq 3}(j-2)t_j(\mathscr{A}_1^*)$ of them are independent. To see this, we consider a j-tuple point V $(j \geq 3)$. Let $F_1, \ldots, F_j$ be the lines of $\mathscr{A}^*$ passing through V. We have $\binom{j}{3}$ equations $(|F_1 F_2 F_3| = 0, \ldots,$ etc). Since $\{F_1, \ldots, F_j\}$ can be linearly generated by F_1 and F_2, the $\binom{j}{3}$ equations is reduced equivalently to $j-2$ equations $|F_1 F_2 F_i| = 0$ for $i = 3, \ldots, j$. Now consider all j-tuple points $(j \geq 3)$. We have a system of $c(\mathscr{A}_1)$ equations, say $\{P_1 = 0, \ldots, P_{c(\mathscr{A}_1)} = 0\}$ which is equivalent to $\{P_1 = 0, \ldots, P_l = 0\}$.

As we observed before, each Pr can be written like

$$\begin{aligned} Pr = {} & a_r y_{ir} x_{jr} x_{kr} + b_r x_{ir} y_{jr} y_{jr} x_{kr} + c_r x_{ir} x_{jr} y_{k}{}^{k}{}_{r} + \alpha_r x_{ir} y_{jr} y_{kr} \\ & + \beta_r y_{ir} x_{jr} y_{kr} + \gamma_r y_{ir} y_{jr} x_{kr} \end{aligned} \tag{3.5}$$

where $a_r = |H_{ir} G_{jr} G_{kr}|$ etc. Replacing $\mathscr{A}_0^*$ by $\varphi(\mathscr{A}_0^*)$ if necessary where $\varphi: \mathbf{CP}^2 \to \mathbf{CP}^2$ is a complex analytic automorphism, we assume without loss of generality that any one (two) line(s) in $\mathscr{A}_0^*$ and any two (one) line(s) in $\mathscr{A}_1^*$ do not intersect at a point. This means that $a_r b_r c_r \alpha_r \beta_r \gamma_r \neq 0$ for all $r = 1, \ldots, c(\mathscr{A}_1)$.

Note that P_r is viewed as polynomial in $((x_1 : y_1), \ldots, (x_n : y_n)) \in (\mathbf{CP}^1)^n$. For each r, indices i_r, j_r, k_r are pairwise distinct and $(i_r, j_r, k_r) \neq (i_s, j_s, k_s)$ for $r \neq s$ where $1 \leq i_r, j_r, k_r, i_s, j_s, k_s \leq n$ and $\leq r, s \leq c(\mathscr{A}_1)$. Before we can continue our proof, we need to introduce the following concept.

DEFINITION 3.4. $(x_i : y_i) \in \mathbf{CP}^1$ is called *irregular* for the following equation

$$\begin{aligned} & a y_i x_j x_k + b x_i y_j x_k + c x_i x_j y_k + \alpha x_i y_j y_k + \beta y_i x_j y_k \\ & \quad + \gamma y_i y_j x_k = 0 \quad (abc\alpha\beta\gamma \neq 0) \end{aligned} \tag{3.6}$$

if $(ay_i)x_jx_k + (bx_i + \gamma y_i)y_jx_k + (cx_i + \beta y_i)x_jy_k + (\alpha x_i)y_jy_k$ is a reducible polynomial of the other two variables $(x_j : y_j)$ and $(x_k : y_k)$. Otherwise we call $(x_i : y_i)$ *regular* for the equation (3.6).

LEMMA 3.5. *Assume* $((x_1 : y_1), (x_2 : y_2), (x_3 : y_3)) \in (\mathbf{CP}^1)^3$ *is a solution of* (3.6). *If* $(x_1 : y_1)$ *is irregular, then either* $(x_2 : y_2)$ or $(x_3 : y_3)$ *is irregular for* (3.6). *If* $(x_1 : y_1)$ *is regular, then* $(x_2 : y_2)$ *and* $(x_3 : y_3)$ *are either both regular or both irregular for* (3.6). *In other words, the number of irregulars cannot be* 1.

Proof. When $y_1 = 0$, (3.6) becomes

$$by_2x_3 + cx_2y_3 + \alpha y_2y_3 = 0$$

which is irreducible. So if $(x_1 : y_1)$ is irregular, then $x_1 \neq 0$ and $y_1 \neq 0$.

Write (3.6) as polynomial of $(x_2 : y_2)$ and $(x_3 : y_3)$

$$(ay_1)x_2x_3 + (bx_1 + \gamma y_1)y_2x_3 + (cx_1 + \beta y_1)x_2y_3 + (\alpha x_1)y_2y_3 = 0.$$

It is reducible if and only if

$$(bx_1 + \gamma y_1)(cx_1 + \beta y_1) = a\alpha x_1y_1$$

or

$$bcx_1^2 + (b\beta + \gamma c - \alpha a)x_1y_1 + \beta\gamma y_1^2 = 0 \tag{3.7}$$

which has at most two roots of $(x_1 : y_1)$. When $(x_1 : y_1)$ is a root of the equation above, (3.6) becomes

$$[(ay_1)x_3 + (cx_1 + \beta y_1)y_3]\left[x_2 + \frac{\alpha x_1}{cx_1 + \beta y_1}\,y_2\right] = 0$$

from which we have the solution to either $(x_2 : y_2) = (-\alpha x_1 : cx_1 + \beta y_1)$ or $(x_3 : y_3) = (-(cx_1 + \beta y_1) : ay_1)$.

In the first case, we have

$$x_1 = -\frac{x_2}{\alpha} \quad \text{and} \quad y_1 = \frac{1}{\beta}\,y_2 + \frac{c}{\alpha\beta}\,x_2.$$

Put these into (3.7) yields

$$bc\beta x_2^2 - (cx_2^2 + \alpha x_2 y_2)(b\beta + \gamma c - \alpha a)\left(y_2 + \frac{c}{\alpha}x_2\right)$$

$$+ \beta\alpha\cdot\frac{1}{\beta^2}\left(y_2^2 + \frac{2c}{\alpha}x_2y_2 + \frac{c^2}{\alpha^2}x_2^2\right) = 0$$

$$\Rightarrow bc\beta_2^2 - (cx_2^2 + \alpha x_2y_2)(b\beta + \gamma c - \alpha a)$$

$$+ (\gamma c^2 x_2^2 + 2x\alpha\gamma x_2 y_2 + \gamma\alpha^2 y_2^2) = 0$$

$$\Rightarrow (bc\beta - cb\beta - \gamma c^2 + \alpha ca + \gamma c^2)x_2^2 + (-\alpha b\beta - \alpha\gamma c + \alpha^2 a)$$

$$+ 2(\alpha\gamma)x_2y_2 + \gamma\alpha^2 y_2^2 = 0$$

$$\Rightarrow cax_2^2 + (a\alpha + \gamma c - b\beta)x_2y_2 + \gamma\alpha y_2^2 = 0.$$

The last equation is a necessary and sufficient condition for $(x_2:y_2)$ being irregular of (3.6). For the second case, we have the same conclusion for $(x_3:y_3)$. □

From the argument above we also have

LEMMA 3.6. *There are at most two irregular* $(x_i:y_i)$ *of* (3.6) *for each* $i = 1, 2, 3$. (0 : 1) *and* (1 : 0) *are regular of* (3.6).

LEMMA 3.7. *For each fixed regular* $(x_1:y_1)$ *of* (3.6), *the following relation prodcuces an automorphism of* $\mathbf{CP}^1$

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = K\begin{pmatrix} -\beta y_1 - cx_1 & -\alpha x_1 \\ ay_1 & bx_1 + \gamma y_1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \quad K \in \mathbf{C}^* \tag{3.8}$$

which sends regular values to regular values of (3.6). *In particular* $(x_1:y_1) = (x_2:y_2) = (0:1)$ (*respectively* (1 : 0)) *corresponds to* $(x_3:y_3) = (0:1)$ (*respectively* (1 : 0)).

Proof.

$$\begin{vmatrix} -\beta y_1 - cx_1 & -\alpha x_1 \\ ay_1 & bx_1 + \gamma y_1 \end{vmatrix} = -bcx_1^2 - (b\beta + \gamma c - \alpha a)x_1y_1 - \beta\gamma y_1^2.$$

Since $(x_1:y_1)$ is a regular value, the above expression is nonzero by (3.7). So (3.8) is an automorphism of $\mathbf{CP}^1$. Clearly (3.8) satisfies equation (3.6). By Lemma 3.5, the mapping (3.8) sends regular values of (3.6) to regular values of (3.6). The last statement of the lemma is obvious. □

REMARK 3.8. Equation (3.8) is equivalent to equation (3.6). For we write (3.6) as

$$(ay_1x_2 + bx_1y_2 + \gamma y_1y_2)x_3 + (\alpha x_1y_2 + \beta y_1x_2 + cx_1x_2) = 0.$$

Then $(x_3, y_3) = K(-\alpha x_1x_2 - \beta y_1y_2 - cx_1x_2, ay_1x_2 + bx_1y^2 + \gamma y_1y_2)$ which is (3.8). So if $(x_1 : y_1)$ and $(x_2 : y_2)$ are regular of (3.6), then there is a unique $(x_3 : y_3)$ solved in terms of $(x_1 : y_1)$ and $(x_2 : y_2)$. We call such procedure "fixing two variables to solve the other" and such $(x_1 : y_1)$, $(x_2 : y_2)$, $(x_3 : y_3)$ "solved variables". Let us now return to the proof of Theorem 3.3. Since $\mathscr{A}_1^*$ is a nice projective arrangement in $\mathbf{CP}^2$, there are $v_1, \ldots, v_m \in VG$, where G is the graph of $\mathscr{A}_1^*$, such that $\mathrm{St}(v_1), \ldots, \mathrm{St}(v_m)$ are disjoint pairwise in G and

$$G' = G - \bigcup_{i=1}^{m} (E(\mathrm{St}(v_i)) \cup \{v_i\})$$

is a forest.

Suppose $m = 1$. Assume that v_1 is a point of multiplicity k in $\mathscr{A}_1^*$. Recall that by the definition of G, $k \geqslant 3$. Then there are k variables appearing in $k - 2$ equations of (3.6). Suppose that these variables are $(x_1 : y_1), \ldots, (x_k : y_k)$ and $(x_1 : y_1)$, $(x_2 : y_2)$ appear in each of these $k - 2$ equations. We can fix $(x_1 : y_1)$, $(x_2 : y_2)$ to solve $(x_3 : y_3), \ldots, (x_k : y_k)$.

The rest of the unsolved variables and equations in (3.5) correspond to the graph G' which is a forest. At each following step, we consider the graph formed by the vertices with unsolved variables. In each component of this graph we pick a vertex which is adjacent (in G) to a vertex whose variables are solved and apply the same procedure to solve its variables. (If there is a connected component of G' which is not connected with any vertices whose variables are solved, we pick any one of its vertices.) The set of vertices whose variables are solved and which lie in a same connected component of G span a subgraph of G which is connected. Thus we can solve all variables in terms of some variables without ambiguity since G' is a forest.

For the case $m = 0$ or $m > 1$ we apply the same procedure. All variables are presented as

$$((x_1 : y_1), \ldots, (x_n : y_n)) = f((x_1 : y_1), \ldots, (x_p : y_p))$$

where each component of f is a composition by some maps as (3.8). So they are homogeneous polynomial of $(x_1 : y_1), \ldots, (x_p : y_p)$. Let $U := (\mathbf{CP}^1)^p - \{((x_1 : y_1), \ldots, (x_p : y_p)) : \text{for some } 1 \leqslant i \leqslant p, (x_i : y_i) \text{ is irregular of some equation}$

of (1)}. By Lemma 3.6, U is an open connected set of $(\mathbf{CP}^1)^p$. By Lemma 3.7, f defines an embedding from $U \subset (\mathbf{CP}^1)^p$ to $(\mathbf{CP}^1)^n$. Since U is irreducible, so is $f(U)$ irreducible. Observe that $(0:1)^n = ((0:1),\dots,(0:1))$ and $(1:0)^n = ((1:0),\dots,(1:0))$ are contained in $f(U)$. We deduce that $(0:1)^n$ and $(1:0)^n$ are in the same irreducible component of $\{P_1 = 0,\ P_2 = 0,\dots, P_{c(\mathscr{A}_1)} = 0\}$. Recall that irreducible variety minus a subvariety is still a connected set. If $((x_1:y_1),\dots,(x_n:y_n)) = ((1:0),\dots,(1:0))$ (respectively $((0:1),\dots,(0:1))$ then $\mathscr{A}^*$ is $\mathscr{A}_0^*$ (respectively $\mathscr{A}_1^*$). Therefore condition (3.22) is satisfied at these two points, so there is a curve from $((1:0),\dots,(1:0))$ to $((0:1),\dots,(0:1))$ such that (3.2) is satisfied for any point lying in the curve. This means that we have constructed a one-parameter family of arrangements $\mathscr{A}^*(t)$ such that $\mathscr{A}^*(0) = \mathscr{A}_0^*$, $\mathscr{A}^*(1) = \mathscr{A}_1^*$ and $L(\mathscr{A}(t)) \equiv L(\mathscr{A}_0)$ for all $t \in [0,1]$.

We now define a stratification of $\mathbf{CP}^2 \times [0,1]$ which consists of three strata W, X and Y only. Let Y be $\{(p,t) \in \mathbf{CP}^2 \times [0,1]$: p is a point of multiplicity $k \geqslant 2$ in $\mathscr{A}^*(t)\}$, X be $\{(p,t) \in \mathbf{CP}^2 \times [0,1]$: p is a point of multiplicity one in $\mathscr{A}^*(t)\}$ and $W = \mathbf{CP}^2 \times [0,1] - \{(p,t) \in \mathbf{CP}^2 \times [0,1]$: p is a point of $\mathscr{A}^*(t)\}$. We can think of $X \cup Y$ as a total space of the family of plane curve singularities $|\mathscr{A}^*(t)|$ (= union of hyperplanes of $\mathscr{A}^*(t)$) in $\mathbf{CP}^2$. Since $L(\mathscr{A}^*(t))$ is isomorphic to $L(\mathscr{A}_0^*)$ for all t, we see easily that this family of plane singularities is a μ^*-constant family. In view of a theorem of Teissier [Te], the stratification satisfies the Whitney condition. Consider $\mathbf{CP}^2 \times [0,1]$ together with the projection map to the second factor. This map is proper since $\mathbf{CP}^2$ is compact. It is also a submersion. Moreover its restriction is a submersion on each stratum. Now we apply Thom's first isotopy theorem (proved by Mather [Ma]) to finish the proof of our Main Theorem. For the convenience of the reader, we recall the statement of Thom's first isotopy theorem which can be found for instance in [Go-Mac].

THOM'S FIRST ISOTOPY THEOREM. *Let $f: Z \to \mathbf{R}^n$ be a proper, smooth map which is a submersion on each stratum of a Whitney stratification of Z. Then there is a stratum-preserving homeomorphism*

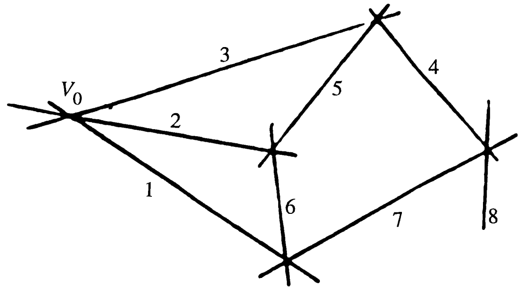
$$h: Z \to \mathbf{R}^n \times (f^{-1}(0) \cap Z).$$

which is smooth on each stratum and commutes with the projection to $\mathbf{R}^n$. In particular, the fibres of f are homeomorphic by a stratum-preserving homeomorphism. □

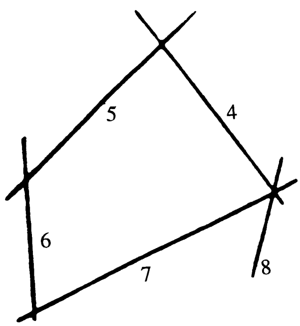
4. Examples

In this section, we shall show an example of an arrangement which is not nice, but the statement of our Main Theorem is still true.

EXAMPLE 4.1. Let G be the following graph



G consists of 8 lines and 5 triple points



$G' = G - (E(\mathrm{St}(V_0)) \cup \{V_0\})$.

Clearly $\mathscr{A}^*$ is not a nice arrangement. However the proof of our Main Theorem still works if we can show that $((1:0), \ldots, (1:0))$ and $((0:1), \ldots, (0:1))$ in $(\mathbf{CP}^1)^8$ is in the same irreducible component of the following variety defined by the following equations

$$a_0 y_1 y_2 x_3 + b_0 y_1 x_2 x_3 + c_0 x_1 y_2 x_3 + \alpha_0 x_1 x_2 y_3 + \beta_0 x_1 y_2 y_3 + \gamma_0 y_1 x_2 y_3 = 0 \quad (4.1)$$

$$a_1 y_3 y_4 x_5 + b_1 y_3 x_4 x_5 + c_1 x_3 y_4 x_5 + \alpha_1 x_3 x_4 y_5 + \beta_1 x_3 y_4 y_5 + \gamma_1 y_3 x_4 y_5 = 0 \quad (4.2)$$

$$a_2 y_2 y_5 x_6 + b_2 y_2 x_5 x_6 + c_2 x_2 y_5 x_6 + \alpha_2 x_2 x_5 y_6 + \beta_2 x_2 y_5 y_6 + \gamma_2 y_2 x_5 y_6 = 0 \quad (4.3)$$

$$a_3 y_1 y_6 x_7 + b_3 y_1 x_6 x_7 + c_3 x_1 y_6 x_7 + \alpha_3 x_1 x_6 y_7 + \beta_3 x_1 y_6 y_7 + \gamma_3 y_1 x_6 y_7 = 0 \quad (4.4)$$

$$a_4 y_4 y_7 x_8 + b_4 y_4 x_7 x_8 + c_4 x_4 y_7 x_8 + \alpha_4 x_4 x_7 y_8 + \beta_4 x_4 y_7 y_8 + \gamma_4 y_4 x_7 y_8 = 0 \quad (4.5)$$

Let $(x_1 : y_1)$, $(x_2 : y_2)$ and $(x_4 : y_4)$ be regular values of (4.1), …, (4.5). Thus we have from (4.1), …, (4.5) respectively the following

$$(x_3:y_3)=(-(\alpha_0x_1x_2+\beta_0x_1y_2+\gamma_0y_1x_2):(a_0y_1y_2+b_0y_1x_2+c_0x_1y_2)) \quad (4.6)$$

$$(x_5:y_5)=(-(\alpha_1x_3x_4+\beta_1x_3y_4+\gamma_1y_3x_4):(a_1y_3y_4+b_1y_3x_4+c_1x_3y_4)) \quad (4.7)$$

$$(x_6:y_6)=(-(\alpha_2x_2x_5+\beta_2x_2y_5+\gamma_2y_2x_5):(a_2y_2y_5+b_2y_2x_5+c_2x_2y_5)) \quad (4.8)$$

$$(x_7:y_7)=(-(\alpha_3x_1x_6+\beta_3x_1y_6+\gamma_3y_1x_6):(a_3y_1y_6+b_3y_1x_6+c_3x_1y_6)) \quad (4.9)$$

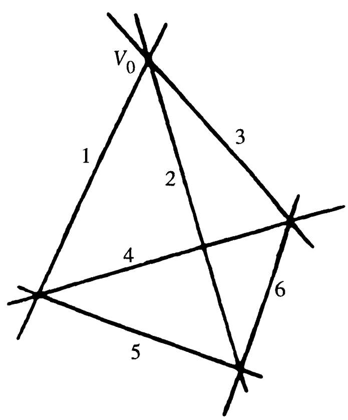
$$(x_8:y_8)=(-(\alpha_4x_4x_7+\beta_4x_4y_7+\gamma_4y_4x_7):(a_4y_4y_7+b_4y_4x_7+c_4x_4y_y)) \quad (4.10)$$

Let $V=\{((x_1:y_1),(x_2:y_2),(x_4:y_4))\in(\mathbf{CP}^1)^3:(x_1:y_1),(x_2:y_2)$ and $(x_4:y_4)$ are regular of (4.1)–(4.5)$\}$. Since for each i, $(x_i:y_i)$ has at most 2 irregular values for each equation. So V is a connected open subset of $(\mathbf{CP}^1)^3$ by taking away finite number of planes and lines from $(\mathbf{CP}^1)^3$. Let $W=\{((x_1:y_1),\ldots,(x_8:y_8))\in(\mathbf{CP}^1)^8$: (4.6)–(4.10) are satisfied and $((x_1:y_1),(x_2:y_2),(x_4:y_4))\in V\}$. Observe that V is homeomorphic to W by the map defined by (4.6)–(4.10). It follows that W is an irreducible variety containing $((0:1),\ldots,(0:1))$ *and* $((1:0),\ldots,(1:0))\in(\mathbf{CP}^1)^8$. So the Main Theorem is still true for this arrangement, i.e. $\mathscr{A}^*$ is a good arrangement.

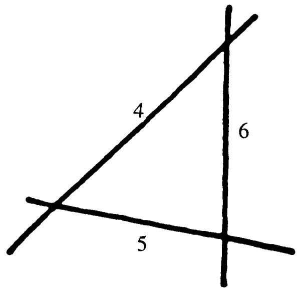
REMARK. Each expression of (4.7)–(4.10) can be written in terms of $(x_1:y_1)$, $(x_2:y_2)$ and $(x_4:y_4)$ by successive substitution of each previous expression.

In the next example, we shall associate to a projective arrangement in $\mathbf{CP}^2$ two varieties in $(\mathbf{CP}^1)^6$. We shall show that these two varieties have 8 irreducible components. In the first variety, $((1:0),\ldots,(1:0))$ and $((0:1),\ldots,(0:1))$ are in the same irreducible component, while in the second variety $((1:0),\ldots,(1:0))$ and $((0:1),\ldots,(0:1))$ are in different irreducible components.

EXAMPLE 4.2. Let $\mathscr{A}^*$ be a projective arrangement with the following graph G.



G consists of 6 lines and 4 triple points



$G' = G - (E(\mathrm{St}(v_0) \cup \{v_0\})$

$$\left.\begin{aligned}
&a_0y_1x_2x_3+b_0x_1y_2x_3+c_0x_1x_2y_3+\alpha_0x_1y_2y_3+\beta_0y_1x_2y_3+\gamma_0y_1y_2x_3=0\\
&a_1y_1x_4x_5+b_1x_1y_4x_5+c_1x_1x_4y_5+\alpha_1x_1y_4y_5+\beta_1y_1x_4y_5+\gamma_1y_1y_4x_5=0\\
&a_2y_2x_5x_6+b_2x_2y_5x_6+c_2x_2x_5y_6+\alpha_2x_2y_5y_6+\beta_2y_2x_5y_6+\gamma_2y_2y_4x_6=0\\
&a_3y_3x_6x_4+b_3x_3y_6x_4+c_3x_3x_6y_4+\alpha_3x_3y_6y_4+\beta_3y_3x_6y_4+\gamma_3y_3y_6x_4=0
\end{aligned}\right\} \quad (4.1)$$

(I). The simplest example is that we take $a_i = b_i = c_i = \alpha_i = \beta_i^i = \gamma_i = 1$ ($i=0$, 1, 2, 3) in (4.11).

If $(x_3 : y_3)$ is irregular of (2), then Lemma 3.5 says that either $(x_1 : y_1)$ *or* $(x_2 : y_2)$ is irregular of (2). Therefore if $(x_1 : y_1)$ and $(x_2 : y_2)$ are regular of (2), so is $(x_3 : y_3)$. Thus we have

$$x_i^2 + y_i^2 + x_iy_i \neq 0 \quad \text{for } i = 1, 2, 3 \tag{4.12}$$

i.e. $(x_i : y_i) \neq (\alpha : 1)$ or $(\beta : 1)$ for $i = 1, 2, 3$ where

$$\alpha = \frac{-1+\sqrt{3}i}{2}, \quad \beta = \frac{-1-\sqrt{3}i}{2}$$

Observe that $\alpha\beta = 1$, $\alpha + \beta = -1$. Write equation (4.11) as follows

$$\begin{aligned}
&(x_3 : y_3) = (-x_1x_2 - x_1y_2 - y_1x_2 : y_1y_2 + y_1x_2 + x_1y_2)\\
&\begin{pmatrix} x_4 \\ y_4 \end{pmatrix} = k_1A_1\begin{pmatrix} x_5 \\ y_5 \end{pmatrix}, \begin{pmatrix} x_5 \\ y_5 \end{pmatrix} = k_2A_2\begin{pmatrix} x_6 \\ y_6 \end{pmatrix}, \begin{pmatrix} x_6 \\ y_6 \end{pmatrix} = k_3A_3\begin{pmatrix} x_4 \\ y_4 \end{pmatrix}
\end{aligned} \tag{4.13}$$

where $k_i \in \mathbf{C}\backslash\{0\}$ and

$$A_i = \begin{pmatrix} -(x_i + y_i) & -x_i \\ y_i & x_i + y_i \end{pmatrix} \quad i = 1, 2, 3.$$

Each $\|A_i\| \neq 0$ by (4.12). It follows from (4.13) that

$$\begin{pmatrix} x_4 \\ y_4 \end{pmatrix} = kA \begin{pmatrix} x_4 \\ y_4 \end{pmatrix}$$

where k is some number if $\mathbf{C}\backslash\{0\}$ and

$$A = (x_1^2 + y_1^2 + x_1 y_1) \begin{pmatrix} x_2^2 - y_2^2 & x_2^2 + 2x_2 y_2 \\ y_2^2 + 2x_2 y_2 & -(x_2^2 - y_2^2) \end{pmatrix}.$$

For a solution $(x_4 : y_4) \in \mathbf{CP}^1$, we must have $\det(kA - I) = 0$. So

$$k = \pm\frac{1}{\Delta} \quad \text{where } \Delta = (x_1^2 + y_1^2 + x_1 y_1)(x_2^2 + y_2^2 + x_2 y_2).$$

For $k = 1/\Delta$, we have a solution of (4.11)

$$\left.\begin{aligned} &(x_4 : y_4) = (x_2 : y_2) \\ &(x_3 : y_3) = (x_5 : y_5) = (-x_1 x_2 - x_1 y_2 - y_1 x_2 : y_1 y_2 + x_1 y_2 + y_1 x_2) \\ &(x_6 : y_6) = (x_1 : y_1) \end{aligned}\right\} \qquad (4.14)$$

It is easy to check that the solution (4.14) of (4.11) is valid for all $((x_1 : y_1), (x_2 : y_2)) \in (\mathbf{CP}^1)^2 - \{P_1, P_2\}$ where $P_1 = ((\alpha : 1), (\beta : 1))$ and $P_2 = ((\beta : 1), (\alpha : 1))$. This solution set (4.14) is isomorphic to $(\mathbf{CP}^1)^2 - \{P_1, P_2\}$. It contains $((0:1), \ldots, (0:1))$ and $((1:0), \ldots, (1:0))$ *of* $(\mathbf{CP}^1)^6$. We denote this solution set U_1'.

For $k = -1/\Delta$, we have another solution set of (4.11) denoted by U_2'

$$\begin{aligned} &(x_3 : y_3) = (-x_1 x_2 - x_1 y_2 - y_1 x_2 : y_1 y_2 + y_1 x_2 + \ + x_1 y_2) \\ &(x_4 : y_4) = (-x_2 - 2y_2 : 2x_2 + y_2) \\ &(x_5 : y_5) = (x_1 x_2 - x_1 y_2 - y_1 x_2 - 2y_1 y_2 : y_1 y_2 - y_1 x_2 - x_1 y_2 - 2x_1 x_2) \\ &(x_6 : y_6) = (-x_1 - 2y_1 : 2x_1 + y_1) \end{aligned} \qquad (4.15)$$

for all $((x_1 : y_1), (x_2 : y_2)) \in (\mathbf{CP}^1)^2 - \{P_1, P_2\}$.

For $(x_1:y_1)$ or $(x_2:y_2)$ being fixed as irregular of (4.11) we get all other six solution sets of (2) in $(\mathbf{CP}^1)^6$.

$$U_3 = \{(x_1:y_1) = (x_6:y_6) = (\alpha:1), (x_2:y_2) = (x_4:y_4) = (\beta:1)\}$$
$$U_4 = \{(x_1:y_1) = (x_6:y_6) = (\beta:1), (x_2:y_2) = (x_4:y_4) = (\alpha:1)\}$$
$$U_5 = \{(x_1:y_1) = (x_6:y_6) = (\alpha:1), (x_3:y_3) = (x_5:y_5) = (\beta:1)\}$$
$$U_6 = \{(x_1:y_1) = (x_6:y_6) = (\beta:1), (x_3:y_3) = (x_5:y_5) = (\alpha:1)\}$$
$$U_7 = \{(x_2:y_2) = (x_4:y_4) = (\alpha:1), (x_3:y_3) = (x_5:y_5) = (\beta:1)\}$$
$$U_8 = \{(x_2:y_2) = (x_4:y_4) = (\beta:1), (x_3:y_3) = (x_5:y_5) = (\alpha:1)\}.$$

Each of these six sets is isomorphic to $(\mathbf{CP}^1)^2$. So they are irreducible components. Furthermore, if we define

$$V_1 = \{(x_3:y_3) = (x_5:y_5)\} \cap (U_3 \cup U_4),$$
$$V_2 = \left\{(x_5:y_5) = (x_3:y_3)\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} \cap (U_3 \cup U_4)\right\}$$

in $(\mathbf{CP}^1)^6$. Let $U_1 = U'_1 \cup V_1$ and $U_2 = U'_2 \cup V_2$. Then both U_1 and U_2 are irreducible components of the algebraic set defined by (4.11). For the proof of the last statement we consider U'_1 which is isomorphic to $(\mathbf{CP}^1)^2 - \{P_1, P_2\}$. Each element of V_1 *is in the closure of* U'_1 since

$$(x_3:y_3) = (x_5:y_5) = (-x_1x_2 - x_2y_1 - x_1y_2 : y_1y_2 + x_1y_2 + y_1x_2)$$

in U'_1 so U_1 is an irreducible component defined by the following equations:

$$x_2y_4 - y_2x_4 = 0$$
$$x_1y_6 - y_1x_6 = 0$$
$$x_3y_5 - y_3x_5 = 0$$
$$(y_1y_2 + x_1y_2 + x_1y_1)x_3 + (x_1x_2 + x_1y_2 + x_2{}^2y_1)y_3 = 0.$$

Similarly we can show that U_2 is also an irreducible component.

The connection among those eight irreducible components can be expressed by the following configurations.

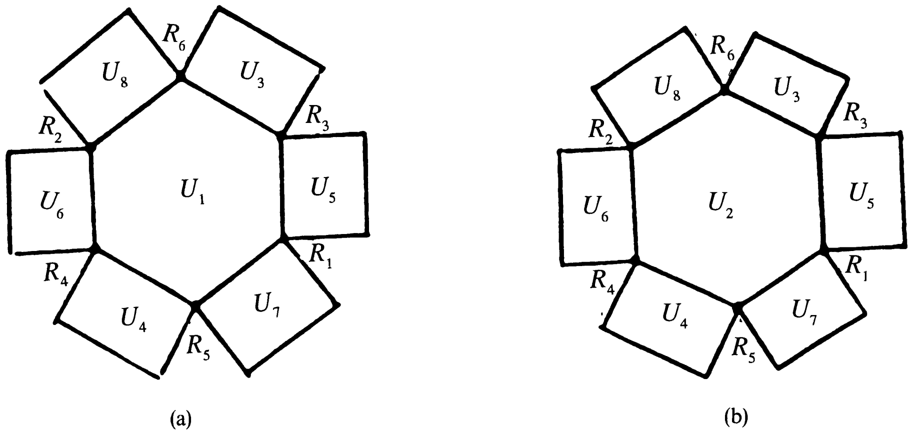


Fig. 4.1.

Figure 4.1(a) and (b) indicates that $U_1 \cap U_2 = \{R_1, R_2, \ldots, R_6\}$ where the six points are as follows:

$$U_5 \cap U_7 = R_1$$
$$= \{(x_1 : y_1) = (x_2 : y_2) = (x_4 : y_4) = (x_6 : y_6) = (\alpha : 1),\ (x_3 : y_3) = (x_5 : y_5) = (\beta : 1)\}$$
$$U_6 \cap U_8 = R_2$$
$$= \{(x_1 : y_1) = (x_2 : y_2) = (x_4 : y_4) = (x_6 : y_6) = (\beta : 1),\ (x_3 : y_3) = (x_5 : y_5) = (\alpha : 1)\}$$
$$U_3 \cap U_5 = R_3$$
$$= \{(x_1 : y_1) = (x_6 : y_6) = (\alpha : 1),\ (x_2 : y_2) = (x_3 : y_3) = (x_4 : y_4) = (x_5 : y_5) = (\beta : 1)\}$$
$$U_4 \cap U_6 = R_4$$
$$= \{(x_1 : y_1) = (x_6 : y_6) = (\beta : 1),\ (x_2 : y_2) = (x_3 : y_3) = (x_4 : y_4) = (x_5 : y_5) = (\alpha : 1)\}$$
$$U_4 \cap U_7 = R_5$$
$$= \{(x_2 : y_2) = (x_4 : y_4) = (\alpha : 1),\ (x_1 : y_1) = (x_6 : y_6) = (x_3 : y_3) = (x_5 : y_5) = (\beta : 1)\}$$
$$U_3 \cap U_8 = R_6$$
$$= \{(x_2 : y_2) = (x_4 : y_4) = (\beta : 1),\ (x_1 : y_1) = (x_6 : y_6) = (x_3 : y_3) = (x_5 : y_5) = (\alpha : 1)\}.$$

The intersection of U_i ($i = 1$ or 2) and $U_j =$ ($j = 3, 4, 5$ or 6) is a line. We list them as follows:

$$U_1 \cap U_3 = \{(x_1 : y_1) = (x_6 : y_6) = (\alpha : 1), (x_2 : y_2) = (x_4 : y_4) = (\beta : 1), (x_3 : y_3) = (x_5 : y_5)\}$$

$$U_1 \cap U_4 = \{(x_1 : y_1) = (x_6 : y_6) = (\beta : 1), (x_2 : y_2) = (x_4 : y_4) = (\alpha : 1), (x_3 : y_3) = (x_5 : y_5)\}$$

$$U_2 \cap U_3 = \left\{(x_1 : y_1) = (x_6 : y_6) = (\alpha : 1), (x_2 : y_2) = (x_4 : y_4) = (\beta : 1), (x_5 : y_5) = (x_3 : y_3) \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}\right\}$$

$$U_2 \cap U_4 = \left\{(x_1 : y_1) = (x_6 : y_6) = (\beta : 1), (x_2 : y_2) = (x_4 : y_4) = (\alpha : 1), (x_5 : y_5) = (x_3 : y_3) \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}\right\}$$

$$U_1 \cap U_5 = \{(x_1 : y_1) = (x_6 : y_6) = (\alpha : 1), (x_3 : y_3) = (x_5 : y_5) = (\beta : 1), (x_2 : y_2) = (x_4 : y_4)\}$$

$$U_1 \cap U_6 = \{(x_1 : y_1) = (x_6 : y_6) = (\beta : 1), (x_3 : y_3) = (x_5 : y_5) = (\alpha : 1), (x_2 : y_2) = (x_4 : y_4)\}$$

$$U_2 \cap U_5 = \left\{(x_1 : y_1) = (x_6 : y_6) = (\alpha : 1), (x_3 : y_3) = (x_5 : y_5) = (\beta : 1), (x_4 : y_4) = (x_2 : y_2) \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}\right\}$$

$$U_2 \cap U_6 = \left\{(x_1 : y_1) = (x_6 : y_6) = (\beta : 1), (x_3 : y_3) = (x_5 : y_5) = (\alpha : 1), (x_4 : y_4) = (x_2 : y_2) \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}\right\}$$

$$U_1 \cap U_7 = \{(x_2 : y_2) = (x_4 : y_4) = (\alpha : 1), (x_3 : y_3) = (x_5 : y_5) = (\beta : 1), (x_1 : y_1) = (x_6 : y_6)\}$$

$$U_1 \cap U_8 = \{(x_2 : y_2) = (x_4 : y_4) = (\beta : 1), (x_3 : y_3) = (x_5 : y_5) = (\alpha : 1), (x_1 : y_1) = (x_6 : y_6)\}$$

$$U_2 \cap U_7 = \left\{(x_2 : y_2) = (x_4 : y_4) = (\alpha : 1), (x_3 : y_3) = (x_5 : y_5) = (\beta : 1), (x_6 : y_6) = (x_1 : y_1) \begin{pmatrix} -2 & 2 \\ -2 & 1 \end{pmatrix}\right\}$$

$$U_2 \cap U_8 = \left\{ (x_2 : y_2) = (x_4 : y_4) = (\beta : 1), (x_3 : y_3) = (x_5 : y_5) = (\alpha : 1), \right.$$

$$\left. (x_6 : y_6) = (x_1 : y_1) \begin{pmatrix} -2 & 2 \\ -2 & 1 \end{pmatrix} \right\}.$$

(II). The second example is that we take $a_i = -3$, $\beta_i = \gamma_i = -1$, $b_i = c_i = \alpha_i = 1$ for $i = 1, 2, 3$. Then we get the following P_r equations in the proof of Theorem 3.3

$$P_1 \quad y_1x_2x_3 + x_1y_2x_3 + x_1x_2y_3 + x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3 = 0$$

$$P_2 \quad -3y_1x_4x_5 + x_1y_4y_5 + x_1x_4y_5 + x_1y_4y_5 - y_1x_4y_5 - y_1y_4x_5 = 0$$

$$P_3 \quad -3y_2x_5y_6 + x_2y_5x_6 + x_2x_5y_6 + x_2y_5x_6 - y_2x_5y_6 - y_2y_5x_6 = 0$$

$$P_4 \quad -3y_3x_6x_4 + x_3y_6x_4 + x_3x_6y_4 + x_3y_6x_4 - y_3x_6y_4 - y_3y_6x_4 = 0.$$

To keep the same lattice, we want to take away the following varieties defined by the following Q_r equations in the proof of Theorem 3.3.

$$Q_1 \quad a_1x_1x_2x_4 + b_1y_1x_2x_4 + c_1x_1y_2x_4 + d_1x_1x_2y_4 + e_1x_1y_2y_4 + f_1y_1x_2y_4 + g_1y_1y_2x_4 + h_1y_1y_2y_4 = 0$$

$$Q_2 \quad a_2x_2x_3x_5 + b_2y_2x_3x_5 + c_2x_2y_3x_5 + d_2x_2x_3y_5 + e_3x_2y_3y_5 + f_2y_2x_3y_5 + g_2y_2y_3x_5 + h_2y_2y_3y_5 = 0$$

$$Q_3 \quad a_3x_3x_1x_6 + b_3y_3x_1x_6 + c_3x_3y_1x_6 + d_3x_3x_1y_6 + e_3x_3y_1y_6 + f_3y_3x_1y_6 + g_3y_3y_1x_6 + h_3y_3y_1y_6 = 0.$$

Similar computations as before shows that the algebraic set

$$W = \{P_i = 0, i = 1, 2, 3, 4\}$$

has eight irreducible components $W_1, W_2, \dots, W_8$ where each W_i is an irreducible component in $(\mathbf{CP}^1)^6$ of dimension 2. In fact the equations for W_1 and W_2 are given respectively

$$W_1 : \begin{cases} 2x_1x_6 + x_1y_6 + y_1x_6 = 0 \\ 2x_2x_4 + x_2y_4 + y_2x_4 = 0 \\ 2x_3x_5 + x_3y_5 + y_3x_5 = 0 \\ (y_1y_2 + x_1y_2 + x_2y_1)x_3 + (x_1x_2 + x_1y_2 + x_2y_1)y_3 = 0 \end{cases}$$

$$W_2: \begin{cases} x_1y_6 + 3y_1x_6 + 2y_1y_6 = 0 \\ x_2y_4 + 3y_2x_4 + 2y_2y_4 = 0 \\ x_3y_5 + 3y_3x_5 + 3y_3y_5 = 0 \\ (y_1y_2 + x_1y_2 + x_2y_2)x_3 + (x_1x_2 + x_1y_2 + x_2y_1)y_3 = 0. \end{cases}$$

Clearly $((0:1),\ldots,(0:1)) \in W_1$ and $((1:0,\ldots,(1:0)) \in W_2$.

We want to show that for some chosen coefficients in equations Q_i,

$$W - \bigcup_{i=1}^{3} \{Q_i = 0\}$$

becomes disconnected in such a way that there is no path joining $((1:0),\ldots,(1:0))$ and $((0:1),\ldots,(0:1))$. We first observe that $W_1 \cap W_2 = \{R_1, R_2,\ldots,R_6\}$ where

$$R_1:((x_1:y_1) = (x_2:y_2) = (\alpha:1), (x_3:y_3) = (\beta:1),$$
$$(x_4:y_4) = (x_6:y_6) = (-\alpha:2a+1), (x_5:y_5) = (-\beta:2\beta+1))$$
$$R_2:((x_1:y_1) = (x_2:y_2) = (\beta:1), (x_3:y_3) = (\alpha:1),$$
$$(x_4:y_4) = (x_6:y_6) = (-\beta:2\beta+1), (x_5:y_5) = (-\alpha:2\alpha+1))$$
$$R_3:((x_1:y_1) = (\alpha:1), (x_6:y_6) = (-\alpha:2\alpha+1),$$
$$(x_2:y_2) = (x_3:y_3) = (\beta:1), (x_4:y_4) = (x_5:y_5) = (-\beta:2\beta+1))$$
$$R_4:((x_1:y_1) = (\beta:1), (x_6:y_6) = (-\beta:2\beta+1),$$
$$(x_2:y_2) = (x_3:y_3) = (\alpha:1), (x_4:y_4) = (x_5:y_5) = (-\alpha:2\alpha+1))$$
$$R_5:((x_2:y_2) = (\alpha:1), (x_4:y_4) = (-\alpha:2\alpha+1),$$
$$(x_1:y_1) = (x_3:y_3) = (\beta:1), (x_5:y_5) = (x_6:y_6) = (-\beta:2\beta+1)$$
$$R_6:((x_2:y_2) = (\beta:1), (x_4:y_4) = (-\beta:2\beta+1),$$
$$(x_1:y_1) = (x_3:y_3) = (\alpha:1), (x_5:y_5) = (x_6:y_6) = (-\alpha:2\alpha+1)).$$

Let us take $a_i = 2$, $b_i = 1$, $c_i = 1$, $d_i = 1$, $e_i = 0$, $g_i = 2$, $h_i = 1$ in equations Q_1, Q_2, and Q_3. Then

$$\{Q_1 = 0\} \supseteq (W_1 \cap W_3) \cup (W_1 \cap W_4) \cup (W_2 \cap W_3) \cup (W_2 \cap W_4)$$
$$\{Q_2 = 0\} \supseteq (W_1 \cap W_7) \cup (W_1 \cup W_8) \cup (W_2 \cap W_7) \cup (W_2 \cap W_8)$$
$$\{Q_3 = 0\} \supseteq (W_1 \cap W_5) \cup (W_1 \cap W_6) \cup (W_2 \cap W_5) \cup (W_2 \cap W_6).$$

So $\{R_1, \ldots, R_6\} \subseteq \bigcup_{i=1}^{3} \{Q_i = 0\}$. The method in Theorem 3.3 does not provide an affirmative answer. However we do not know whether the coefficients so chosen can actually be realized in geometric situation.

In a forthcoming paper, we shall show that the diffeomorphic type of $M(\mathscr{A}^*)$ indeed depends only on the lattice $L(\mathscr{A})$ for a general class of $\mathscr{A}$ which includes Example 4.2 as a special case.

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